Assignment 01: Bayes Rule and Computer Generation of Random Variables

1. **A posteriori probability:** *(10 points)*

   There are $K = 11$ urns labeled by $u \in \{0, 1, 2, \ldots, 10\}$, each containing $L = 10$ balls. Urn u contains u black balls and $10 - u$ white balls. Fred selects an urn u at random and draws N times with replacement from that urn, obtaining N_B blacks and $N - N_B$ whites. Fred's friend, Bill, looks on. If after $N = 10$ draws $n_B = 3$ blacks have been drawn, what is the probability that the urn Fred is using is urn u, from Bill's point of view? (Bill doesn't know the value of u.)

2. **Computer Generation of Random Variables** *(30 points)*

   Suppose that $p(u)$ denotes a valid probability density function (PDF) for a (continuous) real-valued random variable and let $F(u) = \int_{-\infty}^{u} p(v)dv$ denote the corresponding cumulative distribution function. Let X be a random variable that is uniformly distributed over the interval between 0 and 1, i.e., its probability density function (PDF) is given by

   $$p_X(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   Now let $G(\cdot)$ denote a function for which the function $F(\cdot)$ is an inverse[1], i.e. for each $t \in [0, 1]$,

   $$F(G(t)) = t \tag{2}$$

   and define the random variable Y as a function of X by the relation:

   $$Y = G(X) \tag{3}$$

   (a) Show that the PDF $p_Y(y)$ for the random variable Y is given by $p_Y(y) = p(y)$, i.e., the PDF of the random variable matches the probability density function $p(\cdot)$. In order to simplify your proof, you may assume that $G(t)$ is differentiable with a non-zero derivative for all $t \in [0, 1]$ (though this condition is not required).

   Hint: You can solve this problem either by computing the cumulative distribution function for Y and computing the PDF from that or by using the formula for the PDF of a function of a random variable along with the chain rule for derivatives.

   (b) **Matlab:** For computer simulation, the above observation suggests a method for generating a random variable with a desired PDF $p(u)$, using the mapping $Y = G(X)$ on a uniformly distributed random variable X. Consider the PDF

   $$p(u) = \begin{cases} 2u & 0 \leq u \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

   Generate $N = 10^4$ realizations of the random variable with the above PDF using the method suggested above. Plot and compare the *suitably normalized* histogram of these realizations you generate against the above PDF in a single plot. Explain your normalization procedure. Repeat this exercise for $N = 10^6$

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[1] Note that for a reasonable (i.e. measurable) function $p(u)$ that can be the PDF of a continuous random variable, it can be shown that such a $G(\cdot)$ exists.