PHYSICS 340

Majors Electricity & Magnetism

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ELECTROSTATICS

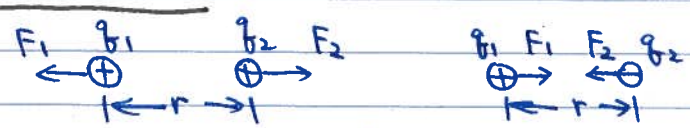
1. The Electric Field

– Coulomb's law

* Electric charge

- Charge is quantized, comes in integer multiples of the elementary charge
$$e \simeq 1.602 \times 10^{-19}\,C$$
- The proton has a charge of $+e$, and the electron has a charge of $-e$.
- We can study the source charges (q_i) through their forces acting on another test charge (Q).
- In general, both the source charges and the test charge are in motion.
  To begin with, we shall consider the special case of electrostatics in which all the source charges are stationary.

* Coulomb's law



$$|\vec{F}_1| = |\vec{F}_2| = k_e \frac{|q_1||q_2|}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{|q_1||q_2|}{r^2}$$

- The force on a point charge q_2 due to another q_1 (at rest) based on experiments is
$$|\vec{F}| = \frac{1}{4\pi\epsilon_0}\frac{|q_1||q_2|}{r^2}$$
- k_e is Coulomb's constant
$$k_e \simeq 8.988 \times 10^9\ N \cdot m^2 \cdot C^{-2}$$

ϵ_0 is the permittivity of free space
$$\epsilon_0 \simeq 8.85 \times 10^{-12}\ \frac{C^2}{N \cdot m^2}$$

- Considering both the magnitude and the direction of vectorial quantities, Coulomb's law can be written

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{R}|^2} \hat{\vec{R}}}$$

$\vec{R}$ is the separation vector from the source charge to the test charge

- The force is along the straight line joining the two charges. If the charges have the same (different) sign(s), the electrostatic force between them is repulsive (attractive).

– The electric field

- We introduce the test charge to manifest the "intrinsic properties" of the source charge by measuring the electric force exerting on the test charge.
- To remove the reference to the test ~~particle~~ charge Q, we can introduce the <u>electric field</u> of the source charge q:

$$\vec{F} \equiv Q\vec{E} = Q\left(\frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{R}|^2} \hat{\vec{R}}\right)$$

where $$\boxed{\vec{E} \equiv \frac{1}{4\pi\epsilon_0} \frac{q}{R^2} \hat{\vec{R}}}.$$

- Note that the electric field is the "physical field" surrounding electric charges, whether there is any test charges or not. It is a physical entity which exerts force on all ~~to~~ other charged particles in the field. That's why electric force is not "action at a distance".

- The electric field is defined as <u>a vector field</u> that associates to each point in space the Coulomb force per unit of charge exerted on an infinitesimal positive test charge at rest at that point. The derived SI unit for the electric field is <u>newton per coulomb (N/C)</u>

– Principle of superposition

* The electric field of a point charge distribution

- If we have several point charges $q_1, q_2, \dots q_n$, at distances $R_1, R_2, \dots R_n$ from Q, the total force is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \dots + \vec{F}_n = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{R_1^2}\hat{\vec{R}}_1 + \frac{q_2 Q}{R_2^2}\hat{\vec{R}}_2 + \dots\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{R_1^2}\hat{\vec{R}}_1 + \frac{q_2}{R_2^2}\hat{\vec{R}}_2 + \dots\right)$$

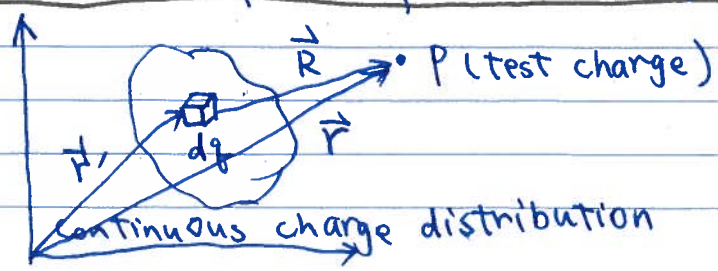
thus the total electric field of the source charges is

- $$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i^2}\hat{\vec{R}}_i$$

* The principle of superposition

- the total electric field, at a point, due to a collection of charges is equal to the vector sum of the electric fields at that point due to the individual charges.
- this is also true for the electric force

* The electric field of a continuous charge distribution,



- Applying the principle of superposition, we can get the total electric field by discretizing the continuous charge distributions into small elements

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{1}{R_i} \hat{\vec{R}}\, dq(\vec{r}\,')$$

$$\xrightarrow{n\to\infty} \frac{1}{4\pi\epsilon_0} \int \frac{1}{R^2} \hat{\vec{R}}\, dq(\vec{r}\,')$$

Warning: • $\vec{R}$ is the vector from dq to the field point $\vec{r}$.
dq is the "~~test particle~~" "source charges" at the source point $\vec{r}\,'$ (thus the notation $dq(\vec{r}\,')$)
• the unit vector $\hat{\vec{R}}$ is not constant, whose direction depends on the source point ($\vec{r}\,'$). Thus $\hat{\vec{R}}$ cannot be taken outside the integrals.

- If the charge is spread out along a line, or smeared out over a surface, or fills a volume then the element

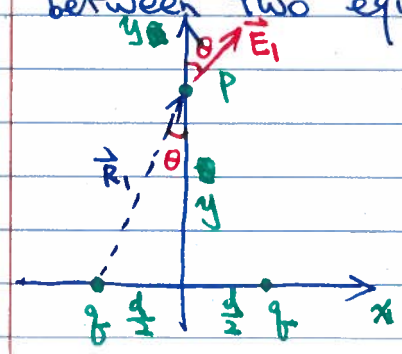
$$dq \to \lambda dl' \sim \sigma da' \sim \rho d\tau'$$

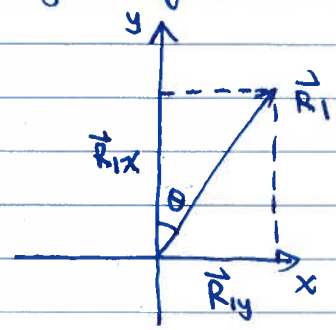
with λ charge-per-unit-length, dl' an element of length along the line
σ charge-per-unit-area, da' an element of area on the surface
ρ charge-per-unit-volumn, $d\tau'$ an element of volumn.

- For the most general and realistic case of a volum charge

$$\boxed{\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{R^2} \hat{\vec{R}}\, d\tau'}$$

Example 1.1 Find the electric field a distance y above the midpoint between two equal charges (q), a distance d apart





Let's start with the electric field of q on the left

$$\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\hat{\vec{R}}_1$$

Now we decompose $\vec{E}_1$ into components in x- and y-directions

- we can write it as

$$\vec{E}_1 = E_{1x}\hat{\vec{x}} + E_{1y}\hat{\vec{y}} = \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\hat{\vec{R}}_1$$

Here we make it a bit complicated to review vector analysis!

considering $\hat{\vec{x}}\cdot\hat{\vec{x}} = 1$, $\hat{\vec{y}}\cdot\hat{\vec{y}} = 1$, $\hat{\vec{x}}\cdot\hat{\vec{R}}_1 = \cos(\frac{\pi}{2}-\theta) = \sin\theta$

$\hat{\vec{y}}\cdot\hat{\vec{R}}_1 = \cos\theta$

we can get $E_{1x} = \hat{\vec{x}}\cdot\vec{E}_1 = E_{1x}\,\hat{\vec{x}}\cdot\hat{\vec{x}}$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\hat{\vec{x}}\cdot\hat{\vec{R}}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\sin\theta$$

$$E_{1y} = \hat{\vec{y}}\cdot\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\hat{\vec{y}}\cdot\hat{\vec{R}}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\cos\theta$$

Thus $\vec{E}_1 = \underbrace{\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\sin\theta\right)}_{\to E_{1x}}\hat{\vec{x}} + \underbrace{\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\cos\theta\right)}_{\to E_{1y}}\hat{\vec{y}}$

- In a different way, we can decompose $\vec{R}_1$, similarly

$$\vec{R}_1 = R_{1x}\hat{\vec{x}} + R_{1y}\hat{\vec{y}} = (R_1\sin\theta)\hat{\vec{x}} + (R_1\cos\theta)\hat{\vec{y}}$$

then $\vec{E}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\hat{\vec{R}}_1 = \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\frac{\vec{R}_1}{R_1}$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\frac{1}{R_1}\left[(R_1\sin\theta)\hat{\vec{x}} + (R_1\cos\theta)\hat{\vec{y}}\right]$$

$$= \left(\frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\sin\theta\right)\hat{\vec{x}} + \left(\frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\cos\theta\right)\hat{\vec{y}}$$

Very similarly, we can get the electric field of the right charge $\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{q}{R_2^2}\hat{\vec{R}}_2$

$$= -\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\sin\theta\right)\hat{\vec{x}} + \left(\frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\cos\theta\right)\hat{\vec{y}}$$

So the net electric field is given by their vectorial sum

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = 2\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R_1^2}\cos\theta\right)\hat{\vec{y}}$$

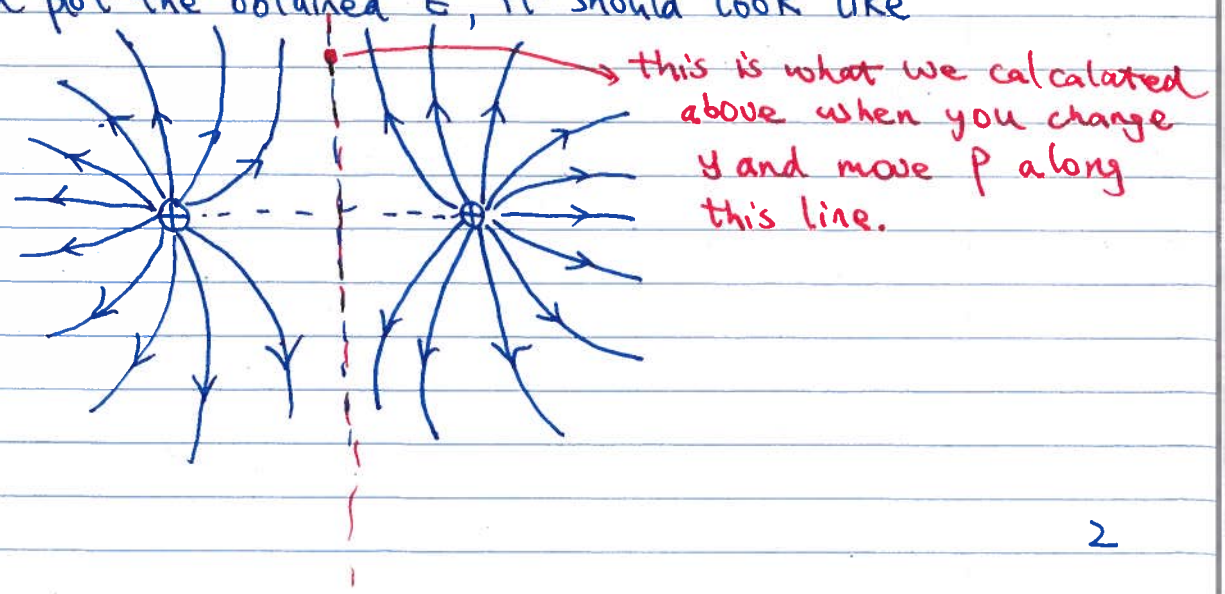
Since $R_1 = \sqrt{y^2+(d/2)^2}$, and $\cos\theta = y/R_1$, we get

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2qy}{R_1^3}\hat{\vec{y}} = \frac{1}{4\pi\epsilon_0}\frac{2qy}{[y^2+(d/2)^2]^{3/2}}\hat{\vec{y}}$$

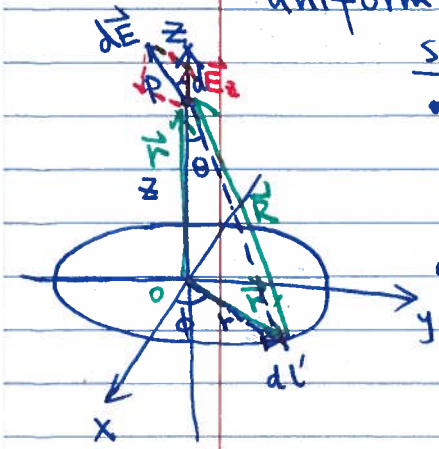
We see that when $y \gg d$, where P is far away from the charges and thus they look like a single charge $2q$, the field does reduce to

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{2q}{y^2}\hat{\vec{y}}$$

- If you calculat $\vec{E}$ for an arbitary point P in the x-y plane, and plot the obtained $\vec{E}$, it should look like



Example 1.2 Find the electric field a distance z above the center of a ~~flat~~ circular loop of radius r that carries a uniform line charge λ.



Solution

- Since the horizontal components will cancel, we only need to consider the component in z of the electric field.
- First let's ~~calculate~~ write down the equation to use

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec{r}')}{R^2}\,\hat{R}\,dl'$$

then we rewrite it more explicitly using the coordinate system we choose, for the element dl'.

This is R^2

- We denote the electric field of the element dl' as $d\vec{E}$,

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\lambda}{z^2+r^2}\,\hat{R}\cdot \underbrace{r\,d\phi}_{=dl'}$$

this notation just means it's the field of an element dl'

magnitude of $d\vec{E}$

unit vector of $d\vec{E}_z$

its z-component is

$$d\vec{E}_z = |d\vec{E}|\cdot\cos\theta\cdot\hat{z} = |d\vec{E}|\cdot\frac{z}{R}\cdot\hat{z} = |d\vec{E}|\frac{z}{\sqrt{z^2+r^2}}\cdot\hat{z}$$

- Then the total electric field is

this notation just means it's the z-component of $d\vec{E}$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda}{z^2+r^2}\,r\,\frac{z}{\sqrt{z^2+r^2}}\,\hat{z}\,d\phi$$

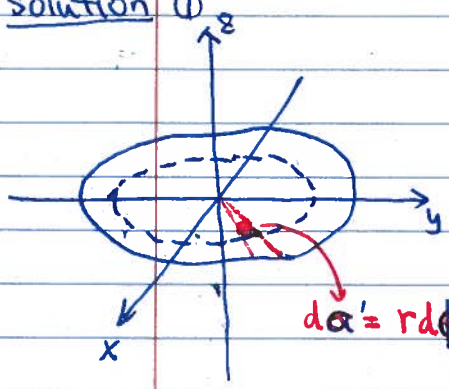
$$= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{\lambda r z}{\left(\sqrt{z^2+r^2}\right)^3}\,d\phi\,\hat{z}$$

constant vector, brought outside the integral.

$$= \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi)rz}{(z^2+r^2)^{3/2}}\,\hat{z}$$

Example 1.3. Find the electric field a distance z above the center of a flat circular disk of radius R that carries a uniform surface charge σ.

solution ①



Following the same method as in the case of the ring, the field of a small surface element is

$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\sigma\, da'}{z^2+r^2}\hat{\vec{R}}$$

$da' = r\,d\phi\,dr$ and its z-component is

$$d\vec{E}_z = \frac{1}{4\pi\epsilon_0}\frac{\sigma\, r\,d\phi\,dr}{z^2+r^2}\frac{z}{\sqrt{z^2+r^2}}\hat{\vec{z}}$$

The total electric field is

$$\vec{E} = \int d\vec{E}_z = \frac{1}{4\pi\epsilon_0}\int_0^R\int_0^{2\pi}\frac{\sigma z r}{(z^2+r^2)^{3/2}}\,d\phi\,dr\,\hat{\vec{z}}$$

$$= \boxed{\frac{1}{4\pi\epsilon_0}\int_0^R\frac{\sigma z(2\pi r)}{(z^2+r^2)^{3/2}}\,dr\,\hat{\vec{z}}}$$

$$= \frac{2\pi\sigma z}{4\pi\epsilon_0}\int_0^R\frac{r}{(z^2+r^2)^{3/2}}\,dr\,\hat{\vec{z}}$$

$$= \frac{2\pi\sigma z}{4\pi\epsilon_0}\left[\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right]\hat{\vec{z}}$$

solution ② we can break the disk into circular rings, and use the results we obtained for the ring. Let's say the ring has radius r, and thickness dr.

From Example 1.2, we know

$$\vec{E}_{ring} = \frac{1}{4\pi\epsilon_0}\frac{\lambda(2\pi r)\,z}{(z^2+r^2)^{3/2}}\hat{\vec{z}}$$

In the current situation, $\lambda = \frac{2\pi r\,dr\cdot\sigma}{2\pi r} = \sigma\,dr$

See how they are related.

Thus $\vec{E}_{ring} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{\sigma(2\pi r)z\,dr}{(z^2+r^2)^{3/2}}\hat{\vec{z}}}$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int_0^R\frac{\sigma(2\pi r)z}{(z^2+r^2)^{3/2}}\,dr\,\hat{\vec{z}} = \text{same as above}$$

Since $\vec{E} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}} \right] \hat{z}$

- For $R \gg z$ the second term $\to 0$, which gives the field of an infinite plate

$$E_{plane} = \frac{1}{4\pi\epsilon_0} 2\pi\sigma \hat{z} = \boxed{\frac{\sigma}{2\epsilon_0} \hat{z}}$$

- For $z \gg R$, $\frac{1}{\sqrt{R^2+z^2}} = \frac{1}{z}\left(1+\frac{R^2}{z^2}\right)^{-\frac{1}{2}} \approx \frac{1}{z}\left(1-\frac{1}{2}\frac{R^2}{z^2}\right)$

So $\vec{E} \to \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left[\frac{1}{z} - \frac{1}{z}\left(1-\frac{1}{2}\frac{R^2}{z^2}\right)\right] \hat{z}$

$$= \frac{1}{4\pi\epsilon_0} 2\pi\sigma z \left(\frac{1}{2}\frac{R^2}{z^3}\right) \hat{z}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{\pi\sigma R^2}{z^2} \hat{z}$$

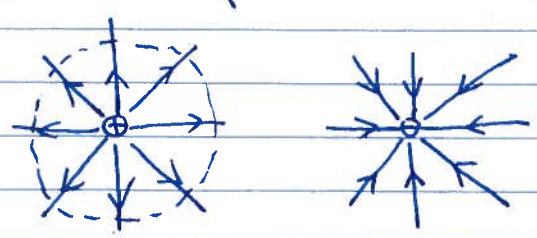
In other words, the field looks like from a point charge of $(\pi R^2 \sigma)$, which makes sense!

2. Gauss's Law

— Field lines and the flux of $\vec{E}$

* Field lines

- A vector field defines a direction and magnitude at each point in space. A field line is a graphical method for visualising vector fields. A field line for a vector field can be constructed by starting at a point and tracing a line through space that follows the direction of the vector field.
- An individual field line shows the direction of the vector field but not the magnitude. In order to depict the magnitude of the field, ~~to~~ one can draw a representative set of neighboring field lines in such a way that each line represents the same quantity of flux. Then the density of field lines (number of field lines per unit perpendicular area) at any location is proportional to the magnitude of the vector field at that point.



- It's important to represent the field-line diagram in three dimensions. In 3-D, the density of lines passing through a sphere of radius r is

$$\frac{n}{4\pi r^2} \propto \frac{1}{r^2}$$

which is consistent with ~~Colum~~ Coulomb's Law. (This example is for a single point charge.)

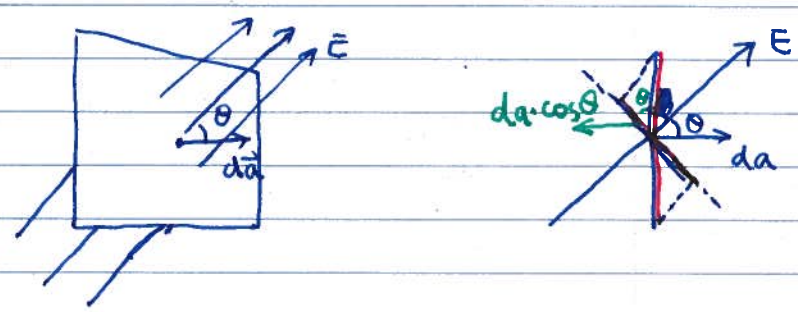
- Field lines begin on positive charges and end on negative ones. They may extend out to infinity. Field lines can never cross.

* The flux of $\vec{E}$

- The flux of $\vec{E}$ through a surface S is defined as

$$\boxed{\Phi_E \equiv \int_S \vec{E}\cdot d\vec{a}}$$ (surface integrals on P26).

Here $d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface.



$$\vec{E}\cdot d\vec{a} = |\vec{E}||d\vec{a}|\cos\theta = |\vec{E}|\cdot(\text{component of } d\vec{a} \text{ along } \vec{E})$$
$$= |\vec{E}|\cdot(\text{area in the plane perpendicular to } \vec{E})$$

- For a given sampling rate of the field lines, since the density of field lines is proportional to the field strength $|\vec{E}|$, the flux should be proportional to the number of lines drawn.

$$\Phi_E \equiv \int_S \vec{E}\cdot d\vec{a} = \int_S |\vec{E}|\cdot(\text{area perpendicular to } \vec{E})$$

Φ_E → number of field lines

$|\vec{E}|$ → density of field lines (number per unit area)

(area perpendicular to $\vec{E}$) → area $\perp \vec{E}$

(Note that the "density of field lines" was defined as number of field lines per unit perpendicular area.)

- On the other hand, because $|\vec{E}|$ is proportional to the charge of the source, we can guess that the flux ~~of~~ through any "closed surface" is a measure of the total charge inside.

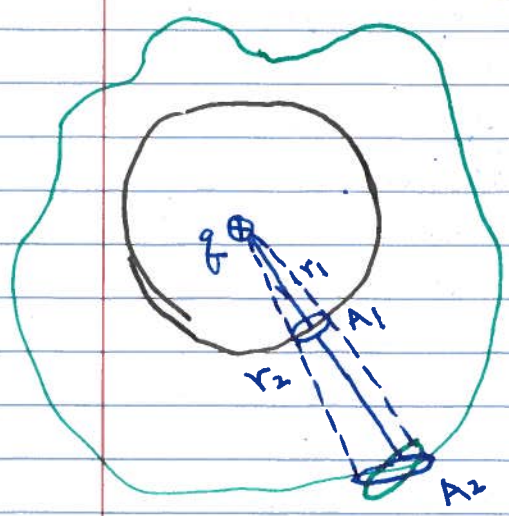
* Gauss's Law (integral form)

- In the case of a point charge q at the origin, the flux of $\vec{E}$ through a surface of radius r is

$$\oint \vec{E}\cdot d\vec{a} = \int \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r^2}\hat{r}\right)\cdot\left(r^2\sin\theta\, d\theta\, d\phi\, \hat{r}\right) = \frac{1}{\epsilon_0} q.$$

Note that the results don't depend on the radius we choose. In fact this relationship doesn't depend on the closed surface we pick, i.e., the flux through any surface enclosing the charge is q/ϵ_0.

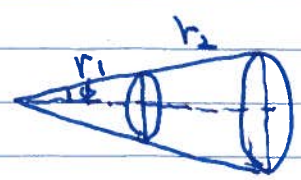
- This makes good sense, in terms of the field-line picture; since the same number of field lines pass through any closed surface.
On the other hand, a charge outside the surface will contribute nothing to the total flux, since its field lines pass in one side and out the other.
- For an arbitary surface?



all we have to prove is that the flux through A_1 is the same as A_2.

- at A_1, $\vec{E}\cdot d\vec{a} = \frac{1}{4\pi\epsilon_0}\frac{q}{r_1^2} A_1$
- at A_2, $\vec{E}\cdot d\vec{a} = \frac{1}{4\pi\epsilon_0}\frac{q}{r_2^2} A_2 \cdot \cos\theta$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r_2^2} A_{2\perp}$$

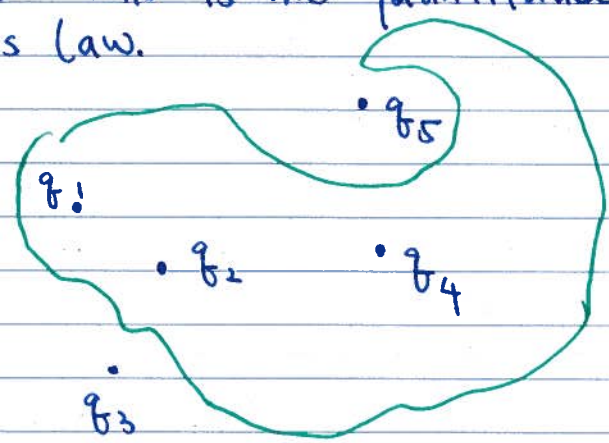


$$\left.\begin{aligned} A_1 &= \pi(r_1\sin\phi)^2 \\ A_{2\perp} &= \pi(r_2\sin\phi)^2 \end{aligned}\right\} \frac{A_1}{r_1^2} = \frac{A_{2\perp}}{r_2^2}$$

- If we have a bunch of charges scattered about, the total flux is

$$\boxed{\oint \vec{E} \cdot d\vec{a} = \sum_{i=1}^{n} \left(\oint \vec{E}_i \cdot d\vec{a} \right) = \sum_{i=1}^{n} \left(\frac{1}{\epsilon_0} q_i \right) = \frac{Q_{enc}}{\epsilon_0}}$$

where Q_{enc} is the total charge enclosed within the surface. This is the quantitative statement of Gauss's Law.



- Note that while the electric flux is not affected by charges that are not within the closed surface, the net electric field $\vec{E}$ in the Gauss's law equation, can be affected by charges that lie outside the closed surface.

— Gauss's Law (differential form)

- we have obtained

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$$

the divergence theorem

$$\int_V (\nabla\cdot\vec{E})\,d\tau = \int_V \rho\, d\tau$$

Since this holds for any volumn, the integrands must be equal:

$$\boxed{\nabla\cdot\vec{E} = \frac{1}{\epsilon_0}\rho}$$

This is Gauss's Law in differential form.

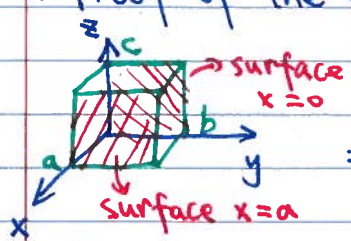
Note: Both sides are functions of spatial coordinates.
~~You ca~~ It describes local relationship between $\vec{E}(x)$ and $\rho(x)$.

- Divergence theorem (P31)

$$\boxed{\int_V (\nabla\cdot\vec{v})\,d\tau = \oint_S \vec{v}\cdot d\vec{a}}$$

- it is also called Gauss's theorem, or Green's theorem.
- it says that the integral of the divergence $(\nabla\cdot\vec{v})$ over a volume V is equal to the value of the function $\vec{v}$ at the boundary, i.e. the surface S that bounds the volumn.

- Proof of the divergence theorem



$$\int_V \vec{\nabla}\cdot\vec{v}\,d\tau = \int_0^c\int_0^b\int_0^a \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\,dy\,dz$$

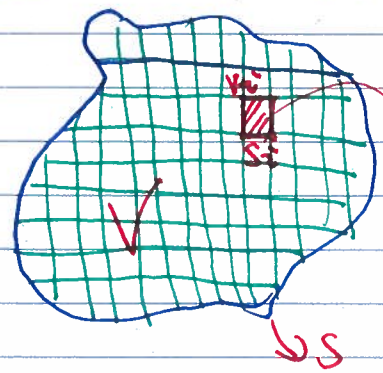
$$= \int_0^c\int_0^b [v_x(a,y,z) - v_x(0,y,z)\,dy\,dz$$

Volume $0\le x\le a$, $0\le y\le b$, $0\le z\le c$

$$+ \int_0^c\int_0^a [v_y(x,b,z) - v_y(x,0,z)\,dx\,dz$$

$$+ \int_0^b\int_0^a [v_z(x,y,c) - v_z(x,y,0)\,dx\,dy$$

- For the surface $x=a$, a surface element is $d\vec{s} = \hat{x}\,dy\,dz$. $\int_0^c \int_0^b v_x(a, y, z)\,dy\,dz$ is apparently the surface integral on this surface $\int_0^c \int_0^b \vec{v}\cdot d\vec{a}$. Similarly, for other surfaces.
- Finally we prove that $\int_V \vec{\nabla}\cdot\vec{v}\,d\tau = \oint_S \vec{v}\cdot d\vec{a}$ for the box.
- To prove this for any arbitary volume, we just need to discretize the volume into small boxes, and apply this to all the boxes.



$$\int_{V_i} (\nabla\cdot\vec{v})\,d\tau = \oint_{S_i} \vec{v}\cdot d\vec{a}$$

$$\sum_i \oint_{S_i} \vec{v}\cdot d\vec{a} = \oint_S \vec{v}\cdot d\vec{a}$$

$$\sum_i \int_{V_i} (\nabla\cdot\vec{v})\,d\tau = \int_V \nabla\cdot\vec{v}\,d\tau$$

- The divergence of $\vec{E}$

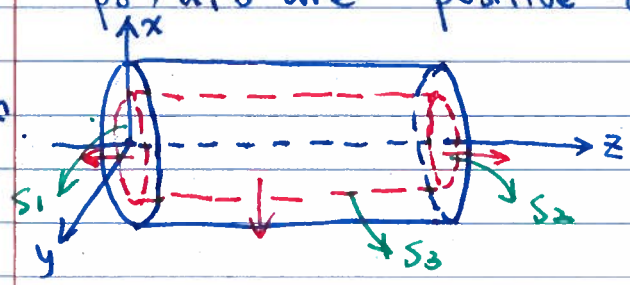
$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\,\rho(\vec{r}\,')\,d\tau' \quad \text{where } \vec{\mathscr{R}} = \vec{r} - \vec{r}\,'$$

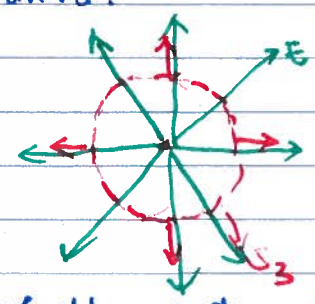
then
$$\nabla\cdot\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \nabla\cdot\left(\frac{\hat{\mathscr{R}}}{\mathscr{R}^2}\right) \rho(\vec{r}\,')\,d\tau'$$
$$= \frac{1}{4\pi\epsilon_0} \int 4\pi\,\delta^3(\vec{r} - \vec{r}\,')\,\rho(\vec{r}\,')\,d\tau'$$
$$= \frac{1}{\epsilon_0}\,\rho(\vec{r}).$$

which is Gauss's law in differential form.

Example 2.1 Determine $\vec{E}$ for an infinitely long cylinder of radius R which has a volume charge density that varies with the radius as $\rho(r) = \rho_0\left(a - \frac{r}{b}\right)$, where ρ_0, a, b are positive constants.

Solution





- First, based on the symmetry of the system, the electric field should be perpendicular to the z-axis, and in the radial direction.

① $r<R$

- We can choose the surface as shown in the plot as the Gaussian surface. $(r<R)$
- Now we apply the Gauss's law

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\epsilon_0}$$

<u>LHS</u> $\oint \vec{E}\cdot d\vec{a} = \int_{S_1} \vec{E}\cdot d\vec{a} + \int_{S_2} \vec{E}\cdot d\vec{a} + \int_{S_3} \vec{E}\cdot d\vec{a}$

On S_1 surface, $d\vec{a}$ is always in $-\hat{z}$, and $\vec{E}$ is in $\hat{r}$ direction, thus $\vec{E}\cdot d\vec{a} = 0$ on S_1

$\int_{S_1} \vec{E}\cdot d\vec{a} = 0$.

Similarly $\int_{S_2} \vec{E}\cdot d\vec{a} = 0$.

On S_3, $\int_{S_3} \vec{E}\cdot d\vec{a} = \int_{S_3} |\vec{E}|\,\hat{r}\cdot(da\,\hat{r}) = \int_{S_3} E\,da$

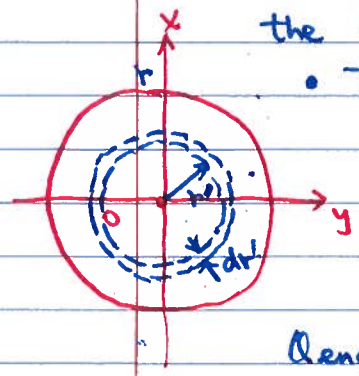
$= E\int_{S_3} da = E\cdot 2\pi r\cdot l$

Thus $\oint \vec{E}\cdot d\vec{a} = E\cdot 2\pi r\cdot l$

RHS • $Q_{enc} = \int \rho(\vec{r}')\, d\tau'$

the volumn of the Gaussian surface.

we need to integrate over the volumn enclosed by the Gaussian surface to get the charge enclosed.



• To do this, we break the cylinder into cylindrical shells.

$$dQ_{enc} = \rho(r')\, 2\pi r' dr' \cdot l$$
$$= l\rho_0 \left(a - \frac{r'}{b}\right) 2\pi r' dr'$$

$$Q_{enc} = \int dQ_{enc} = \int_0^r l\rho_0 \left(a - \frac{r'}{b}\right) 2\pi r' dr'$$
$$= 2\pi l \rho_0 \int_0^r \left(a - \frac{r'}{b}\right) r' dr'$$
$$= 2\pi l \rho_0 \left(\frac{1}{2} a r'^2 - \frac{1}{3b} r'^3\right)\Big|_0^r$$
$$= 2\pi l \rho_0 r^2 \left(\frac{a}{2} - \frac{r}{3b}\right).$$

Thus $E \cdot 2\pi r \cdot l = 2\pi l \rho_0 r^2 \left(\frac{a}{2} - \frac{r}{3b}\right)$

$$\vec{E}(r) = \frac{\rho_0 r}{\epsilon_0}\left(\frac{a}{2} - \frac{r}{3b}\right)\hat{\vec{r}}.$$

② Outside the cylinder.

$r > R$

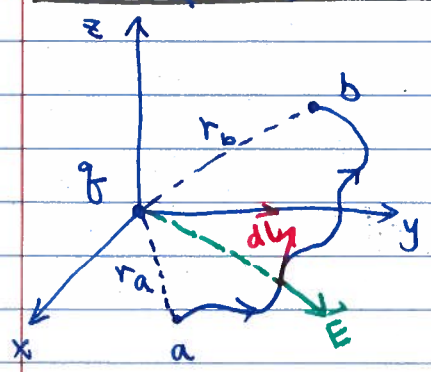
$$Q_{enc} = \int_0^R l\rho_0 \left(a - \frac{r'}{b}\right) 2\pi r' dr'$$
$$= 2\pi l \rho_0 R^2 \left(\frac{a}{2} - \frac{R}{3b}\right).$$

$$\vec{E} = \frac{\rho_0 R^2 \left(\frac{a}{2} - \frac{R}{3b}\right)}{\epsilon_0 r}.$$

3. Electric Potential

— The curl of electric field

* Curl of $\vec{E}$ from a point charge.



This time we put a point charge q at the origin

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

The line element $d\vec{l}$ in spherical coordinates

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

(Page 38).

If we calculate the line integral of $\vec{E}$ from poin $\vec{a}$ to $\vec{b}$ (Page 24).

$$\int_a^b \vec{E}\cdot d\vec{l} = \int_a^b \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr$$

$$= -\frac{1}{4\pi\epsilon_0}\frac{q}{r}\Big|_{r_a}^{r_b} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

Thus the integral around a closed path is zero

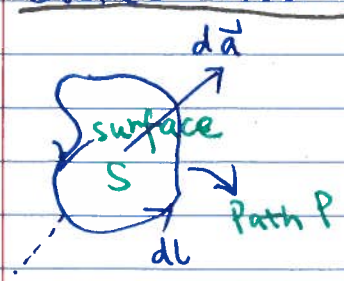
$$\boxed{\oint \vec{E}\cdot d\vec{l} = 0}$$

Hence, applying Stokes' theorem (Page 34).

$$\boxed{\nabla\times\vec{E} = \vec{0}}$$

(Note: Stokes' theorem, $\int_s (\nabla\times\vec{v})\cdot d\vec{a} = \oint_P \vec{v}\cdot d\vec{l}$)

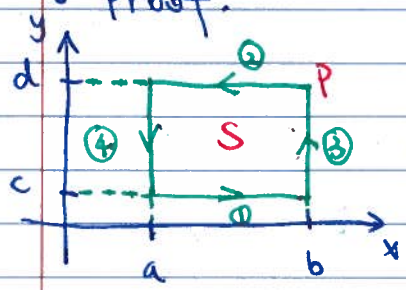
* Stoke's theorem



- The theorem states that the integral of the curl $(\nabla\times\vec{v})$ over a patch of surface (s) is equal to the value of the function $(\vec{v})$ at the boundary, i.e., the perimeter of the patch (P)
- The boundary line and surface follow the right-hand rule: if your fingers point in the ~~line~~ ~~in~~ direction of the line

integral, then your thumb fixed the direction of $d\vec{a}$.

- Note that the theorem indicates that the flux integral of a curl $\int_s (\nabla \times \vec{v}) \cdot d\vec{a}$ doesn't depend on what surface you integrate over, and it depends only on the boundary line.
- Proof:



$$\vec{v} = v_x(x,y)\,\vec{i} + v_y(x,y)\,\vec{j}$$

$$\int_s \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) ds = \int_c^d \int_a^b \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) dx\,dy$$

$$= \int_c^d [\overset{3}{v_y(b,y)} - \overset{4}{v_y(a,y)}]\, dy$$

$$- \int_a^b [\overset{2}{v_x(x,d)} - \overset{1}{v_x(x,c)}]\, dx$$

$$= \oint_c (v_x\, dx + v_y\, dy)$$

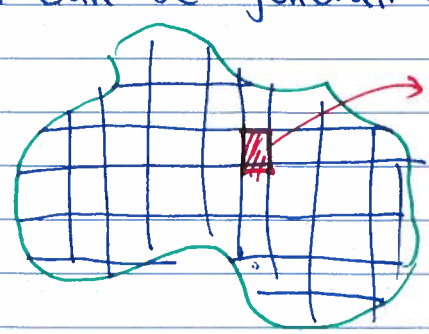
Note that $\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = (\vec{\nabla} \times \vec{v}) \cdot \vec{k}$

$$d\vec{a} = \vec{k}\, da$$

$$v_x\, dx + v_y\, dy = \vec{v} \cdot d\vec{l}$$

thus $\boxed{\int_s (\nabla \times \vec{v}) \cdot d\vec{a} = \oint_p \vec{v} \cdot d\vec{l}}$

This can be generalized to any boundary and surface



Apply the above equation to each grid and then add them up.

* <u>Curl of $\vec{E}$ for any static charge distribution.</u>

- We have proved that $\nabla \times \vec{E} = \vec{0}$ for a point charge at the origin, but it makes no reference to a specific choice of coordinates.
- If we have a static charge distribution, then

$$\nabla \times \vec{E} = \nabla \times (\vec{E}_1 + \vec{E}_2 + \dots) = (\nabla \times \vec{E}_1) + (\nabla \times \vec{E}_2) + \dots = \vec{0}$$

as a result of the principle of superposition.

- The ~~curless~~ curl-free property of $\vec{E}$ indicate that the electric field is a very special kind of vector function; whose three components are ~~ex~~ interrelated.

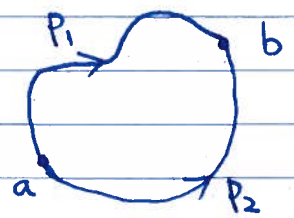
$$\nabla \times \vec{E} = \vec{0} \Rightarrow \frac{\partial E_x}{\partial y} = \frac{\partial E_y}{\partial x}, \quad \frac{\partial E_z}{\partial y} = \frac{\partial E_y}{\partial z}, \quad \frac{\partial E_x}{\partial z} = \frac{\partial E_z}{\partial x}.$$

- Because of this special property, we can reduce a vector problem (finding $\vec{E}$) to a much simpler scalar problem.

— Electric potential

* Defination of potential

- we have shown that the electric field of any static charge distribution is curl free $\nabla \times \vec{E} = \vec{0}$. Hence the line integral of $\vec{E}$ around any closed loop is zero.



For example, around the closed loop P_1 and $(-P_2)$, we have

$$\oint_{P_1 - P_2} \vec{E} \cdot d\vec{l} = 0 = \int_{P_1} \vec{E} \cdot d\vec{l} - \int_{P_2} \vec{E} \cdot d\vec{l}$$

In other words, the line integral of $\vec{E}$, from $\vec{a}$ to $\vec{b}$, is path independent

$$\int_a^b \vec{E} \cdot d\vec{l} = \int_{P_1} \vec{E} \cdot d\vec{l} = \int_{P_2} \vec{E} \cdot d\vec{l} \quad (\text{from } \vec{a} \text{ to } \vec{b}).$$

- Thus we can define a ~~seta~~ scalar function which only depends on the position of $\vec{a}$ and $\vec{b}$

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E} \cdot d\vec{l}$$

Note that (i) the line integral can be calculated along any path from $\vec{a}$ to $\vec{b}$, and (ii) the "−" sign is largely a matter of convention.

- If we make $\vec{a}$ as some standard reference point, $\vec{b}$ as any point in the field, then the electric potential

$$\boxed{V(\vec{r}) = -\int_{\vec{0}}^{\vec{r}} \vec{E} \cdot d\vec{l} + K}$$

where K is $V(\vec{r})$ at the reference point $\vec{0}$, i.e., $V(\vec{0}) \equiv K$

- Note that there is an ambiguity in the definition of potential, since the reference point can be chosen arbitrarily. The potential at the reference point K is also arbitary, as adding a constant to V will not affect the potential

difference between two points.

- Potential as such carries no real physical significance. The only quantity of intrinsic interest is the <u>potential difference</u> between two points.
- There is a natural spot to use for the reference point in electrostatics and that is a point infinitely far from the charge.
  However, when the charge distribution itself extends to infinity, this choice will be in trouble.
- Units of potential
  $$N\cdot m/C \quad \text{or} \quad J/C \equiv \text{Volt}.$$

* <u>Calculate $\vec{E}$ from V</u>

- with the fundamental theorem for gradients (Page 29)
  $$\int_a^b (\nabla T)\cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$$
  we can show the potential difference between $\vec{a}$ and $\vec{b}$ is
  $$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = \int_a^b (\nabla V)\cdot d\vec{l}$$
  Since this is true for any points $\vec{a}$ and $\vec{b}$, the integrands must be equal:
  $$\boxed{\vec{E} = -\nabla V}$$
  which says that the electric field is the gradient of the scalar electric potential.
  Note: This is a local property which holds at any points in a electric field of any static charge configuration
- In some cases, it might be easier to get V first, and the calculate $\vec{E}$ by taking gradient.

* <u>The superposition principle of V</u>

- Since
  $$V(\vec{r}) = -\int_0^{\vec{r}} \vec{E}\cdot d\vec{l} = -\int_0^{\vec{r}} (\vec{E}_1 + \vec{E}_2 + \dots)\cdot d\vec{l}$$
  $$= -\int_0^{\vec{r}} \vec{E}_1\cdot d\vec{l} - \int_0^{\vec{r}} \vec{E}_2\cdot d\vec{l} + \dots$$
  $$= V_1 + V_2 + \dots$$
  That is, the potential at any given point is the sum of the potentials

The potential of a localized charge distribution

- Let's first calculate the potential of a point charge at the origin. We have obtained

$$\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

and the potential difference

$$V(\vec{b}) - V(\vec{a}) = -\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = -\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

Now let's put our reference point $\vec{a}$ to infinity, thus

$$r_a \to \infty, \quad V(\vec{a}) \to 0$$

we get

$$V(\vec{b}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r_b}$$

- Note the advantage of using infinity for the reference point, which kills the lower limit on the integral.
- In general, the potential of a point charge is

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{R}}$$

where R is the distance from q to $\vec{r}$. $R = |\vec{R}| = |\vec{r} - \vec{r}'|$.

- Invoking the superposition principle, the potential of a collection of charges is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i}$$

- or for a continuous distribution

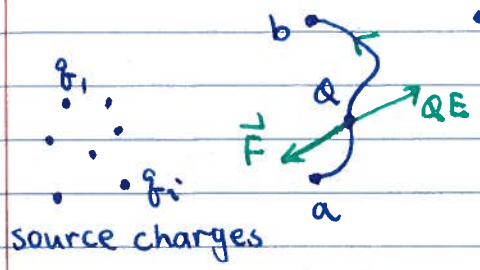
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{R}\, dq$$

$$\sim \underbrace{\frac{1}{4\pi\epsilon_0}\int \frac{\lambda(\vec{r}')}{R}\, dl'}_{\text{line charge}} \sim \underbrace{\frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\vec{r}')}{R}\, da'}_{\text{surface charge}} \sim \underbrace{\frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}')}{R}\, d\tau'}_{\text{volumn charge.}}$$

Note that they are obtained on the assumption that the reference point is at infinity.

4. Work and Energy in Electrostatics

– Potential Energy



- To move a test charge Q from $\vec{a}$ to $\vec{b}$ in the electric field of a stationary configuration of source charges, the minimum force we have to exert on it is

$$\vec{F} = -Q\vec{E}$$

- The corresponding work we have to do is

$$W = \int_{\vec{a}}^{\vec{b}} \vec{F}\cdot d\vec{l} = -Q\int_{\vec{a}}^{\vec{b}} \vec{E}\cdot d\vec{l} = Q[V(\vec{b}) - V(\vec{a})]$$

Unit: $N\cdot m/C$ or $J/C \equiv$ Volt.

i.e. $W = QV(\vec{b}) - QV(\vec{a}) = W_{\vec{b}} - W_{\vec{a}}$ $\quad J = C\cdot V = N\cdot m = kg\cdot m^2\cdot s^{-2}$

where we define the potential energy

$$W_{\vec{r}} = QV(\vec{r}). \quad \text{(reference point at infinity)}.$$

- In other words, the minimum energy we have to do to move the test charge from $\vec{a}$ to $\vec{b}$, is used to change its potential energy. (By "minimum", we mean we don't want to ~~increa~~ change the other kind of energies of Q, such as kinetic energy.).
- OR, the potential difference between points $\vec{a}$ and $\vec{b}$ is equal to the work per unit charge required to carry a particle from $\vec{a}$ to $\vec{b}$.

- If we bring Q in from far away and stick it at point $\vec{r}$, we need to work

$$W = Q[V(\vec{r}) - V(\infty)] = QV(\vec{r}) = W_{\vec{r}}$$

we can also think of it as the work it takes to creat the SYSTEM, or the energy gets stored in the system.

Note, by "system", we mean the test charge Q we bring in, and also the charge configuration already there.

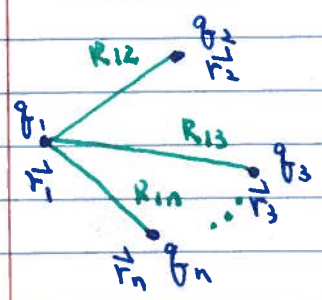
— The energy of a charge distribution

* Point charge distribution

- To calculate the energy of a charge distribution, we can instead calculate the work it would take to assemble the entire collection of point charge from far away.
- Remember that to bring in a point charge Q from far away, the work needed is

$$W = QV(\vec{r})$$

where $V(\vec{r})$ is the potential because of the charge distribution which is already there without Q.



First we can calculate the energy needed to bring in q_1 to $\vec{r}_1$, assuming all the other charges are already at their places, which is

$$W_1 = \frac{1}{4\pi\epsilon_0} q_1 \left(\frac{q_2}{R_{12}} + \frac{q_3}{R_{12}} + \cdots \frac{q_n}{R_{1n}} \right)$$

Then, for q_2,

$$W_2 = \frac{1}{4\pi\epsilon_0} q_2 \left(\frac{q_1}{R_{21}} + \frac{q_3}{R_{23}} + \cdots \frac{q_n}{R_{2n}} \right)$$

$$\vdots$$

Finally we add them all up,

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{q_i q_j}{R_{ij}}$$

Note that in fact we have count the energy of each pair twice, so the total energy is

$$W = \frac{1}{2} \times \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \frac{q_i q_j}{R_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{j \neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{R_{ij}} \right)$$

$$= \boxed{\frac{1}{2} \sum_{i=1}^{n} q_i V(\vec{r}_i)}$$

where $V(\vec{r}_i) \equiv \sum_{j \neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{R_{ij}}$, which is the potential at point $\vec{r}_i$ (the position of q_i) due to all the other charges.

Two equal but opposite charges $E = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{R}$

$= \frac{1}{2}\left[q \times \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{R}\right) + (-q) \times \left(\frac{1}{4\pi\epsilon_0}\frac{q}{R}\right)\right]$

* <u>Continuous charge distribution</u>.

- For a continuous volume charge density ρ, the energy is

$$W = \frac{1}{2}\int_V \rho V \, d\tau = \frac{1}{2}\int_V (\epsilon_0 \nabla \cdot \vec{E}) V \, d\tau \quad (\text{used } \nabla \cdot \vec{E} = \rho/\epsilon_0)$$

$$= \frac{1}{2}\epsilon_0 \left[\int_V \nabla \cdot (\vec{E} V) \, d\tau - \int_V \vec{E} \cdot (\nabla V) \, d\tau\right] \quad (\text{v}$$

$$= \frac{1}{2}\epsilon_0 \left[\oint_S V \vec{E} \cdot d\vec{a} - \int_V \vec{E} \cdot (\nabla V) \, d\tau\right.$$

$$= \frac{1}{2}\epsilon_0 \left[\oint_S V \vec{E} \cdot d\vec{a} + \int_V (\vec{E} \cdot \vec{E}) \, d\tau\right] \quad (\text{used } \vec{E} = -\nabla V)$$

where the volume is the region with charge located.

- Actually we can integrate over any larger volume which encloses all the charge distribution in

$$W = \frac{1}{2}\int_V \rho V \, d\tau \quad (V \text{ can be any larger volume})$$

because the region with no charge would not contribute.

- If we simply integrate over all space, then the surface goes to infinity, where V and $\vec{E}$ would go zero. Thus $\oint_S V \vec{E} \cdot d\vec{a} \to 0$ when S is infinitely large.

- Finally we are left with

$$\boxed{W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau}.$$

- Please notice a perplexing "inconsistency" described on Page 96

why can it be negative?

⊕ ⊖

<u>Example</u> Find the energy of a uniformly charged spherical shell of total charge q and radius R.

- Inside the sphere $\vec{E} = 0$
- outside $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$, $E^2 = \frac{q^2}{(4\pi\epsilon_0)^2 r^4}$

So $$W_{tot} = \frac{\epsilon_0}{2} \frac{1}{(4\pi\epsilon_0)^2} \int_{outside} \left(\frac{q^2}{r^4}\right)(r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

$$= \frac{1}{32\pi^2\epsilon_0} q^2 4\pi \int_R^\infty \frac{1}{r^2} dr$$

$$= \frac{1}{8\pi\epsilon_0} \frac{q^2}{R}$$

5. Conductors

– Electrostatic properties of ideal conductors

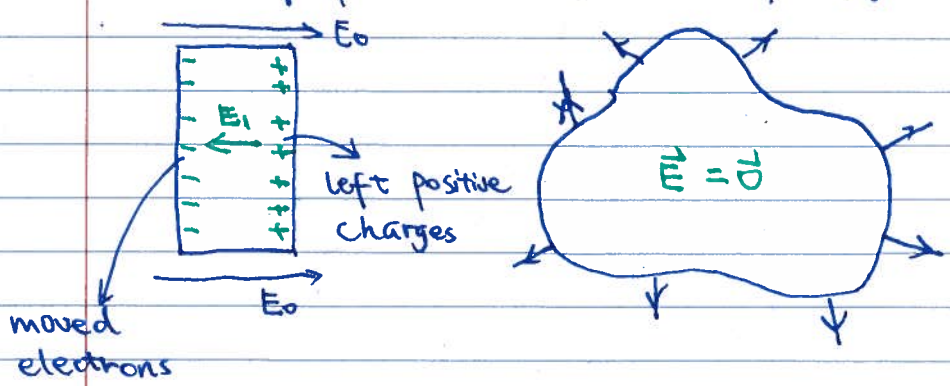
* Insulator vs. conductor

- insulator: each electron is attached to a particular atom. Such as glass or rubber.
- conductor: One or more electrons ~~are~~ per atom are free to roam. Such as metals.

perfect conductor: contain an unlimited supply of free charges.

* Ideal conductors' electrostatic properties

- $\vec{E} = \vec{0}$ inside a conductor
- $\rho = 0$ inside a conductor
- Any net charge resides on the surface (charged conductors)
- A conductor is an equipotential
- $\vec{E}$ is perpendicular to the surface, just outside a conductor.

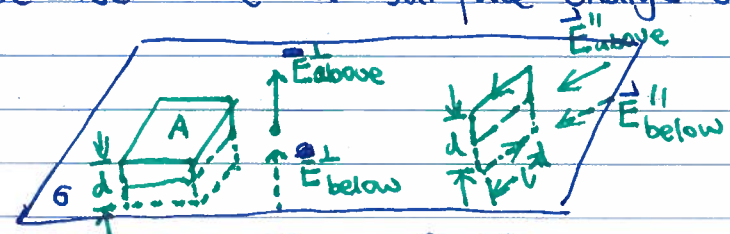


The charge on a conductor will seek the configuration that minimizes its potential energy. The electrostatic energy of a solid object is a minimum when that charge is spread over the surface.

Note: Here we are talking about conductors and electric fields in static equilibrium.

— Boundary conditions

- We noticed that the electric field undergoes a discontinuity when you cross ~~the~~ a surface. We shall find out the amount ~~of~~ by which $\vec{E}$ changes at such a boundary.
- Suppose we have a surface charge σ,



* Boundary conditions of the field

- We decompose the fields into components $\perp$ and $\parallel$ to the surface:

$$\vec{E}_{above} = (\vec{E}^{\perp}_{above}, \vec{E}^{\parallel}_{above})$$
$$\vec{E}_{below} = (\vec{E}^{\perp}_{below}, \vec{E}^{\parallel}_{below})$$

where $E^{\perp}$ denotes the component of $\vec{E}$ that is perpendicular to the surface, and $\vec{E}^{\parallel}$ the components of $\vec{E}$ parallel to the surface.

- First we apply Gauss's law to the area A,

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc} = \frac{\sigma A}{\epsilon_0}$$

where the Gauss surface is a wafer-thin Gaussian pillbox. In the limit of zero thickness d, the sides of the pillbox contributes nothing to the flux. We are left with

$$E^{\perp}_{above}\cdot A - E^{\perp}_{below} A = \frac{\sigma A}{\epsilon_0} \quad \text{(upwards as positive)}$$

or

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$$

- The normal component of $\vec{E}$ is discontinuous by an amount σ/ϵ_0 at any boundary.

where there is no surface charge, $E^{\perp}$ is continuous.

- For the tangetial componets, we apply

$$\oint \vec{E}\cdot d\vec{l} = 0$$

to the thin rectangular loop. In the limit $d \to 0$,

$$\oint \vec{E}\cdot d\vec{l} = E^{\parallel}_{above}\, l - E^{\parallel}_{below}\, l = 0.$$

thus $\vec{E}^{\parallel}_{above} = \vec{E}^{\parallel}_{below}$

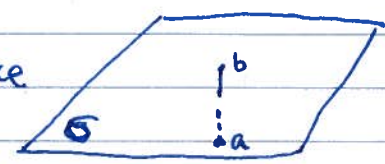
- Combine $\boxed{\begin{cases} E^{\perp}_{above} - E^{\perp}_{below} = \frac{1}{\epsilon_0}\sigma \\ \vec{E}^{\parallel}_{above} = \vec{E}^{\parallel}_{below} \end{cases}}$

we can write the boundary conditions on $\vec{E}$ into a single formula

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing from "below" to "above".

* Boundary conditions of the potential



- The potential difference across the surface

$$V_{above} - V_{below} = -\int_a^b \vec{E}\cdot d\vec{l}$$

when $a \to b$ approach one another

$$\boxed{V_{above} = V_{below}}$$

- On the other hand, $\vec{E} = -\nabla V$, thus

$$(-\nabla V_{above}) - (-\nabla V_{below}) = \frac{\sigma}{\epsilon_0}\hat{n}$$

or $\nabla V_{above} - \nabla V_{below} = -\frac{\sigma}{\epsilon_0}\hat{n}$

Project out the component perpendicular to the surface

$$\boxed{\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{1}{\epsilon_0}\sigma}$$

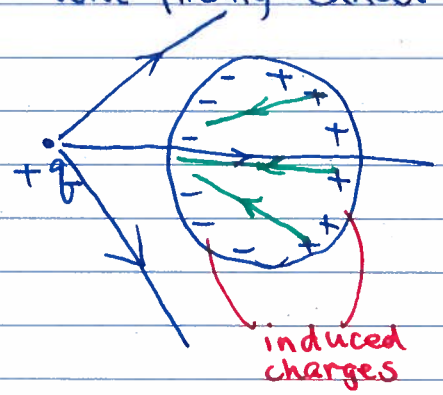
where $\partial V/\partial n = \nabla V\cdot\hat{n}$ denotes the normal derivative of V.

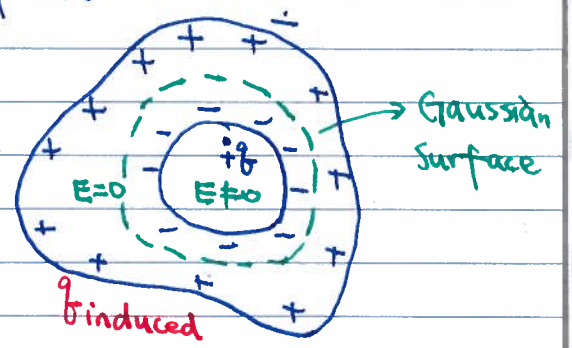
- Note: We here talk about fields and potentials approching the surface.

— Induced charges and surface charges

* Induced charges

- If you hold a charge $+q$ near an uncharged conductor, it will pull minus charges over to the near side and repel plus charges to the far side.
  These are called <u>induced charges</u>, which produce a field of their own. The electric field of the induced charges will finally cancel the field of $+q$.

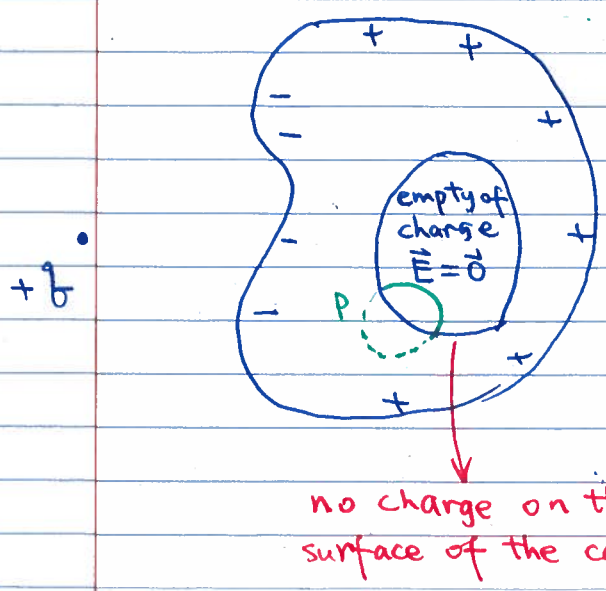




$$Q_{enc}/\epsilon_0 = \oint_S \vec{E}\cdot d\vec{a} = 0$$

$$Q_{enc} = q + (-q_{induced}) = 0$$

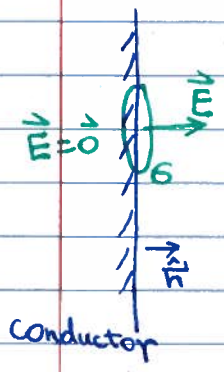
$q_{induced} = q$ on the outer surface



$\oint_P \vec{E}\cdot d\vec{l} = 0 \Rightarrow \vec{E} = \vec{0}$ inside the cavity.

Faraday cage: shields external flux inside a hollow conductor.

* Surface charge



- Apply the boundary condition

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$$

on the surface of a conductor, and because inside the conductor $\vec{E} = \vec{0}$, the electric field right outside the conductor surface is

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n} \quad \left(\text{and } \sigma = -\epsilon_0 \frac{\partial V}{\partial n} \right)$$

If we can determine $\vec{E}$ or V, then we can use these equations to calculate the surface charge on a conductor.

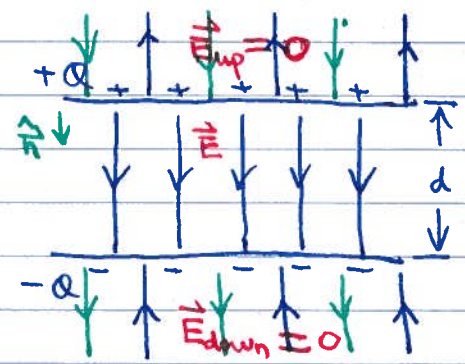
— Capacitors

* Definition

- A capacitor is a device that stores electrical energy in an electric field.
- A capacitor consists of two conductors separated by a non-conductive region. The non-conductive region can either be a vacuum or an electrical insulator material known as a dielectric.
- We can treat a single conductor as a capacitor as well. In this case, the "second conductor" would be an imaginary spherical shell of inifinite radius surrounding the one conductor.

* Parallel-plate capacitor

- Suppose we have a parallel-plate capacitor consisting of two metal surfaces of area A held a distance d apart.



- If we put $+Q$ on the top and $-Q$ on the bottom, they will spread out uniformly over the two surfaces. The surface charge density is

$$\sigma = Q/A$$

- using the boundary condition

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$$

we have

$$|\vec{E}| - (-\cancel{|\vec{E}_{up}|}) = \frac{Q}{A\epsilon_0} \quad \text{for the upper plate}$$

$$(-\cancel{|\vec{E}_{down}|}) - |\vec{E}| = \frac{-Q}{A\epsilon_0} \quad \text{for the bottom plate.}$$

Besides $|\vec{E}_{up}| = |\vec{E}_{down}| = 0$

The field between the plates is

$$\vec{E} = \frac{Q}{A\epsilon_0}\hat{n}$$

- The potential difference between the plates is

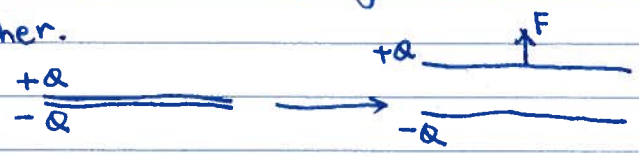
$$V = E \cdot d = d/(A\epsilon_0) \cdot Q$$

- Now Let's calculate how much energy is stored in the capacitor, or how much work ~~dee~~ it needs to charge the capacitor up to a final amount Q.

① we can use $W = \frac{\epsilon_0}{2}\int E^2 d\tau$ (all space). The total energy stored in the field is

$$W = \frac{\epsilon_0}{2} \times \left(\frac{1}{\epsilon_0}\frac{Q}{A}\right)^2 \times (dA) = \frac{1}{2}\left(\frac{d}{A\epsilon_0}\right) Q^2$$

② we can calculate how much energy is needed to pull the two plates away by the distance d, from being together.



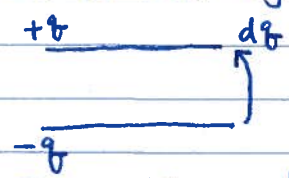
In this case, the force we need to exert on $+Q$ plate is $F = Q \cdot E = Q \cdot \left(\frac{1}{2\epsilon_0}\frac{Q}{A}\right)$

~~Notice~~ Note that the field of $-Q$ plate is $\frac{1}{2\epsilon_0}\frac{Q}{A}$.

Thus work is

$$W = F \cdot d = \frac{1}{2}\left(\frac{d}{\epsilon_0 A}\right) Q^2$$

③ we can also remove electrons from the positive plate and carry them to the negative plate, to charge up a capacitor



suppose that at some intermediate stage, the charge on the positive plate is q. The potential difference is $\left(\frac{1}{\epsilon_0}\right)\frac{q}{A} \cdot d$

work needed to transport the next piece of charge dq

$$dW = \frac{1}{\epsilon_0}\frac{q}{A} \cdot d \cdot dq$$

The total work necessary, to go from $q=0$ to $q=Q$, is

$$W = \int_0^Q dW = \int_0^Q \frac{1}{\epsilon_0}\frac{d}{A}\, q\, dq = \frac{1}{2}\left(\frac{d}{\epsilon_0 A}\right) Q^2$$

* Capacitance

- Now we want to define a measure of the capacity of storing electric charges for capacitors.
  We found that the potential difference between the two plates of a parallel-plate capacitor is

$$V = d/(A\epsilon_0) \cdot Q$$

  we can define the capacitance of a capacitor as

$$\boxed{C \equiv \frac{Q}{V}.}$$

For the case of parallel-plate capacitors,

$$C = \frac{A\epsilon_0}{d}$$

and we see that capacitance is a purely geometrical quantity determined by the sizes, shapes, and separation of the two conductors.

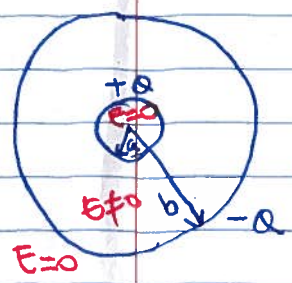
- In SI units, C is measured in farads (F).

$$1\ F = 1\ C/V.$$

  More practical units are the microfarad (10^{-6} F) and the pico farad (10^{-12} F).
- using the defination of $C \equiv Q/V$, we can find the energy stored in a capacitor is

$$W = \frac{1}{2}\frac{1}{C}Q^2 = \frac{1}{2}CV^2$$

Example: Find the capacitance of a spherical ball with radius a and a spherical shell with radius b. Both are conductors.



Solution:

① The electric field can be obtained using Gauss's law. $\oint \vec{E}\cdot d\vec{a} = Q_{enc}/\epsilon_0$

$$E\cdot 4\pi r^2 = Q/\epsilon_0$$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}\ (a<r<b).$$

note this is the same as a point charge.

② The potential difference between the two conductors

$$V(a)-V(b) = -\int_b^a \vec{E}\cdot d\vec{l} = -\int_b^a \vec{E}\cdot(\hat{r}\,dr)$$

$$= -\int_b^a \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dr = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

③ The capacitance is

$$C = \frac{Q}{V} = \left[\frac{1}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)\right]^{-1}$$

$$= 4\pi\epsilon_0\frac{ab}{b-a}$$

Method II: The total energy is

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau = \frac{\epsilon_0}{2}\int\left(\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\right)^2 r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \frac{\epsilon_0}{2}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\times 4\pi\int_a^b\frac{1}{r^2}dr$$

$$= \frac{\epsilon_0}{2}\frac{Q^2}{4\pi\epsilon_0^2}\left(-\frac{1}{r}\right)\Big|_a^b = \frac{Q^2}{2}\frac{1}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\equiv \frac{1}{2}\frac{1}{C}Q^2$$

$$C = \left[\frac{1}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)\right]^{-1} = 4\pi\epsilon_0\frac{ab}{b-a}.$$

Note that the result doesn't chang if the spherical ball become a spherial shell.

ELECTRIC POTENTIALS

1. Laplace's Equation

- Poisson's equation
  - with $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and $\vec{E} = -\nabla V$, we can get

$$\nabla \cdot \vec{E} = \nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho}{\epsilon_0}$$

The equation $\boxed{\nabla^2 V = -\rho/\varepsilon_0}$ is called Poisson's equation. ($\nabla^2 V$ is called the Laplacian of V, P23).

- Poisson's equation relates V and ρ; it would give us ρ, if we knew V.

  In fact, we must invert Poisson's equation: we want V, knowing ρ, so that we can calculate $\vec{E}$ by taking the gradient. We have obtained the "solution" to poisson's equation for a localized charge distribution.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}\,')}{R} d\tau'$$

$$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

- Laplace's equation
  - when there are charges in a region, but if we are interested in finding the potential in a region where $\rho = 0$, Poisson's equation reduces to Laplace's equation:

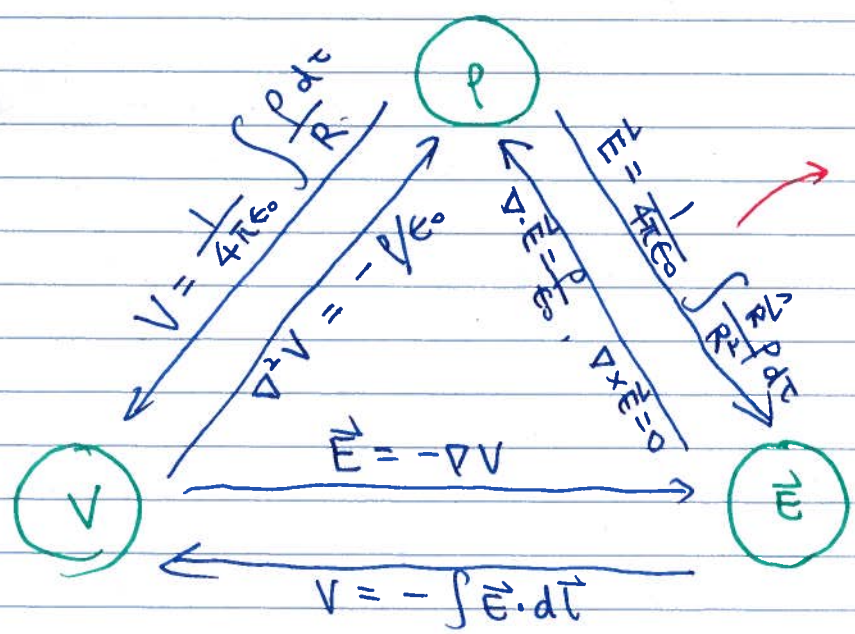
$$\nabla^2 V = 0.$$ Solve it with b.c.

  - The solutions of Laplace's equation are the so-called "harmonic functions". They are all analytic within the domain where the equation is satisfied.
  - If any two functions are solutions to Laplace's equation, their sum (or any linear combination) is also a solution. This property, called the principle of superposition, is very ~~simple~~. useful.

The value of V at point $\vec{r}$ is the average value of V over a spherical surface of radius R centered at $\vec{r}$.

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oint_{\text{sphere}} V \, da.$$

- A solution to Laplace's equation has the property that the average value over a spherical surface is equal to the value at the center of the sphere (Gauss's harmonic function theorem).
- Solutions have no local maxima or minima. Extreme values of the solutions must happen at the end points / boundaries.
- Mathematically, a solution to Laplace's equation is uniquely determined if
  (i) the value of the function is specified on all boundaries (Dirichlet boundary conditions)
  or (ii) the normal derivative of the function is specified on all boundaries (Neumann boundary conditions).

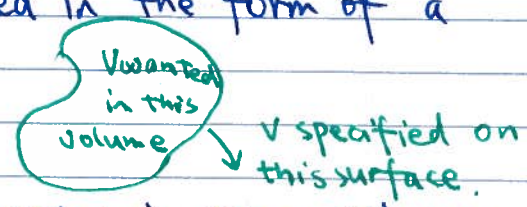


Gauss's Law in the integral form

A potential that doesn't satisfy Laplace's equation together with the boundary condition is an invalid electrostatic potential.

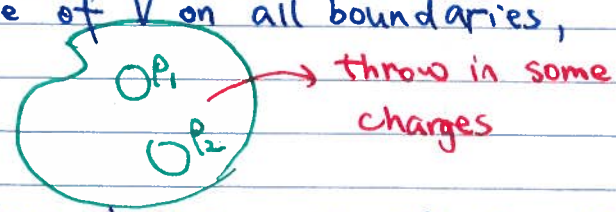
Boundary conditions and uniqueness theorems

- Laplace's equation does not by itself determine V; in addition, suitable boundary conditions must be supplied. The proof that a proposed set of boundary conditions will suffice is usually presented in the form of a uniqueness theorem.



* First uniqueness theorem

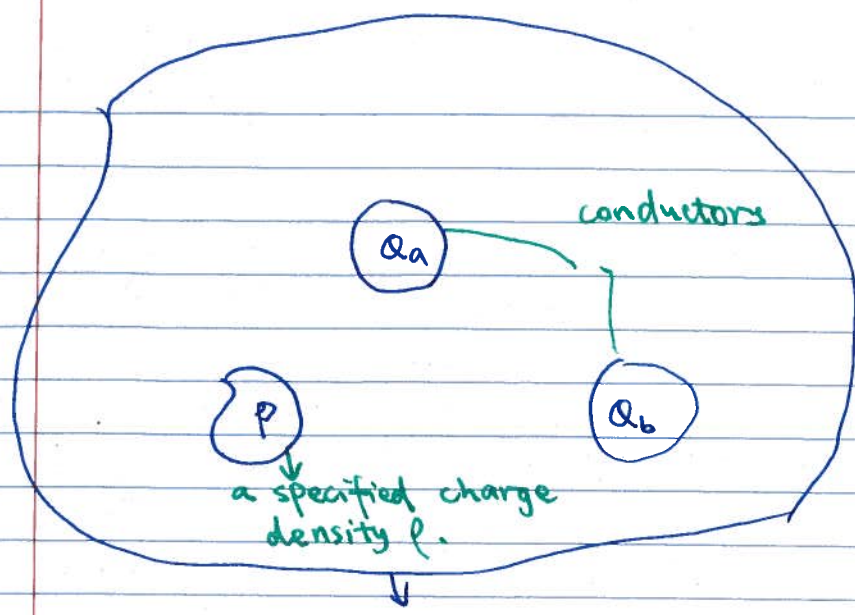
- The solution to Laplace's equation in some volume is uniquely determined if V is ~~set~~ specified on the boundary surface.
- Corollary: The potential in a volume is uniquely determined if (a) the charge density throughout the region, and (b) the value of V on all boundaries, are specified.



* Second uniqueness theorem

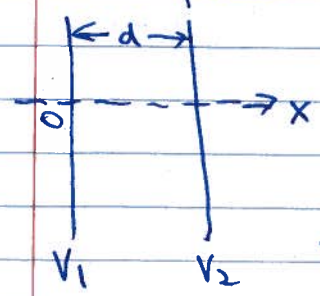
- Suppose there are conductors in the region of interest, and we don't know the potentials on the conductors, but rather the charges on various conducting surfaces, is the electric field now uniquely determined?
- In a volume surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is given. (The region as a whole can be bounded by another conductor, or else unbounded.)

- The uniqueness theorem is a license to your imagination. It doesn't matter how you come by your solution. If it satisfies Laplace's equation and it has the correct value on the boundaries, then it's right.



outer boundary could be at infinity (unbounded) or bounded by anothe conductor.

<u>Example:</u> Find the electric potential between the parallel plate capacitor.



Laplace's equation gives

$$\nabla^2 V = 0 \rightarrow \frac{d^2 V}{dx^2} = 0.$$

the solution is $V(x) = mx + b$.

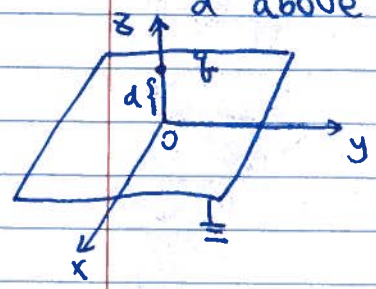
And the constants can be fixed using the boundary conditions

$$\begin{cases} V(0) = b = V_1 \\ V(d) = md + b = V_2 \end{cases}$$

2. The method of images

– The classic Image problem

- Consider an ifinitely large conductor plane which is grounded. A point charge q is located a distance d above the plane. what is V for regions $z>0$.



- Solve $\nabla^2 = -\frac{\rho}{\epsilon_0}$ for $z>0$.

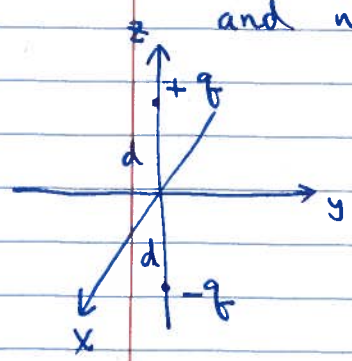
  with boundary conditions

  $$\begin{cases} V(z=0) = 0 & \text{(grounded)} \\ V(\cancel{z=\infty}) = 0 & \text{(far from the system)} \end{cases}$$

  ρ is given by the point charge at $(0,0,d)$.

  $x^2+y^2+z^2 \gg d^2$.

- The first uniqueness theorem's corollary guarantees that there is only one function that meets these requirements.
- Think of a complete different situation. The new configuration consists of two point charges $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$. and no conducting plane.



$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

- It follows that

  $V=0$ when $z=0$

  $V \to 0$ for $x^2+y^2+z^2 \gg d^2$

  and the only charge in the region $z>0$ is the point charge $+q$ at $(0,0,d)$.

- The second configuration happens to produce exactly the same potential as the first one, for $z>0$.
- If it satisfies Poisson's equation in the region of interest, and assumes the correct boundaries, then it's correct

- Now we can calculate the induced surface charge on the conductor. $\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$$

$$= -\epsilon_0 \cdot \frac{1}{4\pi\epsilon_0}\left\{\frac{-q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} + \frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right\}$$

$$= \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

- The total induced charge

$$Q = \int \sigma\, da = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi(r^2+d^2)^{3/2}}\, r\,dr\,d\phi$$

$$= \frac{qd}{\sqrt{r^2+d^2}}\Big|_0^{\infty} = -q.$$

* The total induced charge is $-q$. The potential generated by the induced charges and the point charge is just the same as from the two point charges $\pm q$.
* we can think of the $-q$ point charge as the mirror image of $+q$, and the conductor plane is the "mirror".
* Any stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image
  — method of images.

- Force between the $+q$ and the plate is

$$\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

- Energy is

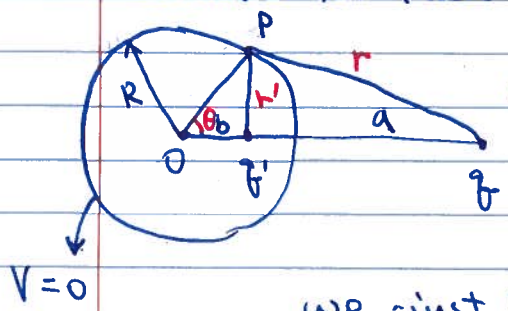
$$W = \frac{1}{2} \times \left(-\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}\right)$$

half of the energy stored in two point charges.
because only the upper region contains a nonzero field.

similar triangles : All the corresponding sides are proportional.

— Other image problems

- A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R. Find the potential outside the sphere.



① $$V(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$$

At any point on the sphere surface $V = 0$. $\Rightarrow \frac{r}{r'} = -\frac{q}{q'} = \text{constant}$.

we just have to require $\triangle OPq'$ and $\triangle OPq$ are similar triangles, then.

$$\frac{r'}{r} = \frac{R}{a} \Rightarrow \text{constant}.$$
$$= \frac{b}{R}$$

To make them similar

$$\frac{R}{a} = \frac{b}{R} \Rightarrow b = \frac{R^2}{a} \quad (\text{the position of } q')$$

$$q' = -\frac{q}{r}r' = -\frac{qR}{a} \quad (\text{the charge of } q').$$

② The potential outside the sphere is (move P outside)

$$V = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$$

③ The force between the charge and the sphere is

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2 Ra}{(a^2-R^2)^2}.$$

3. Separation of Variables.

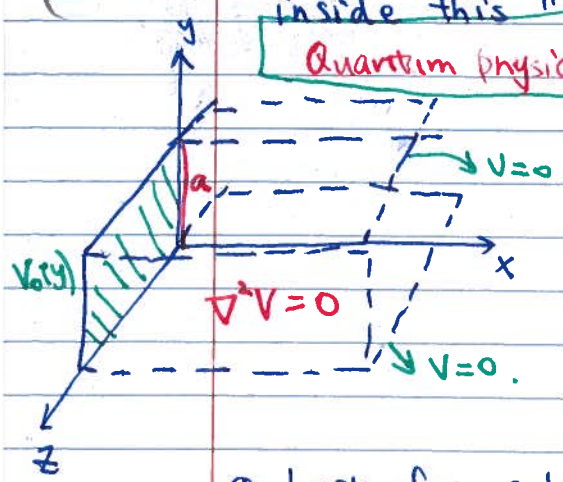
- Solve Laplace's equation directly, using the method of separation of variables.
- Look for solutions that are products of functions, each of which depends on only one of the coordinates.

— Cartesian coordinates

- Two infinite grounded metal plates lie parallel to the xz plane, one at $y=0$, the other at $y=a$. The left end, $x=0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot".

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

Quantum physics: infinite square well, finite square well.



- The solution should be independent of z.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$ partial derivative.

b.c. $\begin{cases} V=0, & \text{at } y=0 \\ V=0, & \text{at } y=a. \\ V=V_0(y), & \text{at } x=0 \\ V=0, & \text{at } x\to\infty. \end{cases}$ (not explicitly stated)

① Look for solutions in the form of products.

$$V(x,y) = X(x)\,Y(y).$$

- [none of them by itself need to satisfy the final boundary condition. But it's possible to combine them in a way that does.]
- $$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$ total derivative

② Separate the variables: Collect all the x-dependence into one term and all the y-dependence into another.

By dividing through $V = XY$, we get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0.$$

There is only one way this could possibly be true:
Both terms should be constants.

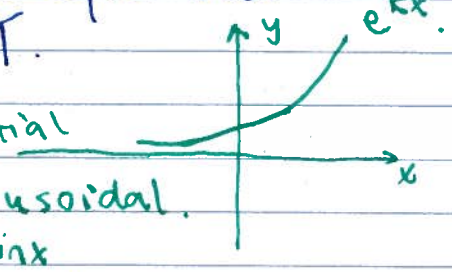
$$\frac{1}{X}\frac{d^2X}{dx^2} = c_1 \quad \text{and} \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = c_2, \quad \text{with } c_1 + c_2 = 0.$$

we need c_1 positive and c_2 negative.

③ Solve two ordinary differential equations.

$$\frac{d^2X}{dx^2} = k^2 X \quad \text{and} \quad \frac{d^2Y}{dy^2} = -k^2 Y.$$

$$\begin{cases} X(x) = Ae^{kx} + Be^{-kx}. & \text{exponential} \\ Y(y) = C\sin ky + D\cos ky. & \text{sinusoidal.} \end{cases}$$



(check: $\sin' x = \cos x$. $\cos' x = -\sin x$

④ construct the special solutions of V.

$$\boxed{V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)}.$$

which remains to impose the boundary conditions.

- $V \to 0$, when $x \to \infty \Rightarrow A = 0$.
- Absorbing B into C and D.
  $$V(x,y) = e^{-kx}(C\sin ky + D\cos ky)$$
- $V = 0$, when $y = 0 \Rightarrow D = 0$.
  $$V(x,y) = Ce^{-kx}\sin ky.$$
- $V = 0$, when $y = a$.
  $$Ce^{-kx}\sin ka = 0 \Rightarrow \sin ka = 0$$
  $$ka = n\pi, \quad (n = 1, 2, 3 \ldots).$$
  $$\Rightarrow k = \frac{n\pi}{a}$$
  Note: $n=0$. would mak $V=0$ everywhere.

Finally, we have

$$V(x,y) = Ce^{-kx}\sin ky \quad \text{with } k = \frac{n\pi}{a}, \quad n = 1, 2, 3 \ldots$$

One more b.c. to impose.

⑤ Construct the general solution:

The linear combination of all the special solutions.

$$\boxed{V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a} \sin(n\pi y/a)}$$

- It still satisfies three of the b.c.s.
- The fourth b.c. requires. any function can be expanded in this way

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y).$$ Fourier sine series.

completeness.

⑥ Determine the coefficients C_n.

Multiply the above equation by $\sin(n'\pi y/a)$ and integrate from 0 to a:

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy.$$

On the left: $\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \begin{cases} 0 & \text{if } n' \neq n \\ \frac{a}{2} & \text{if } n' = n. \end{cases}$

orthogonality.

what is left is only the term $n = n'$

$$\boxed{C_{n'} \cdot \frac{a}{2} = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy}$$

$\boxed{\text{Example}}$ $V_0(y) = V_0$. Then

$$C_n = \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy = \frac{2V_0}{n\pi}(1 - \cos n\pi)$$

$$= \begin{cases} 0, & \text{if } n \text{ is even}, \cos n\pi = 1 \\ \frac{4V_0}{n\pi}, & \text{if } n \text{ is odd}, \cos n\pi = -1 \end{cases}$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right).$$

* Completeness and orthogonality.

- A set of functions $f_n(y)$ is said to be complete if any other function $f(y)$ can be expressed as a linear combination of them:

$$f(y) = \sum_{n=1}^{\infty} c_n f_n(y).$$

- A set of functions is orthogonal if the integral of the product of any two different members of the set is zero:

$$\int_0^a f_n(y) f_{n'}(y)\, dy = 0 \quad \text{for } n' \neq n.$$

some range

- The functions $\sin\left(\frac{n\pi y}{a}\right)$ are both complete and orthogonal on the interval $0 \leq y \leq a$.
  - complete: $V_0(y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi y}{a}\right)$.
  - orthogonal: $\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi y}{a}\right) dy = \begin{cases} 0, & \text{if } n' \neq n. \\ \frac{a}{2} & \text{if } n' = n. \end{cases}$

- The other analogue
  - Aribitary vector can be expanded in orthogonal basis vectors.

$$\vec{A} = A_x \hat{\vec{x}} + A_y \hat{\vec{y}} + A_z \hat{\vec{z}}$$

    The basis vectors are mutually orthogonal.
  - The stationary states of time-independent Schrödinger equation are complete and orthogonal.

* <u>More on separable solutions of $\nabla^2 V = 0$ in Cartesian</u>

• why $\frac{d^2Y}{dy^2} = -k^2 Y$ and $\frac{d^2X}{dx^2} = k^2 X$. (instead of $\frac{d^2Y}{dy^2} = k^2 Y$)

so that $Y(y) = C \sin ky + D \cos ky$

$X(x) = Ae^{kx} + Be^{-kx}$?

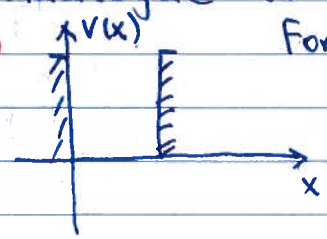
Because we have boundary conditions for $V(x,y) = X(x)Y(y)$

$$\begin{cases} V = 0 \text{ when } y = 0 \\ V = 0 \text{ when } y = a. \end{cases}$$

* $Y(y)$ looks just like the standing waves on a string of length a.

* Analogue to Quantum Physics again:

Ex ①



For the <u>infinite square well potential</u> the time-independent Schrödinger equation inside the well

$$\frac{d^2\psi}{dx^2} = -k^2\psi, \text{ where } k \equiv \sqrt{\frac{2mE}{\hbar}}.$$

with b.c. $\psi(x) = 0$ at $x = 0$ and $x = a$.

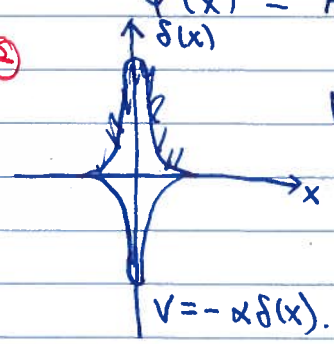
This is just the equation we solve for $Y(y)$ above.

$V \leftrightarrow \psi(x)$

potential wave function.

$\psi(x) = A \sin kx + B \cos kx.$

Ex ②



For the Dirac delta function well. and the bound states $E < 0$.

In the region $x < 0$, $V(x) = 0$,

$$\frac{d^2\psi}{dx^2} = -\cancel{\frac{2mE}{\hbar^2}\psi} = k^2\psi.$$

where $k \equiv \frac{\sqrt{-2mE}}{\hbar}.$

$\psi(x) = Ae^{-kx} + Be^{kx}.$

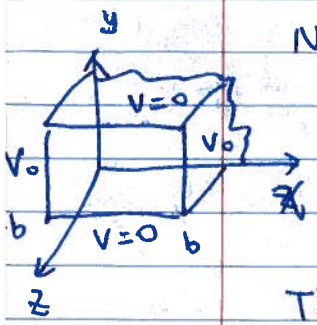
This is just the equation we solve for $X(x)$ above.

- How to solve other problems with different setup?

* The solution we obtain before imposing the b.c.s

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky).$$

Now if the boundary condition changes in x-direction.



$$V(x=-b, y) = V(x=b, y) = V_0.$$

then $A = B$, and $Ae^{kx} + Be^{-kx} \Rightarrow A\,2\cosh kx.$

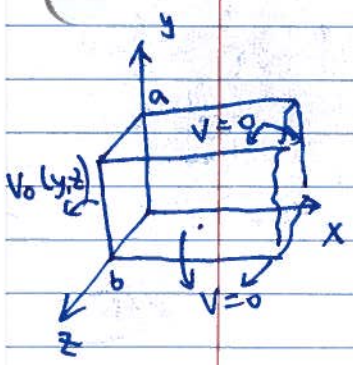
Absorbing $2A$ into C and D, we have

$$V(x,y) = \cosh kx\,(C\sin ky + D\cos ky).$$

The rest will be very similar to what we did before.

* If the system also depends on z, and b.c.s in z direction become

$$\begin{cases} V=0, & \text{when } z=0 \\ V=0, & \text{when } z=b. \end{cases}$$

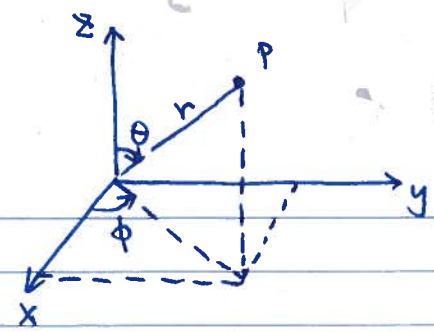


$$V(x,y,z) = X(x)\,Y(y)\,Z(z).$$

$$\Rightarrow \frac{d^2X}{dx^2} = (k^2+l^2)X, \quad \frac{d^2Y}{dy^2} = -k^2Y, \quad \frac{d^2Z}{dz^2} = -l^2Z.$$

$$\begin{cases} X(x) = Ae^{\sqrt{k^2+l^2}\,x} + Be^{-\sqrt{k^2+l^2}\,x} \\ Y(y) = C\sin ky + D\cos ky \\ Z(z) = E\sin lz + F\cos lz. \end{cases}$$

— Spherical Coordinates



In the spherical system, Laplace's equation reads:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0.$$

For a system with azimuthal symmetry, V is independent of ϕ: (ϕ-dependent V in graduate books).

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0.$$

① Solutions in the form of products:

$$V(r,\theta)=R(r)\,\Theta(\theta)$$

Putting into the eq. and dividing by V.

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0.$$

② Separate the variables:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1),\qquad \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1).$$

③ Solving two ordinary differential equations.

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)R.\qquad \frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1)\sin\theta\,\Theta.$$

the solutions are

$$R(r)=Ar^l+\frac{B}{r^{l+1}}\qquad \Theta(\theta)=P_l(\cos\theta).$$

$\frac{dR}{dr}=Alr^{l-1}-B(l+1)\frac{1}{r^{l+2}}$

$r^2\frac{dR}{dr}=Alr^{l+1}-B(l+1)\frac{1}{r^l}$

$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=Al(l+1)r^l+Bl(l+1)\times\frac{1}{r^{l+1}}$

$l(l+1)R$

where $P_l(x)$ are Legendre polynomials: A complete set of functions which are orthogonal

$$\int_{-1}^{1}P_l(x)P_{l'}(x)\,dx=\begin{cases}0, & \text{if } l'\neq l\\ \frac{2}{2l+1}, & \text{if } l'=l.\end{cases}$$

$P_0(x)=1$

$P_1(x)=x$

$P_2(x)=(3x^2-1)/2$

$P_3(x)=(5x^3-3x)/2$

...

- $P_l(x)$ is an lth-order polynomial in x; it contains only even powers, if l is even; and odd powers, if l is odd.

④ Construct the special solutions and the general solution.

- Now the separable solution to Laplace's equation, in the case of azimuthal symmetry, is

$$V(r,\theta) = R(r)\Theta(\theta)$$
$$= \left(Ar^l + \frac{B}{r^{l+1}}\right) P_l(\cos\theta).$$

This is an inifinite set of solutions, one for each l.

- The general solution is the linear combination of separable solutions:

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)}$$. with azimuthal symmetry.

Example The potential $V_0(\theta)$ is specified on the surface of a hollow sphere, of Radius R. Find the potential inside and outside the sphere.

Sol We need to solve Laplace's equation with the following b.c.s

$\nabla^2 V = 0$ inside & outside the sphere.

$$\begin{cases} V \to \text{finite}, & r \to 0 \\ V \to 0, & r \to \infty \\ V = V_0(\theta), & r = R. \end{cases}$$

General solution

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta)$$

① Inside the sphere, $r < R$,

- Since we need V to be finite when $r \to 0$,

$B_l = 0$

Thus $$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)}$$

- we need $V(r=R,\theta) = V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$

this can be satisfied bc $P_l(x)$ functions are complete

using the orthogonality.

$$\int_{-1}^{1} P_l(x) P_{l'}(x)\,dx = \int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta\,d\theta = \begin{cases} 0, & \text{if } l' \neq l \\ \frac{2}{2l+1}, & \text{if } l' = l \end{cases}$$

$x \to \cos\theta$

$x: (-1,+1) \to \theta: (\pi, 0)$

$dx \to d\cos\theta \to -\sin\theta\, d\theta$

- Multiplying the above expansion by $P_{l'}(\cos\theta)\sin\theta$ and integrating over θ, from 0 to π,

$$\int_0^{\pi} P_{l'}(\cos\theta)\sin\theta \left(V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) \right) d\theta$$

we get

$$\int_0^{\pi} P_{l'}(\cos\theta)\sin\theta\, V_0(\theta)\, d\theta = \sum_{l=0}^{\infty} A_l R^l \underline{\int_0^{\pi} P_{l'}(\cos\theta) P_l(\cos\theta)\sin\theta\, d\theta}$$

$$= A_{l'} R^{l'} \left(\frac{2}{2l'+1} \right) \quad \text{(only } l = l' \text{ survives)}$$

thus $\boxed{A_l = \frac{2l+1}{2R^l} \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta.}$

② Outside the sphere, $r > R$.

- Since we need $V \to 0$, when $r \to \infty$,

$A_l = 0$.

Thus $\boxed{V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)}$

- we need $V(r=R, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) = V_0(\theta)$.

similarly, $\boxed{B_l = \frac{2l+1}{2} R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta.}$

$\boxed{\text{Example}}$ If we were told the potential on the sphere is

$V_0(\theta) = k\sin^2\left(\frac{\theta}{2}\right)$, k is constant.

we can solve the problem by observing that

$$V_0(\theta) = k\sin^2\left(\frac{\theta}{2}\right) = \frac{k}{2}(1-\cos\theta) = \frac{k}{2}\left[P_0(\cos\theta) - P_1(\cos\theta)\right]$$

- when $r < R$, $V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$

thus only $l=0$, and $l=1$ will survive.

$A_0 = \frac{1}{2}k$, $A_1 = -\frac{1}{2R}k$.

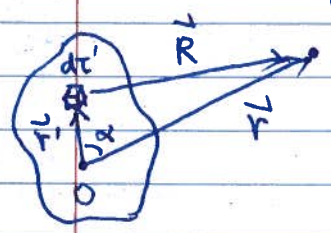
$$V(r,\theta) = \frac{k}{2}\left[r^0 P_0(\cos\theta) - \frac{r^1}{R} P_1(\cos\theta) \right] = \frac{k}{2}\left(1 - \frac{r}{R}\cos\theta\right)$$

4. Multipole Expansion

— Approximate potentials at large distances

- The potential of a localized charge distribution is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{R}\,\rho(\vec{r}\,')\,d\tau'$$



where $|\vec{R}| = |\vec{r} - \vec{r}\,'|$

$$|\vec{R}|^2 = |\vec{r}|^2 + |\vec{r}\,'|^2 - 2\,\vec{r}\cdot\vec{r}\,'$$

$$\Rightarrow R^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\Rightarrow R = r\sqrt{1 + \left[\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)\right]}$$ → denote it as ϵ

- when P is far from the charge distribution, $r \gg r'$,

$$\epsilon \equiv \left[\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)\right] \ll 1. \rightarrow \frac{1}{R} = \frac{1}{r}(1+\epsilon)^{-\frac{1}{2}}$$

Thus we can do a binomial expansion for $\frac{1}{R}$

$$\frac{1}{R} = \frac{1}{r}(1+\epsilon)^{-\frac{1}{2}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \dots\right)$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\alpha - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\alpha - 3\cos\alpha}{2}\right) + \dots\right]$$

$\left(\frac{r'}{r}\right)^0 P_0(\cos\alpha)$ → collect like powers of (r'/r)

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

The terms of $\cos\alpha$ in parentheses are Legendre polynomials.

Thus $$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\,\rho(\vec{r}\,')\,d\tau'}$$

The multipole expansion of V in powers of $\frac{1}{r}$

r is independent of r', and comes outside the integral

We note that:

- Since α is the angle between $\vec{r}$ and $\vec{r}'$, the integrals depend on the direction of the field point.
  (If r is fixed, the expansion would be different if you move P to change the angle α).
- If you keep all the terms in the expansion, then it is exact. However, it is useful primarily as an approximation scheme; the lowest nonzero terms in the expansion provides the approximate potential at large r.

— <u>The first few terms: Monople, dipole, ~~quadrupole~~...</u>

- <u>The monopole term</u>

*when $n=0$, we have

$$V_{mon}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int \rho(\vec{r}')\,d\tau' \equiv \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \quad \boxed{\propto \frac{1}{r}}$$

where $Q = \int \rho(\vec{r}')\,d\tau'$ is the total charge of the configuration.

* This is just what we expect for the approximate potential at large distances from the charge.
* $Q = \int \rho(\vec{r}')\,d\tau'$ is called the <u>monopole moment</u> of the distribution. $\int (r')^0 \rho(\vec{r}')\,d\tau'$ weighted by $(r')^0$.
* <u>Perfect (pure) monopole</u>: For <u>a point charge at the origin</u>, V_{mon} is the exact potential, not merely a first approximation at large r; in this case, all the higher multipoles vanish.

Monopole.
$\left(V \sim \frac{1}{r}\right)$

* If the total charge is zero, then the monopole contribution to $V(\vec{r})$ is vanished.

* Monopole moment doesn't depend on the coordinate system

- <u>The dipole term</u>

* when $n=1$, we have

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\alpha\,\rho(\vec{r}')\,d\tau'$$

$$= \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{r}\cdot\boxed{\int \vec{r}'\rho(\vec{r}')\,d\tau'} \equiv \vec{p} \text{ dipole moment of the distribution}$$

$$= \frac{1}{4\pi\epsilon_0}\cdot\frac{\vec{p}\cdot\hat{r}}{r^2} \quad \boxed{\propto \frac{1}{r^2}}$$

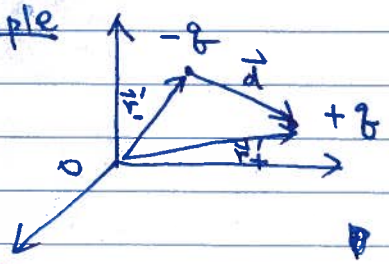
* $\vec{p} \equiv \int \vec{r}'\rho(\vec{r}')\,d\tau'$ is called the dipole moment of <u>the distribution</u>. For a collection of point charges.

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i'$$

* $V_{dip}(\vec{r}) \propto \frac{1}{r^2}$

* The dipole moment is determined by the geometry (size, shape, and density) of the charge distribution.

<u>Example</u>



$$\vec{p} = q\vec{r}_+' - q\vec{r}_-' = q(\vec{r}_+' - \vec{r}_-') = q\vec{d}$$

where $\vec{d}$ is the vector from the negative to the positive charge.

* Physical dipole vs. perfect (pure) dipole.

- The above example, two equal and opposite charges, gives a physical dipole, in the sense that higher multipole terms also contribute to the potential of the system. (not purely dipole contribution)
- To construct a perfect (point) dipole whose potential is given exactly by $V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$,

  we need to make $d \to 0$, $q \to \infty$, with the product $qd = p$ fixed.

* The dipole moment is independent of the choice of origin if and only if the total charge is zero.

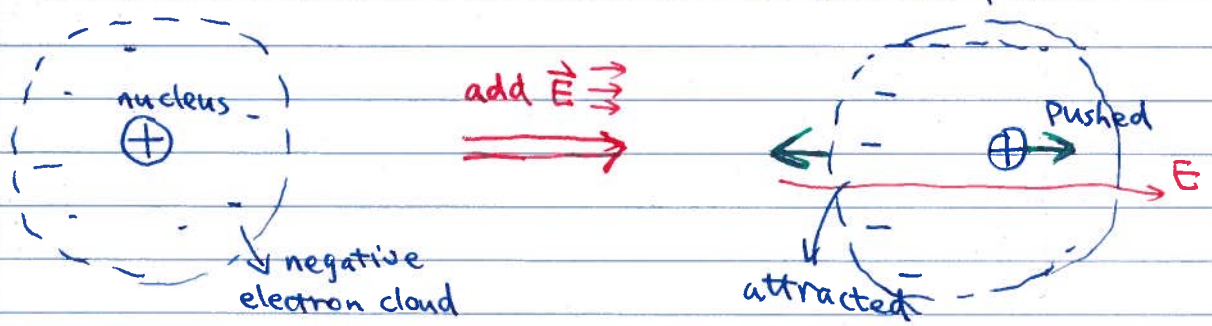
ELECTRIC FIELDS IN MATTER

1. Polarization

- We have talked about electric fields in conductors, a type of matter that contains an "unlimited" supply of charges that are free to move about through the material.
- In this chapter, we focus on electric fields in insulators (or dielectrics).
- Dielectrics: in dielectrics, all charges are attached to specific atoms or molecules; they're on a tight leash.
- Question: what happens if dielectrics is placed in fields?

— Induced dipoles

- Neutral atoms "stretched" in the electric field.



- The nucleus is pushed in the direction of the field, and the electrons the opposite way.
- If the external electric field is not large enough, which pulls the electrons and nucleus apart, the electron cloud and the nucleus attract each other and hold the atom together. (If the external field is large enough, the atom can be pulled apart completely, and the atom gets "ionized". The material will be turned from insulator to a conductor.)
- We say that the atom is "polaried" by the external electric field.

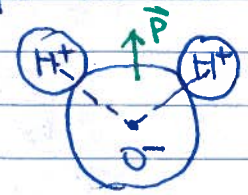
- The atom now has a tiny dipole moment $\vec{p}$, which points in the same direction as $\vec{E}$.
- when the electric field $\vec{E}$ is not too strong, the induced dipole moment is approximately proportional to $\vec{E}$. $\vec{p} = \alpha \vec{E}$. $[q]\cdot[L] = [\alpha][\frac{1}{4\pi\epsilon_0}][q][L^{-2}]$

  The constant of proportionality α is called atomic polarizability.

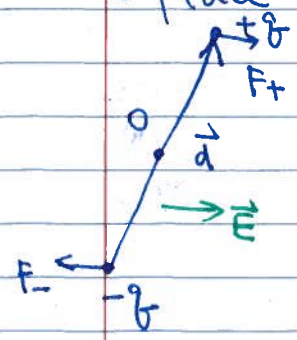
  The dimension of α is the same as $4\pi\epsilon_0 \cdot [L^3]$

— Alignment of polar molecules

- The neutral atom discussed above has no dipole moment to start with.
- Polar molecules: Some molecules have built-in, permanent dipole moment.



- Place a polar molecules in an electric field (uniform)



— $\vec{F}_+ = q\vec{E}$. $\vec{F}_- = -q\vec{E}$.

The net force on the dipole is zero. However there will be a torque.

$$\vec{N} = (\vec{r}_+ \times \vec{F}_+) + (\vec{r}_- \times \vec{F}_-)$$
$$= [(\vec{d}/2) \times (q\vec{E})] + [(-\vec{d}/2) \times (-q\vec{E})]$$
$$= q\vec{d} \times \vec{E}$$
$$= \vec{p} \times \vec{E}$$

— Thus a dipole $\vec{p} = q\vec{d}$ in a uniform field $\vec{E}$ experiences a torque

$$\vec{N} = \vec{p} \times \vec{E}.$$

— $\vec{N}$ is in such a direction as to line $\vec{p}$ up parallel to $\vec{E}$.

dielectrics in $\vec{E}$ $\left\{ \begin{array}{l} \text{neutral atom} \rightarrow \text{stretched} \rightarrow \text{induced dipole} \\ \text{polar molecules} \rightarrow \text{rotated} \rightarrow \text{line up along } \vec{E} \end{array} \right\}$ polarized

– Polarization

- We have considered the two effects of an external electric field on an individual atom or molecule: ① stretching ② rotating.
- What happens if we place a dielectric material in an external field?
  - If it consists of neutral atoms or nonpolar molecules, the field will induce in each a tiny dipole moment, pointing in the same direction as the field.
  - If it consists of polar molecules, each permanent dipole will experience a torque, tending to line it up along the field direction.
  - Since a lot of little dipoles point along the direction of the field, the material becomes polarized.
- Polarization: a measure of the effect is defined

$$\vec{P} \equiv \text{dipole moment per unit volume.}$$
$$= \sum_i \vec{p}_i / V \quad \text{(vector sum of all dipole moments)}$$

Example The energy of an ideal dipole $\vec{p}$ in an electric field $\vec{E}$ is given by $U = -\vec{p}\cdot\vec{E}$.
(not directly related to the dielectrics, just about dipole)

- Choose infinity as the reference point, the energy of a point charge Q is $W = QV(\vec{r})$.
- For a physical dipole, with $-q$ at $\vec{r}$ and $+q$ at $\vec{r}+\vec{d}$.

$$U = qV(\vec{r}+\vec{d}) - qV(\vec{r}) = q[V(\vec{r}+\vec{d}) - V(\vec{r})]$$
$$= q\left[-\int_{\vec{r}}^{\vec{r}+\vec{d}} \vec{E}\cdot d\vec{l}\right]$$

- For an ideal dipole the integral reduces to $\vec{E}\cdot\vec{d}$, and $U = -q\vec{E}\cdot\vec{d} = -\vec{p}\cdot\vec{E}$

2. The Field of A Polarized Object.

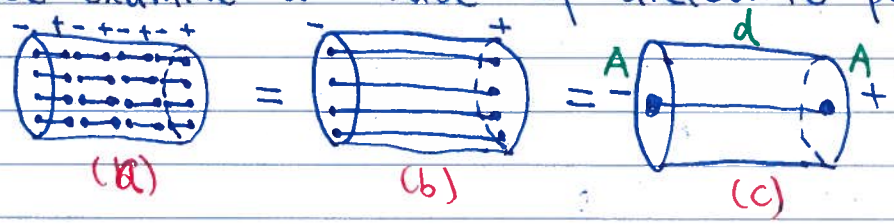
- what is the field of a polarized object whose $\vec{P}$ is given?

— <u>Surface and volume bound charges</u>

- <u>Bound charges</u>: In a dielectric, every electron is attached to a specific atom or molecule. All the charges cannot be moved around freely — they are bounded.
- Now we want to calculate the actual amount of bound charge resulting from a given polarization.

* <u>surface bound charge</u> (uniformly polarized)

we examine a "tube" of dielectric parallel to $\vec{P}$.

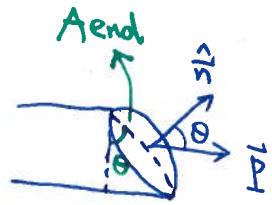


- suppose we have a long string of dipoles,(a) along the line, the head of one effectively cancels the tail of its neighbor, but at the ends there are two charges left over.(b)
- At each end, we add those left charges, and effectively we get a dipole.(c)
- suppose A is the cross-sectional area of the tube and d is the length. Then the dipole moment of the chunk is $P(Ad)$. In terms of the charge q at the end, this same dipole moment can be written qd.

$$P(Ad) = qd \Rightarrow q = PA.$$

thus if the ends have been sliced off perpendicularly, the surface charge density is

$$\sigma_b = \frac{q}{A} = P$$



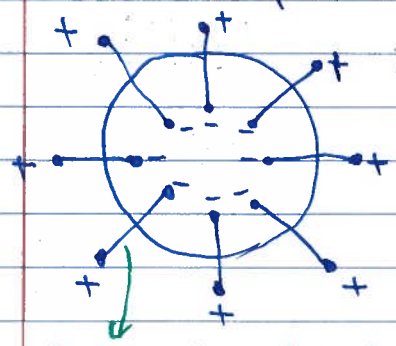
If the end is an oblique cut, the charge is still the same, but $A = A_{end}\cos\theta$. Thus

$$\sigma_b = \frac{q}{A_{end}} = P\cos\theta = \vec{P}\cdot\hat{n}.$$

- The effect of the polarization is to paint a bound charge $\boxed{\sigma_b = \vec{P}\cdot\hat{n}}$ over the surface of the material.

* <u>Volume bound charges</u>

- If the polarization is nonuniform, besides the surface bound charges, we also get accumulations of bound charge within the material.
- To calculate the bound charge's volume density, we can pick a spherical surface.



In the plot, the polarization $\vec{P}$ is diverging.

- The net bound charge in a volume is equal and opposite to the amount that has been pushed out through the surface.

This is alread a long string of dipoles.

- The net bound charge in the volume is

$$\int_V \rho_b \, d\tau.$$

The amount that has been push out through the surface is

$$\oint_S \sigma_b \, da = \oint_S \vec{P}\cdot\hat{n}\, da = \underbrace{\oint_S \vec{P}\cdot d\vec{a} = \int_V (\nabla\cdot\vec{P})\, d\tau}_{\text{divergence theorem}}$$

Thus we have $\int_V \rho_b \, d\tau = -\int_V (\nabla\cdot\vec{P})\, d\tau$.

Since this is ~~true for any~~ volume, we have

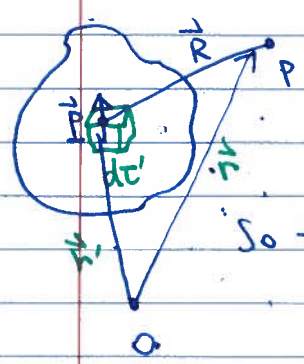
$$\boxed{\rho_b = -\nabla\cdot\vec{P}}$$

— The potential of a polarized object

- The potential (and hence also the field) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\nabla\cdot\vec{P}$ plus a surface charge density $\sigma_b = \vec{P}\cdot\vec{n}$.

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\sigma_b}{R}\,da' + \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho_b}{R}\,d\tau'}$$

- Of course, one can integrate the contributions of all the infinitesimal dipoles.



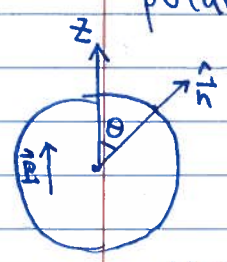
The dipole moment $\vec{p}$ of each volume element is $\vec{p} = \vec{P}\,d\tau'$. The potential of such small volume element is

$$dV(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{R}}{R^2} = \frac{1}{4\pi\epsilon_0}\frac{\vec{P}\cdot\hat{R}}{R^2}\,d\tau'$$

So the total potential is

$$V(\vec{r}) = \int dV(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{r}')\cdot\hat{R}}{R^2}\,d\tau'$$

$\boxed{\text{Example}}$ Find the electric field produced by a uniformly polarized sphere of radius R.



$\boxed{\text{Solution}}$ • Since ~~th~~ $\vec{P}$ is uniform, the volume bound charge density ρ_b is 0.

• The surface bound charge density is

$$\sigma_b = \vec{P}\cdot\hat{n} = P\cos\theta$$

Now the problem becomes calculating the potential due to a spherical shell with charge density

$$\sigma_b(\theta) = P\cos\theta.$$

- For the region $r \leq R$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta).$$

- For $r \geq R$, $V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$

The boundary conditions at $r=R$,

$$\begin{cases} V_{in} = V_{out} & \text{(i)} \\ \left(\dfrac{\partial V_{out}}{\partial r} - \dfrac{\partial V_{in}}{\partial r}\right)\Big|_{r=R} = -\dfrac{1}{\epsilon_0}\sigma_0(\theta) & \text{(ii)} \end{cases}$$

For (i) b.c., we have

$$\sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta)$$

$\Rightarrow$ the ~~like~~ coefficients of like Legendre polynomials are equal $\dfrac{B_l}{R^{l+1}} = A_l R^l \Rightarrow B_l = A_l R^{2l+1}$

For (ii) b.c., we have

$$-\sum_{l=0}^{\infty} (l+1)\frac{B_l}{R^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta)$$

$$= -\sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = -\frac{1}{\epsilon_0}\sigma_0(\theta)$$

Thus $\displaystyle\sum_{l=0}^{\infty} (2l+1) A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\epsilon_0} P\cos\theta = \frac{1}{\epsilon_0} P P_1(\cos\theta)$

so only $l=1$ survives.

$$3 A_1 R^0 P_1(\cos\theta) = \frac{1}{\epsilon_0} P P_1(\cos\theta)$$

$$\Rightarrow A_1 = \frac{P}{3\epsilon_0}, \quad B_1 = \frac{P}{3\epsilon_0} R^3$$

So $V(r,\theta) = \begin{cases} A_1 r P_1(\cos\theta) = \dfrac{P}{3\epsilon_0} r \times \cos\theta, & r \le R \\ \dfrac{B_1}{r^2} P_1(\cos\theta) = \dfrac{P R^3}{3\epsilon_0}\dfrac{1}{r^2}\cos\theta, & r \ge R. \end{cases}$

- Since $r\cos\theta = z$, we can obtain the field:
  - Inside the sphere ($r<R$)

$$\vec{E} = -\nabla V = -\frac{P}{3\epsilon_0}\hat{z} = -\frac{1}{3\epsilon_0}\vec{P}, \text{ for } r<R.$$

  - outside the sphere ($r>R$)

$$\vec{E} = -\nabla V = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

where $\vec{p} = \frac{4}{3}\pi R^3 \vec{P}$, which is equal to the total dipole moment of the sphere.

3. The Electric Displacement

— Gauss's Law in the presence of dielectrics.

- Two types of charges:
  - Bound charge: The charge due to polarization.
    Volume density $\rho_b = -\nabla\cdot\vec{P}$
    Surface density $\sigma_b = \vec{P}\cdot\hat{n}$
  - Free charge: the charge which is not a result of polarization. The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever that is not a result of polarization.
- The total electric field due to both bound charge and free charge in the presence of dielectrics is what we are interested in this section.
- The Gauss's law in differential form reads

$$\epsilon_0 \nabla\cdot\vec{E} = \rho$$

where $\vec{E}$ is the total field, due to both free charge (ρ_f) and bound charge (ρ_b) and the total charge density within the dielectric is

$$\rho = \rho_b + \rho_f.$$

And the bound charge density $\rho_b = -\nabla\cdot\vec{P}$.

So $\epsilon_0 \nabla\cdot\vec{E} = \rho = \rho_b + \rho_f = -\nabla\cdot\vec{P} + \rho_f$.

$$\Rightarrow \nabla\cdot(\epsilon_0\vec{E} + \vec{P}) = \rho_f$$

- The electric displacement is introduced as

$$\boxed{\vec{D} = \epsilon_0\vec{E} + \vec{P}}$$

Now the Gauss's law in differential form reads

$$\boxed{\nabla\cdot\vec{D} = \rho_f}$$

(Note: $\oint_S \vec{P}\cdot d\vec{a} = \int_V (\nabla\cdot\vec{P})d\tau$
$= \int_V -\rho_b\, d\tau$
$= -Q_{b\,enc}$)

and in integral form

$$\boxed{\oint \vec{D}\cdot d\vec{a} = Q_{f\,enc}}$$

where $Q_{f\,enc}$ denotes the total free charge enclosed in the volume.

- whenever the requisite symmetry is present, we can calculate $\vec{D}$.

by the standard Gauss's law method.

- When we put the free charge in place, its electric field causes a certain polarization in the presence of dielectrics, and this polarization produces the bound charge.

  The Gauss's law in the form of $\nabla\cdot\vec{D}=\rho_f$ and $\oint D\cdot d\vec{a} = Q_{fenc}$ is particularly useful, in the context of dielectrics, because it makes reference only to free charges.

- Summary

$$\boxed{\begin{aligned}\epsilon_0\,\nabla\cdot E &= \rho_b+\rho_f\\ \oint \vec{E}\cdot da &= \frac{(Q_b+Q_f)_{enc}}{\epsilon_0}\end{aligned}} \xrightarrow[\vec{D}\equiv\epsilon_0\vec{E}+\vec{P}]{\rho_b=-\nabla\cdot\vec{P}} \boxed{\begin{aligned}\nabla\cdot\vec{D} &= \rho_f\\ \oint\vec{D}\cdot d\vec{a} &= Q_{fenc}\end{aligned}}$$

with no reference ~~to~~ bound charge

Note that the curl of electrostatic fields is alway $\vec{0}$.

$$\nabla\times\vec{E}=\vec{0}.$$

- but the curl of $\vec{D}$ is not always $\vec{0}$.

$$\nabla\times\vec{D} = \nabla\times(\epsilon_0\vec{E}+\vec{P}) = \epsilon_0(\nabla\times\vec{E})+(\nabla\times\vec{P}) = \nabla\times\vec{P}$$

The curl of $\vec{P}$ generally is not $\vec{0}$.

- Because $\nabla\times\vec{D}\neq\vec{0}$, $\vec{D}$ cannot be expressed as the gradient of a scalar — there is no "potential" for $\vec{D}$.

— Boundary conditions

- Recall from chapter 2.

(i) $\oint\vec{E}\cdot da = \frac{1}{\epsilon_0}Q_{enc} \rightarrow \boxed{E^{\perp}_{above}-E^{\perp}_{below}=\frac{1}{\epsilon_0}\sigma}$ in the current context. $\sigma=\sigma_f+\sigma_b$

(ii) $\oint\vec{E}\cdot d\vec{l}=0 \rightarrow \boxed{\vec{E}^{\parallel}_{above}=\vec{E}^{\parallel}_{below}}$

The (i) boundary condition can be obtained for $\vec{D}$ as follows

$$\boxed{D^{\perp}_{above}-D^{\perp}_{below}=\sigma_f}$$

The (ii) boundary condition now is different for $\vec{D}$, as $\nabla\times\vec{D}\neq\vec{0}$ and thus $\oint\vec{D}\cdot d\vec{l}\neq 0$.

$$\nabla\times\vec{D}=\nabla\times\vec{P} \rightarrow (\vec{D}^{\parallel}_{above}-\vec{P}^{\parallel}_{above}) = (\vec{D}^{\parallel}_{below}-\vec{P}^{\parallel}_{below})$$

$$(\nabla\times(\vec{D}-\vec{P})=0) \rightarrow \boxed{\vec{D}^{\parallel}_{above}-\vec{D}^{\parallel}_{below}=\vec{P}^{\parallel}_{above}-\vec{P}^{\parallel}_{below}}$$

$\boxed{\text{Example}}$ A thick spherical shell (inner a, outer b). is made of dielectric material with a frozen-in polarization

$$\vec{P}(\vec{r}) = k/r \ \hat{r}.$$

where k is a constant and r is the distance from the center. Find the electric fied. (Assume no free charge).

$\boxed{\text{Sol 1}}$ Find the bound charges:

$$\rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} +\vec{P}\cdot\hat{r} = k/b & (\text{at } r=b) \\ -\vec{P}\cdot\hat{r} = -k/a & (\text{at } r=a). \end{cases}$$

Gauss's law $\quad E\cdot 4\pi r^2 = Q_{enc}/\epsilon_0$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{r^2}\hat{r}.$$

For $r<a$, $Q_{enc}=0$, so $\vec{E}=\vec{0}$.

$r>b$, $Q_{enc}=0$, $\vec{E}=\vec{0}$.

$a<r<b$, $\quad Q_{enc} = \left(-\frac{k}{a}\right)\cdot 4\pi a^2 + \int_a^r\left(-\frac{k}{r'^2}\right)4\pi r'^2 dr'$

$$= -4\pi k a - 4\pi k(r-a)$$
$$= -4\pi k r.$$

so $\vec{E} = -\left(\frac{k}{\epsilon_0 r}\right)\hat{r}$

$\boxed{\text{Sol 2}}$. $\oint \vec{D}\cdot d\vec{a} = Q_{fenc} = 0$, so $\vec{D}=\vec{0}$ everywhere.

$$\vec{D} = \epsilon_0\vec{E}+\vec{P} = \vec{0} \Rightarrow \vec{E} = -\frac{1}{\epsilon_0}\vec{P}.$$

thus $\vec{E} = \begin{cases} \vec{0}, & (r>b \text{ and } r<a) \\ -\left(\frac{k}{\epsilon_0 r}\right)\hat{r}, & (a<r<b) \end{cases}$.

4. Linear Dielectrics

— Permittivity of linear dielectrics

* Linear dielectrics

- Recall that the induced dipole moment is typically proportional to the field (as long as the latter is not too strong):

$$\vec{p} = \alpha \vec{E}.$$

α is the atomic polarizability $[\alpha] = [4\pi\epsilon_0][L^3]$.

- Polarization of a material is

$$\vec{P} \equiv \text{dipole moment per unit volume.} \quad \frac{[\alpha][E]}{[L^3]} = [\epsilon_0][E]$$

- For many dielectric materials, the polarization is proportional to the field, provided $\vec{E}$ is not too strong.

$$\vec{P} = \epsilon_0 \chi_e \vec{E}. \quad (\epsilon_0 \text{ extracted to make } \chi_e \text{ dimensionless})$$

- The constant of proportionality, χ_e, is called the electric susceptibility of the medium.
- χ_e depends on the microscopic structure of the material and also external conditions such as temperature.

- Linear dielectrics: Materials that obey $\vec{P} = \epsilon_0 \chi_e \vec{E}$.
- Note: $\vec{E}$ here is the total field which gets contributions from both free charges and the polarization itself. Thus $\vec{P}$ cannot be calculated directly from external field $\vec{E}_0$, ~~dire~~ using $\vec{P} = \epsilon_0 \chi_e \vec{E}$.

* Permittivity

- We deal with the problem by calculating $\vec{D}$ first which can be deduced directly from the free charge distribution.
- In linear dielectrics, we have

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 \vec{E} + \epsilon_0 \chi_e \vec{E} = \epsilon_0 (1+\chi_e) \vec{E}$$
$$\equiv \epsilon \vec{E}, \quad \text{where } \epsilon \equiv \epsilon_0 (1+\chi_e).$$

- $\vec{D}$ is proportional to $\vec{E}$ for linear media.
- ϵ is called the permittivity of the material.

$\chi_e > 0$, $\epsilon > \epsilon_0$ for ordinary matter

- In vacuum, where there is no matter to polarize, the susceptibility is zero, and the permittivity $\epsilon = \epsilon_0$. That's why ϵ_0 is called the permittivity of free space.

- Relative permittivity / dielectric constant is defined as $\epsilon_r \equiv 1 + \chi_e = \epsilon/\epsilon_0$. $\epsilon_r > 1$ for ordinary matter.
- The polarization

$$\vec{P} = \epsilon_0 \chi_e \vec{E} = \epsilon_0 \frac{\chi_e}{\epsilon} \vec{D} = \epsilon_0 \frac{\chi_e}{\epsilon_0(1+\chi_e)} \vec{D} = \frac{\chi_e}{1+\chi_e} \vec{D}.$$

($\vec{D}$ can be obtained from the free charge).

* Homogeneous linear dielectrics

- A homogeneous medium is one whose properties do not vary with position
- A homogeneous linear dielectric has the same susceptibility.
- If the space is entirely filled with a homogeneous linear dielectric, considering $\vec{D} = \epsilon \vec{E}$, we have

$\nabla \cdot \vec{D} = \rho_f$ and $\underbrace{\nabla \times \vec{D} = \nabla \times (\epsilon \vec{E}) = \vec{0}}_{\text{(special case)}}$ (holds for this special case)

On the othe hand, denote the field the same free charge distribution would produce in the absence of any dielectric (vacuum!) as $\vec{E}_{vac}$, we have

$\nabla \cdot \vec{E}_{vac} = \rho_f / \epsilon_0$ and $\nabla \times \vec{E}_{vac} = \vec{0}$.

$\vec{D} = \epsilon_0 \vec{E}_{vac}$

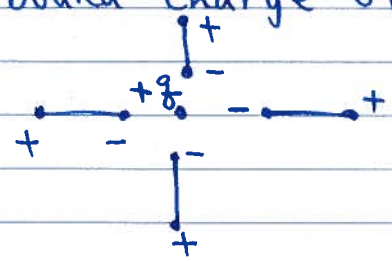
- In other words, in this rather special circumstance, $\vec{D}$ can be found from the free charge just as though the dielectric were not there.
- Further, we have

$$\vec{E} = \frac{1}{\epsilon} \vec{D} = \frac{1}{\epsilon} \epsilon_0 \vec{E}_{vac} = \frac{1}{\epsilon_r} \vec{E}_{vac}.$$

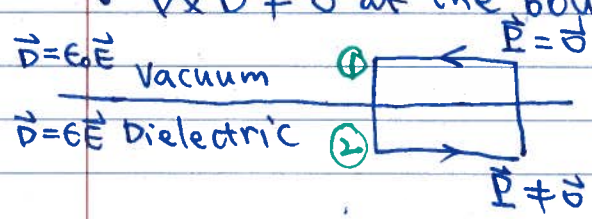
- For example, if a free charge q is embedded in a large dielectric, the field it produces is

$$\vec{E} = \frac{1}{\epsilon_r}\vec{E}_{vac} = \frac{1}{\epsilon_r}\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{r}.$$

- Note that $\epsilon_r = \epsilon/\epsilon_0$ is greater than 1 for all ordinary materials. $|\vec{E}| < |\vec{E}_{vac}|$, and thus the force it exerts on nearby charges is reduced accordingly
- This is because the polarization of the medium partially "shields" the charge, by surrounding it with bound charge of the opposite sign.



- <u>Boundary conditions</u>
  - $\nabla\times\vec{D} \neq \vec{0}$ at the boundary



Although $\nabla\times\vec{E}=0$, $\oint\vec{E}\cdot d\vec{l}=0$, $\nabla\times\vec{D}\neq\vec{0}$, $\oint\vec{D}\cdot d\vec{l}\neq 0$ because the line integral of $\vec{P}$ around a closed path that straddles the boundary between one type of material and anothe need not be zero.

$$\oint\vec{D}\cdot d\vec{l} = \oint(\epsilon_0\vec{E}+\vec{P})\cdot d\vec{l} = \oint\vec{P}\cdot d\vec{l}$$
$$= \underbrace{\int_{①}\vec{P}\cdot d\vec{l}}_{=0} + \underbrace{\int_{②}\vec{P}\cdot d\vec{l}}_{\neq 0} \neq \vec{0}.$$

- We have obtained that, in a linear dielectric,
$$\vec{P} = [\chi_e/(1+\chi_e)]\vec{D}.$$
The bound charge density is then proportional to the free charge density, as
$$\rho_b = -\nabla\cdot\vec{P} = -\nabla\left[\left(\frac{\chi_e}{1+\chi_e}\right)\vec{D}\right] = -\frac{\chi_e}{1+\chi_e}\rho_f.$$

- Unless free charge is actually embedded in the material, $\rho = 0$ (including both free and bound charge), and thus any net charge must reside at the surface.
- within such a dielectric (with no free charge embedded), the potential obeys Laplace's equation, and all the machinery of chapter 3 carries over.

* <u>Boundary conditions</u>

For an interface in the presence of dielectrics, we have obtained:

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

Apply this to linear dielectrics, ($\vec{D} = \epsilon \vec{E}$), we have

$$\epsilon_{above} E^{\perp}_{above} - \epsilon_{below} E^{\perp}_{below} = \sigma_f$$

or, in terms of the potential,

$$\boxed{\epsilon_{above} \frac{\partial V_{above}}{\partial n} - \epsilon_{below} \frac{\partial V_{below}}{\partial n} = -\sigma_f}$$

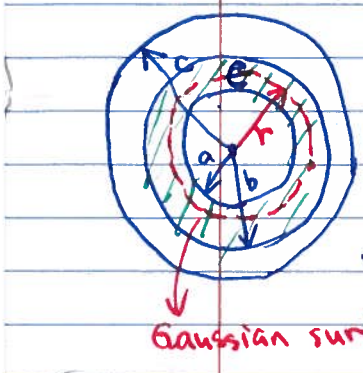
and, the potential is continuous

$$\boxed{V_{above} = V_{below}}$$

$$V \leftarrow \vec{E} \leftarrow \rho_b, \sigma_b \leftarrow \vec{P}$$

$$V \leftarrow \vec{E} \leftarrow \vec{D} \leftarrow \rho_f$$

Example A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ϵ. They are further surrounded by a spherical shell, out to radius c, carrying charge $-Q$. ① Find the capacitance of the system. ② Find the bound charge densities.



Solution we can easily know that the charge Q on the inner sphere will be on its surface at $r=a$. And the charge $-Q$ will be on the inner surface of the outer spherical shell.

- Calculate $\vec{D}$.

$$\oint \vec{D}\,da = Q_{fenc} \Rightarrow D \cdot 4\pi r^2 = Q$$

$$\Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\hat{r}, \text{ when } a<r<b.$$

- $\vec{E} = \vec{P} = \vec{D} = \vec{0}$ for $r<a$ and $r>b$.
- $\vec{E} = \frac{\vec{D}}{\epsilon} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$, for $a<r<b$.

electric field is decreased.

where are the bound charges? on the surfaces.

- The potential

$$V = V_a - V_b = -\int_b^a \vec{E}\cdot d\vec{l} = -\int_b^a \frac{Q}{4\pi\epsilon r^2}\hat{r}\cdot(dr\,\hat{r})$$

$$= -\int_b^a \frac{Q}{4\pi\epsilon_0 r^2}dr = +\frac{Q}{4\pi\epsilon r}\Big|_b^a = \frac{Q}{4\pi\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)$$

- The capacitance

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon}\left(\frac{1}{a}-\frac{1}{b}\right)} = 4\pi\epsilon\left(\frac{ab}{b-a}\right)$$

capacitance is increased.

$$= \epsilon_r\left[4\pi\epsilon_0\left(\frac{ab}{b-a}\right)\right] = \epsilon_r C_{vac},$$

② $\sigma_b = \vec{P}\cdot\hat{n}$ $\rho_b = -\nabla\cdot\vec{P}$.

The polarization

$$\vec{P} = \cancel{\epsilon_0 \chi_e} = \vec{D} - \epsilon_0\vec{E} = (\epsilon-\epsilon_0)\vec{E}$$

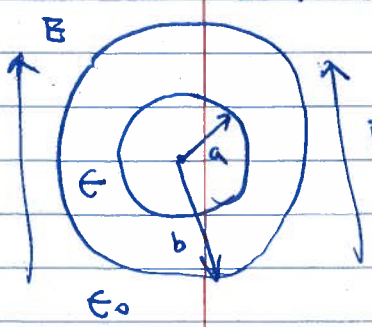
$$= (\epsilon-\epsilon_0)\frac{Q}{4\pi\epsilon r^2}\hat{r} = \frac{\epsilon-\epsilon_0}{\epsilon}\frac{Q}{4\pi r^2}\hat{r}$$

$$\rho_b = -\nabla\cdot\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(\cancel{r^2}\cdot\frac{\epsilon-\epsilon_0}{\epsilon}\frac{Q}{4\pi\cancel{r^2}}\right) = 0.$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} \frac{\epsilon-\epsilon_0}{\epsilon}\frac{Q}{4\pi r^2}\Big|_{r=b} = \frac{\epsilon-\epsilon_0}{\epsilon}\frac{Q}{4\pi b^2}, & \text{at outer surface} \\ -\frac{\epsilon-\epsilon_0}{\epsilon}\frac{Q}{4\pi a^2}, & \text{at inner surface.} \end{cases}$$

The surface charge at a is negative, at b positive.
- These negative charge produce electric field which reduces the field, within the dielectric, from $\frac{Q}{4\pi\epsilon r^2}\hat{r}$ to $\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$.

$\boxed{\text{Example}}$ An uncharge conducting sphere of radius a is coated with a thick insulating shell (dielectric constant ϵ_r) out to radius b. The object is placed in an otherwise uniform electric field $\vec{E}_0$. Find the potential in the insulator.



$\boxed{\text{solution}}$ $$V(r,\theta) = \sum_{n=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).$$

Potentials:
$$\begin{cases} V_{out} = -E_0 r\cos\theta + \sum_l \frac{B_l}{r^{l+1}}P_l(\cos\theta) & (r>b) \\ V_{med} = \sum_l \left(A_l r^l + \frac{B'_l}{r^{l+1}}\right)P_l(\cos\theta) & (a<r<b) \\ V_{in} = 0, & r<a. \end{cases}$$

Boundry conditions:
$$\begin{cases} \text{(i)}\ V_{out} = V_{med}, & (r=b) \\ \text{(ii)}\ \epsilon_0\frac{\partial V_{out}}{\partial r} = \epsilon\frac{\partial V_{med}}{\partial r} & (r=b) \\ \text{(iii)}\ V_{med} = 0. & (r=a). \end{cases}$$

$$(i) \Rightarrow -E_0 b\cos\theta + \sum \frac{B_l}{b^{l+1}} P_l(\cos\theta) = \sum \left(A_l b_l + \frac{B_l'}{b^{l+1}} \right) P_l(\cos\theta).$$

$$(ii) \Rightarrow \epsilon_r \sum \left[l A_l b^{l-1} - (l+1) \frac{B_l'}{b^{l+2}} \right] P_l(\cos\theta)$$

$$= -E_0 \cos\theta - \sum (l+1) \frac{B_l}{b^{l+2}} P_l(\cos\theta).$$

$$(iii) \Rightarrow A_l a^l + \frac{B_l'}{a^{l+1}} = 0 \Rightarrow B_l' = -a^{2l+1} A_l.$$

For $l \neq 1$

$$(i) \quad \frac{B_l}{b^{l+1}} = \left(A_l b^l - \frac{a^{2l+1} A_l}{b^{l+1}} \right) \Rightarrow B_l = A_l \left(b^{2l+1} - a^{2l+1} \right)$$

$$(ii) \quad \epsilon_r \left[l A_l b^{l-1} + (l+1) \frac{a^{2l+1} A_l}{b^{l+2}} \right] = -(l+1) \frac{B_l}{b^{l+2}}$$

$$\Rightarrow B_l = -\epsilon_r A_l \left[\left(\frac{l}{l+1} \right) b^{2l+1} + a^{2l+1} \right] \Rightarrow A_l = B_l = 0.$$

For $l=1$.

$$(i) \quad -E_0 b + \frac{B_1}{b^2} = A_1 b - \frac{a^3 A_1}{b^2} \Rightarrow B_1 - E_0 b^3 = A_1 (b^3 - a^3).$$

$$(ii) \quad \epsilon_r \left(A_1 + 2 \frac{a^3 A_1}{b^3} \right) = -E_0 - 2 \frac{B_1}{b^3} \Rightarrow -2B_1 - E_0 b^3 = \epsilon_r A_1 (b^3 + 2a^3).$$

$$A_1 = \frac{-3E_0}{2[1-(a/b)^3] + \epsilon_r [1+2(a/b)^3]}$$

$$V_{med} = A_1 \left(r - \frac{a^3}{r^2} \right) \cos\theta.$$

Review (Midterm 2)

- Poisson's equation and Laplace's equation
  - $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ and $\vec{E} = -\nabla V \rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0}$ (Poisson's eqn)
  - $\nabla^2 V = 0$, in a region $\rho = 0$ (Laplace's equation).

- Uniqueness theorems
  - Tell us what boundary conditions are sufficient for us to find unique solutions of Laplace's equation.

- The method of images
  - If we want to find the potential in a region of a system, and the Laplace's equation is not easy to solve
  - And if we know from the uniqueness theorems that the potential to be found ~~is~~ is unique given the boundary conditions.
  - Then we can see if we can find anthek system, which shares the same boundary conditions and thus the same solution in the region we are interested in,
    The new system should have the same boundary conditions, and not change the charge distribution in the region we are interested in.
  - If there are multiple charges in the region, then we may need multiple "image" charges for the new system.

- Separation of variables.

$\nabla^2 V = 0$, $V(x,y,z) = X(x)Y(y)Z(z)$ Cartesian

$V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ Spherical.

* Cartesian Coordinates

For a 2-dimensional problem

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0.$$

- The separable solutions depend on the boundary conditions. Say, if in x-direction, the potential is 0 at both boundaries, then $X(x)$ needs to be sinusoidal. Thus

$$\frac{d^2X}{dx^2} = \underbrace{-k^2}_{\text{negative}}x, \quad \frac{d^2Y}{dy^2} = \underbrace{k^2}_{\text{positive}}y,$$

$$\begin{cases} X(x) = A\sin kx + B\cos kx & \text{sinusoidal} \\ Y(y) = Ce^{ky} + De^{-ky} & \text{exponential.} \end{cases}$$

- The coefficients need to be determined using all the boundary conditions.

$$V(x,y) = \sum_k (A\sin kx + B\cos kx)(Ce^{ky} + De^{-ky}).$$

* Spherical coordinates

The general solution for a system with azimuthal symmetry.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta).$$

- At $r \to 0$, if there is no charge at the center, such as point charge or dipole, then $V(r,\theta)$ is finite. $\to B_l = 0$.
- If there is a point charge or a dipole, then $V(r,\theta) \to$ potential of point charge or dipole. there will be nonzero B_l terms.

Note that $l=0$, $B_0/r\, P_0(\cos\theta) = \frac{B_0}{r}$, (point charge)

$l=1$, $\frac{B_1}{r^2}P_1(\cos\theta) = \frac{B_1\cos\theta}{r^2}$, (dipole's potential).

- At $r \to \infty$, if there is external field $\vec{E} = E_0 \hat{z}$, then $V(r,\theta) \to -E_0 r\cos\theta$.
  when $l=1$, $A_1 r^1 P_1(\cos\theta) = A_1 r\cos\theta$.
- If there is no external field, then $V(r,\theta) \to 0$, then $A_l = 0$.
- At $r = R$ of a sphere, there can be different boundary conditions:
  - continuous condition: $V_{in}\big|_{r=R} = V_{out}\big|_{r=R}$ (always true).
  - If V is given $V_0(\theta)$ at $r=R$, then $V_{in}(R,\theta) = V_{out}(R,\theta) = V_0(\theta)$
  - If the surface charge density is given as $\sigma_0(\theta)$, then $\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)\Big|_{r=R} = -\frac{1}{\epsilon_0}\sigma_0(\theta)$.
  - If the sphere is a charged conductor with Q. (or unchanged, then $Q=0$). then the boundary condition is ~~not used to determine unkown coefficients~~, ~~but~~ used to calculate surface charge density. first

$$\sigma(\theta) = -\epsilon_0 \frac{\partial V}{\partial r}\Big|_{r=R}$$

potential outside the sphere.

and $\int \sigma(\theta)\, da = Q$ (total surface charge).

- However, if the charge of the conductor is unkown, then we need to know the potential on the conductor. In this case

$$V(r,\theta)\big|_{r=R} = V_0 \text{ (potential on the conductor)}$$

- To summarize, if we have a conductor, we either need to know its potential, or its total charge. Either one of them can give us a constraint on the coefficients.

- <u>Multipole expansion.</u>
  - Monopole moment $Q = \int \rho \, d\tau$, total charge.

$$V_{mon}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

  - dipole moment $\vec{p} \equiv \int \vec{r}' \rho(\vec{r}') d\tau'$ (continuous)

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i' \quad \text{(discrete)}.$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \xrightarrow{\vec{p}=p\hat{z}} \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}$$

If the total charge is zero, then the dipole moment is independent of the choice of origin.

$$E_{dip}(r,\theta) = -\nabla V_{dip}(\vec{r}) = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right).$$

- <u>Polarization of dielectric in fields.</u>
  - induced dipole moment of neutral atom

$$\vec{p} = \alpha \vec{E}, \quad \alpha \text{ is called atomic polarizability.}$$

  - Polarization $\vec{P} \equiv$ dipole moment per unit volume.
  - surface charge $\sigma_b \equiv \vec{P}\cdot\hat{n}$ ($\hat{n}$ is normal unit vector)

    volume charge $\rho_b \equiv -\nabla\cdot\vec{P}$

- <u>Gauss's Law in the presence of dielectrics.</u>

$\vec{E}$:
$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_b + \rho_f}{\epsilon_0}$$
$$\oint \vec{E}\cdot d\vec{a} = \frac{(Q_b + Q_f)_{enc}}{\epsilon_0}$$

$\vec{D}$:
$$\nabla\cdot\vec{D} = \rho_f$$
$$\oint \vec{D}\cdot d\vec{a} = (Q_f)_{enc}$$
(charge with no reference to bound)

Electric displacement $\vec{D} = \epsilon_0\vec{E} + \vec{P}$.

- <u>Boundary conditions</u>
  - In terms of $\vec{E}$

$$E^{\perp}_{above} - E^{\perp}_{below} = (\sigma_f + \sigma_b)/\epsilon_0$$
$$\vec{E}^{\parallel}_{above} = \vec{E}^{\parallel}_{below}$$

- In terms of $\vec{D}$

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

$$\vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below}$$

- In terms of V.

$V_{above} = V_{below}$ (continuous condition) (alway true)

Two ways to express the condition for their derivatives

- $E^{\perp}_{above} - E^{\perp}_{below} = (\sigma_f + \sigma_b)/\epsilon_0$

$$\Rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{(\sigma_f + \sigma_b)}{\epsilon_0}$$

- $D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$ and $\vec{D} = \epsilon \vec{E}$ (for linear dielectric)

$$\Rightarrow \epsilon_{above} \frac{\partial V_{above}}{\partial n} - \epsilon_{below} \frac{\partial V_{below}}{\partial n} = -\sigma_f \text{ (linear only).}$$

- special cases
  - when there is vacuum, simply treat it as a special linear dielectric of permittivity ϵ_0.
  - when there is conductor.
    - If the potential is given, then on the side of sphere, $V = V_{sphere}$ and $\partial V_{sphere}/\partial n = 0$.
    - If the charge is given, then the surface charge density is

$$\epsilon \frac{\partial V}{\partial n} = -\sigma_f \quad \text{and} \quad \int \sigma_f \, da = Q$$

the other dielectric at the boundary (can be vacuum as well)

surface charge on the conductor (free charge).

total charge

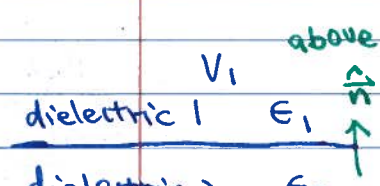
  - If there is no conductor, and there is no embedded free charge, then $\sigma_f = 0$, thus

$$\epsilon_{above} \frac{\partial V_{above}}{\partial n} = \epsilon_{below} \frac{\partial V_{below}}{\partial n} \text{ (linear)}$$

- <u>Linear dielectrics</u>
  - susceptibility χ_e
    $$\vec{P} = \epsilon_0 \chi_e \vec{E}.$$
    $\vec{E}$ is the total field.
  - permittivity ϵ
    $$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}$$
    $\epsilon \equiv \epsilon_0 (1+\chi_e)$. For vacuum, $\epsilon = \epsilon_0$ $(\chi_e = 0)$.
  - $\vec{P} = \dfrac{\chi_e}{1+\chi_e} \vec{D}$.

- <u>Homogeneous linear dielectrics</u>
  - If a space is filled with linear dielectrics homogeneously.
    $$\vec{D} = \epsilon \vec{E} \text{ everywhere.} \rightarrow \nabla \times \vec{D} = \vec{0}.$$
    - Assume $\vec{E}_{vac}$ is the field due to free charge distribution in vacuum, then
      $$\vec{D} = \epsilon_0 \vec{E}_{vac}, \quad \vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{1}{\epsilon}\epsilon_0 \vec{E}_{vac} = \frac{1}{\epsilon_r}\vec{E}_{vac}$$
    - The field produced by a point charge q embedded in the dielectric is
      $$\vec{E} = \frac{1}{4\pi\epsilon}\frac{q}{r^2}\hat{\vec{r}}$$

- <u>At the boundaries</u>
  - $\nabla \times \vec{D} \neq \vec{0}$
  - In a dielectric
    $$\vec{P} = [\chi_e/(1+\chi_e)]\,\vec{D}$$
    $$\rho_b = -\nabla\cdot\vec{P} = -\frac{\chi_e}{1+\chi_e}\nabla\cdot\vec{D} = -\frac{\chi_e}{1+\chi_e}\rho_f.$$
    - Unless free charge is embedded in the material.
      $$\rho_b = \rho_f = 0, \quad \rho = \rho_b + \rho_f = 0.$$
    - Any net charge must reside at the boundries (or surfacers).
  - Boundary conditions: see the previous page.



At this boundary, we have obtained two different boundary conditions for the normal derivative of potential:

$$\frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} = -\frac{(\sigma_f + \sigma_b)}{\epsilon_0} \quad \text{---- (i)}$$

$$\text{and} \quad \epsilon_1 \frac{\partial V_1}{\partial n} - \epsilon_2 \frac{\partial V_2}{\partial n} = -\sigma_f \quad \text{---- (ii)}$$

(i) is from $E_1^\perp - E_2^\perp = (\sigma_f + \sigma_b)/\epsilon_0$

(ii) is from $D_1^\perp - D_2^\perp = \sigma_f$

- using $D_1^\perp = \epsilon_0 E_1^\perp + P_1^\perp$ and $D_2^\perp = \epsilon_0 E_2^\perp + P_2^\perp$ we can see that, to make (i) and (ii) consistent, we need $P_1^\perp - P_2^\perp = -\sigma_b$
- Indeed, this is true.
  - consider $\sigma_b = \vec{P} \cdot \hat{n}$, ($\hat{n}$ normal vector) — this is a general notation, to be determined.
  - For dielectric 1, its normal vector should be downward $(= -\hat{n})$, and thus
    $$\sigma_{b1} = \vec{P}_1 \cdot (-\hat{n}) = -P_1^\perp$$
  - For dielectric 2, its normal vector should be upward $(= \hat{n})$, and thus
    $$\sigma_{b2} = \vec{P}_2 \cdot \hat{n} = P_2^\perp.$$
  - Thus at the boundary, the total surface charge
    $$\sigma_b = \sigma_{b1} + \sigma_{b2} = -P_1^\perp + P_2^\perp$$
    or $P_1^\perp - P_2^\perp = -\sigma_b$.

Note:
- we pick $\hat{n}$ to be upward, then dielectric 1 becomes "above" and 2 "below".
  If you pick $\hat{n}$ to be downward, the result will stay the same.
- The surface bound charge density is different at the two sides. The σ_b in the boundary condition is the total surface charge.