# Particle Sampler in iS3D

Right now, what we have implemented in the particle sampler code so far is the sampling of an ideal distribution for each particle species:

$$f_{i,\text{eq}}(x,p) = g_i \left[\exp\left(\frac{p \cdot u(x) - b_i\,\mu_B(x)}{T(x)}\right) + \Theta_i\right]^{-1} \tag{1}$$

where $g_i$ is the spin degeneracy and $\Theta_i = (1,-1,0)$ for fermions, bosons, and Boltzmann particles. This simplification will be relaxed later when we include viscous corrections from the linearized $\delta f$ or modified equilibrium distribution $f_{\text{eq}}^{(\text{mod})}$.

Here we outline the steps for sampling particles from the Cooper Frye formula:

$$dN_i(x,p) = \frac{p \cdot d^3\sigma(x)}{(2\pi\hbar)^3}\,\frac{d^3p}{E}\,f_{i,\text{eq}}(x,p) \tag{2}$$

It mostly follows the sampling procedure from Long-Gang's CLVisc paper (Phys. Rev. C 97, 064918)

1) For each freezeout cell, compute the mean number of hadrons

$$\Delta N = \sum_i \frac{g_i}{(2\pi\hbar)^3} \int \frac{d^3p}{E}\, p \cdot d^3\sigma\, f_{i,\text{eq}}(u \cdot p) = u \cdot d^3\sigma \sum_i n_{i,\text{eq}} \tag{3}$$

where

$$n_{i,\text{eq}} = \frac{g_i}{(2\pi\hbar)^3} \int d^3p_{\text{LRF}} f_{i,\text{eq}}(u \cdot p) = \frac{g_i}{2\pi^2\hbar^3} \int_0^\infty \frac{p_{\text{LRF}}^2\, dp_{\text{LRF}}}{\exp\left[\dfrac{\sqrt{p_{\text{LRF}}^2 + m_i^2} - b_i\,\mu_B(x)}{T(x)}\right] + \Theta_i} \tag{4}$$

is the LRF equilibrium number density of each species and $p_{\text{LRF}}$ is the LRF radial momentum. To compute radial momentum integral, we expand the denominator as a geometric series, keeping only the first term for heavy hadrons and first 10 terms for pions (alternatively, you could do Gauss-quadrature as a future option).

We skip the rest of the sampling procedure for any cells with $u \cdot d^3\sigma < 0$ ($\Delta N < 0$), which rarely occurs.

2) Then sample the number of hadrons $N$ in that cell with a Poisson distribution

$$P(N) = \frac{(\Delta N)^N \exp(-\Delta N)}{N!} \tag{5}$$

This assumes the number of each species can be individually sampled with a Poisson distribution.

3) Out of $N$ hadrons, sample the number of each species $N_i$ with a discrete probability distribution

$$P(N_i) = \frac{\Delta N_i}{\Delta N} \tag{6}$$

4) For each sampled particle, sample its LRF momentum $p^\mu_{\rm LRF} = (u\cdot p, -X\cdot p, -Y\cdot p, -Z\cdot p)$ from the Cooper Frye formula (2). We further assume the equilibrium distribution (1) is Boltzmann-like or $\Theta_i = 0$ (will relax for pions in the immediate future). We outline the momentum sampling procedure for light and heavy particles separately.

**Light Boltzmann particles ($\mathbf{T/m > 0.6}$):**

The Cooper Frye formula (2) in spherical coordinates takes the form (drop constant factors)

$$dN_i \sim \frac{p\cdot d^3\sigma}{u\cdot p}\exp\left(\frac{p_{\rm LRF} - u\cdot p}{T}\right)\left[p^2_{\rm LRF}\exp\left(-\frac{p_{\rm LRF}}{T}\right)dp_{\rm LRF}\,d\phi_{\rm LRF}\,d\cos\theta_{\rm LRF}\right] \tag{7}$$

First we draw $p^\mu_{\rm LRF}$ from the massless Boltzmann distribution in the brackets [...] in Eq. (7). This is done with Scott Pratt's trick: one introduces the coordinates $(r_1, r_2, r_3) \in (0, 1]$

$$p_{\rm LRF} = -T(\log r_1 + \log r_2 + \log r_3) \tag{8a}$$

$$\phi_{\rm LRF} = 2\pi\left(\frac{\log r_1 + \log r_2}{\log r_1 + \log r_2 + \log r_3}\right)^2 \tag{8b}$$

$$\cos\theta_{\rm LRF} = \frac{\log r_1 - \log r_2}{\log r_1 + \log r_2} \tag{8c}$$

The determinant of the Jacobian is $|J| = \left|\frac{\partial p}{\partial r}\right| = \frac{8\pi T^3\exp(p_{\rm LRF}/T)}{p^2_{\rm LRF}}$. The massless distribution reduces to

$$p^2_{\rm LRF}\exp\left(-\frac{p_{\rm LRF}}{T}\right)dp_{\rm LRF}\,d\phi_{\rm LRF}\,d\cos\theta_{\rm LRF} = 8\pi T^3 dr_1 dr_2 dr_3 \tag{9}$$

This means $(r_1, r_2, r_3)$ can be sampled uniformly. After we have sampled the $r$-coordinates, we get the spherical coordinates using Eq. (8). Then, we can compute the components of the proposed $p^\mu_{\rm LRF}$

$$E_{\rm LRF} = u\cdot p = \sqrt{p^2_{\rm LRF} + m_i^2} \tag{10a}$$

$$p^x_{\rm LRF} = p_{\rm LRF}\sin\theta_{\rm LRF}\cos\phi_{\rm LRF} \tag{10b}$$

$$p^y_{\rm LRF} = p_{\rm LRF}\sin\theta_{\rm LRF}\sin\phi_{\rm LRF} \tag{10c}$$

$$p^z_{\rm LRF} = p_{\rm LRF}\cos\theta_{\rm LRF} \tag{10d}$$

The Cooper Frye formula (7) still has an additional weight factor $w_{\rm light}$ outside the brackets [...], which is now constant. To make $0 \leq w_{\rm light} \leq 1$, we rescale it by a constant factor. In this case, we use the Minkowski inequality:

$$\left|p\cdot d^3\sigma\right| \leq (u\cdot p)\left(\left|u\cdot d^3\sigma\right| + \sqrt{(u\cdot d^3\sigma)^2 - d^3\sigma\cdot d^3\sigma}\right) \tag{11}$$

The weight now becomes

$$w_{\rm light} = \frac{\left|p\cdot d^3\sigma\right|\exp\left[\frac{p_{\rm LRF} - u\cdot p}{T}\right]}{(u\cdot p)\left(\left|u\cdot d^3\sigma\right| + \sqrt{(u\cdot d^3\sigma)^2 - d^3\sigma\cdot d^3\sigma}\right)} \tag{12}$$

which lies between 0 and 1. The dot product $\left|p\cdot d^3\sigma\right|$ is computed in the LRF.

Finally, we draw a random number $m \in [0, 1]$ from a uniform distribution. If $m < w_{\rm light}$, then $p^\mu_{\rm LRF}$ is accepted. If rejected, we repeat the above steps (7) - (12) until it gets accepted. (I feel like there's a difference from my version and Long-Gang's version in the weight step)

**Heavy Boltzmann particles ($\mathbf{T/m < 0.6}$):**

For heavy Boltzmann particles, we replace the radial momentum $p_{\mathrm{LRF}}$ with the variable $k = E_{\mathrm{LRF}} - m$, to rewrite the Cooper Frye formula (2) as (drop out constant factors)

$$dN_i \sim \frac{p \cdot d^3\sigma}{u \cdot p} \frac{p_{\mathrm{LRF}}}{E_{\mathrm{LRF}}} \left[(m^2 + 2mk + k^2) \exp\left(-k/T\right) \, dk \, d\phi_{\mathrm{LRF}} \, d\cos\theta_{\mathrm{LRF}}\right] \tag{13}$$

Then we sample from either one of the three distributions in the parentheses (...) based on their $k$-integrated weights.

$$I_1 = \int_0^\infty dk \, m^2 \exp(-k/T) = m^2 T \tag{14a}$$
$$I_2 = \int_0^\infty dk \, 2mk \exp(-k/T) = 2mT^2 \tag{14b}$$
$$I_3 = \int_0^\infty dk \, k^2 \exp(-k/T) = 2T^3 \tag{14c}$$
$$I_{\mathrm{tot}} = I_1 + I_2 + I_3 \tag{14d}$$

The probability of sampling from the first distribution is approximately $I_1/I_{\mathrm{tot}}$, etc. We draw a random number $q \in [0, 1]$ from a uniform distribution and look at each of the possible outcomes:

**Case 1: $\mathbf{e^{-k/T}}$**

If $0 \leq q \leq I_1/I_{\mathrm{tot}}$ (most likely scenario for heavy hadrons), we draw $p^\mu_{\mathrm{LRF}}$ from the distribution

$$\exp\left(-k/T\right) \, dk \, d\phi_{\mathrm{LRF}} \, d\cos\theta_{\mathrm{LRF}} \tag{15}$$

To sample $k$, we only need to sample one $r$-coordinate $r_1 \in (0, 1]$, where

$$k = -T \log r_1 \tag{16}$$

One can easily check that $\exp(-k/T) \, dk = T dr_1$, so we can sample $r_1$ uniformly. Once we sample $k$, $E_{\mathrm{LRF}}$ and $p_{\mathrm{LRF}}$ are determined. Next, the angles $(\phi_{\mathrm{LRF}}, \cos\theta_{\mathrm{LRF}})$ are sampled uniformly. The $p^\mu_{\mathrm{LRF}}$ components can then be computed like before, and we check its acceptance against the weight factor

$$w_{\mathrm{heavy}} = \frac{\left|p \cdot d^3\sigma\right| \, p_{\mathrm{LRF}}/E_{\mathrm{LRF}}}{(u \cdot p) \left(|u \cdot d^3\sigma| + \sqrt{(u \cdot d^3\sigma)^2 - d^3\sigma \cdot d^3\sigma}\right)} \tag{17}$$

If rejected, repeat the steps for **Case 1** until accepted.

**Case 2: $\mathbf{k \, e^{-k/T}}$**

If $I_1/I_{\mathrm{tot}} < q \leq (I_1 + I_2)/I_{\mathrm{tot}}$, then we draw $p^\mu_{\mathrm{LRF}}$ from the distribution

$$k \exp\left(-k/T\right) \, dk \, d\phi_{\mathrm{LRF}} \, d\cos\theta_{\mathrm{LRF}} \tag{18}$$

For this case, we need to draw $(k, \phi_{\mathrm{LRF}})$ by sampling two variables $(r_1, r_2) \in (0, 1]$, where

$$k = -T(\log r_1 + \log r_2) \tag{19a}$$
$$\phi_{\mathrm{LRF}} = 2\pi \frac{\log r_2}{\log r_1 + \log r_2} \tag{19b}$$

One can check that $k \exp(-k/T) \, dk \, d\phi_{\mathrm{LRF}} = 4\pi T^2 dr_1 dr_2$, so we can sample $(r_1, r_2)$ uniformly. The angle $\cos\theta_{\mathrm{LRF}}$ is sampled separately. Then compute $p^\mu_{\mathrm{LRF}}$ and check its acceptance against the weight $w_{\mathrm{heavy}}$ in (17). If rejected, repeat the steps for **Case 2** until accepted.

**Case 3: $\mathbf{k^2\, e^{-k/T}}$**

If $(I_1 + I_2)/I_{\rm tot} < q \leq 1$, then we draw $p^\mu_{\rm LRF}$ from the distribution

$$k^2 \exp\left(-k/T\right)\, dk\, d\phi_{\rm LRF}\, d\cos\theta_{\rm LRF} \tag{20}$$

The sampling procedure is the same as the massless distribution [...] in (7) after one swaps out $p_{\rm LRF}$ for $k$. After computing $p^\mu_{\rm LRF}$, check its acceptance against $w_{\rm heavy}$ in (17). If rejected, repeat the steps for **Case 3** until accepted.

5) After sampling the particle's LRF momentum, compute the lab frame momentum in Milne coordinates by using the decomposition:

$$p^\mu = (u \cdot p) u^\mu + (-X \cdot p) X^\mu + (-Y \cdot p) Y^\mu + (-Z \cdot p) Z^\mu \tag{21}$$

The components of the spatial basis vectors are

$$X^\mu = \left(u_\perp \cosh\kappa_L, \frac{u^x\, u^\tau_\perp}{u_\perp}, \frac{u^y\, u^\tau_\perp}{u_\perp}, \frac{u_\perp \sinh\kappa_L}{\tau}\right) \tag{22a}$$

$$Y^\mu = \left(0, -\frac{u^y}{u_\perp}, \frac{u^x}{u_\perp}, 0\right) \tag{22b}$$

$$Z^\mu = \left(\sinh\kappa_L, 0, 0, \frac{\cosh\kappa_L}{\tau}\right) \tag{22c}$$

where $\kappa_L = \tanh^{-1}(\tau u^\eta / u^\tau)$, $u_\perp = \sqrt{(u^x)^2 + (u^y)^2}$, and $u^\tau_\perp = \sqrt{1 + u^2_\perp}$. We also compute the Cartesian lab frame energy $E$ and longitudinal momentum $p^z$.

$$E = p^\tau \cosh\eta + \tau p^\eta \sinh\eta \tag{23a}$$

$$p^z = \tau p^\eta \cosh\eta + p^\tau \sinh\eta \tag{23b}$$

6) If $p \cdot d^3\sigma \geq 0$, amend the sample particle's MC ID number, position and momentum to the output particle list file (current file format):

$$(\mathrm{MC}, \quad \tau, \quad x, \quad y, \quad \eta, \quad E, \quad p^x, \quad p^y, \quad p^z)$$

Otherwise, we discard it.