

Project 2

Bottle Rocket Modeling

Assigned: Oct 30, 2020

*Due: Presentation, slides, and code submitted to Canvas by 11:59 pm, **Friday, Dec. 4, 2020.***

1. Background and Introduction

1.1. Project Motivation

The goal of this project is to model the trajectory of the bottle rocket launch, using numerical integration of a system of ordinary differential equations. Numerical, computational, or mathematical model is a common tool used by engineers to understand or predict the behavior of a physical system. Further, engineers can use the results of their numerical simulations to determine how to best design the physical system.

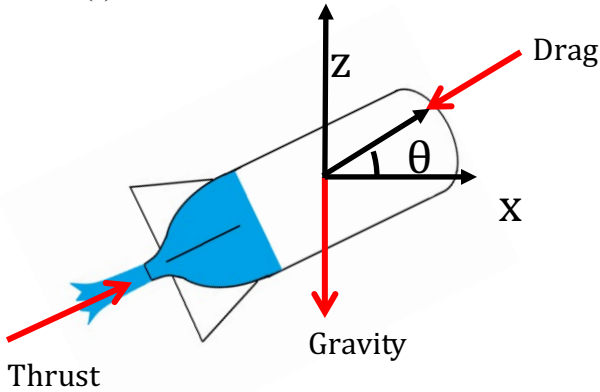
A bottle rocket is a very simple rocket and consists of a plastic bottle (typically a two-liter soda bottle) filled partially with a liquid (usually water) and pressurized by air. When the stopper is removed, the water is forced out leading to a reactionary force that propels the bottle according to Newton's laws of motion. The photograph below shows a typical bottle rocket launch.

To prepare you for the Bottle Rocket Design and Performance Analysis Lab in ASEN 2004 in spring, you will numerically simulate a bottle rocket flight trajectory to understand the functional dependence of bottle rocket performance on the design parameters, such as the volumetric fraction of water in the bottle, the initial pressure of air, drag of the rocket, and the launch angle. Your task is to use the knowledge you have gained to date to develop a MATLAB code to determine the bottle rocket thrust as a function of time, and predict the resulting height and range of the rocket. You are then asked to use the code to explore the parameter space in order to determine how each of the parameters affect the height and the range of the rocket, and what combination of parameters will allow the rocket to land within 1 meter of a 80meter marker.



1.2. Trajectory of a Bottle Rocket

To determine the bottle rocket trajectory, we will apply Newton's laws of motion, using a free body diagram. We will simplify this problem to only look at motion in 2 dimensions, in the horizontal (x) and vertical (z) directions:



$$\sum \text{Forces} = m_r \vec{a} = m_r \begin{bmatrix} a_x \\ a_z \end{bmatrix} = m_r \vec{V} = \vec{F} - \vec{D} + m_r \vec{g} \quad (1)$$

m_r is the mass of the rocket. \vec{a} is the acceleration vector of the rocket. \vec{F} is the thrust vector of the rocket (note that F is used instead of T to avoid confusion with temperature below). \vec{g} is the gravity vector, with only a z component equal to 9.8 m/s^2 . \vec{V} is the velocity vector of the rocket. Note that m_r , V , F , and D are all functions of time t .

To find the components of the F and D forces in the x and z directions, you can multiply the magnitude of the force by the heading. The heading of the rocket at any instant can be found by the velocity components at that instant:

$$\vec{h} = \frac{\vec{v}_h}{|\vec{v}_h|} \text{ or } \frac{v_x}{\sqrt{v_x^2 + v_z^2}} \text{ and } \frac{v_z}{\sqrt{v_x^2 + v_z^2}}$$

These components are equivalent to $\cos\theta$ and $\sin\theta$ respectively by trigonometry and can be used to determine the respective components of the forces acting on the rocket.

The magnitude of the drag force is a function of dynamic pressure $q = \frac{1}{2}\rho V^2$, where ρ is density and V is

the magnitude of the rocket's velocity, the drag coefficient C_D , and the cross-sectional area of the front of the bottle A_B :

$$D = q C_D A_B = \frac{1}{2} \rho V^2 C_D A_B \quad (2)$$

The drag coefficient, C_D , depends on a variety of factors, including the Reynolds number of the flow, any flow separation toward the rear of the bottle, the drag on the stabilizing surfaces and the shape of the nose cone, and is hard to determine theoretically. It is usually measured in a wind tunnel, but we can expect $C_D = 0.3$ to 0.5 .

The most important term in Eq. (1) is the thrust F . It can be estimated by applying the laws of thermodynamics and aerodynamics you have been taught to the expansion of air in the bottle (see Section

1.3). Note that the nomenclature “F” was chosen to reduce confusion with "T" used for temperature. We also need to keep track of the mass of the rocket, as water and later air is expelled through its mouth (we will call it the throat in conformity with rocket terminology).

1.3. Bottle Rocket Thermodynamics

Let p_{air}^i , v_{air}^i and T_{air}^i be the initial pressure, volume and temperature of air inside the bottle. Then the initial mass of air in the bottle is $m_{air}^i = \frac{p_{air}^i v_{air}^i}{RT_{air}^i}$ where $R = 287 \text{ J kg}^{-1} \text{ K}^{-1}$. The bottle rocket thrust phase can be divided into two phases: 1. Before the water is exhausted and 2. After the water is exhausted.

1.3.1. Before the water is exhausted

During this phase, the mass of air m_{air} remains constant but the air volume v increases as water is expelled, and therefore the air density is inversely proportional to its volume. We will assume the expansion of air during the rocket operation is isentropic, meaning the process is adiabatic (no heat transfer to or from the air mass) and there are no frictional losses. This is a good approximation and the air pressure p at any future time t is then given by:

$$\frac{p}{p_{air}^i} = \left(\frac{v_{air}^i}{v} \right)^g \quad (3)$$

where the specific heat ratio $g = 1.4$.

The mass flow rate of water out the throat of the bottle is:

$$\dot{m} = c_d \rho_w A_t V_e \quad (4)$$

where ρ_w is the density of water, V_e is the velocity of the exhaust, A_t is the throat area and c_d is the discharge coefficient, which is less than 1. The thrust of a rocket (F) is given by:

$$F = \dot{m} V_e + (p_e - p_a) A_t \quad (5)$$

where p_a is the ambient pressure and p_e is the pressure at the exit. Since the water is incompressible, applying the Bernoulli equation for incompressible flows using the air pressure (p):

$$(p - p_a) = \frac{\rho_w}{2} V_e^2 \quad (6)$$

so that the exhaust velocity:

$$V_e = \sqrt{\frac{2(p - p_a)}{\rho_w}} \quad (7)$$

Since $p_e = p_a$ (the exit pressure equals the ambient air pressure), Eq. (5)'s second term is zero and the thrust, when plugging in Eq. (7) for the exit velocity (V_e) is:

$$F = \dot{m} V_e = 2 c_d A_t (p - p_a) \quad (8)$$

Note that the thrust is independent of the liquid density! The air pressure p decreases with time as the air volume expands and therefore the thrust decreases with time. The air pressure at time t can be computed if the volume is known from Eq. (3). The rate of change of volume of air with time is:

$$\frac{dv}{dt} = c_d A_t V_e = c_d A_t \sqrt{\frac{2(p-p_a)}{\rho_w}} = c_d A_t \sqrt{\frac{2}{\rho_w} \left(p_o \left(\frac{v_o}{v} \right)^g - p_a \right)} \quad (9)$$

This formulation involves first inserting Eq. (7) and then the relationship in Eq. (3), but now using initial condition nomenclature for the volume ($v_0 = v_{air}^i$) and air pressure ($p_0 = p_{air}^i$). From Eq. (9) the volume $v(t)$ of air can be determined. Eq. (9) needs to be solved with the initial condition $v = v_{air}^i$ at time $t = 0$.

Because of its nonlinear nature, this has to be done numerically using an ODE solver such as the 4th order Runge-Kutta. The integration stops when $v = v_B$, the volume of the bottle, and all the water has been expelled.

Eq. (3) can be used to determine $p(t)$. From this $F(t)$ can be determined from Eq. (8).

The rocket mass changes with time as the water mass flows out of the bottle, according to:

$$\dot{m}_r = -\dot{m} = -c_d \rho_w A_t V_e = -c_d A_t \sqrt{2 \rho_w (p - p_a)} \quad (10)$$

The initial mass of the rocket is the sum of the masses of the bottle, the initial pressurized air, and the initial water:

$$m_R^i = m_B + m_{water}^i + m_{air}^i \quad (11)$$

Substituting in, this can be formulated in terms of the density of the water (ρ_w), the initial volume of the water (v_{water}^i) which is the difference between the total volume of the bottle (v_B) and that of the air initially (v_{air}^i), and finally, the pressure and temperature of the air.

$$m_r^i = m_B + \rho_w (v_B - v_{air}^i) + \frac{p_{air}^i v_{air}^i}{RT_{air}^i} \quad (12)$$

1.3.2. After the water is exhausted:

Let p_{end} be the pressure and T_{end} be the temperature of air in the bottle at the time all the water is expelled:

$$p_{end} = p_{air}^i \left(\frac{v_{air}^i}{v_B} \right)^g ; T_{end} = T_{air}^i \left(\frac{v_{air}^i}{v_B} \right)^{g-1} \quad (13)$$

Once the water is exhausted, the volume of air remains constant but its mass decreases, and therefore the density is proportional to the mass. Again, we assume air expands isentropically, until the air pressure p drops to the ambient pressure p_a . Then the pressure at any time t is given by:

$$\frac{p}{p_{end}} = \left(\frac{m_{air}}{m_{air}^i} \right)^g \quad (14)$$

and the corresponding density and temperature are:

$$\rho = \frac{m_{air}}{v_B} ; T = \frac{p}{\rho R} \quad (15)$$

To determine the exit velocity of the air, define critical pressure:

$$p_* = p \left(\frac{2}{g+1} \right)^{g/(g-1)} \quad (16)$$

1. If $p_* > p_a$, the flow is choked (exit Mach number $M_e = 1$) and the exit velocity is:

$$V_e = \sqrt{gRT_e} \quad (17)$$

Where:

$$T_e = \left(\frac{2}{g+1}\right)T; \rho_e = \frac{p_e}{RT_e}; p_e = p_* \quad (18)$$

2. If $p_* < p_a$, the flow is not choked and the exit Mach number is obtained from:

$$\frac{p}{p_a} = \left(1 + \frac{g-1}{2}M_e^2\right)^{g/g-1} \quad (19)$$

With:

$$\frac{T}{T_e} = \left(1 + \frac{g-1}{2}M_e^2\right); \rho_e = \frac{p_a}{RT_e}; p_e = p_a \quad (20)$$

And the exit velocity is:

$$V_e = M_e \sqrt{gRT_e} \quad (21)$$

The thrust in both of the above cases is given by:

$$F = \dot{m}_{air} V_e + (p_a - p_e)A_t \quad (22)$$

Where:

$$\dot{m}_{air} = c_d \rho_e A_t V_e \quad (23)$$

During this phase, the rocket mass decreases according to:

$$\dot{m}_R = -\dot{m}_{air} = -c_d \rho_e A_t V_e \quad (24)$$

1.3.3. Ballistic Phase:

Thrust is generated by the bottle rocket until the air pressure in the bottle p falls to the ambient pressure p_a . After that the thrust is zero and the rocket enters its free ballistic phase under the influence of gravity:

$$F = 0 \text{ and } m_R \sim m_B. \quad (25)$$

The rocket is on a ballistic trajectory for the rest of the flight (until the rocket hits the ground), with initial conditions corresponding to those at the end of the thrust phase.

As you can see, the bottle rocket flight consists of three distinct phases: 1. From the moment the stopper is removed until the water is exhausted, 2. After the water is exhausted until the air pressure drops to the ambient value and the thrust phase ends, and 3. Ballistic phase. The first two comprise the thrust phase, which is usually a small fraction of the total flight time.

2. Your Assignment: Determine the Rocket Trajectory

Your assignment is to determine the flight parameters that will allow the bottle rocket to land within 1 meter of an 80-meter marker. Please include a plot of the trajectory and a plot of your thrust profile with time over the flight, and markers on the plot to indicate where the transition between the three phases of flight occur. The rocket trajectory in the first phase of flight can be estimated using Eqs. (1) and (8), in the second phase of flight using Eqs. (1) and (22), and Eqs. (1) and (25) in the ballistic phase of flight. To convert force to acceleration, you must also keep track of rocket mass using Eq. (10) and (24). To launch, the rocket is set

on a test stand, which guides the rocket during the initial portion of flight to ensure a straight trajectory. The initial angle of the test stand is the launch angle θ^i and the initial velocity $\vec{V}^i = 0$. You may assume the test stand is 0.5m in length, and after the rocket clears the test stand it is free to change heading. Putting the rocket on the test stand elevates it off the ground by 0.25m.

To check your code, you will be given a “Test Case”, showing the range and height for particular parametric combination. You should aim to replicate the flight trajectory (z vs. x position) and thrust profile (F vs. time). Once you have reproduced this result, you will know your code is working and you can then move on to explore the parameter space.

The performance of the bottle rocket depends on four parameters: p_{air}^i the initial pressure of air (the limit being the burst pressure of the bottle, with some factor of safety), the initial volume fraction of water (or equivalently initial mass), the drag coefficient, and the launch angle. Part 2 of the assignment is to investigate the 4 design parameters and determine how varying each will affect range and height.

Note that you should do some independent research on water bottle rocket parameters to develop your code prior to matching your code to the “Test Case”. You are advised to use consistent units. We prefer SI units.

Submission details will be provided in the coming weeks.

Please note: All projects will be graded exactly as you submitted. There will be no exceptions to this policy, so please ensure your submission is accurate and completed on time. No late work will be accepted.

References

- Anderson, J. D., Jr., **Introduction to Flight**, 7th Ed., McGraw-Hill (2009).
Sutton, G. and Biblarz, O., **Rocket Propulsion Elements**, 8th Ed., Wiley (2010).