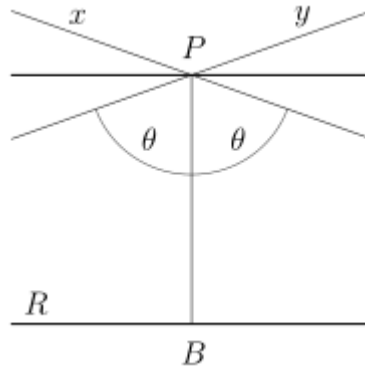
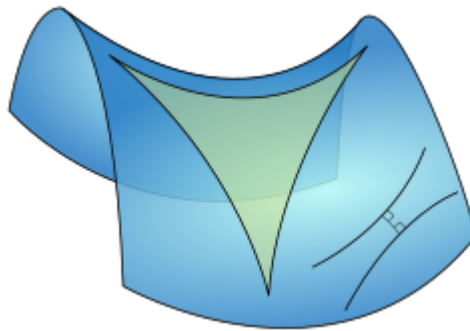


Hyperbolic geometry

In [mathematics](#), **hyperbolic geometry** (also called **Lobachevskian geometry** or **Bolyai–Lobachevskian geometry**) is a non-Euclidean geometry. The [parallel postulate](#) of [Euclidean geometry](#) is replaced with:



Lines through a given point P and asymptotic to line R



A triangle immersed in a saddle-shape plane (a [hyperbolic paraboloid](#)), along with two diverging ultra-parallel lines

For any given line R and point P not on R , in the plane containing both line R and point P there are at least two distinct lines through P that do not intersect R .

(Compare the above with [Playfair's axiom](#), the modern version of [Euclid's parallel postulate](#).)

Hyperbolic plane [geometry](#) is also the geometry of [saddle surfaces](#) and [pseudospherical surfaces](#), surfaces with a constant negative [Gaussian curvature](#).

A modern use of hyperbolic geometry is in the theory of [special relativity](#), particularly the [Minkowski model](#) and [gyrovector space](#).

When geometers first realised they were working with something other than the standard Euclidean geometry, they described their geometry under many different names; [Felix Klein](#) finally gave the subject the name **hyperbolic geometry** to include it in the now rarely used sequence [elliptic](#)

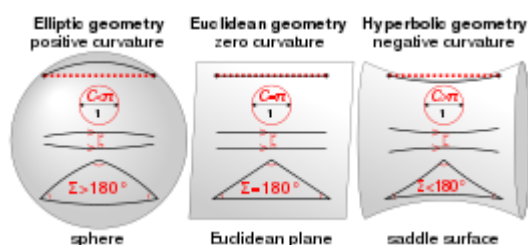
geometry ([spherical geometry](#)), parabolic geometry ([Euclidean geometry](#)), and hyperbolic geometry. In the [former Soviet Union](#), it is commonly called Lobachevskian geometry, named after one of its discoverers, the Russian geometer [Nikolai Lobachevsky](#).

This page is mainly about the 2-dimensional (planar) hyperbolic geometry and the differences and similarities between Euclidean and hyperbolic geometry.

Hyperbolic geometry can be extended to three and more dimensions; see [hyperbolic space](#) for more on the three and higher dimensional cases.

Properties

Relation to Euclidean geometry



Comparison of elliptic, Euclidean and hyperbolic geometries in two dimensions

Hyperbolic geometry is more closely related to Euclidean geometry than it seems: the only [axiomatic](#) difference is the [parallel postulate](#). When the parallel postulate is removed from Euclidean geometry the resulting geometry is [absolute geometry](#). There are two kinds of absolute geometry, Euclidean and hyperbolic. All theorems of absolute geometry, including the first 28 propositions of book one of [Euclid's Elements](#), are valid in Euclidean and hyperbolic geometry. Propositions 27 and 28 of Book One of Euclid's *Elements* prove the existence of parallel/non-intersecting lines.

This difference also has many consequences: concepts that are equivalent in Euclidean geometry are not equivalent in hyperbolic geometry; new concepts need to be introduced. Further, because of the [angle of parallelism](#), hyperbolic geometry has an [absolute scale](#), a relation between distance and angle measurements.

Lines

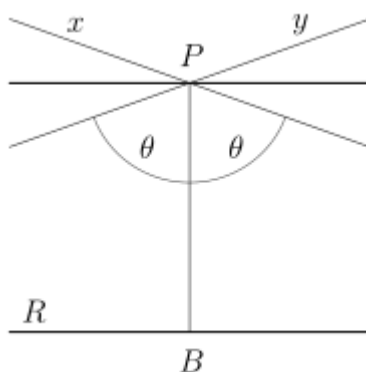
Single lines in hyperbolic geometry have exactly the same properties as single straight lines in Euclidean geometry. For example, two points uniquely define a line, and line segments can be infinitely extended.

Two intersecting lines have the same properties as two intersecting lines in Euclidean geometry. For example, two distinct lines can intersect in no more than one point, intersecting lines form equal opposite angles, and adjacent angles of intersecting lines are [supplementary](#).

When a third line is introduced, then there can be properties of intersecting lines that differ from intersecting lines in Euclidean geometry. For example, given two intersecting lines there are infinitely many lines that do not intersect either of the given lines.

These properties are all independent of the [model](#) used, even if the lines may look radically different.

Non-intersecting / parallel lines



Lines through a given point P and asymptotic to line R .

Non-intersecting lines in hyperbolic geometry also have properties that differ from non-intersecting lines in [Euclidean geometry](#):

For any line R and any point P which does not lie on R , in the plane containing line R and point P there are at least two distinct lines through P that do not intersect R .

This implies that there are through P an infinite number of coplanar lines that do not intersect R .

These non-intersecting lines are divided into two classes:

- Two of the lines (x and y in the diagram) are [limiting parallels](#) (sometimes called critically parallel, horoparallel or just parallel): there is one in the direction of each of the [ideal points](#) at the "ends" of R , asymptotically approaching R , always getting closer to R , but never meeting it.
- All other non-intersecting lines have a point of minimum distance and diverge from both sides of that point, and are called *ultraparallel*, *diverging parallel* or sometimes *non-intersecting*.

Some geometers simply use the phrase "*parallel* lines" to mean "*limiting parallel* lines", with *ultraparallel* lines meaning just *non-intersecting*.

These [limiting parallels](#) make an angle θ with PB ; this angle depends only on the [Gaussian curvature](#) of the plane and the distance PB and is called the [angle of parallelism](#).

For ultraparallel lines, the [ultraparallel theorem](#) states that there is a unique line in the hyperbolic plane that is perpendicular to each pair of ultraparallel lines.

Circles and disks

In hyperbolic geometry, the circumference of a circle of radius r is greater than $2\pi r$.

Let $R = \frac{1}{\sqrt{-K}}$, where K is the [Gaussian curvature](#) of the plane. In hyperbolic geometry, K is negative, so the square root is of a positive number.

Then the circumference of a circle of radius r is equal to:

$$2\pi R \sinh \frac{r}{R}.$$

And the area of the enclosed disk is:

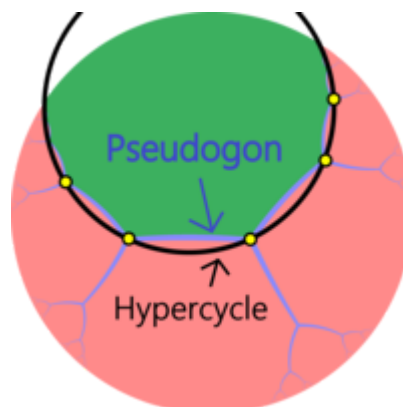
$$4\pi R^2 \sinh^2 \frac{r}{2R} = 2\pi R^2 \left(\cosh \frac{r}{R} - 1 \right).$$

Therefore, in hyperbolic geometry the ratio of a circle's circumference to its radius is always strictly greater than 2π , though it can be made arbitrarily close by selecting a small enough circle.

If the Gaussian curvature of the plane is -1 then the [geodesic curvature](#) of a circle of radius r is:

$$\frac{1}{\tanh(r)} \quad [1]$$

Hypercycles and horocycles



Hypercycle and pseudogon in the [Poincaré disk model](#)

In hyperbolic geometry, there is no line all of whose points are equidistant from another line. Instead, the points that all have the same orthogonal distance from a given line lie on a curve called a [hypercycle](#).

Another special curve is the [horocycle](#), a curve whose [normal](#) radii ([perpendicular](#) lines) are all [limiting parallel](#) to each other (all converge asymptotically in one direction to the same [ideal point](#), the centre of the horocycle).

Through every pair of points there are two horocycles. The centres of the horocycles are the [ideal points](#) of the [perpendicular bisector](#) of the line-segment between them.

Given any three distinct points, they all lie on either a line, [hypercycle](#), [horocycle](#), or circle.

The **length** of the line-segment is the shortest length between two points. The arc-length of a hypercycle connecting two points is longer than that of the line segment and shorter than that of a horocycle, connecting the same two points. The arclength of both horocycles connecting two points are equal. The arc-length of a circle between two points is larger than the arc-length of a horocycle connecting two points.

If the Gaussian curvature of the plane is -1 then the [geodesic curvature](#) of a horocycle is 1 and of a hypercycle is between 0 and 1.^[1]

Triangles

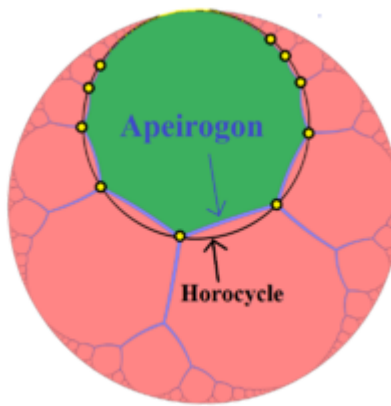
Unlike Euclidean triangles, where the angles always add up to π [radians](#) (180° , a [straight angle](#)), in hyperbolic geometry the sum of the angles of a hyperbolic triangle is always strictly less than π [radians](#) (180° , a [straight angle](#)). The difference is referred to as the [defect](#).

The area of a hyperbolic triangle is given by its defect in radians multiplied by R^2 . As a consequence, all hyperbolic triangles have an area that is less than or equal to $R^2\pi$. The area of a hyperbolic [ideal triangle](#) in which all three angles are 0° is equal to this maximum.

As in [Euclidean geometry](#), each hyperbolic triangle has an [incircle](#). In hyperbolic geometry, if all three of its vertices lie on a [horocycle](#) or [hypercycle](#), then the triangle has no [circumscribed circle](#).

As in [spherical](#) and [elliptical geometry](#), in hyperbolic geometry if two triangles are similar, they must be congruent.

Regular apeirogon



an [apeirogon](#) and circumscribed [horocycle](#) in the [Poincare disk model](#)

A special polygon in hyperbolic geometry is the regular [apeirogon](#), a [uniform polygon](#) with an infinite number of sides.

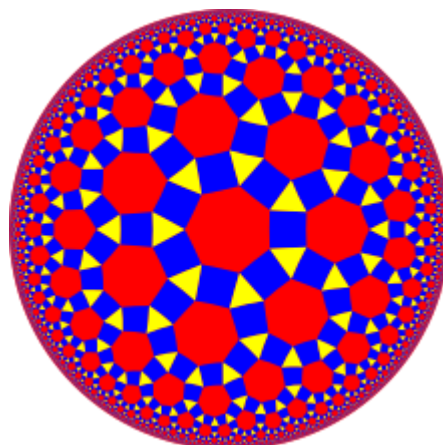
In [Euclidean geometry](#), the only way to construct such a polygon is to make the side lengths tend to zero and the apeirogon is indistinguishable from a circle, or make the interior angles tend to 180 degrees and the apeirogon approaches a straight line.

However, in hyperbolic geometry, a regular apeirogon has sides of any length (i.e., it remains a polygon).

The side and angle [bisectors](#) will, depending on the side length and the angle between the sides, be limiting or diverging parallel (see [lines above](#)). If the bisectors are limiting parallel the apeirogon can be inscribed and circumscribed by concentric [horocycles](#).

If the bisectors are diverging parallel then a pseudogon (distinctly different from an apeirogon) can be inscribed in [hypercycles](#) (all vertices are the same distance of a line, the axis, also the midpoint of the side segments are all equidistant to the same axis.)

Tessellations



Rhombitriheptagonal tiling of the hyperbolic plane, seen in the Poincaré disk model

Like the Euclidean plane it is also possible to tessellate the hyperbolic plane with regular polygons as faces.

There are an infinite number of uniform tilings based on the Schwarz triangles $(p\ q\ r)$ where $1/p + 1/q + 1/r < 1$, where p, q, r are each orders of reflection symmetry at three points of the fundamental domain triangle, the symmetry group is a hyperbolic triangle group. There are also infinitely many uniform tilings that cannot be generated from Schwarz triangles, some for example requiring quadrilaterals as fundamental domains.^[2]

Standardized Gaussian curvature

Though hyperbolic geometry applies for any surface with a constant negative Gaussian curvature, it is usual to assume a scale in which the curvature K is -1 .

This results in some formulas becoming simpler. Some examples are:

- The area of a triangle is equal to its angle defect in radians.
- The area of a horocyclic sector is equal to the length of its horocyclic arc.
- An arc of a horocycle so that a line that is tangent at one endpoint is limiting parallel to the radius through the other endpoint has a length of 1.^[3]
- The ratio of the arc lengths between two radii of two concentric horocycles where the horocycles are a distance 1 apart is $e : 1$.^[3]

Cartesian-like coordinate systems

In hyperbolic geometry, the sum of the angles of a quadrilateral is always less than 360 degrees, and hyperbolic rectangles differ greatly from Euclidean rectangles since there are no equidistant lines, so a proper Euclidean rectangle would need to be enclosed by two lines and two hypercycles. These all complicate coordinate systems.

There are however different coordinate systems for hyperbolic plane geometry. All are based around choosing a point (the origin) on a chosen directed line (the x-axis) and after that many choices exist.

The Lobachevski coordinates x and y are found by dropping a perpendicular onto the x-axis. x will be the label of the foot of the perpendicular. y will be the distance along the perpendicular of the given

point from its foot (positive on one side and negative on the other).

Another coordinate system measures the distance from the point to the [horocycle](#) through the origin centered around $(0, +\infty)$ and the length along this horocycle.^[4]

Other coordinate systems use the Klein model or the Poincare disk model described below, and take the Euclidean coordinates as hyperbolic.

Distance

Construct a Cartesian-like coordinate system as follows. Choose a line (the x-axis) in the hyperbolic plane (with a standardized curvature of -1) and label the points on it by their distance from an origin ($x=0$) point on the x-axis (positive on one side and negative on the other). For any point in the plane, one can define coordinates x and y by dropping a perpendicular onto the x-axis. x will be the label of the foot of the perpendicular. y will be the distance along the perpendicular of the given point from its foot (positive on one side and negative on the other). Then the distance between two such points will be

$$\text{dist}(\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle) = \text{arcosh}(\cosh y_1 \cosh(x_2 - x_1) \cosh y_2 - \sinh y_1 \sinh y_2).$$

This formula can be derived from the formulas about [hyperbolic triangles](#).

The corresponding metric tensor is: $(ds)^2 = \cosh^2 y (dx)^2 + (dy)^2$.

In this coordinate system, straight lines are either perpendicular to the x-axis (with equation $x = a$ constant) or described by equations of the form

$$\tanh y = A \cosh x + B \sinh x \quad \text{when} \quad A^2 < 1 + B^2$$

where A and B are real parameters which characterize the straight line.

History

Since the publication of [Euclid's Elements](#) circa 300 BCE, many [geometers](#) made attempts to prove the [parallel postulate](#). Some tried to prove it by [assuming its negation and trying to derive a contradiction](#). Foremost among these were [Proclus](#), [Ibn al-Haytham](#) (Alhacen), [Omar Khayyám](#),^[5] [Nasīr al-Dīn al-Tūsī](#), [Witelo](#), [Gersonides](#), [Alfonso](#), and later [Giovanni Gerolamo Saccheri](#), [John Wallis](#), [Johann Heinrich Lambert](#), and [Legendre](#).^[6] Their attempts were doomed to failure (as we now know, the parallel postulate is not provable from the other postulates), but their efforts led to the discovery of hyperbolic geometry.

The theorems of Alhacen, Khayyam and al-Tūsī on [quadrilaterals](#), including the [Ibn al-Haytham–Lambert quadrilateral](#) and [Khayyam–Saccheri quadrilateral](#), were the first theorems on hyperbolic geometry. Their works on hyperbolic geometry had a considerable influence on its development among later European geometers, including Witelo, Gersonides, Alfonso, John Wallis and Saccheri.^[7]

In the 18th century, [Johann Heinrich Lambert](#) introduced the [hyperbolic functions](#)^[8] and computed the area of a [hyperbolic triangle](#).^[9]

19th-century developments

In the 19th century, hyperbolic geometry was explored extensively by [Nikolai Ivanovich Lobachevsky](#), [János Bolyai](#), [Carl Friedrich Gauss](#) and [Franz Taurinus](#). Unlike their predecessors, who just wanted to eliminate the parallel postulate from the axioms of Euclidean geometry, these authors realized they had discovered a new geometry.^{[10][11]} Gauss wrote in an 1824 letter to [Franz Taurinus](#) that he had constructed it, but Gauss did not publish his work. Gauss called it "non-Euclidean geometry"^[12] causing several modern authors to continue to consider "non-Euclidean geometry" and "hyperbolic geometry" to be synonyms. Taurinus published results on hyperbolic trigonometry in 1826, argued that hyperbolic geometry is self consistent, but still believed in the special role of Euclidean geometry. The complete system of hyperbolic geometry was published by Lobachevsky in 1829/1830, while Bolyai discovered it independently and published in 1832.

In 1868, [Eugenio Beltrami](#) provided [models \(see below\)](#) of hyperbolic geometry, and used this to prove that hyperbolic geometry was consistent [if and only if](#) Euclidean geometry was.

The term "hyperbolic geometry" was introduced by [Felix Klein](#) in 1871.^[13] Klein followed an initiative of [Arthur Cayley](#) to use the transformations of [projective geometry](#) to produce [isometries](#). The idea used a [conic section](#) or [quadric](#) to define a region, and used [cross ratio](#) to define a [metric](#). The projective transformations that leave the conic section or quadric [stable](#) are the isometries. "Klein showed that if the [Cayley absolute](#) is a real curve then the part of the projective plane in its interior is isometric to the hyperbolic plane..."^[14]

For more history, see article on non-Euclidean geometry, and the references [Coxeter](#)^[15] and [Milnor](#).^[16]

Philosophical consequences

The discovery of hyperbolic geometry had important [philosophical](#) consequences. Before its discovery many philosophers (for example [Hobbes](#) and [Spinoza](#)) viewed philosophical rigour in

terms of the "geometrical method", referring to the method of reasoning used in [Euclid's Elements](#).

[Kant](#) in the [Critique of Pure Reason](#) came to the conclusion that space (in [Euclidean geometry](#)) and time are not discovered by humans as objective features of the world, but are part of an unavoidable systematic framework for organizing our experiences.^[17]

It is said that [Gauss](#) did not publish anything about hyperbolic geometry out of fear of the "uproar of the [Boeotians](#)", which would ruin his status as *princeps mathematicorum* (Latin, "the Prince of Mathematicians").^[18] The "uproar of the Boeotians" came and went, and gave an impetus to great improvements in [mathematical rigour](#), [analytical philosophy](#) and [logic](#). Hyperbolic geometry was finally proved consistent and is therefore another valid geometry.

Geometry of the universe (spatial dimensions only)

Because Euclidean, hyperbolic and elliptic geometry are all consistent, the question arises: which is the real geometry of space, and if it is hyperbolic or elliptic, what is its curvature?

Lobachevsky had already tried to measure the curvature of the universe by measuring the [parallax](#) of [Sirius](#) and treating Sirius as the ideal point of an [angle of parallelism](#). He realised that his measurements were [not precise enough](#) to give a definite answer, but he did reach the conclusion that if the geometry of the universe is hyperbolic, then the [absolute length](#) is at least one million times the diameter of the [earth's orbit](#) (2 000 000 [AU](#), 10 [parsec](#)).^[19] Some argue that his measurements were methodologically flawed.^[20]

[Henri Poincaré](#), with his [sphere-world thought experiment](#), came to the conclusion that everyday experience does not necessarily rule out other geometries.

The [geometrization conjecture](#) gives a complete list of eight possibilities for the fundamental geometry of our space. The problem in determining which one applies is that, to reach a definitive answer, we need to be able to look at extremely large shapes – much larger than anything on Earth or perhaps even in our galaxy.^[21]

Geometry of the universe (special relativity)

[Special relativity](#) places space and time on equal footing, so that one considers the geometry of a unified [spacetime](#) instead of considering space and time separately.^{[22][23]} [Minkowski geometry](#) replaces [Galilean geometry](#) (which is the three-dimensional Euclidean space with time of [Galilean relativity](#)).^[24]

In relativity, rather than considering Euclidean, elliptic and hyperbolic geometries, the appropriate geometries to consider are [Minkowski space](#), [de Sitter space](#) and [anti-de Sitter space](#),^{[25][26]} corresponding to zero, positive and negative curvature respectively.

Hyperbolic geometry enters special relativity through [rapidity](#), which stands in for [velocity](#), and is expressed by a [hyperbolic angle](#). The study of this velocity geometry has been called [kinematic geometry](#). The space of relativistic velocities has a three-dimensional hyperbolic geometry, where the distance function is determined from the relative velocities of "nearby" points (velocities).^[27]

Physical realizations of the hyperbolic plane

The hyperbolic plane is a plane where every point is a [saddle point](#). There exist various [pseudospheres](#) in Euclidean space that have a finite area of constant negative Gaussian curvature.

By [Hilbert's theorem](#), it is not possible to isometrically [immerse](#) a complete hyperbolic plane (a complete regular surface of constant negative [Gaussian curvature](#)) in a three-dimensional Euclidean space.

Other useful [models](#) of hyperbolic geometry exist in Euclidean space, in which the metric is not preserved. A particularly well-known paper model based on the [pseudosphere](#) is due to [William Thurston](#).



A collection of crocheted hyperbolic planes, in imitation of a coral reef, by the [Institute For Figuring](#)



A coral with similar geometry on the [Great Barrier Reef](#)

The art of [crochet](#) has been used (see [Mathematics and fiber arts § Knitting and crochet](#)) to demonstrate hyperbolic planes with the first being made by [Daina Taimiņa](#).^[28]

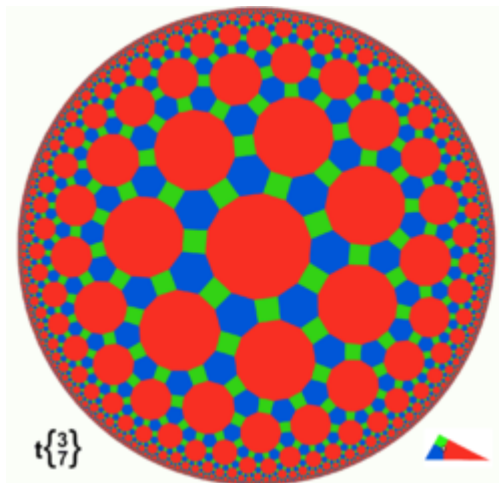
In 2000, Keith Henderson demonstrated a quick-to-make paper model dubbed the "[hyperbolic soccerball](#)" (more precisely, a [truncated order-7 triangular tiling](#)).^{[29][30]}

Instructions on how to make a hyperbolic quilt, designed by [Helaman Ferguson](#),^[31] have been made available by [Jeff Weeks](#).^[32]

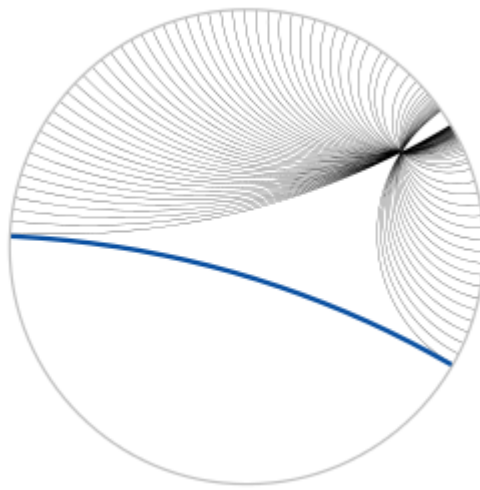
Models of the hyperbolic plane

There are different [pseudospherical surfaces](#) that have for a large area a constant negative Gaussian curvature, the [pseudosphere](#) being the best well known of them.

But it is easier to do hyperbolic geometry on other models.



Poincaré disk model with [truncated triheptagonal tiling](#)



Lines through a given point and parallel to a given line, illustrated in the Poincaré disk model

There are four [models](#) commonly used for hyperbolic geometry: the [Klein model](#), the [Poincaré disk model](#), the [Poincaré half-plane model](#), and the Lorentz or [hyperboloid model](#). These models define a hyperbolic plane which satisfies the axioms of a hyperbolic geometry. Despite their names, the first three mentioned above were introduced as models of hyperbolic space by [Beltrami](#), not by [Poincaré](#) or [Klein](#). All these models are extendable to more dimensions.

The Beltrami–Klein model

The [Beltrami–Klein model](#), also known as the projective disk model, Klein disk model and [Klein model](#), is named after [Eugenio Beltrami](#) and [Felix Klein](#).

For the two dimensions this model uses the interior of the [unit circle](#) for the complete hyperbolic [plane](#), and the [chords](#) of this circle are the hyperbolic lines.

For higher dimensions this model uses the interior of the [unit ball](#), and the [chords](#) of this n -ball are the hyperbolic lines.

- This model has the advantage that lines are straight, but the disadvantage that [angles](#) are distorted (the mapping is not [conformal](#)), and also circles are not represented as circles.
- The distance in this model is half the logarithm of the [cross-ratio](#), which was introduced by [Arthur Cayley](#) in [projective geometry](#).

The Poincaré disk model

The [Poincaré disk model](#), also known as the conformal disk model, also employs the interior of the [unit circle](#), but lines are represented by arcs of circles that are [orthogonal](#) to the boundary circle,

plus diameters of the boundary circle.

- This model preserves angles, and is thereby [conformal](#). All isometries within this model are therefore [Möbius transformations](#).
- Circles entirely within the disk remain circles although the Euclidean center of the circle is closer to the center of the disk than is the hyperbolic center of the circle.
- [Horocycles](#) are circles within the disk which are [tangent](#) to the boundary circle, minus the point of contact.
- [Hypercycles](#) are open-ended chords and circular arcs within the disc that terminate on the boundary circle at non-orthogonal angles.

The Poincaré half-plane model

The [Poincaré half-plane model](#) takes one-half of the Euclidean plane, bounded by a line B of the plane, to be a model of the hyperbolic plane. The line B is not included in the model.

The Euclidean plane may be taken to be a plane with the [Cartesian coordinate system](#) and the x -axis is taken as line B and the half plane is the upper half ($y > 0$) of this plane.

- Hyperbolic lines are then either half-circles orthogonal to B or rays perpendicular to B .
- The length of an interval on a ray is given by [logarithmic measure](#) so it is invariant under a [homothetic transformation](#) $(x, y) \mapsto (x, \lambda y)$, $\lambda > 0$.
- Like the Poincaré disk model, this model preserves angles, and is thus [conformal](#). All isometries within this model are therefore [Möbius transformations](#) of the plane.
- The half-plane model is the limit of the Poincaré disk model whose boundary is tangent to B at the same point while the radius of the disk model goes to infinity.

The hyperboloid model

The [hyperboloid model](#) or Lorentz model employs a 2-dimensional [hyperboloid](#) of revolution (of two sheets, but using one) embedded in 3-dimensional [Minkowski space](#). This model is generally credited to Poincaré, but Reynolds^[33] says that [Wilhelm Killing](#) used this model in 1885

- This model has direct application to [special relativity](#), as Minkowski 3-space is a model for [spacetime](#), suppressing one spatial dimension. One can take the hyperboloid to represent the events that various moving observers, radiating outward in a spatial plane from a single point, will reach in a fixed [proper time](#).

- The hyperbolic distance between two points on the hyperboloid can then be identified with the relative [rapidity](#) between the two corresponding observers.
- The model generalizes directly to an additional dimension, where three-dimensional hyperbolic geometry relates to Minkowski 4-space.

The hemisphere model

The [hemisphere](#) model is not often used as model by itself, but it functions as a useful tool for visualising transformations between the other models.

The hemisphere model uses the upper half of the [unit sphere](#): $x^2 + y^2 + z^2 = 1, z > 0$.

The hyperbolic lines are half-circles orthogonal to the boundary of the hemisphere.

The hemisphere model is part of a [Riemann sphere](#), and different projections give different models of the hyperbolic plane:

- [Stereographic projection](#) from $(0, 0, -1)$ onto the plane $z = 0$ projects corresponding points on the [Poincaré disk model](#)
- [Stereographic projection](#) from $(0, 0, -1)$ onto the surface $x^2 + y^2 - z^2 = -1, z > 0$ projects corresponding points on the [hyperboloid model](#)
- [Stereographic projection](#) from $(-1, 0, 0)$ onto the plane $x = 1$ projects corresponding points on the [Poincaré half-plane model](#)
- [Orthographic projection](#) onto a plane $z = C$ projects corresponding points on the [Beltrami–Klein model](#).
- [Central projection](#) from the centre of the sphere onto the plane $z = 1$ projects corresponding points on the [Gans Model](#)

See further: [Connection between the models \(below\)](#)

The Gans model

In 1966 David Gans proposed a [flattened hyperboloid model](#) in the journal [American Mathematical Monthly](#).^[34] It is an [orthographic projection](#) of the hyperboloid model onto the xy-plane. This model is not as widely used as other models but nevertheless is quite useful in the understanding of hyperbolic geometry.

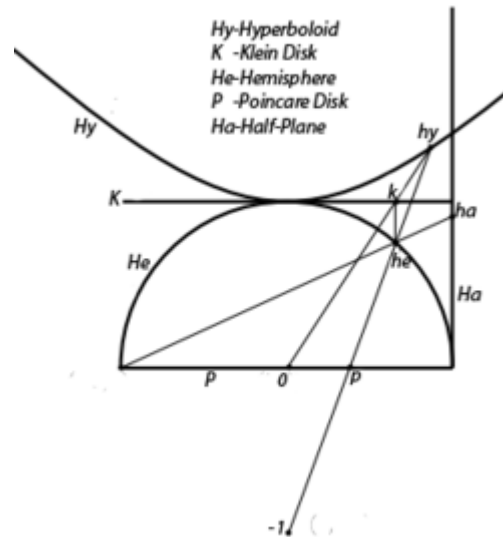
- Unlike the Klein or the Poincaré models, this model utilizes the entire [Euclidean plane](#).

- The lines in this model are represented as branches of a [hyperbola](#).^[35]

The band model

The band model employs a portion of the Euclidean plane between two parallel lines.^[36] Distance is preserved along one line through the middle of the band. Assuming the band is given by $\{z \in \mathbb{C} : |\operatorname{Im} z| < \pi/2\}$, the metric is given by $|dz| \sec(\operatorname{Im} z)$.

Connection between the models



Poincaré disk, hemispherical and hyperboloid models are related by [stereographic projection](#) from -1 . [Beltrami-Klein model](#) is [orthographic projection](#) from hemispherical model. [Poincaré half-plane model](#) here projected from the hemispherical model by rays from left end of Poincaré disk model.

All models essentially describe the same structure. The difference between them is that they represent different [coordinate charts](#) laid down on the same [metric space](#), namely the hyperbolic plane. The characteristic feature of the hyperbolic plane itself is that it has a constant negative [Gaussian curvature](#), which is indifferent to the coordinate chart used. The [geodesics](#) are similarly invariant: that is, geodesics map to geodesics under coordinate transformation. Hyperbolic geometry generally is introduced in terms of the geodesics and their intersections on the hyperbolic plane.^[37]

Once we choose a coordinate chart (one of the "models"), we can always [embed](#) it in a Euclidean space of same dimension, but the embedding is clearly not isometric (since the curvature of Euclidean space is 0). The hyperbolic space can be represented by infinitely many different charts;

but the embeddings in Euclidean space due to these four specific charts show some interesting characteristics.

Since the four models describe the same metric space, each can be transformed into the other.

See, for example:

- [the Beltrami–Klein model's relation to the hyperboloid model](#),
- [the Beltrami–Klein model's relation to the Poincaré disk model](#),
- and [the Poincaré disk model's relation to the hyperboloid model](#).

Isometries of the hyperbolic plane

Every [isometry](#) ([transformation](#) or [motion](#)) of the hyperbolic plane to itself can be realized as the composition of at most three [reflections](#). In n -dimensional hyperbolic space, up to $n+1$ reflections might be required. (These are also true for Euclidean and spherical geometries, but the classification below is different.)

All the isometries of the hyperbolic plane can be classified into these classes:

- Orientation preserving
 - the [identity isometry](#) — nothing moves; zero reflections; zero [degrees of freedom](#).
 - [inversion through a point \(half turn\)](#) — two reflections through mutually perpendicular lines passing through the given point, i.e. a rotation of 180 degrees around the point; two [degrees of freedom](#).
 - [rotation](#) around a normal point — two reflections through lines passing through the given point (includes inversion as a special case); points move on circles around the center; three degrees of freedom.
 - "rotation" around an [ideal point](#) (horolation) — two reflections through lines leading to the ideal point; points move along horocycles centered on the ideal point; two degrees of freedom.
 - translation along a straight line — two reflections through lines perpendicular to the given line; points off the given line move along hypercycles; three degrees of freedom.
- Orientation reversing
 - reflection through a line — one reflection; two degrees of freedom.

- combined reflection through a line and translation along the same line — the reflection and translation commute; three reflections required; three degrees of freedom.

Hyperbolic geometry in art

M. C. Escher's famous prints *Circle Limit III* and *Circle Limit IV* illustrate the conformal disc model (Poincaré disk model) quite well. The white lines in *III* are not quite geodesics (they are [hypercycles](#)), but are close to them. It is also possible to see quite plainly the negative [curvature](#) of the hyperbolic plane, through its effect on the sum of angles in triangles and squares.

For example, in *Circle Limit III* every vertex belongs to three triangles and three squares. In the Euclidean plane, their angles would sum to 450° ; i.e., a circle and a quarter. From this, we see that the sum of angles of a triangle in the hyperbolic plane must be smaller than 180° . Another visible property is [exponential growth](#). In *Circle Limit III*, for example, one can see that the number of fishes within a distance of n from the center rises exponentially. The fishes have an equal hyperbolic area, so the area of a ball of radius n must rise exponentially in n .

The art of [crochet](#) has [been used](#) to demonstrate hyperbolic planes (pictured above) with the first being made by Daina Taimiņa,^[28] whose book *Crocheting Adventures with Hyperbolic Planes* won the 2009 Bookseller/Diagram Prize for Oddest Title of the Year.^[38]

[HyperRogue](#) is a [roguelike](#) game set on various tilings of the [hyperbolic plane](#).

Higher dimensions

Hyperbolic geometry is not limited to 2 dimensions; a hyperbolic geometry exists for every higher number of dimensions.

Homogeneous structure

[Hyperbolic space](#) of dimension n is a special case of a Riemannian [symmetric space](#) of noncompact type, as it is [isomorphic](#) to the quotient

$$\mathbf{O}(1, n)/(\mathbf{O}(1) \times \mathbf{O}(n)).$$

The [orthogonal group](#) $\mathbf{O}(1, n)$ acts by norm-preserving transformations on [Minkowski space](#) $\mathbf{R}^{1, n}$, and it acts [transitively](#) on the two-sheet hyperboloid of norm 1 vectors. Timelike lines (i.e., those with positive-norm tangents) through the origin pass through antipodal points in the hyperboloid, so the space of such lines yields a model of hyperbolic n -space. The [stabilizer](#) of any particular line is

isomorphic to the [product](#) of the orthogonal groups $O(n)$ and $O(1)$, where $O(n)$ acts on the tangent space of a point in the hyperboloid, and $O(1)$ reflects the line through the origin. Many of the elementary concepts in hyperbolic geometry can be described in [linear algebraic](#) terms: geodesic paths are described by intersections with planes through the origin, dihedral angles between hyperplanes can be described by inner products of normal vectors, and hyperbolic reflection groups can be given explicit matrix realizations.

In small dimensions, there are exceptional isomorphisms of Lie groups that yield additional ways to consider symmetries of hyperbolic spaces. For example, in dimension 2, the isomorphisms $SO^+(1, 2) \cong \text{PSL}(2, \mathbf{R}) \cong \text{PSU}(1, 1)$ allow one to interpret the upper half plane model as the quotient $\text{SL}(2, \mathbf{R})/\text{SO}(2)$ and the Poincaré disc model as the quotient $\text{SU}(1, 1)/\text{U}(1)$. In both cases, the symmetry groups act by fractional linear transformations, since both groups are the orientation-preserving stabilizers in $\text{PGL}(2, \mathbf{C})$ of the respective subspaces of the Riemann sphere. The Cayley transformation not only takes one model of the hyperbolic plane to the other, but realizes the isomorphism of symmetry groups as conjugation in a larger group. In dimension 3, the fractional linear action of $\text{PGL}(2, \mathbf{C})$ on the Riemann sphere is identified with the action on the conformal boundary of hyperbolic 3-space induced by the isomorphism $O^+(1, 3) \cong \text{PGL}(2, \mathbf{C})$. This allows one to study isometries of hyperbolic 3-space by considering spectral properties of representative complex matrices. For example, parabolic transformations are conjugate to rigid translations in the upper half-space model, and they are exactly those transformations that can be represented by [unipotent upper triangular](#) matrices.

See also

- [Constructions in hyperbolic geometry](#)
- [Hyperbolic 3-manifold](#)
- [Hyperbolic manifold](#)
- [Hyperbolic set](#)
- [Hjelmslev transformation](#)
- [Hyperbolic tree](#)
- [Kleinian group](#)
- [Lambert quadrilateral](#)
- [Open universe](#)
- [Poincaré metric](#)

- [Saccheri quadrilateral](#)
- [Systolic geometry](#)
- [Uniform tilings in hyperbolic plane](#)
- [\$\delta\$ -hyperbolic space](#)
- [Band model](#)

Notes

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"Three scientists, Ibn al-Haytham, Khayyam and al-Tūsī, had made the most considerable contribution to this branch of geometry whose importance came to be completely recognized only in the 19th century. In essence their propositions concerning the properties of quadrangles which they considered assuming that some of the angles of these figures were acute or obtuse, embodied the first few theorems of the hyperbolic and the elliptic geometries. Their other proposals showed that various geometric statements were equivalent to the Euclidean postulate V. It is extremely important that these scholars established the mutual connection between this postulate and the sum of the angles of a triangle and a quadrangle. By their works on the theory of parallel lines Arab mathematicians directly influenced the relevant investigations of their European counterparts. The first European attempt to prove the postulate on parallel lines – made by Witelo, the Polish scientist of the 13th century, while revising Ibn al-Haytham's *Book of Optics* (*Kitab al-Manazir*) – was undoubtedly prompted by Arabic sources. The proofs put forward in the 14th century by the Jewish scholar [Levi ben Gerson](#), who lived in southern France, and by the above-mentioned Alfonso from Spain directly border on Ibn al-Haytham's demonstration. Above, we have demonstrated that *Pseudo-Tusi's Exposition of Euclid* had stimulated both J. Wallis's and G. Saccheri's studies of the theory of parallel lines."

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External links

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