The Ribik's Cube Explained

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1 Exposition

In this video we explain one of the world's most popular puzzles - the Rubik's cube. While there are many tutorials out there that can teach you how to solve the puzzle, they all require memorizing about half a dozen of sequences of moves, without explaining why they work or how could anyone come up with them in the first place. In this video we focus instead on conceptually understanding the Rubik's cube, by analysing it from a mathematical perspective. So, if you pay attention, by the end of this video, not only that you'll have a practical method for solving the puzzle in just a few minutes, you will also understand why and how this solution works and even be able to explore further variations, tricks and shortcuts by yourself. Along the way, you will learn some basic concepts from a mathematical field known as "group theory" and the way those concepts can be applied to solve intricate algorithmic problems.

2 The puzzle

A standard Rubik's cube is a cube, with faces colored by: Red, Blue, Yellow, Green, Orange and White. The cube is cut in three in each dimension allowing each of the three layers to rotate independently. Applying several of these layer rotation moves in a row scrambles the colors. To solve the puzzle, one has to find a sequence of moves that gets you back to a state where each face consists of 9 little squares of the same color. If you play around with it long enough, high chances you'll figure out how to solve "the first layer". This means that you have the top face in a uniform color *and* the top row on each of the four sides matches the color of the center square. To solve "the second layer", that is to get also the middle row of each of the four sides to be of the same color as the top row, is more difficult. The problem is that one needs to somehow move the pieces around but without ruining the already solved first layer. To solve "the third layer", and hence the whole puzzle, is even more difficult as it gets increasingly harder to make any changes that would not ruin the already solved part of the cube. To do that, most solvers apply some "algorithms", which are given sequences of moves that generate a specific desired effect, such as, swapping two corners and two edges, or, say, rotating two adjacent corners in opposite directions. Memorizing half a dozen of these algorithms will allow you to solve any configuration. By memorizing a few dozens of them, you might even be able, with enough practice, to solve any cube in under 30 seconds. That being said, these so-called algorithms are usually presented as black boxes with no explanation for why they work, let alone for how you could find them by yourself. This, and more, is what we are set to explain in this video.

3 Preliminary discussion

When tackling a mathematical problem, it is often useful to try and break it into smaller problems. That is, to formulate some intermediate goals that are potentially easier to achieve and together provide a solution to the whole problem. The layer-by-layer approach is an example of such a strategy for solving the Rubik's cube. But, while being a very natural thing to do, it concentrates most of the difficulty in the final steps without any indication of how to overcome it. Instead, we shall try to devise a divide-and-conquer approach that takes advantage of the mathematical structure and symmetry of the cube. (We shall come back to layer-by-layer methods in the end of the video, once we have gained enough insights.)

To systematically analyze the problem at hand, we first observe that the cube consists of 26 pieces that fall into 3 categories: The 6 center pieces with only one colored square each, the 12 edge pieces with 2 colored square each, and the 8 corner pieces with 3 colored squares each. Each piece has both a spatial position, and an "orientation". That is, edges can be flipped, and corners can be rotated. Each move rearranges the pieces in each category among themselves and may also effect their orientation. In fact, one can somewhat simplify the problem with the following observation: The only moves that affect the center pieces are the rotations of the middle layers. But, a cw quarter-turn of the middle layer has the same effect as a simultaneous ccw quarter-turn of the two outer layers, up to a rotation of the entire cube. Since rotating the entire cube is harmless, we can do without the middle layer rotations. This fixes the center pieces, so we have to worry only about the edges and corners. Of course, allowing middle layer rotations might lead to a more efficient solution, but discarding them makes the problem conceptually simpler.

4 Strategy

This way of thinking about the problem leads to two divide-and-conquer strategies. One, is that we might try to solve the edges first, ignoring the corners, and only then try to solve the corners, without messing up the edges (or vice-versa). Two, that we might try to first put the pieces into their correct spatial position, ignoring their orientation (i.e., if they are flipped or rotated), and then try to flip/rotate them into the correct orientation without effecting their position. These two strategies can also be mixed. For example, we can try to

(1) Put the edges into their correct positions, ignoring their orientations and the effect on corners.

(2) Put the corners into their correct positions without moving the edges, ignoring orientations.

(3) Flipping the edges into their correct orientation, without moving the corners.

(4) Rotating the corners into their correct orientation, without effecting the edges.

Of course, to carry out this strategy we need to come up with sequences of moves that have the restricted effect needed to execute steps (1)-(4).

5 Permutations

We begin by concentrating on steps (1) and (2) of the above strategy, which are concerned only with the position of the pieces (edges and corners), ignoring the orientations. Each move rearranges the edges and corners in some way, and we want to figure out how to generate a sequences of these rearrangements that will put all the pieces into their correct places. Such rearrangements are called in mathematics "permutations". So, each move permutes, say, the corners according to a certain rule. For example, if we number the corners pieces from 1 to 8, the corner-permutation of the move $R =$ "ccw quarter-turn of the Red face", can be described as follows [some permutation]. Similarly, the corner-permutation of the move $B =$ "cw quarter-turn of the Blue face", can be described as follows [some permutation]. Permutations can be *composed*. So the operation of doing first R and then B, which we write as RB, has a corner-permutation which is the composition [some permutation]. Similarly, each move permutes the 12 edges, which again compose when several moves are executed in a sequence. To get the edges and corners to the correct position, we need to understand the world of permutations a bit better.

6 Groups

The collection of permutations on an n element set, which is usually taken to be simply the set of numbers 1 to n, with the composition operation is a prominent example of an abstract mathematical structure known as a "group". The characteristic features of this structure are

(1) We have the "do nothing" permutation I, which is *neutral* for composition. Meaning, composing any permutation A with I, in either order, is again A. That is

$$A \circ I = I \circ A = I.$$

(2) Every permutation A has an *inverse permutation* A^{-1}, such that composing A with A^{-1}, in either order, does nothing. That is

$$A \circ A^{-1} = A^{-1} \circ A = I.$$

(3) Composition is *associative*. Meaning, for every three permutations A, B and C, the two ways to compute the composition of A with B with C are the same

$$A \circ (B \circ C) = (A \circ B) \circ C.$$

A group is any set with a specified binary operation, which satisfies the analogs of the above properties (called "group axioms"). For example, we also have the group of integers

$$\ldots, -2, -1, 0, 1, , 2, \ldots$$

with the addition operation. The number 0 is neutral to addition. Each number a has an inverse under addition $-a$ and it is a well known fact that addition is associative

$$(a + b) + c = a + (b + c).$$

"Group theory" is the subject that studies both the abstract properties of groups and the specific examples of groups and the way they interact. It is a corner stone of modern algebra with many applications to other branches of mathematics, computer science, physics and more. In what follows, we will touch upon some of the basic ideas of this theory and the way they are used in understanding and solving the Rubik's cube puzzle.

7 Commutators

A crucial difference between the two examples of groups we had, is that addition of integers is *commutative*

$$a + b = b + a,$$

while composition of permutations is usually not

$$A \circ B \neq B \circ A.$$

Namely, the order in which we sum two integers does not matter, but the order in which we preform two permutations one after the other (usually) does matter.

This non-commutativity phenomenon is what makes permutations complicated. But, if we are clever, we can also use it to our advantage. Metaphorically speaking, we can use the opponent's force against them.

Consider,

$$C = RBR^{-1}B^{-1}$$