Magnetostatics

1. The Lorentz Force Law

— currents and magnetic fields

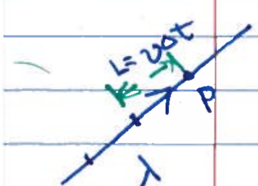
- In electrostatics, we considered the simple case in which the source charge is at rest.
- The time has come to consider the forces between charges in motion.

* Currents

- The current in a wire is the charge per unit time passing a given point. current is measured in coulombs-per-second, or amperes (A):

$$1\,A = 1\,c/s.$$

- In practice, it is ordinarily the negatively charged electrons that do the moving — in the direction opposite to the electric current.
  - By definition, negative charges moving to the left count the same as positive ones to the right.
  - We shall often pretend it's the positive charges that move.
- A line charge λ travels down a wire at speed v. The charge that passes point P is (in a time interval Δt)



$$\Delta Q = \lambda L = \lambda v \Delta t.$$

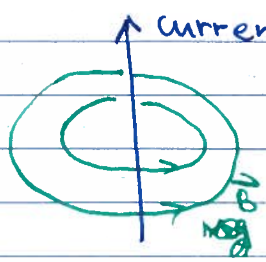
Then the current is

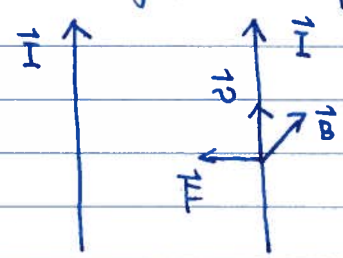
$$I = \Delta Q/\Delta t = \lambda \vec{v}.$$ (currents are vectors).

  - A neutral wire contains as many stationary positive charges as mobile negative ones. The former do not contribute to the current — the charge density λ refers only to the moving charges.
  - In the unusual situation where both types move, $\vec{I} = \lambda_+ \vec{v}_+ + \lambda_- \vec{v}_-$

* Magnetic fields

- Currents in opposite directions repel.
  Currents in same directions attract.
- Note that the wires are in fact electrically neutral. So the force is not electrostatic in nature.
- whereas a stationary charge produces only an electric field $\vec{E}$ in the space around it, a moving charge generates, in addition, a magnetic field $\vec{B}$.
- A compass can be used to detect the direction of the local magnetic field.
  - The field circles around the wire. If you grab the wire with your right hand — thumb in the direction of the current — your fingers curl around in the direction of the magnetic field.





* Magnetic forces

- The Lorentz force law: the magnetic force on a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is

$$\vec{F}_{mag} = Q(\vec{v} \times \vec{B}).$$

  It is a fundamental axiom of the theory, whose justification is to be found in experiments.
- In the presence of both electric and magnetic fields, the net force on Q would be

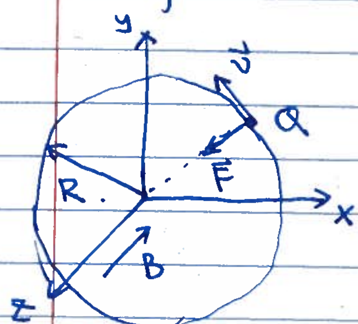
$$\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})].$$

- Magnetic forces do no work. For if Q moves an amount $d\vec{l} = \vec{v}\,dt$, the work done is

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v}\,dt = 0.$$

because $(\vec{v}\times\vec{B}) \perp \vec{v}$, and $(\vec{v}\times\vec{B})\cdot\vec{v} = 0$.

- Magnetic forces may alter the direction in which a particle moves, but they cannot speed it up or slow it down.
- Cyclotron motion



— A uniform magnetic field $\vec{B} = -B\hat{z}$, the magnetic force points inward,

$$F = QvB = \frac{mv^2}{R}$$

which provides the centripetal acceleration.

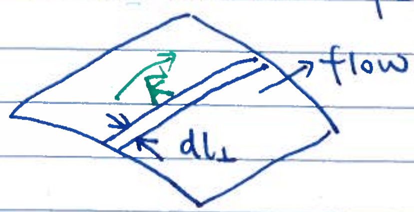
— The above equation is know as the cyclotron formula because it describes the motion of a particle in a cyclotron.

* Magnetic force on currents

- Three types of currents

— line current: $\vec{I} = \lambda\vec{v}$

— surface current: when charges flow over a surface, we describe it by the surface current density $\vec{K}$, the current per unit width.

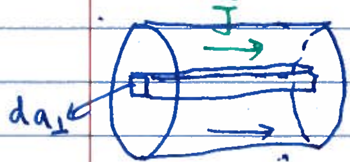


Consider a ribbon of infinitesimal width $dl_\perp$, runing parallel to the flow. If the current in this ribbon is $d\vec{I}$, then the surface ~~charge~~ current density is $\vec{K} \equiv \frac{d\vec{I}}{dl_\perp}$.

If the surface charge density is σ and its velocity is $\vec{v}$, then $\vec{K} = \sigma\vec{v}$.

In general, $\vec{K}$ will differ from point to point over the surface, reflecting variations in σ and/or $\vec{v}$.

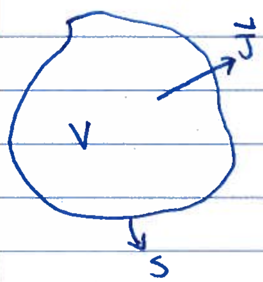
→ Volume current: when the flow of charge is distributed throughout a 3-D region, we describe it by the volume charge density, J, the current per unit area.



- Consider a "tube" of infinitesimal cross section $da_\perp$, runing parallel to the flow. If the current in this tube is $d\vec{I}$, the volume current density is

$$\vec{J} \equiv d\vec{I}/da_\perp .$$

- If the volume charge density is ρ and the velocity is $\vec{v}$, then $\vec{J} = \rho\vec{v}$.



- The total current crossing a closed surface S can be written as

$$I = \oint_S \vec{J}\cdot d\vec{a} = \int_V (\nabla\cdot\vec{J})\, d\tau .$$

This is also the charge per unit time leaving the volume V. Because charge is conserved, whatever flows out through the surface must come at the expense of what remains inside: ↗ total derivative . ↗ partial derivative

$$I = -\frac{d}{dt}\int_V \rho\, d\tau = -\int_V \left(\frac{\partial\rho}{\partial t}\right) d\tau .$$

Thus $\int_V (\nabla\cdot\vec{J})\, d\tau = -\int_V \left(\frac{\partial\rho}{\partial t}\right) d\tau$

since this applies to any volume, we conclude that

$$\boxed{\nabla\cdot\vec{J} = -\frac{\partial\rho}{\partial t}}$$

This equation is called the continuity equation, which is the mathematical statement of local charge conservation.

- The magnetic force on a current.
  - On a segment of current-carrying wire is
    $$\vec{F}_{mag} = \int (\vec{v}\times\vec{B})\,dq = \int (\vec{v}\times\vec{B})\,\lambda\, dl = \int (\vec{I}\times\vec{B})\,dl$$
    $$= \int I\,(d\vec{l}\times\vec{B}).$$
    Typically, the current is constant in magnitude along the wire, and in that case
    $$\vec{F}_{mag} = I\int (d\vec{l}\times\vec{B}).$$
  - The magnetic force on the surface current is
    $$\vec{F}_{mag} = \int (\vec{v}\times\vec{B})\,\sigma\, da = \int (\vec{K}\times\vec{B})\,da.$$
  - The magnetic force on a volume current is
    $$\vec{F}_{mag} = \int (\vec{v}\times\vec{B})\,\rho\, d\tau = \int (\vec{J}\times\vec{B})\,d\tau.$$

In analogue to $q \sim \lambda dl \sim \sigma da \sim \rho d\tau$ for the various charge distributions, the equations for point, line, surface, and volume currents:
$$\sum_{i=1}^{n} (\;)\, q_i \vec{v}_i \sim \int_{line} (\;)\,\vec{I}\,dl \sim \int_{surface} (\;)\,\vec{K}\,da \sim \int_{volume} (\;)\,\vec{J}\,d\tau$$

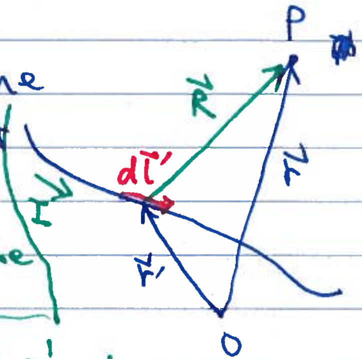
* <u>steady currents and magnetostatics</u>

- stationary charges produce $\vec{E}$ that are constant in time — electrostatics; steady currents produce magnetic fields that are constant in time — magnetostatics.
- By steady currents, we mean a continuous flow that has been going on forever.
- Formally, electro/magnetostatics is the regime
  $$\partial\rho/\partial t = 0, \quad \partial\vec{J}/\partial t = \vec{0}.$$
  at all places and all times.
- Since $\partial\rho/\partial t = 0$ in magnetostatics, the continuity equation becomes
  $$\nabla\cdot\vec{J} = 0.$$

2. The magnetic field of a steady Current.

– The Biot-Savart Law

Electrostatics : ~~Colum~~ Coulomb's Law
Magnetostatics : Biot-Savart law.



• The magnetic field of a steady line current is given by the Biot-Savart Law

The path of the flow is dictated by the shape of the wire

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I} \times \hat{R}}{R^2}\, dl'$$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{R^2}\hat{R}\, dq$$

$$= \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{R}}{R^2} \rightarrow \left(\text{Note the } \frac{1}{R^2} \text{ dependence}\right)$$

The integration is along the current path, in the direction of the flow; $d\vec{l}'$ is an element of length along the wire, and $\vec{R}$ is the vector from the source to the field point $\vec{r}$.

• The unit of $\vec{B}$ is newtons per ampere-meter, or teslas (T):

$$\vec{F} = Q(\vec{v} \times \vec{B})$$

$$1\,T = 1\,N/(A\cdot m)$$

$$N = C\left(\frac{m}{s}\right)[\vec{B}] = A\cdot m\cdot[\vec{B}]$$

• The constant μ_0 is called the permeability of free space:

$$\mu_0 = 4\pi \times 10^{-7}\, N/A^2.$$

$$\underset{N/(A\cdot m)}{\vec{B}} = \underset{N/A^2}{\frac{\mu_0}{4\pi}}\ \underset{A}{I} \underset{m^{-1}}{\int \frac{d\vec{l}' \times \hat{R}}{R^2}}$$

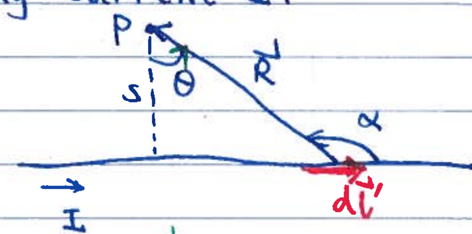
• For surface and volume currents,

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}') \times \hat{R}}{R^2}\, da' \quad \text{and} \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}') \times \hat{R}}{R^2}\, d\tau'.$$

• The superposition principle applies to magnetic fields just as it does to electric fields: if you have a collection of source currents, the net field is the vector sum of the fields due to each of them taken separately.

Example Find the magnetic field ~~of~~ a distance s from a long straight wire carrying a steady current I.

Sol $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}' \times \hat{R}}{R^2}$



- we should first turn it to be an integral over scalars.

Note that

$$|d\vec{l}' \times \hat{R}| = dl' \sin\alpha = dl' \cos\theta$$

and $d\vec{l}' \times \hat{R}$ points out of the page.

Since $l' = s\tan\theta$, $dl' = \frac{s}{\cos^2\theta} d\theta$.

and $s = R\cos\theta$, $\frac{1}{R^2} = \frac{\cos^2\theta}{s^2}$

To integrate over $d\vec{l}'$, we can integrate over θ, instead.

- The field of any ~~stra~~ straight segment of wire, in terms of the initial and final angles θ_1 and θ_2,

$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right) \cos\theta \, d\theta$$

$$= \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta \, d\theta = \frac{\mu_0 I}{4\pi s} (\sin\theta_2 - \sin\theta_1).$$

- In the case of an infinite wire,

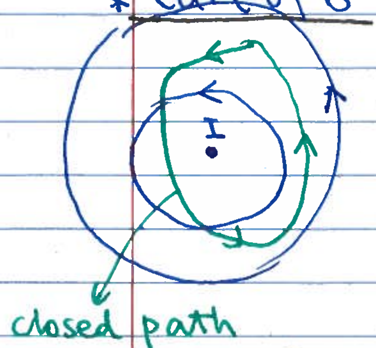
$$\theta_1 = -\pi/2 \text{ and } \theta_2 = \pi/2.$$

So we obtain $B = \frac{\mu_0 I}{2\pi s}$. $\Rightarrow \vec{B} = \frac{\mu_0 I}{2\pi s} \hat{\phi}$.

Note that the field is inversely proportional to the distance from the wire.

– The divergence and Curl of $\vec{B}$.

* **Curl of $\vec{B}$**



• To calculate the divergence of $\vec{B}$, we first calculate its path integral.
Choose a loop that encloses the wire, and in the cylindrical coordinates (s, ϕ, z), with the current flowing along the z axis,

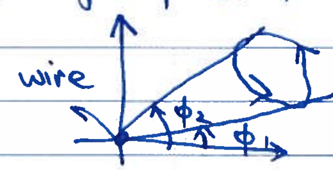
$$\vec{B} = (\mu_0 I / 2\pi s)\,\hat{\phi}.$$

and $d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}$.

So $\oint \vec{B}\cdot d\vec{l} = \frac{\mu_0 I}{2\pi}\oint \frac{1}{s}\, s\,d\phi = \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I.$

– This assumes the loop encircles the wire exactly once.

– If it didn't enclose the wire at all, then ϕ would go from ϕ_1 to ϕ_2 and back again, with $\int d\phi = 0$.



• Now suppose we have a bundle of straight wires. Each wire that passes through our loop contributes $\mu_0 I$, and those outside contribute nothing. The line integral will then be

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}} \quad \text{(Ampère's law)}$$

where I_{enc} stands for the total current enclosed by the integration path. (Amperian loop)

• If the flow of charge is represented by a volume current density $\vec{J}$, the enclosed current is

$$I_{enc} = \int \vec{J}\cdot d\vec{a}.$$

with the integral taken over any surface bounded by the loop. Applying Stoke's theorem,

$$\int (\nabla\times\vec{B})\cdot d\vec{a} = \mu_0 \int \vec{J}\cdot d\vec{a}. \rightarrow \boxed{\nabla\times\vec{B} = \mu_0\vec{J}}$$

• Although our derivation is restricted to infinite straight line currents (and combinations thereof), the curl of $\vec{B}$ we obtained is general.

* Divergence of $\vec{B}$

- "Gauss's law" for magnetic field

$$\oint \vec{B}\cdot d\vec{a} = 0 \iff \nabla\cdot\vec{B} = 0.$$

(Integral form) (differential form).

- The magnetic field $\vec{B}$ has divergence equal to 0,
  - in other words, it is a solenoidal vector field. Divergence-less
- The name "Gauss's law for magnetism" is not universally used. Griffiths's textbook explicitly says the law has "no name".
- It is also equivalent to the statement that magnetic monopoles do not exist. [Jackson].

there is no point sources for $\vec{B}$.

Thus the law is also called "Absence of free magnetic poles". [still in extensive search].

- Gauss's Law for magnetism is equivalent to the statement that the field lines have neither a beginning nor an end: Each one either forms a closed loop, winds around forever without ever quite joining back up to itself exactly, or extends to infinity.

— Maxwell's equations for electrostatics and magnetostatics

Maxwell's equations:

$$\begin{cases} \nabla\cdot E = \frac{1}{\epsilon_0}\rho & \text{(Gauss's law)} \\ \nabla\times\vec{E} = \vec{0} & \text{(no name)} \\ \nabla\cdot\vec{B} = 0 & \text{(no name)} \\ \nabla\times\vec{B} = \mu_0\vec{J} & \text{(Ampère's law).} \end{cases}$$

The Lorentz force Law $\vec{F} = Q(\vec{E} + \vec{v}\times\vec{B})$.

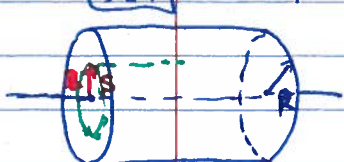
- Maxwell's equations and the force law constitute the most elegant formulation of electrostatics and magnetostatics.

— <u>Applications of Ampère's Law</u>

- Ampère's law is always true for steady currents.
- For currents with appropriate symmetry, Ampère's law in integral form offers an efficient way of calculating the magnetic field.
- So it's not always useful.
  Only when the symmetry of the problem enables you to pull $\vec{B}$ outside the integral $\oint \vec{B}\cdot d\vec{l}$, can you calculate the magnetic field from Ampère's law. When it doesn't work, we have to use Biot–Savart law.
- The current configurations that can be handled by Ampère's law are:
  - Infinite straight lines
  - Infinite planes
  - Infinite solenoids
  - Toroids.

$\boxed{\text{Example}}$ A steady current I flows down a long cylindrical wire of radius R. Find the magnetic field, both inside and outside the wire, if the current is distributed in such a way that J is proportional to s, the distance from the axis.

$\boxed{\text{Sol}}$ $J = ks$, k to be determined.



$$I = \int_0^R J\,da = \int_0^R ks(2\pi s)\,ds = 2\pi k R^3/3 \Rightarrow k = 3I/(2\pi R^3)$$

Choose an Amperian loop of radius s, then

- $I_{enc} = \int_0^s J\,da = \int_0^s ks'(2\pi s')\,ds' = \dfrac{2\pi k s^3}{3} = I\dfrac{s^3}{a^3}$
  (when $s < a$)
- $I_{enc} = I$, when $s > a$.

The Ampère's law gives

$$\oint \vec{B}\cdot d\vec{l} = B\cdot 2\pi s = \mu_0 I_{enc}$$

so $\vec{B} = \begin{cases} \mu_0 I s^2/(2\pi a^3)\,\hat{\phi}, & s < a \\ \mu_0 I/(2\pi s)\,\hat{\phi} & s > a. \end{cases}$

3. Magnetic Vector Potential

— The vector potential

- If the divergence of a vector field ($\vec{F}$) vanishes everywhere then $\vec{F}$ can be expressed as the curl of a vector potential $\vec{A}$.

$$\nabla \cdot \vec{F} = 0 \iff \vec{F} = \nabla \times \vec{A} \quad \text{(equivalent)}.$$

- Since $\nabla \cdot \vec{B} = 0$, we can introduce a vector potential $\vec{A}$ in magnetostatics

$$\vec{B} = \nabla \times \vec{A}.$$

  - Since the divergence of a curl is always zero, the potential formulation atomatically takes care of $\nabla \cdot \vec{B} = 0$.
  - The Ampère's law requires

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}.$$

- Built-in ambiguity of $\vec{A}$
  - We can add to $\vec{A}$ any function whose curl vanishes (i.e. the gradient of any scalar), with no effect on the magnetic field $\vec{B}$.

$$\vec{B} = \nabla \times (\vec{A} + \text{gradient of any scalar}) = \nabla \times \vec{A}$$

  - It's always possible to elimate the divergence of $\vec{A}$, such that ~~[illegible]~~ $\nabla \cdot \vec{A} = 0$.
  - Proof: ① Suppose that our original potential, $\vec{A}_0$, is not divergenceless, and then we add to it the gradient of λ

$$\vec{A} = \vec{A}_0 + \nabla \lambda \quad (\nabla \cdot \vec{A}_0 \neq 0).$$

The new divergence is

$$\nabla \cdot \vec{A} = \nabla \cdot \vec{A}_0 + \nabla^2 \lambda.$$

then we can have $\nabla \cdot \vec{A} = 0$, if

$$\nabla^2 \lambda = - \nabla \cdot \vec{A}_0$$

② Since this equation is mathematically identical to Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \quad \left(\text{solution } V = \frac{1}{4\pi\epsilon_0} \int \frac{\rho}{R} d\tau'\right).$$

similar to how to solve poisson's equation, if $\nabla \cdot A_0$ goes to zero at infinity, then

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \vec{A}_0}{R}\, d\tau'.$$

If $\nabla \cdot \vec{A}_0$ doesn't go to zero at infinity, then we'll have to use other means to discover the appropriate λ.

③ The essential point is: It is always possible to make the vector potential divergenceless.

- The definition $\vec{B} = \nabla \times \vec{A}$ specifies the curl of $\vec{A}$, but it doesn't say anything about the divergence. There is freedom to eliminate the divergence of $\vec{A}$.

• Ampère's law in terms of $\vec{A}$.

- With $\nabla \cdot \vec{A} = 0$, we have

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = -\nabla^2 \vec{A} = \mu_0 \vec{J}$$

or $$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

This is three Poisson's equations, one for each Cartesian component.

- Assuming $\vec{J}$ goes to zero at infinity, we can read off the solution

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')}{R}\, d\tau'$$

• For line and surface currents,

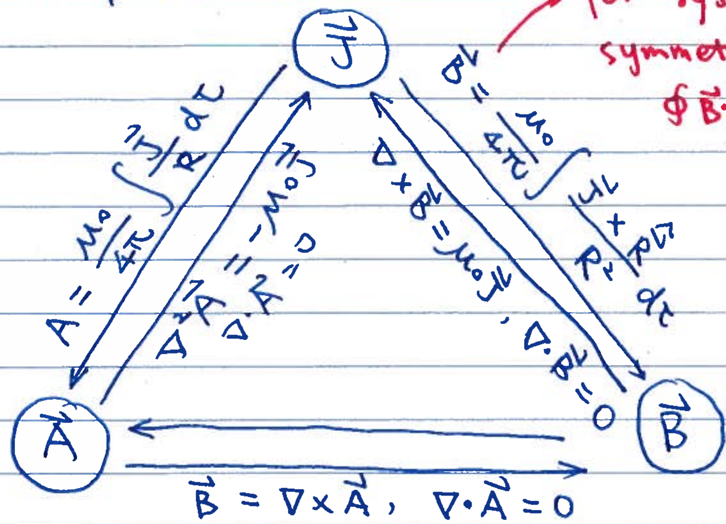
$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{I}}{R}\, dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{R}\, d\vec{l}'$$

and $$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{R}\, da'$$

• If the current does not go to zero at infinity, we have to find other ways to get $\vec{A}$.

• Ordinarily the direction of $\vec{A}$ would match that of the current.

- The relations among the fundamental quantites of magnetostatics: the current density, the field and the potential.

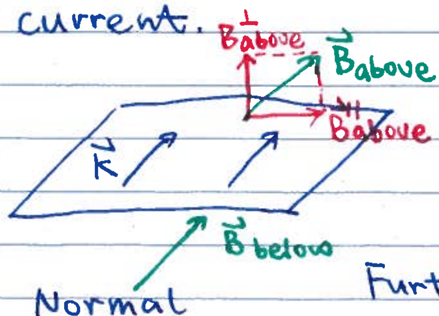


→ for systems with appropriate symmetry, we can also use $\oint \vec{B}\cdot d\vec{l} = \mu_0 \int \vec{J}\cdot d\vec{a}$

Always write down both the divergence and the curl.

— Boundary condition

The ~~dis~~ magnetic field is discontinuous at a surface current.



We denote the fields above and below the surface current as

$$\vec{B}_{above} = (B^{\perp}_{above}, \vec{B}^{\parallel}_{above})$$
$$\vec{B}_{below} = (B^{\perp}_{below}, \vec{B}^{\parallel}_{below}).$$

Further $\vec{B}^{\parallel} = (B^{\parallel}\perp K,\ B^{\parallel} \parallel K) = (B^{\parallel}, B^{\parallel}_{p})$

- ~~Tangential~~ Normal components: We apply $\oint \vec{B}\cdot d\vec{a} = 0$ to a wafer-thin pillbox straddling the surface, we get

$$B^{\perp}_{above} = B^{\perp}_{below}$$

- Tangential components (which is $\perp \vec{K}$)

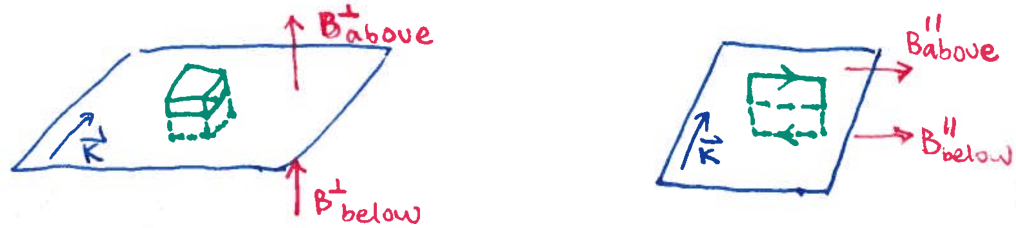
we apply $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$ to an Amperian loop running perpendicular to the current,

$$\oint \vec{B}\cdot d\vec{l} = (B^{\parallel}_{above} - B^{\parallel}_{below})\, l = \mu_0 I_{enc} = \mu_0 K l.$$

thus $B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$.

- Tangential components (which is $\parallel \vec{K}$) → $B^{\parallel}_{p}$ component.

A similar Amperian loop running parallel to the



current reveals that the parallel component is continous

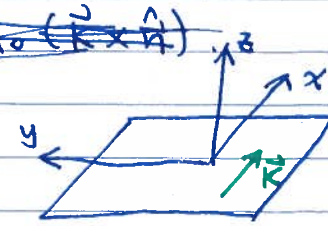
- Combine the above boundary conditions, we have
$$\boxed{\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n})},$$
where $\hat{n}$ is a unit vector perpendicular to the surface pointing "upward".

* Boundary conditions for $\vec{A}$

Suppose we choose divergenceless $\vec{A}$, such that $\nabla \cdot \vec{A} = 0$.

- Then $\nabla \cdot \vec{A} = 0 \rightarrow \oint \vec{A} \cdot d\vec{a} = 0 \rightarrow A^{\perp}_{above} = A^{\perp}_{below}$
- Since $\nabla \times \vec{A} = \vec{B}$, we have
$$\oint \vec{A} \cdot d\vec{l} = \int (\nabla \times \vec{A}) \cdot d\vec{a} = \int \vec{B} \cdot d\vec{a} = \text{flux of } \vec{B}$$
Thus if we pick an Amperian loop running parallel to $\vec{A}''$, then
$$(A''_{above} - A''_{below})\, l = \text{flux of } \vec{B} \text{ through the surface}$$
l bounded by the Amperian loop.
Because the flux through an Amperian loop of vanishing thickness is zero,
$$\vec{A}''_{above} = \vec{A}''_{below}.$$
- Since both the normal and the tangential components are continuous, the vector potential is continuous across any boundary:
$$\boxed{\vec{A}_{above} = \vec{A}_{below}.}$$
- Besides, since $\begin{cases} \vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n}) \\ \vec{A}_{above} = \vec{A}_{below} \text{ and } \nabla \cdot \vec{A} = 0 \end{cases}$

we have $\cancel{\vec{A}_{above} - \vec{A}_{below} = \mu_0 (\vec{K} \times \hat{n})}$
$$\boxed{\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}.}$$



Proof: Set up the Cartesian coordinates at the surface, with $z \perp$ surface and $x \parallel \vec{K}$.

- Because $\vec{A}_{above} = \vec{A}_{below}$ at every point on the surface, $(z=0)$
i.e. $\vec{A}_{above}(x, y, 0) = \vec{A}_{below}(x, y, 0)$
thus $\partial \vec{A}/\partial x$ and $\partial \vec{A}/\partial y$ are the same above and below,

• Since $\vec{B} = \nabla \times \vec{A} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$
$+ \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right).$

$$\vec{B}_{above} - \vec{B}_{below} = \left(-\frac{\partial A^y_{above}}{\partial z} + \frac{\partial A^y_{below}}{\partial z}\right)\hat{x} + \left(\frac{\partial A^x_{above}}{\partial z} - \frac{\partial A^x_{below}}{\partial z}\right)\hat{y}$$

where we have used the fact that $\partial A/\partial x$ and $\partial A/\partial y$ are continuous across the surface.

On the other hand
$\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K} \times \hat{n}) = \mu_0 K (-\hat{y}).$

Thus $\begin{cases} -\partial A^y_{above}/\partial z + \partial A^y_{below}/\partial z = 0 \\ \partial A^x_{above}/\partial z - \partial A^x_{below}/\partial z = -\mu_0 K. \end{cases}$

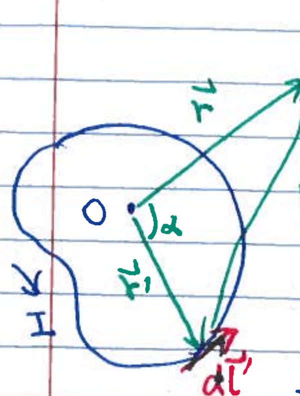
• Thus the normal derivative of the component of $\vec{A}$ parallel to $\vec{K}$ suffers a discontinuity $-\mu_0 K$;

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}.$$

— Multiple Expansion of the vector potential.

• We will write the vector potential of a localized current distribution in the form of a power series in $1/r$, where r is the distance to the point in question.

• If r is sufficiently large, the series will be dominated by the lowest nonvanishing contribution, and the higher terms can be ignored.



• Again, recall

$$\frac{1}{R} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'\cos\alpha}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha).$$

The vector potential of a current loop can be written as

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{1}{R}\, d\vec{l}' = \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha)\, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi}\Big[\underbrace{\frac{1}{r}\oint d\vec{l}'}_{\text{monopole term}} + \underbrace{\frac{1}{r^2}\oint r'\cos\alpha\, d\vec{l}'}_{\text{dipole term}} + \underbrace{\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right) d\vec{l}'}_{\text{quadrupole term}} + \cdots$$

- <u>Monopole term:</u> Alway zero; as the integral is just the total vector displacement around a closed loop. $\oint d\vec{l}' = 0$.
  This also reflects the fact that there are no magnetic monopoles in nature.
- <u>Dipole term:</u> The dominant term in the absence of any monopole contribution.

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha\, d\vec{l}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}')\, d\vec{l}'$$

using equation $\oint_P (\vec{c}\cdot\vec{r})\, d\vec{l}' = \left(\int_S d\vec{a}'\right)\times\vec{c}$, we have

$(\vec{c}=\hat{r}) \rightarrow \oint (\hat{r}\cdot\vec{r}')\, d\vec{l}' = \int d\vec{a}' \times \hat{r}$

– Then $\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\int d\vec{a}' \times \hat{r} \equiv \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$

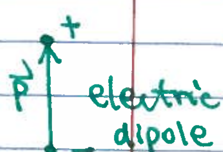
where $\vec{m}$ is the magnetic dipole moment

$$\vec{m} \equiv I\int d\vec{a} = I\vec{a}.$$

Here $\vec{a}$ is the "vector area" of the loop.

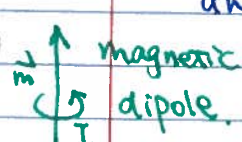
– The magnetic dipole moment is independent of the choice of origin.

– <u>Perfect dipole:</u> To devise a current distribution whose potential is pure dipole, we must take an infinitesimally small loop, and an infinite current, with the product $m = Ia$ held fixed.
If we put a perfect dipole $\vec{m}$ at the origin and let it point in the z-direction, the the potential at point (r, θ, ϕ) is



$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

and hence $\vec{B}_{dip}(\vec{r}) = \nabla\times\vec{A} = \frac{\mu_0 m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$.



Magnetic dipoles are tiny current loops, due to moving electric charges, not due to magnetic monopoles.

MAGNETIC FIELDS IN MATTER

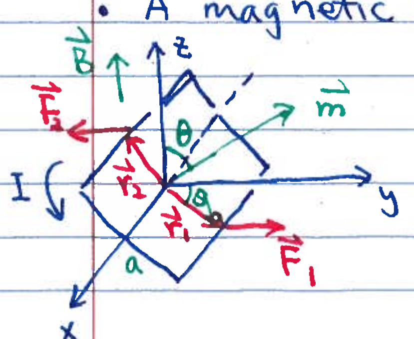
1. Magnetization

— Diamagnets, paramagnets, Ferromagnets.

- All magnetic phenomena are due to eletric charges in motion.
- Magnetic material: On an atomic scale there are tiny currents due to eletrons orbiting around nuclei and spinning about their axes.
  - These current loops are so small that we may treat them as magnetic dipoles on macroscopic level.
  - Because of randomness due to thermal motion, these dipoles cancel each other.
  - When an external $\vec{B}$ is applied, a net alignment of these magnetic dipoles occurs, and the medium becomes magnetically polarized, or magnetized.
- Three types of magnetic materials
  - Paramagnets: acquire a magnetization parallel to $\vec{B}$.
  - Diamagnets: ... opposite to $\vec{B}$.
  - Ferromagnets: Retain their magnetization after the external $\vec{B}$ is removed. The magnetization of this type of materials is not determined by the present field but by the whole magnetic history of the objext. (iron, nickel, and cobalt).

— Torques on Magnetic dipoles and paramagnetism

- A magnetic dipole experiens a torque in a magnetic field



  - Center the rectangular current loop at the orign, and tilt it an angle θ from the z axis towards the y axis.
  - Let $\vec{B}$ points in the z direction.

- The forces on the two sloping sides tend to stretch the loop, but they don't rotate it.
- The forces on the horizontal sides are equal and opposite, but the net torque is

$$\vec{N} = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2$$
$$= \left(\frac{a}{2}\sin\theta\, F + \frac{a}{2}\sin\theta\, F\right)\hat{x}$$
$$= aF\sin\theta\,\hat{x}$$

The magnitude of the force on each of these segments is $F = IbB$. (note $\vec{F}_{mag} = I\int(d\vec{l}\times\vec{B})$).

Thus $\vec{N} = aF\sin\theta\,\hat{x} = IabB\sin\theta\,\hat{x}$
$= mB\sin\theta\,\hat{x}$ ($m = Iab$ dipole moment of the loop).
$= \vec{m}\times\vec{B}$

- The torque on any localized current distribution, in the presence of a uniform field.
  Or the torque for a perfect dipole of infinitesimal size: $\vec{N} = \vec{m}\times\vec{B}$.
- The net force on a current loop (not necessarily rectangle) in a uniform field is

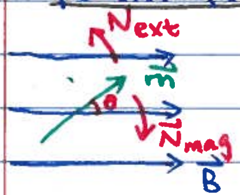
$$\vec{F} = I\oint(d\vec{l}\times\vec{B}) = I\underbrace{\left(\oint d\vec{l}\right)}_{\text{the net displacement around a closed loop} = 0}\times\vec{B} = \vec{0}.$$

- It is the torque above that accouts for paramagnetism.
  - Although every electron constitutes a magnetic dipole, paramagnetism is not a universal phenomenon.
    + This is because the Pauli exclusion principle tends to lock the electrons within a given atom together in pairs with opposing spins, and this effectively neutralizes the torque on the combination.
    + Thus, paramagnetism most often occures in atoms or molecules with an odd number of electrons, where the "extra" unpaired member is subject to the magnetic torque.
  - Besides, random thermal collisions tend to destroy

the order, and thus the alignment is far from complete.

Hydrogen $\boxed{\uparrow}$ Helium $\boxed{\uparrow\downarrow}$ Lithium $\boxed{\uparrow\downarrow}\ \boxed{\uparrow}$

• Potential energy of a magnetic dipole in $\vec{B}$.



- The magnetic torque tends to align $\vec{m}$ with the field direction.

$$\vec{N}_{mag} = \vec{m} \times \vec{B} = mB \sin\theta \ (\text{inward}).$$

- Thus some work needs to be done to rotate it by $d\theta$ against the torque.

$$N_{ext} = N_{mag} \quad (\text{Magnitude of external torque} = \text{magnetic torque}).$$

$$dW = N_{ext}\, d\theta = mB \sin\theta\, d\theta.$$

Note that if we rotate the dipole counter-clockwisely by a small angle $d\theta$ ($d\theta$ is positive, as θ increases), we do positive work, and indeed $mB\sin\theta\, d\theta$ is positive.

- The total work needed to rotate the dipole from θ_1 to θ_2

$$W = \int dW = \int_{\theta_1}^{\theta_2} mB\sin\theta\, d\theta = mB(-\cos\theta)\Big|_{\theta_1}^{\theta_2}$$
$$= -mB(\cos\theta_2 - \cos\theta_1).$$

- The change in potential energy is (from θ_1 to θ_2)

$$\Delta U = W = -mB(\cos\theta_2 - \cos\theta_1) = U_2 - U_1$$

we can define the potential energy as

$$U = -mB\cos\theta = -\vec{m}\cdot\vec{B}.$$

- Note that using this defination

$$\theta = 0, \quad \vec{m} \parallel \vec{B}, \quad U = -mB$$
$$\theta = \pi/2, \quad \vec{m} \perp \vec{B}, \quad U = 0$$
$$\theta = \pi, \quad \vec{m} \parallel (-\vec{B}), \quad U = +mB.$$

— Effect of a Magnetic field on Atomic orbits and diamagnetism

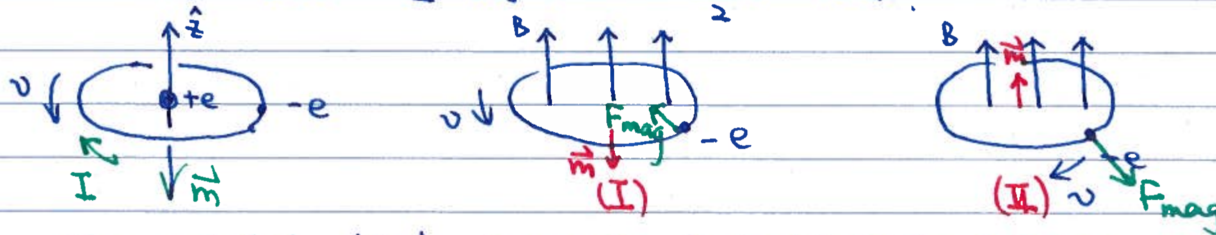
• Empirical effect: In diamagnetic materials the induced dipole moments point opposite to the magnetic field. Diamagnetism is a quantum phenomenon. Here we only give a qualitative account of diamagnetism.

- Electrons revolve around the nucleus.
  - Assume the orbit is a circle of radius R, the period of its circular motion is $T = 2\pi R/v$.

    The current produced is
    $$I = -e/T = -\frac{ev}{2\pi R} \quad (e \text{ is positive}).$$

    I is opposite to v of electron.

    The orbital dipole moment is
    $$\vec{m} = I\vec{a} = I\pi R^2 \hat{z} = -\frac{1}{2} e v R \hat{z}$$



  - The orbital dipole moment is subject to a torque $\vec{m} \times \vec{B}$ when you turn on a magnetic field.

    But it's a lot harder to tilt the entire orbit than it is the spin, so the orbital contribution to paramagnetism is small.

    However, there is a more significant effect on the orbital motion: The electron speeds up or slows down, depending on the orientation of $\vec{B}$.

    Case I: When $\vec{B}$ is turned on, the electron speeds up. A change in orbital speed means a change in the dipole moment $\Delta\vec{m} \approx -\frac{1}{2} e \Delta v R \hat{z}$

    The change in $\vec{m}$ is opposite to the direction of $\vec{B}$.

    Case II: when the electron circles the other way, the dipole moment would point upward. Such an orbit would be slowed down by the field. The ~~change in~~ dipole moment would be reduced, and thus the change in $\vec{m}$ is still opposite to $\vec{B}$.
  - The induced dipole moment $\Delta\vec{m}$ is opposite to the magnetic field.

- Ordinarily, the electron orbits are randomly oriented, and the orbital dipole moments cancel out.
  In the presence of a magnetic field, each atom picks up a little "extra" dipole moment ("induced dipole moment), and these increments are all antiparallel to the field.
  This is the mechanism responsible for ~~diag~~ diamagnetism.
  It is a universal phonomenon, affecting all atoms.
  However, it is typically much weaker than paramagnetism, and is therefore observed mainly in atoms with even numbers of electrons, where paramagnetism is usually absent.

- <u>Something about Quantum physics</u>

- As we have discussed, the magnetic dipole moment of an electron in an atomic orbital arises from both the electron spin and the orbital angular momentum.
- Put the system in a magnetic field, both will introduce an energy shift ($u = -\vec{m} \cdot \vec{B}$).
  - The energy shift of the atomic orbital is determined by the magnetic quantum number.
  - Associated with the electron spin, it is expressed in the spin quantum number.

– Magnetization

- Put a matter in a magnetic field, the tiny dipoles inside will ~~be~~ obtain a net alignment along some direction, and the matter becomes magnetized/magnetically polarized.
- Two mechanisms:
  - Paramagnetism, the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to the field.
  - Diamagnetism, the orbital speed of the electrons is altered in such a way as to change the orbital dipole moment in a direction opposite to the field.
- Whatever the cause, we describe the state of magnetic polarization by the Magnetization

$$\vec{M} \equiv \text{magnetic dipole moment per unit volume.}$$

- In general, when a sample is placed in a region of nonuniform field, the paramagnet is attracted into the field, whereas the diamagnet is repelled away. However, the actual forces are very weak, because the diamagnetism and paramagnetism are extremely weak.

2. The field of a magnetized object

— Bound currents

- In magnetized objects, note that every charge is attached to a particular atom, and thus each charge moves only in a tiny little loop within a single atom.

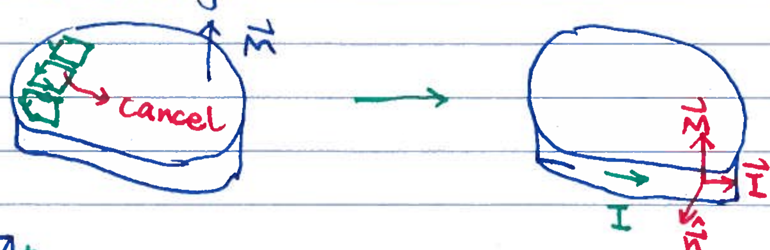
  Thus we call the induced currents because of magnetization "bound currents".

  A bound current is a genuine current and it produces a magnetic field in the same way any other current does.

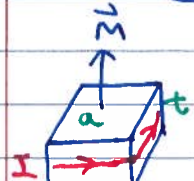
* Bound surface current.

- Inside a thin slab of uniformly magnetized material, with the dipoles represented by tiny currents loops, all the "internal currents cancel.

  However, at the edge there is no adjacent loop to do the canceling. The whole thing is equivalent to a single ribbon of current I flowing around the boundary.



$\vec{I}$ is in the direction of $\vec{M} \times \hat{n}$.



For a tiny volume which has area a and thickness t; Its dipole moment is, in terms of $\vec{M}$,

$$m = MV = Mat$$

and in terms of the circulating current I,

$$m = Ia.$$

Thus $I = Mt$, and the surface current is

$$K_b = I/t = M.$$

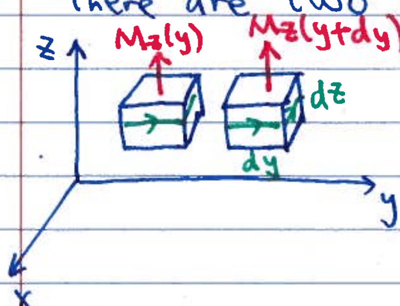
Note that there is no current on the top or bottom surface of the slab.

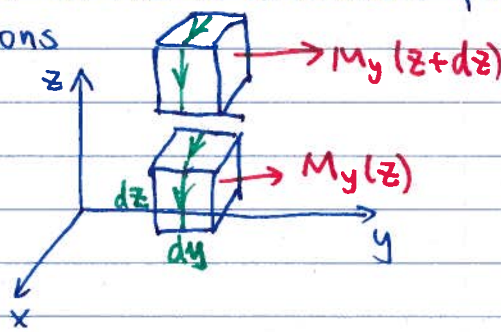
$$\rho_b = -\nabla \cdot \vec{P} \qquad \sigma_b = \vec{P} \cdot \hat{n}$$

Thus the surface current can be conveniently written as
$$\boxed{\vec{K}_b = \vec{M} \times \hat{n}}$$

* Bound volume current.

- Inside nonuniformly magnetized material, the internal currents no longer cancel.
- Focus on the net current in the x direction first, there are two contributions





- For the left plot, on the surface where they join, using $I = Mt$ we obtained above, we have a net current in the x direction:
$$I_x = [M_z(y+dy) - M_z(y)]\,dz = \frac{\partial M_z}{\partial y}\,dy\,dz$$

The corresponding volume current density is therefore
$$(J_b)_x = \partial M_z/\partial y = I_x/(dy\,dz)$$

- Similarly, a nonuniform magnetization in the y direction would contribute an amount $-\partial M_y/\partial z$. (Right plot)
- The total current in x direction is
$$(J_b)_x = \frac{\partial M_z}{\partial y} - \frac{\partial M_y}{\partial z}.$$

In general, $\boxed{\vec{J}_b = \nabla \times \vec{M}}$

- Note that as a steady current, $\nabla \cdot \vec{J}_b = 0$, obeying the conservation law. (The divergence of a curl is always zero: $\nabla \cdot \vec{J}_b = \nabla \cdot (\nabla \times \vec{M}) = 0$.

Ampère's law in magnetized materials

- The magnetic field inside matter is attributable to bound currents and free currents:
  - bound currents: the bound volume current within the material $\vec{J}_b = \nabla \times \vec{M}$ and bound surface current on the surface $\vec{K}_b = \vec{M} \times \hat{n}$, due to magnetization
  - free currents: currents due to everything else. It might flow through wires imbedded in the magnetized substance, or, if the latter is a conductor, through the material itself.
    It ~~evolves~~ involves actual transport of charge.
- In any event, the total current can be written as

$$\vec{J} = \vec{J}_b + \vec{J}_f .$$

Ampère's law can be written

$$\frac{1}{\mu_0}(\nabla \times \vec{B}) = \vec{J} = \vec{J}_b + \vec{J}_f = (\nabla \times \vec{M}) + \vec{J}_f$$

collecting together the two curls

$$\nabla \times \left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = J_f$$

Thus, we get $\boxed{\nabla \times \vec{H} = J_f}$, $\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$.

- In integral form, $\oint \vec{H} \cdot d\vec{l} = I_{f\,enc}$, with $I_{f\,enc}$ the total free current passing through the Amperian loop.
- $\vec{H}$ permits us to express Ampère's law in terms of free current alone.
- Griffiths: $\vec{H}$ has no sensible name, just call it "$\vec{H}$".

- Although $\nabla \times \vec{H} = \vec{J}_f$, $\nabla \times \vec{B} = \mu_0(\vec{J}_f + \vec{J}_b)$ (a lot similar), note that the curl alone doesn't determine a vector field, you must also know the divergence.

- $\nabla \cdot \vec{B} = 0$, but $\nabla \cdot \vec{H} = \nabla \cdot \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = -\nabla \cdot \vec{M} \neq 0$.

  Only when the divergence of $\vec{M}$ vanishes is the parallel between $\vec{B}$ and $\mu_0 \vec{H}$ faithful.

- Boundary conditions

• Recall for $\vec{B}$ at the boundaries.

1. $\oint \vec{B} \cdot d\vec{a} = 0 \rightarrow B^{\perp}_{above} = B^{\perp}_{below}$
2. $\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \rightarrow B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$.
3. The component of $\vec{B}$ that is parallel to the surface and also parallel to the current is continuous.

   The last two can be combined to

$$\vec{B}^{\parallel}_{above} - \vec{B}^{\parallel}_{below} = \mu_0 (\vec{K} \times \hat{n}).$$

• Using the same argument, we can get

1. $\oint \vec{H} \cdot d\vec{a} = -\oint \vec{M} \cdot d\vec{a} \rightarrow H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below})$
2. $\oint \vec{H} \cdot d\vec{l} = I_{f\,enc} \rightarrow \vec{H}^{\parallel}_{above} - \vec{H}^{\parallel}_{below} = \vec{K}_f \times \hat{n}$.

3. Linear media

- In paramagnetic and ~~diag~~ diamagnetic materials, the magnetization is sustained by the field. When $\vec{B}$ is removed, $\vec{M}$ disappears.
- In fact, for most substances the magnetization is proportional to the field, provided the field is not too strong.

$$\vec{M} = \chi_m \vec{H} \quad (\chi_m \text{ magnetic susceptibility})$$

χ_m is dimensionless, positive for paramagnets and negative for diamagnets.

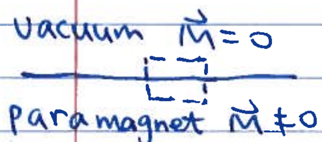
- Materials that obey $\vec{M} = \chi_m \vec{H}$ is called linear media.

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1+\chi_m)\vec{H} = \mu \vec{H}$$

· where $\mu \equiv \mu_0 (1+\chi_m)$. called the permeability of the material.

- In a vacuum, where there is no matter to magnetize, the susceptibility χ_m vanishes, and the permeability is μ_0. That's why μ_0 is called the permeability of free space.

- <u>At the boundaries</u>



· At the boundary of linear paramagnetic material, $\vec{M}$ is zero on one side but not on the other.

For the Gaussian surface, $\oint \vec{M}\cdot da \neq 0$, and thus, by the divergence theorem, $\nabla\cdot\vec{M}$ cannot vanish everywhere in it.

$$\nabla\cdot\vec{H} = -\nabla\cdot M \neq 0 \text{ in general}$$

Although $\vec{B} = \mu\vec{H}$, $\vec{H} = \frac{1}{\mu}\vec{B}$ and $\nabla\cdot\vec{B} = 0$!

(Another argument, $\nabla\cdot\vec{H} = \nabla\cdot\left(\frac{1}{\mu}\vec{B}\right) = \frac{1}{\mu}\nabla\cdot\vec{B} + \vec{B}\cdot\nabla\left(\frac{1}{\mu}\right)$
$= B\cdot\nabla\left(\frac{1}{\mu}\right)$. So $\vec{H}$ is not divergenceless at points where μ is changing).

- In a homogeneous linear material, the volume bound current density is proportional to the free current density:

$$\vec{J}_b = \nabla \times \vec{M} = \nabla \times (\chi_m \vec{H}) = \chi_m \vec{J}_f$$

In particular, unless free current actually flows through the material, all bound current will be at the surface.