In der Mathematik ist die Kunst Fragen zu stellen wertvoller als Probleme zu lösen

— Georg Cantor

1. For each given relation $R_i \subseteq {M_i}^2$, determine whether it is *reflexive, irreflexive, coreflexive, symmetric, antisymmetric, asymmetric, transitive, left/right Euclidean, connex*. Provide a counterexample for each non-complying property (*e.g.*, "transitivity does not hold for $x, y, z = (3, 1, 2)$"). Organize your answer in a table (*e.g.*, columns — relations, rows — properties).
   (a) $M_1 = \mathbb{R}$
   $x \; R_1 \; y \leftrightarrow |x - y| \leq 1$
   (b) $M_2 = \mathcal{P}(\{a, b, c\})$
   $R_2 = \text{“}\subseteq\text{”}$
   (c) $M_3 = \{a, b, c, d\} \quad \|R_3\| = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
   (d) $M_4 = \{\text{“rock”}, \text{“scissors”}, \text{“paper”}\}$
   $R_4 = \{\langle x, y \rangle \mid x \text{ beats } y\}$

2. Prove (rigorously) or disprove (by providing a counterexample) the following statements about arbitrary homogeneous relations $R \subseteq M^2$ and $S \subseteq M^2$:
   (a) If R and S are *reflexive*, then $R \cap S$ is so.
   (b) If R and S are *symmetric*, then $R \cap S$ is so.
   (c) If R and S are *transitive*, then $R \cap S$ is so.
   (d) If R and S are *reflexive*, then $R \cup S$ is so.
   (e) If R and S are *symmetric*, then $R \cup S$ is so.
   (f) If R and S are *transitive*, then $R \cup S$ is so.

3. An equinumerosity relation $\sim$ over sets is defined as follows: $A \sim B \leftrightarrow |A| = |B|$.
   (a) Show that $\sim$ is an equivalence relation over finite sets.
   (b) Show that $\sim$ is an equivalence relation over infinite sets[1].
   (c) Find the quotient set of $\mathcal{P}(\{a, b, c, d\})$ by $\sim$.

4. Let R_θ be a relation of θ-similarity (clearly, $\theta \in [0; 1] \subseteq \mathbb{R}$) of finite non-empty sets defined as follows: a set A is said to be *θ-similar* to B *iff* the Jaccard index $\text{Jac}(A, B) = \frac{|A \cap B|}{|A \cup B|}$ for these sets is at least θ, *i.e.* $\langle A, B \rangle \in R_\theta \leftrightarrow \text{Jac}(A, B) \geq \theta$.
   (a) Determine whether θ-similarity is a tolerance relation[2].
   (b) Determine whether θ-similarity is an equivalence relation.
   (c) Draw the graph of a relation $R_\theta \subseteq \{A_i\}^2$, where $\theta = 0.25$, $A_1 = \{1, 2, 5, 6\}$, $A_2 = \{2, 3, 4, 5, 7, 9\}$, $A_3 = \{1, 4, 5, 6\}$, $A_4 = \{3, 7, 9\}$, $A_5 = \{1, 5, 6, 8, 9\}$.

5. Any binary relation $R \subseteq M^2$ can be represented as a zero-one matrix $\|R\| = [r_{ij}]$, where the element r_{ij} is equal to 1 if $\langle m_i, m_j \rangle \in R$ and 0 otherwise. Boolean product of two square matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ is a matrix $C = A \odot B = [c_{ij}]$ defined as follows: $c_{ij} = \bigvee_k (a_{ik} \wedge b_{kj})$. A composition of relations R and S is a relation $S \circ R$ defined as follows: $\langle a, b \rangle \in S \circ R \leftrightarrow \exists c : \langle a, c \rangle \in R \wedge \langle c, b \rangle \in S$. Show that the matrix representation of the composition of relations R and S is equal to the Boolean product of the corresponding matrices, *i.e.* $\|S \circ R\| = \|R\| \odot \|S\|$.

6. Find the error in the "proof" of the following "theorem".
   "*Theorem*": Let R be a relation on a set A that is symmetric and transitive. Then R is reflexive.
   "*Proof*": Let $a \in A$. Take an element $b \in A$ such that $\langle a, b \rangle \in R$. Because R is symmetric, we also have $\langle b, a \rangle \in R$. Now using the transitive property, we can conclude that $\langle a, a \rangle \in R$ because $\langle a, b \rangle \in R$ and $\langle b, a \rangle \in R$.

7. Give an example of a relation R on the set $\{a, b, c\}$ such that the symmetric closure of the reflexive closure of the transitive closure of R is not transitive.

[1] For infinite sets, $|A| = |B|$ means there is a bijection between A and B.
[2] A tolerance relation is a *reflexive* and *symmetric* relation.

8. Let R be the relation on the set of all colorings of the 2×2 checkerboard where each of the four squares is colored either *red* or *blue* so that $\langle C_1, C_2 \rangle$, where C_1 and C_2 are 2×2 checkerboards with each of their four squares colored blue or red, belongs to R if and only if C_2 can be obtained from C_1 either by rotating the checkerboard or by rotating it and then reflecting it.
   (a) Show that R is an equivalence relation.
   (b) What are the equivalence classes of R?

9. Consider two relations $R \subseteq A \times B$ and $S \subseteq B \times C$. Prove that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

10. Prove or disprove the following statements about the functions f and g:
    (a) If f and g are injections, then $g \circ f$ is also an injection.
    (b) If f and g are surjections, then $g \circ f$ is also a surjection.
    (c) If f and $f \circ g$ are injections, then g is also an injection.
    (d) If f and $f \circ g$ are surjections, then g is also a surjection.

11. Let $H = \{1, 2, 4, 5, 10, 12, 20\}$. Consider a divisibility relation $R \subseteq H^2$ defined as follows: $x \mathrel{R} y \leftrightarrow y \vdots x$.
    (a) Sort R (as a set of pairs) lexicographically[3].
    (b) Show that R is a partial order.
    (c) Determine whether R is a linear (total) order.
    (d) Draw the Hasse diagram for a graded poset $\langle H, R, \rho \rangle$, where $\rho \colon H \longrightarrow \mathbb{N}_0$ is a grading function which maps a number $n \in H$ to the sum of all exponents appearing in its prime factorization, *e.g.*, $\rho(20) = \rho(2^2 \cdot 5^1) = 2 + 1 = 3$, so the number 20 would have the 3rd rank (bottom-up).
    (e) Find the minimal, minimum (least), maximal and maximum (greatest) elements in the poset $\langle H, R \rangle$. If there are multiple or none, explain why.
    (f) Perform a topological sort of the poset $\langle H, R \rangle$.

12. Prove that the transitive closure R^+ is in fact transitive.

    Definition. *$R^+ = \bigcup_{n \in \mathbb{N}^+} R^n$ is a transitive closure of $R \subseteq M^2$, where*
    * *$R^{k+1} = R^k \circ R$ is a compositional (functional) power*[4]*,*
    * *$R^1 = R$,*
    * *$S \circ R = \{\langle x, y \rangle \mid \exists z : (x \mathrel{R} z) \land (z \mathrel{S} y)\}$ is a composition (relative product) of relations R and S.*

13. Given a set S and two partitions P_1 and P_2 of S, we define the relation $P_1 \preceq P_2$ as follows: partition P_1 is considered a *refinement* of the partition P_2 if every set in P_1 is a subset of one of the sets in P_2. Show that the set of all partitions of a set S with the refinement relation $\preceq$ is a lattice.

14. A poset $\langle R, \preceq \rangle$ is *well-founded* if there is no infinite decreasing sequence of elements in the poset, that is, elements $x_1, x_2, \dots, x_n$ such that $\cdots \prec x_n \prec \cdots \prec x_2 \prec x_1$. Determine whether the set of strings of lowercase English letters with lexicographic order is well-founded.

[3] Lexicographic order for pairs: $\langle a, b \rangle \preceq \langle a', b' \rangle \leftrightarrow (a < a') \lor ((a = a') \land (b \leq b'))$. For example, $\langle 1, 2 \rangle \preceq \langle 1, 3 \rangle \preceq \langle 2, 1 \rangle$.
[4] Note: this *is not a Cartesian power*, despite of the same notation R^n. Another possible notation for compositional power is $R^{\circ n}$, but it is too wild to use it here.