

Dynamics and Control

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1 Maths for 1D Motion

PID Controller

$$\begin{aligned} k_c &= m_{eq}\omega_c^2\sqrt{\alpha}, & \tau_z &= \frac{\sqrt{\alpha}}{\omega_c} \\ \tau_i &= \beta\tau_z, & \tau_p &= \alpha\tau_z \end{aligned} \quad (1)$$

A skew-sine reference profile has a maximum jerk at:

$$\ddot{r}_{max} = \frac{4\pi^2 h_m}{t_m^3} \quad (2)$$

$$\omega_c = \left(\frac{\ddot{r}_{max}\beta}{\alpha e_{LF}} \right)^{\frac{1}{3}} \quad (3)$$

$$= \left(\frac{4\pi^2 h_m \beta}{\alpha e_{LF}} \right)^{\frac{1}{3}} \frac{1}{t_m} \quad (4)$$

Our values.

Based off: <https://new.abb.com/products/robotics/industrial-robots/irb-360>. See IRB 360-1/1600 model.

Cycle from 0 mm \rightarrow d_m \rightarrow 0 mm

$$m_{eq} = 1 \text{ kg}$$

$$e_{max} = 0.1 \text{ mm}$$

$$d_m = 280 \text{ mm}$$

$$h_m = \arctan\left(\frac{d_m}{L_u + L_t}\right)$$

$$= 0.1391 \text{ rad} = 8^\circ$$

$$t_m = 0.35 \text{ s}$$

2 Maths for 2D motion

2.1 Definitions

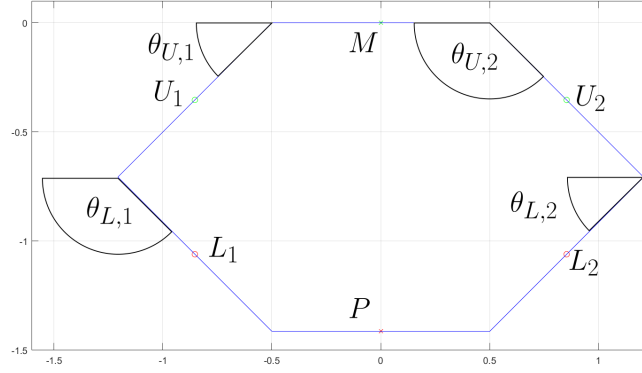


Figure 1: Initial configuration of the 2D Delta robot.

Constraint equations

$$\Phi = \begin{bmatrix} M_x \mp \frac{L_b}{2} - U_{j,x} - \frac{L_u}{2} \cos(\theta_{U,j}) \\ M_y - U_{j,y} - \frac{L_u}{2} \sin(\theta_{U,j}) \\ U_{j,x} - \frac{L_u}{2} \cos(\theta_{U,j}) - L_{j,x} - \frac{L_l}{2} \cos(\theta_{L,j}) \\ U_{j,y} - \frac{L_u}{2} \sin(\theta_{U,j}) - L_{j,x} - \frac{L_l}{2} \sin(\theta_{L,j}) \\ L_{j,x} - \frac{L_l}{2} \cos(\theta_{L,j}) - P_x \pm \frac{L_e}{2} \\ L_{j,y} - \frac{L_l}{2} \sin(\theta_{L,j}) - P_y \\ \vdots \\ \vdots \end{bmatrix} \quad (5)$$

$$\Phi^{driving} = \begin{bmatrix} \theta_{U,1} - \omega_1 t - (\theta_{U,1})_0 \\ \theta_{U,2} - \omega_2 t - (\theta_{U,2})_0 \end{bmatrix} \quad (6)$$

Independent and Dependent Co-ordinates

$$q_i = \begin{bmatrix} \theta_{U,1} \\ \theta_{U,2} \end{bmatrix} \quad q_d = \begin{bmatrix} \vdots \end{bmatrix} \quad (7)$$

2.2 Kinematics

$$\Phi = 0$$

$$\implies \frac{d\Phi}{dt} = \Phi_q \frac{dq}{dt} + \Phi_t = 0 \quad (8)$$

$$\therefore \frac{dq}{dt} = -\Phi_q^{-1} \Phi_t \quad (9)$$

$$\implies \frac{d^2\Phi}{dt^2} = ([\Phi_q \frac{dq}{dt}]_q \frac{dq}{dt} + [\Phi_q \frac{dq}{dt}]_t) + (\Phi_{tq} \frac{dq}{dt} + \Phi_{tt}) = 0 \quad (10)$$

$$\begin{aligned} \therefore \frac{d^2q}{dt^2} &= -\Phi_q^{-1} (\Phi_{qq} \frac{dq}{dt}^2 + 2\Phi_{qt} \frac{dq}{dt} + \Phi_{tt}) \\ &= \Phi_q^{-1} \gamma \end{aligned} \quad (11)$$

2.3 Inverse Kinematics

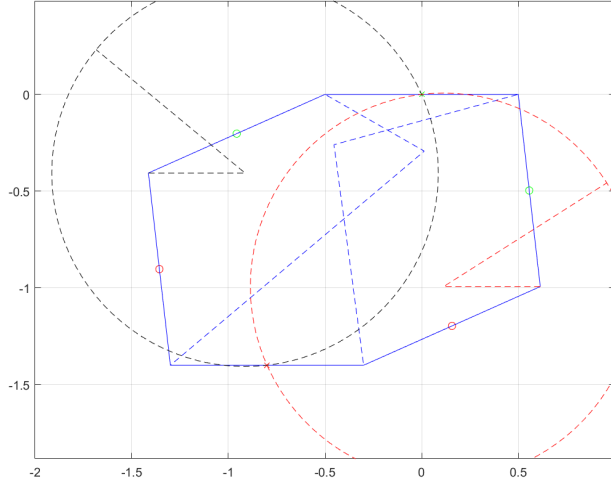


Figure 2: Interpretation of the inverse kinematics equation. The circles are centered at $x = \mp \frac{1}{2}L_b - L_u \cos(\theta_{U,j}) \pm \frac{1}{2}L_e$ and $y = -L_u \sin(\theta_{U,j})$

$$\Phi^{IK} = \begin{bmatrix} (\mp \frac{L_b}{2} - L_u \cos(\theta_{U,j}) \pm \frac{L_e}{2} - P_x)^2 + (-P_y - L_u \sin(\theta_{U,j}))^2 - L_l^2 \\ \vdots \end{bmatrix} \quad (12)$$

$$\Phi^{IK} = 0$$

$$\implies \frac{d\Phi}{dt} = \Phi_{q_i} \frac{dq_i}{dt} + \Phi_P \frac{dP}{dt} = 0 \quad (13)$$

$$\begin{aligned} \implies \frac{d^2\Phi}{dt^2} &= ([\Phi_{q_i} \frac{dq_i}{dt}]_{q_i} \frac{dq_i}{dt} + [\Phi_{q_i} \frac{dq_i}{dt}]_P \frac{dP}{dt}) + \Phi_{q_i} \frac{d^2q_i}{dt^2} + \\ &([\Phi_P \frac{dP}{dt}]_{q_i} \frac{dq_i}{dt} + [\Phi_P \frac{dP}{dt}]_P \frac{dP}{dt}) + \Phi_P \frac{d^2P}{dt^2} = 0 \end{aligned} \quad (14)$$

$$\therefore \frac{d^2q_i}{dt^2} = -\Phi_{q_i}^{-1} (\Phi_{q_i q_i} \frac{dq_i}{dt}^2 + \Phi_{PP} \frac{dP}{dt}^2 + 2\Phi_{q_i P} \frac{dq_i}{dt} \frac{dP}{dt} + \Phi_P \frac{d^2P}{dt^2}) \quad (15)$$

2.4 Kinetics

The Lagrange multiplier method has been used to calculate the forces. See [1] for more detail.

The primary equation is:

$$M\ddot{q} + \Phi_q^T \lambda = Q_A \quad (16)$$

Where Q_A are the applied forces and λ the Lagrange multipliers. The λ can be solved for directly if Φ_q has full rank, which will be the case if it kinematically driven. Otherwise, Φ_q will have more columns for co-ordinates q than rows for constraint equations, and will not be invertible. They must then be solved for simultaneously with the accelerations:

$$\begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} Q_A \\ \gamma \end{bmatrix} \quad (17)$$

See the appendix for more detail.

Each λ can be related to constraint forces at a particular point P from a centroid with co-ordinates (X, Y, Θ) with:

$$\begin{aligned} F_{P,x} &= -\Phi_{P,X}^T \lambda_P \\ F_{P,y} &= -\Phi_{P,Y}^T \lambda_P \\ M_P &= -\Phi_{P,\Theta}^T \lambda_P + y_{P/Y} - F_{P,x} - x_{P/X} F_{P,y} \end{aligned} \quad (18)$$

Where $\Phi_{P,\Theta}$ is the Jacobian of the relevant constraints at P with respect to (X, Y, Θ) . Equivalently, it is the sub-matrix of Φ_q with the same indices as the relevant constraint equations and co-ordinates.

The constraint forces can also be found by solving:

$$\begin{aligned} m_u \ddot{x}_u &= F_{A,x} + F_{B,x} \\ m_u \ddot{y}_u &= F_{A,y} + F_{B,y} - m_u g \\ I_u \ddot{\theta}_u &= -(F_{B,x} - F_{A,x}) \frac{1}{2} L_u \sin(\theta_U) + (F_{B,y} - F_{A,y}) \frac{1}{2} L_u \cos(\theta_U) + M_a \\ m_l \ddot{x}_l &= -F_{B,x} + F_{C,x} \\ m_l \ddot{y}_l &= -F_{B,y} + F_{C,y} - m_l g \end{aligned} \quad (19)$$

References: [1] [2]

3 Maths for 3D motion

3.1 Definitions

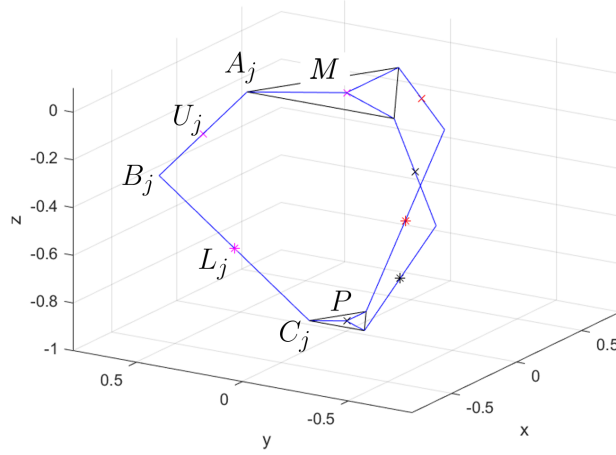


Figure 3: Initial configuration of the 3D Delta robot.

Independent and dependent co-ordinates

$$q_i = \begin{bmatrix} (\theta_{U,j})_x \\ (\theta_{U,j})_y \\ (\theta_{U,j})_z \\ \vdots \end{bmatrix} \quad q_a = \begin{bmatrix} \theta_{a,1} \\ \theta_{a,2} \\ \theta_{a,3} \end{bmatrix} \quad (20)$$

$$\begin{aligned} \omega_a^{O,U} &= R_U^O \omega_a^{U,U} \\ &= \begin{bmatrix} \cos(\theta_a) \cos(\psi_0) & -\sin(\psi_0) & \cos(\psi_0) \sin(\theta_a) \\ \cos(\theta_a) \sin(\psi_0) & \cos(\psi_0) & \sin(\psi_0) \sin(\theta_a) \\ -\sin(\theta_a) & 0 & \cos(\theta_a) \end{bmatrix} \begin{bmatrix} 0 \\ \omega_a \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -\sin(\psi_0) \omega_a \\ \cos(\psi_0) \omega_a \\ 0 \end{bmatrix} \\ \Rightarrow \int \omega_a^{O,U} dt &= \int R_U^O \omega_a^{U,U} dt \\ \therefore \theta_U - \theta_{U,0} &= \begin{bmatrix} -\sin(\psi_0) (\theta_a - \theta_{a,0}) \\ \cos(\psi_0) (\theta_a - \theta_{a,0}) \\ \psi_0 \end{bmatrix} \end{aligned} \quad (21)$$

Constraint equations for the position

$$\Phi = \begin{bmatrix} M + R_{A,j} \begin{bmatrix} \frac{L_b}{2} \\ 0 \\ 0 \end{bmatrix} - U_j - R_{U,j} \begin{bmatrix} 0 \\ 0 \\ \frac{L_u}{2} \end{bmatrix} \\ U_j + R_{U,j} \begin{bmatrix} 0 \\ 0 \\ -\frac{L_u}{2} \end{bmatrix} - L_j - R_{L,j} \begin{bmatrix} 0 \\ 0 \\ \frac{L_l}{2} \end{bmatrix} \\ L_j + R_{L,j} \begin{bmatrix} 0 \\ 0 \\ -\frac{L_l}{2} \end{bmatrix} - P - R_{A,j} \begin{bmatrix} \frac{L_p}{2} \\ 0 \\ 0 \end{bmatrix} \\ [0 \quad 1 \quad 0] R_{L,j}^T (R_{L,j})_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \vdots \end{bmatrix}, \quad j = 1, 2, 3 \quad (22)$$

$$\Phi^{driving} = \begin{bmatrix} (\theta_{U,j})_x + \sin((\theta_{a,j})_0) \omega_j t - 0 \\ (\theta_{U,j})_y - \cos((\theta_{a,j})_0) \omega_j t - (\theta_{a,j})_0 \\ (\theta_{U,j})_z - (\theta_{U,j})_{z,0} \\ \vdots \end{bmatrix} \quad (23)$$

3.2 Kinematics

The following is true for all rotation matrices:

$$RR^T = I \quad (24)$$

$$\implies \dot{R}R^T + R\dot{R}^T = 0$$

$$\therefore \dot{R}R^T = -R\dot{R}^T = \tilde{\omega} \quad (25)$$

Written with reference frame superscripts and subscripts:

$$\dot{R}_C^O = \tilde{\omega}^O R_C^O \quad (26)$$

$$= R_C^O \tilde{\omega}^C \quad (27)$$

The velocity and acceleration for a point on a rigid body is then:

$$r_P^{O,O} = r_C^{O,O} + R_C^O r_P^{C,C} \quad (28)$$

$$\begin{aligned} \implies v_P^{O,O} &= v_C^{O,O} + \dot{R}_C^O r_P^{C,C} \\ &= v_C^{O,O} + \tilde{\omega}_C^{O,O} R_C^O r_P^{C,C} \end{aligned} \quad (29)$$

$$\begin{aligned} \implies a_P^{O,O} &= a_C^{O,O} + \tilde{\alpha}_C^{O,O} R_C^O r_P^{C,C} + \tilde{\omega}_C^{O,O} \dot{R}_C^O r_P^{C,C} \\ &= a_C^{O,O} + \tilde{\alpha}_C^{O,O} R_C^O r_P^{C,C} + \tilde{\omega}_C^{O,O} \tilde{\omega}_C^{O,O} R_C^O r_P^{C,C} \end{aligned} \quad (30)$$

The following identity is useful:

$$\begin{aligned}
\tilde{\omega}^O R_C^O r^C &= \tilde{\omega}^O r^O \\
&= (\tilde{r}^O)^T \omega^O && \text{property of anti-symmetric matrices} \\
&= R_C^O (\tilde{r}^C)^T R_C^O \omega^O && \text{equations 26 and 27}
\end{aligned}$$

$\frac{d\Phi}{dt}$ can be found using equations 29 and 30 and can then be compared with equations 8 and 11:

$$\begin{aligned}
\frac{d\Phi}{dt} &= \begin{bmatrix} I & -I & -R_{U,j}(\tilde{r}_A^{U,U})^T R_{U,j}^T & 0 & 0 & \dots \\ 0 & I & +R_{U,j}(\tilde{r}_B^{U,U})^T R_{U,j}^T & -I & -R_{L,j}(\tilde{r}_L^{L,B})^T R_{L,j}^T & \dots \\ 0 & 0 & 0 & I & +R_{L,j}(\tilde{r}_L^{L,C})^T R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & e_y^0 R_{L,j}^T \tilde{e}_x^U & \dots \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} \dot{M} \\ \dot{U}_j \\ \omega_{U,j} \\ \dot{L}_J \\ \omega_{L,j} \\ \vdots \end{bmatrix} \\
&= \Phi_q \frac{dq}{dt}
\end{aligned} \tag{31}$$

Similarly, for the accelerations

$$\begin{aligned}
\frac{d^2\Phi}{dt^2} &= \begin{bmatrix} I & 0 & -R_{U,j}\tilde{r}_A^{U,U} R_{U,j}^T & 0 & 0 & \dots \\ 0 & I & +R_{U,j}\tilde{r}_B^{U,U} R_{U,j}^T & -I & -R_{L,j}\tilde{r}_L^{L,B} R_{L,j}^T & \dots \\ 0 & 0 & 0 & I & +R_{L,j}\tilde{r}_L^{L,C} R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & e_y^0 R_{L,j}^T \tilde{e}_x^U & \dots \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} \ddot{M} \\ \ddot{U}_j \\ \alpha_{U,j} \\ \dot{L}_J \\ \alpha_{L,j} \\ \vdots \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & -\tilde{\omega}_U^{O,O} R_{U,j}\tilde{r}_A^{U,U} R_{U,j}^T & 0 & 0 & \dots \\ 0 & 0 & +\tilde{\omega}_U^{O,O} R_{U,j}\tilde{r}_B^{U,U} R_{U,j}^T & 0 & -\tilde{\omega}_L^{O,O} R_{L,j}\tilde{r}_L^{L,B} R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & +\tilde{\omega}_L^{O,O} R_{L,j}\tilde{r}_L^{L,C} R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & e_y^0 (\tilde{\omega}_L^{O,O} R_{L,j})^T \tilde{e}_x^U & \dots \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} \dot{M} \\ \dot{U}_j \\ \omega_{U,j} \\ \dot{L}_J \\ \omega_{L,j} \\ \vdots \end{bmatrix} \\
&= \Phi_q \ddot{q} + [\Phi_q \frac{dq}{dt}]_q \frac{dq}{dt} \\
&= \Phi_q \ddot{q} - \gamma
\end{aligned} \tag{32}$$

3.3 Inverse kinematics

Add the first 3 rows in equation 22:

$$\begin{aligned}
r &= R_{L,j} \begin{bmatrix} 0 \\ 0 \\ L_l \end{bmatrix} = R_{A,j} \begin{bmatrix} \frac{L_b - L_e}{2} \\ 0 \\ 0 \end{bmatrix} - R_{U,j} \begin{bmatrix} 0 \\ 0 \\ L_u \end{bmatrix} - P \\
\therefore r^T r &= r^T R^T R r
\end{aligned} \tag{33}$$

$$\begin{aligned} \Rightarrow L_l^2 = & \left(\left(\frac{1}{2}(L_b - L_e) - L_u \sin(\theta_{a,j}) \right) \cos(\psi_0) - P_x \right)^2 + \\ & \left(\left(\frac{1}{2}(L_b - L_e) - L_u \sin(\theta_{a,j}) \right) \sin(\psi_0) - P_y \right)^2 + (-L_u \cos(\theta_{a,j}) - P_z)^2 \end{aligned} \quad (34)$$

This means the 2D equations 15 can be used:

$$\begin{aligned} \frac{d^2 q_a}{dt^2} &= -\Phi_{q_a}^{-1} \left([\Phi_{q_a} \frac{dq_a}{dt}]_{q_a} \frac{dq_a}{dt} + [\Phi_P \frac{dP}{dt}]_P \frac{dP}{dt} + 2[\Phi_{q_a} \frac{dq_a}{dt}]_P \frac{dP}{dt} + \Phi_P \frac{d^2 P}{dt^2} \right) \\ &= -\Phi_{q_a}^{-1} \left(\Phi_{q_a q_a} \frac{dq_a}{dt}^2 + \Phi_{PP} \frac{dP}{dt}^2 + 2\Phi_{q_a P} \frac{dq_a}{dt} \frac{dP}{dt} + \Phi_P \frac{d^2 P}{dt^2} \right) \\ &= \Phi_{q_a}^{-1} \gamma^{IK} \end{aligned} \quad (35)$$

3.4 Kinetics

The 3D equations of motion for a centre of mass C are:

$$\begin{aligned} m a_C^{O,O} &= F^O \\ I^C \alpha_C^{O,O} + \tilde{\omega}_C^{O,O} I^C \omega_C^{O,O} &= \tau^C \end{aligned} \quad (36)$$

Using equation 26, the moments can also be expressed as:

$$\begin{aligned} I^C R_O^C \alpha_C^{O,O} + \tilde{\omega}_C^{O,O} R_O^C I^C R_O^C \omega_C^{O,O} &= \tau^O \\ \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} - \begin{bmatrix} (I_y - I_z) \omega_y \omega_z \\ (I_z - I_x) \omega_x \omega_z \\ (I_x - I_y) \omega_x \omega_y \end{bmatrix} &= \tau^C \end{aligned}$$

The equations of motion given by 16 and 17 can still be used, as long as Q_A takes into account the extra gyroscopic forces as well.

Finally, the constraint forces can also be found by solving:

$$\begin{aligned} m_U a_U^{O,O} &= F_A + F_B + Q_{A,U} \\ I_U \alpha_C^{O,O} + \tilde{\omega}_C^{O,O} I_U \omega_C^{O,O} &= \tilde{r}_A^{O,U} R_O^C F_A + \tilde{r}_B^{O,U} R_O^C F_B + \tau_i^U \\ m_L a_L^{O,O} &= -F_B + F_C + Q_{A,L} \end{aligned} \quad (37)$$

References: [1] [2]

References

- [1] J. Schilder, *Dynamics 3*. University of Twente, 2018.
- [2] R. C. Hibbeler, *Engineering Mechanics: Dynamics*, 13th ed. Prentice Hall, 2013, Chapters 16-17 and 20-21.