

# Mathematics

**Mathematics** (from [Greek](#): [μάθημα](#), *máthēma*, 'knowledge, study, learning') includes the study of such topics as [quantity](#) ([number theory](#)),<sup>[1]</sup> [structure](#) ([algebra](#)),<sup>[2]</sup> [space](#) ([geometry](#)),<sup>[1]</sup> and [change](#) ([analysis](#)).<sup>[3][4][5]</sup> It has no generally accepted [definition](#).<sup>[6][7]</sup>



Greek mathematician [Euclid](#) (holding [calipers](#)), 3rd century BC, as imagined by [Raphael](#) in this detail from *The School of Athens* (1509–1511)<sup>[a]</sup>

Mathematicians seek and use [patterns](#)<sup>[8][9]</sup> to formulate new [conjectures](#); they resolve the [truth](#) or falsity of such by [mathematical proof](#). When [mathematical structures](#) are good models of real phenomena, mathematical reasoning can be used to provide insight or predictions about nature. Through the use of [abstraction](#) and [logic](#), mathematics developed from [counting](#), [calculation](#), [measurement](#), and the systematic study of the [shapes](#) and [motions](#) of [physical objects](#). Practical mathematics has been a human activity from as far back as [written records](#) exist. The [research](#) required to solve mathematical problems can take years or even centuries of sustained inquiry.

[Rigorous arguments](#) first appeared in [Greek mathematics](#), most notably in [Euclid's](#) *[Elements](#)*.<sup>[10]</sup> Since the pioneering work of [Giuseppe Peano](#) (1858–1932), [David Hilbert](#) (1862–1943), and others [on axiomatic systems in the late 19th century](#), it has become customary to view mathematical research as establishing truth by [rigorous deduction](#) from appropriately chosen [axioms](#) and [definitions](#). Mathematics developed at a relatively slow pace until the [Renaissance](#), when mathematical innovations interacting with new [scientific discoveries](#) led to a rapid increase in the rate of mathematical discovery that has continued to the present day.<sup>[11]</sup>

Mathematics is essential in many fields, including [natural science](#), [engineering](#), [medicine](#), [finance](#), and the [social sciences](#). [Applied mathematics](#) has led to entirely new mathematical disciplines, such as [statistics](#) and [game theory](#). Mathematicians engage in [pure mathematics](#) (mathematics for its own sake) without having any application in mind, but practical applications for what began as pure mathematics are often discovered later.<sup>[12][13]</sup>

# History

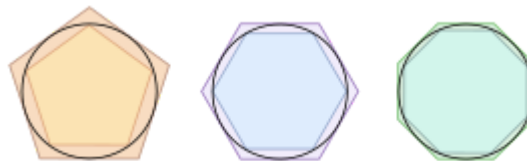
The history of mathematics can be seen as an ever-increasing series of [abstractions](#). The first abstraction, which is shared by many animals,<sup>[14]</sup> was probably that of numbers: the realization that a collection of two apples and a collection of two oranges (for example) have something in common, namely the quantity of their members.

As evidenced by [tallies](#) found on bone, in addition to recognizing how to [count](#) physical objects, [prehistoric](#) peoples may have also recognized how to count abstract quantities, like time—days, seasons, or years.<sup>[15][16]</sup>



The Babylonian mathematical tablet  
Plimpton 322, dated to 1800 BC.

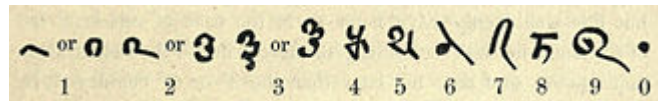
Evidence for more complex mathematics does not appear until around 3000 BC, when the [Babylonians](#) and Egyptians began using [arithmetic](#), [algebra](#) and geometry for taxation and other financial calculations, for building and construction, and for [astronomy](#).<sup>[17]</sup> The oldest mathematical texts from [Mesopotamia](#) and [Egypt](#) are from 2000 to 1800 BC.<sup>[18]</sup> Many early texts mention [Pythagorean triples](#) and so, by inference, the [Pythagorean theorem](#) seems to be the most ancient and widespread mathematical development after basic arithmetic and geometry.<sup>[19]</sup> It is in [Babylonian mathematics](#) that [elementary arithmetic](#) ([addition](#), [subtraction](#), [multiplication](#) and [division](#)) first appear in the archaeological record. The Babylonians also possessed a place-value system and used a [sexagesimal](#) numeral system<sup>[19]</sup> which is still in use today for measuring angles and time.<sup>[20]</sup>



Archimedes used the [method of exhaustion](#) to  
approximate the value of [pi](#).

Beginning in the 6th century BC with the [Pythagoreans](#), with [Greek mathematics](#) the [Ancient Greeks](#) began a systematic study of mathematics as a subject in its own right.<sup>[21]</sup> Around 300 BC, [Euclid](#)

introduced the [axiomatic method](#) still used in mathematics today, consisting of definition, axiom, theorem, and proof. His book, *Elements*, is widely considered the most successful and influential textbook of all time.<sup>[22]</sup> The greatest mathematician of antiquity is often held to be [Archimedes](#) (c. 287–212 BC) of [Syracuse](#).<sup>[23]</sup> He developed formulas for calculating the surface area and volume of [solids of revolution](#) and used the [method of exhaustion](#) to calculate the [area](#) under the arc of a [parabola](#) with the [summation of an infinite series](#), in a manner not too dissimilar from modern calculus.<sup>[24]</sup> Other notable achievements of Greek mathematics are [conic sections](#) ([Apollonius of Perga](#), 3rd century BC),<sup>[25]</sup> [trigonometry](#) ([Hipparchus of Nicaea](#), 2nd century BC),<sup>[26]</sup> and the beginnings of algebra ([Diophantus](#), 3rd century AD).<sup>[27]</sup>



The numerals used in the [Bakhshali manuscript](#), dated between the 2nd century BC and the 2nd century AD.

The [Hindu–Arabic numeral system](#) and the rules for the use of its operations, in use throughout the world today, evolved over the course of the first millennium AD in [India](#) and were transmitted to the [Western world](#) via [Islamic mathematics](#).<sup>[28]</sup> Other notable developments of Indian mathematics include the modern definition and approximation of [sine](#) and [cosine](#),<sup>[28]</sup> and an early form of [infinite series](#).



A page from al-Khwārizmī's *Algebra*

During the [Golden Age of Islam](#), especially during the 9th and 10th centuries, mathematics saw many important innovations building on Greek mathematics. The most notable achievement of [Islamic mathematics](#) was the development of [algebra](#). Other achievements of the Islamic period

include advances in [spherical trigonometry](#) and the addition of the [decimal point](#) to the Arabic numeral system.<sup>[29][30]</sup> Many notable mathematicians from this period were Persian, such as [Al-Khwarismi](#), [Omar Khayyam](#) and [Sharaf al-Dīn al-Ṭūsī](#).

During the [early modern period](#), mathematics began to develop at an accelerating pace in [Western Europe](#). The development of [calculus](#) by Newton and Leibniz in the 17th century revolutionized mathematics.<sup>[31]</sup> [Leonhard Euler](#) was the most notable mathematician of the 18th century, contributing numerous theorems and discoveries.<sup>[32]</sup> Perhaps the foremost mathematician of the 19th century was the German mathematician [Carl Friedrich Gauss](#),<sup>[33]</sup> who made numerous contributions to fields such as [algebra](#), [analysis](#), [differential geometry](#), [matrix theory](#), [number theory](#), and [statistics](#). In the early 20th century, [Kurt Gödel](#) transformed mathematics by publishing his [incompleteness theorems](#), which show in part that any consistent axiomatic system—if powerful enough to describe arithmetic—will contain true propositions that cannot be proved.<sup>[34]</sup>

Mathematics has since been greatly extended, and there has been a fruitful interaction between mathematics and [science](#), to the benefit of both. Mathematical discoveries continue to be made today. According to Mikhail B. Sevryuk, in the January 2006 issue of the [Bulletin of the American Mathematical Society](#), "The number of papers and books included in the [Mathematical Reviews](#) database since 1940 (the first year of operation of MR) is now more than 1.9 million, and more than 75 thousand items are added to the database each year. The overwhelming majority of works in this ocean contain new mathematical [theorems](#) and their [proofs](#)."<sup>[35]</sup>

## Etymology

The word *mathematics* comes from [Ancient Greek](#) *máthēma* ([μάθημα](#)), meaning "that which is learnt,"<sup>[36]</sup> "what one gets to know," hence also "study" and "science". The word for "mathematics" came to have the narrower and more technical meaning "mathematical study" even in Classical times.<sup>[37]</sup> Its [adjective](#) is *mathēmatikós* ([μαθηματικός](#)), meaning "related to learning" or "studious," which likewise further came to mean "mathematical." In particular, *mathēmatikḗ tékhnē* ([μαθηματικὴ τέχνη](#); [Latin](#): *ars mathematica*) meant "the mathematical art."

Similarly, one of the two main schools of thought in [Pythagoreanism](#) was known as the *mathēmatikoi* ([μαθηματικοί](#))—which at the time meant "learners" rather than "mathematicians" in the modern sense.<sup>[38]</sup>

In Latin, and in English until around 1700, the term *mathematics* more commonly meant "[astrology](#)" (or sometimes "[astronomy](#)") rather than "mathematics"; the meaning gradually changed to its present one from about 1500 to 1800. This has resulted in several mistranslations. For example,

[Saint Augustine](#)'s warning that Christians should beware of *mathematici*, meaning astrologers, is sometimes mistranslated as a condemnation of mathematicians.<sup>[39]</sup>

The apparent [plural](#) form in English, like the French plural form *les mathématiques* (and the less commonly used singular [derivative](#) *la mathématique*), goes back to the Latin [neuter](#) plural *mathematica* ([Cicero](#)), based on the Greek plural *ta mathēmatiká* (τὰ μαθηματικά), used by [Aristotle](#) (384–322 BC), and meaning roughly "all things mathematical", although it is plausible that English borrowed only the adjective *mathematic(al)* and formed the noun *mathematics* anew, after the pattern of [physics](#) and [metaphysics](#), which were inherited from Greek.<sup>[40]</sup> In English, the noun *mathematics* takes a singular verb. It is often shortened to *maths* or, in North America, *math*.<sup>[41]</sup>

## Definitions of mathematics

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[Leonardo Fibonacci](#), the Italian mathematician who introduced the [Hindu–Arabic numeral system](#) invented between the 1st and 4th centuries by Indian mathematicians, to the Western World.

Mathematics has no generally accepted definition.<sup>[6][7]</sup> [Aristotle](#) defined mathematics as "the science of quantity" and this definition prevailed until the 18th century. However, Aristotle also noted a focus on quantity alone may not distinguish mathematics from sciences like physics; in his view, abstraction and studying quantity as a property "separable in thought" from real instances set mathematics apart.<sup>[42]</sup>

In the 19th century, when the study of mathematics increased in rigor and began to address abstract topics such as [group theory](#) and [projective geometry](#), which have no clear-cut relation to



quantity and measurement, mathematicians and philosophers began to propose a variety of new definitions.<sup>[43]</sup>

A great many professional mathematicians take no interest in a definition of mathematics, or consider it undefinable.<sup>[6]</sup> There is not even consensus on whether mathematics is an art or a science.<sup>[7]</sup> Some just say, "Mathematics is what mathematicians do."<sup>[6]</sup>

## Three leading types

Three leading types of definition of mathematics today are called [logician](#), [intuitionist](#), and [formalist](#), each reflecting a different philosophical school of thought.<sup>[44]</sup> All have severe flaws, none has widespread acceptance, and no reconciliation seems possible.<sup>[44]</sup>

### Logician definitions

An early definition of mathematics in terms of logic was that of [Benjamin Peirce](#) (1870): "the science that draws necessary conclusions."<sup>[45]</sup> In the *Principia Mathematica*, [Bertrand Russell](#) and [Alfred North Whitehead](#) advanced the philosophical program known as [logicism](#), and attempted to prove that all mathematical concepts, statements, and principles can be defined and proved entirely in terms of [symbolic logic](#). A logicist definition of mathematics is Russell's (1903) "All Mathematics is Symbolic Logic."<sup>[46]</sup>

### Intuitionist definitions

[Intuitionist](#) definitions, developing from the philosophy of mathematician [L. E. J. Brouwer](#), identify mathematics with certain mental phenomena. An example of an intuitionist definition is "Mathematics is the mental activity which consists in carrying out constructs one after the other."<sup>[44]</sup> A peculiarity of intuitionism is that it rejects some mathematical ideas considered valid according to other definitions. In particular, while other philosophies of mathematics allow objects that can be proved to exist even though they cannot be constructed, intuitionism allows only mathematical objects that one can actually construct. Intuitionists also reject the [law of excluded middle](#) (i.e.,  $P \vee \neg P$ ). While this stance does force them to reject one common version of [proof by contradiction](#) as a viable proof method, namely the inference of  $P$  from  $\neg P \rightarrow \perp$ , they are still able to infer  $\neg P$  from  $P \rightarrow \perp$ . For them,  $\neg(\neg P)$  is a strictly weaker statement than  $P$ .<sup>[47]</sup>

### Formalist definitions

[Formalist](#) definitions identify mathematics with its symbols and the rules for operating on them. [Haskell Curry](#) defined mathematics simply as "the science of formal systems".<sup>[48]</sup> A [formal system](#) is a set of symbols, or *tokens*, and some *rules* on how the tokens are to be combined into *formulas*.

In formal systems, the word *axiom* has a special meaning different from the ordinary meaning of "a self-evident truth", and is used to refer to a combination of tokens that is included in a given formal system without needing to be derived using the rules of the system.

## Mathematics as science



Carl Friedrich Gauss, known  
as the prince of  
mathematicians

The German mathematician [Carl Friedrich Gauss](#) referred to mathematics as "the Queen of the Sciences".<sup>[49]</sup> More recently, [Marcus du Sautoy](#) has called mathematics "the Queen of Science ... the main driving force behind scientific discovery".<sup>[50]</sup> The philosopher [Karl Popper](#) observed that "most mathematical theories are, like those of [physics](#) and [biology](#), [hypothetico-deductive](#): pure mathematics therefore turns out to be much closer to the natural sciences whose hypotheses are conjectures, than it seemed even recently."<sup>[51]</sup> Popper also noted that "I shall certainly admit a system as empirical or scientific only if it is capable of being tested by experience."<sup>[52]</sup>

Several authors consider that mathematics is not a science because it does not rely on [empirical evidence](#).<sup>[53][54][55][56]</sup>

Mathematics shares much in common with many fields in the physical sciences, notably the [exploration of the logical consequences](#) of assumptions. [Intuition](#) and experimentation also play a role in the formulation of [conjectures](#) in both mathematics and the (other) sciences. [Experimental mathematics](#) continues to grow in importance within mathematics, and computation and simulation are playing an increasing role in both the sciences and mathematics.

The opinions of mathematicians on this matter are varied. Many mathematicians<sup>[57]</sup> feel that to call their area a science is to downplay the importance of its aesthetic side, and its history in the traditional seven [liberal arts](#); others feel that to ignore its connection to the sciences is to turn a blind eye to the fact that the interface between mathematics and its applications in science and

engineering has driven much development in mathematics.<sup>[58]</sup> One way this difference of viewpoint plays out is in the philosophical debate as to whether mathematics is *created* (as in art) or *discovered* (as in science). In practice, mathematicians are typically grouped with scientists at the gross level but separated at finer levels. This is one of many issues considered in the [philosophy of mathematics](#).<sup>[59]</sup>

## Inspiration, pure and applied mathematics, and aesthetics

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[Isaac Newton](#) (left) and [Gottfried Wilhelm Leibniz](#) developed infinitesimal calculus.

Mathematics arises from many different kinds of problems. At first these were found in commerce, [land measurement](#), architecture and later [astronomy](#); today, all sciences suggest problems studied by mathematicians, and many problems arise within mathematics itself. For example, the [physicist Richard Feynman](#) invented the [path integral formulation](#) of [quantum mechanics](#) using a combination of mathematical reasoning and physical insight, and today's [string theory](#), a still-developing scientific theory which attempts to unify the four [fundamental forces of nature](#), continues to inspire new mathematics.<sup>[60]</sup>

Some mathematics is relevant only in the area that inspired it, and is applied to solve further problems in that area. But often mathematics inspired by one area proves useful in many areas, and joins the general stock of mathematical concepts. A distinction is often made between [pure mathematics](#) and [applied mathematics](#). However pure mathematics topics often turn out to have applications, e.g. [number theory](#) in [cryptography](#).

This remarkable fact, that even the "purest" mathematics often turns out to have practical applications, is what the physicist [Eugene Wigner](#) has named "[the unreasonable effectiveness of mathematics](#)".<sup>[13]</sup> The [philosopher of mathematics Mark Steiner](#) has written extensively on this matter and acknowledges that the applicability of mathematics constitutes "a challenge to naturalism."<sup>[61]</sup> For the philosopher of mathematics [Mary Leng](#), the fact that the physical world acts in accordance with the dictates of non-causal mathematical entities existing beyond the universe is "a happy coincidence".<sup>[62]</sup> On the other hand, for some [anti-realists](#), connections, which are acquired



among mathematical things, just mirror the connections acquiring among objects in the universe, so that there is no "happy coincidence".<sup>[62]</sup>

As in most areas of study, the explosion of knowledge in the scientific age has led to specialization: there are now hundreds of specialized areas in mathematics and the latest [Mathematics Subject Classification](#) runs to 46 pages.<sup>[63]</sup> Several areas of applied mathematics have merged with related traditions outside of mathematics and become disciplines in their own right, including statistics, [operations research](#), and [computer science](#).

For those who are mathematically inclined, there is often a definite aesthetic aspect to much of mathematics. Many mathematicians talk about the *elegance* of mathematics, its intrinsic [aesthetics](#) and inner beauty. [Simplicity](#) and generality are valued. There is beauty in a simple and elegant [proof](#), such as [Euclid's](#) proof that there are infinitely many [prime numbers](#), and in an elegant [numerical method](#) that speeds calculation, such as the [fast Fourier transform](#). [G. H. Hardy](#) in [A Mathematician's Apology](#) expressed the belief that these aesthetic considerations are, in themselves, sufficient to justify the study of pure mathematics. He identified criteria such as significance, unexpectedness, inevitability, and economy as factors that contribute to a mathematical aesthetic.<sup>[64]</sup> Mathematical research often seeks critical features of a mathematical object. A theorem expressed as a [characterization](#) of the object by these features is the prize. Examples of particularly succinct and revelatory mathematical arguments have been published in [Proofs from THE BOOK](#).

The popularity of [recreational mathematics](#) is another sign of the pleasure many find in solving mathematical questions. And at the other social extreme, philosophers continue to find problems in [philosophy of mathematics](#), such as the nature of [mathematical proof](#).<sup>[65]</sup>

## Notation, language, and rigor

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[Leonhard Euler](#) created and popularized much of the

Most of the mathematical notation in use today was not invented until the 16th century.<sup>[66]</sup> Before that, mathematics was written out in words, limiting mathematical discovery.<sup>[67]</sup> Euler (1707–1783) was responsible for many of the notations in use today. Modern notation makes mathematics much easier for the professional, but beginners often find it daunting. According to Barbara Oakley, this can be attributed to the fact that mathematical ideas are both more *abstract* and more *encrypted* than those of natural language.<sup>[68]</sup> Unlike natural language, where people can often equate a word (such as *cow*) with the physical object it corresponds to, mathematical symbols are abstract, lacking any physical analog.<sup>[69]</sup> Mathematical symbols are also more highly encrypted than regular words, meaning a single symbol can encode a number of different operations or ideas.<sup>[70]</sup>

**Mathematical language** can be difficult to understand for beginners because even common terms, such as *or* and *only*, have a more precise meaning than they have in everyday speech, and other terms such as *open* and *field* refer to specific mathematical ideas, not covered by their laymen's meanings. Mathematical language also includes many technical terms such as *homeomorphism* and *integrable* that have no meaning outside of mathematics. Additionally, shorthand phrases such as *iff* for "if and only if" belong to **mathematical jargon**. There is a reason for special notation and technical vocabulary: mathematics requires more precision than everyday speech. Mathematicians refer to this precision of language and logic as "rigor".

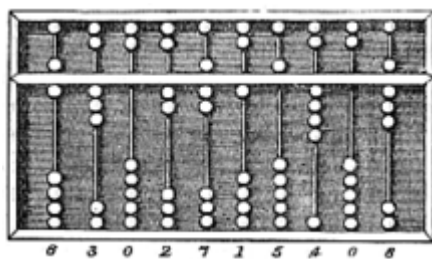
**Mathematical proof** is fundamentally a matter of **rigor**. Mathematicians want their theorems to follow from axioms by means of systematic reasoning. This is to avoid mistaken "**theorems**", based on fallible intuitions, of which many instances have occurred in the history of the subject.<sup>[b]</sup> The level of rigor expected in mathematics has varied over time: the Greeks expected detailed arguments, but at the time of Isaac Newton the methods employed were less rigorous. Problems inherent in the definitions used by Newton would lead to a resurgence of careful analysis and formal proof in the 19th century. Misunderstanding the rigor is a cause for some of the common misconceptions of mathematics. Today, mathematicians continue to argue among themselves about **computer-assisted proofs**. Since large computations are hard to verify, such proofs may be erroneous if the used computer program is erroneous.<sup>[c][71]</sup> On the other hand, **proof assistants** allow verifying all details that cannot be given in a hand-written proof, and provide certainty of the correctness of long proofs such as that of the **Feit–Thompson theorem**.<sup>[d]</sup>

**Axioms** in traditional thought were "self-evident truths", but that conception is problematic.<sup>[72]</sup> At a formal level, an axiom is just a string of symbols, which has an intrinsic meaning only in the context of all derivable formulas of an **axiomatic system**. It was the goal of **Hilbert's program** to put all of mathematics on a firm axiomatic basis, but according to **Gödel's incompleteness theorem** every (sufficiently powerful) axiomatic system has **undecidable** formulas; and so a final **axiomatization** of

mathematics is impossible. Nonetheless mathematics is often imagined to be (as far as its formal content) nothing but [set theory](#) in some axiomatization, in the sense that every mathematical statement or proof could be cast into formulas within set theory.<sup>[73]</sup>

## Fields of mathematics

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The [abacus](#) is a simple calculating tool used since ancient times.

Mathematics can, broadly speaking, be subdivided into the study of quantity, structure, space, and change (i.e. [arithmetic](#), [algebra](#), geometry, and [analysis](#)). In addition to these main concerns, there are also subdivisions dedicated to exploring links from the heart of mathematics to other fields: to [logic](#), to [set theory](#) ([foundations](#)), to the empirical mathematics of the various sciences ([applied mathematics](#)), and more recently to the rigorous study of [uncertainty](#). While some areas might seem unrelated, the [Langlands program](#) has found connections between areas previously thought unconnected, such as [Galois groups](#), [Riemann surfaces](#) and [number theory](#).

[Discrete mathematics](#) conventionally groups together the fields of mathematics which study mathematical structures that are fundamentally discrete rather than continuous.

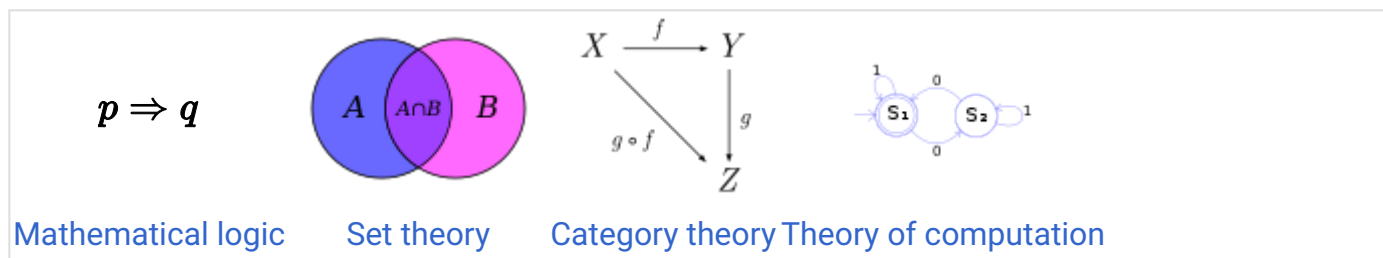
## Foundations and philosophy

In order to clarify the [foundations of mathematics](#), the fields of [mathematical logic](#) and [set theory](#) were developed. Mathematical logic includes the mathematical study of [logic](#) and the applications of formal logic to other areas of mathematics; set theory is the branch of mathematics that studies [sets](#) or collections of objects. The phrase "crisis of foundations" describes the search for a rigorous foundation for mathematics that took place from approximately 1900 to 1930.<sup>[74]</sup> Some disagreement about the foundations of mathematics continues to the present day. The crisis of foundations was stimulated by a number of controversies at the time, including the [controversy over Cantor's set theory](#) and the [Brouwer–Hilbert controversy](#).

Mathematical logic is concerned with setting mathematics within a rigorous [axiomatic](#) framework, and studying the implications of such a framework. As such, it is home to [Gödel's incompleteness theorems](#) which (informally) imply that any effective [formal system](#) that contains basic arithmetic, if

*sound* (meaning that all theorems that can be proved are true), is necessarily *incomplete* (meaning that there are true theorems which cannot be proved *in that system*). Whatever finite collection of number-theoretical axioms is taken as a foundation, Gödel showed how to construct a formal statement that is a true number-theoretical fact, but which does not follow from those axioms. Therefore, no formal system is a complete axiomatization of full number theory. Modern logic is divided into [recursion theory](#), [model theory](#), and [proof theory](#), and is closely linked to [theoretical computer science](#),<sup>[75]</sup> as well as to [category theory](#). In the context of recursion theory, the impossibility of a full axiomatization of number theory can also be formally demonstrated as a [consequence of the MRDP theorem](#).

[Theoretical computer science](#) includes [computability theory](#), [computational complexity theory](#), and [information theory](#). Computability theory examines the limitations of various theoretical models of the computer, including the most well-known model—the [Turing machine](#). Complexity theory is the study of tractability by computer; some problems, although theoretically solvable by computer, are so expensive in terms of time or space that solving them is likely to remain practically unfeasible, even with the rapid advancement of computer hardware. A famous problem is the "[P = NP?](#)" problem, one of the [Millennium Prize Problems](#).<sup>[76]</sup> Finally, information theory is concerned with the amount of data that can be stored on a given medium, and hence deals with concepts such as [compression](#) and [entropy](#).



## Pure mathematics

### Number systems and number theory

The study of quantity starts with numbers, first the familiar [natural numbers](#)  $\mathbb{N}$  and [integers](#)  $\mathbb{Z}$  ("whole numbers") and arithmetical operations on them, which are characterized in [arithmetic](#). The deeper properties of integers are studied in [number theory](#), from which come such popular results as [Fermat's Last Theorem](#). The [twin prime](#) conjecture and [Goldbach's conjecture](#) are two unsolved problems in number theory.

As the number system is further developed, the integers are recognized as a [subset](#) of the [rational numbers](#)  $\mathbb{Q}$  ("fractions"). These, in turn, are contained within the [real numbers](#),  $\mathbb{R}$  which are used to represent limits of sequences of rational numbers and [continuous](#) quantities. Real numbers are generalized to the [complex numbers](#)  $\mathbb{C}$ . According to the [fundamental theorem of algebra](#), all

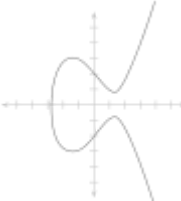

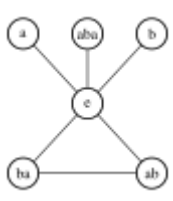
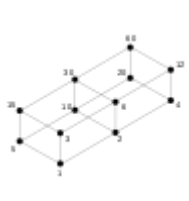

polynomial [equations](#) in one unknown with complex coefficients have a solution in the complex numbers, regardless of degree of the polynomial.  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  are the first steps of a hierarchy of numbers that goes on to include [quaternions](#) and [octonions](#). Consideration of the natural numbers also leads to the [transfinite numbers](#), which formalize the concept of "infinity". Another area of study is the size of sets, which is described with the [cardinal numbers](#). These include the [aleph numbers](#), which allow meaningful comparison of the size of infinitely large sets.

$(0), 1, 2, 3, \dots, -2, -1, 0, 1, 2 \dots -2, \frac{2}{3}, 1.21 -e, \sqrt{2}, 3, \pi 2, i, -2 + 3i, 2e^{i\frac{4\pi}{3}} \aleph_0, \aleph_1, \aleph_2, \dots, \aleph_\alpha, \dots$					
Natural numbers	Integers	Rational numbers	Real numbers	Complex numbers	Infinite cardinals

## Structure

Many mathematical objects, such as [sets](#) of numbers and [functions](#), exhibit internal structure as a consequence of [operations](#) or [relations](#) that are defined on the set. Mathematics then studies properties of those sets that can be expressed in terms of that structure; for instance [number theory](#) studies properties of the set of [integers](#) that can be expressed in terms of [arithmetic](#) operations. Moreover, it frequently happens that different such structured sets (or [structures](#)) exhibit similar properties, which makes it possible, by a further step of [abstraction](#), to state [axioms](#) for a class of structures, and then study at once the whole class of structures satisfying these axioms. Thus one can study [groups](#), [rings](#), [fields](#) and other abstract systems; together such studies (for structures defined by algebraic operations) constitute the domain of [abstract algebra](#).

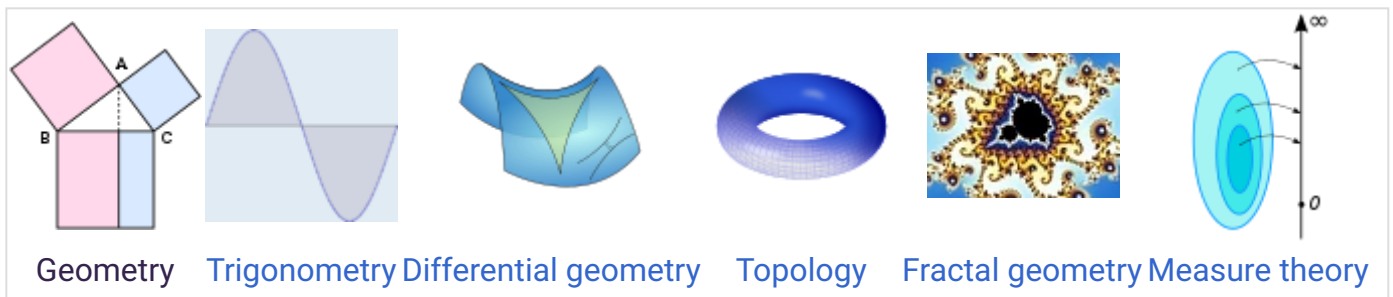
By its great generality, abstract algebra can often be applied to seemingly unrelated problems; for instance a number of ancient problems concerning [compass and straightedge constructions](#) were finally solved using [Galois theory](#), which involves field theory and group theory. Another example of an algebraic theory is [linear algebra](#), which is the general study of [vector spaces](#), whose elements called [vectors](#) have both quantity and direction, and can be used to model (relations between) points in space. This is one example of the phenomenon that the originally unrelated areas of [geometry](#) and [algebra](#) have very strong interactions in modern mathematics. [Combinatorics](#) studies ways of enumerating the number of objects that fit a given structure.

$(1, 2, 3) \quad (1, 3, 2)$ $(2, 1, 3) \quad (2, 3, 1)$ $(3, 1, 2) \quad (3, 2, 1)$					
Combinatorics	Number theory	Group theory	Graph theory	Order theory	Algebra

## Space



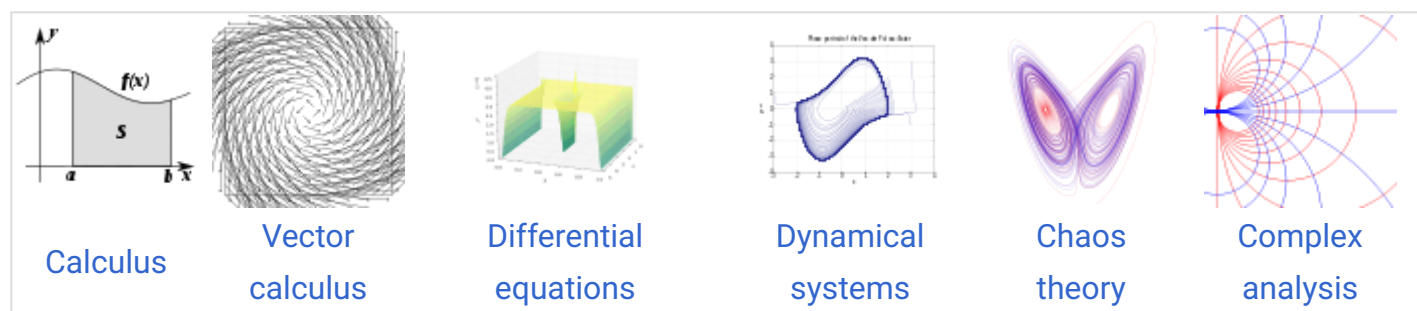
The study of space originates with geometry—in particular, [Euclidean geometry](#), which combines space and numbers, and encompasses the well-known [Pythagorean theorem](#). [Trigonometry](#) is the branch of mathematics that deals with relationships between the sides and the angles of triangles and with the trigonometric functions. The modern study of space generalizes these ideas to include higher-dimensional geometry, [non-Euclidean geometries](#) (which play a central role in [general relativity](#)) and [topology](#). Quantity and space both play a role in [analytic geometry](#), [differential geometry](#), and [algebraic geometry](#). [Convex](#) and [discrete geometry](#) were developed to solve problems in [number theory](#) and [functional analysis](#) but now are pursued with an eye on applications in [optimization](#) and [computer science](#). Within differential geometry are the concepts of [fiber bundles](#) and calculus on [manifolds](#), in particular, [vector](#) and [tensor calculus](#). Within algebraic geometry is the description of geometric objects as solution sets of [polynomial](#) equations, combining the concepts of quantity and space, and also the study of [topological groups](#), which combine structure and space. [Lie groups](#) are used to study space, structure, and change. [Topology](#) in all its many ramifications may have been the greatest growth area in 20th-century mathematics; it includes [point-set topology](#), [set-theoretic topology](#), [algebraic topology](#) and [differential topology](#). In particular, instances of modern-day topology are [metrizability theory](#), [axiomatic set theory](#), [homotopy theory](#), and [Morse theory](#). Topology also includes the now solved [Poincaré conjecture](#), and the still unsolved areas of the [Hodge conjecture](#). Other results in geometry and topology, including the [four color theorem](#) and [Kepler conjecture](#), have been proven only with the help of computers.



## Change

Understanding and describing change is a common theme in the [natural sciences](#), and [calculus](#) was developed as a tool to investigate it. [Functions](#) arise here as a central concept describing a changing quantity. The rigorous study of [real numbers](#) and functions of a real variable is known as [real analysis](#), with [complex analysis](#) the equivalent field for the [complex numbers](#). [Functional analysis](#) focuses attention on (typically infinite-dimensional) [spaces](#) of functions. One of many applications of functional analysis is [quantum mechanics](#). Many problems lead naturally to relationships between a quantity and its rate of change, and these are studied as [differential equations](#). Many phenomena in nature can be described by [dynamical systems](#); [chaos theory](#)

makes precise the ways in which many of these systems exhibit unpredictable yet still **deterministic** behavior.



## Applied mathematics

**Applied mathematics** concerns itself with mathematical methods that are typically used in science, **engineering**, **business**, and **industry**. Thus, "applied mathematics" is a **mathematical science** with specialized **knowledge**. The term *applied mathematics* also describes the professional specialty in which mathematicians work on practical problems; as a profession focused on practical problems, *applied mathematics* focuses on the "formulation, study, and use of mathematical models" in science, engineering, and other areas of mathematical practice.

In the past, practical applications have motivated the development of mathematical theories, which then became the subject of study in pure mathematics, where mathematics is developed primarily for its own sake. Thus, the activity of applied mathematics is vitally connected with research in **pure mathematics**.

## Statistics and other decision sciences

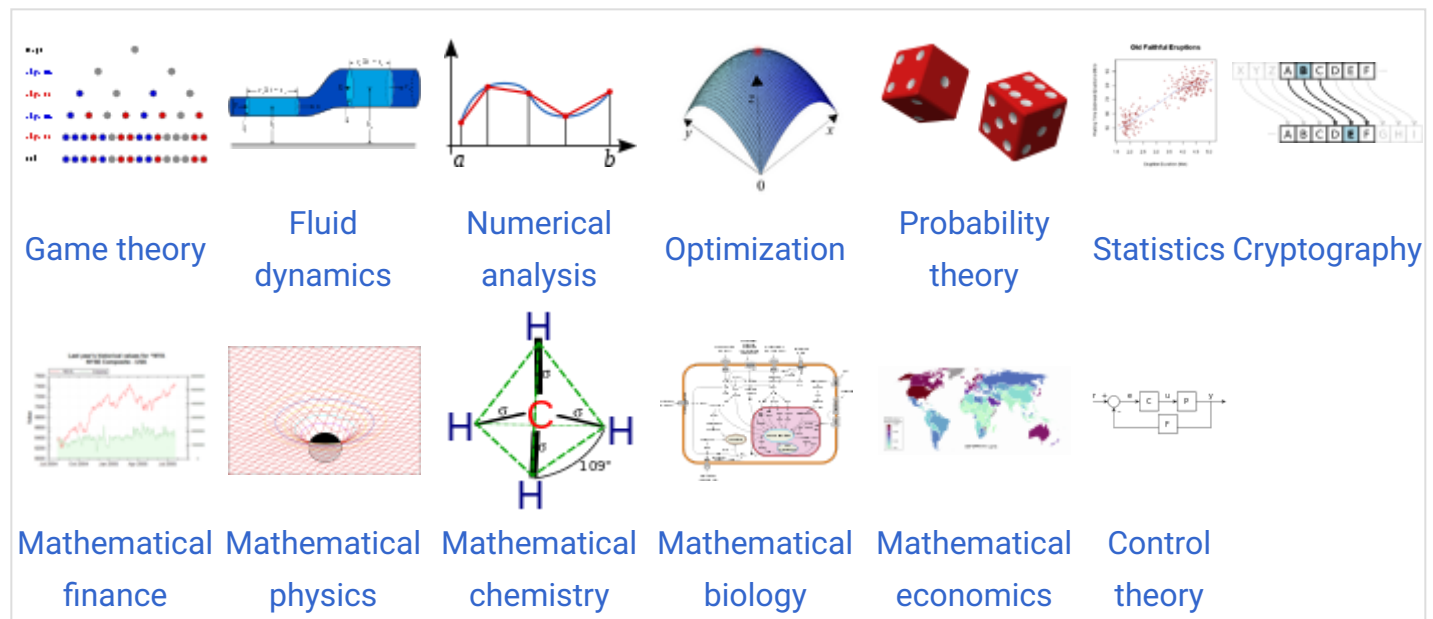
Applied mathematics has significant overlap with the discipline of statistics, whose theory is formulated mathematically, especially with **probability theory**. Statisticians (working as part of a research project) "create data that makes sense" with **random sampling** and with randomized **experiments**;<sup>[77]</sup> the design of a statistical sample or experiment specifies the analysis of the data (before the data becomes available). When reconsidering data from experiments and samples or when analyzing data from **observational studies**, statisticians "make sense of the data" using the art of **modelling** and the theory of **inference**—with **model selection** and **estimation**; the estimated models and consequential **predictions** should be **tested** on **new data**.<sup>[e]</sup>

**Statistical theory** studies **decision problems** such as minimizing the **risk** (**expected loss**) of a statistical action, such as using a **procedure** in, for example, **parameter estimation**, **hypothesis testing**, and **selecting the best**. In these traditional areas of **mathematical statistics**, a statistical-decision problem is formulated by minimizing an **objective function**, like expected loss or **cost**, under specific constraints: For example, designing a survey often involves minimizing the cost of

estimating a population mean with a given level of confidence.<sup>[78]</sup> Because of its use of [optimization](#), the mathematical theory of statistics shares concerns with other [decision sciences](#), such as [operations research](#), [control theory](#), and [mathematical economics](#).<sup>[79]</sup>

## Computational mathematics

[Computational mathematics](#) proposes and studies methods for solving [mathematical problems](#) that are typically too large for human numerical capacity. [Numerical analysis](#) studies methods for problems in [analysis](#) using [functional analysis](#) and [approximation theory](#); numerical analysis includes the study of [approximation](#) and [discretisation](#) broadly with special concern for [rounding errors](#). Numerical analysis and, more broadly, scientific computing also study non-analytic topics of mathematical science, especially [algorithmic matrix](#) and [graph theory](#). Other areas of computational mathematics include [computer algebra](#) and [symbolic computation](#).



## Mathematical awards

Arguably the most prestigious award in mathematics is the [Fields Medal](#),<sup>[80][81]</sup> established in 1936 and awarded every four years (except around World War II) to as many as four individuals. The Fields Medal is often considered a mathematical equivalent to the Nobel Prize.

The [Wolf Prize in Mathematics](#), instituted in 1978, recognizes lifetime achievement, and another major international award, the [Abel Prize](#), was instituted in 2003. The [Chern Medal](#) was introduced in 2010 to recognize lifetime achievement. These accolades are awarded in recognition of a particular body of work, which may be innovational, or provide a solution to an outstanding problem in an established field.

A famous list of 23 [open problems](#), called "[Hilbert's problems](#)", was compiled in 1900 by German mathematician [David Hilbert](#). This list achieved great celebrity among mathematicians, and at least nine of the problems have now been solved. A new list of seven important problems, titled the "[Millennium Prize Problems](#)", was published in 2000. Only one of them, the [Riemann hypothesis](#), duplicates one of Hilbert's problems. A solution to any of these problems carries a 1 million dollar reward. Currently, only one of these problems, the [Poincaré conjecture](#), has been solved.

## See also

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- [International Mathematical Olympiad](#)
- [List of mathematical jargon](#)
- [Lists of mathematics topics](#)
- [Mathematical sciences](#)
- [Mathematics and art](#)
- [Mathematics education](#)
- [National Museum of Mathematics](#)
- [Philosophy of mathematics](#)
- [Relationship between mathematics and physics](#)
- [Science, technology, engineering, and mathematics](#)

## Notes

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- a. No likeness or description of Euclid's physical appearance made during his lifetime survived antiquity. Therefore, Euclid's depiction in works of art depends on the artist's imagination (see [Euclid](#)).
- b. See [false proof](#) for simple examples of what can go wrong in a formal proof.
- c. For considering as reliable a large computation occurring in a proof, one generally requires two computations using independent software
- d. The book containing the complete proof has more than 1,000 pages.
- e. Like other mathematical sciences such as [physics](#) and [computer science](#), statistics is an autonomous discipline rather than a branch of applied mathematics. Like research physicists and computer scientists, research statisticians are mathematical scientists. Many statisticians have a degree in mathematics, and some statisticians are also mathematicians.

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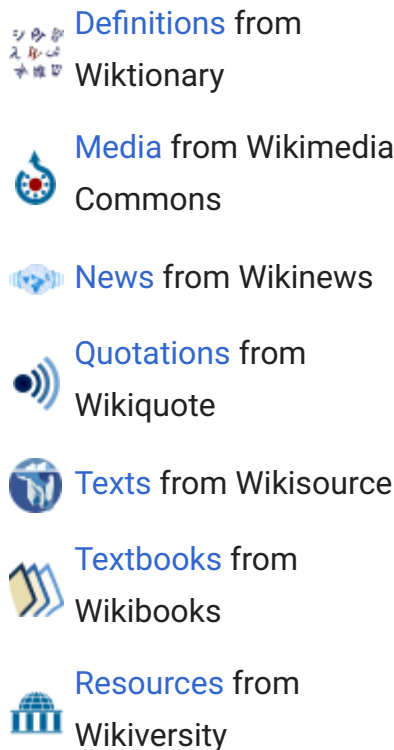
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