Chapter 3

State space modelling and the Kalman filter

State space modelling was originated in 1960s. And in 1960, R.E.Kalman brought up the Kalman filter for the first time.[reference, kalman 1960] The state space model, also known as the hidden Markov model [reference, Rabiner 1989], is a powerful modelling method and applied widely in engineering, statistics, economics and etc. We shall introduce this model and explain it with some examples in section [*.1].

For SMMs or HMMs, there are many methods to extract hidden states from our observations and the Kalman filter is the most widely used one. The first two sections will introduce state space modelling briefly and give some examples. Section 3.3 shows the theory of how the Kalman filter works given a common state space model, which is also what we used in our research. Different from the Kalman filter, the particle filter is a more general method for SMMs. We will introduce this method briefly in the last section. But in our research problem, seasonal adjustment, we mainly focus on the Kalman filter instead of the latter.

3.1 Introduction to state space modelling

State space modelling was first proposed to solve the problems in the area of control theory in 1960s. Then in 1980s and 1990s, with the gradual development of related theories, this model became more and more popular.

For a state space model, the observation is usually composed by one or more components, which is called state in our model. For each state space model, both of the observation and the state could be multivariate or univariate. But in practice, at least in seasonal adjustment, we usually deal with cases in which the observation is univariate and the state space is multivariate. In SSMs, states are usually unobserved, and this is the reason why we call SSMs as hidden Markov models. In general cases, what we know about the whole system are our observed measurements, the relation between observations and states, and the relation of two adjacent states.

Figure 3.1 illustrates the pattern of SMMs vividly. In this figure, $y_{0:T}$ is the observation and $x_{0:T}$ is our hidden state, which behaves as a Markov chain, that is, the current state only depends on the last state. Generally, we use two equations as

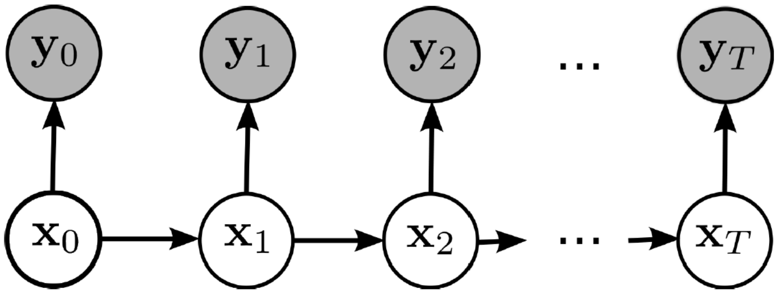

Figure 3.1: State space models

following to express a linear gaussian state space model:

$$y_t = Z_t X_t + \epsilon_t \qquad \epsilon_t \sim NID(0, H_t) \tag{3.1}$$

$$X_{t+1} = T_t X_t + R_t \eta_t \qquad \eta_t \sim NID(0, Q_t) \tag{3.2}$$

Equation 3.1 is called the *measurement equation* and equation 3.2 is called the *transition equation*. Z_t reflects the relation between observations and states at time t, T_t is the transition matrix of states from time t to $t+1$. Suppose the dimension of our observation is $p \times 1$ and state is $m \times 1$, then dimensions of above matrices are given in the table 3.1

Vector	Dimension	Matrix	Dimension
y_t	$p \times 1$	Z_t	$p \times m$
X_t	$m \times 1$	T_t	$m \times m$
ϵ_t	$p \times 1$	R_t	$m \times r$
η_t	$r \times 1$	H_t	$p \times p$
		Q_t	$r \times r$

Table 3.1: Caption

Example 3.1.1. In chapter 2, we have talked about the ARIMA models. Here we will show how to transform a $AR(2)$ model to a state space form at first and then introduce the SSM form for $AR(p)$ models.

Suppose our model is

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t \tag{3.3}$$

where $\epsilon_t \sim NID(0, \sigma^2)$, then we may find a new observation is related to the previous two values, therefore when defining this state space model, the transition equation 3.2 should have at least two states to achieve iterations.

Based on this, we will get the following result

$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t \tag{3.4}$$

$$x_t = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} x_{t-1} + \omega_t \tag{3.5}$$

where $x_t = \begin{bmatrix} y_t & y_{t-1} \end{bmatrix}^T$ and $\omega_t = \begin{bmatrix} \epsilon_t & 0 \end{bmatrix}^T$.

More generally, suppose our model is $AR(p)$, that is

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t \tag{3.6}$$

where $\epsilon_t \sim NID(0, \sigma^2)$. Then the state space form will be

$$y_t = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix} x_t \tag{3.7}$$

$$x_t = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{p-1} & \phi_p \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} x_{t-1} + \omega_t \tag{3.8}$$

where $x_t = \begin{bmatrix} y_t & y_{t-1} & \dots & y_{t-p+1} \end{bmatrix}^T_{1\times p}$ and $\omega_t = \begin{bmatrix} \epsilon_t & 0 & \dots & 0 \end{bmatrix}^T_{1\times p}$.

3.2 Common state space models

Since state space modelling is a general method, many different models could be transformed into state space forms. Durbin and Koopman have detailedly showed in their book[reference, book 2012], so here we shall briefly introduce three common models.

3.2.1 Structural time series models

The structural time series model is the main one we used in SSM due to its structural characteristic. For one time series data, if we model it as a combination of the trend, seasonal, cycle and irregular components, then we call it a structural time series model. But in many research papers, the cycle component is combined with the trend, and we shall take the same strategy in this paper. The structural time series model is usually written in two ways:

$$y_t = T_t + S_t + I_t \tag{3.9}$$

$$y_t = T_t \times S_t \times I_t \tag{3.10}$$

If the model is multiplicative, then we usually take the logarithm before transforming.

The simplest case is the *local level model*, where we do not have any seasonal or other explanatory variables:

$$\begin{aligned} y_t &= T_t + \varepsilon_t \\ T_{t+1} &= T_t + \eta_t \end{aligned} \tag{3.11}$$

where $\varepsilon_t \sim NID(0, \sigma_y^2)$ and $\eta_t \sim NID(0, \sigma_T^2)$. If we add a slope to the trend component, the model will be:

$$\begin{aligned} y_t &= T_t + \varepsilon_t \\ T_{t+1} &= T_t + v_t + \eta_t \\ v_{t+1} &= v_t + \zeta_t \end{aligned} \tag{3.12}$$

we call it *the local linear trend model*, which is what we will use to replace the trend in equation 3.9 later. As for the seasonal component, the simple way to model it is:

$$S_{t+1} = -\sum_{j=1}^{s-1} S_{t+1-j} + \omega_t \tag{3.13}$$

where $\omega_t \sim NID(0, \sigma_S^2)$ and s is the seasonal frequency of our data, that is, for weekly and monthly data, $s = 7$ and 12 separately. But sometimes people prefer to use the trigonometric form to express seasonal components:

$$\begin{aligned} S_t &= \sum_{j=1}^{[s/2]} (\tilde{S}_{jt} cos\lambda_j t + \tilde{S}^*_{jt} sin\lambda_j t) \\ \tilde{S}_{j,t+1} &= \tilde{S}_{jt} + \tilde{\omega}_{jt} \\ \tilde{S}^*_{j,t+1} &= \tilde{S}^*_{jt} + \tilde{\omega}^*_{jt} \end{aligned} \tag{3.14}$$

where $\lambda_j = \frac{2\pi j}{s}$, $j = 1, \ldots, [s/2]$ and $\tilde{\omega}_{jt}$, $\tilde{\omega}^*_{jt}$ are normally and independently distributed variables with variance σ_ω^2.[reference, Young, Lane, Ng and Palmer 1991] Generally the irregular component in equation 3.9 is treated as a normally-distributed noise directly.

Therefore, if we put all above information together, we could obtain the state space form of structural time series models:

$$\begin{aligned} y_t &= T_t + S_t + I_t \\ T_{t+1} &= T_t + v_t + \eta_t \\ v_{t+1} &= v_t + \zeta_t \\ S_{t+1} &= -\sum_{j=1}^{s-1} S_{t+1-j} + \omega_t \end{aligned} \tag{3.15}$$

In terms of the equations 3.1 and 3.2, we could get the following expression:

$$\begin{aligned} X_t &= \begin{bmatrix} T_t & v_t & S_t & S_{t-1} & \cdots & S_{t-s+2} \end{bmatrix}^T \\ Z_t &= \begin{bmatrix} Z_{[T]} & Z_{[S]} \end{bmatrix} \\ T_t &= diag \begin{bmatrix} T_{[T]} & T_{[S]} \end{bmatrix} \\ R_t &= diag \begin{bmatrix} R_{[T]} & R_{[S]} \end{bmatrix} \\ Q_t &= diag \begin{bmatrix} Q_{[T]} & Q_{[S]} \end{bmatrix} \end{aligned} \tag{3.16}$$

where

$$Z_{[T]} = [1 \quad 0] \qquad Z_{[S]} = [1 \quad 0 \quad \cdots \quad 0]_{s-1}$$

$$T_{[T]} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad T_{[S]} = \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

$$R_{[T]} = I_2 \qquad R_{[S]} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

$$Q_{[T]} = \begin{bmatrix} \sigma_\eta^2 & 0 \\ 0 & \sigma_\zeta^2 \end{bmatrix} \qquad Q_{[S]} = \begin{bmatrix} \sigma_\omega^2 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

3.2.2 ARIMA models

We have introduced the ARIMA model in [refer to arima section] and showed an AR(2) example in Section 3.1 above. In this section, we will show how to transform an arbitrary ARIMA model into state space form.

When encountering a stationary time series data, not only could we model it into an ARMA model but also into a state space model. Suppose we now have an ARMA(p,q) model

$$\begin{aligned} y_t &= \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} \\ &= \sum_{i=1}^{p} \phi_i y_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} \\ &= \sum_{i=1}^{r} \phi_i y_{t-i} + \varepsilon_t + \sum_{j=1}^{r-1} \theta_j \varepsilon_{t-j} \end{aligned} \tag{3.17}$$

where $r = max(p, q+1)$ and $\varepsilon_t \sim NID(0, \sigma_\varepsilon^2)$. To transform it into state space form, we can define the measurement equation as

$$\begin{aligned} y_t &= \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} x_t \\ where \qquad x_t &= \begin{pmatrix} y_t \\ \phi_2 y_{t-1} + \cdots + \phi_r y_{t-r+1} + \theta_1 \varepsilon_t + \cdots + \theta_{r-1} \varepsilon_{t-r+2} \\ \phi_3 y_{t-1} + \cdots + \phi_r y_{t-r+2} + \theta_2 \varepsilon_t + \cdots + \theta_{r-1} \varepsilon_{t-r+3} \\ \vdots \\ \phi_r y_{t-1} + \theta_{r-1} \varepsilon_t \end{pmatrix} \end{aligned} \tag{3.18}$$

The matrices in the transition equation are:

$$T_t = T = \begin{bmatrix} \phi_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ \phi_{r-1} & 0 & \cdots & 1 \\ \phi_r & 0 & \cdots & 0 \end{bmatrix} \qquad R_t = R = \begin{pmatrix} 1 \\ \theta_1 \\ \vdots \\ \theta_r \end{pmatrix} \tag{3.19}$$

By 3.18 and 3.19, we have the capacity to transform every known ARMA model to a corresponding state space model. Similarly, we could put any ARIMA model into a SSM, see [reference, Durbin and Koopman 2012 section 3.4].

Therefore, mathematically we are able to transform every ARIMA and ARMA model to a state space form, which just confirms that state space modelling is a more general and practical method. Simultaneously, with the development of techniques in SSM, we could handle these ARIMA models better. On the other hand, many but not all state space models have their corresponding ARIMA models. Example 3.2.1 is one simple case and more related work could be referred to [reference harvey 1989].

Example 3.2.1. In the local linear trend model 3.12, if we take two difference of observations, we shall get

$$\Delta^2 y_t = \varepsilon_t - 2\varepsilon_{t-1} + \varepsilon_{t-2} + \eta_{t-1} - \eta_{t-2} + \zeta_{t-2}$$

It is not hard to notice only the first two autocorrelations are nonzero, so we can use a *MA(2)* model to express the right hand side equivalently, that is

$$\Delta^2 y_t = \delta_t + \theta_1^* \delta_{t-1} + \theta_2^* \delta_{t-2}$$

which is the expression of one *ARIMA(0,2,2)* model.

We have been aware of the relation between ARIMA modelling and state space modelling. In the example above, although we did transform the local linear trend model to an ARIMA model, the information with regard to the slope v_t and the level/trend $T_t$is lost in this process. And this is the reason why we would like to apply the structural time series SSM in our research instead of the ARIMA model-based methods.

3.2.3 Regression models

The regression model is one of the most fundamental concept in statistics. The interesting thing is if we consider the *measurement equation* ignoring the subscript t, it is exactly a regression model, which means we could perhaps view a linear regression model as a SSM. Suppose we have a simple regression model for a univariate variable y:

$$y = X\beta + \varepsilon \qquad where \quad \varepsilon \sim N(0, H) \tag{3.20}$$

corresponding to 3.1, suppose $t = 1, 2, \cdots, n$ and n=number of measurements, then we have

$$Z_t = X_t \qquad T_t = I_t \qquad R_t = Q_t = 0 \tag{3.21}$$

If the coefficient β_t is changeable, then we could model it based on equation 3.2. For example, if each element in β follows a random walk, then it is the multivariate version of the transition equation in 3.11, that is

$$T_t = R_t = I_t \qquad Q_t = \Sigma_t \tag{3.22}$$

where Σ is the diagonal variance matrix of coefficients. For regression problems, one of goals is to determine these coefficients, which is exactly what we shall compute with the above SSM. From this perspective, we can use the techniques in SSM to solve a regression problem.