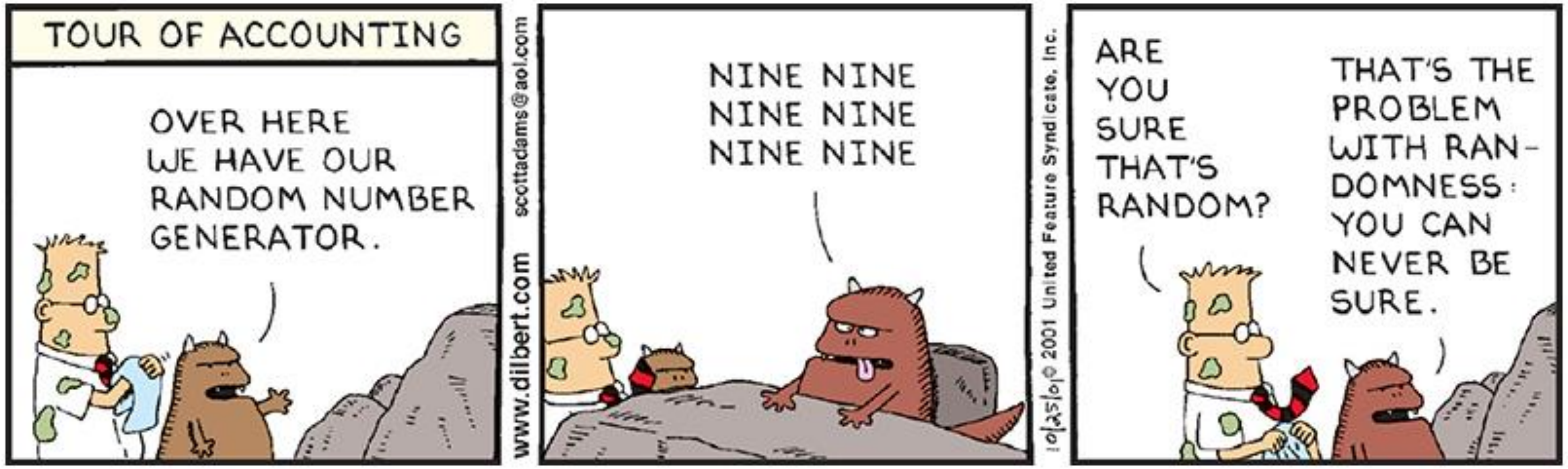
Probability, simulations, sampling, & distributions

Introduction to data analysis: Lecture 5

Ori Plonsky

Spring 2023



Slides partially based on Berkeley data8

Re: Data science: What?

Extracting useful insights from data using computation, statistics, and visualization

- Exploration
  - Play with the data, let it guide the questions
  - Identify patterns in information
  - Uses visualizations
- Inference
  - Start with a formal question
  - Quantify whether patterns are reliable
  - Uses randomization and statistical decision theory
- Prediction
  - Make informed guesses
  - Uses machine learning

Domain knowledge!

Goal

Reminder: Data typically includes a sample from a larger population

- Population includes all the elements (individuals) of interest
- Sample is a subset if observations from a population

Our goal (inference):

Answer questions that concern the population using a sample taken from it

Reminder

- In an experiment, we compare a treatment group with the control group
- Establishing causality: If the two groups are similar apart from the treatment, differences between outcomes can be ascribed to the treatment.
- If assignment of individuals to treatment and control is done at random, then the two groups are *likely* to be similar apart from the treatment

Reminder

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- Establishing causality: If the two groups are similar apart from the treatment, differences between outcomes can be ascribed to the treatment.
- If assignment of individuals to treatment and control is done at random, then the two groups are *likely* to be similar apart from the treatment
- We need a randomization device!
    - Jupyter notebook

Probability: Basic definitions

- Random trial – a trial (experiment) the exact outcome(s) of which cannot be known in advance
  - Rolling a die
- Sample space – a set of possible outcomes of an experiment
  - {1, 2, 3, 4, 5, 6}

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- Event – subset of the sample space
  - A = {5}: “a five” (elementary event)
  - B = {1, 3, 5}: “an uneven number”
  - C = {1, 2}: “a number under 3”
  - D = {1, 2, 3, 4, 5, 6}

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- Complementary event – the complementary set of an event
  - what are the complementary events A, B, C, D?

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  - Highest probability value
- If an event has probability p,
  the probability of its complementary event is $1-p$

Equally likely outcomes

- If each possible outcome is equally likely (all elementary events in the sample space have the same chance of happening), then the probability of event A is:

$$P\ \ A\ = \frac{Number\,of\,outcomes\,that\,make\,A\,occur}{Total\,number\,of\,possible\,outcomes}$$

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- What are the probabilities of A, B, C, D? Of their complementary events?

- A = {5}
- B = {1, 3, 5}
- C = {1, 2}
- D = {1, 2, 3, 4, 5, 6}

Conditional probability

- Think of the following game-of-chance:
  - There are three tickets in a hat: Red, Blue, and Green
  - Stage 1: You draw a ticket from the hat.
    - If it is Green, you move to Stage 2;
    - otherwise, you lose
  - Stage 2: without putting the Green ticket back in the hat, you draw another ticket
    - If it is Red, you win the game
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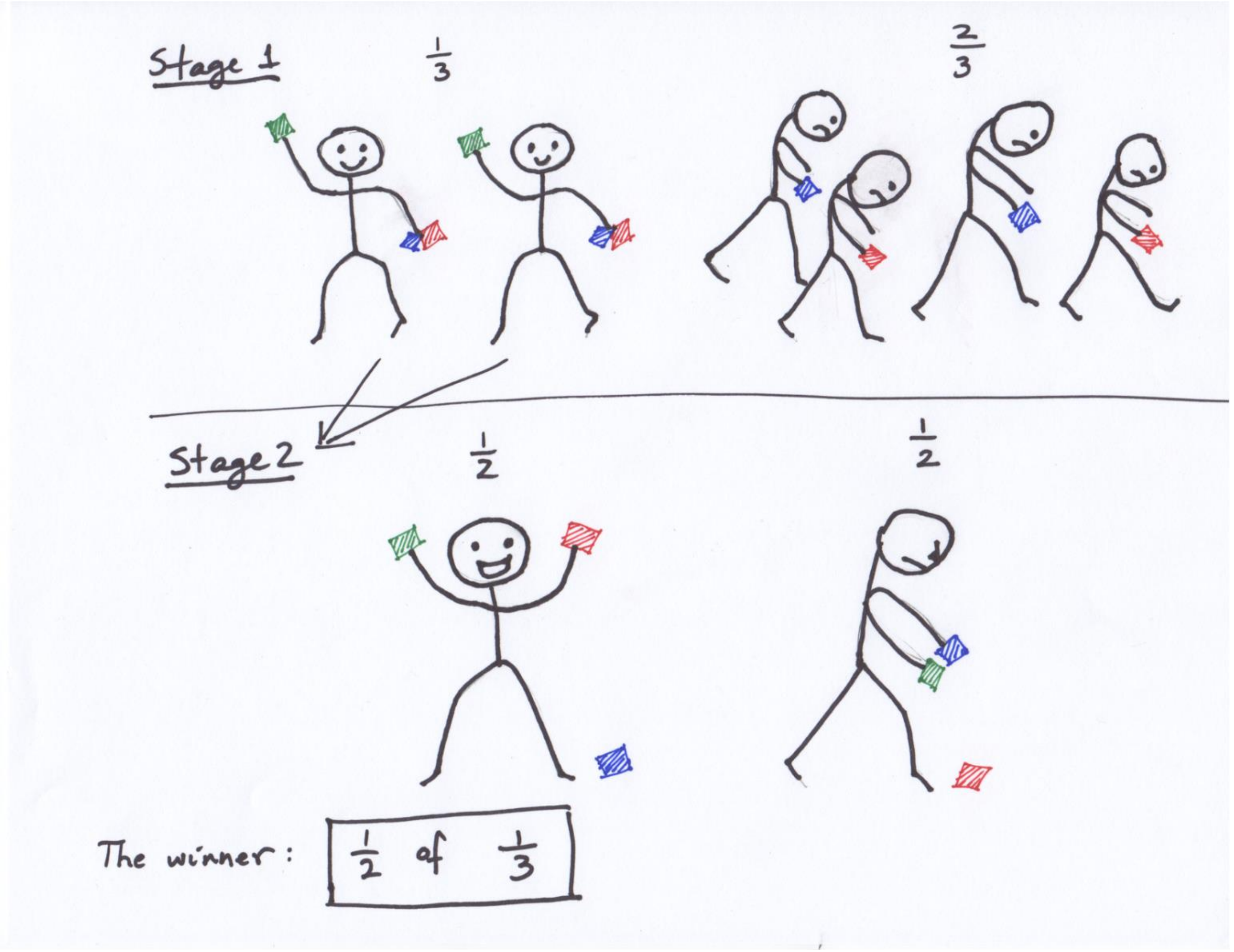
- What are the chances of winning the game?



Stage 1
1/3
2/3
Stage 2
1/2
1/2
The winner:
1/2 of 1/3

Multiplication rule

Chance that two events *A* and *B* both happen

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- The more conditions you have to satisfy, the less likely you are to satisfy them all

Yoram Lass is at **Neve Avivim**.

2 hrs • Tel Aviv, Israel •

לא נפטרים ממנה

בתגובה על "מעקב קורונה – ישראל, כל הנתונים",
("הארץ", 25.4)

תוחלת החיים בישראל היא 82 שנה. זה הגיל הממוצע שבו נפטרים בישראל. הגיל ממוצע של הנפטרים בגלל מחלת הקורונה הוא גם כן כ-82 שנה. מכאן שבממוצע, בכלל לא מתים ממחלת הקורונה (לידיעת כל מי שמדבר על ימי הביניים ועל השואה).

יורם לס, תל אביב

376 162 Comments • 53 Shares

Problem 1

Please choose ‘A’ or ‘B’:

A:
300 with certainty

B:
400 with chance of 4/5
0 with chance of 1/5

A

B

Problem 2

Please choose ‘A’ or ‘B’:

A:
300 with chance of 1/4
0 with chance of 3/4

B:
400 with chance of 1/5
0 with chance of 4/5

A

B

Problem 2

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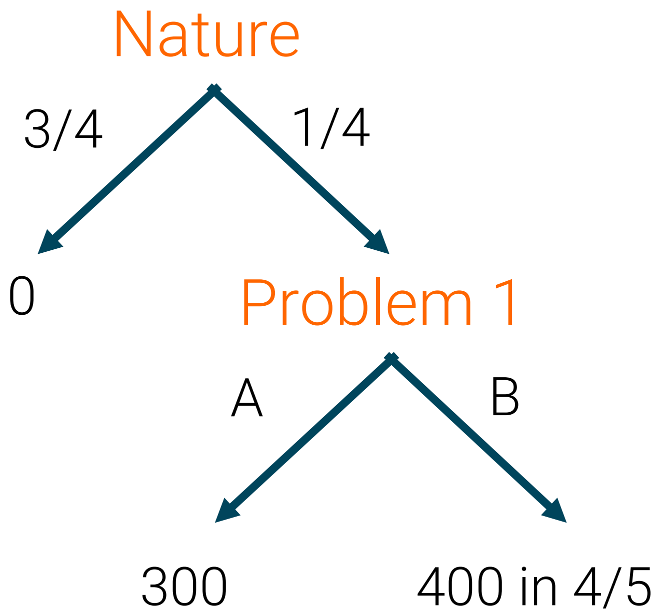
A

B



Probability of disjoint events

- Think of the following game-of-chance:
  - There are three tickets in a hat: Red, Blue, and Green
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- Would you prefer playing this game or the previous game?

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- Would you prefer playing this game or the previous game?
- What are the chances of winning the game?

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If event A can happen in exactly one of two ways, and these two ways cannot happen at the same time (disjoint):

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Say you have 2 coin tosses.

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- What is the probability of getting <u>exactly</u> one H?
- What is the probability of getting <u>at least</u> one H?

- Possible outcomes: HH, HT, TH, TT

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    - Any outcome except TTT
    - $P(TTT) = (\frac{1}{2}) \times (\frac{1}{2}) \times (\frac{1}{2}) = \frac{1}{8}$
    - $P(\text{at least one head}) = 1 - P(TTT) = 1 - (\frac{1}{2})^3 = \frac{7}{8} = 87.5\%$

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- In 10 tosses?
  - $1 - (\frac{1}{2})^{10}$ = 99.9%

(Notebook: Probabilities)

Simulating experiments

- Using Probability Theory, we can compute probabilities and other statistical quantities mathematically
  - Like we just did for “at least one success”
- In this course, we focus on simulating such quantities
- We will tell Python to simulate a random trial that gives us what we want and repeat it many times
- We can then summarize the results of all simulations to see the value of the quantity we want

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4. Code the simulation
   - Create a “collection array”: an empty array in which we’ll collect the simulated values
   - Create a *for* loop with as many iterations as you defined in Step 3.
   - In each iteration, simulate one value based on the code from Step 2, and append the simulated value to the collection array
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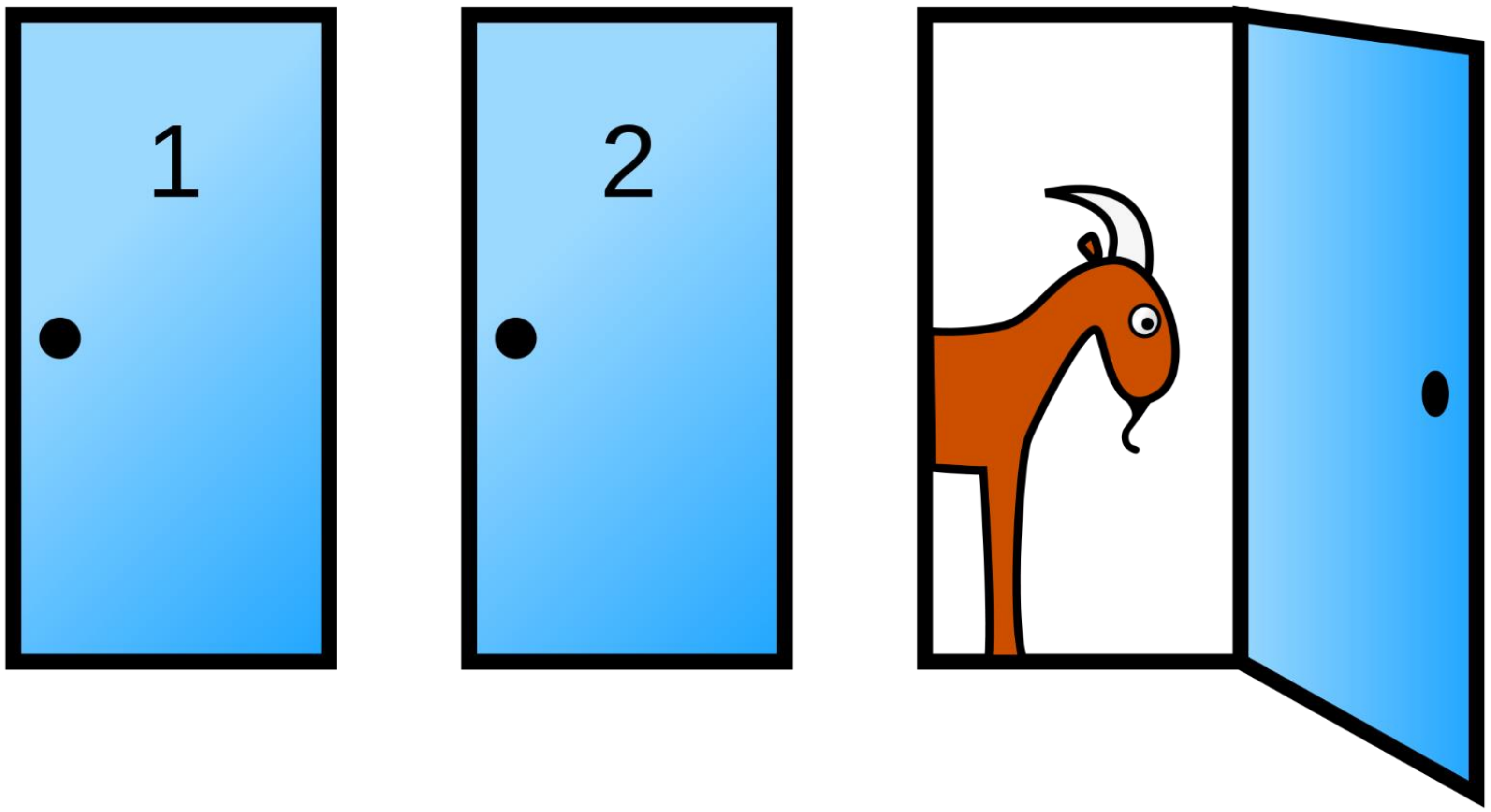
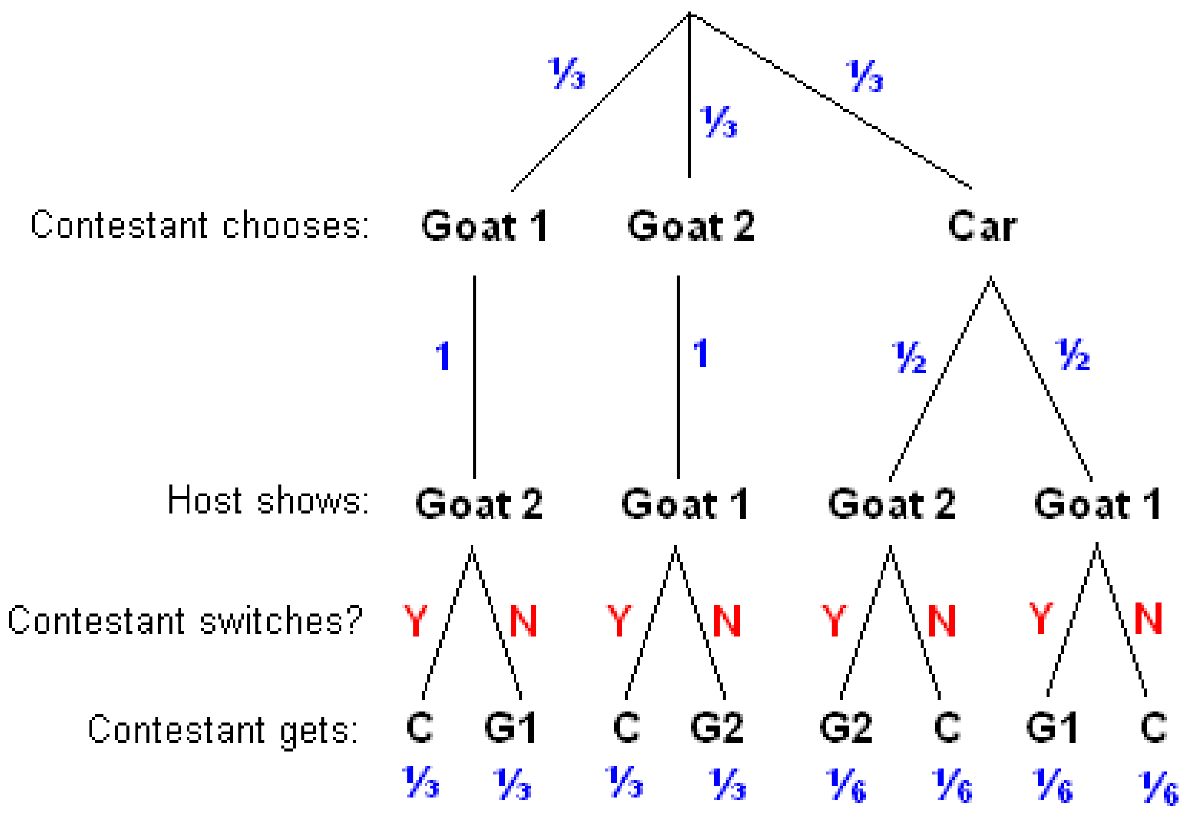
(Notebook: Simulation)


1
2



Switch
- Car: ⅔
- Goat 1: ⅙
- Goat 2: ⅙

No switch
- Car: ⅓
- Goat 1: ⅓
- Goat 2: ⅓

Sampling

Deterministic sample:

- Sampling does not involve chance

Probability sample:

- Each individual has a non-zero probability of being selected into the sample
- Before sample is drawn, we need to know the probability of selecting each group of individuals from the population
    - It is not required that each individual will have the same probability of being sampled
    - But it can help if they do
        - If in addition the selection of each individual is independent on the selection of all other individuals, such sampling is called *simple random sampling*

(notebook)

Convenience sampling

- Example: sample consists of whoever walks by
- Just because you think you’re sampling “at random”, doesn’t mean you are

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- Example: sample consists of whoever walks by
- Just because you think you're sampling "at random", doesn't mean you are
- If you can't figure out ahead of time
    1. what's the population
    2. what's the chance of selection, for each group in the population

then you don't have a random sample

Probability Distribution

- Random quantity with various possible values
- "Probability distribution":
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- If you can do the math, you can work out the probability distribution without ever simulating the random quantity

Empirical Distribution

- Based on observations
- Observations can be from repetitions of an experiment
- "Empirical Distribution":
  - All observed values
  - The proportion of counts of each value

Law of Averages

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As a result:

If the sample size is large, then the empirical distribution of a **simple random sample** resembles the distribution of the population, with high probability