Statistics, models, & hypothesis testing

Introduction to data analysis: Lecture 6

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Source: xkcd

Inference

- **Statistical Inference:** Making conclusions based on data in random samples
- **Example:**

fixed

Use the data to guess the value of an unknown number

depends on the random sample

Create an **estimate** of the unknown quantity

Terminology

Parameter

A number associated with the population

- Unknowable (unless we have data for the whole population)

Statistic

A number which is a function of (only) the sample

- Can be exactly calculated from the sample

A statistic can be used as an **estimate** of a parameter

(notebook)

Probability distribution of a Statistic

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- "Sampling distribution" or "probability distribution" of the statistic:
  - All possible values of the statistic,
  - and all the corresponding probabilities
- Can be hard to calculate
  - Either have to do the math
  - Or have to generate all possible samples and calculate the statistic based on each sample

Empirical distribution of a Statistic

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- Empirical distribution of the statistic:
  - **Based on simulated values** of the statistic
  - Consists of all the observed values of the statistic,
  - and the proportion of times each value appeared
- Good approximation to the probability distribution of the statistic
  - if the number of repetitions in the simulation is large
  - According to?

Simulating a Statistic

- Figure out the code to generate *one* value of the statistic
  - Based on a fixed sample size
- Create an empty array in which you will collect all the simulated values
- For each repetition of the process:
  - Simulate one value of the statistic
  - Append this value to the collection array
- At the end of all the repetitions, the collection array will contain all the simulated values

(notebook)

Models and Hypotheses

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  - To simulate the coin, we need to assume we know the (unknowable!) parameter

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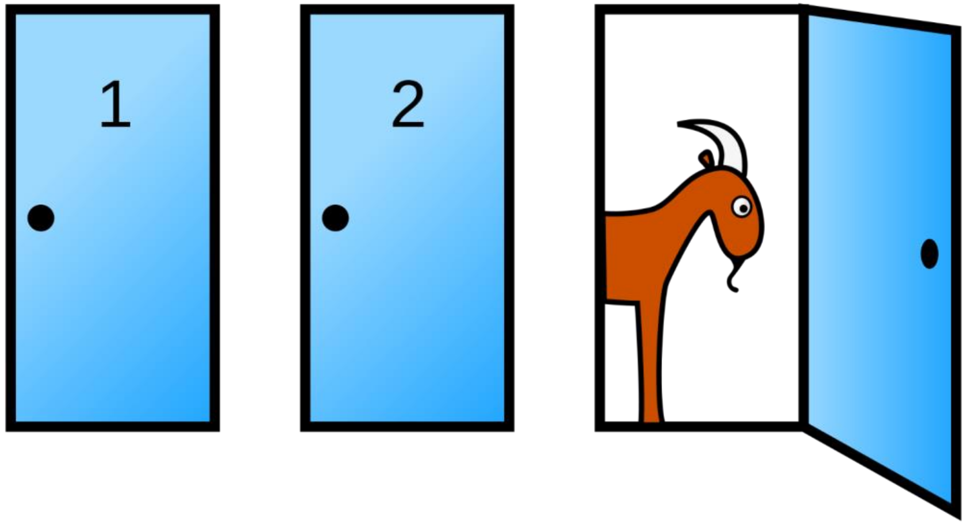
A model is a set of assumptions about the world

- Often, models include assumptions about the chance processes used to **generate data**

Lying Monty?

- Parameter: chance for win given switch
- Statistic: #wins/#shows
  - (Given switch)
- To simulate the statistic, we need a model



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  - E.g., the two groups are equal

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- We have access to data that gives us **one** estimate of this parameter
- We know the parameter has some theoretical value
    - E.g., The coin is unbiased
    - E.g., Monty never lies
    - E.g., the two groups are equal
- We want to ask whether the data - an estimate from **one** random sample - is **consistent or inconsistent** with the theoretical value of the parameter

Approach to Assessment

- If we can simulate data according to the assumptions of the model, we can learn what the model predicts as likely values of the statistic.
- Then, we compare the predictions of the model to the observed data
- If the observed data and the model's predictions are not consistent, it is evidence **against the model**.

Approach to Assessment

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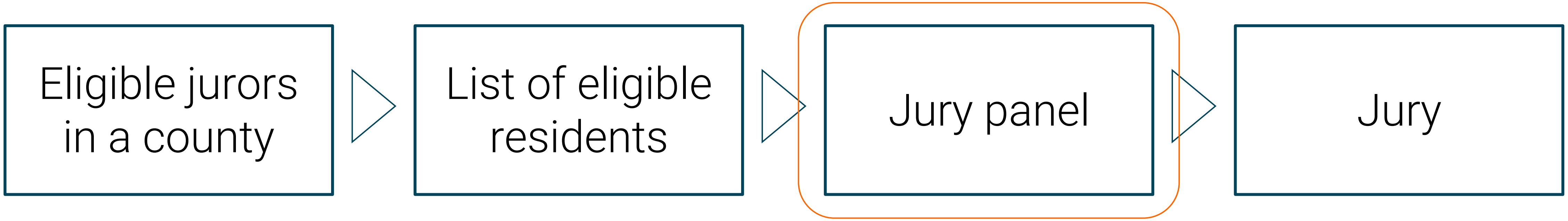
- If the observed data and the model's predictions are not consistent, it is evidence **against the model**.
  - We say we can **reject the** model (with some probability)
    - E.g., the coin is not unbiased
    - E.g., Monty may sometime lie
    - E.g., the two groups are not equal

Example inference problem: Jury Panels



Section 197 of California's Code of Civil Procedure:
All persons selected for jury service shall be selected **at random**, from a source or sources inclusive of a representative cross section of the population of the area served by the court.

Sixth Amendment to the US Constitution:
… the accused shall enjoy the right to a speedy and public trial, by an impartial jury of the state and district wherein the crime shall have been committed.

Example inference problem

- Talladega County, Alabama, 1965
- Robert Swain, black man convicted of crime
- Appeal: one factor was all-white jury
- Population as required by law: all men 21 years or older
- 26% of this population were black
- Swain's jury panel consisted of 100 men
- 8 men on the panel were black

https://en.wikipedia.org/wiki/Swain_v._Alabama

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- Swain's jury panel consisted of 100 men
- 8 men on the panel were black
    - Is this likely when the jury panel is selected at random from the population?

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Example inference problem

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  "... the overall percentage disparity has been small and reflects no studied attempt to include or exclude a specified number of Negroes.“

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- That is, the court claims:
  - The panel was selected at random
  - Difference between 8% and 26% is due to chance
- Model: “Selection is made at random”
  - We can simulate data according to the assumptions of the model
  - We’ll get an empirical distribution of how many black jurors we can expect
    - Under the assumptions of the model!
  - We’ll check if the observed data (8% black jurors) is **consistent** with random selection (from a population of which 26% is black)

(notebook)

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Assessing models using simulations

We want to know whether our model is correct, but we only have a sample!

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The truth
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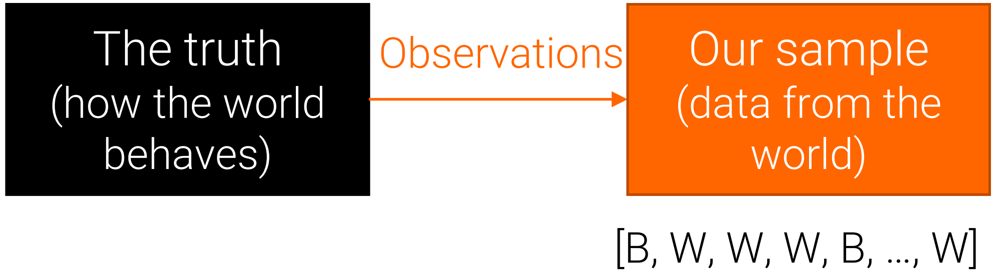

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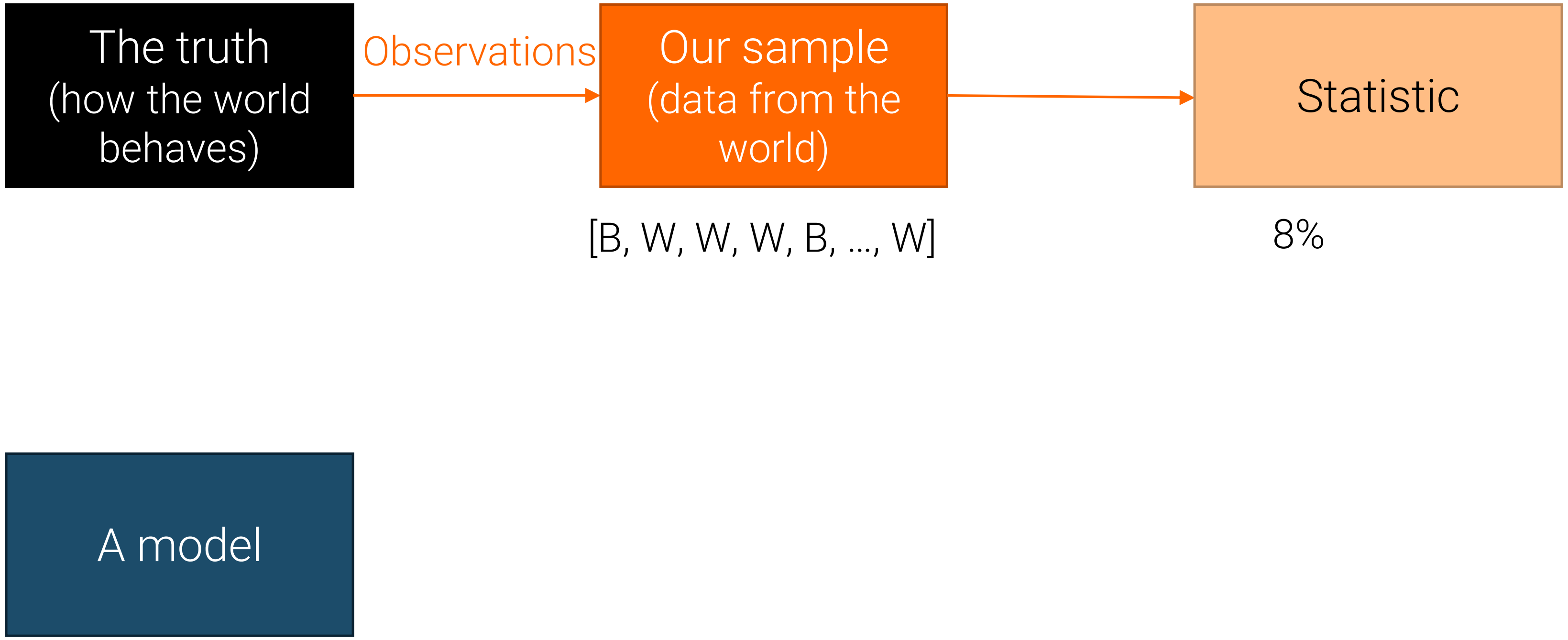

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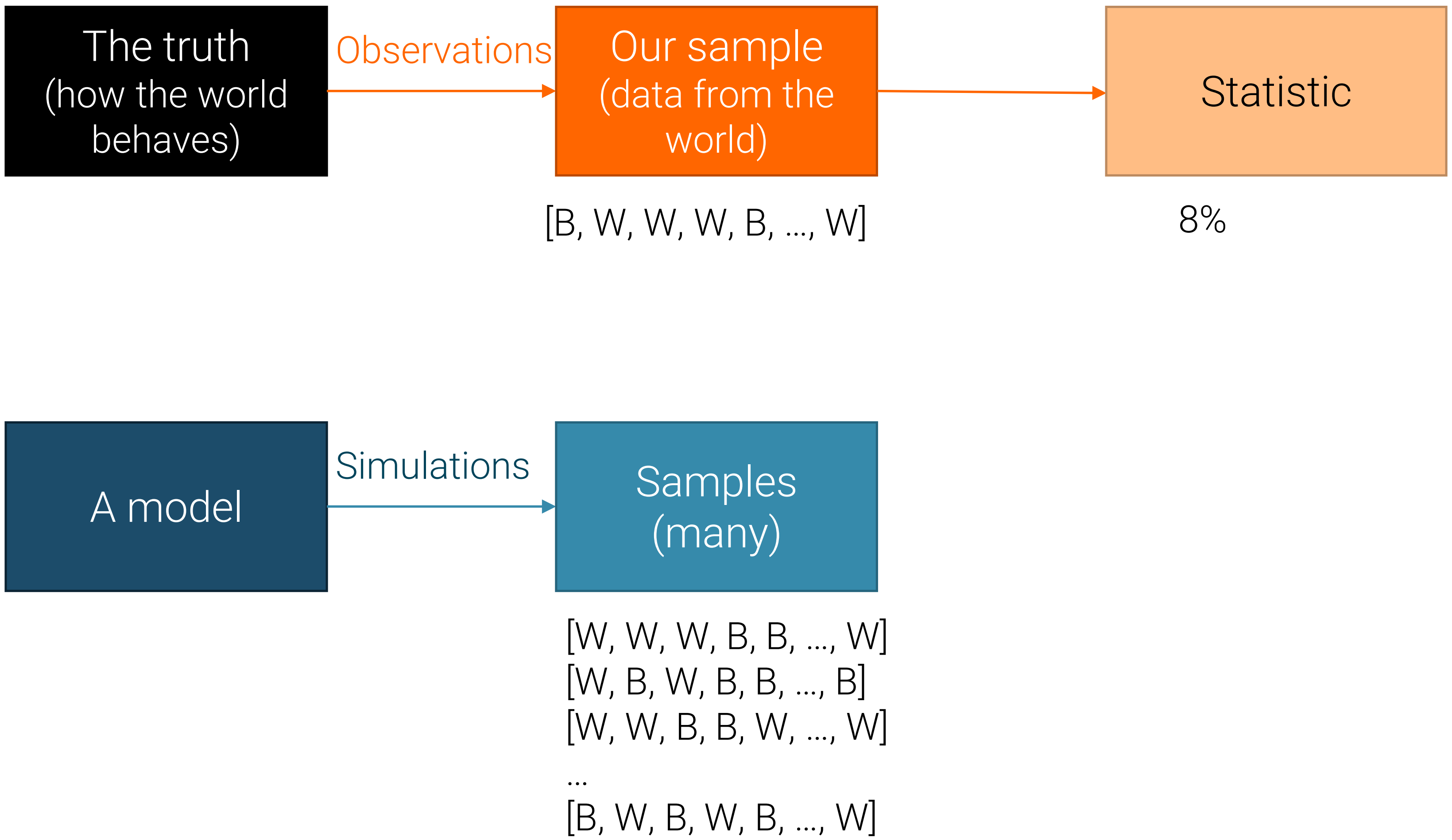

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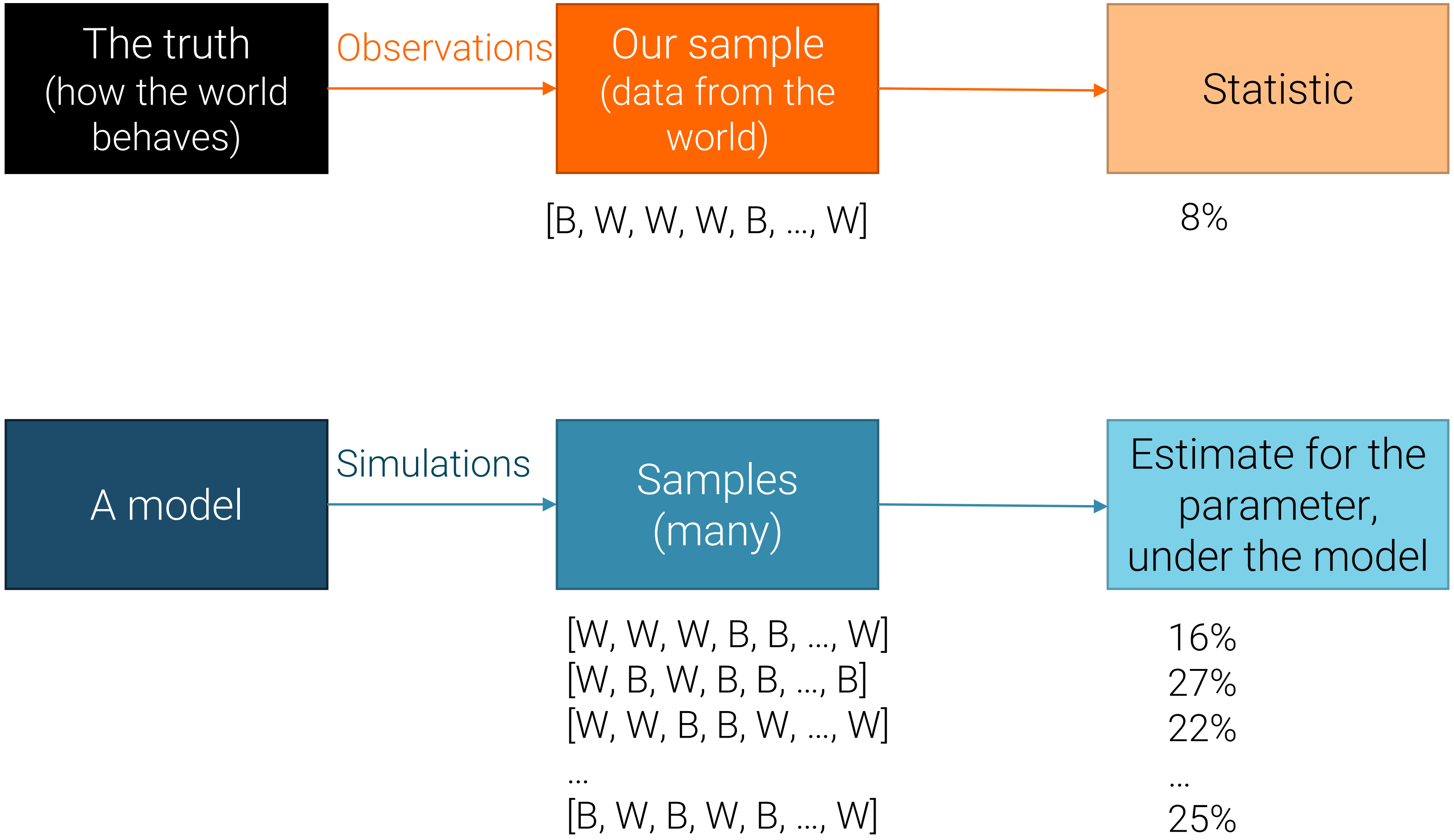

Assessing models using simulations

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The truth (how the world behaves) → Observations → Our sample (data from the world) → Statistic

[B, W, W, W, B, ..., W]

8%

A model → Simulations → Samples (many) → Estimate for the parameter, under the model

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[W, B, W, B, B, ..., B]
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...
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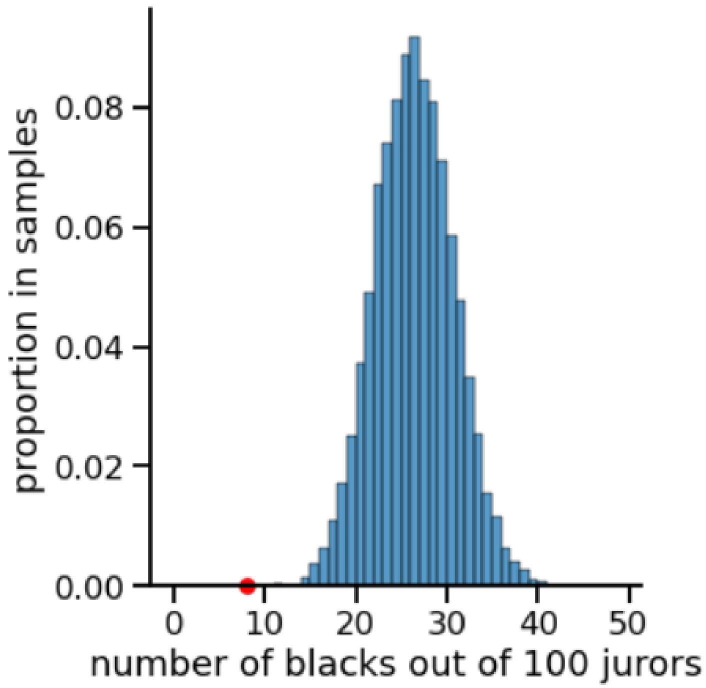
16%
27%
22%
...
25%



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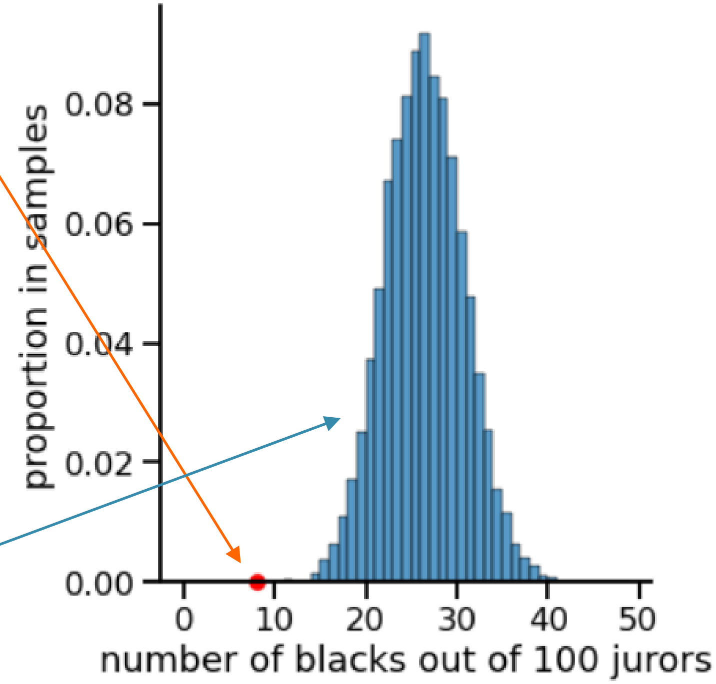

Incomplete information

- Two views of the world
  - “The model reflects the truth”
  - “The model does not reflect the truth”
- We want to choose between them based on data in a sample.
  - But random samples can turn out to be extreme (perhaps unlikely, but possible)
    - So both views *can* be consistent with the data
- We need a formal test to decide between the two views, using the incomplete information we have

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Step 1: The Hypotheses

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  - And any variation from the model which is observed in the data is due to chance

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    - And also, the coin is biased with P(H) = 0.75
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  - We need to be able to simulate data under the assumptions of the null hypothesis
- Alternative hypothesis – H_1
  - There is an effect other than chance: The model is incorrect
  - Variation from the model observed in the data is not only because of chance

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  - We need to be able to simulate data under the assumptions of the null hypothesis
- Alternative hypothesis – H_1
  - There is an effect other than chance: The model is incorrect
  - Variation from the model observed in the data is not only because of chance
  - We may not have a clue on how the data *is* generated

Testing Hypotheses

Step 2: The Test Statistic

- A value that can be computed for the data and for samples
  - And that we can therefore simulate/compute analytically
- Questions to ask:
  - What values of the test statistic will make us favor the null hypothesis?
  - What values of the test statistic will make us favor the alternative?

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Step 3: The Sampling distribution of the Test Statistic

- What the test statistic might be **if** the null hypothesis were true
- Approximate the sampling distribution by an empirical distribution

Conclusion of the Test

Resolve choice between null and alternative hypotheses

- Compare the observed value of the test statistic with its empirical distribution **under the null hypothesis**
- If the observed value is **not consistent** with the distribution, then the test favors the alternative – “rejects the null hypothesis”

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Consistency with the distribution:

- A visualization may be enough
- There are conventions about what is "consistent enough" (in a few slides)

Back to the jury example

- What are the two hypotheses?
- What is the value of the test statistic?
- Was the statistic consistent with the distribution?
- What will be the conclusion from the test?

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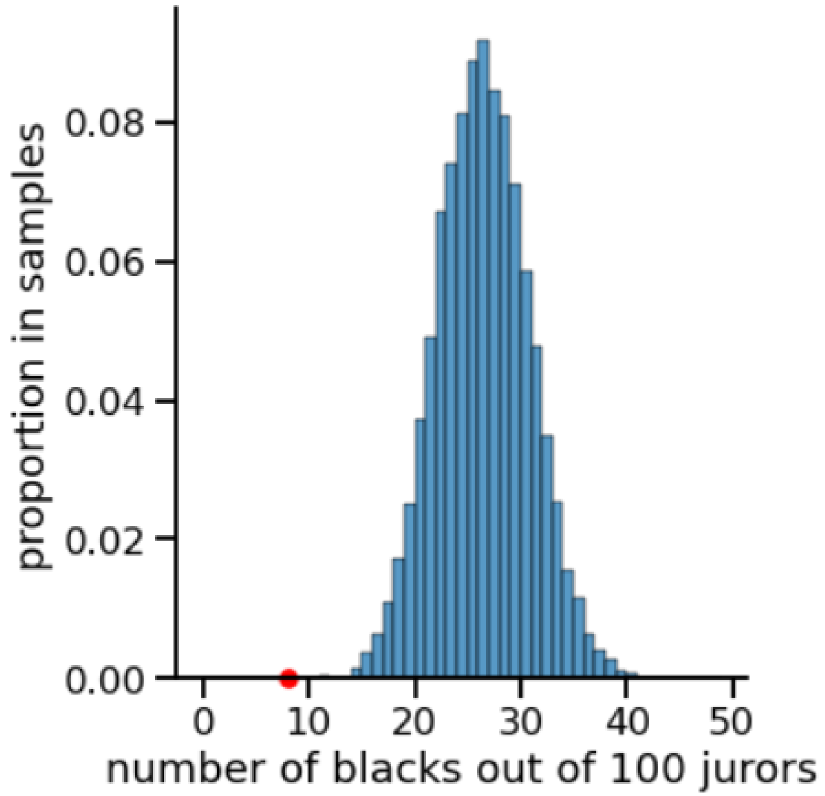

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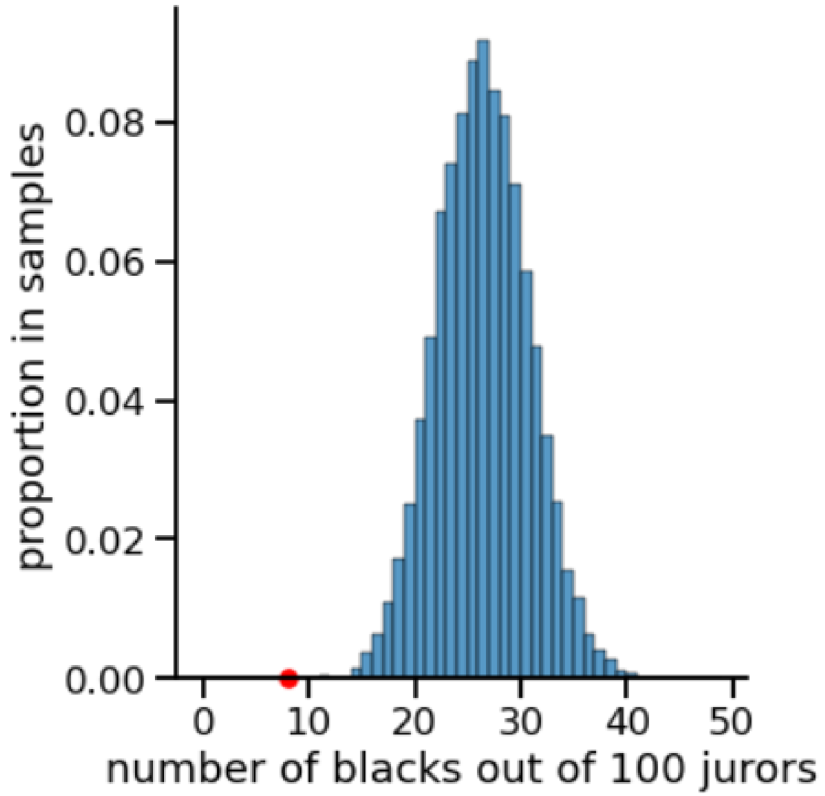

- But what if it is not this clear?

Conventions about inconsistency

- “Inconsistent”: The test statistic is in the tail of the empirical distribution under the null hypothesis
- “In the tail,” convention:
  - The area in the tail is less than 5%
  - The result is then “statistically significant”

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    - The area in the tail is less than 5%
    - The result is then “statistically significant”
        - (Note this does not mean it is necessarily important)

Definition of the *P*-value

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- under the null hypothesis,
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ההסתברות לקבל במדגם (עבור סטטיסטי המבחן) ערך השווה לזה שקיבלנו בפועל, או קיצוני יותר בכיוון ההשערה האלטרנטיבית, **תחת ההנחה שהשערת האפס נכונה.**

(notebook)

Can the conclusion be wrong?

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- Yes.

	Null is true	Null is false
Test rejects the null	✗	✓
Test doesn't reject the null	✓	✗

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Different worlds, and we do not know in which of these worlds we live

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Different samples may lead to different conclusions

	Null is true	Null is false
Test rejects the null	X	✓
Test doesn’t reject the null	✓	X

An error probability

- The cutoff we use for the *P*-value **is** an error probability.

- If
  - your cutoff is 5%
  - and you live in a world in which the null hypothesis is true

- then there is about a 5% chance that your test will (wrongly) reject the null hypothesis.

An error probability

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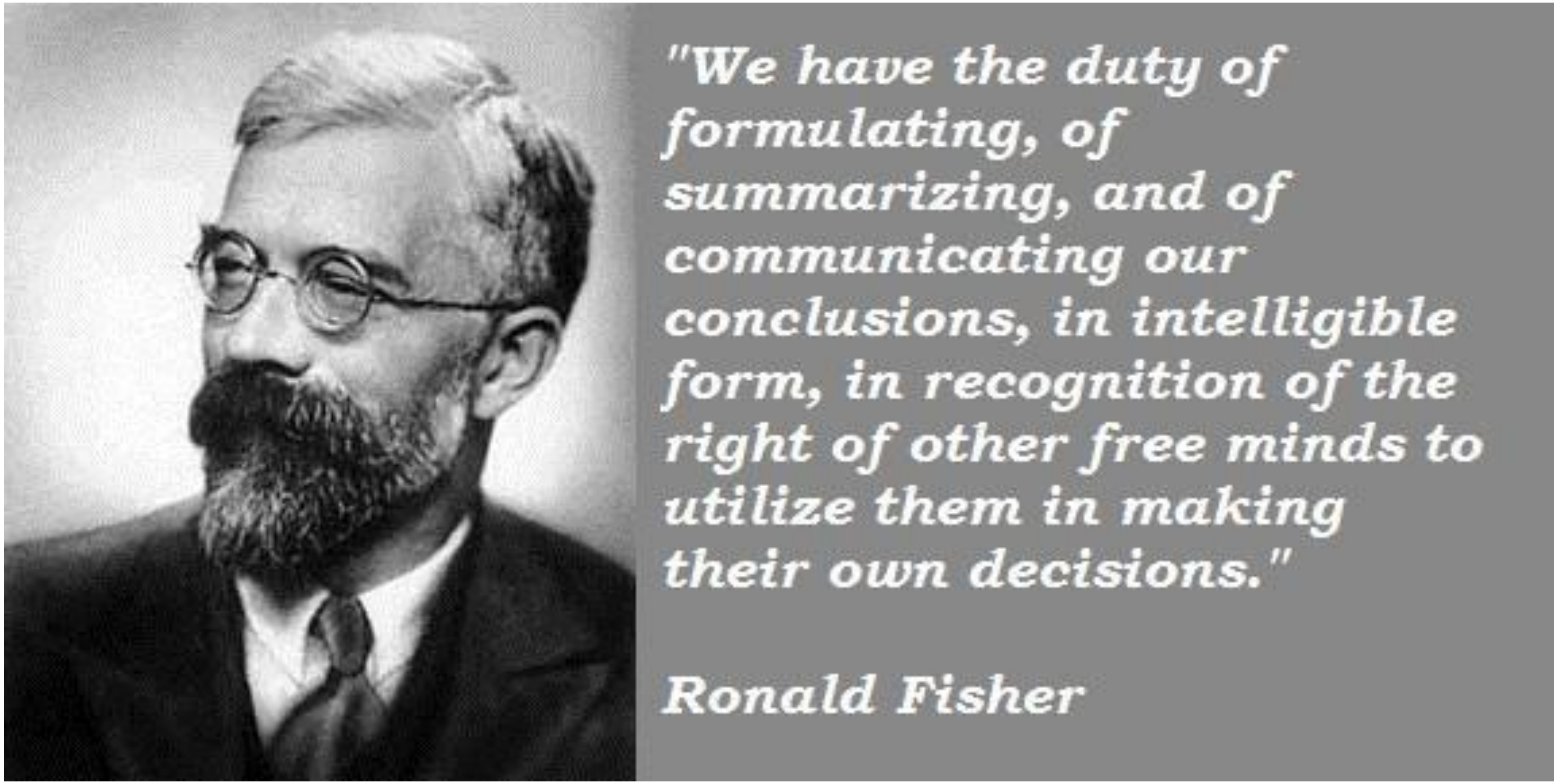
- If
  - your cutoff is 5%
  - and you live in a world in which the null hypothesis is true
- then there is about a 5% chance that your test will (wrongly) reject the null hypothesis.
- (there is also another type of possible error)

The origin of the 5% convention



Sir Ronald Fisher

“It is **convenient** to take this point [5%] as a limit in judging whether a deviation is to be considered significant or not.”

–– *Statistical Methods for Research Workers*

“If one in twenty does not **seem** high enough odds, **we may, if we prefer it**, draw the line at one in fifty (the 2 percent point), or one in a hundred (the 1 percent point). **Personally**, the author **prefers** to set a low standard of significance at the 5 percent point …”