Contents

The Time Value of Money

Interest is the compensation received for lending a certain asset. For instance, suppose you deposit some money into a bank account for a year. Then, the bank can utilize that money for a year. To compensate you for this, it pays you some interest.

The asset being lent out is called the *capital*. Usually, both the capital and the interest are expressed in monetary terms. However, this is not necessary. For instance, a farmer may lend his tractor to a neighbor and receive 10% of the grain harvested in return. In this course, the capital is always expressed in money, and in that case, it is also referred to as the *principal*.

Simple Interest

Interest is the reward for lending the capital to somebody for a period of time. There are various methods for computing interest. As the name implies, *simple interest* is easy to understand, and that is the main reason why we discuss it here. The concept behind simple interest is that the amount of interest is the product of three quantities: the rate of interest, the principal, and the period of time. However, as we will see at the end of this section, simple interest suffers from a major limitation. For this reason, its practical use is limited.

Definition 1.1.1 (. The **simple interest** earned on a capital C lent over a period n years at a rate i per annum is niC.

Example 1.1.2. How much interest do you earn if you deposit Rs 1000 for two years in a savings account that pays simple interest at a rate of 9% per annum? And if you leave it in the account for only half a year?

Answer. If you leave it for two years, you earn

$$2 \times 0.09 \times 1000 = 180$$

rupees in interest. If you leave it for only half a year, then you earn

$$0.5 \times 0.09 \times 1000 = 45$$

rupees.

As this example shows, the rate of interest is usually quoted as a percentage; 9% corresponding to a factor of 0.09. Furthermore, you have to ensure that the rate of interest is quoted using the same time unit as the period. In this example, the period is measured in years, and the interest rate is quoted per annum ("per annum" is Latin for "per year"). These are the units that are used by default. In Section 1.5, we will consider other possibilities.

Example 1.1.3. Suppose you deposit £1000 in a savings account paying simple interest at 9% per annum for one year. Then, you withdraw the money with interest and deposit it for one year in another account paying simple interest at 9%. How much do you have in the end?

Answer. In the first year, you would earn £90 in interest, so you have £1090 after one year. In the second year, you earn £98.10 in interest, so you have £1188.10 (= £1090 + £98.10) at the end of two years.

Now compare Examples 1.1.2 and 1.1.3. The first example shows that if you invest £1000 for two years, the capital grows to £1180. But the second example shows that you can have £1188.10 by switching accounts after a year. Even better is to open a new account every month.

This inconsistency means that simple interest is not frequently used in practice. Instead, savings accounts in banks pay compound interest, which will be introduced in the next section. Nevertheless, simple interest is sometimes used, especially in short-term investments.

Exercises

1. How many days does it take for £1450 to accumulate to £1500 under 4% p.a. simple interest?
2. A bank charges simple interest at a rate of 7% p.a. on a 90-day loan of £1500. Compute the interest.

Compound Interest

Most bank accounts use *compound interest*. The idea behind compound interest is that in the second year, you should receive interest on the interest you earned in the first year. In other words, the interest you earn in the first year is combined with the principal, and in the second year, you earn interest on the combined sum.

What happens with the example from the previous section, where the investor deposited £1000 for two years in an account paying 9%, if we consider compound interest? In the first year, the investor would receive £90 interest (9% of £1000). This would be credited to his account, so he now has £1090. In the second year, he would get £98.10 interest (9% of £1090), so that he ends up with £1188.10; this is the same number as we found before. The capital is multiplied by 1.09 every year: $1.09 \times 1000 = 1090$ and $1.09 \times 1090 = 1188.10$.

More generally, the interest over one year is iC, where i denotes the interest rate and C the capital at the beginning of the year. Thus, at the end of the year, the capital has grown to $C + iC = (1+i)C$. In the second year, the principal is $(1+i)C$ and the interest is computed over this amount, so the interest is $i(1+i)C$ and the capital has grown to $(1+i)C + i(1+i)C = (1+i)^2C$. In the third year, the interest is $i(1+i)^2C$ and the capital has grown to $(1+i)^3C$.

This reasoning, which can be made more formal by using complete induction, leads to the following definition.

Definition 1.2.1 Under **Compound interest**, a capital C lent over a period n years at a rate i p.a. grows to $(1+i)^nC$.

Example 1.2.2. How much do you have after you put 1000 rupees for two years in a savings account that pays compound interest at a rate of 9% per annum? And if you leave it in the account for only half a year?

Answer. If you leave it in the account for two years, then at the end you have $(1+0.09)^2 \cdot 1000 = 1188.10$ rupees, as we computed above. If you leave it in the account for only half a year, then at the end you have $(1+0.09)^{1/2} \cdot 1000 = \sqrt{1.09} \cdot 1000 = 1044.03$ rupees (rounded to the nearest cents). This is 97 cents less than the 45 rupees interest you get if the account would pay simple interest at the same rate (see Example 1.1.2).

Example 1.2.3. Suppose that a capital of 500 dollars earns 150 dollars of interest in 6 years. What was the interest rate if compound interest is used? What if simple interest is used?

Answer. The capital accumulated to \$650, so in the case of compound interest we have to solve the rate i from the equation $(1+i)^6 \cdot 500 = 650 \Rightarrow (1+i)^6 = 1.3 \Rightarrow 1+i = 1.3^{1/6} = 1.044698$. Thus, the interest rate is 4.47%, rounded to the nearest basis point (a basis point is 0.01%). In the case of simple interest, the equation to solve is $6 \cdot i \cdot 500 = 150$, so $i = \frac{150}{6 \cdot 500} = 0.05$.

Example 1.2.4. How long does it take to double your capital if you put it in an account paying compound interest at a rate of 7.1%? What if the account pays simple interest?

Answer. The question is for what value of n does a capital C accumulate to $2C$ if $i = 0.075$. So we have to solve the equation $1.075^nC = 2C$. Taking logarithms: $\log(1.075^n) = \log(2) \Rightarrow n\log(1.075) = \log(2) \Rightarrow n = \frac{\log(2)}{\log(1.075)} = 9.58$. So, it takes 9.58 years to double your capital. The computation is simpler for simple interest. We have to solve the equation $n \cdot 0.075 \cdot C = C$, so $n = \frac{1}{0.075} = 13.33$, so with simple interest it takes 13.33 years to double your capital.

More generally, if the interest rate is i, then the time required to double your capital is $\frac{\log(2)}{\log(1+i)}$. We can approximate the denominator by $\frac{\log(1+i)}{i}$ for small i; this is the first term of the Taylor series of $\log(1+i)$ around $i = 0$ (note that, as is common in mathematics, "log" denotes the natural logarithm). Thus, we get $\frac{\log(2)}{i}$. If instead of the interest rate i we use the percentage $p = 100i$, and we approximate $\log(2) = 0.693$ by 0.72, we get $\frac{72}{p}$.

This is known as the *rule of 72*: To calculate how many years it takes you to double your money, you divide 72 by the interest rate expressed as a percentage. Let us return to the above example with a rate of 7.1%. We have $p = 7.1$ so we compute $\frac{72}{7.1} = 10.14$, which is very close to the actual value of 9.58 we computed before.

The rule of 72 can already be found in an Italian book from 1494: *Summa de Arithmetica* by Luca Pacioli. The use of the number 72 instead of 69.3 has two advantages: many numbers divide 72, and it gives a better approximation for rates above 4% (remember that the Taylor approximation is centered around $i = 0$; it turns out that it is slightly too small for rates of 5–10% and using 72 instead of 69.3 compensates for this).

Remember that with simple interest, you could increase the interest you earn by withdrawing your money from the account halfway. Compound interest has the desirable property that this does not make a difference. Suppose that you put your money m years in one account and then n years in another account, and that both account pay compound interest at a rate i. Then, after the first m years, your capital has grown to $(1+i)^m C$. You withdraw that and put it in another account for n years, after which your capital has grown to $(1+i)^n(1+i)^m C$. This is the same as what you would get if you had kept the capital in the same account for $m+n$ years, because $(1+i)^n(1+i)^m C = (1+i)^{m+n} C$.

This is the reason why compound interest is used so much in practice. Unless noted otherwise, interest will always refer to compound interest.

Exercises

1. The rate of interest on a certain bank deposit is 10% compounded annually. What is the effective annual rate of interest if the interest is compounded semiannually? Quarterly? Monthly?

2. You borrow \$10,000 for 6 months at 10% annual interest. How much do you have to repay if the interest is compounded monthly?

3. A certain bank offers you an interest rate of 4% compounded quarterly on a savings account. If you deposit \$1000 in this account, how much money will you have after 5 years?

Comparing simple and compound interest

Simple interest is defined by the formula interest $= niC$. Thus, in n years, the capital grows from C to $C + niC = (1 + ni)C$. Simple interest and compound interest compare as follows:

$$\begin{aligned} \text{simple interest: capital after } n \text{ years} &= (1 + ni)C \\ \text{compound interest: capital after } n \text{ years} &= (1 + i)^n C \end{aligned}$$

These formulas are compared in Figure 1.1. The left plot shows how a principal of 1 pound grows under interest at 9%. The dashed line is for simple interest and the solid curve for compound interest. We see that compound interest pays out more in the long term. A careful comparison shows that for periods less than a year, simple interest pays out more, while compound interest pays out more if the period is longer than a year. This agrees with what we found before. A capital of £1000, invested for half a year at 9%, grows to £1045 under simple interest and to £1044.03 under compound interest, while the same capital invested for two years grows to £1180 under simple interest and £1188.10 under compound interest. The difference between compound and simple interest gets bigger as the period gets longer.

This follows from the following algebraic inequalities: if i is positive, then

$$(1 + i)^n < 1 + ni \quad \text{if } n < 1,$$

$$n(1 + i) > 1 + ni \quad \text{if } n > 1.$$

These will not be proven here. However, it is easy to see that the formulas for simple and compound interest give the same results if $n = 0$ and $n = 1$.

Now consider the case $n = 2$. A capital C grows to $(1 + 2i)C$ under simple interest and to $(1 + i)^2 C = (1 + 2i + i^2)C$ under compound interest. We have $(1 + 2i + i^2)C > (1 + 2i)C$ (because C is positive), so compound interest pays out more than simple interest.

The right plot in Figure 1.1 shows the final capital after putting a principal of 1 pound away for five years at varying interest rates. Again, the dashed line corresponds to simple interest and the solid curve corresponds to compound interest. We see again that compound interest pays out more, as $n = 5$ is greater than 1. However, the plot also shows that the difference is smaller if the interest rate is small.

This can be explained with the theory of Taylor series. A capital C will grow in n years to $(1 + i)^n C$. The Taylor series of $f(i) = (1 + i)^n C$ around $i = 0$ is

$$f(0) + f'(0)i + \frac{1}{2}f''(0)i^2 + \cdots = C + niC + \frac{1}{2}n(n-1)i^2 C + \cdots .$$

The first two terms are $C + niC = (1 + ni)C$, which is precisely the formula for simple interest. Thus, you can use the formula for simple interest as an approximation for compound interest; this approximation is especially good if the rate of interest is small. Especially in the past, people often used simple interest instead of compound interest, notwithstanding the inconsistency of simple interest, to simplify the computations.

Discounting

The formula for compound interest relates four quantities: the capital C at the start, the interest rate i, the period n, and the capital at the end. We have seen how to calculate the interest rate (Example 1.2.3), the period (Example 1.2.4), and the capital at the end (Example 1.2.2). The one remaining possibility is covered in the next example.

Example 1.4.1. How much do you need to invest now to get Rs 2000 after five years if the rate of interest is 4.25%? **Answer.** One pound will accumulate to $(1 + 0.0425)^5 = 1.2313466$ in five years, so you need to invest $2000/1.2313466 = 1624.24$ pounds.

We say that Rs 1624.24 now is equivalent to £2000 in five years at a rate of 4.25%. We call Rs 1624.24 the present value and Rs 2000 the future value. When you move a payment forward in time, it accumulates; when you move it backward, it is discounted (see Figure 1.2).

Example 1.4.2. Suppose that the interest rate is 7%. What is the present value of a payment of €70 in a year's time? **Answer.** The discount factor is $v = 1/1.07 = 0.934579$, so the present value is $0.934579 \times 70 = 65.42$ euro (to the nearest cent).

Usually, interest is paid in arrears. If you borrow money for a year, then at the end of the year, you have to pay the money back plus interest. However, there are also some situations in which the interest is paid in advance. The rate of discount is useful in these situations, as the following example shows.

Example 1.4.3. Suppose that the interest rate is 7%. If you borrow €1000 for a year and you have to pay interest at the start of the year, how much do you have to pay? **Answer.** If interest were to be paid in arrears, then you would have to pay $0.07 \times 1000 = 70$ euros at the end of the year. However, you have to pay the interest one year earlier. As we saw in Example 1.4.2, the equivalent amount is $v \times 70 = 65.42$ euros.

There is another way to arrive at the answer. At the start of the year, you get €1000 from the lender, but you have to pay interest immediately, so in effect, you get less from the lender. At the end of the year, you pay €1000 back. The amount you should get at the start of the year should be equivalent to the €1000 you pay at the end of the year. The discount factor is $v = 1/1.07 = 0.934579$, so the present value of the €1000 at the end of the year is €934.58. Thus, the interest you have to pay is €1000 - €934.58 = €65.42.

In terms of the interest rate $i = 0.07$ and the capital $C = 1000$, the first method calculates ivC and the second method calculates $C - vC = (1 - v)C = dC$.

Both methods yield the same answer, so we arrive at the important relation $d = iv$.

We can check this relation algebraically. We found before, in equation (1.1), that the discount factor is

$$v = \frac{1}{1+i}.$$

The rate of discount is

$$d = 1 - v = 1 - \frac{1}{1+i} = \frac{i}{1+i}.$$

Comparing these two formulas, we find that indeed $d = iv$.

We summarize this discussion with a formal definition of the three quantities d, i, and v.

Definition 1.4.4. The rate of interest i is the interest paid at the end of a time unit divided by the capital at the beginning of the time unit. The rate of discount d is the interest paid at the beginning of a time unit divided by the capital at the end of the time unit. The discount factor v is the amount of money one needs to invest to get one unit of capital after one time unit.

This definition concerns periods of one year (assuming that time is measured in years). In Example 1.4.1, we found that the present value of a payment of £2000 due in five years is £1624.24 if compound interest is used at a rate of 4.25%. This was computed as $2000/(1 + 0.0425)^5$. The same method can be used to find the present value of a payment of C due in n years if compound interest is used at a rate i. The question is: which amount x accumulates to C in n years? The formula for compound interest yields that $(1 + i)^n x = C$, so the present value x is

$$x = \frac{C}{(1+i)^n} = vnC = (1 - d)^n C.$$

This is called compound discounting, analogous with compound interest.

There is another method, called simple discounting (analogous to simple interest) or commercial discounting. This is defined as follows. The present value of a payment of C due in n years, at a rate of simple discount of d, is $(1 - nd)C$.

Simple discounting is not the same as simple interest. The present value of a payment of C due in n years, at a rate of simple interest of i, is the amount x that accumulates to C over n years. Simple interest is defined by $C = (1 + ni)x$, so the present value is $x = (1 + ni)^{-1}C$.

Example 1.4.5. What is the present value of Rs 6000 due in a month assuming 8% p.a. simple discount? What is the corresponding rate of compound discount? And the rate of compound interest? And the rate of simple interest?

Answer. One month is $\frac{1}{12}$ year, so the present value is $(1-\frac{1}{12}\times 0.08)\times 6000 = 5960$ rupees. We can compute the rate of (compound) discount d from the formula "present value $= (1-d)^n C$":

$$5960 = (1-d)^{\frac{1}{12}} \times 6000 \Rightarrow (1-d)^{\frac{1}{12}} = \sqrt[12]{\frac{5960}{6000}} = 0.993333$$

$$\Rightarrow 1 - d = 0.993333^{12} = 0.922869 \Rightarrow d = 0.077131.$$

Thus, the rate of discount is 7.71%. The rate of (compound) interest i follows from $1+i = 1-d = 0.922869 \Rightarrow 1+i = 1.083577$, so the rate of (compound) interest is 8.36%. Finally, to find the rate of simple interest, solve $5960 = (1+\frac{1}{12}i)^{-1}\times 6000$ to get $i = 0.080537$, so the rate of simple interest is 8.05%.

One important application for simple discount is U.S. Treasury Bills. However, it is used even less in practice than simple interest.

Exercises

1. In return for a loan of £100 a borrower agrees to repay £110 after seven months.
   a. Find the rate of interest per annum.
   b. Find the rate of discount per annum.
   c. Shortly after receiving the loan the borrower requests that he be allowed to repay the loan by a payment of £50 on the original settlement date and a second payment six months after this date. Assuming that the lender agrees to the request and that the calculation is made on the original interest basis, find the amount of the second payment under the revised transaction.
2. The commercial rate of discount per annum is 18% (this means that simple discount is applied with a rate of 18%).
   a. We borrow a certain amount. The loan is settled by a payment of £1000 after three months.

Interest payable monthly, quarterly, etc.

Up to now, we assumed that interest is paid once a year. In practice interest is often paid more frequently, for instance quarterly (four times a year). This is straightforward if the interest rate is also quoted per quarter, as the following example shows.

Example 1.5.1. Suppose that you save £1000 in an account that pays 2% interest every quarter. How much do you have in one year, if the interest is paid in the same account?

Answer. We can use the formula for compound interest in Definition 1.2.1, which says that a capital C accumulates to $(1+i)^n C$ over a period n, if the rate is i. The rate $i = 0.02$ is measured in quarters, so we also have to measure the period n in quarters. One year is four quarters, so the capital accumulates to $1.02^4 \times 1000 = 1082.43$ rupees.

However, interest rates are usually not quoted per quarter even if interest is paid quarterly. The rate is usually quoted per annum (p.a.). In the above example, with 2% per quarter, the interest rate would be quoted as 8% p.a. payable quarterly. This rate is called the nominal interest rate payable quarterly. You may also see the words "convertible" or "compounded" instead of "payable".

It may seem more logical to quote the rate as 8.243%. After all, we computed that £1000 accumulates to £1082.43 in a year. The rate of 8.243% is called the effective interest rate. It often appears in advertisements in the U.K. as the Annual Equivalent Rate (AER). The effective interest rate corresponds to the interest rate i as defined in Definition 1.4.4: the interest paid at the end of a time unit divided by the capital at the beginning of the time unit.

Definition 1.5.2. The interest conversion period is the period between two successive interest payments. Denote the quotient of the time unit and the interest conversion period by p. Let $i_{[p]}$ denote the interest rate per conversion period. The nominal interest rate, denoted $i^{(p)}$, is then p times $i_{[p]}$.

Common values for p include $p = 365$ (interest payable daily) and $p = 12$ (interest payable monthly). The term "interest payable *pthly*" is used if we do not want to specify the conversion period. In the example, the interest conversion period is a quarter and the time unit is a year, so $p = 4$. The interest rate per quarter is 2%, meaning that $i_{[4]} = 0.02$, so the nominal interest rate is $i^{(4)} = 4 \times 0.02 = 0.08$ or 8%, and the effective interest rate is $i = 0.08243$.

To compute the effective interest rate from the nominal interest rate $i^{(p)}$, remember that the interest rate per conversion period is $i_{[p]} = i^{(p)}/p$. There are p conversion periods in a time unit. Thus, by the formula for compound interest, a capital C accumulates to $(1+i_{[p]})^p C = (1+i^{(p)}/p)^p C$ in a time unit. However, if the effective interest rate is i, then a capital C accumulates to $(1+i)C$ in a time unit. Thus, a nominal interest rate $i^{(p)}$ payable *pthly* corresponds to an effective interest rate i as follows:

$$(1+i) = \left(1 + \frac{i^{(p)}}{p}\right)^p.$$

Interest payable continuously

Suppose we are going to borrow \$50 for 3 years with an interest rate of 10%compounded quarterly.

Then after 3 years, we will have to pay back is

$$50 * \left(1 + \frac{0.1}{4}\right)^{3*4}$$

In general, a Principal, of 1, at time t=0, accumulates after n years at a nominal interest rate, $i^{(p)}$, compounded *pthly* per year, to FV where

$$FV = \left(1 + \frac{i^{(p)}}{p}\right)^{n*p}$$

Now, we want to see what happens when p goes to infinity; i.e. when we compound continuously.

Exponential limit

We know that

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e$$

So, if we let $x = \frac{p}{i^{(p)}}$, ans substitute in the formula for FV, then as p goes to infinity, x also goes to infinity. So, the future value at time, t is

$$FV = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e^{i^{(\infty)} * t}$$

We denote $i^{(\infty)}$ as the force of interest, δ.

Annuities and loans

An *annuity* is a sequence of payments with fixed frequency. The term “annuity” originally referred to annual payments (hence the name), but it is now also used for payments with any frequency. Annuities appear in many situations; for example, the monthly payments to repay a loan, the monthly interest payments from a bond investment, or the annual payments from a pension plan.

Annuities are the sum of a series of payments made at fixed intervals of time. In the context of loans, an annuity series refers to a series of equal-sized payments made at regular intervals to repay the original loan amount plus interest. These payments can be monthly, quarterly, annually, or at any other regular interval.

Annuities Immediate

Annuities Immediate are series of payments that are made at the end of each period. The future value and present value of annuities immediate are generally calculated using the Geometric progression formula:

$$1 + r + r^2 + \ldots + r^n = \sum_{k=0}^{n} r^k = \frac{r^{n+1} - 1}{r - 1}$$

Example 2.1.1: You deposit \$100 into a savings account at the end of each year for 8 years with an annual interest rate of 5%. What is the accumulated value of the annuity?

Solution: The first payment is made at the end of the first year and the last payment is eigth at the end of the eighth year. Thus, the first payment accumulates interest for seven years, so it grows to $(1 + 0.05)^7$ů$100 = 140.71$ USD. The second payment accumulates interest for six years, so it grows to 1.05^6ů$100 = 134.01$ USD. And so on, until the last payment which does not accumulate any interest. The accumulated value of the eight payments is

$$1.05^7 \times 100 + 1.05^6 \times 100 + \ldots + 100 =$$
$$100\left(1 + \ldots + 1.05^6 + 1.05^7\right) =$$
$$100 \sum_{k=0}^{7} 1.05^k.$$

This sum can be evaluated with the formula for a geometric sum. Substitute $r = 1.05$ and $n = 7$ in (2.1) to get

$$\sum_{k=0}^{7} 1.05^k = \frac{1.05^8 - 1}{1.05 - 1} = 9.5491.$$

Thus, the accumulated value of the eight payments is \$954.91.

In the above example, we computed the accumulated value of an annuity. More precisely, we considered an annuity with payments made at the end of every year. Such an annuity is called an annuity immediate (the term is unfortunate because it does not seem to be related to its meaning).

Definition 2.1.2. An *annuity immediate* is a regular series of payments at the end of every period. Consider an annuity immediate paying one unit of capital at the end of every period for n periods. The accumulated value of this annuity at the end of the nth period is denoted $s_{\overline{n}|}$.

The accumulated value depends on the interest rate i, but the rate is usually only implicit in the symbol $s_{\overline{n}|}$. If it is necessary to mention the rate explicitly, the symbol $s_{\overline{n}|i}$ is used.

Let us derive a formula for $s_{\overline{n}|}$. The situation is depicted in Figure 2.1. The annuity consists of payments of 1 at $t = 1, 2, \ldots, n$ and we wish to compute the accumulated value at $t = n$. The accumulated value of the first payment is $(1+i)^{n-1}$, the accumulated value of the second payment is $(1+i)^{n-2}$, and so on till the last payment which has accumulated value 1. Thus, the accumulated values of all payments together is

$$(1+i)^{n-1} + (1+i)^{n-2} + \ldots + 1 = \sum_{k=0}^{n-1} (1+i)^k$$

.

The formula for a geometric sum, cf. (2.1), yields

$$\sum_{k=0}^{n-1} (1+i)^k = \frac{(1+i)^n - 1}{(1+i) - 1}$$

.

We arrive at the following formula for the accumulated value of an annuity immediate:

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i} \qquad (2.2)$$

This formula is not valid if $i = 0$. In that case, there is no interest, so the accumulated value of the annuities is just the sum of the payments: $s_n = n$.

The accumulated value is the value of the annuity at $t = n$. We may also be interested in the value at $t = 0$, the present value of the annuity. This is denoted by a_n, as shown in Figure 2.1.

Definition 2.1.3. Consider an annuity immediate paying one unit of capital at the end of every period for n periods. The value of this annuity at the start of the first period is denoted $a_{\overline{n}|}$.

A formula for $a_{\overline{n}|}$ can be derived as above. The first payment is made after a year, so its present value is the discount factor $v = \frac{1}{1+i}$. The present value of the second value is v^2, and so on till the last payment which has a present value of v^n. Thus, the present value of all payments together is

$$v + v^2 + \ldots + v^n = v(1 + v + \ldots + v^{n-1}) = v\sum_{k=0}^{n-1} v^k$$

.

Now, use the formula for a geometric sum:

$$v\sum_{k=0}^{n-1} v^k = v\frac{v^n - 1}{v - 1} = \frac{v}{1-v}(1 - v^n)$$

The fraction $\frac{v}{1-v}$ can be simplified to $\frac{1}{i}$ and we get the present value of an annuity immediate is

$$a_{\overline{n}|} = \frac{1 - v^n}{i} \qquad (2.3)$$

The formulas (2.2) and (2.3) are due to Jacob Bernoulli (1654-1705). Similar to equation (2.2) for $s_{\overline{n}|}$, the equation for $a_{\overline{n}|}$ is not valid for $i = 0$, in which case $a_{\overline{n}|} = n$.

There is a simple relation between the present value $a_{\overline{n}|}$ and the accumulated value $s_{\overline{n}|}$. They are value of the same sequence of payments, but evaluated at different times: $a_{\overline{n}|}$ is the value at $t = 0$ and $s_{\overline{n}|}$ is the value at $t = n$ (see Figure 2.1). Thus, $a_{\overline{n}|}$ equals $s_{\overline{n}|}$ discounted by n years:

$$a_{\overline{n}|} = v^n s_{\overline{n}|} \qquad (2.4)$$

This relation is easily checked. According to (2.2), the right-hand side evaluates to

$v^n s_{\overline{n}|} = v^n \left(\frac{(1+i)^n-1}{i}\right)$

$= v^n \frac{v^n(1+i)^n-v^n}{i}$

$= \frac{1-v^n}{i}$

$= a_{\overline{n}|}$

where the last-but-one equality follows from $v = \frac{1}{1+i}$ and the last equality from (2.3). This proves (2.4).

One important application of annuities is the repayment of loans. This is illustrated in the following example.

Example 2.1.4. A loan of €2500 at a rate of 6.5% p.a. is paid off in ten years, by paying ten equal installments at the end of every year. How much is each installment?

Answer. Suppose that each installment is x euros. Then the loan is paid off by a 10-year annuity immediate. The present value of this annuity is $xa_{\overline{10}|0.065}$.

We compute $v = (1+i)^{-1} = 0.938967$ and

$a_{\overline{10}|0.06} = \frac{1-v^{10}}{i} = \frac{1-0.938967}{0.065} = 7.18830$

The present value should be equal to €2500, so the size of each installment is

$x = \frac{2500}{a_{\overline{10}|0.065}} = \frac{2500}{7.18830} = 347.7617$ euros. Rounded to the nearest cent, this is €347.76.

Every installment in the above example is used to both pay interest and pay back a part of the loan. This is studied in more detail in Chapter 3. Another possibility is to only pay interest every year, and to pay back the principal at the end. If the principal is one unit of capital which is borrowed for n years, then the borrower pays i at the end of every year and 1 at the end of the n years. The payments of i form an annuity with present value $ia_{\overline{n}|}$. The present value of the payment of 1 at the end of n years is v^n. These payments are equivalent to the payment of the one unit of capital borrowed at the start. Thus, we find

$$1 = ia_{\overline{n}|} + v^n$$

This gives another way to derive formula (2.3). Similarly, if we compare the payments at $t = n$, we find

$$(1+i)^n = is_{\overline{n}|} + 1$$

and (2.2) follows.

Exercise On 15 November in each of the years 1964 to 1979 inclusive an investor deposited £500 in a special bank savings account. On 15 November 1983 the investor withdrew his savings. Given that over the entire period, the bank used an annual interest rate of 7% for its special savings accounts, find the sum withdrawn by the investor.

Annuities Due

The previous section considered annuities immediate, in which the payments are made in arrears (that is, at the end of the year). Another possibility is to make the payments at advance. Annuities that pay at the start of each year are called annuities due.

Definition 2.2.1. An *annuity due* is a regular series of payments at the beginning of every period. Consider an annuity immediate paying one unit of capital at the beginning of every period for n periods. The value of this annuity at the start of the first period is denoted $\ddot{a}_{\overline{n}|}$, and the accumulated value at the end of the nth period is denoted $\ddot{s}_{\overline{n}|}$.

Both $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ are measured at $t = 0$, while $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ are both measured at $t = n$. The present value of an annuity immediate ($a_{\overline{n}|}$) is measured one period before the first payment, while the present value of an annuity due ($\ddot{a}_{\overline{n}|}$) is measured at the first payment. On the other hand, the accumulated value of an annuity immediate ($s_{\overline{n}|}$) is at the last payment, while the accumulated value of an annuity due ($\ddot{s}_{\overline{n}|}$) is measured one period after the last payment.

We can easily derive formulas for $\ddot{a}_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$. One method is to sum a geometric series. An annuity due consists of payments at $t = 0$, $t = 1$, . . . , $t = n - 1$, so its value at $t = 0$ is

$$\ddot{a}_{\overline{n}|} = 1 + v + \ldots + v^{n-1} = \sum_{k-0}^{n-1} v^k = \frac{1-v^n}{1-v} = \frac{1-v^n}{d} \tag{2.5}$$

The value at $t = n$ is

$$\ddot{s}_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + \ldots + (1+i) = \sum_{k=1}^{n}(1+i)^k = \frac{(1+i)^n - 1}{d} \tag{2.6}$$

If we compare these formulas with the formulas for $a_{\overline{n}|}$ and $s_{\overline{n}|}$, given in (2.3) and (2.2), we see that they are identical except that the denominator is d instead of i. In other words,

$$d\ddot{a}_{\overline{n}|} = ia_{\overline{n}|}$$

$$d\ddot{s}_{\overline{n}|} = is_{\overline{n}|}$$

There a simple explanation for this. An annuity due is an annuity immediate with all payments shifted one time period in the past (compare Figures 2.1 and 2.2). Thus, the value of an annuity due at $t = 0$ equals the value of an annuity immediate at $t = 1$. We know that an annuity immediate is worth $a_{\overline{n}|}$ at $t = 0$, so its value at $t = 1$ is $(1 + i)a_{\overline{n}|}$ and this is to equal $\frac{i}{d}a_{\overline{n}|} = \ddot{a}_{\overline{n}|}$. Similarly, $\ddot{s}_{\overline{n}|}$ is not only the value of an annuity due at $t = n$ but also the value of an annuity immediate at $t = n + 1$. Annuities immediate and annuities due refer to the same sequence of payments evaluated at different times.

There is another relationship between annuities immediate and annuities due. An annuity immediate over n years has payments at $t = 1, \ldots, t = n$ and an annuity due over $n+1$ years has payments at $t = 0, t = 1, \ldots, t = n$. Thus, the difference is a single payment at $t = 0$. It follows that

$$\ddot{a}_{\overline{n+1}|} = a_{\overline{n}|} + 1 \qquad (2.7)$$

Similarly, $\ddot{s}_{\overline{n}|}$ is the value at $t = n + 1$ of a series of $n + 1$ payments at times $t = 1, \ldots, n + 1$, which is the same as the value at $t = n$ of a series of $n + 1$ payments at $t = 0, \ldots, n$. On the other hand, $s^{\ddot{n}}$ is the value at $t = n$ of a series of n payments at $t = 0, \ldots, n - 1$. The difference is a single payment at $t = n$, so

$$s_{\overline{n+1}|} = \ddot{s}_{\overline{n}|} + 1 \qquad (2.8)$$

The relations (2.7) and (2.8) can be checked algebraically by substituting (2.2), (2.3), (2.5), and (2.6) in them.

There is an alternative method to derive the formulas for $\ddot{a}_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$, analogous to the discussion at the end of the previous section. Consider a loan of one unit of capital over n years, and suppose that the borrower pays interest *in advance* and repays the principal after n years. As discussed in Section 1.4, the interest over one unit of capital is d if paid in advance, so the borrower pays an annuity due of size d over n years and a single payment of 1 after n years. These payments should be equivalent to the one unit of capital borrowed at the start. By evaluating this equivalence at $t = 0$ and $t = n$, respectively, we find that

$$1 = d\ddot{a}_{\overline{n}|} + v^n \quad \text{and} \quad (1 + i)^n = d\ddot{s}_{\overline{n}|} + 1$$

and the formulas (2.5) and (2.6) follow immediately.

As a final example, we consider continuous annuities, which are annuities continuing payable continuously.

Continuous Annuities