MSA08 – 02 TURUNAN

PERINGATAN HAK CIPTA

SubCPMK

- Mahasiswa mampu menerapkan konsep turunan dalam menyelesaikan soal perhitungan. (C3, A3)

Materi

1. Turunan.
2. Aturan pencarian turunan.
3. Turunan fungsi trigonometri.
4. Aturan rantai.
5. Turunan tingkat tinggi.
6. Diferensiasi implisit.
7. Laju yang berkaitan.
8. Diferensiasi dan Aproksimasi.

1. Turunan

Turunan (1)

$y = (m)x + c$

$m = f'(x)$

- Turunan merupakan dasar dari pencarian gradien garis singgung, kecepatan sesaat, kecepatan pertumbuhan organisme dan lain-lain.
- Berdadarkan konsep limit, turunan suatu fungsi f adalah suatu fungsi yang dinotasikan sebagai f' yang didefinisikan sebagai

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Contoh 1.1. Diketahui $f(x) = 13x - 6$. Tentukan $f'(4)$.

Solusi.

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{[13(4+h) - 6] - [13(4) - 6]}{h}$$

$$= \lim_{h \to 0} \frac{13h}{h} = \lim_{h \to 0} 13 = 13$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x) = 13x - 6$$

$f'(x)$

$f'(a) = \ldots ?$

$$f(x+h) = 13(x+h) - 6$$

$$= 13x + 13h - 6 \checkmark$$

$$\lim_{h \to 0} \frac{13x + 13h - 6 - (13x - 6)}{h}$$

$$\lim_{h \to 0} \frac{\cancel{13x} + 13h \cancel{-6} \cancel{-13x} \cancel{+6}}{h}$$

$$\lim_{h \to 0} \frac{13\cancel{h}}{\cancel{h}} = 13$$

$$f'(x) = 13$$

$$f'(a) = 13$$

Turunan (2)

Contoh 1.2. Diketahui $f(x) = x^3 + 7x$. Tentukan $f'(x)$.

Solusi.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3 + 7(x+h)] - [x^3 + 7x]}{h}$$

$$= \lim_{h\to 0} \frac{3x^{2h} + 3xh^2 + h^3 + 7h}{h} = \lim_{h\to 0} (3x^2 + 3xh + h^2 + 7)$$

$$= 3x^2 + 7$$

Contoh 1.3. Diketahui $f(x) = 1/x$. Tentukan $f'(x)$.

Solusi.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \left[\frac{x - (x+h)}{(x+h)x} \cdot \frac{1}{h}\right]$$

$$= \lim_{h\to 0} \left[\frac{-h}{(x+h)x} \cdot \frac{1}{h}\right] = \lim_{h\to 0} \frac{-1}{(x+h)x} = -\frac{1}{x^2}$$

$$f(x) = x^3 + 7x \qquad f'(x)$$

1 1
1 2 1
1 3 3 1

$$f(x+h) = (x+h)^3 + 7(x+h)$$

$$= 1x^3 + 3x^2h + 3xh^2 + 1h^3 + 7x + 7h$$

$$= x^3 + 3x^2h + 3xh^2 + h^3 + 7x + 7h$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 + \cancel{7x} + 7h - (\cancel{x^3} + \cancel{7x})}{h}$$

$$\lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 7h}{h}$$

$$\lim_{h \to 0} \frac{\cancel{h}(3x^2 + 3xh + h^2 + 7)}{\cancel{h}}$$

$$3x^2 + 3x(0) + 0^2 + 7$$

$$3x^2 + 7$$

$$f(x) = \frac{1}{x} \qquad f'(x)$$

$$f(x+h) = \frac{1}{x+h}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{1 \cdot x - 1(x+h)}{x(x+h)}}{\frac{h}{1}}$$

$$\lim_{h \to 0} \frac{\cancel{x} - \cancel{x} - h}{x(x+h)} \cdot \frac{1}{h}$$

$$\lim_{h \to 0} \frac{-\cancel{h}}{x(x+h)} \cdot \frac{1}{\cancel{h}}$$

$$\lim_{h \to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)}$$

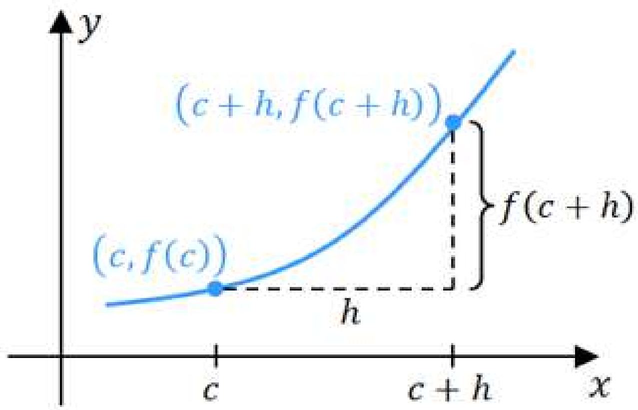
$$= \frac{-1}{x \cdot x} = \frac{-1}{x^2}$$

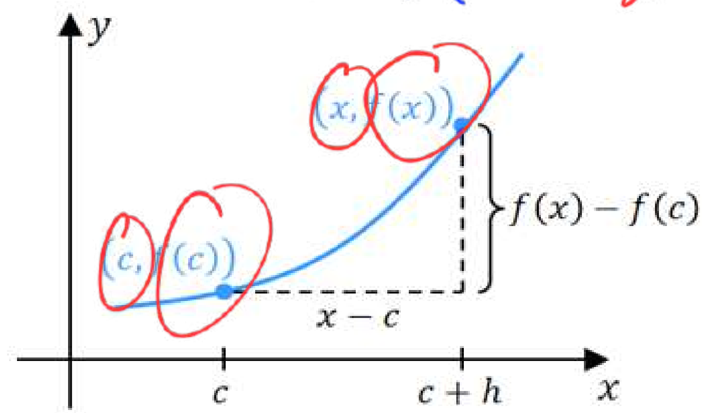
1.1. Bentuk Ekuivalen dari Turunan

- Untuk mencari nilai dari $f'(c)$ dengan menggunakan definisi turunan, diperoleh $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$.

$$y' = \frac{\Delta y}{\Delta x} = \frac{f(x) - f(c)}{x - c}$$

- Perhatikan dua gambar berikut.





- Dengan substitusi $c + h = x$ sehingga $h = x - c$, diperoleh

$$\boxed{f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}}$$

CONTOH SOAL

Contoh 1.4. Gunakan definisi turunan dalam kotak untuk menentukan $g'(c)$ dari $g(x) = 2/(x+3)$.

Solusi.

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{\dfrac{2}{x+3} - \dfrac{2}{c+3}}{x - c}$$

$$= \lim_{x \to c} \left[\frac{2(c+3) - 2(x+3)}{(x+\cancel{2}3)(c+3)} \cdot \frac{1}{x-c} \right]$$

$$= \lim_{x \to c} \left[\frac{-2(x-c)}{(x+\cancel{2}3)(c+3)} \cdot \frac{1}{x-c} \right]$$

$$= \lim_{x \to c} \frac{-2}{(x+3)(c+3)} = \frac{-2}{(c+3)^2}$$

$$g(x) = \frac{2}{x+3} \qquad g'(c)$$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

$$\lim_{x \to c} \frac{\frac{2}{x+3} - \frac{2}{c+3}}{x - c} = \lim_{x \to c} \frac{\frac{2(c+3) - 2(x+3)}{(x+3)(c+3)}}{\frac{x-c}{1}}$$

$$\lim_{x \to c} \frac{2c + \cancel{6} - 2x - \cancel{6}}{(x+3)(c+3)} \cdot \frac{1}{x - c}$$

$$\lim_{x \to c} \frac{2c - 2x}{(x+3)(c+3)} \cdot \frac{1}{x - c}$$

$$\lim_{x \to c} \frac{-2\cancel{(x-c)}}{(x+3)(c+3)} \cdot \frac{1}{\cancel{(x-c)}}$$

$$\lim_{x \to c} \frac{-2}{(x+3)(c+3)} = \frac{-2}{(c+3)(c+3)}$$

$$= \frac{-2}{(c+3)^2}$$

2. Aturan Pencarian Turunan

$f'(x) = \sqrt{3}$ $f'(x) = 0$

Aturan Pencarian Turunan (1)

$f(x) = -0,5$

$f'(x) = 0$ $f(x) = 8$ $f'(x) = 0$

- **Aturan Fungsi Konstan**

  Jika $f(x) = k$, dengan k merupakan konstanta sembarang, maka $f'(x) = 0$.

  $f(x) = 5x$ $f'(x) = 5$

  $f(x) = -6,2x$ $f'(x) = -6,2$

- **Aturan Fungsi Identitas**

  Jika $f(x) = x$, maka $f'(x) = 1$.

  $f(x) = x^3$ $f'(x) = 3x^2$

- **Aturan Pangkat**

  Jika $f(x) = x^n$, maka $f'(x) = nx^{n-1}$. (n bilangan bulat positif)

  $f(x) = x^5$ $f'(x) = 5x^4$

- **Aturan Perkalian Konstan**

  Jika k merupakan konstanta sembarang dan fungsi f memiliki turunan, maka $(kf)'(x) = k \cdot f'(x)$.

  $f(x) = 2x^3$ $f'(x) = 6x^2$

$f(x) = 8x^5$ $f'(x) = 40x^4$

① $f(x) = 6x^{-2}$ $\quad -2-1$

$f'(x) = -12x^{-3}$ $\quad -4-1$

② $f(x) = -2x^{-4}$

$f'(x) = \cancel{-4x^{4}}$ $\quad 8x^{-5}$

③ $f'(x) = \dfrac{5}{x^{6}} = \cancel{5(x)^{6}}$ $\quad 5x^{-6}$

$f'(x) = \cancel{\dfrac{5}{6x^{5}}}$ $\quad -30x^{-7} = \dfrac{-30}{x^{7}}$

④ $f(x) = \sqrt{x} = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2x^{\frac{1}{2}}} = \dfrac{1}{2\sqrt{x}}$

Aturan Pencarian Turunan (2)

$f(x) = 2x^3$
$g(x) = 4x^5$
$f(x) + g(x) = 2x^3 + 4x^5$
$f'(x) + g'(x) = 6x^2 + 20x^4$

- **Aturan Penjumlahan dan Pengurangan**

  Jika fungsi f dan g memiliki turunan, maka

$$(f \pm g)'(x) = f'(x) \pm g'(x)$$

- **Aturan Perkalian**

  Jika fungsi f dan g memiliki turunan, maka

$$(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x)f'(x)$$

- **Aturan Pembagian**

  Jika fungsi f dan g memiliki turunan, maka

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

$$\boxed{\begin{aligned} f(x) &= u \cdot v \\ f'(x) &= u' \cdot v + u \cdot v' \end{aligned}}$$

$$f(x) = (3x^2 - 5) \cdot (8 - 4x)$$

misal : $u = 3x^2 - 5 \rightarrow u' = 6x$

$v = 8 - 4x \rightarrow v' = -4$

$$f'(x) = u' \cdot v + u \cdot v'$$

$$= 6x \cdot (8 - 4x) + (3x^2 - 5)(-4)$$

$$= 48x - 24x^2 + -12x^2 + 20$$

$$= -36x^2 + 48x + 20$$

$$f(x) = \frac{u}{v}$$

$$f'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$f(x) = \frac{6 - x^2}{3x + 5}$$

Misal: $u = 6 - x^2 \rightarrow u' = -2x$

$v = 3x + 5 \rightarrow v' = 3$

$$f'(x) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

$$= \frac{-2x(3x+5) - (6 - x^2) \cdot 3}{(3x+5)^2}$$

$$= \frac{-6x^2 - 10x - 18 + 3x^2}{(3x+5)^2}$$

$$= \frac{-3x^2 - 10x - 18}{(3x+5)^2}$$

CONTOH SOAL (1)

Contoh 2.1. Carilah turunan dari:

(a) $f(x) = 5x^2 + 7x - 6,$ $f'(x) = 10x + 7$

(b) $g(x) = 3x^4 + 8x^3 - 2x^2 - 13,$ dan $g'(x) = 12x^3 + 24x^2 - 4x$

(c) $h(x) = 4x^6 - 3x^5 - 10x^2 + 5x + 16.$

Solusi. $h'(x) = 24x^5 - 15x^4 - 20x + 5$

(a) $f'(x) = 5(2x^{2-1}) + 7(1) - 0 = 10x + 7.$

(b) $g'(x) = 3(4x^{4-1}) + 8(3x^{3-1}) - 2(2x^{2-1}) - 0$

$= 12x^3 + 24^2 - 4x$

(c) $h'(x) = 4(6x^{6-1}) - 3(5x^{5-1}) - 10(2x^{2-1}) + 5(1) + 16$

$= 24x^5 - 15x^4 - 20x + 5.$

CONTOH SOAL (2)

Contoh 2.2. Carilah turunan dari $h(x) = (3x^2 - 5)(2x^4 - x)$ dengan menggunakan aturan perkalian.

u v

$u' : 6x$ $v' : 8x^3 - 1$

Solusi.

Misalkan $f(x) = 3x^2 - 5$ dan $g(x) = 2x^4 - x$, maka $f'(x) = 6x$ dan $g'(x) = 8x^3 - 1$; sehingga

$$\begin{aligned} h'(x) &= (f \cdot g)'(x) = f(x) \cdot g'(x) + g(x)f'(x) \\ &= (3x^2 - 5)(8x^3 - 1) + (2x^4 - x)(6x) \\ &= 24x^5 - 3x^2 - 40x^3 + 5 + 12x^5 - 6x^2 \\ &= 36x^5 - 40x^3 - 9x^2 + 5. \end{aligned}$$

CONTOH SOAL (3)

Contoh 2.3. Carilah turunan dari $h(x) = (3x - 5)/(x^2 + 7)$

u v

$u' = 3$ $v' = 2x$

Solusi.

Misalkan $f(x) = 3x - 5$ dan $g(x) = x^2 + 7$, maka $f'(x) = 3$ dan $g'(x) = 2x$; sehingga

$$h'(x) = \left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

$$= \frac{(x^2 + 7)(3) - (3x - 5)(2x)}{(x^2 + 7)^2}$$

$$= \frac{3x^2 + 21 - 6x^2 + 10x}{(x^2 + 7)^2}$$

$$= \frac{-3x^2 + 10x + 21}{(x^2 + 7)^2}$$

3. Turunan Fungsi Trigonometri

Turunan Fungsi Trigonometri

Turunan dari fungsi-fungsi trigonometri didefinisikan sebagai berikut.

- Jika $f(x) = \sin x$, maka $f'(x) = \cos x$.
- Jika $f(x) = \cos x$, maka $f'(x) = -\sin x$.
- Jika $f(x) = \tan x$, maka $f'(x) = \sec^2 x$.
- Jika $f(x) = \sec x$, maka $f'(x) = \sec x \tan x$.
- Jika $f(x) = \cot x$, maka $f'(x) = -\csc^2 x$.
- Jika $f(x) = \csc x$, maka $f'(x) = -\csc x \cot x$.

Contoh 3.1. Tentukan turunan dari $f(x) = 3\sin x - 2\cos x$.

Solusi.

$$f'(x) = 3(\cos x) - 2(-\sin x) = 3\cos x + 2\sin x$$

1. $f(x) = 5\cos x - 4\sin x$

$$f'(x) = 5.(-\sin x) - 4.(\cos x)$$
$$= -5\sin x - 4\cos x$$

2. $f(x) = 3\sin x + 8\cos x$

$$f'(x) = 3(\cos x) + 8(-\sin x)$$
$$f'(x) = 3\cos x - 8\sin x$$

3. $h(x) = x^n . \tan x$

Misal: $u = x^n \rightarrow u' = n.x^{n-1}$

$v = \tan x \rightarrow v' = \sec^2 x$

$$h'(x) = u'.v + u.v'$$
$$= n.x^{n-1}.\tan x + x^n.\sec^2 x$$

CONTOH SOAL

Contoh 3.2. Tentukan turunan dari $h(x) = x^n \tan x$, untuk $n \geq 1$.

Solusi. Misalkan $f(x) = x^n$ dan $g(x) = \tan x$, maka $f'(x) = nx^{n-1}$ dan $g'(x) = \sec^2 x$; sehingga

$$h'(x) = x^n(\sec^2 x) + (\tan x)(nx^{n-1}) = x^n \sec^2 x + nx^{n-1} \tan x$$

Contoh 3.3. Carilah turunan dari $h(x) = (1 + \sin x)/(\cos x)$.

Solusi. Misalkan $f(x) = 1 + \sin x$ dan $g(x) = \cos x$, maka $f'(x) = \cos x$ dan $g'(x) = -\sin(x)$; sehingga

$u' = \cos x$ $\quad v' = -\sin x$

$-\sin x$

$$h'(x) = \frac{(\cos x)(\cos x) - (1 + \sin x)(\cancel{\sin x})}{\cos^2 x}$$

$$\frac{u'v - u.v'}{v^2}$$

$\sin^2 x + \cos^2 x = 1$

$$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x} = \frac{1 + \sin x}{\cos^2 x}$$

4. Aturan Rantai

Aturan Rantai

Misalkan $y = f(u)$ dan $u = g(x)$. Jika fungsi g dapat diturunkan terhadap x dan fungsi f dapat diturunkan terhadap u, maka fungsi komposisi $f \circ g$ yang didefinisikan sebagai $(f \circ g)(x) = f\big(g(x)\big)$ memiliki turunan terhadap x dan

$$(f \circ g)'(x) = f'\big(g(x)\big) \cdot g'(x)$$

atau

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Contoh 4.1. Jika $y = (2x^2 - 4x + 1)^{60}$, carilah turunan dari y.

Solusi. Misalkan $u = 2x^2 - 4x + 1$ dan $y = u^{60}$, maka

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (60u^{59})(4x - 4) = 60(2x^2 - 4x - 1)^{59}(4x - 4)$$

4.1 $y = (2x^2 - 4x + 1)^{60}$

$y' = 60(2x^2 - 4x + 1)^{59} \cdot (4x - 4)$

$y' = 60(4x - 4) \cdot (2x^2 - 4x + 1)^{59}$

4.2 a) $y = \dfrac{1}{(2x^5 - 7)^3} = 1 \cdot (2x^5 - 7)^{-3}$

$y' = -3 \cdot (2x^5 - 7)^{-4} \cdot (10x^4)$

$y' = -30x^4 (2x^5 - 7)^{-4}$

$y' = \dfrac{-30x^4}{(2x^5 - 7)^4}$

b) $y = \sin 2x$

$y' = \cos 2x \cdot 2 = \cancel{\cos 4x}$

$y' = 2\cos 2x$

CONTOH SOAL

Contoh 4.2. Tentukan dy/dx dari fungsi-fungsi berikut:

(a) $y = 1/(2x^5 - 7)^3$ dan

(b) $y = \sin 2x$.

Solusi.

(a) Misalkan $u = 2x^5 - 7$ dan $y = 1/u^3 = u^{-3}$, maka

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (-3u^{-4})(10x^4) = \frac{-3}{u^4} \cdot 10x^4 = \frac{-30x^4}{(2x^5 - 7)}$$

(b) Misalkan $u = 2x$ dan $y = \sin u$, maka

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (\cos u)(2) = 2\cos 2x$$

4.1. Menerapkan Aturan Rantai Lebih dari Satu Kali

Contoh 4.3. Carilah turunan dari:

(a) $y = \sin^3(4x)$

(b) $y = \sin[\cos(x^2)]$

Solusi.

(a) Misalkan $u = 4x$, $v = \sin u$, dan $y = v^3$, maka

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = (3v^2)(\cos u)(4) = 12\sin^2(4x)\cos(4x)$$

(b) Misalkan $u = x^2$, $v = \cos u$, dan $y = \sin v$, maka

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx} = (\cos v)(-\sin u)(2x)$$
$$= -2x\sin(x^2)\cos[\cos(x^2)]$$

4.3 a) $y = \sin^3(4x)$

$y' = 3\sin^2(4x) \cdot \cos(4x) \cdot 4$

$y' = 12\sin^2 4x \cdot \cos 4x$ ✓

$\boxed{2\sin\alpha \cdot \cos\alpha = \sin 2\alpha}$

$y' = 6 \cdot 2\sin 4x \cdot \cos 4x \cdot \sin 4x$

$y' = 6\sin 8x \cdot \sin 4x$

b) $y = \sin[\cos(x^2)]$

$y' = \cos[\cos(x^2)] \cdot -\sin(x^2) \cdot 2x$

$y' = -2x\sin(x^2) \cdot \cos[\cos(x^2)]$

5. Turunan Tingkat Tinggi

$y = 6x^3$

$y' = 18x^2$

$y'' = 36x$

$y''' = 36$

$y^{iv} = 0$