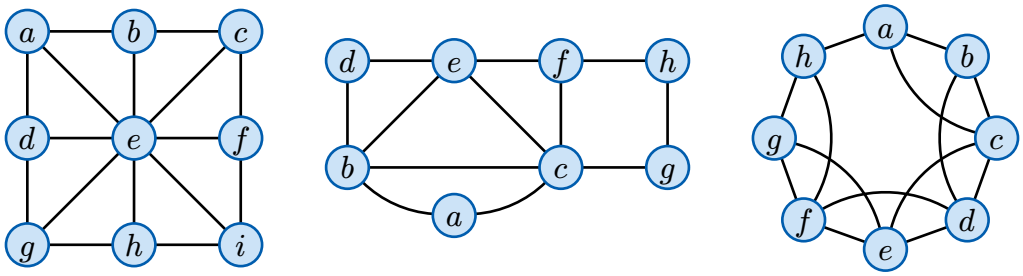
Problem 1: Graph Invariants Analysis

Core

For each of the following graphs, compute the requested metrics and properties.



1. **Basic connectivity metrics:**
   - Minimum degree $\delta(G)$, maximum degree $\Delta(G)$
   - Vertex connectivity $\kappa(G)$, edge connectivity $\lambda(G)$
   - All $(\kappa+1)$-connected components, $(\lambda+1)$-edge-connected components
   - Verify Whitney's inequality: $\kappa(G) \leq \lambda(G) \leq \delta(G)$
2. **Distance metrics:**
   - $\text{ecc}(v)$ for every vertex $v \in V(G)$
   - $\text{rad}(G)$, $\text{diam}(G)$, $\text{center}(G)$, $\text{girth}(G)$
   - Verify the bounds: $\text{rad}(G) \leq \text{diam}(G) \leq 2 \cdot \text{rad}(G)$
3. **Structural properties:**
   - Is the graph Eulerian? Hamiltonian? Bipartite? Justify each answer.
   - Find: maximum clique $Q \subseteq V$, maximum stable set $S \subseteq V$, minimum dominating set $D \subseteq V$
   - Find: maximum matching $M \subseteq E$. Is M perfect?
   - Find: minimum vertex cover $R \subseteq V$, minimum edge cover $F \subseteq E$
   - Find: minimum vertex coloring $C : V \to \{1, 2, ..., \chi(G)\}$
   - Find: minimum edge coloring $C : E \to \{1, 2, ..., \chi'(G)\}$

Problem 2: Degree Sequences

Core

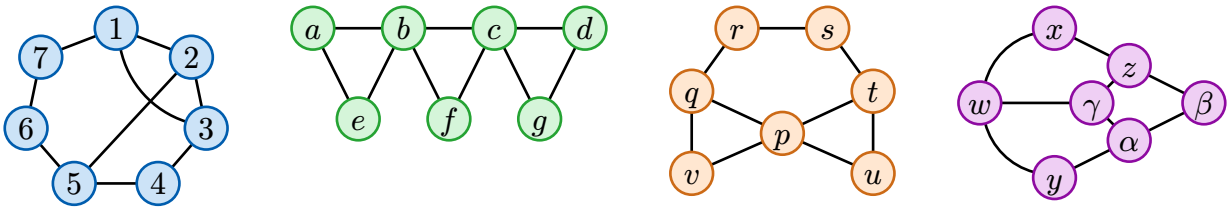
A sequence $d = (d_1, d_2, ..., d_n)$ with $d_1 \geq d_2 \geq ... \geq d_n \geq 0$ is *graphical* if there exists a simple graph G with this degree sequence.

1. For each sequence below, determine whether it is graphical. If yes, construct a graph realizing it. If no, explain why (cite the Erdős–Gallai criterion or parity/sum arguments).
   (a) $(5, 4, 3, 2, 2, 2)$
   (b) $(3, 3, 3, 3, 3, 3)$
   (c) $(4, 4, 3, 2, 1)$
   (d) $(6, 3, 3, 3, 3, 2, 2)$
   (e) $(1, 1, 1, 1, 1, 1)$
2. The total number of non-isomorphic simple undirected graphs on n vertices is given by the sequence $(1, 2, 4, 11, 34, 156, ...)$ (OEIS A000088). The number of distinct degree sequences of length n is given by the sequence $(1, 2, 4, 11, 31, 102, ...)$ (OEIS A004251). Explain why the second sequence grows more slowly than the first, and determine which degree sequences of length $n = 5$ are realizable by more than one graph.

Problem 3: Graph Isomorphism

Core

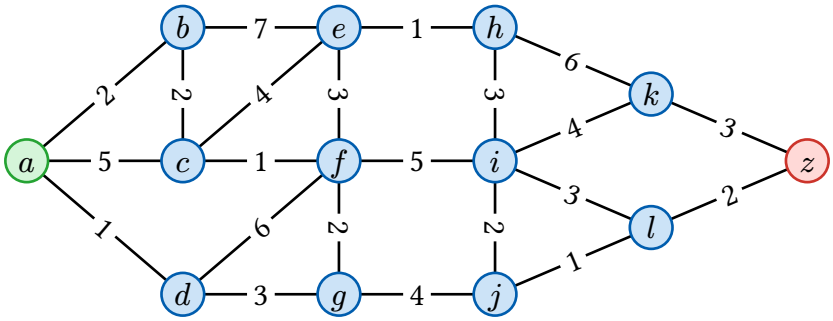
Determine which pairs of graphs below are *isomorphic*. For each isomorphic pair (G, H), exhibit an explicit bijection $f : V(G) \to V(H)$ that preserves adjacency. For each non-isomorphic pair, identify a distinguishing invariant (degree sequence, girth, number of triangles, *etc.*).



Problem 4: Dijkstra's Shortest Path Algorithm

Essential

Apply **Dijkstra's algorithm** to find a shortest path from a to z in the weighted graph below.



1. **Trace the algorithm** step-by-step:
   - Maintain a table showing, after each iteration: the visited set V_{visited}, the current distance labels $d(v)$ for all $v \in V$, and the predecessor pointers $\pi(v)$.
   - Reconstruct a shortest path from a to z, and compute its total weight.
2. **Correctness & Structure**:
   (a) State the key *invariant* that Dijkstra maintains during execution: what is guaranteed to be true about the distances $d(v)$ for visited or yet unvisited vertices?
   (b) In a graph where some shortest paths are not unique, the predecessor pointers form a **shortest path DAG** rooted at a. Draw this DAG (include only edges corresponding to some $\pi(v)$, and mark vertex z). How many different shortest paths from a to z are in this DAG?
   (c) Dijkstra fails on graphs with negative-weight edges. Give a small example (3–4 vertices, one negative edge) where the algorithm produces an incorrect result, and explain *why* the invariant from *(a)* is violated.

Problem 5: Find the Error

Core

The following "proof" contains a subtle error. Identify the error and explain why the conclusion is false and the claim is not valid.

> **False Claim:** Every tree with n vertices has a path of length $n - 1$.
>
> **"Proof:"**
>
> *Base case:* A tree with one vertex clearly has a path of length $0 = 1 - 1$.

> *Inductive step:* Assume that every tree with n vertices has a path of length $n-1$, which terminate at some leaf u. Add a new vertex v and connect it to u with an edge. The resulting tree has $n+1$ vertices and contains a path of length n, which is $(n+1)-1$. ■

Problem 6: Trees

Essential

The following are six fundamental characterizations of trees. Prove that they are all equivalent for a graph $G = \langle V, E \rangle$ with $n = |V|$ vertices and $m = |E|$ edges by establishing a cycle of implications:

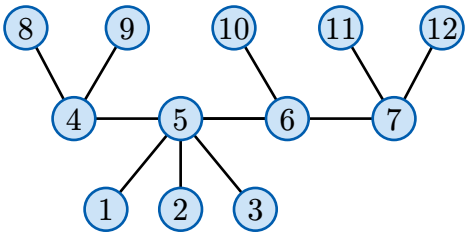
$$(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1)$$

> **Theorem:** The following are equivalent:
> 1. G is a tree (connected and acyclic).
> 2. G is connected and $m = n - 1$.
> 3. G is acyclic and $m = n - 1$.
> 4. For every pair of vertices $u, v \in V$, there exists a *unique* path from u to v.
> 5. G is connected, but removing any edge disconnects it.
> 6. G is acyclic, but adding any new edge creates exactly one cycle.

Problem 7: Prüfer Code

Core

The Prüfer sequence provides a bijection between labeled trees on n vertices and sequences of length $n-2$ with entries in $\{1, 2, \ldots, n\}$.



1. Encode the labeled tree above into its Prüfer sequence.
2. Decode the Prüfer sequence $(3, 3, 3, 7, 7, 5)$ back into a labeled tree. Draw the result.
3. Prove that in the Prüfer sequence the number i appears exactly $\deg(i) - 1$ times.

Problem 8: Minimum Spanning Trees

Essential

Consider the weighted graph from Problem 4.

1. Use **Kruskal's algorithm** to find a minimum spanning tree (MST). Show the edges in the order they are considered, and mark which are added to the MST.
2. Use **Prim's algorithm** starting from vertex a. Show how the tree grows step by step.
3. Prove: If all edge weights in a connected graph are *distinct*, then the MST is unique.

Problem 9: Eulerian Graphs

Essential

1. For which values of $n \geq 3$ does the complete graph K_n have an Euler circuit? An Euler trail (but not a circuit)? Prove your answers.

2. For which pairs (m, n) with $m, n \geq 1$ does the complete bipartite graph $K_{m,n}$ have an Euler circuit? An Euler trail?
3. State and prove Euler's criterion: A connected graph has an Euler circuit if and only if every vertex has even degree.

Problem 10: Hamiltonian Graphs

Challenge

1. Prove that $K_{m,n}$ is Hamiltonian if and only if $m = n \geq 2$.
2. The n-dimensional *hypercube graph* Q_n has vertex set $\{0,1\}^n$ (all binary strings of length n), and two vertices are adjacent iff they differ in exactly one coordinate. Prove that Q_n has a Hamiltonian cycle for all $n \geq 2$.
3. Construct a 2-connected non-Hamiltonian graph with at least 6 vertices.

Problem 11: Bipartite Graphs

Essential

Prove the following fundamental properties of bipartite graphs.

1. A graph G is bipartite if and only if it contains no odd cycle.
2. If G is bipartite and d-regular with $d \geq 1$, then G has a perfect matching.
3. In any bipartite graph, the size of a maximum matching equals the size of a minimum vertex cover.

Problem 12: Hall's Marriage Theorem

Essential

A dance school has 6 leaders and 6 followers. Each leader is willing to dance with certain followers, as shown in the table below ($\times$ indicates willingness).

	F_1	F_2	F_3	F_4	F_5	F_6
L_1	$\times$	$\times$				$\times$
L_2	$\times$		$\times$			
L_3		$\times$		$\times$		
L_4			$\times$		$\times$	
L_5				$\times$	$\times$	
L_6	$\times$					$\times$

1. Determine whether a perfect matching exists by verifying Hall's condition.
2. Find a maximum matching. Is it perfect?
3. Now suppose L_2 becomes more selective and will *only* dance with F_1. Does a perfect matching still exist? If Hall's condition fails, identify the violating subset S and explain why no perfect matching exists.

Problem 13: Graceful Graphs

Challenge

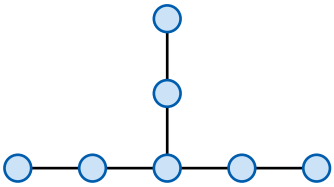
A *graceful labeling* of a graph G with m edges is an injective function $f : V(G) \to \{0, 1, 2, \ldots, m\}$ such that the induced edge labels $|f(u) - f(v)|$ for each edge $(u, v) \in E(G)$ are distinct and cover

the set $\{1, 2, \ldots, m\}$. In other words, the absolute differences of the labels of adjacent vertices are all different and cover the integers from 1 to m. A graph is called *graceful* if it has a graceful labeling.

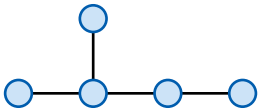
Show that the following graphs are graceful (by explicitly constructing a graceful labeling for each).

(a)

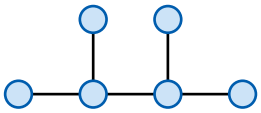
(b)



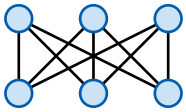
(c)



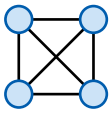
(d)



(e)



(f)



Problem 14: Fundamental Theorems

Challenge

Prove each of the following theorems *rigorously*.

1. **(Harary)** Every block of a block graph is a clique.

   A *block graph* $H = B(G)$ is an intersection graph of all blocks (biconnected components) of G, *i.e.* each vertex $v \in V(H)$ corresponds to a block of G, and there is an edge $\{u, v\} \in E(H)$ iff "blocks" u and v share a cut vertex.

2. **(Gallai)** For every finite graph G, we have:

   $$\alpha(G) + \tau(G) = |V(G)|$$

   where $\alpha(G)$ is the size of a maximum independent set and $\tau(G)$ is the minimum vertex cover size.

3. **(Erdős–Gallai)** The sequence $d_1 \geq d_2 \geq \ldots \geq d_n \geq 0$ is *graphical* iff $d_1 + \ldots + d_n$ is even and

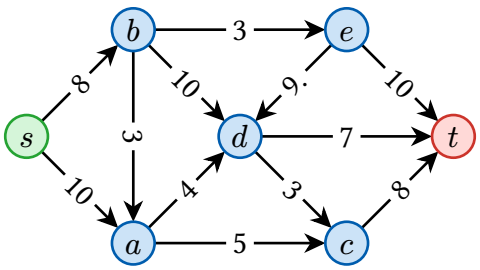
   $$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{j=k+1}^{n} \min(d_j, k)$$

   holds for every k in $1 \leq k \leq n$.

Problem 15: Network Flows

Essential

Consider the following flow network $N = (V, E, s, t, c)$ with source s and sink t.



1. **Ford–Fulkerson execution.** Starting from zero flow, perform augmentations until no augmenting path exists.
   - At each iteration, specify the chosen augmenting path and its bottleneck.
   - Give the updated flow value $|f|$ and describe all changed residual capacities.
   - Your solution must include at least one iteration that uses a backward residual edge.

2. **Max-flow/min-cut certificate.**
   - Compute a maximum flow value.
   - Extract one minimum s-t cut (S, T) from the final residual network.
   - Verify numerically that $|f| = c(S, T)$.
3. **Sensitivity analysis.**
   - Increase the capacity of edge (d, t) by 2 and recompute the new maximum flow value.
   - Identify one edge whose +1 capacity increase does **not** change the maximum flow, and justify via minimum cuts.
4. Prove the **integrality theorem**: if all capacities are integers, then there exists a maximum flow with integer values on all edges.

Notes on Tags System

The problems in this assignment are *tagged* with three levels of difficulty:

Core are mandatory problems that must be completed *before the test*.

Essential are main problems that have to be submitted *before the announced deadline*.

Challenge are slightly more challenging problems with *a flexible deadline*.

Optional Bonus Problems

> The following problems are *"optional"*. They are more challenging and are intended for students who wish to explore deeper topics in graph theory. These problems will *not* count toward your grade but may earn bonus points.

Problem A: Ramsey Theory

Bonus

The *Ramsey number* $R(r, s)$ is the minimum n such that any 2-coloring of the edges of K_n (complete graph) contains either a red K_r or a blue K_s.

1. Prove that $R(3, 3) = 6$ by showing:
   - Any edge-coloring of K_5 with two colors can avoid monochromatic triangles.
   - Any edge-coloring of K_6 with two colors must contain a monochromatic triangle.
2. Show that $R(3, 4) \leq 10$ by using a similar argument.
3. Prove the *Ramsey recurrence*: For all $r, s \geq 2$,
$$R(r, s) \leq R(r - 1, s) + R(r, s - 1)$$
4. Use the recurrence to show that $R(4, 4) \leq 18$.

Problem B: The Friendship Theorem

Bonus

The *Friendship Theorem* (Erdős, Rényi, Sós, 1966) states:

> If G is a finite simple graph in which every pair of *distinct* vertices has *exactly one* common neighbor, then G is a *windmill graph*: n triangles all sharing a single common vertex.

1. Verify the theorem for small cases. Try to construct non-windmill graphs satisfying the friendship condition with $n \leq 8$ vertices.
2. Prove that any graph G satisfying the friendship condition must be regular (all vertices have the same degree).
3. Using regularity, show that if G is r-regular and satisfies the friendship condition, then either $r = 2$ or there exists a universal vertex (a vertex adjacent to all others).

Problem C: Max-Flow Solver

Bonus

Design and implement a production-quality max-flow solver.

1. Implement **Edmonds–Karp** (BFS-based Ford–Fulkerson) for directed graphs with non-negative integer edge capacities.

   Your code must produce three deliverables per run:
   - (1) **Maximum flow value** $|f^*|$ from source s to sink t
   - (2) **One minimum cut** (S, T) extracted from the residual network
   - (3) **Flow decomposition:** express the solution as a sum of edge-disjoint s-t paths (with multiplicities) and identify any residual cycles

2. Test on three progressively complex instances:
   - The flow network from Problem 15
   - Random sparse network: $|V| = 30, |E| \approx 60$ with random integer capacities $c(e) \in [1..20]$
   - Random dense network: $|V| = 30, |E| \approx 300$ with same capacity distribution

   For each test, verify the *max-flow/min-cut duality*: $|f^*| = c(S, T)$ numerically.

3. Across varying sizes ($|V| \in \{10, 20, \ldots, 100\}$), measure and report:
   - **Execution time** (wall-clock seconds, average of 3 runs per size)
   - **Iteration count:** number of augmenting paths found until termination
   - **Empirical scaling:** does runtime grow as $\mathcal{O}(|V| \cdot |E|^2)$ or faster? Compare fit quality to theoretical bound.
   - **Runtime plots:** present as a figure (runtime vs $|V|$ with fitted curve)

4. Implement **Dinic's algorithm** (blocking flows; theoretical bound $O(|V|^2\ |E|)$) and benchmark against Edmonds–Karp on the same test suite. Which dominates on sparse vs dense graphs? Explain the observed crossover point.

5. Formulate one real-world problem (*e.g.*, airline crew scheduling, data routing, bipartite matching) as a max-flow instance with 20+ vertices. Solve it using your implementation and interpret the flow decomposition.