Dynamics and Control

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1 Maths for 1D Motion

PID Controller

$$
\begin{aligned}
k_c &= m_{eq}\omega_c^2\sqrt{\alpha}, \qquad & \tau_z &= \frac{\sqrt{\alpha}}{\omega_c} \\
\tau_i &= \beta\tau_z, & \tau_p &= \alpha\tau_z
\end{aligned} \tag{1}
$$

A skew-sine reference profile has a maximum jerk at:

$$
\dddot{r}_{max} = \frac{4\pi^2 h_m}{t_m^3} \tag{2}
$$

$$
\omega_c = \left(\frac{\dddot{r}_{max}\beta}{\alpha e_{LF}}\right)^{\frac{1}{3}} \tag{3}
$$

$$
= \left(\frac{4\pi^2 h_m \beta}{\alpha e_{LF}}\right)^{\frac{1}{3}} \frac{1}{t_m} \tag{4}
$$

Our values.
Based off: `https://new.abb.com/products/robotics/industrial-robots/irb-360`. See IRB 360-1/1600 model.
Cycle from 0 mm$\rightarrow d_m \rightarrow$ 0 mm

$$
\begin{aligned}
m_{eq} &= 1\ kg \\
e_{max} &= 0.1\ mm \\
d_m &= 280\ mm \\
h_m &= arctan\left(\frac{d_m}{L_u + L_l}\right) \\
&= 0.1391\ rad\ = 8^\circ \\
t_m &= 0.35\ s
\end{aligned}
$$

2 Maths for 2D motion

2.1 Definitions

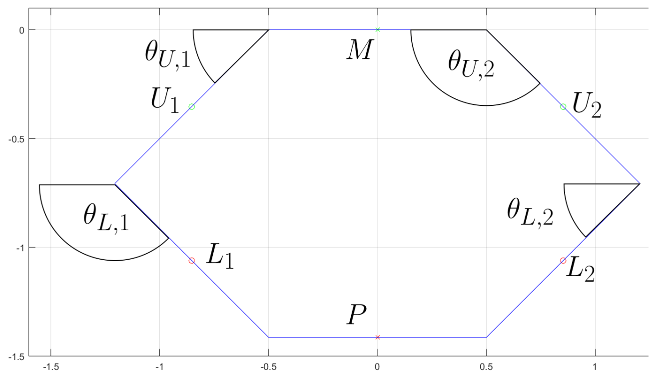

Figure 1: Initial configuration of the 2D Delta robot.

Constraint equations

$$\Phi = \begin{bmatrix} M_x \mp \frac{L_b}{2} - U_{j,x} - \frac{L_u}{2} cos(\theta_{U,j}) \\ M_y - U_{j,y} - \frac{L_u}{2} sin(\theta_{U,j}) \\ U_{j,x} - \frac{L_u}{2} cos(\theta_{U,j}) - L_{j,x} - \frac{L_l}{2} cos(\theta_{L,j}) \\ U_{j,y} - \frac{L_u}{2} sin(\theta_{U,j}) - L_{j,x} - \frac{L_l}{2} sin(\theta_{L,j}) \\ L_{j,x} - \frac{L_l}{2} cos(\theta_{L,j}) - P_x \pm \frac{L_e}{2} \\ L_{j,y} - \frac{L_l}{2} sin(\theta_{L,j}) - P_y \\ \vdots \\ \vdots \end{bmatrix} \tag{5}$$

$$\Phi^{driving} = \begin{bmatrix} \theta_{U,1} - \omega_1 t - (\theta_{U,1})_0 \\ \theta_{U,2} - \omega_2 t - (\theta_{U,2})_0 \end{bmatrix} \tag{6}$$

Independent and Dependent Co-ordinates

$$q_i = \begin{bmatrix} \theta_{U,1} \\ \theta_{U,2} \end{bmatrix} \qquad q_d = \begin{bmatrix} \vdots \end{bmatrix} \tag{7}$$

2.2 Kinematics

$$\begin{aligned}
&\Phi = 0 \\
\implies &\tfrac{d\Phi}{dt} = \Phi_q \tfrac{dq}{dt} + \Phi_t = 0 && (8)\\
\therefore &\tfrac{dq}{dt} = -\Phi_q^{-1}\Phi_t && (9)\\
\implies &\tfrac{d^2\Phi}{dt^2} = ([\Phi_q \tfrac{dq}{dt}]_q \tfrac{dq}{dt} + [\Phi_q \tfrac{dq}{dt}]_t) + (\Phi_{tq}\tfrac{dq}{dt} + \Phi_{tt}) = 0 && (10)\\
\therefore &\tfrac{d^2 q}{dt^2} = -\Phi_q^{-1}(\Phi_{qq}\tfrac{dq}{dt}^2 + 2\Phi_{qt}\tfrac{dq}{dt} + \Phi_{tt}) \\
&= \Phi_q^{-1}\gamma && (11)
\end{aligned}$$

2.3 Inverse Kinematics

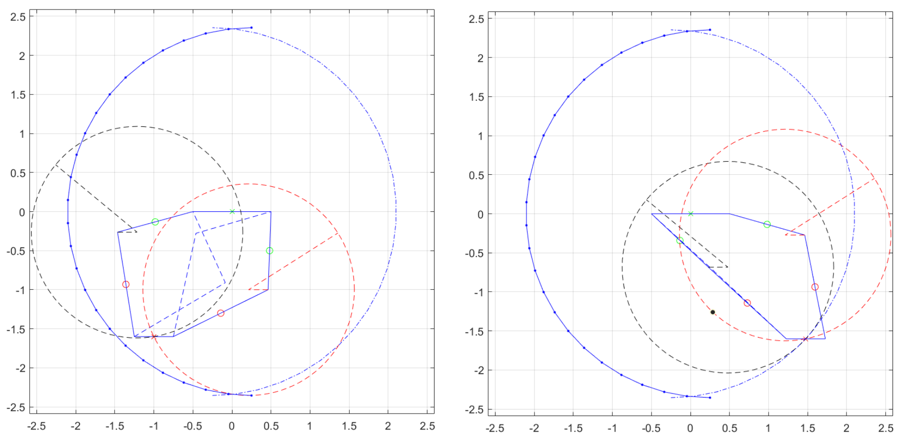

Figure 2: Interpretation of the inverse kinematics equation. The circles are centered at $x = \mp\frac{1}{2}L_b - L_u cos(\theta_{U,j}) \pm \frac{1}{2}L_e$ and $y = -L_u sin(\theta_{U,j})$

By eliminating the dependent x and y co-ordinates in equation 5, the following equations can be obtained:

$$\Phi^{IK} = \begin{bmatrix} (\mp\frac{L_b}{2} - L_u cos(\theta_{U,j}) \pm \frac{L_e}{2} - P_x)^2 + (-P_y - L_u sin(\theta_{U,j}))^2 - L_l^2 \\ \vdots \end{bmatrix} \quad (12)$$

The boundary is obtained by analysing the case where one arm is fully extended. That is, where $\theta_{U,j} = \theta_{L,j}$. See figure 2.:

$$\begin{aligned}
P_x &= \mp\tfrac{L_b}{2} - (L_u + L_l)cos(\theta_{U,j}) \pm \tfrac{L_e}{2} \\
P_y &= -(L_u + L_l)sin(\theta_{U,j})
\end{aligned}$$

Eliminating $\theta_{U,j}$:

$$(P_x \pm \tfrac{L_b}{2} \mp \tfrac{L_e}{2})^2 + P_y^2 = (L_u + L_l)^2 \quad (13)$$

The workspace is the intersection of the two circles. When $L_e = L_b$, the two circles will merge into one full circle.

$$
\begin{aligned}
\Phi^{IK} =& 0 \\
\implies \tfrac{d\Phi}{dt} =& \Phi_{q_i}\tfrac{dq_i}{dt} + \Phi_P\tfrac{dP}{dt} = 0 && (14) \\
\implies \tfrac{d^2\Phi}{dt^2} =& ([\Phi_{q_i}\tfrac{dq_i}{dt}]_{q_i}\tfrac{dq_i}{dt} + [\Phi_{q_i}\tfrac{dq_i}{dt}]_P\tfrac{dP}{dt}) + \Phi_{q_i}\tfrac{d^2q_i}{dt^2} + \\
& ([\Phi_P\tfrac{dP}{dt}]_{q_i}\tfrac{dq_i}{dt} + [\Phi_P\tfrac{dP}{dt}]_P\tfrac{dP}{dt}) + \Phi_P\tfrac{d^2P}{dt^2} \quad = 0 && (15) \\
\therefore \tfrac{d^2q_i}{dt^2} =& -\Phi_{q_i}^{-1}(\Phi_{q_iq_i}\tfrac{dq_i}{dt}^2 + \Phi_{PP}\tfrac{dP}{dt}^2 + 2\Phi_{q_iP}\tfrac{dq_i}{dt}\tfrac{dP}{dt} + \Phi_P\tfrac{d^2P}{dt}) && (16)
\end{aligned}
$$

2.4 Kinetics

The Lagrange multiplier method has been used to calculate the forces. See [1] for more detail.

The primary equation is:

$$M\ddot{q} + \Phi_q^T\lambda = Q_A \tag{17}$$

Where Q_A are the applied forces and λ the Lagrange multipliers. The λ can be solved for directly if Φ_q has full rank, which will be the case if it kinematically driven. Otherwise, Φ_q will have more columns for co-ordinates q than rows for constraint equations, and will not be invertible. They must then be solved for simultaneously with the accelerations:

$$\begin{bmatrix} M & \Phi_q^T \\ \Phi_q & 0 \end{bmatrix} \begin{bmatrix} \ddot{q} \\ \lambda \end{bmatrix} = \begin{bmatrix} Q_A \\ \gamma \end{bmatrix} \tag{18}$$

See the appendix for more detail.

Each λ can be related to constraint forces at a particular point P from a centroid with co-ordinates (X, Y, Θ) with:

$$
\begin{aligned}
F_{P,x} &= -\Phi_{P,X}^T\lambda_P \\
F_{P,y} &= -\Phi_{P,Y}^T\lambda_P \\
M_P &= -\Phi_{P,\Theta}^T\lambda_P + y_{P/Y} - F_{P,x} - x_{P/X}F_{P,y}
\end{aligned} \tag{19}
$$

Where $\Phi_{P,\Theta}$ is the Jacobian of the relevant constraints at P with respect to (X, Y, Θ). Equivalently, it is the sub-matrix of Φ_q with the same indices as the relevant constraint equations and co-ordinates.

The constraint forces can also be found by solving:

$$\begin{aligned} m_u\ddot{x}_u &= F_{A,x} + F_{B,x} \\ m_u\ddot{y}_u &= F_{A,y} + F_{B,y} - m_u g \\ I_u\ddot{\theta}_u &= -(F_{B,x} - F_{A,x})\frac{1}{2}L_u sin(\theta_U) + (F_{B,y} - F_{A,y})\frac{1}{2}L_u cos(\theta_U) + M_a \\ m_l\ddot{x}_l &= -F_{B,x} + F_{C,x} \\ m_l\ddot{y}_l &= -F_{B,y} + F_{C,y} - m_l g \end{aligned} \tag{20}$$

References: [1] [2]

3 Maths for 3D motion

3.1 Definitions

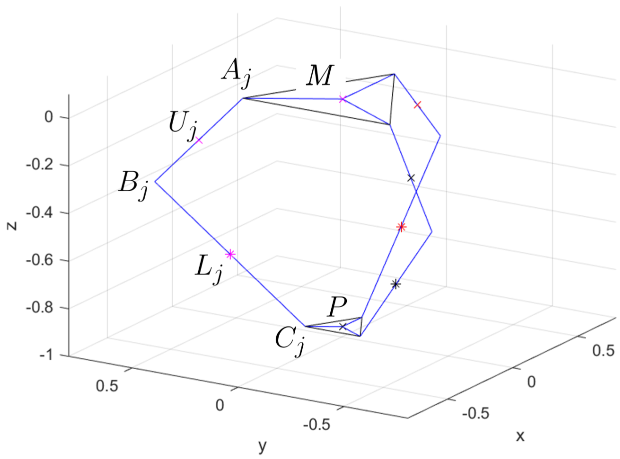

Figure 3: Initial configuration of the 3D Delta robot.

Independent and dependent co-ordinates

$$q_i = \begin{bmatrix} (\theta_{U,j})_x \\ (\theta_{U,j})_y \\ (\theta_{U,j})_z \\ \vdots \end{bmatrix} \qquad q_a = \begin{bmatrix} \theta_{a,1} \\ \theta_{a,2} \\ \theta_{a,3} \end{bmatrix} \tag{21}$$

Rotation Matrices. $R_{A,j}$ and $R_{U,j}$ are calculated using Rodriguez's formula:

$$R(n, \theta) = I + \tilde{n} sin\theta + \tilde{n}^2(1 - cos\theta), \; |n| = 1 \tag{22}$$

$$\begin{aligned} R_A &= R([0,0,1]^T, \psi_0) \\ &= \begin{bmatrix} cos\psi_0 & -sin\psi_0 & 0 \\ sin\psi_0 & cos\psi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \tag{23}$$

$$\begin{aligned} R_U &= R_A R([0,1,0]^T, \theta_a) \\ &= \begin{bmatrix} cos\theta_a cos\psi_0 & -sin\psi_0 & cos\psi_0 sin\theta_a \\ cos\theta_a sin\psi_0 & cos\psi_0 & sin\psi_0 sin\theta_a \\ -sin\theta_a & 0 & cos\theta_a \end{bmatrix} \end{aligned} \tag{24}$$

Relation between actuator co-ordinates and independent co-ordinates.

$$\begin{aligned}
\omega_a^{O,U} &= R_U^O \omega_a^{U,U} = R_U^O \begin{bmatrix} 0 \\ \omega_a \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -sin(\psi_0)\omega_a \\ cos(\psi_0)\omega_a \\ 0 \end{bmatrix} \\
\implies \int \omega_a^{O,U} dt &= \int R_U^O \omega_a^{U,U} dt \\
\therefore \theta_U - \theta_{U,0} &= \begin{bmatrix} -sin(\psi_0)(\theta_a - \theta_{a,0}) \\ cos(\psi_0)(\theta_a - \theta_{a,0}) \\ \psi_0 \end{bmatrix}
\end{aligned} \tag{25}$$

Constraint equations for the position

$$\Phi = \begin{bmatrix} M + R_{A,j}\begin{bmatrix} \frac{L_b}{2} \\ 0 \\ 0 \end{bmatrix} - U_j - R_{U,j}\begin{bmatrix} 0 \\ 0 \\ \frac{L_u}{2} \end{bmatrix} \\ U_j + R_{U,j}\begin{bmatrix} 0 \\ 0 \\ -\frac{L_u}{2} \end{bmatrix} - L_j - R_{L,j}\begin{bmatrix} 0 \\ 0 \\ \frac{L_l}{2} \end{bmatrix} \\ L_j + R_{L,j}\begin{bmatrix} 0 \\ 0 \\ \frac{-L_l}{2} \end{bmatrix} - P - R_{A,j}\begin{bmatrix} \frac{L_e}{2} \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} R_{L,j}^T (R_{L,j})_0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \vdots \end{bmatrix}, \ j = 1, 2, 3 \tag{26}$$

$$\Phi^{driving} = \begin{bmatrix} (\theta_{U,j})_x + sin((\theta_{a,j})_0)\omega_j t - 0 \\ (\theta_{U,j})_y - cos((\theta_{a,j})_0)\omega_j t - (\theta_{a,j})_0 \\ (\theta_{U,j})_z - (\theta_{U,j})_{z,0} \\ \vdots \end{bmatrix} \tag{27}$$

3.2 Kinematics

The following is true for all rotation matrices:

$$RR^T = I \tag{28}$$

$$\implies \dot{R}R^T + R\dot{R^T} = 0$$

$$\therefore \dot{R}R^T = -R\dot{R^T} = \tilde{\omega} \tag{29}$$

Written with reference frame superscripts and subscripts:

$$\dot{R}_C^O = \tilde{\omega}^O R_C^O \tag{30}$$
$$= R_C^O \tilde{\omega}^C \tag{31}$$

The velocity and acceleration for a point on a rigid body is then:

$$
\begin{aligned}
r_P^{O,O} &= r_C^{O,O} + R_C^O r_P^{C,C} && (32)\\
\implies v_P^{O,O} &= v_C^{O,O} + \dot{R}_C^O r_P^{C,C} \\
&= v_C^{O,O} + \tilde{\omega}_C^{O,O} R_C^O r_P^{C,C} && (33)\\
\implies a_P^{O,O} &= a_C^{O,O} + \tilde{\alpha}_C^{O,O} R_C^O r_P^{C,C} + \tilde{\omega}_C^{O,O} \dot{R}_C^O r_P^{C,C} \\
&= a_C^{O,O} + \tilde{\alpha}_C^{O,O} R_C^O r_P^{C,C} + \tilde{\omega}_C^{O,O} \tilde{\omega}_C^{O,O} R_C^O r_P^{C,C} && (34)
\end{aligned}
$$

The following identity is useful:

$$
\begin{aligned}
\tilde{\omega}^O R_C^O r^C &= \tilde{\omega}^O r^O \\
&= (\tilde{r}^O)^T \omega^O && \text{property of anti-symmetric matrices} \\
&= R_C^O (\tilde{r}^C)^T R_C^O \omega^O && \text{equations 30 and 31}
\end{aligned}
$$

$\frac{d\Phi}{dt}$ can be found using equations 33 and 34 and can then be compared with equations 8 and 11:

$$
\begin{aligned}
\frac{d\Phi}{dt} &= \begin{bmatrix}
I & -I & -R_{U,j}(\tilde{r}_A^{U,U})^T R_{U,j}^T & 0 & 0 & \cdots \\
0 & I & +R_{U,j}(\tilde{r}_B^{U,U})^T R_{U,j}^T & -I & -R_{L,j}(\tilde{r}_L^{L,B})^T R_{L,j}^T & \cdots \\
0 & 0 & 0 & I & +R_{L,j}(\tilde{r}_L^{L,C})^T R_{L,j}^T & \cdots \\
0 & 0 & 0 & 0 & e_y^0 R_{L,j}^T \tilde{e}_x^U & \cdots \\
 & & & \vdots & &
\end{bmatrix}
\begin{bmatrix} \dot{M} \\ \dot{U}_j \\ \omega_{U,j} \\ \dot{L}_J \\ \omega_{L,j} \\ \vdots \end{bmatrix} \\
&= \Phi_q \frac{dq}{dt}
\end{aligned} \tag{35}
$$

Similarly, for the accelerations

$$
\frac{d^2\Phi}{dt^2} = \begin{bmatrix} I & 0 & -R_{U,j}\tilde{r}_A^{U,U}R_{U,j}^T & 0 & 0 & \dots \\ 0 & I & +R_{U,j}\tilde{r}_B^{U,U}R_{U,j}^T & -I & -R_{L,j}\tilde{r}_L^{L,B}R_{L,j}^T & \dots \\ 0 & 0 & 0 & I & +R_{L,j}\tilde{r}_L^{L,C}R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & e_y^0 R_{L,j}^T \tilde{e}_x^U & \dots \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} \ddot{M} \\ \ddot{U}_j \\ \alpha_{U,j} \\ \dot{L}_J \\ \alpha_{L,j} \\ \vdots \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & 0 & -\tilde{\omega}_U^{O,O}R_{U,j}\tilde{r}_A^{U,U}R_{U,j}^T & 0 & 0 & \dots \\ 0 & 0 & +\tilde{\omega}_U^{O,O}R_{U,j}\tilde{r}_B^{U,U}R_{U,j}^T & 0 & -\tilde{\omega}_L^{O,O}R_{L,j}\tilde{r}_L^{L,B}R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & +\tilde{\omega}_L^{O,O}R_{L,j}\tilde{r}_L^{L,C}R_{L,j}^T & \dots \\ 0 & 0 & 0 & 0 & e_y^0(\tilde{\omega}_L^{O,O}R_{L,j})^T\tilde{e}_x^U & \dots \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} \dot{M} \\ \dot{U}_j \\ \omega_{U,j} \\ \dot{L}_J \\ \omega_{L,j} \\ \vdots \end{bmatrix}
$$

$$
= \Phi_q\ddot{q} + [\Phi_q \tfrac{dq}{dt}]_q \tfrac{dq}{dt}
$$

$$
= \Phi_q\ddot{q} - \gamma \tag{36}
$$

3.3 Inverse kinematics

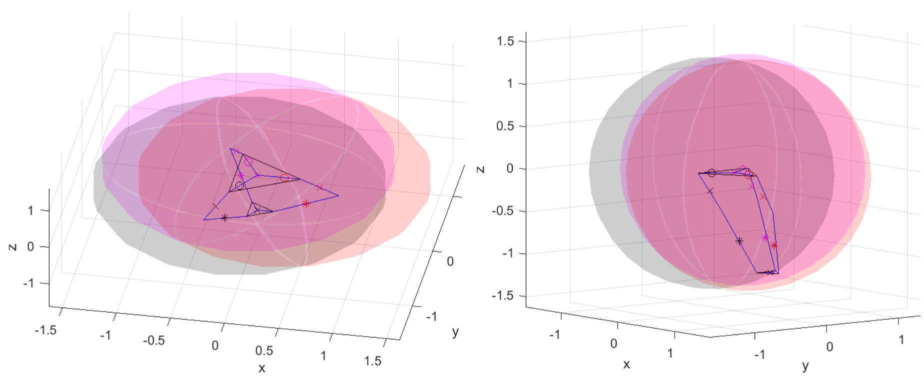

Figure 4: Boundaries of the 3D Delta robot. The robot is bounded to the intersection of the 3 spheres, which is the dark shaded region.

Add the first 3 rows in equation 26:

$$
r = R_L \begin{bmatrix} 0 \\ 0 \\ L_l \end{bmatrix} = R_A \begin{bmatrix} \frac{L_b - L_e}{2} \\ 0 \\ 0 \end{bmatrix} - R_U \begin{bmatrix} 0 \\ 0 \\ L_u \end{bmatrix} - P \tag{37}
$$

$$
\therefore r^T r = r^T R^T R r
$$

$$\implies L_l^2 = \left(\left(\tfrac{1}{2}(L_b - L_e) - L_u sin\theta_a\right) cos\psi_0 - P_x\right)^2 + \\ \left(\left(\tfrac{1}{2}(L_b - L_e) - L_u sin\theta_a\right) sin\psi_0 - P_y\right)^2 + (-L_u cos\theta_a - P_z)^2 \tag{38}$$

As with the 2D case in section 2.3, the boundary is defined by the case where 1 arm is fully extended, $R_L = R_U$:

$$P = \begin{bmatrix} \left(\frac{1}{2}(L_b - L_e) - (L_u + L_l)sin\theta_a\right) cos\psi_0 \\ \left(\frac{1}{2}(L_b - L_e) - (L_u + L_l)sin\theta_a\right) sin\psi_0 \\ -(L_u + L_l)cos\theta_a \end{bmatrix} \tag{39}$$

This is a sphere. Eliminating θ_a shows this more clearly:

$$(P_x - \tfrac{1}{2}(L_b - L_e)cos\psi_0)^2 + (P_y - \tfrac{1}{2}(L_b - L_e)sin\psi_0)^2 + P_z^2 = (L_u + L_l)^2 \tag{40}$$

The delta robot is confined to the intersection of the 3 spheres.

For the relationship of the end effector velocity and acceleration, a similar derivation for equation 16 for the 2D case is used:

$$\begin{aligned} \frac{d^2q_a}{dt^2} &= -\Phi_{q_a}^{-1}([\Phi_{q_a}\frac{dq_a}{dt}]_{q_a}\frac{dq_a}{dt} + [\Phi_P\frac{dP}{dt}]_P\frac{dP}{dt} + 2[\Phi_{q_a}\frac{dq_a}{dt}]_P\frac{dP}{dt} + \Phi_P\frac{d^2P}{dt^2}) \\ &= -\Phi_{q_a}^{-1}(\Phi_{q_aq_a}\frac{dq_a}{dt}^2 + \Phi_{PP}\frac{dP}{dt}^2 + 2\Phi_{q_aP}\frac{dq_a}{dt}\frac{dP}{dt} + \Phi_P\frac{d^2P}{dt^2}) \\ &= \quad \Phi_{q_a}^{-1}\gamma^{IK} \end{aligned} \tag{41}$$

3.4 Kinetics

The 3D equations of motion for a centre of mass C are:

$$\begin{aligned} &ma_C^{O,O} = F^O \\ &I^C\alpha_C^{O,O} + \tilde{\omega}_C^{C,O}I^C\omega_C^{C,O} = \tau^C \end{aligned} \tag{42}$$

Using equation 30, the moments can also be expressed as:

$$I^C R_O^C \alpha_C^{O,O} + \tilde{\omega}_C^{O,O} R_C^O I^C R_O^C \omega_C^{O,O} = \tau^O$$

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} - \begin{bmatrix} (I_y - I_z)\omega_y\omega_z \\ (I_z - I_x)\omega_x\omega_z \\ (I_x - I_y)\omega_x\omega_y \end{bmatrix} = \tau^C$$

The equations of motion given by 17 and 18 can still be used, as long as Q_A takes into account the extra gyroscopic forces as well.

Finally, the constraint forces can also be found by solving:

$$\begin{aligned} &m_U a_U^{O,O} = F_A + F_B + Q_{A,U} \\ &I_U\alpha_C^{O,O} + \tilde{\omega}_C^{C,O}I_U\omega_C^{C,O} = \tilde{r}_A^{O,U}R_O^C F_A + \tilde{r}_B^{O,U}R_O^C F_B + \tau_i^U \\ &m_L a_L^{O,O} = -F_B + F_C + Q_{A,L} \end{aligned} \tag{43}$$

References: [1] [2]

References

[1] J. Schilder, *Dynamics 3*. University of Twente, 2018.

[2] R. C. Hibbeler, *Engineering Mechanics: Dynamics*, 13th ed. Prentice Hall, 2013, Chapters 16-17 and 20-21.