Dynamics and Control

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1 Maths for 1D Motion

PID Controller

$$
\begin{aligned}
k_c &= m_{eq}\omega_c^2\sqrt{\alpha}, \qquad & \tau_z &= \frac{\sqrt{\alpha}}{\omega_c} \\
\tau_i &= \beta\tau_z, & \tau_p &= \alpha\tau_z
\end{aligned} \tag{1}
$$

A skew-sine reference profile has a maximum jerk at:

$$\dddot{r}_{max} = \frac{4\pi^2 h_m}{t_m^3} \tag{2}$$

$$
\begin{aligned}
\omega_c &= \left(\frac{\dddot{r}_{max}\beta}{\alpha e_{LF}}\right)^{\frac{1}{3}} \qquad (3) \\
&= \left(\frac{4\pi^2 h_m\beta}{\alpha e_{LF}}\right)^{\frac{1}{3}} \frac{1}{t_m} \qquad (4)
\end{aligned}
$$

Our values.
Based off: `https://new.abb.com/products/robotics/industrial-robots/irb-360`. See IRB 360-1/1600 model.
Cycle from 0 mm$\rightarrow$ d_m $\rightarrow$ 0 mm

$$
\begin{aligned}
m_{eq} &= 1\ kg \\
e_{max} &= 0.1\ mm \\
d_m &= 280\ mm \\
h_m &= arctan\left(\frac{d_m}{L_u + L_l}\right) \\
&= 0.1391\ rad\ = 8^\circ \\
t_m &= 0.35\ s
\end{aligned}
$$

2 Maths for 2D motion

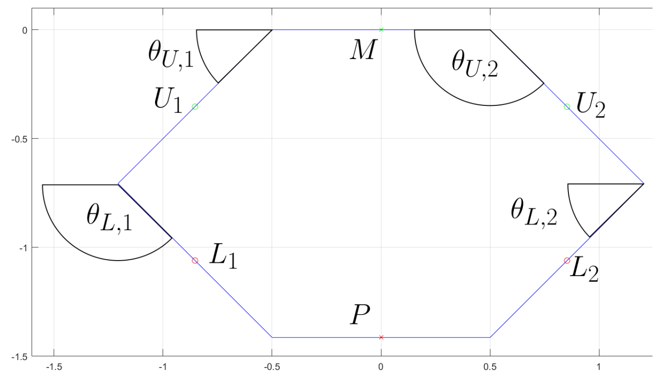

Figure 1: Initial configuration of the 2D Delta robot.

Constraint equations

$$\Phi = \begin{bmatrix} M_x \mp \frac{D}{2} - U_{j,x} - \frac{L_u}{2} cos(\theta_{U,j}) \\ M_y - U_{j,y} - \frac{L_u}{2} sin(\theta_{U,j}) \\ U_{j,x} - \frac{L_u}{2} cos(\theta_{U,j}) - L_{j,x} - \frac{L_l}{2} cos(\theta_{L,j}) \\ U_{j,y} - \frac{L_u}{2} sin(\theta_{U,j}) - L_{j,x} - \frac{L_l}{2} sin(\theta_{L,j}) \\ L_{j,x} - \frac{L_l}{2} cos(\theta_{L,j}) - P_x \pm \frac{L_e}{2} \\ L_{j,y} - \frac{L_l}{2} sin(\theta_{L,j}) - P_y \\ \vdots \\ \vdots \end{bmatrix} \tag{5}$$

$$\Phi^{driving} = \begin{bmatrix} \theta_{U,1} - \omega_1 t - (\theta_{U,1})_0 \\ \theta_{U,2} - \omega_2 t - (\theta_{U,2})_0 \end{bmatrix} \tag{6}$$

Independent and Dependent Co-ordinates

$$q_i = \begin{bmatrix} \theta_{U,1} \\ \theta_{U,2} \end{bmatrix} \qquad q_d = \begin{bmatrix} \vdots \end{bmatrix} \tag{7}$$

Kinematics

$$\Phi = 0$$

$$\implies \frac{d\Phi}{dt} = \Phi_q \frac{dq}{dt} + \Phi_t = 0 \tag{8}$$

$$\therefore \frac{dq}{dt} = -\Phi_q^{-1} \Phi_t \tag{9}$$

$$\implies \frac{d^2\Phi}{dt^2} = ([\Phi_q \frac{dq}{dt}]_q \frac{dq}{dt} + [\Phi_q \frac{dq}{dt}]_t) + (\Phi_{tq} \frac{dq}{dt} + \Phi_{tt}) = 0 \tag{10}$$

$$\therefore \frac{d^2q}{dt^2} = -\Phi_q^{-1} (\Phi_{qq} \frac{dq}{dt}^2 + 2\Phi_{qt} \frac{dq}{dt} + \Phi_{tt})$$

$$= \Phi_q^{-1} \gamma \tag{11}$$

Embedded form:

$$J = \Phi_{q_d}^{-1}\Phi_{q_i} \tag{12}$$
$$\dot{q}_d = -J\dot{q}_i \tag{13}$$
$$\hat{M} = M_{ii} + J^T M_{dd} J \tag{14}$$
$$\hat{Q} = -J^T Q_{A,d} + J^T M_{dd}\Phi_{q_d}^{-1}\gamma \tag{15}$$

Inverse Kinematics

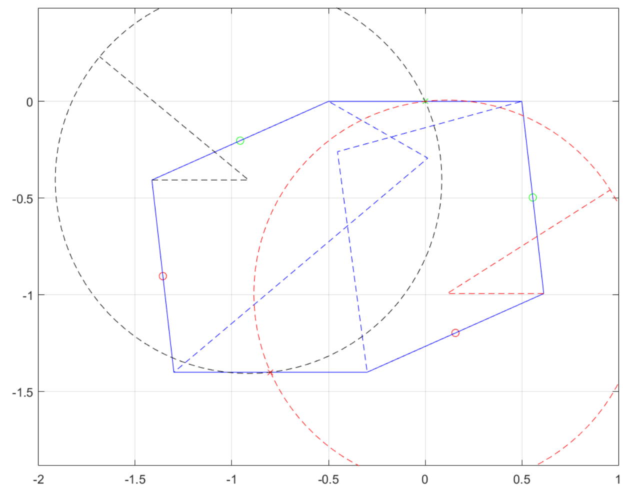

Figure 2: Interpretation of the inverse kinematics equation. The circles are centered at $x = \mp\frac{1}{2}D - L_u cos(\theta_{U,j}) \pm \frac{1}{2}L_e$ and $y = -L_u sin(\theta_{U,j})$

$$\Phi^{IK} = \begin{bmatrix} (\mp\frac{D}{2} - L_u cos(\theta_{U,j}) \pm \frac{L_e}{2} - P_x)^2 + (-P_y - L_u sin(\theta_{U,j}))^2 - L_l^2 \\ \vdots \end{bmatrix} \tag{16}$$

$$\Phi^{IK} = 0$$
$$\implies \frac{d\Phi}{dt} = \Phi_{q_i}\frac{dq_i}{dt} + \Phi_P\frac{dP}{dt} = 0 \tag{17}$$
$$\begin{aligned}\implies \frac{d^2\Phi}{dt^2} =& ([\Phi_{q_i}\frac{dq_i}{dt}]_{q_i}\frac{dq_i}{dt} + [\Phi_{q_i}\frac{dq_i}{dt}]_P\frac{dP}{dt}) + \Phi_{q_i}\frac{d^2q_i}{dt^2} + \\ & ([\Phi_P\frac{dP}{dt}]_{q_i}\frac{dq_i}{dt} + [\Phi_P\frac{dP}{dt}]_P\frac{dP}{dt}) + \Phi_P\frac{d^2P}{dt^2} = 0\end{aligned} \tag{18}$$
$$\therefore \frac{d^2q_i}{dt^2} = -\Phi_{q_i}^{-1}(\Phi_{q_iq_i}\frac{dq_i}{dt}^2 + \Phi_{PP}\frac{dP}{dt}^2 + 2\Phi_{q_iP}\frac{dq_i}{dt}\frac{dP}{dt} + \Phi_P\frac{d^2P}{dt}) \tag{19}$$

3 Maths for 3D motion

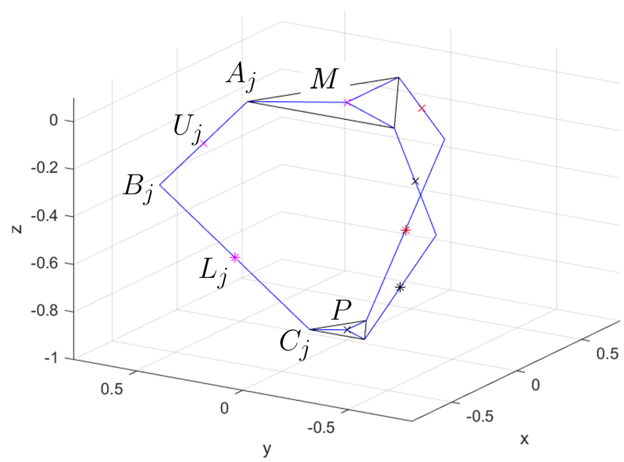

Figure 3: Initial configuration of the 3D Delta robot.

Independent and dependent co-ordinates

$$q_i = \begin{bmatrix} (\theta_{U,j})_x \\ (\theta_{U,j})_y \\ (\theta_{U,j})_z \\ \vdots \end{bmatrix} \qquad q_a = \begin{bmatrix} \theta_{a,1} \\ \theta_{a,2} \\ \theta_{a,3} \end{bmatrix} \tag{20}$$

$$\begin{aligned}
\omega_a^{O,U} &= R_U^O \omega_a^{U,U} \\
&= \begin{bmatrix} cos(\theta_a)cos(\psi_0) & -sin(\psi_0) & cos(\psi_0)sin(\theta_a) \\ cos(\theta_a)sin(\psi_0) & cos(\psi_0) & sin(\psi_0)sin(\theta_a) \\ -sin(\theta_a) & 0 & cos(\theta_a) \end{bmatrix} \begin{bmatrix} 0 \\ \omega_a \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -sin(\psi_0)\omega_a \\ cos(\psi_0)\omega_a \\ 0 \end{bmatrix} \\
\implies \int \omega_a^{O,U} dt &= \int R_U^O \omega_a^{U,U} dt \\
\therefore \theta_U - \theta_{U,0} &= \begin{bmatrix} -sin(\psi_0)(\theta_a - \theta_{a,0}) \\ cos(\psi_0)(\theta_a - \theta_{a,0}) \\ \psi_0 \end{bmatrix}
\end{aligned} \tag{21}$$

Constraint equations for the position

$$\Phi = \begin{bmatrix} M + R_{A,j}\begin{bmatrix}\frac{D}{2}\\0\\0\end{bmatrix} - U_j - R_{U,j}\begin{bmatrix}0\\0\\\frac{L_u}{2}\end{bmatrix} \\ U_j + R_{U,j}\begin{bmatrix}0\\0\\-\frac{L_u}{2}\end{bmatrix} - L_j - R_{L,j}\begin{bmatrix}0\\0\\\frac{L_l}{2}\end{bmatrix} \\ L_j + R_{L,j}\begin{bmatrix}0\\0\\\frac{-L_l}{2}\end{bmatrix} - P - R_{A,j}\begin{bmatrix}\frac{L_e}{2}\\0\\0\end{bmatrix} \\ \begin{bmatrix}0 & 1 & 0\end{bmatrix} R_{L,j}^T (R_{L,j})_0 \begin{bmatrix}1\\0\\0\end{bmatrix} \\ \vdots \end{bmatrix}, \ j = 1,2,3 \tag{22}$$

$$\Phi^{driving} = \begin{bmatrix} (\theta_{U,j})_x + sin((\theta_{a,j})_0)\omega_j t - 0 \\ (\theta_{U,j})_y - cos((\theta_{a,j})_0)\omega_j t - (\theta_{a,j})_0 \\ (\theta_{U,j})_z - (\theta_{U,j})_{z,0} \\ \vdots \end{bmatrix} \tag{23}$$

Kinematics The following is true for all rotation matrices:

$$RR^T = I \tag{24}$$

$$\implies \dot{R}R^T + R\dot{R^T} = 0$$

$$\therefore \dot{R}R^T = -R\dot{R^T} = \tilde{\omega} \tag{25}$$

Written with reference frame subscripts:

$$\dot{R}_C^O = \tilde{\omega}^O R_C^O \tag{26}$$

$$= R_C^O \tilde{\omega}^C \tag{27}$$

The velocity and acceleration for a point on a rigid body is then:

$$r_P^{O,O} = r_C^{O,O} + R_C^O r_P^{C,C} \tag{28}$$

$$\implies v_P^{O,O} = v_C^{O,O} + \dot{R}_C^O r_P^{C,C}$$

$$= v_C^{O,O} + \tilde{\omega}_C^{O,O} R_C^O r_P^{C,C} \tag{29}$$

$$\implies a_P^{O,O} = a_C^{O,O} + \tilde{\alpha}_C^{O,O} R_C^O r_P^{C,C} + \tilde{\omega}_C^{O,O} \dot{R}_C^O r_P^{C,C}$$

$$= a_C^{O,O} + \tilde{\alpha}_C^{O,O} R_C^O r_P^{C,C} + \tilde{\omega}_C^{O,O} \tilde{\omega}_C^{O,O} R_C^O r_P^{C,C} \tag{30}$$

In order to compare the above equations with equation 8 and 11, the following identity is used:

$$
\begin{aligned}
\tilde{\omega}^O R_C^O r^C &= \tilde{\omega}^O r^O \\
&= (\tilde{r}^O)^T \omega^O && \text{property of anti-symmetric matrices} \\
&= R_C^O (\tilde{r}^C)^T R_C^O \omega^O && \text{equations 26 and 27}
\end{aligned}
$$

This results in:

$$
\begin{aligned}
\frac{d\Phi}{dt} &= \begin{bmatrix} I & -I & -R_{U,j}(\tilde{r}_A^{U,U})^T R_{U,j}^T & 0 & 0 & \dots \\ 0 & I & +R_{U,j}(\tilde{r}_B^{U,U})^T R_{U,j}^T & -I & -R_{L,j}(\tilde{r}_L^{L,B})^T R_{L,j}^T & \dots \\ 0 & 0 & 0 & I & +R_{L,j}(\tilde{r}_L^{L,C})^T R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & e_y^0 R_{L,j}^T \tilde{e}_x^U & \dots \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} \dot{M} \\ \dot{U}_j \\ \omega_{U,j} \\ \dot{L}_J \\ \omega_{L,j} \\ \vdots \end{bmatrix} \\
&= \Phi_q \tfrac{dq}{dt}
\end{aligned}
\tag{31}
$$

Similarly, for the accelerations

$$
\begin{aligned}
\frac{d^2\Phi}{dt^2} &= \begin{bmatrix} I & 0 & -R_{U,j}\tilde{r}_A^{U,U} R_{U,j}^T & 0 & 0 & \dots \\ 0 & I & +R_{U,j}\tilde{r}_B^{U,U} R_{U,j}^T & -I & -R_{L,j}\tilde{r}_L^{L,B} R_{L,j}^T & \dots \\ 0 & 0 & 0 & I & +R_{L,j}\tilde{r}_L^{L,C} R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & e_y^0 R_{L,j}^T \tilde{e}_x^U & \dots \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} \ddot{M} \\ \ddot{U}_j \\ \alpha_{U,j} \\ \dot{L}_J \\ \alpha_{L,j} \\ \vdots \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & -\tilde{\omega}_U^{O,O} R_{U,j}\tilde{r}_A^{U,U} R_{U,j}^T & 0 & 0 & \dots \\ 0 & 0 & +\tilde{\omega}_U^{O,O} R_{U,j}\tilde{r}_B^{U,U} R_{U,j}^T & 0 & -\tilde{\omega}_L^{O,O} R_{L,j}\tilde{r}_L^{L,B} R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & +\tilde{\omega}_L^{O,O} R_{L,j}\tilde{r}_L^{L,C} R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & e_y^0 (\tilde{\omega}_L^{O,O} R_{L,j})^T \tilde{e}_x^U & \dots \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} \dot{M} \\ \dot{U}_j \\ \omega_{U,j} \\ \dot{L}_J \\ \omega_{L,j} \\ \vdots \end{bmatrix} \\
&= \Phi_q \ddot{q} + [\Phi_q \tfrac{dq}{dt}]_q \tfrac{dq}{dt} \\
&= \Phi_q \ddot{q} - \gamma
\end{aligned}
\tag{32}
$$

Inverse kinematics

$$
\begin{aligned}
r = R_{L,j} \begin{bmatrix} 0 \\ 0 \\ L_l \end{bmatrix} &= R_{A,j} \begin{bmatrix} \frac{D-L_e}{2} \\ 0 \\ 0 \end{bmatrix} - R_{U,j} \begin{bmatrix} 0 \\ 0 \\ L_u \end{bmatrix} - P \\
\therefore r^T r &= r^T R^T R r
\end{aligned}
\tag{33}
$$

$$\implies L_l^2 = \left(\left(\frac{1}{2}(D - L_e) - L_u sin(\theta_{a,j})\right) cos(\psi_0) - P_x\right)^2 + \left(\left(\frac{1}{2}(D - L_e) - L_u sin(\theta_{a,j})\right) sin(\psi_0) - P_y\right)^2 + (-L_u cos(\theta_{a,j}) - P_z)^2 \tag{34}$$

$$\frac{d^2 q_a}{dt^2} = -\Phi_{q_a}^{-1}(\Phi_{q_a q_a} \frac{dq_a}{dt}^2 + \Phi_{PP} \frac{dP}{dt}^2 + 2\Phi_{q_a P} \frac{dq_a}{dt} \frac{dP}{dt} + \Phi_P \frac{d^2 P}{dt^2}) \tag{35}$$