Light front QCD in a transverse harmonic oscillator basis

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We develop a framework for solving Hamiltonian QCD on the light-front (LFQCD) in a basis space consisting of transverse harmonic oscillator states and longitudinal plane waves.

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I. TWO-DIMENSIONAL HARMONIC OSCILLATOR

Consider the 2-dimensional harmonic oscillator

$$\left[\frac{p_x^2+p_y^2}{2\,M}+\tfrac{1}{2}M\Omega^2\,(x^2+y^2)\right]\Psi \;=\; E\,\Psi \tag{1}$$

Explicitly, in Cartesian and in 2-dimensional polar coordinates and with units such that $\hbar = c = 1$

$$\Big[-\frac{1}{2\,M}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\tfrac{1}{2}k(x^2+y^2)\Big]\Psi(x,y) \;=\; E\,\Psi(x,y) \tag{2}$$

$$\Big[-\frac{1}{2\,M}\left(\frac{\partial^2}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}\right)+\tfrac{1}{2}k\,\rho^2\Big]\Psi(\rho,\phi) \;=\; E\,\Psi(\rho,\phi) \tag{3}$$

where

$$k \;=\; M\Omega^2 \tag{4}$$

$$\rho \;=\; \sqrt{x^2+y^2} \tag{5}$$

and the polar angle ϕ running from 0 to 2π.

A. Separation of variables in polar coordinates

Let $\Psi(\rho,\phi) = f(\rho)\chi(\phi)$, then

$$\left[\frac{\rho^2}{f(\rho)}\,\frac{\partial^2 f(\rho)}{\partial \rho^2}+\frac{\rho}{f(\rho)}\,\frac{\partial f(\rho)}{\partial \rho}+2\,M\,\rho^2\,\left(E-\tfrac{1}{2}\,k\,\rho^2\right)\right]+\frac{1}{\chi(\phi)}\frac{\partial^2\chi(\phi)}{\partial\phi^2} \;=\; 0 \tag{6}$$

provided that neither $f(\rho)$ nor $\chi(\phi)$ vanishes. The angular wavefunction $\chi(\phi)$ is the solution of

$$\frac{\partial^2\chi(\phi)}{\partial\phi^2} \;=\; -C\,\chi(\phi) \tag{7}$$

furthermore, it has to be periodic in ϕ: $\chi(\phi+2\pi)=\chi(\phi)$. A solution of the angular differential equation is

$$\chi(\phi) \;=\; \mathcal{A}\,\mathrm{e}^{i\,m_l\,\phi} \tag{8}$$

with $m_l^2 = C$; furthermore, m_l has to be integer due to the periodicity of χ. The constant $\mathcal{A}$ is (at this point) an arbitrary nonzero constant.

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B. Radial wavefunction

With this solution for the angular part of the wavefunction, the equation for the radial part becomes

$$\left[\frac{\rho^2}{f(\rho)}\,\frac{\partial^2 f(\rho)}{\partial\rho^2}+\frac{\rho}{f(\rho)}\,\frac{\partial f(\rho)}{\partial\rho}+2\,M\,\rho^2\,\left(E-\tfrac{1}{2}\,k\,\rho^2\right)\right]-m_l^2\;=\;0 \tag{9}$$

or equivalently

$$\frac{\partial^2 f(\rho)}{\partial\rho^2}+\frac{1}{\rho}\,\frac{\partial f(\rho)}{\partial\rho}+2\,M\,\left(E-\tfrac{1}{2}\,k\,\rho^2\right)\,f(\rho)-\frac{m_l^2}{\rho^2}\,f(\rho)\;=\;0 \tag{10}$$

Anticipating that there will be discrete energy levels, we can define these levels by
[at this point, n is arbitrary, and this is merely a definition for later convenience]

$$E_n\;=\;(2\,n+|m_l|+1)\Omega \tag{11}$$

and then the radial differential equation can be written as

$$\frac{\partial^2 f(\rho)}{\partial\rho^2}+\frac{1}{\rho}\,\frac{\partial f(\rho)}{\partial\rho}+(4\,n+2\,|m_l|+2)\,M\,\Omega\,f(\rho)-M^2\,\Omega^2\,\rho^2\,f(\rho)-\frac{m_l^2}{\rho^2}\,f(\rho)\;=\;0 \tag{12}$$

Dividing by $M\,\Omega$ gives

$$\frac{\partial^2 f(\rho)}{M\,\Omega\,\partial\rho^2}+\frac{1}{M\,\Omega\,\rho}\,\frac{\partial f(\rho)}{\partial\rho}+(4\,n+2\,|m_l|+2)\,f(\rho)-M\,\Omega\,\rho^2\,f(\rho)-\frac{m_l^2}{M\,\Omega\,\rho^2}\,f(\rho)\;=\;0 \tag{13}$$

Defining $t=\sqrt{M\,\Omega}\;\rho$ we obtain

$$\frac{\partial^2 f(t)}{\partial t^2}+\frac{1}{t}\,\frac{\partial f(t)}{\partial t}+(4\,n+2\,|m_l|+2)\,f(t)-t^2\,f(t)-\frac{m_l^2}{t^2}\,f(t)\;=\;0 \tag{14}$$

which is the same as the equation at the bottom of page 3 of JV's notes with $t=\sqrt{\Omega}\;x$.

Next, define

$$f(t)\;=\;t^{-\frac{1}{2}}\;h(t) \tag{15}$$

such that

$$f'(t)\;=\;-\tfrac{1}{2}t^{-\frac{3}{2}}\;h(t)+t^{-\frac{1}{2}}\;h'(t) \tag{16}$$

$$f''(t)\;=\;\tfrac{3}{4}t^{-\frac{5}{2}}\;h(t)-t^{-\frac{3}{2}}\;h'(t)+t^{-\frac{1}{2}}\;h''(t) \tag{17}$$

and thus

$$f''(t)+\frac{1}{t}f'(t)\;=\;\tfrac{1}{4}t^{-\frac{5}{2}}\;h(t)+t^{-\frac{1}{2}}\;h''(t) \tag{18}$$

In this notation, the differential equation Eq. (14) can be re-written as

$$\frac{\partial^2 h(t)}{\partial t^2}+(4\,n+2\,|m_l|+2)\,h(t)-t^2\,h(t)+\frac{1-4\,m_l^2}{4\,t^2}\,h(t)\;=\;0 \tag{19}$$

which we can find in Abramowitz and Stegun, 22.6.18

$$y''(x)+\left((4\,n+2\,\alpha+2-x^2+\frac{1-4\,\alpha^2}{4\,x^2}\right)y(x)\;=\;0 \tag{20}$$

Here is where n has to be integer, or rather, a natural number; furthermore, $\alpha>-1$, which is okay for any (integer) value of m_l. The solution of this differential equation can be given in terms of Generalized Laguerre polynomials

$$y(x)\;=\;\mathrm{e}^{-x^2/2}\;x^{\alpha+\frac{1}{2}}\;L_n^{(\alpha)}(x^2) \tag{21}$$

Thus the solution of Eq. (14) is

$$f(t)\;=\;t^{-\frac{1}{2}}\;h(t)\;=\;\mathcal{N}\,\mathrm{e}^{-t^2/2}\;\;t^{|m_l|}\;\;L_n^{|m_l|}(t^2) \tag{22}$$

or in terms of $\rho=t/\sqrt{M\,\Omega}$

$$f(\rho)\;=\;\mathcal{N}\,\mathrm{e}^{-M\,\Omega\,\rho^2/2}\;\left(\sqrt{M\,\Omega}\;\rho\right)^{|m_l|}\;\;L_n^{|m_l|}(M\,\Omega\,\rho^2) \tag{23}$$

C. Orthonormality

The wavefunctions $\Psi_{n,m}(\rho,\phi) = f_{n,m}(\rho)\chi_m(\phi)$ should be orthogonal to each other, and they should be normalized:

$$\langle nm|n'm'\rangle \;=\; \int_0^\infty \int_0^{2\pi} \rho\, d\rho\; d\phi\; \Psi_{n,m}(\rho,\phi)^*\; \Psi_{n',m'}(\rho,\phi) \;=\; \delta_{n,n'}\; \delta_{m,m'} \tag{24}$$

After the separation into the radial wavefunction

$$f_{n,m}(\rho) \;=\; \mathcal{N}\; \mathrm{e}^{-M\,\Omega\,\rho^2/2}\; \left(\sqrt{M\,\Omega}\;\rho\right)^{|m|}\; L_n^{|m|}(M\,\Omega\,\rho^2) \tag{25}$$

and the angular wavefunction

$$\chi_m(\phi) \;=\; \frac{1}{\sqrt{2\pi}}\; \mathrm{e}^{i\,m\,\phi} \tag{26}$$

it is straightforward to prove orthogonality: First consider the angular wavefunction

$$\int_0^{2\pi} d\phi\; \chi_m(\phi)^*\; \chi_{m'}(\phi) \;=\; \delta_{m,m'} \tag{27}$$

Next consider the radial wavefunction. We know already that the angular wavefunctions are orthogonal for $m \neq m'$ so we can safely take those quantum numbers equal for the radial wavefunction

$$\int_0^\infty \rho\, d\rho\; f_{n,m}(\rho)^*\; f_{n',m}(\rho) \;=\; \int_0^\infty \rho\, d\rho\; \mathcal{N}\mathcal{N}^*\; \mathrm{e}^{-M\,\Omega\,\rho^2}\; \left(M\,\Omega\,\rho^2\right)^{|m|}\; L_{n'}^{|m|}(M\,\Omega\,\rho^2)\; L_n^{|m|}(M\,\Omega\,\rho^2)^* \tag{28}$$

$$\;=\; \frac{\mathcal{N}\mathcal{N}^*}{2\,M\,\Omega}\; \int_0^\infty ds\; \mathrm{e}^{-s}\; s^{|m|}\; L_{n'}^{|m|}(s)\; L_n^{|m|}(s)^* \tag{29}$$

$$\;=\; \frac{\mathcal{N}\mathcal{N}^*}{2\,M\,\Omega}\; \frac{(|m|+n)!}{n!}\; \delta_{n,n'} \tag{30}$$

using the standard normalization of the Generalized Laguerre polynomials, see Abramowitz and Stegun 22.2.12.

Thus the properly normalized wavefunctions $\Psi_{n,m}(\rho,\phi) = f_{n,m}(\rho)\chi_m(\phi)$ are given by

$$f_{n,m}(\rho) \;=\; \sqrt{2\,M\,\Omega}\; \sqrt{\frac{n!}{(n+|m|)!}}\; \mathrm{e}^{-M\,\Omega\,\rho^2/2}\; \left(\sqrt{M\,\Omega}\;\rho\right)^{|m|}\; L_n^{|m|}(M\,\Omega\,\rho^2) \tag{31}$$

$$\chi_m(\phi) \;=\; \frac{1}{\sqrt{2\pi}}\; \mathrm{e}^{i\,m\,\phi} \tag{32}$$

up to an arbitrary phase factor: the normalization is fixed by Eq. (24), which allows for an arbitrary phase factor $\mathrm{e}^{i\alpha}$.

D. Momentum space

Notice the symmmetry of the harmonic oscillator wave equation

$$\left[\frac{p_x^2+p_y^2}{M\,\Omega} + (M\,\Omega)\,(x^2+y^2)\right]\Psi \;=\; \frac{2\,E}{\Omega}\,\Psi \tag{33}$$

under the transformation $p/\sqrt{M\Omega} \longleftrightarrow x\sqrt{M\Omega}$. Thus the solution of the 2-d harmonic oscillator in momentum space can be written as $\tilde{\Psi}_{n,m}(q,\theta) = \tilde{f}_{n,m}(q)\chi_m(\theta)$ with q the conjugate of ρ, θ the conjugate of ϕ (i.e. angle in momentum space), and

$$\tilde{f}_{n,m}(q) \;=\; \tilde{\mathcal{N}}\; \mathrm{e}^{-q^2/(2\,M\,\Omega)}\; \left(\frac{q}{\sqrt{M\,\Omega}}\right)^{|m|}\; L_n^{|m|}(q^2/(M\,\Omega)) \tag{34}$$

The normalization of the momentum-space wavefunction is fixed by an orthonormality condition similar to Eq. (24), but in momentum space there is an extra factor $1/(2\pi)^2$ for the integration measure

$$\langle nm|n'm'\rangle \;=\; \int \frac{d^2p}{(2\pi)^2}\;\tilde{\Psi}_{nm}(p)^*\;\tilde{\Psi}_{n'm'}(p) \;=\; \delta_{nn'}\;\delta mm' \tag{35}$$

Using the same angular wavefunction

$$\chi_m(\theta) \;=\; \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{i\,m\,\theta} \tag{36}$$

as in coordinate space, the normalization factor $\tilde{\mathcal{N}}$ follows from

$$\begin{aligned}\int_0^\infty \frac{q\,dq}{(2\pi)^2}\;\tilde{f}_{n,m}(q)^*\;\tilde{f}_{n',m}(q) \;&=\; \int_0^\infty \frac{q\,dq}{(2\pi)^2}\;\tilde{\mathcal{N}}\tilde{\mathcal{N}}^*\;\mathrm{e}^{-q^2/(M\,\Omega)}\;\left(\frac{q^2}{M\,\Omega}\right)^{|m|}\;L_{n'}^{|m|}(q^2/(M\,\Omega))\;L_n^{|m|}(q^2/(M\,\Omega))^* \\ &=\; \tilde{\mathcal{N}}\tilde{\mathcal{N}}^*\;\frac{M\,\Omega}{2\,(2\pi)^2}\;\int_0^\infty ds\;\mathrm{e}^{-s}\;s^{|m|}\;L_{n'}^{|m|}(s)\;L_n^{|m|}(s)^* \qquad (37)\\ &=\; \tilde{\mathcal{N}}\tilde{\mathcal{N}}^*\;\frac{M\,\Omega}{2\,(2\pi)^2}\;\frac{(|m|+n)!}{n!}\;\delta_{n,n'} \qquad (38)\end{aligned}$$

Thus the properly normalized radial wavefunction in momentum space is

$$\tilde{f}_{n,m}(q) \;=\; 2\pi\,\sqrt{\frac{2}{M\,\Omega}}\,\sqrt{\frac{n!}{(|m|+n)!}}\,\mathrm{e}^{-q^2/(2\,M\,\Omega)}\;\left(\frac{q}{\sqrt{M\,\Omega}}\right)^{|m|}\;L_n^{|m|}(q^2/(M\,\Omega)) \tag{39}$$

Again, the normalization allows for an arbitrary phase factor $\mathrm{e}^{i\tilde{\alpha}}$. However, the momentum-space wavefunction is the Fourier transform of the the coordinate-space wavefunction

$$\tilde{\Psi}(p) \;=\; \int d^2x\;\mathrm{e}^{i\,\vec{p}\cdot\vec{x}}\;\Psi(x) \tag{40}$$

and thus, the phase of the momentum-space wavefunction is fixed once a phase factor is choosen for the coordinate-space wavefunction. With the coordinate-space wavefunction given by Eqs. (31) and (32), we have

$$\tilde{\Psi}_{n,m}(q,\theta) \;=\; \sqrt{\frac{2\,M\,\Omega}{2\pi}}\,\sqrt{\frac{n!}{(n+|m|)!}}\;\int d^2x\;\mathrm{e}^{i\,\vec{p}\cdot\vec{x}}\;\mathrm{e}^{-M\,\Omega\,\rho^2/2}\;\left(\sqrt{M\,\Omega}\,\rho\right)^{|m|}\;L_n^{|m|}(M\,\Omega\,\rho^2)\;\mathrm{e}^{i\,m\,\phi} \tag{41}$$

The **relative phase factor** should be fixed by above eqns, though I have not (yet) succeeded ...
However, following JV's example and suggestion, it is relatively straightforward to show that the lowest solution

$$\Psi_{0,0}(\rho,\phi) \;=\; \sqrt{\frac{M\,\Omega}{\pi}}\,\mathrm{e}^{-M\,\Omega\,\rho^2/2} \tag{42}$$

does not need a nontrivial relative phase factor

$$\begin{aligned}\tilde{\Psi}_{0,0}(q,\theta) \;&=\; \sqrt{\frac{M\,\Omega}{\pi}}\;\int d^2x\;\mathrm{e}^{i\,\vec{p}\cdot\vec{x}}\;\mathrm{e}^{-M\,\Omega\,\rho^2/2} \\ &=\; \sqrt{\frac{4\pi}{M\,\Omega}}\,\mathrm{e}^{-q^2/(2\,M\,\Omega)} \qquad (43)\end{aligned}$$

(see the appendix for more details)

E. Enumeration of energy levels and quantum numbers in Cartesian coordinates and polar coordinates

In polar coordinates, we have found discrete energy levels, characterized by a magnetic projection quantum number m_l and a radial quantum number n

$$E_{n,m} \;=\; (2\,n+|m_l|+1)\Omega \tag{44}$$

with n a natural number (i.e. integer $n \geq 0$) and m_l integer. I have found no further restristrictions on the allowed range of either n or m_l, though it might be useful/necessary to check the asymptotic behavior of the radial wavefunction

$$f(\rho) \;=\; \mathcal{N}\, \mathrm{e}^{-M\,\Omega\,\rho^2/2}\; \left(\sqrt{M\,\Omega}\;\rho\right)^{|m_l|}\; L_n^{|m_l|}(M\,\Omega\,\rho^2) \tag{45}$$

in the limit $\rho \to \infty$.

We should have the same energy levels and degeneracy for each level in either Cartesian coordinates or in polar coordinates. In order to make a comparison, let us define a principal quantum number

$$N \;=\; 2\,n + |m_l| \;=\; n_x + n_y \tag{46}$$

and tabulate the energy levels

TABLE I: Energy levels and their degeneracy for the 2-d H.O.

N	n_x	n_y	degeneracy	n	m_l
0	0	0	1	0	0
1	1	0	2	0	1
1	0	1	2	0	-1
2	2	0	2+1	0	2
2	0	2	2+1	0	-2
2	1	1	2+1	1	0
3	3	0	2+2	0	3
3	0	3	2+2	0	-3
3	2	1	2+2	1	1
3	1	2	2+2	1	-1
4	4	0	2+2+1	0	4
4	0	4	2+2+1	0	-4
4	3	1	2+2+1	1	2
4	1	3	2+2+1	1	-2
4	2	2	2+2+1	2	0

For comparison, let us also make a similar table for the more familiar 3-d H.O. with the energy levels given by

$$E_{n_r,l} \;=\; (2\,n_r + l + \tfrac{3}{2})\Omega \;=\; (n_x + n_y + n_z + \tfrac{3}{2})\Omega \tag{47}$$

where l is the eigenvalue of L (not the magnetic projection m_l, eigenvalue of L_z), with a $2l+1$ degeneracy for each l value: $-l \leq m_l \leq l$.

TABLE II: Energy levels and their degeneracy for the 3-d H.O. together with the conventional spectroscopic name

N	n_x	n_y	n_z	degeneracy		n_r	m_l
0	0	0	0	1	0s	0	0
1	1	0	0	3	0p	0	1
1	0	1	0	3	0p	0	0
1	0	0	1	3	0p	0	-1
2	2	0	0	5+1	0d	0	2
2	0	2	0	5+1	0d	0	1
2	0	0	2	5+1	0d	0	0
2	1	1	0	5+1	0d	0	-1
2	1	0	1	5+1	0d	0	-2
2	0	1	1	5+1	1s	1	0

Notice that in two dimensions, there is no total L, only L_z with eigenvalues m_l. Furthermore, in two dimensions we have a two-fold degeneracy for each value of $|m_l|$ (except for $m_l = 0$ of course), whereas in three dimensions we have a $2l+1$ degeneracy for each value of l. In DLCQ, the 2-d H.O. quantum numbers combine with quantum numbers from the light-front direction to form 3-d quantum numbers. I am not quite certain how that works in practice (in particular since we do not use a H.O. basis in that dimension).

II. KINETIC ENERGY OPERATOR

A. Single-particle kinetic energy operator

Consider the transverse kinetic energy operator

$$T_{\text{2-d transv}} = \frac{p_x^2 + p_y^2}{2\,M} \tag{48}$$

$$= -\frac{1}{2\,M}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right) \tag{49}$$

Single particle matrix elements of the 2-dimensional transverse kinetic energy operator are orthogonal in the 2-dimensional magnetic projection quantum number m (earlier called m_l, but we dropped the subscript $_l$ somewhere)

$$\langle nm|T_{\text{2-d transv}}|n'm'\rangle = \delta_{m,m'}\langle nm|T_{\text{2-d transv}}|n'm\rangle \tag{50}$$

but they are not orthogonal in the radial quantum number n. Using the fact that the states are (single-particle) solutions of the 2-d H.O., we can write these matrix elements as

$$\langle nm|T_{\text{2-d transv}}|n'm\rangle = \delta_{n,n'}\,E_n - \langle nm|V_{\text{2-d transv}}|n'm\rangle \tag{51}$$

with $E_n = (2\,n + |m| + 1)\Omega$. Thus we need to calculate (again using the notation $s = M\,\Omega\,\rho^2$)

$$\langle nm|V_{\text{2-d transv}}|n'm\rangle = \tfrac{1}{2}\,M\,\Omega^2\,\langle nm|\rho^2|n'm\rangle$$

$$= \tfrac{1}{2}\Omega\,\sqrt{\frac{n!}{(n+|m|)!}}\,\sqrt{\frac{n'!}{(n'+|m|)!}}\,\int_0^\infty ds\,\mathrm{e}^{-s}\,s^{|m|+1}\,L_n^{|m|}(s)\,L_{n'}^{|m|}(s) \tag{52}$$

which we can do using 22.7.30 from Abramowitz and Stegun

$$L_n^{|m|}(s) = L_n^{|m|+1}(s) - L_{n-1}^{|m|+1}(s) \tag{53}$$

Inserting this relation we get

$$\langle nm|V_{\text{2-d transv}}|n'm\rangle = \tfrac{1}{2}\Omega\,\sqrt{\frac{n!}{(n+|m|)!}}\,\sqrt{\frac{n'!}{(n'+|m|)!}}\,\int_0^\infty ds\,\mathrm{e}^{-s}\,s^{|m|+1}$$

$$\left(L_n^{|m|+1}(s)\,L_{n'}^{|m|+1}(s) - L_n^{|m|+1}(s)\,L_{n'-1}^{|m|+1}(s) - L_{n-1}^{|m|+1}(s)\,L_{n'}^{|m|+1}(s) + L_{n-1}^{|m|+1}(s)\,L_{n'-1}^{|m|+1}(s)\right) \tag{54}$$

Now we can use the orthogonality of the Generalized Laguerre polynomials to obtain

$$\langle nm|V_{\text{2-d transv}}|n'm\rangle = \tfrac{1}{2}\Omega\,\sqrt{\frac{n!}{(n+|m|)!}}\,\sqrt{\frac{n'!}{(n'+|m|)!}}$$

$$\left(\delta_{n,n'}\frac{(n+|m|+1)!}{n!} - \delta_{n,n'-1}\frac{(n+|m|+1)!}{n!} - \delta_{n-1,n'}\frac{(n+|m|)!}{(n-1)!} + \delta_{n-1,n'-1}\frac{(n+|m|)!}{(n-1)!}\right)$$

$$= \tfrac{1}{2}\Omega\,\left(\delta_{n,n'}\,(2n+|m|+1) - \delta_{n+1,n'}\,\sqrt{(n+1)\,(n+|m|+1)} - \delta_{n-1,n'}\,\sqrt{n\,(n+|m|)}\right) \tag{55}$$

Thus we arrive at the following single-particle matrix elements

$$\langle nm|H_{\text{2-d H.O.}}|n'm'\rangle = \delta_{m,m'}\,\delta_{n,n'}\,(2\,n+|m|+1)\,\Omega \tag{56}$$

$$\langle nm|V_{\text{2-d H.O.}}|n'm'\rangle = \tfrac{1}{2}\,\delta_{m,m'}\,\left(\delta_{n,n'}\,(2n+|m|+1) - \delta_{n+1,n'}\,\sqrt{(n+1)\,(n+|m|+1)} - \delta_{n-1,n'}\,\sqrt{n\,(n+|m|)}\right)\,\Omega$$

$$\langle nm|T_{\text{2-d transv}}|n'm'\rangle = \tfrac{1}{2}\,\delta_{m,m'}\,\left(\delta_{n,n'}\,(2n+|m|+1) + \delta_{n+1,n'}\,\sqrt{(n+1)\,(n+|m|+1)} + \delta_{n-1,n'}\,\sqrt{n\,(n+|m|)}\right)\,\Omega$$

Using $n_> = \max(n, n')$, we can combine the last terms in these expressions

$$\langle nm|V_{\text{2-d H.O.}}|n'm'\rangle = \tfrac{1}{2}\,\delta_{m,m'}\,\left(\delta_{n,n'}\,(2n+|m|+1) - \delta_{n_>-n_<,1}\,\sqrt{n_>\,(n_>+|m|)}\right)\,\Omega \tag{57}$$

$$\langle nm|T_{\text{2-d transv}}|n'm'\rangle = \tfrac{1}{2}\,\delta_{m,m'}\,\left(\delta_{n,n'}\,(2n+|m|+1) + \delta_{n_>-n_<,1}\,\sqrt{n_>\,(n_>+|m|)}\right)\,\Omega \tag{58}$$

B. Two-body kinetic energy operator

The two-body kinetic energy operator is simply the sum of the product of two one-particle kinetic energy operators

$$T_{\text{two-body}} \;=\; T_1 \otimes \mathbf{1} + \mathbf{1} \otimes T_2 \tag{59}$$

and thus the corresponding matrix elements factorize and reduce to

$$\langle n_1 m_1, n_2 m_2 | T_{\text{two-body}} | n_3 m_3, n_4 m_4 \rangle \;=\; \langle n_1 m_1 | T | n_3 m_3 \rangle \; \delta_{n_2,n_4} \; \delta_{m_2,m_4} + \; \delta_{n_1,n_3} \; \delta_{m_1,m_3} \langle n_2 m_2 | T | n_4 m_4 \rangle \tag{60}$$

(see the appendix for some more details).

However, for MFD we need to split the kinetic energy operator into kinetic energy operators associated with the relative motion and with a "center-of-mass-like motion" (not necessarily the true center-of-mass motion)

$$T \;=\; T_{\text{relative}} + T_{\text{CM}} \;=\; \sum_{i=1}^{A} \frac{p_i^2}{2\,M_i} \tag{61}$$

$$T_{\text{CM}} \;=\; \frac{P^2}{2\,M_{\text{total}}} \tag{62}$$

$$T_{\text{relative}} \;=\; T - T_{\text{CM}} \tag{63}$$

with $P = \sum p_i$ the total momentum of the system, and $M_{\text{total}} = \sum M_i$ is the total mass of the system[1], or rather, the sum of the constituent masses. For simplicity let us consider constituents of equal mass, $M_i = M$, so that the total mass reduces reduces to $M_{\text{total}} = A\,M$.

$$T_{CM} \;=\; \frac{1}{2\,A\,M} \left(\sum_i \vec{p_i} \right)^2 \tag{64}$$

$$\;=\; \frac{1}{2\,A\,M} \sum_{i,j} \vec{p_i} \cdot \vec{p_j} \tag{65}$$

$$T_{\text{rel}} \;=\; \frac{1}{2\,M} \sum_i p_i^2 - \frac{1}{2\,A\,M} \sum_{i,j} \vec{p_i} \cdot \vec{p_j} \tag{66}$$

$$\;=\; \frac{1}{2\,A\,M} \sum_{i,j} \left(p_i^2 - \vec{p_i} \cdot \vec{p_j} \right) \tag{67}$$

$$\;=\; \frac{1}{2\,A\,M} \sum_{i,j} \left(\tfrac{1}{2} p_i^2 - \vec{p_i} \cdot \vec{p_j} + \tfrac{1}{2} p_j^2 \right) \tag{68}$$

$$\;=\; \frac{1}{2\,A\,M} \sum_{i<j} \left(p_i^2 - 2\,\vec{p_i} \cdot \vec{p_j} + p_j^2 \right) \tag{69}$$

$$\;=\; \frac{1}{2\,A\,M} \sum_{i<j} \left(\vec{p_i} - \vec{p_j} \right)^2 \tag{70}$$

The difficult part is the calculations of the contribution from the dot-product to $\vec{p_i} \cdot \vec{p_j}$ to the kinetic energy matrix elements

$$\begin{aligned} \mathcal{M} \;&=\; \langle n_1 m_1, n_2 m_2 | \vec{p_1} \cdot \vec{p_2} | n_1' m_1', n_2' m_2' \rangle \\ &=\; \int \frac{d^2 p_1}{(2\pi)^2} \int \frac{d^2 p_1}{(2\pi)^2} \; \Psi_{n_1 m_1}(p_1)^* \; \Psi_{n_2 m_2}(p_2)^* \; (\vec{p_1} \cdot \vec{p_2}) \; \Psi_{n_1' m_1'}(p_1) \; \Psi_{n_2' m_2'}(p_2) \end{aligned} \tag{71}$$

The evaluation of such matrix elements is given in detail in the appendix.

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[1] Note that this is **not** necessarily the physical mass of many-body system; in the case of (deeply-bound) bound states $M_{\text{bound state}} = M_{\text{total}} - E_b = \sum m_i - E_b$, where E_b is the binding energy. In nuclear physics, we can savely neglect this binding energy compared to the nucleon and nuclei masses; for QCD however, we cannot assume that the binding energy is negligible compared to the hadron masses and the quark masses. In addition, there is the issue of the definition of the quark mass (current quark mass? constituent quark mass?) and the question of confinement complicates the picture even further.

III. LIGHT-FRONT KINETIC ENERGY OPERATOR

The single-particle kinetic energy operator for a parton with light-front momentum fraction $k^+ = xP^+$ and transverse (2-dimensional) momentum $k_\perp$ is given by

$$M_0^2 = T = \frac{M^2 + k_\perp^2}{x} \tag{72}$$

$$= \frac{M^2}{x} + \left(\frac{2\,M}{x}\right)\,\left(\frac{k_\perp^2}{2M}\right) \tag{73}$$

Here P^+ is the +-component of the total momentum $P = (P^+, P^-, P_\perp)$ of the system; the mass M is the mass of the parton (assuming that there is such a thing as a well-defined quark mass ...); both the mass M and the light-front momentum fraction x are c-numbers, rather than operators[2]; on the RHS only $k_\perp^2$ is an operator in the 2-d transverse subspace, and has to be treated as such. Thus the non-relativistic 2-dimensional kinetic energy operator of the previous subsection

$$T_{\text{2-d}} = \frac{k_x^2 + k_y^2}{2\,M} \tag{74}$$

gives us basically the "non-trivial" part of the relativistic single-particle kinetic energy in the 2-d transverse direction of a parton with light-front momentum fraction $k^+ = xP^+$ and (2-dimensional) transverse momentum $k_\perp = (k_x, k_y)$.

For a system of multiple particles, or rather, partons, we have

$$T = \sum \frac{M_i^2 + k_{i,\perp}^2}{x_i} \tag{75}$$

$$= \sum \frac{M_i^2}{x_i} + \sum \left(\frac{2\,M_i}{x_i}\right)\,\left(\frac{k_{i,\perp}^2}{2M_i}\right) \tag{76}$$

$$= M^2 \sum\,\frac{1}{x_i}\,\left[1 + \frac{2}{M}\,T_{\text{2-d}}\right] \tag{77}$$

where the last line holds for equal-mass partons only. This resembles what appears on page 13 of JV's notes, except that there it is called M_0^2 rather than T. Now we define

$$t_i = \frac{1}{x_i}\,\left[1 + \frac{2}{M}\,T_{\text{2-d}}\right] \tag{78}$$

so that

$$T = M^2 \sum\,t_i \tag{79}$$

Next, we can define a number operator

$$\hat{N} = \sum_{j=1}^{\infty} a_j^\dagger\,a_j \tag{80}$$

such that for states with a fixed number of A partons

$$\hat{N}\,|\Psi\rangle = A\,|\Psi\rangle \tag{81}$$

In the sub-space of states with A partons[3], we can write the unity operator as

$$1_A = \frac{1}{A}\sum_{j=1}^{\infty} a_j^\dagger\,a_j \tag{82}$$

---

[2] At least in the 2-dimensional transverse subspace; in the full DLCQ description the light-front momentum fraction x is an operator, and the states are labeled by the light-fron fraction as well as by the quantum numbers of the 2-dimensional transverse direction, $|nmx\rangle$.

[3] This is possible for a system with a fixed number of partons, but in field theory we are dealing with systems with an arbitrary number of particles (and anti-particles). In our calculations we might start with a fixed number of partons ($q\bar{q}$ for mesons, qqq for baryons), but eventually, we will deal with systems with a variable number of partons (e.g. qqq, $qqqg$, $qqqgg$, $qqqq\bar{q}$, ... for baryons).

and therefore

$$t = \frac{1}{A^2} \sum_{i,k} a_i^\dagger \, a_i \, t \, a_k^\dagger \, a_k \tag{83}$$

$$= \frac{1}{A^2} \sum_{i,k} \langle i| \, t \, |k\rangle \, a_i^\dagger \, a_k \tag{84}$$

After acting on a state with A partons once with a_k we obviously are in the subspace with $A-1$ partons, and we can insert Eq. (82) once again, but this time for $A-1$ partons

$$t = \frac{1}{A^2} \sum_{i,k} \frac{1}{A-1} \sum_{j} \langle i| \, t \, |k\rangle \, a_i^\dagger \, a_j^\dagger \, a_j \, a_k \tag{85}$$

and once again

$$t = \frac{1}{A^2} \sum_{i,k} \frac{1}{(A-1)^2} \sum_{j,l} \langle i| \, t \, |k\rangle \, a_i^\dagger \, a_j^\dagger \, a_j \, a_l^\dagger \, a_l \, a_k \tag{86}$$

$$= \frac{1}{A^2} \sum_{i,k} \frac{1}{(A-1)} \sum_{j,l} \langle i| \, t \, |k\rangle \, \delta_{jl} \, a_i^\dagger \, a_j^\dagger \, a_l \, a_k \tag{87}$$

Work in progress ...

IV. INITIAL (FREE) QCD HAMILTONIAN

Let us consider as an initial Hamiltonian a model without interaction, but with two Lagrange multipliers to remove (I) center-of-mass excitations and (II) states with global color charge[4] (color-nonsinglets) from the low-lying spectrum

$$M^2 \;=\; M_0^2 + \lambda_I \; M \left[H_{CM} - \Omega\right] + \lambda_{II} \; M^2 \; \hat{Q}_C \tag{88}$$

The removal of the CM motion is standard, at least in the context of nuclear physics shell-model calculations; the removal of the color-nonsinglets need more explanation.
Question The factors M going with the Lagrange multipliers are such that the multipliers themselves are dimensionless?

A. Center-of-mass motion

to be written – similar to nuclear physics MFD approach

B. Color charges

Ideally (or maybe I should say, "Naïvely" ?), we would restrict ourselves to color-singlet states from the very beginning[5]. In principle this is possible: in the construction of the many-body states out of single-particle state we could restrict ourselves explicitly to color-singlet many-body states only. However, MFD uses as basis states many-body states that are direct products of single-particle states

$$|\psi_1 \ldots \psi_n\rangle \;=\; |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle \tag{89}$$

there are **no** linear combinations of products of single-particle states in our basis. This is a fundamental feature of MFD, and has many advantages (or so I have been told); however, a disadvantage is that we cannot restrict ourselves to a color-singlet subspace.

1. Intermezzo

This is closely related to the fact that we cannot restrict ourselves to a subspace with a specific total spin J, but only to subspaces with specific magnetic projection m_j (referred to as m-scheme in nuclear physics shell-model terminology). Consider e.g. two spin $\frac{1}{2}$-particles, which can couple to either a spin-0 or a spin-1 state

$$|\uparrow\uparrow\rangle \;:\; j=1 \quad j_z=1 \tag{90}$$

$$\frac{1}{\sqrt{2}}|\uparrow\downarrow - \downarrow\uparrow\rangle \;:\; j=1 \quad j_z=0 \tag{91}$$

$$|\downarrow\downarrow\rangle \;:\; j=1 \quad j_z=-1 \tag{92}$$

$$\frac{1}{\sqrt{2}}|\uparrow\downarrow + \downarrow\uparrow\rangle \;:\; j=0 \quad j_z=0 \tag{93}$$

(JL: Above in second equation, it should be '+' for spin-1 state. In the last equation, it should be '-' for spin-0 state.) However, as mentioned above, in MFD we do not use any linear combinations of products of single-particle states in

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[4] This second Lagrange multiplier is **not to be confused** with the confining interaction; it is a method to seperate states with global color charge from color-singlet states; the analogon in QED is to separate states with globale electric charge from states which are neutral.

[5] This is not the same as imposing confinement by hand: e.g. take QED and restrict yourself to the subspace of eletrically neutral systems: this subspace contains states such as e^+e^- pairs, the hydrogen atom, etc. Whether or not such electrically neutral systems form a bound state depends on the strength of the interaction (i.e. electric charge), the masses, the spin, etc, and has nothing to do with confinement.

our basis, so our basis would be

$$|\uparrow\uparrow\rangle \;:\; j=1 \quad j_z=1 \tag{94}$$
$$|\downarrow\downarrow\rangle \;:\; j=1 \quad j_z=-1 \tag{95}$$
$$|\uparrow\downarrow\rangle \;:\; \quad j_z=0 \tag{96}$$
$$|\downarrow\uparrow\rangle \;:\; \quad j_z=0 \tag{97}$$

We can restrict our basis to include only $j_z = 0$ states, in which case we will find both the $j = 0$, $j_z = 0$ and the $j = 1$, $j_z = 0$ eigenstates of the Hamiltonian, but not the $j = 1$, $j_z = \pm 1$ states; or we can restrict our basis to include only $j_z = 1$ states, in which case we will find the $j = 1$ states, but not the $j = 0$ states. However, we cannot restrict our basis to include only the $j = 0$ states.

In order to remove e.g. the $j = 1$ states from the low-lying spectrum, and keep only the $j = 0$ states, we could add a Lagrange multiplier times an operator which (ideally) is zero for the $j = 0$ states and with a (large) positive eigenvalue for the $j = 0$ (JL:instead of $j = 0$, it should be $j \neq 0$ here) states; in practice it may be more convenient to use an operator whose eigenvalue for $j = 0$ states is (much) lower than the eigenvalues for the $j > 0$ states. This is the basic idea that we implement in order to remove the color-nonsinglet states from the low-lying spectrum.

2. Back to color charges – SU(3) basics

How does this translate to color degrees of freedom?

Color degree of freedom is based on an SU(3) symmetry, whereas the spin in the example above is an SU(2) symmetry. The generators F^a of SU(3) satisfy the Lie agebra

$$[F^a, F^b] \;=\; i\sum_{c=1}^{8} f^{abc}\, F^c \tag{98}$$

with f^{abc} the (totally anti-symmetric) structure constants of $SU(3)$. For the fundamental representation (quarks), the generators are the eight traceless hermitian 3×3 Gell-Mann matrices λ^a; these are the SU(3) analogon of the 2×2 Pauli spin matrices which form the generators of SU(2).

An important difference between SU(2) and SU(3) is the number of diagonal generators. Of the three Pauli spin matrices, only one is diagonal

$$\sigma^3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{99}$$

corresponding to the spin-projection in the z direction with eigenvalues j_z. On the other hand, **two** of the eight Gell-Mann matrices are diagonal

$$\lambda^3 \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \lambda^8 \;=\; \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{100}$$

corresponding to two generators F^3 and F^8. This means that in SU(3) we can always **diagonalize two generators simultaneously**, whereas in SU(2) only one can be diagonalized, conventionally J_z with eigenvalue j_z, the spin projection in the z-direction. Thus states in an SU(3) irreducable representation have to be labeled by **two quantum numbers**, conventionally[6] called t_3 (color isospin) and y (color hypercharge), corresponding to the eigenvalues of the two operators

$$T_3 \;=\; F^3 \qquad \text{and} \qquad Y \;=\; \frac{2}{\sqrt{3}}\, F^8 \tag{101}$$

In addition, there are three sets of raising and lowering operators:

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[6] The naming comes from $\mathrm{SU(3)_{flavor}}$, which was discovered before $\mathrm{SU(3)_{color}}$.

- $T_+ = F_1 + iF_2$ which raises t_3 by 1 unit
- $U_+ = F_6 + iF_7$ which lowers t_3 by $\frac{1}{2}$ unit and raises y by 1 unit
- $V_+ = F_4 + iF_5$ which raises t_3 by $\frac{1}{2}$ unit and raises y by 1 unit

and similarly for the lowering operators. (Together,T_3 and Y plus the raising and lowering operators completely define the Lie algebra.) We can characterize each irreducable representation of SU(3) by a set of two integers (p, q). Graphically, each representation forms a hexagonal figure on the $t_3 - y$ plane with three sides of length p units and three sides of length q units. The figure is symmetric under reflections in the y-axis, but not necessarily in the t_3-axis. Sites on the boundary are singly occupied, sites in the interior are doubly (second layer) occupied, triply (third layer) occupied etc. Some commonly occuring irreducable representations are

- triplet 3 representation $(1, 0)$ (quarks)
- antitriplet $\bar{3}$ representation $(0, 1)$ (antiquarks)
- octet 8 representation $(1, 1)$ (gluons, pseudoscalar mesons, baryon octet containing n and p)
- decuplet 10 representation $(3, 0)$ (baryon decuplet containing Δ's)

3. Color charges – Quarks and antiquarks

Let us first consider quarks and antiquarks. Quarks are in the fundamental representation(triplet 3), so the generators are given by the appropriate Gell-Mann matrices: $T_3 = \frac{1}{2}\lambda^3$ and $Y = \frac{1}{\sqrt{3}}\lambda^8$. Explicitly

$$T_3 \;=\; \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Y \;=\; \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{102}$$

Below we have tabulated the T_3 and Y eigenvalues

$$\langle i|T_3|j\rangle \;=\; \delta_{ij}\, t_3 \tag{103}$$

$$\langle i|Y|j\rangle \;=\; \delta_{ij}\, y \tag{104}$$

for the quarks in the fundamental representation. In addition (and more as a consistency check than anything else), we have also calculated the total T quantum number

$$\langle i|T^2|j\rangle \;=\; \delta_{ij}\, t\,(t+1) \tag{105}$$

using

$$T^2 \;=\; \tfrac{1}{2}\,(T_+\,T_- + T_-\,T_+) + T_3^2 \tag{106}$$

TABLE III: T_3, Y, and T^2 eigenvalues for quarks and antiquarks

quarks	eigenstate	t_3	y	t	antiquarks	t_3	y	t
r	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\bar{r}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$
g	$\begin{pmatrix}0\\1\\0\end{pmatrix}$	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$\bar{g}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{2}$
b	$\begin{pmatrix}0\\0\\1\end{pmatrix}$	0	$-\frac{2}{3}$	0	$\bar{b}$	0	$\frac{2}{3}$	0

Question How about the conjugate representation, the antitriplet $\bar{3}$?
Do we get a minus sign in the representation of the states, or a minus sign in the generators?
I suspect the latter, $F^a = -\frac{1}{2}\lambda^a$ for antiquarks in the antitriplet $\bar{3}$ representation, so that the T_3 and Y values change sign. (Or maybe both representation and generators change sign . . . ???)

4. Color charges – gluons

For the gluons, which are in the adjoint representation (octet 8), we need the generators in the adjoint representation

$$\left(C^a\right)_{bc} \;=\; -if^{abc} \tag{107}$$

which satisfy the SU(3) Lie algebra, Eq. (98), just like the Gell-Mann matrices do. In this representation we have for the isospin and hypercharge operators

$$C^3 \;=\; -i\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad C^8 \;=\; -i\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

These are clearly not diagonal matrices, but it is straightforward to diagonalize them and to obtain the corresponding eigenvalues and eigenvectors. The matrices

$$U^\dagger \;=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad U \;=\; \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ i & -i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & i & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

diagonalize C^3 and C^8 and simultaneously: both $U^\dagger\, C^3\, U$ and $U^\dagger\, C^8\, U$ are diagonal. Thus a more convenient basis for the octet representation may be

$$\tilde{C}^a \;=\; U^\dagger\, C^a\, U \tag{108}$$

which also satisfy the SU(3) Lie algebra, Eq. (98), but with the added benefit that $\tilde{C}^3$ and $\tilde{C}^8$ are diagonal

$$\tilde{C}^3 \;=\; \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \tilde{C}^8 \;=\; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

making it easier to track the quantum numbers t_3 and y.

C. Removing color non-singlets from the low-lying spectrum

As a first step in removing all colored states from the low-lying spectrum, we implement the color-equivalent of the restriction to $j_z = 0$ basis states in MFDn: in our many-parton basis for MFDq, we retain only those states that have $T_3 = 0 = Y$. This does remove a lot of the colored states from our basis, though it is obvious that it does not remove all colored states: e.g. at the one-parton level it would keep two single-gluon states! So we need some additionaly

TABLE IV: T_3, Y and T^2 eigenvalues for gluons.

gluon eigenstates in C^a basis	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$	$\begin{pmatrix}0\\ 0\\ 1\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 1\\ i\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 1\\ -i\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 1\\ i\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 1\\ -i\\ 0\\ 0\end{pmatrix}$	$\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 1\end{pmatrix}$
t_3	1	-1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
y	0	0	0	1	-1	1	-1	0
$t(t+1)$	2	2	2	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0
t	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

TABLE V: T_3, Y and T^2 eigenvalues for gluons(JL: This modified new table is based on the original table above.)

gluon eigenstates in C^a basis	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$	$\begin{pmatrix}0\\ 0\\ 1\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 0\\ 1\\ i\\ 0\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 0\\ 1\\ -i\\ 0\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 1\\ i\\ 0\\ 0\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 1\\ -i\\ 0\\ 0\end{pmatrix}$	$\begin{pmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 1\end{pmatrix}$
t_3	1	-1	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0
y	0	0	0	1	-1	1	-1	0
$t(t+1)$	2	2	2	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	0
t	1	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0

constraint in order to remove the unwanted colored states from the low-lying spectrum, and that is where we use the trick with the Lagrange multiplier.

(Note: above I have made a little bit correction about original table.)

So the next question is: what is the appropriate operator to use in combination with the Lagrange multiplier

$$M^2 \;=\; M_0^2 + \lambda_I\, M\, \left[H_{CM} - \Omega\right] + \lambda_{II}\, M^2\, \hat{Q}_C \tag{109}$$

in order to remove the redisual colored states from the low-lying spectrum. The first operator that comes to mind is basically a many-body generalization of the Casimir operator for SU(3)

$$C \;=\; \sum_{a=1}^{8} \left(F^a\right)^2 \tag{110}$$

For particles in the fundamental representation (quarks), this Casimir operator reduces to a 3×3 unit matrix times C_F

$$\begin{aligned} \delta_{ij}\, C_F \;&=\; \left[\sum_{a=1}^{8} \left(\frac{\lambda^a}{2}\right)^2\right]_{ij} \\ &=\; \sum_{a=1}^{8}\sum_{k=1}^{3} \left(\frac{\lambda^a}{2}\right)_{ik} \left(\frac{\lambda^a}{2}\right)_{kj} \end{aligned} \tag{111}$$

with λ^a the usual eight Gell-Mann matrices. For particles in the adjoint representation (gluons), the Casimir operator

can be written as an 8×8 unit matrix times C_A

$$\begin{aligned}
\delta_{cd}\, C_A &= \Big[\sum_{a=1}^{8} \big(C^a\big)^2\Big]_{cd} \\
&= \sum_{a=1}^{8}\sum_{b=1}^{8} \big(C^a\big)_{cb}\, \big(C^a\big)_{bd} \\
&= \sum_{a=1}^{8}\sum_{b=1}^{8} \big(-i\, f^{acb}\big)\, \big(-i\, f^{abd}\big) \\
&= \sum_{a=1}^{8}\sum_{b=1}^{8} f^{cab}\, f^{dab}
\end{aligned} \tag{112}$$

If we use the convention of upper indices for the color labels, $a = 1, \ldots, 8$, and lower indices for the parton labels, $i = 1, \ldots, n$ for an n-parton state, we can define a many-parton version of this Casimir operator as follows

$$\sum_{a=1}^{8} \left[\sum_{i=1}^{n} F_i^a\right]^2 \tag{113}$$

which is independent of the representation. This operator gives zero when acting on color singlet states, and positive for all color-nonsinglet states. **I have not checked this last statement, but I believe it to be true.** However, this is an operator that connects to all single-particle states in our many-body basis, which would cause tremendous amount of CPU time; for practical purposes we typically limit ourselves to rank-2 or rank-3 operators[7] in MFD.

However, a related operator, which is rank-2 (or rather, it is the sum of a number of rank-2 operators) and does have the property that it has its lowest eigenvalue for a colorsinglet, is the following

$$\sum_{i<j}^{n}\sum_{a=1}^{8} F_i^a\, F_j^a \;=\; \tfrac{1}{2}\sum_{a=1}^{8}\left[\sum_{i=1}^{n} F_i^a\right]^2 - \tfrac{1}{2}\sum_{a=1}^{8}\left[\sum_{i}^{n}\big(F_i^a\big)^2\right] \tag{114}$$

where I have used

$$\sum_{a=1}^{8}\left[\sum_{i=1}^{n} F_i^a\right]^2 = \sum_{a=1}^{8}\left[\sum_{i=1}^{n}\sum_{i=j}^{n} F_i^a\, F_j^a\right] \tag{115}$$

$$= \sum_{a=1}^{8}\left[2\sum_{i<j}^{n} F_i^a\, F_j^a + \sum_{i=j}^{n} F_i^a\, F_j^a\right] \tag{116}$$

similar to the manipulations of $T_{\rm relative}$. I believe that this is a suitable operator to use in conjunction with the Lagrange multiplier, and I think it is the proper generalization of the operator Richard Lloyd uses in his thesis for quarks and antiquarks. However, this should be checked via a handful examples, and ideally it should be checked analytically as well ...

1. *Two-body matrix elements for color Lagrange multiplier*

Now we need to evaluate the various 2-parton matrix elements for this (sum of) rank-2 operator. For calculations involving only quarks and antiquarks, we need to evaluate the following matrix elements

$$\tfrac{1}{4}\langle q_1 q_2|\lambda_1^a\lambda_2^a|q_1' q_2'\rangle = \tfrac{1}{4}\langle q_1|\lambda^a|q_1'\rangle\, \langle q_2|\lambda^a|q_2'\rangle \tag{117}$$

$$-\tfrac{1}{4}\langle q_1 \bar{q}_2|\lambda_1^a\lambda_2^a|q_1' \bar{q}_2'\rangle = -\tfrac{1}{4}\langle q_1|\lambda^a|q_1'\rangle\, \langle \bar{q}_2|\lambda^a|\bar{q}_2'\rangle \tag{118}$$

$$\tfrac{1}{4}\langle \bar{q}_1 \bar{q}_2|\lambda_1^a\lambda_2^a|\bar{q}_1' \bar{q}_2'\rangle = \tfrac{1}{4}\langle \bar{q}_1|\lambda^a|\bar{q}_1'\rangle\, \langle \bar{q}_2|\lambda^a|\bar{q}_2'\rangle \tag{119}$$

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[7] I.e. 2-body and 3-body forces in nuclear physics; 1-to-1, 1-to-2, 1-to-3, and 2-to-2 parton interactions in QCD.

where summation over the repeated color index a is implicitly understood. Let's first consider λ^a sandwiched between an incoming and an outgoing quark

$$\langle q,i|\lambda^a|q',j\rangle \;=\; \left(\lambda^a\right)_{ij}\langle q|q'\rangle \tag{120}$$

where on the LHS i and j are the color labels of the quark and on the RHS they are the labels of a matrix-element of the 3×3 Gell-Mann matrix λ^a; the matrix element $\langle q|q'\rangle$ carries all the other quantum numbers of the quarks (n, m, and x for DLCQ). Thus we have

$$\langle q_1,i;q_2,k|\lambda_1^a\lambda_2^a|q_1',j;q_2',l\rangle \;=\; \left(\lambda^a\right)_{ij}\left(\lambda^a\right)_{kl}\langle q_1|q_1'\rangle\,\langle q_2|q_2'\rangle \tag{121}$$

$$\;=\; \left(\lambda^a\right)_{ij}\left(\lambda^a\right)_{kl}\langle q_1q_2|q_1'q_2'\rangle \tag{122}$$

Now we can use some (more or less) standard SU(N) algebra (see e.g. Eq. (4.134) of Cheng & Li, and also Lloyd's thesis Eq. 2.5, though I believe Richard Lloyd mixed up λ and $\lambda/2$ in his thesis)

$$\left(\lambda^a\right)_{ij}\left(\lambda^a\right)_{kl} \;=\; 2\left(\delta_{il}\,\delta_{jk}-\frac{1}{N}\,\delta_{ij}\,\delta_{kl}\right) \tag{123}$$

so that we finally arrive at

$$\tfrac{1}{4}\langle q_1,i;q_2,k|\lambda_1^a\lambda_2^a|q_1',j;q_2',l\rangle \;=\; \tfrac{1}{2}\left(\delta_{il}\,\delta_{jk}-\frac{1}{N}\,\delta_{ij}\,\delta_{kl}\right)\langle q_1|q_1'\rangle\,\langle q_2|q_2'\rangle \tag{124}$$

Next consider λ^a sandwiched between an incoming and an outgoing anti-quark

$$\langle \bar q,k|\lambda^a|\bar q',l\rangle \;=\; \left(\lambda^a\right)_{lk}\langle q|q'\rangle \tag{125}$$

which leads to

$$-\langle q_1,i;\bar q_2,k|\lambda_1^a\lambda_2^a|q_1',j;\bar q_2',l\rangle \;=\; -\left(\lambda^a\right)_{ij}\left(\lambda^a\right)_{lk}\langle q_1|q_1'\rangle\,\langle \bar q_2|\bar q_2'\rangle \tag{126}$$

$$\;=\; -2\left(\delta_{ik}\,\delta_{jl}-\frac{1}{N}\,\delta_{ij}\,\delta_{kl}\right)\langle q_1|q_1'\rangle\,\langle \bar q_2|\bar q_2'\rangle \tag{127}$$

Finally, the matrix elements involving one or two gluons

$$\langle q_1g_2|F_1^a\,F_2^a|q_1'g_2'\rangle \;=\; \tfrac{1}{2}\langle q_1g_2|\lambda_1^a\,C_2^a|q_1'g_2'\rangle \tag{128}$$

$$\langle \bar q_1g_2|F_1^a\,F_2^a|\bar q_1'g_2'\rangle \;=\; -\tfrac{1}{2}\langle \bar q_1g_2|\lambda_1^a\,C_2^a|\bar q_1'g_2'\rangle \tag{129}$$

$$\langle g_1g_2|F_1^a\,F_2^a|g_1'g_2'\rangle \;=\; \langle g_1g_2|C_1^a\,C_2^a|g_1'g_2'\rangle \tag{130}$$

For these two-parton matrix elements, we first need to evaluate C^a sandwiched between two single-gluon states

$$\langle g,b|C^a|g',c\rangle \;=\; -if^{abc}\langle g|g'\rangle \tag{131}$$

With this, we arrive at

$$\langle q_1,i;g_2,b|F_1^a\,F_2^a|q_,'j;g_2',c\rangle \;=\; -if^{abc}\,\left(\frac{\lambda^a}{2}\right)_{ij}\,\langle q_1|q_1'\rangle\,\langle g|g'\rangle \tag{132}$$

$$\;=\; -\left[\frac{\lambda^b}{2},\,\frac{\lambda^c}{2}\right]_{ij}\,\langle q_1|q_1'\rangle\,\langle g|g'\rangle \tag{133}$$

APPENDIX A: ONE-BODY MATRIX ELEMENTS, WAVEFUNCTIONS, BRA'S AND KETS

A simple reminder of bra's, kets, wavefunctions, and matrix elements.

- Wavefunctions

$$\Psi_{nm}(x) = \langle x|nm\rangle \tag{A1}$$
$$\Psi_{nm}(x)^* = \langle nm|x\rangle \tag{A2}$$
$$\tilde{\Psi}_{nm}(p) = \langle p|nm\rangle \tag{A3}$$
$$\tilde{\Psi}_{nm}(p)^* = \langle nm|p\rangle \tag{A4}$$
$$\tag{A5}$$

  all normalized to one

$$\langle nm|n'm'\rangle = \int d^2x\ \Psi_{nm}(x)^*\ \Psi_{n'm'}(x) = \delta_{nn'}\ \delta mm' \tag{A6}$$
$$= \int \frac{d^2p}{(2\pi)^2}\ \tilde{\Psi}_{nm}(p)^*\ \tilde{\Psi}_{n'm'}(p) \tag{A7}$$

- Fourier transform

$$\tilde{\Psi}(p) = \int d^2x\ \mathrm{e}^{i\,\vec{p}\cdot\vec{x}}\ \Psi(x) \tag{A8}$$
$$\Psi(x) = \int \frac{d^2p}{(2\pi)^2}\ \mathrm{e}^{-i\,\vec{p}\cdot\vec{x}}\ \tilde{\Psi}(p) \tag{A9}$$

  such that the wavefunctions are indeed normalized to one in both momentum space and in coordinate space, with the additional factor of $1/(2\pi)^2$ for the measure in momentum space

$$\langle nm|nm\rangle = \int d^2x\ \Psi_{nm}(x)^*\ \Psi_{nm}(x) \tag{A10}$$
$$= \int d^2x \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2}\ \mathrm{e}^{i\,(\vec{p}-\vec{q})\cdot\vec{x}}\ \tilde{\Psi}_{nm}(p)^*\ \tilde{\Psi}_{nm}(q) \tag{A11}$$
$$= \int \frac{d^2p}{(2\pi)^2} \int d^2q\ \tilde{\Psi}_{nm}(p)^*\ \tilde{\Psi}_{nm}(q)\ \delta^2(\vec{p}-\vec{q}) \tag{A12}$$
$$= \int \frac{d^2p}{(2\pi)^2}\ \tilde{\Psi}_{nm}(p)^*\ \tilde{\Psi}_{nm}(p) = 1 \tag{A13}$$

- Completeness

$$\mathbf{1} = \int d^2x\ \ |x\rangle\langle x| \tag{A14}$$
$$\mathbf{1} = \int \frac{d^2p}{(2\pi)^2}\ \ |p\rangle\langle p| \tag{A15}$$

- Matrix elements

$$\mathcal{M} = \langle n_1m_1|\mathcal{O}|n_2m_2\rangle \tag{A16}$$
$$= \int d^2x \int d^2y\ \langle n_1m_1|x\rangle\ \langle x|\mathcal{O}|y\rangle\ \langle y|n_2m_2\rangle \tag{A17}$$
$$= \int \frac{d^2p}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2}\ \langle n_1m_1|p\rangle\ \langle p|\mathcal{O}|q\rangle\ \langle q|n_2m_2\rangle \tag{A18}$$

  For a local operator

$$\langle x|\mathcal{O}|y\rangle = \delta^2(x-y)\ \ \langle x|\mathcal{O}|x\rangle \tag{A19}$$
$$\langle p|\mathcal{O}|q\rangle = \delta^2(p-q)\ \ \langle p|\mathcal{O}|p\rangle \tag{A20}$$

and thus we have for matrix elements of local operators

$$\mathcal{M} \;=\; \int d^2x\, \langle n_1 m_1|x\rangle\, \langle x|\mathcal{O}|x\rangle\, \langle x|n_2 m_2\rangle \;=\; \int d^2x\;\; \mathcal{O}_x\;\; \Psi_{n_1 m_1}(x)^*\, \Psi_{n_2 m_2}(x) \tag{A21}$$

$$= \int \frac{d^2p}{(2\pi)^2}\, \langle n_1 m_1|p\rangle\, \langle p|\mathcal{O}|p\rangle\, \langle p|n_2 m_2\rangle \;=\; \int \frac{d^2p}{(2\pi)^2}\;\; \mathcal{O}_p\;\; \tilde{\Psi}_{n_1 m_1}(p)^*\, \tilde{\Psi}_{n_2 m_2}(p) \tag{A22}$$

- Phase convention – to be discussed
  Note: the relative phase between the coordinate-space wavefunction and the momentum-space wavefuntion is fixed by the Fourier transform. With the coordinate-space wavefunction given by Eqs. (31) and (32), we have

$$\tilde{\Psi}_{n,m}(q,\theta) \;=\; \sqrt{\frac{2\,M\,\Omega}{2\pi}}\, \sqrt{\frac{n!}{(n+|m|)!}}\, \int d^2x\, \mathrm{e}^{i\,\vec{p}\cdot\vec{x}}\, \mathrm{e}^{-M\,\Omega\,\rho^2/2}\;\; \left(\sqrt{M\,\Omega}\;\rho\right)^{|m|}\;\; L_n^{|m|}(M\,\Omega\,\rho^2)\, \mathrm{e}^{i\,m\,\phi} \tag{A23}$$

$$= \sqrt{\frac{2\,M\,\Omega}{2\pi}}\, \sqrt{\frac{n!}{(n+|m|)!}}\, (2\pi)\, \mathrm{e}^{i\,m\,\theta}\;\; \int_0^\infty \rho\, d\rho\, \mathrm{e}^{i\,q\,\rho}\, \mathrm{e}^{-M\,\Omega\,\rho^2/2}\;\; \left(\sqrt{M\,\Omega}\;\rho\right)^{|m|}\;\; L_n^{|m|}(M\,\Omega\,\rho^2) \tag{A24}$$

$$= \sqrt{\frac{2}{2\pi\,M\,\Omega}}\, \sqrt{\frac{n!}{(n+|m|)!}}\, (2\pi)\, \mathrm{e}^{i\,m\,\theta}\;\; \int_0^\infty t\, dt\, \mathrm{e}^{i\,q\,t/\sqrt{M\,\Omega}}\, \mathrm{e}^{-t^2/2}\;\; (t)^{|m|}\;\; L_n^{|m|}(t^2) \tag{A25}$$

$$= \sqrt{\frac{2}{2\pi\,M\,\Omega}}\, \sqrt{\frac{n!}{(n+|m|)!}}\, (2\pi)\, \mathrm{e}^{i\,m\,\theta}\;\; \int_0^\infty \tfrac{1}{2} ds\, \mathrm{e}^{i\,q\,\sqrt{s}/\sqrt{M\,\Omega}}\, \mathrm{e}^{-s/2}\;\; (s)^{|m|/2}\;\; L_n^{|m|}(s) \tag{A26}$$

The **relative phase factor** should be fixed by above eqns, though I have not (yet) succeeded ...
Following JV's example and suggestion, let's try the lowest solution

$$\Psi_{0,0}(\rho,\phi) \;=\; \sqrt{\frac{M\,\Omega}{\pi}}\, \mathrm{e}^{-M\,\Omega\,\rho^2/2} \tag{A27}$$

in which case

$$\tilde{\Psi}_{0,0}(q,\theta) \;=\; \sqrt{\frac{M\,\Omega}{\pi}}\, \int d^2x\, \mathrm{e}^{i\,\vec{p}\cdot\vec{x}}\, \mathrm{e}^{-M\,\Omega\,\rho^2/2} \tag{A28}$$

which should be equal to (up to a phase factor?)

$$\tilde{\Psi}_{0,0}(q,\theta) \;=\; \sqrt{\frac{4\pi}{M\,\Omega}}\, \mathrm{e}^{-q^2/(2\,M\,\Omega)} \tag{A29}$$

This is easiest shown in Cartesian coordinates

$$\tilde{\Psi}_{0,0}(p_1,p_2) \;=\; \sqrt{\frac{M\,\Omega}{\pi}}\;\; \left[\int_{-\infty}^{\infty} dx_1 \mathrm{e}^{i\,p_1\,x_1}\;\mathrm{e}^{-M\,\Omega\,x_1^2/2}\right]\;\; \left[\int_{-\infty}^{\infty} dx_2 \mathrm{e}^{i\,p_2\,x_2}\;\mathrm{e}^{-M\,\Omega\,x_2^2/2}\right]$$

$$= \sqrt{\frac{M\,\Omega}{\pi}}\;\; \left[\sqrt{\frac{\pi}{M\,\Omega/2}}\;\mathrm{e}^{-p_1^2/(2\,M\,\Omega)}\right]\;\; \left[\sqrt{\frac{\pi}{M\,\Omega/2}}\;\mathrm{e}^{-p_2^2/(2\,M\,\Omega)}\right]$$

$$= \sqrt{\frac{4\pi}{M\,\Omega}}\;\mathrm{e}^{-(p_1^2+p_2^2)/(2\,M\,\Omega)} \tag{A30}$$

i.e. no aditional relative phase factor

APPENDIX B: TWO-BODY MATRIX ELEMENTS

Consider a generic two-body matrix element in the 2-dimensional transverse subspace

$$\mathcal{M} \;=\; \langle n_1 m_1, n_2 m_2 | \mathcal{O} | n_3 m_3, n_4 m_4 \rangle \tag{B1}$$

in a "product space" notation

$$\langle n_1 m_1, n_2 m_2 | \;=\; \langle n_1 m_1 | \otimes \langle n_2 m_2 | \tag{B2}$$

and similarly for the kets. Following the same procedure as for a one-body matrix element discussed in the previous section, we can write a two-body matrix element as coordinate space integrals

$$\mathcal{M} \;=\; \int d^2x_1 \int d^2x_2 \int d^2x_3 \int d^2x_4 \; \Psi_{n_1 m_1}(x_1)^* \; \Psi_{n_2 m_2}(x_2)^* \;\; \langle x_1, x_2 | \mathcal{O} | x_3, x_4 \rangle \;\; \Psi_{n_3 m_3}(x_3) \; \Psi_{n_4 m_4}(x_4) \tag{B3}$$

or equivalently as momentum space integrals

$$\mathcal{M} \;=\; \int \frac{d^2p_1}{(2\pi)^2} \int \frac{d^2p_2}{(2\pi)^2} \int \frac{d^2p_2}{(2\pi)^2} \int \frac{d^2p_4}{(2\pi)^2} \; \tilde{\Psi}_{n_1 m_1}(p_1)^* \; \tilde{\Psi}_{n_2 m_2}(p_2)^* \;\; \langle p_1, p_2 | \mathcal{O} | p_3, p_4 \rangle \;\; \tilde{\Psi}_{n_3 m_3}(p_3) \; \tilde{\Psi}_{n_4 m_4}(p_4) \tag{B4}$$

The two-body kinetic energy operator is simply the sum of the product of two one-particle kinetic energy operators

$$T_{\text{two-body}} \;=\; T_1 \otimes \mathbf{1} + \mathbf{1} \otimes T_2 \tag{B5}$$

and thus

$$\langle p_1, p_2 | T_{\text{two-body}} | p_3, p_4 \rangle \;=\; \left(\frac{p_1^2}{2\,M_1} + \frac{p_2^2}{2\,M_2} \right) \; \delta^2(p_1 - p_3) \; \delta^2(p_2 - p_4) \tag{B6}$$

The corresponding matrix elements factorize and reduce to

$$\begin{aligned} \langle n_1 m_1, n_2 m_2 | T_{\text{two-body}} | n_3 m_3, n_4 m_4 \rangle \;&=\; \left(\int \frac{d^2p_1}{(2\pi)^2} \; \tilde{\Psi}_{n_1 m_1}(p_1)^* \; \frac{p_1^2}{2\,M_1} \; \tilde{\Psi}_{n_3 m_3}(p_1) \right) \left(\int \frac{d^2p_2}{(2\pi)^2} \; \tilde{\Psi}_{n_2 m_2}(p_2)^* \; \tilde{\Psi}_{n_4 m_4}(p_2) \right) \\ &\quad + \left(\int \frac{d^2p_1}{(2\pi)^2} \; \tilde{\Psi}_{n_1 m_1}(p_1)^* \; \tilde{\Psi}_{n_3 m_3}(p_1) \right) \left(\int \frac{d^2p_2}{(2\pi)^2} \; \tilde{\Psi}_{n_2 m_2}(p_2)^* \; \frac{p_2^2}{2\,M_2} \; \tilde{\Psi}_{n_4 m_4}(p_2) \right) \\ &=\; \langle n_1 m_1 | T | n_3 m_3 \rangle \; \delta_{n_2, n_4} \; \delta_{m_2, m_4} + \; \delta_{n_1, n_3} \; \delta_{m_1, m_3} \langle n_2 m_2 | T | n_4 m_4 \rangle \end{aligned} \tag{B7}$$

where I have used the fact that the wavefunctions are orthonormal.

Intrinsic two-body interactions correspond to operators of the type

$$\langle p_1, p_2 | \mathcal{O} | p_3, p_4 \rangle \;=\; \mathcal{V}_{12 \to 34} \; \delta^2(p_1 + p_2 - p_3 - p_4) \tag{B8}$$

and give rise to matrix elements of the type

$$\begin{aligned} \mathcal{M} \;=\; \langle n_1 m_1, n_2 m_2 | \mathcal{O} | n_3 m_3, n_4 m_4 \rangle \;=\; \int \frac{d^2p_1}{(2\pi)^2} \int \frac{d^2p_2}{(2\pi)^2} \int \frac{d^2p_3}{(2\pi)^2} \int \frac{d^2p_4}{(2\pi)^2} \\ \delta^2(p_1 + p_2 - p_3 - p_4) \; \Psi_{n_1 m_1}(p_1)^* \; \Psi_{n_2 m_2}(p_2)^* \;\; \mathcal{V}_{12 \to 34} \;\; \Psi_{n_3 m_3}(p_3) \; \Psi_{n_4 m_4}(p_4) \end{aligned} \tag{B9}$$

The δ-function eliminates one of the four 2-dimensional integrals, and we are left with (at most) three 2-dimensional integrals to be done numerically.

APPENDIX C: SPECIFIC MATRIX ELEMENTS NECESSARY FOR $T_{\rm relative}$

For the two-body matrix elements of the relative kinetic energy operator $T_{\rm relative}$ we need to evaluate matrix elements of the type

$$\begin{aligned}
\mathcal{M} &= \langle n_1 m_1, n_2 m_2 | \vec{p_1} \cdot \vec{p_2} | n_1' m_1', n_2' m_2' \rangle \\
&= \int \frac{d^2 p_1}{(2\pi)^2} \int d^2 p_2 \frac{d^2 p_1}{(2\pi)^2} \; \Psi_{n_1 m_1}(p_1)^* \; \Psi_{n_2 m_2}(p_2)^* \; (\vec{p_1} \cdot \vec{p_2}) \; \Psi_{n_1' m_1'}(p_1) \; \Psi_{n_2' m_2'}(p_2) \qquad \text{(C1)} \\
&= \frac{1}{(2\pi)^6} \int d\theta_1 \int d\theta_2 \; \mathrm{e}^{i\,(m_1' - m_1)\,\theta_1} \; \mathrm{e}^{i\,(m_2' - m_2)\,\theta_2} \; \cos(\theta_1 - \theta_2) \\
&\quad \times \int q_1^2 dq_1 \; \tilde{f}_{n_1, m_1}(q_1) \; \tilde{f}_{n_1', m_1'}(q_1) \; \times \int q_2^2 dq_2 \; \tilde{f}_{n_2, m_2}(q_2) \; \tilde{f}_{n_2', m_2'}(q_2) \qquad \text{(C2)} \\
&= \frac{1}{2(2\pi)^6} \left[\int d\theta_1 \; \mathrm{e}^{i\,(m_1' - m_1 + 1)\,\theta_1} \; \times \int d\theta_2 \; \mathrm{e}^{i\,(m_2' - m_2 - 1)\,\theta_2} \; + \int d\theta_1 \; \mathrm{e}^{i\,(m_1' - m_1 - 1)\,\theta_1} \; \times \int d\theta_2 \; \mathrm{e}^{i\,(m_2' - m_2 + 1)\,\theta_2} \right] \\
&\quad \times \int q_1^2 dq_1 \; \tilde{f}_{n_1, m_1}(q_1) \; \tilde{f}_{n_1', m_1'}(q_1) \; \times \int q_2^2 dq_2 \; \tilde{f}_{n_2, m_2}(q_2) \; \tilde{f}_{n_2', m_2'}(q_2) \qquad \text{(C3)} \\
&= \frac{1}{2} \Big[\delta_{m_1 + 1, m_1'} \; \delta_{m_2 - 1, m_2'} + \delta_{m_1 - 1, m_1'} \; \delta_{m_2 + 1, m_2'} \Big] \; I_{n_1, m_1, n_1', m_1'} \; I_{n_2, m_2, n_2', m_2'} \qquad \text{(C4)}
\end{aligned}$$

with

$$I_{n,m,n',m'} \;=\; \frac{1}{(2\pi)^2} \int q^2 dq \; \tilde{f}_{n,m}(q) \; \tilde{f}_{n',m'}(q) \qquad \text{(C5)}$$

Note the magnetic projection conservation: $m_1 + m_2 = m_1' + m_2'$

Next, we need to evaluate $I_{n,m,n',m'}$ for general n and n', but for $m' = m \pm 1$ only due to the Kronecker δ's on the magnetic projection quantum numbers

$$\begin{aligned}
I_{n,m,n',m\pm1} \;=\; & \frac{2}{M\,\Omega} \sqrt{\frac{n!}{(n+|m|)!}} \sqrt{\frac{n'!}{(n'+|m\pm1|)!}} \int dq \; q^2 \; \mathrm{e}^{-q^2/(M\,\Omega)} \\
& \left(\frac{q}{\sqrt{M\,\Omega}} \right)^{|m|} \left(\frac{q}{\sqrt{M\,\Omega}} \right)^{|m\pm1|} \; L_n^{|m|}(q^2/(M\,\Omega)) \; L_{n'}^{|m\pm1|}(q^2/(M\,\Omega)) \qquad \text{(C6)}
\end{aligned}$$

1. Case A

First, consider $I_{n,m,n',m+1}$ with $m \geq 0$

$$\begin{aligned}
I_{n,m,n',m+1} \;=\; & \frac{1}{\sqrt{M\,\Omega}} \sqrt{\frac{n!}{(n+m)!}} \sqrt{\frac{n'!}{(n'+m+1)!}} \int d(q^2) \; \mathrm{e}^{-q^2/(M\,\Omega)} \\
& \left(\frac{q}{\sqrt{M\,\Omega}} \right)^{2m+2} \; L_n^{|m|}(q^2/(M\,\Omega)) \; L_{n'}^{|m+1|}(q^2/(M\,\Omega)) \qquad \text{(C7)}
\end{aligned}$$

and change integration variables to $s = q^2/(M\Omega)$

$$I_{n,m,n',m+1} \;=\; \sqrt{M\,\Omega} \sqrt{\frac{n!}{(n+m)!}} \sqrt{\frac{n'!}{(n'+m+1)!}} \int ds \; \mathrm{e}^{-s} \; s^{m+1} \; L_n^{|m|}(s) \; L_{n'}^{|m+1|}(s) \qquad \text{(C8)}$$

and use 22.7.30 from Abramowitz and Stegun

$$L_n^{|m|}(s) \;=\; L_n^{|m|+1}(s) - L_{n-1}^{|m|+1}(s) \qquad \text{(C9)}$$

so we get

$$\begin{aligned}
I_{n,m,n',m+1} \;=\; & \sqrt{M\,\Omega} \sqrt{\frac{n!}{(n+m)!}} \sqrt{\frac{n'!}{(n'+m+1)!}} \\
& \left[\int ds \; \mathrm{e}^{-s} \; s^{m+1} \; L_n^{|m|+1}(s) \; L_{n'}^{|m+1|}(s) \; - \int ds \; \mathrm{e}^{-s} \; s^{m+1} \; L_{n-1}^{|m|+1}(s) \; L_{n'}^{|m+1|}(s) \right] \qquad \text{(C10)}
\end{aligned}$$

Now we can apply the orthogonality relations for the Generalized Laguerre polynomials

$$\begin{aligned} I_{n,m,n',m+1} &= \sqrt{M\,\Omega}\,\sqrt{\frac{n!}{(n+m)!}}\,\sqrt{\frac{n'!}{(n'+m+1)!}}\,\left[\delta_{n,n'}\,\frac{(n+m+1)!}{n!} - \delta_{n-1,n'}\,\frac{(n+m)!}{(n-1)!}\right] \\ &= \sqrt{M\,\Omega}\,\left[\delta_{n,n'}\,\sqrt{n+m+1} - \delta_{n-1,n'}\,\sqrt{n}\right] \end{aligned} \tag{C11}$$

APPENDIX D: GELL-MANN MATRICES ACTING ON QUARK COLOR STATES

We have obtained the conclusion that the operator F^2 gives zero when acting on the color singlet states, and positive for all color-nonsinglet states. In this appendix, we will see what we can obtain when eight Gell-Mann matrices acting on quark color states seperately. In the future, we will use these to construct color singlet state.

At first we are going to use these definitions.

$$\begin{aligned} I_+ &= F_1 + iF_2, I_- = F_1 - iF_2, \\ I_3 &= F_3, \\ U_+ &= F_6 + iF_7, U_- = F_6 - iF_7, \\ V_+ &= F_4 + iF_5, V_- = F_4 - iF_5, \\ Y &= F_8 \end{aligned} \tag{D1}$$

Also let us define the quark color state by using two quantum number t_3 and y. Therefore using $SU(3)$ algebra, we have

$$\begin{aligned} F_3|t_3, y\rangle &= t_3|t_3, y\rangle \\ F_8|t_3, y\rangle &= \frac{\sqrt{3}}{2}y|t_3, y\rangle \\ I_+|t_3, y\rangle &= \sqrt{(t - t_3)(t + t_3 + 1)}|t_3 + 1, y\rangle \\ I_-|t_3, y\rangle &= \sqrt{(t + t_3)(t - t_3 + 1)}|t_3 - 1, y\rangle \end{aligned} \tag{D2}$$

Using the relation between F_1, F_2 and I_+, I_- and also the result $t = \frac{1}{2}$, we obtain

$$\begin{aligned} F_1|t_3, y\rangle &= \frac{1}{2}(\sqrt{(\frac{1}{2} - t_3)(\frac{1}{2} + t_3 + 1)}|t_3 + 1, y\rangle \\ &\quad + \sqrt{(\frac{1}{2} - t_3)(\frac{1}{2} + t_3 + 1)}|t_3 + 1, y\rangle)\delta_{|t_3|,\frac{1}{2}} \\ F_2|t_3, y\rangle &= \frac{1}{2i}(\sqrt{(\frac{1}{2} - t_3)(\frac{1}{2} + t_3 + 1)}|t_3 + 1, y\rangle \\ &\quad - \sqrt{(\frac{1}{2} - t_3)(\frac{1}{2} + t_3 + 1)}|t_3 + 1, y\rangle)\delta_{|t_3|,\frac{1}{2}} \end{aligned} \tag{D3}$$

Then we can define two more new quantum numbers

$$V_3 = \frac{3}{4}y + \frac{1}{2}t_3,\ U_3 = \frac{3}{4}y - \frac{1}{2}t_3 \tag{D4}$$

Therefore three quarks carry these two quantum numbers seperately. We have

$$\begin{aligned} V_3^R &= \frac{1}{2}, V_3^G = 0, V_3^B = -\frac{1}{2}; \\ U_3^R &= 0, U_3^G = \frac{1}{2}, U_3^B = -\frac{1}{2}. \end{aligned} \tag{D5}$$

Finally if we do a similar calculation as what we have done for I_+ and I_-, we can obtain the result

$$\begin{aligned} F_4|t_3, y\rangle &= \frac{1}{2}(\sqrt{(\frac{1}{2} - \frac{1}{2}t_3 - \frac{3}{4}y)(\frac{1}{2} + \frac{1}{2}t_3 + \frac{3}{4}y + 1)}|t_3 + \frac{1}{2}, y + 1\rangle \\ &\quad + \sqrt{(\frac{1}{2} + \frac{1}{2}t_3 + \frac{3}{4}y)(\frac{1}{2} - \frac{1}{2}t_3 - \frac{3}{4}y + 1)}|t_3 - \frac{1}{2}, y - 1\rangle)\delta_{|\frac{1}{2}t_3 + \frac{3}{4}y|,\frac{1}{2}} \\ F_5|t_3, y\rangle &= \frac{1}{2i}(\sqrt{(\frac{1}{2} - \frac{1}{2}t_3 - \frac{3}{4}y)(\frac{1}{2} + \frac{1}{2}t_3 + \frac{3}{4}y + 1)}|t_3 + \frac{1}{2}, y + 1\rangle \\ &\quad - \sqrt{(\frac{1}{2} + \frac{1}{2}t_3 + \frac{3}{4}y)(\frac{1}{2} - \frac{1}{2}t_3 - \frac{3}{4}y + 1)}|t_3 - \frac{1}{2}, y - 1\rangle)\delta_{|\frac{1}{2}t_3 + \frac{3}{4}y|,\frac{1}{2}} \end{aligned}$$

$$
\begin{aligned}
F_6|t_3,y\rangle &= \frac{1}{2}(\sqrt{(\frac{1}{2}+\frac{1}{2}t_3-\frac{3}{4}y)(\frac{1}{2}-\frac{1}{2}t_3+\frac{3}{4}y+1)}|t_3-\frac{1}{2},y+1\rangle \\
&\quad +\sqrt{(\frac{1}{2}-\frac{1}{2}t_3+\frac{3}{4}y)(\frac{1}{2}+\frac{1}{2}t_3-\frac{3}{4}y+1)}|t_3+\frac{1}{2},y-1\rangle)\delta_{|-\frac{1}{2}t_3+\frac{3}{4}y|,\frac{1}{2}} \\
F_7|t_3,y\rangle &= \frac{1}{2i}(\sqrt{(\frac{1}{2}+\frac{1}{2}t_3-\frac{3}{4}y)(\frac{1}{2}-\frac{1}{2}t_3+\frac{3}{4}y+1)}|t_3-\frac{1}{2},y+1\rangle \\
&\quad -\sqrt{(\frac{1}{2}-\frac{1}{2}t_3+\frac{3}{4}y)(\frac{1}{2}+\frac{1}{2}t_3-\frac{3}{4}y+1)}|t_3+\frac{1}{2},y-1\rangle)\delta_{|-\frac{1}{2}t_3+\frac{3}{4}y|,\frac{1}{2}}. \qquad \text{(D6)}
\end{aligned}
$$

APPENDIX E: COLOR MATRICES ACTING ON GLUON COLOR STATES

In this section, we are continuing on and show what we can obtain when the eight color matrices act on gluon color states.

Since in adjoint representation, gluon eigenstats g_3 and g_8 can not be completely distinguished by two quantum numbers t_3 and y (both of them have the same quantum $t_3 = 0$ and $y = 0$), we have to add one more quantum number t when we express the color eigenstates. Therefore we can see that g_3 has quantum number $t = 1$ and g_8 has quantum number $t = 0$.

In the previous appendix, we have used the following operators:I_+, I_-, V_+, V_-, U_+ and U_-. In this appendix, we still take advantage of these operators. And we notise that for I_+ and I_-, we have

$$\begin{aligned}
I_+g_2 &\rightarrow g_3, I_+g_3 \rightarrow g_1, I_+g_1 = 0;\\
I_-g_1 &\rightarrow g_3, I_-g_3 \rightarrow g_2, I_-g_2 = 0;\\
I_+g_6 &\rightarrow g_4, I_+g_4 = 0;\\
I_-g_4 &\rightarrow g_6, I_-g_6 = 0;\\
I_+g_5 &\rightarrow g_7, I_+g_7 = 0;\\
I_-g_7 &\rightarrow g_5, I_-g_5 = 0.
\end{aligned} \tag{E1}$$

For V_+ and V_-, we have

$$\begin{aligned}
V_+g_7 &\rightarrow g_1, V_+g_1 = 0;\\
V_-g_1 &\rightarrow g_7, V_-g_7 = 0;\\
V_+g_2 &\rightarrow g_6, V_+g_6 = 0;\\
V_-g_6 &\rightarrow g_2, V_-g_2 = 0;\\
V_+g_5 &\rightarrow g_3+g_8, V_+(g_3+g_8) \rightarrow g_4, V_+g_4 = 0;\\
V_-g_4 &\rightarrow g_3+g_8, V_-(g_3+g_8) \rightarrow g_5, V_-g_5 = 0;
\end{aligned} \tag{E2}$$

For U_+ and U_-, we have

$$\begin{aligned}
U_+g_1 &\rightarrow g_4, U_+g_4 = 0;\\
U_-g_4 &\rightarrow g_1, U_-g_1 = 0;\\
U_+g_5 &\rightarrow g_2, U_+g_2 = 0;\\
U_-g_2 &\rightarrow g_5, U_-g_5 = 0;\\
U_+g_7 &\rightarrow g_3+g_8, U_+(g_3+g_8) \rightarrow g_6, U_+g_6 = 0;\\
U_-g_6 &\rightarrow g_3+g_8, U_-(g_3+g_8) \rightarrow g_7, U_-g_7 = 0;
\end{aligned} \tag{E3}$$

Using the same way we used when we calculated the quark color states, we can obtain the results which are given below.

$$\begin{aligned}
F_1|t_3,t,y\rangle &= \frac{y}{2}\delta_{|t_3|,\frac{1}{2}}(\sqrt{(\frac{1}{2}-t_3)(\frac{1}{2}+t_3+1)}|t_3+1,t,y\rangle + \sqrt{(\frac{1}{2}+t_3)(\frac{1}{2}-t_3+1)}|t_3-1,t,y\rangle)\\
&\quad +\frac{1}{2}\delta_{t,1}\delta_{y,0}((-1)^{1-t_3}(1-t_3)|t_3+1,t,y\rangle + (-1)^{t_3}(1+t_3)|t_3-1,t,y\rangle)\\
&\quad +\frac{1}{2}\delta_{t,0}\delta_{y,0}(t_3+y)|t_3,t,y\rangle\\
F_2|t_3,t,y\rangle &= \frac{y}{2i}\delta_{|t_3|,\frac{1}{2}}(\sqrt{(\frac{1}{2}-t_3)(\frac{1}{2}+t_3+1)}|t_3+1,t,y\rangle - \sqrt{(\frac{1}{2}+t_3)(\frac{1}{2}-t_3+1)}|t_3-1,t,y\rangle)\\
&\quad +\frac{1}{2i}\delta_{t,1}\delta_{y,0}((-1)^{1-t_3}(1-t_3)|t_3+1,t,y\rangle - (-1)^{t_3}(1+t_3)|t_3-1,t,y\rangle)\\
&\quad +\frac{1}{2i}\delta_{t,0}\delta_{y,0}(t_3-y)|t_3,t,y\rangle\\
F_3|t_3,t,y\rangle &= t_3|t_3,t,y\rangle\\
F_4|t_3,t,y\rangle &= \delta_{|\frac{3}{4}y+\frac{1}{2}t_3|,\frac{1}{2}}(\frac{1}{2}t_3-\frac{1}{4}y)(\sqrt{(\frac{1}{2}-\frac{3}{4}y-\frac{1}{2}t_3)(\frac{1}{2}+\frac{3}{4}y+\frac{1}{2}t_3+1)}|t_3+\frac{1}{2},t,y+1\rangle
\end{aligned}$$

$$
\begin{aligned}
&+\sqrt{(\frac{1}{2}+\frac{3}{4}y+\frac{1}{2}t_3)(\frac{1}{2}-\frac{3}{4}y-\frac{1}{2}t_3+1)}|t_3-\frac{1}{2},t,y-1\rangle)\\
&+\delta_{t_3,0}\delta_{y,0}\delta_{t,1}((-\frac{1}{4})|t_3+\frac{1}{2},t-\frac{1}{2},y+1\rangle+\frac{1}{4}|t_3-\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,\frac{1}{2}}\delta_{y,1}\delta_{t,\frac{1}{2}}((-\frac{1}{2})|-t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle-\frac{\sqrt{3}}{2}|-t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,-\frac{1}{2}}\delta_{y,-1}\delta_{t,\frac{1}{2}}(\frac{1}{2}|-t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle+\frac{\sqrt{3}}{2}|-t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,0}\delta_{y,0}\delta_{t,0}((-\frac{\sqrt{3}}{4})|t_3+\frac{1}{2},t+\frac{1}{2},y+1\rangle+\frac{\sqrt{3}}{4}|t_3-\frac{1}{2},t+\frac{1}{2},y-1\rangle)\\
F_5|t_3,t,y\rangle &= \delta_{|\frac{3}{4}y+\frac{1}{2}t_3|,\frac{1}{2}}(\frac{1}{2i}t_3-\frac{1}{4i}y)(\sqrt{(\frac{1}{2}-\frac{3}{4}y-\frac{1}{2}t_3)(\frac{1}{2}+\frac{3}{4}y+\frac{1}{2}t_3+1)}|t_3+\frac{1}{2},t,y+1\rangle\\
&-\sqrt{(\frac{1}{2}+\frac{3}{4}y+\frac{1}{2}t_3)(\frac{1}{2}-\frac{3}{4}y-\frac{1}{2}t_3+1)}|t_3-\frac{1}{2},t,y-1\rangle)\\
&+\delta_{t_3,0}\delta_{y,0}\delta_{t,1}(\frac{i}{4}|t_3+\frac{1}{2},t-\frac{1}{2},y+1\rangle+\frac{i}{4}|t_3-\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,\frac{1}{2}}\delta_{y,1}\delta_{t,\frac{1}{2}}((-\frac{i}{2})|-t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle-\frac{\sqrt{3}i}{2}|-t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,-\frac{1}{2}}\delta_{y,1}\delta_{t,\frac{1}{2}}(-\frac{i}{2}|-t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle-\frac{\sqrt{3}i}{2}|-t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,0}\delta_{y,0}\delta_{t,0}((\frac{\sqrt{3}i}{4})|t_3+\frac{1}{2},t+\frac{1}{2},y+1\rangle+\frac{\sqrt{3}i}{4}|t_3-\frac{1}{2},t+\frac{1}{2},y-1\rangle)\\
F_6|t_3,t,y\rangle &= \delta_{|\frac{3}{4}y-\frac{1}{2}t_3|,\frac{1}{2}}(-\frac{1}{2}t_3-\frac{1}{4}y)(\sqrt{(\frac{1}{2}-\frac{3}{4}y+\frac{1}{2}t_3)(\frac{1}{2}+\frac{3}{4}y-\frac{1}{2}t_3+1)}|t_3-\frac{1}{2},t,y+1\rangle\\
&+\sqrt{(\frac{1}{2}+\frac{3}{4}y-\frac{1}{2}t_3)(\frac{1}{2}-\frac{3}{4}y+\frac{1}{2}t_3+1)}|t_3+\frac{1}{2},t,y-1\rangle)\\
&+\delta_{t_3,0}\delta_{y,0}\delta_{t,1}(\frac{1}{4}|t_3-\frac{1}{2},t-\frac{1}{2},y+1\rangle-\frac{1}{4}|t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,-\frac{1}{2}}\delta_{y,1}\delta_{t,\frac{1}{2}}(\frac{1}{2}|t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle-\frac{\sqrt{3}}{2}|t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,\frac{1}{2}}\delta_{y,-1}\delta_{t,\frac{1}{2}}((-\frac{1}{2})|t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle+\frac{\sqrt{3}}{2}|t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,0}\delta_{y,0}\delta_{t,0}((-\frac{\sqrt{3}}{4})|t_3-\frac{1}{2},t+\frac{1}{2},y+1\rangle+\frac{\sqrt{3}}{4}|t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle)\\
F_7|t_3,t,y\rangle &= \delta_{|\frac{3}{4}y-\frac{1}{2}t_3|,\frac{1}{2}}(-\frac{1}{2i}t_3-\frac{1}{4i}y)(\sqrt{(\frac{1}{2}-\frac{3}{4}y+\frac{1}{2}t_3)(\frac{1}{2}+\frac{3}{4}y-\frac{1}{2}t_3+1)}|t_3-\frac{1}{2},t,y+1\rangle\\
&-\sqrt{(\frac{1}{2}+\frac{3}{4}y-\frac{1}{2}t_3)(\frac{1}{2}-\frac{3}{4}y+\frac{1}{2}t_3+1)}|t_3+\frac{1}{2},t,y-1\rangle)\\
&+\delta_{t_3,0}\delta_{y,0}\delta_{t,1}((-\frac{i}{4}|t_3-\frac{1}{2},t-\frac{1}{2},y+1\rangle-\frac{i}{4}|t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,-\frac{1}{2}}\delta_{y,1}\delta_{t,\frac{1}{2}}(\frac{i}{2}|t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle-\frac{\sqrt{3}i}{2}|t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,\frac{1}{2}}\delta_{y,-1}\delta_{t,\frac{1}{2}}((-\frac{i}{2})|t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle-\frac{\sqrt{3}i}{2}|t_3+\frac{1}{2},t-\frac{1}{2},y-1\rangle)\\
&+\delta_{t_3,0}\delta_{y,0}\delta_{t,0}((\frac{\sqrt{3}i}{4})|t_3-\frac{1}{2},t+\frac{1}{2},y+1\rangle+\frac{\sqrt{3}i}{4}|t_3+\frac{1}{2},t+\frac{1}{2},y-1\rangle)\\
F_8|t_3,t,y\rangle &= \frac{\sqrt{3}}{2}y|t_3,t,y\rangle
\end{aligned}
\tag{E4}
$$

APPENDIX F: DEALING WITH COLOR ALGEBRA

1. Nov 2007

About the treatment of the color space:

Introduce a color operator H_λ such that the desired eigenvalue spectrum of the color operator $\sum_{i,j}\lambda_i\lambda_j - D_0$ is be such that $E_\lambda = 0$ for color singlet states and $E_\lambda > 0$ for color non-singlet states. (D_0 is a constant shift to make the eigenvalue zero for color-singlet states.) This color operator is basically a sum of 2-body operators $\langle g_1 g_2|\lambda_1\lambda_2|g_1' g_2'\rangle = \langle g_1|\lambda_1|g_1'\rangle\langle g_2|\lambda_2|g_2'\rangle$. Therefore, H_λ between two many-body states can be evaluated as a sum over 2-body matrix elements.

Suggested sample problems:

- $|q_1 q_2\rangle$
- $|g_1 g_2\rangle$
- $|q_1 q_2 g$, $|q_1 q_2 \bar{q}_3 \bar{q}_4\rangle$
- $|q_1 q_2 q_3\rangle$
- $|q_1 q_2 q_3 g\rangle$, $|q_1 q_2 q_3 \bar{q}_4 \bar{q}_5\rangle\rangle$

Now we can use a Lagrange multiplier to remove the color-nonsinglet states from the low-lying spectrum (similar to the way center-of-mass excitations are removed from the low-lying spectrum in MFD): The role of Lagrange multiplier is to shift the eigenvalues of the Hamiltonian H, such that for the modified Hamiltonian

$$H' = H + c\left(\sum_{i,j}\lambda_i\lambda_j - D_0\right), \tag{F1}$$

the eigenvalues for the color-singlets are those of the original H, but for non-singlet states the eigenvalues of H' are shifted by $c \gg 0$.

Another possibility (suggested by James) might be to diagonalize H_λ explicitly, and retain only color-singlet states in our MB-basis. Let $\alpha = (b, f, n, m, x, s_z, Y, t_3, t)$, where b is the baryon number, f is the flavour and Y, t_3, t are the indices associated with the color, and $\tilde{\alpha} = (b, f, n, m, x, s_z)$, i.e. all the indices except for the color ones. The idea is to diagonalize $\lambda_i\lambda_j - D_0$ for configurations $a^\dagger_\alpha \cdots a^\dagger_\delta b^\dagger_{\tilde{\alpha}} \cdots b^\dagger_{\tilde{\delta}}|0\rangle$, and retain only color-singlet states $|\tilde{\psi}\rangle = \sum_i C_i|\psi_i\rangle$.

The gain for diagonalizing color space separately is that we greatly reduce our basis space: we only keep true color-singlet states, instead of having all states with zero color-projection: $\sum t_3 = \sum y_3 = 0$. The price we pay is that the MB-states $|\tilde{\psi}\rangle$ are superpositions of "simple" states, which greatly complicates the construction of the MB-Hamiltonian; it also leads to a MB-matrix that is much more dense than the "original" MB-matrix. This last point is not necessarily a bad thing, but we may need to use a different algorithm/implementation for the diagonalization. However, that is something we can worry about once we know whether or not this approach works in principle...

The key thing is whether or not the color degrees-of-freedom (DoF) factorize, and it is not completely obvious that they do. As a test, we could try out this idea for $\text{MFD}_{\text{nuclear}}$: constructing a MB-basis with fixed total spin J (rather than using a MB-basis with fixed spin projection m_j, as we do now). However, now that I (PM) think about it, the SU(2) spin depends on the intrinsic spin AND the spatial wavefunctions, whereas the SU(3) color depends only on the intrinsic color. So I suspect that the factorization of color degrees of freedom is doable, but the factorization of the spin D.o.F. may not be...

2. Dec 2007

The general consensus seems to be that the color algebra factorizes completely from the other degrees-of-freedom (in contrast to the spin, which couples with the orbital angular momentum!). So we are not going to test any ideas about factorizing SU(3) color by looking at SU(2) spin degrees-of-freedom in $\text{MFD}_{\text{nuclear}}$ (though we still are interested in a fixing total J in MFD). Rather, let us go aehad with factorizing the color algebra.

First, some notation/convention for an n-particle state

$$|\psi\rangle = |\alpha_1 \dots \alpha_n\rangle \tag{F2}$$
$$= |C_1 \dots C_n\rangle \otimes |a_1 \dots a_n\rangle \tag{F3}$$

where $|\alpha\rangle$ refers to the complete set of quantum numbers, $|C\rangle$ to the color quantum numbers only, and $|a\rangle$ to the rest of the quantum numbers.

Jun Li has (or rather, is working on) a program to generate all color-singlet states for a given Fock space sector; that is, it generates color-singlet state-vectors

$$|C_1 \dots C_n\rangle \tag{F4}$$

for a given number of quarks, antiquarks, and gluons. More or less independently, MFD can (or should be able to) generate a colorless Many-Body basis space

$$|a_1 \dots a_n\rangle \tag{F5}$$

without any SU(3) color indices. The actual Many-Body basis space needed in MFD is simply the direct product of the color vectors times the spin-space-time vectors.

A subtle difference with $\text{MFD}_{\text{nuclear}}$ is in the (anti-)symmetrization. For identical quarks (and for identical antiquarks), the total wave function

$$|\psi\rangle = |\alpha_1 \dots \alpha_n\rangle \tag{F6}$$

should be antisymmetric in all its indices.

I suspect (based on a handfull examples we're familiar with) that for quarks (and for antiquarks) the color-singlet states are completely antisymmetric: that is a color-singlet $|q_1 q_2 q_3\rangle$ state is antisymmetric in its color indices, and should therefore be symmetric in its spin-space-time indices. We believe that this observation holds for multi-quark states in general, but I don't have a proof. We can start by checking a few "simple" cases.

On the other hand I believe that color-singlets of multiple gluons are totally symmetric in its color indices. Note that I have even less proof for this statement than for the statement about quarks: it is true that the 2-gluon color-singlet is totally symmetric in its color indices, and that's about all I know right now.

Therefore, the spin-space-time vectors should be totally symmetric, both gluons and for quarks (and antiquarks); also, we can have identical fermions with the same spin-space-time quantum numbers, since they will have different color quantum numbers. Thus, as far as (anti-)symmetrization and Bose–Einstein vs. Fermi–Dirac statistics is concerned, we have the treat the spin-space-time part of the wave function the same for both fermions (i.e. quarks and antiquarks) and bosons (i.e. gluons).

Checks to be done:
Are we correct in the assertion that any color-singlet state is anti-symmetric in the color quantum numbers of (identical) quarks (and antiquarks), and symmetric in the color quantum numbers of the gluons?

A test of a handful of "simple" case should give us a clue.

APPENDIX G: COLOR SINGLET STATES

As we know, we can not observe the free quarks in reality. What we can see is mesons and baryons which are colorless particles. Therefore we need to consider how to obtain the color singlet states in the color space. Here we take advantage of the operator F^2 which will gives zero eigenvalue when it acts on color singlet states. On the contrary, it will give positive eigenvalues. In the following sections, we will consider different configurations formed either by quarks or gluons. Then we will show how to obtain the color singlet states by using the operator F^2. At first, we list out all the color configurations of the system we are considering. The color singlet state will be the linear combination of some of these color configurations. Then we can get the representation of operator F^2 under the color configurations. Using this method, usually we will get very large matrix. In order to reduce the size of the matrix F^2, we can use two additional operators t_3 and y which will give zero when they are acting on the candidates which could form the color singlet states. Therefore we can firstly use there two operators to prescribe each color configuration before we find the representation of F^2. On this way, it is more efficient since we would get the smaller size matrix of F^2. Finally we will diagonalize the matrix of F^2 to find if there are zero eigenvalues which correspond to color singlet states.

In the following, we will consider the different systems and find the color singlet states if they exist.

1. Mesons

As we know that one quark and one antiquark can form color singlet state, which is called meson. Operator F^2 acting on the color singlet state gives zero eigenvalue. On the other hand, F^2 gives positive eigenvalues when it acts on color non-singlet states. In order to obtain the representation of F^2, we need to find out all the color configuration for one quark and one antiquark. Each quark can have one of three colors. Totally we have $3 \times 3 = 9$ color configurations. Therefore we have $F^2_{9\times 9}$ matrix. In order to reduce the size of matrix, we can take advantage of t_3 and y. As a result, we only consider three color configurations $R\bar{R}$,$G\bar{G}$ and $B\bar{B}$ because it gives zero when t_3 and y act on each of them. Finally we only need to diagonalize $F^2_{3\times 3}$.

After diagonalizing F^2, we obtain one color singlet state, which is given by

$$\frac{1}{\sqrt{3}}(R\bar{R} + G\bar{G} + B\bar{B}) \tag{G1}$$

2. Baryons

Three quarks can also form the color singlet state which is called baryon. Using the same method, we obtain $F^2_{6\times 6}$ instead of $F^2_{27\times 27}$.

After diagonalizing $F^2_{6\times 6}$, we obtain one color singlet state which is given by.

$$\frac{1}{\sqrt{6}}(RGB - RBG + BRG - BGR + GBR - GRB) \tag{G2}$$

From the expression above, we can see that the wavefunction is completely antisymmetic.

3. Glueball

Pure gluons can form the color singlet states which are well known as glueballs.

We can find this kind of color singlet states by using F^2. However there are 64 color configurations for two-gluon system. In order to reduce the dimensions of the representation of F^2, we can use the operator t_3 and y which reduce the original matrix into $F^2_{10\times 10}$ matrix.

After the diagonalization, we found there only exists one color singlet state for two gluons. The color singlet state corresponding to eigenvalue equal to zero is given by

$$\frac{1}{\sqrt{8}}(g^1g^2 + g^2g^1 + g^3g^3 + g^4g^5 + g^5g^4 + g^6g^7 + g^7g^6 + g^8g^8) \tag{G3}$$

And we notice that the wavefunction is completely symmetric if we exchange the first gluon with the second gluon.

4. one quark, one antiquark and one gluon system

In the previous sections, we have seen that pure quark, antiquark and gluons can form the color singlet states. Now we need to consider more general cases: states with mixed quarks, antiquark and gluons.

At first, we consider the state with one quark, one antiquark and one gluon. Taking advantage of operator t_3 and y, we have 12 by 12 matrix for operator F^2. After the diagonalization, we found there exists one color singlet state which is given by the following linear combination of color configurations.

$$\begin{aligned}&\frac{1}{4}(R\bar{R}g^3 - G\bar{G}g^3) + \frac{1}{\sqrt{48}}(R\bar{R}g^8 + G\bar{G}g^8 - 2B\bar{B}g^8)\\ +&\frac{1}{\sqrt{8}}(R\bar{G}g^2 + R\bar{B}g^5 + G\bar{R}g^1 + G\bar{B}g^7 + B\bar{R}g^4 + B\bar{G}g^6)\end{aligned} \tag{G4}$$

5. two quarks and two antiquarks

Next we consider more complicated case: two quarks and two antiquarks. In this case, we have two color singlet states. One of these color singlet state has complete symmetric wavefunction:

$$\begin{aligned}&\frac{1}{\sqrt{6}}(RR\bar{R}\bar{R} + GG\bar{G}\bar{G} + BB\bar{B}\bar{B})\\ +&\frac{1}{\sqrt{24}}(RG\bar{R}\bar{G} + RG\bar{G}\bar{R} + RB\bar{R}\bar{B} + RB\bar{B}\bar{R} + GR\bar{R}\bar{G} + GR\bar{G}\bar{R}\\ &+GB\bar{G}\bar{B} + GB\bar{B}\bar{G} + BR\bar{R}\bar{B} + BR\bar{B}\bar{R} + BG\bar{G}\bar{B} + BG\bar{B}\bar{G})\end{aligned} \tag{G5}$$

The other one is completely antisymmetric:

$$\begin{aligned}&\frac{1}{\sqrt{12}}(RG\bar{G}\bar{R} - RG\bar{R}\bar{G} + RB\bar{B}\bar{R} - RB\bar{R}\bar{B} + GR\bar{R}\bar{G} - GR\bar{G}\bar{R}\\ &+GB\bar{B}\bar{G} - GB\bar{G}\bar{B} + BR\bar{R}\bar{B} - BR\bar{B}\bar{R} + BG\bar{G}\bar{B} - BG\bar{B}\bar{G})\end{aligned} \tag{G6}$$

With these two color singlet states in hand, we can use them to find two-meson state as the linear combination of symmetric and antisymmetric color singlet states which is given by

$$\begin{aligned}&\sqrt{\frac{2}{3}}|sym\rangle - \sqrt{\frac{1}{3}}antisym\rangle\\ =\ &\frac{1}{3}(R\bar{R} + G\bar{G} + B\bar{B})(R\bar{R} + G\bar{G} + B\bar{B})\end{aligned} \tag{G7}$$

Then the other color singlet state which is orthogonal to two-meson state is given by

$$\sqrt{\frac{1}{3}}|sym\rangle + \sqrt{\frac{2}{3}}|antisym\rangle \tag{G8}$$

From the example above, we have already seen that within two quark and two antiquark system, we can build up the intrinsic structure,two-meson state. In the next example, we will see how to construct the intrinsic structure again.

6. four quarks and one antiquark

From the previous section, we know that three quarks can form one color singlet state, baryon. One quark and one antiquark can form meson. Therefore within the four quark and one antiquark system, we expect to find the intrinsic structure.

At first, using the exactly same method, we can find 36 by 36 matrix which represents operator F^2. After we diagonalize the matrix, we obtain three color singlet states. At first, we list out their color configurations. The first

color singlet state is given by

$$\begin{aligned}
&0.288673(RRGB\bar{R} - RRBG\bar{R} - GGRB\bar{G} + GGBR\bar{G} + BBRG\bar{B} - BBGR\bar{B}) \\
&+0.145076(RBRG\bar{R} - RGRB\bar{R} + GRGB\bar{G} - GBGR\bar{G} - BRBG\bar{B} + BGBR\bar{B}) \\
&+0.143597(RGGB\bar{G} - GRRB\bar{R} - RBBG\bar{B} + GBBR\bar{B} - BGGR\bar{G} + BRRG\bar{R}) \\
&+0.145164(RGBR\bar{R} - RBGR\bar{R} + GBRG\bar{G} - GRBG\bar{G} + BRGB\bar{B} - GBRB\bar{B}) \\
&+0.143509(GRBR\bar{R} - BRGR\bar{R} + BGRG\bar{G} - RGBG\bar{G} + RBGB\bar{B} - GBRB\bar{B}) \\
&+0.000088(GBRR\bar{R} - BGRR\bar{R} + BRGG\bar{G} - RBGG\bar{G} + RGBB\bar{B} - GRBB\bar{B})
\end{aligned} \tag{G9}$$

The second color singlet state is given by

$$\begin{aligned}
&0.000062(RRGB\bar{R} - RRBG\bar{R} - GGRB\bar{G} + GGBR\bar{G} + BBRG\bar{B} - BBGR\bar{B}) \\
&+0.149055(RBRG\bar{R} - RGRB\bar{R} + GRGB\bar{G} - GBGR\bar{G} - BRBG\bar{B} + BGBR\bar{B}) \\
&-0.148993(RGGB\bar{G} - GRRB\bar{R} - RBBG\bar{B} + GBBR\bar{B} - BGGR\bar{G} + BRRG\bar{R}) \\
&-0.139543(RGBR\bar{R} - RBGR\bar{R} + GBRG\bar{G} - GRBG\bar{G} + BRGB\bar{B} - GBRB\bar{B}) \\
&+0.139605(GRBR\bar{R} - BRGR\bar{R} + BGRG\bar{G} - RGBG\bar{G} + RBGB\bar{B} - GBRB\bar{B}) \\
&-0.288598(GBRR\bar{R} - BGRR\bar{R} + BRGG\bar{G} - RBGG\bar{G} + RGBB\bar{B} - GRBB\bar{B})
\end{aligned} \tag{G10}$$

And the third color singlet state is given by

$$\begin{aligned}
&-0.001110(RRGB\bar{R} - RRBG\bar{R} - GGRB\bar{G} + GGBR\bar{G} + BBRG\bar{B} - BBGR\bar{B}) \\
&+0.200172(RBRG\bar{R} - RGRB\bar{R} + GRGB\bar{G} - GBGR\bar{G} - BRBG\bar{B} + BGBR\bar{B}) \\
&-0.201282(RGGB\bar{G} - GRRB\bar{R} - RBBG\bar{B} + GBBR\bar{B} - BGGR\bar{G} + BRRG\bar{R}) \\
&+0.206854(RGBR\bar{R} - RBGR\bar{R} + GBRG\bar{G} - GRBG\bar{G} + BRGB\bar{B} - GBRB\bar{B}) \\
&-0.207964(GRBR\bar{R} - BRGR\bar{R} + BGRG\bar{G} - RGBG\bar{G} + RBGB\bar{B} - GBRB\bar{B}) \\
&+0.006682(GBRR\bar{R} - BGRR\bar{R} + BRGG\bar{G} - RBGG\bar{G} + RGBB\bar{B} - GRBB\bar{B})
\end{aligned} \tag{G11}$$

We need to eliminate color configurations in which the first two quarks have the same colors. After we have done this, we have the reduced matrix for F^2 which has dimension 30 by 30. Finally we obtain only two color singlet states corresponding to the requirement that the first two quarks can not have the same color. We list out these two color singlet states.

The first one is given by

$$\begin{aligned}
&0.015734(-RGRB\bar{R} - RGGB\bar{G} - GBGR\bar{G} - GBBR\bar{B} - BRRG\bar{R} - BRBG\bar{B} \\
&\quad +RBRG\bar{R} + RBBG\bar{B} + GRRB\bar{R} + GRGB\bar{G} + BGGR\bar{G} + BGBR\bar{B}) \\
&+0.229990(-RGBR\bar{R} - RGBG\bar{G} - GBRG\bar{G} - GBRB\bar{B} - BRGR\bar{R} - BRGB\bar{B} \\
&\quad +RBGR\bar{R} + RBGB\bar{B} + GRBR\bar{R} + GRBG\bar{G} + BGRG\bar{G} + BGRB\bar{B}) \\
&+0.245724(-RGBB\bar{B} + RBGG\bar{G} + GRBB\bar{B} - GBRR\bar{R} - BRGG\bar{G} + BGRR\bar{R})
\end{aligned} \tag{G12}$$

The second one is given by

$$\begin{aligned}
&0.249504(RGRB\bar{R} + RGGB\bar{G} + GBGR\bar{G} + GBBR\bar{B} + BRRG\bar{R} + BRBG\bar{B} \\
&\quad -RBRG\bar{R} - RBBG\bar{B} - GRRB\bar{R} - GRGB\bar{G} - BGGR\bar{G} - BGBR\bar{B}) \\
&+0.098002(-RGBR\bar{R} - RGBG\bar{G} - GBRG\bar{G} - GBRB\bar{B} - BRGR\bar{R} - BRGB\bar{B} \\
&\quad +RBGR\bar{R} + RBGB\bar{B} + GRBR\bar{R} + GRBG\bar{G} + BGRG\bar{G} + BGRB\bar{B}) \\
&+0.151502(RGBB\bar{B} - RBGG\bar{G} - GRBB\bar{B} + GBRR\bar{R} + BRGG\bar{G} - BGRR\bar{R})
\end{aligned} \tag{G13}$$

We can change these two singlet states into the new forms. The first singlet state is

$$\begin{aligned}
&0.015734(-RGRB\bar{R} - RGGB\bar{G} - GBGR\bar{G} - GBBR\bar{B} - BRRG\bar{R} - BRBG\bar{B} \\
&\quad +RBRG\bar{R} + RBBG\bar{B} + GRRB\bar{R} + GRGB\bar{G} + BGGR\bar{G} + BGBR\bar{B} \\
&\quad -RGBB\bar{B} + RBGG\bar{G} + GRBB\bar{B} - GBRR\bar{R} - BRGG\bar{G} + BGRR\bar{R}) \\
&+0.229990(-RGBR\bar{R} - RGBG\bar{G} - GBRG\bar{G} - GBRB\bar{B} - BRGR\bar{R} - BRGB\bar{B}
\end{aligned}$$

$$
\begin{aligned}
&+RBGR\bar{R}+RBGB\bar{B}+GRBR\bar{R}+GRBG\bar{G}+BGRG\bar{G}+BGRB\bar{B}\\
&-RGBB\bar{B}+RBGG\bar{G}+GRBB\bar{B}-GBRR\bar{R}-BRGG\bar{G}+BGRR\bar{R})\\
=\ & 0.015734(-RG(R\bar{R}+G\bar{G}+B\bar{B})B+RB(R\bar{R}+G\bar{G}+B\bar{B})G-BR(R\bar{R}+G\bar{G}+B\bar{B})G\\
&+BG(R\bar{R}+G\bar{G}+B\bar{B})R-GB(R\bar{R}+G\bar{G}+B\bar{B})R+GR(R\bar{R}+G\bar{G}+B\bar{B})B)\\
&+0.229990(-RGB+RBG-BRG+BGR-GBR+GRB)(R\bar{R}+G\bar{G}+B\bar{B}) \qquad \text{(G14)}
\end{aligned}
$$

Using the same method, we can get the very similar structure for the second singlet state. Therefore we can construct two color singlet states which have the intrinsic structure. The first one is

$$(-RGB+RBG-BRG+BGR-GBR+GRB)(R\bar{R}+G\bar{G}+B\bar{B}) \qquad \text{(G15)}$$

The second one is

$$
\begin{aligned}
&(-RG(R\bar{R}+G\bar{G}+B\bar{B})B+RB(R\bar{R}+G\bar{G}+B\bar{B})G-BR(R\bar{R}+G\bar{G}+B\bar{B})G\\
&+BG(R\bar{R}+G\bar{G}+B\bar{B})R-GB(R\bar{R}+G\bar{G}+B\bar{B})R+GR(R\bar{R}+G\bar{G}+B\bar{B})B) \qquad \text{(G16)}
\end{aligned}
$$

7. three quark and three antiquark system

We consider the case with three quarks and three antiquarks.

Using the operator F^2, we have the conclusion that there exists six color singlet states which have the zero eigenvalues.

And we have only one color singlet state with completely symmetric wavefunction which is given by

$$
\begin{aligned}
&\frac{1}{\sqrt{10}}(RRR\bar{R}\bar{R}\bar{R}+GGG\bar{G}\bar{G}\bar{G}+BBB\bar{B}\bar{B}\bar{B})\\
&+\frac{1}{\sqrt{90}}(RRG\bar{R}\bar{R}\bar{G}+c.s.p)\\
&+\frac{1}{\sqrt{90}}(RRB\bar{R}\bar{R}\bar{B}+c.s.p)\\
&+\frac{1}{\sqrt{90}}(GGR\bar{G}\bar{G}\bar{R}+c.s.p)\\
&+\frac{1}{\sqrt{90}}(BBR\bar{B}\bar{B}\bar{R}+c.s.p)\\
&+\frac{1}{\sqrt{90}}(GGB\bar{G}\bar{G}\bar{B}+c.s.p)\\
&+\frac{1}{\sqrt{90}}(BBG\bar{B}\bar{B}\bar{G}+c.s.p)\\
&+\frac{1}{\sqrt{360}}(RGB\bar{R}\bar{G}\bar{B}+c.s.p) \qquad \text{(G17)}
\end{aligned}
$$

where c.s.p represents the complete symmetric partners.

Other five color singlet states are either antisymmetic or mixed symmetric(antisymmetic).