

# Combinatorics

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# Combinatorics

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# Introduction to Combinatorics

**Definition 1:** Combinatorics is the branch of discrete mathematics that deals with *counting*, *arranging*, and analyzing *discrete structures*.

## Three basic problems of Combinatorics:

1. Existence: *Is there at least one arrangement of a particular kind?*
2. Counting: *How many arrangements are there?*
3. Optimization: *Which one is best according to some criteria?*

## Discrete structures

- Graphs, sets, multisets, sequences, patterns, coverings, partitions...

## Enumeration

- Permutations, combinations, inclusion/exclusion, generating functions, recurrence relations...

## Algorithms and optimization

- Sorting, eulerian circuits, hamiltonian cycles, planarity testing, graph coloring, spanning trees, shortest paths, network flows, bipartite matchings, chain partitions...

# Discrete Structures

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We investigate the *building blocks* of combinatorics:

- Sets and multisets
- Sequences and strings
- Arrangements
- Graphs, networks, trees
- Posets and lattices
- Partitions
- Patterns, coverings, designs, configurations
- Schedules, assignments, distributions

*Used in data modeling, logic, cryptography, and the design of data structures.*

# Enumerative Combinatorics

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We learn how to count *without explicit listing*:

- Permutations and combinations
- Inclusion–Exclusion Principle
- Set partitions, integer partitions, Stirling numbers, Catalan numbers
- Recurrence relations
- Generating functions

*Used in probability theory, complexity theory, coding theory, computational biology.*

# Algorithmic and Optimization Methods

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Combinatorics powers *algorithm design* and complexity analysis:

- Sorting
- Searching
- Eulerian paths and Hamiltonian cycles
- Planarity, colorings, cliques, coverings
- Spanning trees
- Shortest paths
- Network flows
- Bipartite matchings
- Dilworth's theorem, chain and antichain partitions

*Used in logistics, scheduling, routing, and complexity optimization.*

# Basic Counting Principles

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## Basic Counting Rules

**PRODUCT RULE:** If something can happen in  $n_1$  ways, *and* no matter how the first thing happens, a second thing can happen in  $n_2$  ways, then the two things *together* can happen in  $n_1 \cdot n_2$  ways.

**SUM RULE:** If one event can occur in  $n_1$  ways and a second event in  $n_2$  (different) ways, then there are  $n_1 + n_2$  ways in which *either* the first event *or* the second event can occur (*but not both*).



## Addition Principle

**Definition 2:** We say a finite set  $S$  is *partitioned* into *parts*  $S_1, \dots, S_k$  if the parts are pairwise disjoint and their union is  $S$ . In other words,  $S_i \cap S_j = \emptyset$  for  $i \neq j$  and  $S_1 \cup S_2 \cup \dots \cup S_k = S$ . In that case:

$$|S| = |S_1| + |S_2| + \dots + |S_k|$$

*Example:* Let  $S$  be the set of students attending the combinatorics lecture. It can be partitioned into parts  $S_1$  and  $S_2$  where

$S_1$  = set of students that *like* easy examples.

$S_2$  = set of students that *don't like* easy examples.

If  $|S_1| = 22$  and  $|S_2| = 8$ , then we can conclude  $|S| = |S_1| + |S_2| = 30$ .

## Multiplication Principle

**Definition 3:** If  $S$  is a finite set that is the *product* of  $S_1, \dots, S_k$ , that is,  $S = S_1 \times \dots \times S_k$ , then

$$|S| = |S_1| \times \dots \times |S_k|$$

*Example:* TODO: example with car plates

## Subtraction Principle

**Definition 4:** Let  $S$  be a subset of a finite set  $T$ . We define the *complement* of  $S$  as  $\overline{S} = T \setminus S$ . Then

$$|\overline{S}| = |T| - |S|$$

*Example:* If  $T$  is the set of students studying at KIT and  $S$  the set of students studying neither math nor computer science. If we know  $|T| = 23905$  and  $|S| = 20178$ , then we can compute the number  $|\overline{S}|$  of students studying either math or computer science:

$$|\overline{S}| = |T| - |S| = 23905 - 20178 = 3727$$

## Bijection Principle

**Definition 5:** If  $S$  and  $T$  are sets, then

$$|S| = |T| \iff \text{there exists a bijection between } S \text{ and } T$$

*Example:* Let  $S$  be the set of students attending the combinatorics lecture and  $T$  the set of homework submissions (unique per student) for the first problem sheet. If the number of students and the number of submissions coincide, then there is a bijection between students and submissions.

**Note:** The bijection principle works both for *finite* and *infinite* sets.

## Pigeonhole Principle

**Definition 6:** Let  $S_1, \dots, S_k$  be finite sets that are pairwise disjoint and  $|S_1| + |S_2| + \dots + |S_k| = n$ .

$$\exists i \in \{1, \dots, k\} : |S_i| \geq \left\lfloor \frac{n}{k} \right\rfloor \quad \text{and} \quad \exists j \in \{1, \dots, k\} : |S_j| \leq \left\lceil \frac{n}{k} \right\rceil$$

*Example:* Assume there are 5 holes in the wall where pigeons nest. Say there is a set  $S_i$  of pigeons nesting in hole  $i$ . Assume there are  $n = 17$  pigeons in total. Then we know:

- There is some hole with at least  $d = 4$  pigeons.
- There is some hole with at most  $b = 3$  pigeons.

## Double Counting

If we count the same quantity in *two different ways*, then this gives us a (perhaps non-trivial) identity.

*Example (Handshaking Lemma):* Assume there are  $n$  people at a party and everybody will shake hands with everybody else. How many handshakes will occur? We count this number in two ways:

1. Every person shakes  $n - 1$  hands and there are  $n$  people. However, two people are involved in a handshake so if we just multiply  $n \cdot (n - 1)$ , then every handshake is counted twice. The total number of handshakes is therefore  $\frac{n \cdot (n-1)}{2}$ .
2. We number the people from 1 to  $n$ . To avoid counting a handshake twice, we count for person  $i$  only the handshakes with persons of lower numbers. Then the total number of handshakes is:

$$\sum_{i=1}^n (i-1) = \sum_{i=0}^{\{n-1\}} i = \sum_{i=1}^{n-1} i$$

The identity we obtain is therefore:  $\sum_{i=1}^{n-1} i = \frac{n \cdot (n-1)}{2}$

# **Arrangements, Permutations, Combinations**

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# Ordered Arrangements

**Definition 7:** Denote by  $[n] = \{1, \dots, n\}$  the set of natural numbers from 1 to  $n$ .

Hereinafter, let  $X$  be a finite set.








**Definition 8:** An *ordered arrangement* of  $n$  elements of  $X$  is a *map*  $s : [n] \rightarrow X$ .

- Here,  $[n]$  is the *domain* of  $s$ , and  $s(i)$  is the *image* of  $i \in [n]$  under  $s$ .
- The set  $\{x \in X \mid s(i) = x \text{ for some } i \in [n]\}$  is the *range* of  $s$ .

Other common names for ordered arrangements are:

- *string* (or *word*), e.g. “Banana”
- *sequence*, e.g. “0815422372”
- *tuple*, e.g.  $(3, 5, 2, 5, 8)$

Example:

$i$	1	2	3	4	5	6	7
$s(i)$							



# Permutations

**Definition 9:** A *permutation* of  $X$  is a *bijective* map  $\pi : [n] \rightarrow X$ .

Usually,  $X = [n]$ , and the set of all permutations of  $[n]$  is denoted by  $S_n$ .

Example:

$i$	1	2	3	4	5	6	7
$\pi(i)$	2	7	1	3	5	4	6

**Definition 10:** *k-permutation* of  $X$  is an ordered arrangement of  $k$  *distinct* elements of  $X$ , that is, an *injective* map  $\pi : [k] \rightarrow X$ .

The set of all  $k$ -permutations of  $X = [n]$  is denoted by  $P(n, k)$ . In particular,  $S_n = P(n, n)$ .

TODO: circular permutations

## Counting Permutations

**Theorem 1:** For any natural numbers  $0 \leq k \leq n$ , we have

$$|P(n, k)| = n \cdot (n - 1) \cdot \dots \cdot (n - k + 1) = \frac{n!}{(n - k)!}$$

This formula is also called the *falling factorial* and denoted  $n^{\underline{k}}$  or  $(n)_k$ .

**Proof:** A permutation is an injective map  $\pi : [k] \rightarrow [n]$ . We count the number of ways to pick such a map, picking the images one after the other. There are  $n$  ways to choose  $\pi(1)$ . Given a value for  $\pi(1)$ , there are  $(n - 1)$  ways to choose  $\pi(2)$  (since we may not choose  $\pi(1)$  again). Continuing like this, there are  $(n - i + 1)$  ways to pick  $\pi(i)$ , and the last value we pick is  $\pi(k)$  with  $(n - k + 1)$  possibilities.

Every  $k$ -permutation can be constructed like this in *exactly one way*. The total number of  $k$ -permutations is therefore given as the product:

$$|P(n, k)| = n \cdot (n - 1) \cdot \dots \cdot (n - k + 1) = \frac{n!}{(n - k)!}$$

□

## Counting Circular Permutations

**Theorem 2:** For any natural numbers  $0 \leq k \leq n$ , we have

$$|P_c(n, k)| = \frac{n!}{k \cdot (n - k)!}$$

**Proof:** We doubly count  $P(n, k)$ :

1.  $|P(n, k)| = \frac{n!}{(n-k)!}$  which we proved before.
2.  $|P(n, k)| = |P_c(n, k)| \cdot k$  because every equivalence class in  $P_c(n, k)$  contains  $k$  permutations from  $P(n, k)$  since there are  $k$  ways to rotate a  $k$ -permutation.

From this we get  $\frac{n!}{(n-k)!} = |P_c(n, k)| \cdot k$ , which implies  $|P_c(n, k)| = \frac{n!}{k \cdot (n-k)!}$ . □

# Unordered Arrangements

**Definition 11:** An *unordered arrangement* of  $k$  elements of  $X$  is a *multiset*  $S = \langle X, r \rangle$  of size  $k$ .

In a multiset,  $X$  is the set of *types*, and for each type  $x \in X$ ,  $r_x$  is its *repetition number*.

*Example:* Let  $X = \{ \text{👉}, \text{🐼}, \text{🐱}, \text{🎴}, \text{🌵} \}$ .

- An unordered arrangement of 7 elements could be  $S = \{ \text{👉}, \text{👉}, \text{🐼}, \text{🐱}, \text{🐱}, \text{🐱}, \text{🌵} \}^*$ .
- The same multiset could be written as  $S = \{ 2 \text{👉}, 1 \text{🐼}, 3 \text{🐱}, 0 \text{🎴}, 1 \text{🌵} \}$ .

## Subsets

The most important special case of unordered arrangements is where all repetitions are 1, that is,  $r_x = 1$  for all  $x \in X$ . Then  $S$  is simply a *subset* of  $X$ , denoted  $S \subseteq X$ .

**Definition 12:** A *k-combination* of  $X$  is an unordered arrangement of  $k$  *distinct* elements of  $X$ .

**Note:** The more standard term is *subset*. The term “combination” is only used to emphasize the selection process.

The set of all  $k$ -subsets of  $X$  is denoted  $\binom{X}{k} = \{A \subseteq X \mid |A| = k\}$ . If  $|X| = n$ , then

$$\binom{n}{k} := \left| \binom{X}{k} \right|$$

*Example:* The set of edges in a simple undirected graph consists of 2-subsets of its vertices:  $E \subseteq \binom{V}{2}$ .

## Counting $k$ -Combinations

**Theorem 3:** For  $0 \leq k \leq n$ , we have

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

**Proof:**  $|P(n, k)| = \frac{n!}{(n-k)!} = \binom{n}{k} \cdot k!$

□

# Multisets

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# Multiset

**Definition 13:** A *multiset* is a modification of the concept of a set that allows for *repetitions* of its elements. Formally, it is denoted as a pair  $M = \langle X, r \rangle$ , where  $X$  is the *groundset* (the set of *types*) and  $r : X \rightarrow \mathbb{N}_0$  is the *multiplicity function*.

*Example:* When the multiset is defined by enumeration, it is advisable to use the notation with the star:

$$M = \{a, b, a, a, b\}^* = \{3 \cdot a, 2 \cdot b\} \quad X = \{a, b\} \quad r_a = 3, r_b = 2$$

*Example:* Prime factorization of a natural number  $n$  is a multiset, e.g.  $120 = 2^3 \cdot 3^1 \cdot 5^1$ .



## $k$ -Combinations of a Multiset

**Definition 14:** Let  $X$  be a finite set of types, and let  $M = \langle X, r \rangle$  be a finite multiset with repetition numbers  $r_1, \dots, r_{|X|}$ . A  *$k$ -combination of  $M$*  is a multiset  $S = \langle X, s \rangle$  with types in  $X$  and repetition numbers  $s_1, \dots, s_{|X|}$  such that  $s_i \leq r_i$  for all  $1 \leq i \leq |X|$ , and  $\sum_{i=1}^{|X|} s_i = k$ .

*Example:* Consider  $M = \{2 \text{ 🦊}, 1 \text{ 🥤}, 3 \text{ 🍉}, 1 \text{ 💎}\}$ .

- $T = \{1 \text{ 🦊}, 2 \text{ 🍉}\}$  is a 3-combination of  $M$ .
- $T' = \{3 \text{ 💎}\}$  is not.

*Counting  $k$ -combinations of a multiset is not as simple as it might seem...*

## $k$ -Permutations of a Multiset

**Definition 15:** Let  $M$  be a finite multiset with set of types  $X$ . A  $k$ -permutation of  $M$  is an ordered arrangement of  $k$  elements of  $M$  where different orderings of elements of the same type are *not distinguished*. This is an ordered multiset with types in  $X$  and repetition numbers  $s_1, \dots, s_{|X|}$  such that  $s_i \leq r_i$  for all  $1 \leq i \leq |X|$ , and  $\sum_{i=1}^{|X|} s_i = k$ .

**Note:** There might be several elements of the same type compared to a permutation of a set (where each repetition number equals 1).

*Example:* Let  $M = \{2 \text{ 🦊}, 1 \text{ 🥤}, 3 \text{ 🍉}, 1 \text{ 💎}\}$ , then  $T = (\text{💎}, \text{🍉}, \text{🍉}, \text{🦊})$  is a 4-permutation of multiset  $M$ .

## Binomial Theorem

**Theorem 4:** The expansion of any non-negative integer power  $n$  of the binomial  $(x + y)$  is a sum

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} \cdot x^k \cdot y^{n-k}$$

where each  $\binom{n}{k}$  is a positive integer known as a *binomial coefficient*, defined as

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+1)}{k \cdot (k-1) \cdot (k-2) \cdot \dots \cdot 2 \cdot 1}$$

# Multinomial Theorem

**Theorem 5:** The generalization of the binomial theorem:

$$(x_1 + \dots + x_r)^n = \sum_{\substack{0 \leq k_1, \dots, k_r \leq n \\ k_1 + \dots + k_r = n}}^n \binom{n}{k_1, \dots, k_r} \cdot x_1^{k_1} \cdot \dots \cdot x_r^{k_r}$$

*Multinomial coefficients* are defined as

$$\binom{n}{k_1, \dots, k_r} = \frac{n!}{k_1! \cdot \dots \cdot k_r!}$$

**Note:** Binomial coefficients are special cases of multinomial coefficients ( $r = 2$ ):

$$\binom{n}{k} = \binom{n}{k_1, k_2} = \binom{n}{k, n-k} = \frac{n!}{k! \cdot (n-k)!}$$

**Proof:** TODO



## Permutations of a Multiset

**Theorem 6:** Let  $S$  be a finite multiset with  $k$  different types and repetition numbers  $r_1, \dots, r_k$ . Let the size of  $S$  be  $n = r_1 + \dots + r_k$ . Then the number of  $n$ -permutations of  $S$  equals

$$\binom{n}{r_1, \dots, r_k}$$

**Proof:** In an  $n$ -permutation there are  $n$  positions that need to be assigned a type.

First, choose the  $r_1$  positions for the first type, there are  $\binom{n}{r_1}$  ways to do so. Then, assign  $r_2$  positions for the second type, out of the  $(n - r_1)$  positions that are still available, there are  $\binom{n-r_1}{r_2}$  ways to do so. Continue for all  $k$  types. The total number of choices will be:

$$\binom{n}{r_1} \cdot \binom{n-r_1}{r_2} \cdot \dots \cdot \binom{n-r_1-r_2-\dots-r_{k-1}}{r_k} = \binom{n}{r_1, \dots, r_k}$$

□

## *k*-Combinations of an *Infinite Multiset*

*Example:* Suppose you have a *sufficiently large* amount of each type of fruit (🍌, 🍎, 🍐) in the supermarket, and you want to buy *two* fruits. How many choices do you have?

There are exactly *six* combinations: {🍌, 🍌}, {🍌, 🍎}, {🍌, 🍐}, {🍎, 🍎}, {🍎, 🍐}, {🍐, 🍐}.

Note that your selection is *not ordered*, so {🍐, 🍎} and {🍎, 🍐} are considered the *same* choice.

## $k$ -Combinations of an *Infinite Multiset* [2]

**Theorem 7:** Let  $k, s \in \mathbb{N}$  and let  $S$  be a multiset with  $s$  types and large repetition numbers (each  $r_1, \dots, r_s$  is *at least  $k$* ), then the number of  $k$ -combinations of  $S$  equals

$$\binom{k+s-1}{k} = \binom{k+s-1}{s-1}$$

**Proof:** Let  $S = \langle X, r_\infty \rangle = \{\infty \text{ 🍌}, \infty \text{ 🍎}, \infty \text{ 🍇}\}$  with  $r_x = \infty$  and  $|X| = s = 3$ .

- Let  $k = 5$  (as an example). Consider a 5-combination of  $S$ : { 🍌 🍌 🍌 🍌 🍌, 🍎, 🍌, 🍇, 🍇 }.
- Reorder and group: { 🍌 🍌 🍌 | 🍎 | 🍇 🍇 }.
- Convert to *dots* and *bars*:  $\bullet\bullet\bullet | \bullet | \bullet\bullet$
- Represent as a 2-type multiset:  $M = \{k \cdot \bullet, (s-1) \cdot | \}$
- Observe: each *permutation* of  $k$  dots and  $(s-1)$  bars corresponds *uniquely* to a  *$k$ -combination* of  $S$ .
- Permute the 2-type multiset:  $\binom{k+s-1}{k, s-1}$  ways, by Theorem 5.

This method is also known as *Stars and Bars*. □

# Compositions

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## Weak Compositions

**Definition 16:** A *weak composition* of a non-negative integer  $k \geq 0$  into  $s$  parts is a *solution* to the equation  $b_1 + \dots + b_s = k$ , where each  $b_i \geq 0$ .

*Example:* Let  $k = 5$ ,  $s = 3$ . Possible non-negative integer solutions for  $b_1 + b_2 + b_3 = 5$  are:

- $(b_1, b_2, b_3) = (1, 1, 3)$
- $(b_1, b_2, b_3) = (1, 3, 1)$
- $(b_1, b_2, b_3) = (2, 0, 3)$
- $(b_1, b_2, b_3) = (0, 5, 0)$
- ... (total 21 solutions)

**Note:** If  $M$  is a multiset over groundset  $\{1, \dots, s\}$  with all multiplicities infinite ( $r_i = \infty$ ), then for  $k \geq 0$ , the number of sub-multisets of  $M$  of size  $k$  is exactly the number of weak compositions of  $k$  into  $s$  parts.

## Counting Weak Compositions

**Theorem 8:** There are  $\binom{k+s-1}{k, s-1}$  *weak compositions* of  $k > 0$  into  $s$  parts.

**Proof:** Observe that  $k = \overbrace{1+1+\dots+1+1}^{k \text{ ones}}$   
 $\qquad\qquad\qquad \underbrace{\qquad}_{b_1} \quad \underbrace{\qquad}_{b_i} \quad \underbrace{\qquad}_{b_s}$

Use the *stars-and-bars* method to count the number of  $s$  groups composed of  $k$  “ones”. □

*Example:* Let  $k = 3$ . There are  $\binom{3+3-1}{3, 3-1} = \binom{5}{3} = \binom{5}{2} = 10$  ways to decompose  $k = 3$  into  $s = 3$  parts:

$$\begin{aligned} k = 3 &= \\ &= 0 + 1 + 2 = 0 + 2 + 1 \\ &= 1 + 0 + 2 = 1 + 2 + 0 = 1 + 1 + 1 \\ &= 2 + 0 + 1 = 2 + 1 + 0 \\ &= 3 + 0 + 0 = 0 + 3 + 0 = 0 + 0 + 3 \end{aligned}$$

## Compositions

**Definition 17:** A *composition* of a positive integer  $k \geq 1$  into  $s$  *positive* parts is a *solution* to the equation  $b_1 + \dots + b_s = k$ , where each  $b_i > 0$ .

**Theorem 9:** There are  $\binom{k-1}{s-1}$  *compositions* of  $k > 0$  into  $s$  positive parts.

**Theorem 10:** The total number of compositions of  $k > 0$  into *some* number of positive parts is

$$\sum_{s=1}^k \binom{k-1}{s-1} = 2^{k-1}$$

## Parallel Summation Identity

Q: How many integer solutions are there to the *inequality*  $b_1 + \dots + b_s \leq k$ , where each  $b_i \geq 0$ ?

**Theorem 11:** 
$$\sum_{m=0}^k \binom{m+s-1}{m} = \binom{k+s}{k}$$

**Proof** (*hint*): Introduce a “dummy” variable  $b_{s+1}$  to take up the *slack* between  $b_1 + \dots + b_s$  and  $k$ . Construct a bijection with the solutions to  $b_1 + \dots + b_s + b_{s+1} = k$ , where  $b_i \geq 0$ . □

# Set Partitions

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## Set Partitions

**Definition 18:** A *partition* of a set  $X$  is a set of non-empty subsets of  $X$  such that every element of  $X$  belongs to exactly one of these subsets.

Equivalently, a family of sets  $P$  is a partition of  $X$  iff:

1. The family  $P$  does not contain the empty set:  $\emptyset \notin P$ .
2. The union of  $P$  is  $X$ , that is,  $\bigcup_{A \in P} A = X$ . The sets in  $P$  are said to *cover*  $X$ .
3. The intersection of any two distinct sets in  $P$  is empty:  $\forall A, B \in P. (A \neq B) \rightarrow (A \cap B = \emptyset)$ .

The sets in  $P$  are said to be *pairwise disjoint* or *mutually exclusive*.

The sets in  $P$  are called *blocks*, *parts*, or *cells*, of the partition.

The block in  $P$  containing an element  $x \in X$  is denoted by  $[x]$ .

## Examples of Set Partitions

*Example:* The empty set  $X = \emptyset$  has exactly one partition,  $P = \emptyset$ .

*Example:* Any singleton set  $X = \{x\}$  has exactly one partition,  $P = \{\{x\}\}$ .

*Example:* For any non-empty proper subset  $A \subset U$ , the set  $A$  and its complement form a partition of  $U$ , namely  $P = \{A, U - A\}$ .

*Example:* The set  $X = \{1, 2, 3\}$  has five partitions:

1.  $\{\{1\}, \{2\}, \{3\}\}$  or  $1 \mid 2 \mid 3$
2.  $\{\{1\}, \{2, 3\}\}$  or  $1 \mid 2 \ 3$
3.  $\{\{1, 2\}, \{3\}\}$  or  $1 \ 2 \mid 3$
4.  $\{\{1, 3\}, \{2\}\}$  or  $1 \ 3 \mid 2$
5.  $\{\{1, 2, 3\}\}$  or  $1 \ 2 \ 3$

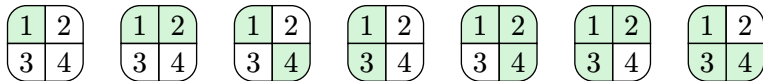
*Example:* The following are *not* partitions of  $\{1, 2, 3\}$ :

- $\{\{\}, \{1, 3\}, \{2\}\}$ , because it contains the empty set.
- $\{\{1, 2\}, \{2, 3\}\}$ , because the element 2 is contained in more than one block.
- $\{\{1\}, \{3\}\}$ , because no block contains the element 3.

## Counting Set Partitions

**Definition 19:** The number of partitions of a set  $X$  (of size  $n = |X|$ ) into  $k$  non-empty blocks (“unlabeled subsets”) is called a *Stirling number of the second kind* and denoted  $S(n, k)$  or  $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ .

*Example:* Let  $X = \{1, 2, 3, 4\}$ ,  $k = 2$ . There are 7 possible partitions:



**Theorem 12:** Let  $\left\{ \begin{smallmatrix} n \\ 0 \end{smallmatrix} \right\} = 0$  for  $n \geq 1$ ,  $\left\{ \begin{smallmatrix} 0 \\ k \end{smallmatrix} \right\} = 0$  for  $k \geq 1$ , and  $\left\{ \begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \right\} = 1$ . For  $n, k \geq 1$ , we have:

$$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} = \left\{ \begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right\} + k \cdot \left\{ \begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right\}$$

**Proof** (*informal*): TODO

□



## Bell Numbers

**Definition 20:** The total number of partitions of a set  $X$  of size  $n = |X|$  (into an arbitrary number of non-empty blocks) is called a *Bell number* and denoted  $B_n$ .

$$B_n = \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$$

**Note:** Consider the special case of  $n = 0$ . There is exactly *one* partition of  $\emptyset$  into non-empty parts:  $\emptyset = \bigcup_{A \in \emptyset} A \in \emptyset$ . Every  $A \in \emptyset$  is non-empty, since no such  $A$  exists. Thus, we have  $B_0 = S(0, 0) = 1$ .

## Bell Numbers [2]

**Theorem 13:** For  $n \geq 1$ , we have the recursive identity for Bell numbers:

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

**Proof:** Every partition of  $[n]$  has one part that contains the number  $n$ . In addition to  $n$ , this part also contains  $k$  other numbers (for some  $0 \leq k \leq n-1$ ). The remaining  $n-1-k$  elements are partitioned arbitrarily. From this correspondence, we obtain the desired identity:

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_{n-1-k} = \sum_{k=0}^{n-1} \binom{n-1}{n-1-k} B_{n-1-k} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k$$

□

# Integer Partitions

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# Integer Partitions

**Definition 21:** An *integer partition* of a positive integer  $n \geq 1$  into  $k$  *positive* parts is a *solution* to the equation  $n = a_1 + \dots + a_k$ , where  $a_1 \geq a_2 \geq \dots \geq a_k \geq 1$ .

- The number of integer partitions of  $n$  into  $k$  positive non-decreasing parts is denoted  $p_k(n)$  and defined recursively:

$$p_k(n) = \begin{cases} 0 & \text{if } k > n \\ 0 & \text{if } n \geq 1 \text{ and } k = 0 \\ 1 & \text{if } n = k = 0 \\ p_k(n - k) + p_{k-1}(n - 1) & \text{if } 1 \leq k \leq n \end{cases}$$

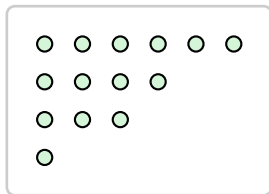
- The number of partitions of  $n$  (into an arbitrary number of parts) is the *partition function*  $p(n)$ :

$$p(n) = \sum_{k=0}^n p_k(n)$$

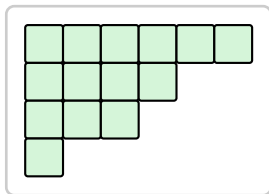
# Ferrer Diagrams and Young Tableaux

*Example:* Consider an integer partition:  $14 = 6 + 4 + 3 + 1$ .

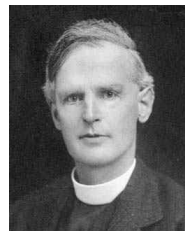
**Ferrer Diagram**



**Young Tableaux**



Norman Ferrer



Alfred Young

# Inclusion–Exclusion

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# The Inclusion–Exclusion Principle

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TODO: small example of PIE with 2 or 3 sets

## Principle of Inclusion–Exclusion (PIE)

**Theorem 14:** Let  $X$  be a finite set and  $P_1, \dots, P_m$  *properties*.

- Define  $X_i = \{x \in X \mid x \text{ has } P_i\}$ , i.e. the set of all elements from  $X$  having a property  $P_i$ .
- Define for  $S \subseteq [m]$  the set  $N(S) = \{x \in X \mid \forall i \in S : x \text{ has } P_i\}$ . Observe:  $N(S) = \bigcap_{i \in S} X_i$ .

The number of elements of  $X$  that satisfy *none* of the properties  $P_1, \dots, P_m$  is given by

$$|X \setminus (X_1 \cup \dots \cup X_m)| = \sum_{S \subseteq [m]} (-1)^{|S|} |N(S)| \quad (1)$$

**Proof:** Consider any  $x \in X$ . If  $x \in X$  has none of the properties, then  $x \in N(\emptyset)$  and  $x \notin N(S)$  for any other  $S \neq \emptyset$ . Hence  $x$  *contributes 1* to the sum (1).

If  $x \in X$  has exactly  $k \geq 1$  of the properties, call this set  $T \in \binom{[m]}{k}$ . Then  $x \in N(S)$  iff  $S \subseteq T$ .

The *contribution of  $x$*  to the sum (1) is  $\sum_{S \subseteq T} (-1)^{|S|} = \sum_{i=0}^k \binom{k}{i} (-1)^i = 0$ , i.e. *zero*. □

**Note:** In the last step, we used that for any  $y \in \mathbb{R}$  we have  $(1 - y)^k = \sum_{i=0}^k \binom{k}{i} (-y)^i$  which implies (for  $y = 1$ ) that  $0 = \sum_{i=0}^k \binom{k}{i} (-1)^i$ .



## Very Useful Corollary of PIE

**Corollary 14.1:** 🦊

$$\left| \bigcup_{i \in [m]} X_i \right| = |X| - \sum_{S \subseteq [m]} (-1)^{|S|} |N(S)| = \sum_{\emptyset \neq S \subseteq [m]} (-1)^{|S|-1} |N(S)|$$

## Applications of PIE

Let's state the principle of inclusion-exclusion using a rigid pattern:

1. *Define “bad” properties.*

Identify the things to count as the elements of some universe  $X$  except for the whose having *at least one* of the “bad” properties  $P_1, \dots, P_m$ . In other words, we want to count  $X \setminus (X_1 \cup \dots \cup X_m)$ .

2. *Count  $N(S)$ .*

For each  $S \subseteq [m]$ , determine  $N(S)$ , the number of elements of  $X$  having *all* bad properties  $P_i$  for  $i \in S$ .

3. *Apply PIE.*

Use Theorem 14 to obtain a closed formula for  $|X \setminus (X_1 \cup \dots \cup X_m)|$ .

## Counting Surjections via PIE

**Theorem 15:** The number of surjections from  $[k]$  to  $[n]$  is given by

$$\left| \left\{ f : [k] \xrightarrow{\text{surj.}} [n] \right\} \right| = \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^k$$

**Proof:** Let  $X$  be the set of all maps from  $[k]$  to  $[n]$ .

1. *Define bad properties:* Define the “bad” property  $P_i$  for  $i \in [n]$  as “ $i$  is not in the image of  $f$ ”, i.e.

$$f : [k] \rightarrow [n] \text{ has property } P_i \iff \forall j \in [k] : f(j) \neq i$$

The *surjective* functions are exactly those functions that *do not* have any of the “bad” properties.

2. *Count  $N(S)$ :* We claim  $N(S) = (n - |S|)^k$  for any  $S \subseteq [n]$ . To see this, observe that  $f$  has all properties with indices from  $S$  if and only if  $f(i) \notin S$  for all  $i \in [k]$ . In other words,  $f$  must be a function from  $[k]$  to  $[n] \setminus S$ , and there are  $(n - |S|)^k$  of those.

## Counting Surjections via PIE [2]

3. *Apply PIE:* Using Theorem 14, the number of surjections from  $[k]$  to  $[n]$  is

$$\begin{aligned} |X \setminus (X_1 \cup \dots \cup X_n)| &\stackrel{\text{PIE}}{=} \sum_{S \subseteq [n]} (-1)^{|S|} |N(S)| \\ &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)^k \\ &= \sum_{i=0}^n (-1)^i \binom{n}{i} (n - i)^k \end{aligned}$$

In the last step, we used that  $(-1)^{|S|} (n - |S|)^k$  only depends on the size of  $S$ , and there are  $\binom{n}{i}$  sets  $S \subseteq [n]$  of size  $i$ .

□

## More Useful Corollaries

**Corollary 15.1:** Consider the case  $n = k$ . A function from  $[n]$  to  $[n]$  is a *surjection* iff it is a *bijection*. Since there are  $n!$  bijections on  $[n]$  (namely, all permutations), we have the following identity:

$$n! = \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^n$$

**Corollary 15.2:** A surjection from  $[k]$  to  $[n]$  can be seen as a partition of  $[k]$  into  $n$  non-empty distinguishable (labeled) parts (the map assigns a part to each  $i \in [n]$ ).

Since the partition of  $[k]$  into  $n$  non-empty indistinguishable parts is denoted  $s_n^{\text{II}}(k)$ , and there are  $n!$  ways to assign labels to  $n$  parts, we obtain that the number of surjections is equal to  $n!s_n^{\text{II}}(k)$ :

$$n!s_n^{\text{II}}(k) = \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)^k$$

## Derangements

**Theorem 16:** The *derangements*  $D_n$  on  $n$  elements are permutations of  $[n]$  without fixed points.

The number of derangements is given by

$$|D_n| = \sum_{i=0}^n (-1)^i \binom{n}{i} (n-i)!$$

**Proof:** Let  $X$  be the set of all permutations of  $[n]$ .

1. Define the “bad” property  $P_i$  to mean “ $\pi$  has a fixpoint  $i$ ” ( $i \in [n]$ ):

$$\pi \in X \text{ has property } P_i \iff \pi(i) = i$$

2. We claim  $N(S) = (n - |S|)!$  for any  $S \subseteq [n]$ .

Indeed,  $\pi \in X$  has all properties with indices from  $S$  if and only if all  $i \in S$  are fixed points of  $\pi$ . On the other elements, i.e. on  $[n] \setminus S$ ,  $\pi$  may be an arbitrary bijection, so there are  $(n - |S|)!$  choices for  $\pi$ .

## Derangements [2]

3. Using Theorem 14, the number of derangements is given by

$$\begin{aligned} |X \setminus (X_1 \cup \dots \cup X_n)| &\stackrel{\text{PIE}}{=} \sum_{S \subseteq [n]} (-1)^{|S|} |N(S)| \\ &= \sum_{S \subseteq [n]} (-1)^{|S|} (n - |S|)! \\ &= \sum_{i=0}^n (-1)^i \binom{n}{i} (n - i)! \end{aligned}$$

In the last step, we used that  $(-1)^{|S|} (n - |S|)!$  only depends on the size of  $S$ , and there are  $\binom{n}{i}$  sets  $S \subseteq [n]$  of size  $i$ .

□

# Generating Functions

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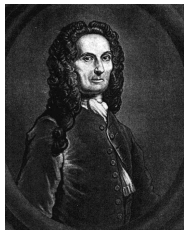
# Generating Functions

*A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag.*

— George Pólya, Mathematics and Plausible Reasoning [1]

*A generating function is a clothesline on which we hang up a sequence of numbers for display.*

— Herbert Wilf, generatingfunctionology [2]



Abraham  
de Moivre



George Pólya



Herbert Wilf

## Ordinary Generating Functions

**Definition 22:** An *ordinary generating function* (OGF) of a sequence  $a_n$  is a *power series*

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n$$

*Example:* The *sequence*  $a_n = (a_0, a_1, a_2, \dots)$  is *generated* by the OGF  $G(x) = a_0 + a_1x + a_2x^2 + \dots$

*Example:*  $G(x) = 3 + 8x^2 + x^3 + \frac{1}{7}x^5 + 100x^6 + \dots$  *generates* the sequence  $(3, 0, 8, 1, 0, \frac{1}{7}, 100, 0, \dots)$

*Example:* Consider a long division of 1 by  $(1 - x)$ , the result is an infinite power series

$$\frac{1}{1-x} = 1 + x^1 + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n$$

Note that all coefficients are 1. Thus, the generating function of  $(1, 1, 1, \dots)$  is  $G(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ .

## The Core Generating Function

Another proof that  $(1, 1, 1, \dots)$  is generated by  $G(x) = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = S$ :

$$S = 1 + x + x^2 + x^3 + \dots$$

$$x \cdot S = x + x^2 + x^3 + \dots$$

---

$$S - x \cdot S = 1$$

$$\text{Thus, } S = \frac{1}{1-x}.$$

The generating function  $G(x) = 1 + x + x^2 + \dots$  is also known as the *Maclaurin series* of  $\frac{1}{1-x}$ .

## More Examples of Generating Functions

Formula	Power series	Sequence	Description
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$	$(1, 1, 1, \dots)$	constant 1
$\frac{2}{1-x}$	$\sum_{n=0}^{\infty} 2x^n = 2 + 2x + 2x^2 + 2x^3 + \dots$	$(2, 2, 2, \dots)$	constant 2
$\frac{x}{1-x}$	$\sum_{n=1}^{\infty} x^n = 0 + x + x^2 + x^3 + \dots$	$(0, 1, 1, 1, \dots)$	right shift
$\frac{1}{1+x}$	$\sum_{n=0}^{\infty} (-1)^n x^n = 0 + 1 - x + x^2 - x^3 + \dots$	$(1, -1, 1, \dots)$	sign-alternating 1's
$\frac{1}{1-3x}$	$\sum_{n=0}^{\infty} 3^n x^n = 1 + 3x + 9x^2 + 27x^3 + \dots$	$(1, 3, 9, \dots)$	powers of 3

## More Examples of Generating Functions [2]

Formula	Power series	Sequence	Description
$\frac{1}{1-x^2}$	$\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \dots$	$(1, 0, 1, 0, \dots)$	regular gaps
$\frac{1}{(1-x)^2}$	$\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \dots$	$(1, 2, 3, 4, \dots)$	natural numbers

$$\begin{aligned}
 \frac{1-x^{n+1}}{1-x} &= \frac{1}{1-x} - \frac{x^{n+1}}{1-x} = \\
 &\triangleq (1, 1, 1, \dots) - \underbrace{(0, 0, \dots, 0, 1, 1, \dots)}_{n+1 \text{ zeros}} = \\
 &= \underbrace{(1, 1, \dots, 1, 0, 0, \dots)}_{n+1 \text{ ones}} = \\
 &\triangleq 1 + x + x^2 + \dots + x^n
 \end{aligned}$$

## Exercises

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*Example:* Find GF for odd numbers:  $(1, 3, 5, \dots)$ .

*Example:* Find GF for  $(1, 3, 7, 15, 31, 63)$ , which satisfies  $a_n = 3a_{n-1} - 2a_{n-2}$  with  $a_0 = 1, a_1 = 3$ .

## Solving Combinatorial Problems via Generating Functions

*Example:* Find the number of integer solutions to  $y_1 + y_2 + y_3 = 12$  with  $0 \leq x_i \leq 6$ .

- Possible values for  $y_1$  are  $0 \leq y_1 \leq 6$ .
  - ▶ There is a *single* way to select  $y_1 = 0$ . The same for other values among  $1, \dots, 6$ .
  - ▶ There are *no* ways to select any value of  $y_1$  higher than 6.
  - ▶ The *number of ways to select  $y_1$  to be equal to  $n$*  forms a sequence  $(1, 1, 1, 1, 1, 1, 0, \dots)$ .
  - ▶ Write this sequence as a polynomial  $x^0 + x^1 + \dots + x^6$ .
  - ▶ Do the same for  $y_2$  and  $y_3$  (*in isolation!*).
- Since all combinations of  $y_1, y_2$  and  $y_3$  are valid non-conflicting solutions, we can multiply those polynomials and obtain the *generating function*  $G(x) = (1 + x + x^2 + \dots + x^6)^3$ .
  - ▶ For each  $n$ , the coefficient of  $x^n$  in  $G(x)$  is the number of integer solutions to  $x_1 + x_2 + x_3 = n$ .
  - ▶ In particular, we are interested in the coefficient of  $x^{12}$  in  $G(x)$ , denoted  $[x^{12}]G(x)$ .
  - ▶ Use ~~pen and paper~~ Wolfram Alpha to expand  $G(x)$ :

$$G(x) = x^{18} + 3x^{17} + 6x^{16} + \dots + \underline{28x^{12}} + \dots + 6x^2 + 3x + 1$$

- The *answer* is  $[x^{12}]G(x) = 28$  solutions.

## Slightly More Complex Combinatorial Problem

*Example:* Suppose we have marbles of three different colors (●, ●, ●), and we want to *count* the number of ways to select 20 marbles, such that:

- There are an even number of ●:  $1 + x^2 + x^4 + \dots + x^{20}$ .
- There are at least 12 ●:  $x^{12} + x^{13} + \dots + x^{20}$ .
- There are at most 5 ●:  $1 + x + x^2 + x^3 + x^4 + x^5$ .

Multiply polynomials and find  $[x^{20}]G(x)$ :

$$\begin{aligned} & [x^{20}](1 + x^2 + x^4 + \dots + x^{20})(x^{12} + x^{13} + \dots + x^{20})(1 + x + x^2 + x^3 + x^4 + x^5) = \\ & = [x^{20}](x^{45} + 2x^{44} + \dots + \underline{21x^{20}} + \dots + 2x^{13} + x^{12}) \\ & = 21 \end{aligned}$$



## Using Power Series in Combinatorial Problems

*Example:* Find the number of integer solutions to  $a_1 + a_2 + a_3 = 12$  with  $a_1 \geq 2, 3 \leq a_2 \leq 6, 0 \leq a_3 \leq 9$ .

- Compose the generating function:

$$G(x) = (x^2 + x^3 + \dots) \cdot (x^3 + x^4 + x^5 + x^6) \cdot (1 + x + x^2 + \dots + x^9)$$

- Substitute the power series with the corresponding simple forms:

$$G(x) = \left(x^2 \cdot \frac{1}{1-x}\right) \cdot \left(x^3 \cdot \frac{1-x^4}{1-x}\right) \cdot \left(\frac{1-x^{10}}{1-x}\right)$$

- Expand the series:

$$G(x) = x^5 + 3x^6 + 6x^7 + 10x^8 + 14x^9 + 18x^{10} + 22x^{11} + \underline{26x^{12}} + 30x^{13} + \\ 34x^{14} + 37x^{15} + 39x^{16} + 40x^{17} + \dots + 40x^n + \dots$$

- Sequence:  $(g_n) = (0, 0, 0, 0, 0, 1, 3, 6, 10, 14, 18, 22, 26, 30, 34, 37, 39, \overline{40}, \dots)$
- Answer for  $n = 12$  is  $[x^{12}]G(x) = 26$ .

## Operations on Generating Functions

Let  $F(x) = \sum_{n=0}^{\infty} a_n x^n$  and  $G(x) = \sum_{n=0}^{\infty} b_n x^n$  be ordinary generating functions.

Operation	Result
Differentiate $F(x)$ term-wise	$F'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$
Multiply $F(x)$ by a scalar $\lambda \in \mathbb{R}$ term-wise	$\lambda F(x) = \sum_{n=0}^{\infty} \lambda a_n x^n$
Add $F(x)$ and $G(x)$ term-wise	$F(x) + G(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$
Multiply $F(x)$ and $G(x)$ term-wise ( <i>Cauchy product</i> , or <i>convolution</i> )	$F(x) \cdot G(x) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) x^n$

## Well-Formed Paranthesis Expressions

*Example:* Find the number of *well-formed parenthesis expressions* with  $n$  pairs of parenthesis.

For example, “( ( ) ( ) )” is a well-formed parenthesis expression with 4 pairs of parenthesis.

Formally, a permutation of the multiset  $\{n \cdot "(", n \cdot ")"\}$  is *well-formed* if reading it from left to right and counting “+1” for every opening parenthesis “(“ and “−1” for every closing parenthesis “)” never yields a negative number at any time.

Every well-formed expression with  $n \geq 1$  pairs of paranthesis starts with “(“ and there is a unique matching “)” such that the sequence in between and the sequence after are well-formed. For example:

$$()()() \quad (())() \quad ()(())()$$

In other words, a well-formed expression with  $n$  pairs of parenthesis is obtained by putting a well-formed expression with  $k$  pairs in between “(“ and “)” and then appending a well-formed expression with  $n - k - 1$  pairs of parenthesis. This gives the equation:

$$a_n = \sum_{k=0}^{n-1} a_k a_{n-k-1}$$

## Well-Formed Paranthesis Expressions [2]

Let  $F(x)$  be a generating function for  $a_n$ , then we know:

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \left( \sum_{k=0}^{n-1} a_k a_{n-k-1} \right) x^n = 1 + \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k a_{n-k} \right) x^{n+1} \\ &= 1 + x \cdot \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k a_{n-k} \right) x^n = 1 + x \cdot F(x)^2 \end{aligned}$$

Solving the equation  $xF^2 - F + 1 = 0$  for  $F$  gives:

$$F(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

We are only interested in the root with the *minus* sign, since  $\lim_{x \rightarrow 0} F(x) = a_0 = 1$ :

$$\lim_{x \rightarrow 0} F(x) = \lim_{x \rightarrow 0} \frac{1 - \sqrt{1-4x}}{2x} = \lim_{x \rightarrow 0} \frac{\frac{2}{\sqrt{1-4x}}}{2} = 1$$

# Newton's Binomial Theorem

Let's revisit the binomial theorem:

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k = \sum_{k=0}^{\infty} \binom{n}{k} x^k \quad \forall n \in \mathbb{N}$$

where  $\binom{n}{k} = 0$  for  $k > n$ .

**Note:** This shows that  $(1+x)^n$  is the generating function for the series  $(a_k)_{k \in \mathbb{N}}$  with  $a_k = \binom{n}{k}$ .

We can extend this result from natural numbers  $n \in \mathbb{N}$  to any *real* number  $n \in \mathbb{R}$ .

## Binomial Coefficients for Real Numbers

**Definition 23:** Let  $p(n, k) = n \cdot (n - 1) \cdot \dots \cdot (n - k + 1)$ , also called the *falling factorial*  $n^{\underline{k}}$ .

Extend the definition of binomial coefficients for real numbers  $n, k \in \mathbb{R}$ :

$$\binom{n}{k} = \frac{p(n, k)}{k!}$$

**Note:** This definition aligns with the definition of binomial coefficients for natural numbers:

$$\binom{n}{k} = \frac{n!}{k! \cdot (n - k)!} = \frac{n \cdot (n - 1) \cdot \dots \cdot (n - k + 1)}{k!}$$

*Example:* Consider the number “ $-7 / 2$  choose 5”:

$$\binom{-7 / 2}{5} = \frac{-\frac{7}{2} \cdot -\frac{9}{2} \cdot -\frac{11}{2} \cdot -\frac{13}{2} \cdot -\frac{15}{2}}{5!} = -\frac{9009}{256}$$

**Note:**  $p(n, 0) = 1$  and for  $k \geq 1$ , we have  $p(n, k) = (n - k + 1) \cdot p(n, k - 1) = n \cdot p(n - 1, k - 1)$ . (★)

## Extended Newton's Binomial Theorem

**Theorem 17:** For all non-zero  $n \in \mathbb{R}$ , we have:

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

*Example:* Let  $n = 1/2$ , then we have an identity for  $\sqrt{1+x}$ :

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k$$

To actually *use* this fact, we need some *lemma*...

**Lemma 18:** For any integer  $n \geq 1$ , we have:

$$\binom{1/2}{n} = (-1)^{n+1} \cdot \binom{2n-2}{n-1} \cdot \frac{1}{2^{2n-1}} \cdot \frac{1}{n}$$

## Extended Newton's Binomial Theorem [2]

**Proof:** By induction on  $n$ .

**Base:**  $n = 1$ . 
$$\binom{1/2}{1} = \frac{1/2}{1!} = \frac{1}{2} = 1 \cdot 1 \cdot \frac{1}{2} \cdot 1 = \underbrace{(-1)^2}_1 \cdot \underbrace{\binom{2-2}{1-1}}_1 \cdot \underbrace{\frac{1}{2^{2-1}}}_{\frac{1}{2}} \cdot \underbrace{\frac{1}{1}}_1$$

**Induction step:**  $n$  to  $n + 1$  for  $n > 1$ . We use the recursion  $(\star) p(n, k) = n \cdot p(n - 1, k - 1)$ :

$$\begin{aligned} \binom{1/2}{n+1} &= \frac{p(1/2, n+1)}{(n+1)!} = \frac{(1/2 - (n+1) + 1) \cdot p(1/2, n)}{(n+1) \cdot n!} = -\frac{n-1/2}{n+1} \binom{1/2}{n} \quad \square \\ &\stackrel{\text{IH}}{=} -\frac{n-1/2}{n+1} (-1)^{n+1} \cdot \binom{2n-2}{n-1} \cdot \frac{1}{2^{2n-1}} \cdot \frac{1}{n} \\ &= \frac{2n}{2n} \cdot \frac{2n-1}{2n} \cdot (-1)^{n+2} \cdot \binom{2n-2}{n-1} \cdot \frac{1}{2^{2n-1}} \cdot \frac{1}{n+1} \\ &= (-1)^{n+2} \cdot \underbrace{\frac{(2n-2)! \cdot (2n-1) \cdot (2n)}{(n-1)! \cdot (n-1) \cdot n \cdot n}}_{\binom{2n}{n}} \cdot \frac{1}{2^{2n+1}} \cdot \frac{1}{n+1} \end{aligned}$$



## Catalan Numbers

**Proposition 19:** Now we can expand  $\sqrt{1+n}$  into the following series:

$$\sqrt{1+n} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \sum_{n=1}^{\infty} -2 \cdot \binom{2n-2}{n-1} \cdot (-1)^n \cdot \frac{1}{2^{2n}} \cdot \frac{1}{n} \cdot x^n$$

*Example:* Going back to the example with the number of well-formed parenthesis expressions, we get:

$$\begin{aligned} F(x) &= \frac{1 - \sqrt{1-4x}}{2x} = \frac{1}{2x} \sum_{n=1}^{\infty} 2 \cdot \binom{2n-2}{n-1} \cdot (-1)^n \cdot \frac{1}{2^{2n}} \cdot \frac{1}{n} \cdot (-4x)^n \\ &= \frac{1}{x} \sum_{n=1}^{\infty} \binom{2n-2}{n-1} \frac{1}{n} x^n = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n+1} x^n \end{aligned}$$

The numbers  $C_n := \binom{2n}{n} \frac{1}{n+1}$  are called *Catalan numbers*.

# Recurrence Relations

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# Recurrence Relations

*Example:*

- *Recurrent relation* defining a sequence  $(a_n)$ :

$$a_n = \begin{cases} a_0 = \text{const} & \text{if } n = 0 \\ a_{n-1} + d & \text{if } n > 0 \end{cases}$$

- *Solving* it results in a non-recursive *closed* formula:

$$a_n = a_0 + n \cdot d$$

- *Checking* it confirms that the formula is correct:

$$a_n = \underbrace{a_{n-1}} + d = a_0 + \underbrace{(n-1)d}_{a_{n-1}} + d = a_0 + n \cdot d \quad \blacksquare$$

# Linear Homogeneous Recurrence Relations

**Definition 24:** A *linear homogeneous* recurrence relation *of degree  $k$*  with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k},$$

where  $c_1, c_2, \dots, c_k$  are constants (real or complex numbers), and  $c_k \neq 0$ .

*Examples:*

- $b_n = 2.71b_{n-1}$  is a linear homogeneous recurrence relation of degree 1.
- $F_n = F_{n-1} + F_{n-2}$  is a linear homogeneous recurrence relation of degree 2.
- $g_n = 2g_{n-5}$  is a linear homogeneous recurrence relation of degree 5.
- The recurrence relation  $a_n = a_{n-1} + a_{n-2}^2$  is *not linear*.
- The recurrence relation  $H_n = 2H_{n-1} + 1$  is *not homogeneous*.
- The recurrence relation  $B_n = nB_{n-1}$  does *not* have *constant* coefficients.

## Characteristic Equations

Hereinafter,  $(*)$  denotes a linear homogeneous recurrence relation  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ .

**Theorem 20:**  $a_n = r^n$  is a solution to  $(*)$  if and only if  $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$ .

**Definition 25:** A *characteristic equation* for  $(*)$  is the algebraic equation in  $r$  defined as:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$$

The sequence  $(a_n)$  with  $a_n = r^n$  (with  $r_n \neq 0$ ) is a solution if and only if  $r$  is a solution of the characteristic equation. Such solutions are called *characteristic roots* of  $(*)$ .

## Distinct Roots Case

**Theorem 21:** Let  $c_1$  and  $c_2$  be real numbers. Suppose that  $r^2 - c_1r - c_2 = 0$  has two *distinct* roots  $r_1$  and  $r_2$ . Then the sequence  $(a_n)$  is a solution of the recurrence relation  $a_n = c_1a_{n-1} + c_2a_{n-2}$  if and only if  $a_n = \alpha_1r_1^n + \alpha_2r_2^n$  for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants.

**Proof (sketch):** Since  $r_1$  and  $r_2$  are roots, then  $r_1^2 = c_1r_1 + c_2$  and  $r_2^2 = c_1r_2 + c_2$ . Next, we can see:

$$\begin{aligned}c_1a_{n-1} + c_2a_{n-2} &= c_1(\alpha_1r_1^{n-1} + \alpha_2r_2^{n-1}) + c_2(\alpha_1r_1^{n-2} + \alpha_2r_2^{n-2}) \\&= \alpha_1r_1^{n-2}(c_1r_1 + c_2) + \alpha_2r_2^{n-2}(c_1r_2 + c_2) \\&= \alpha_1r_1^{n-2}r_1^2 + \alpha_2r_2^{n-2}r_2^2 \\&= \alpha_1r_1^n + \alpha_2r_2^n \\&= a_n\end{aligned}$$

To show that every solution  $(a_n)$  of the recurrence relation  $a_n = c_1a_{n-1} + c_2a_{n-2}$  has  $a_n = \alpha_1r_1^n + \alpha_2r_2^n$  for some constants  $\alpha_1$  and  $\alpha_2$ , suppose that the initial condition are  $a_0 = C_0$  and  $a_1 = C_1$ , and show that there exist constants  $\alpha_1$  and  $\alpha_2$  such that  $a_n = \alpha_1r_1^n + \alpha_2r_2^n$  satisfies the same initial conditions.  $\square$

## Solving Recurrence Relations using Characteristic Equations

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*Example:* Solve  $a_n = a_{n-1} + 2a_{n-2}$  with  $a_0 = 2$  and  $a_1 = 7$ .

- The *characteristic* equation is  $r^2 - r - 2 = 0$ .
- It has two *distinct* roots  $r_1 = 2$  and  $r_2 = -1$ .
- The sequence  $(a_n)$  is a solution iff  $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$  for  $n = 0, 1, 2, \dots$  and some constants  $\alpha_1$  and  $\alpha_2$ .

$$\begin{cases} a_0 = 2 = \alpha_1 + \alpha_2 \\ a_1 = 7 = \alpha_1 \cdot 2 + \alpha_2 \cdot (-1) \end{cases}$$

- Solving these two equations gives  $\alpha_1 = 3$  and  $\alpha_2 = -1$ .
- Hence, the *solution* to the recurrence equation with given initial conditions is the sequence  $(a_n)$  with

$$a_n = 3 \cdot 2^n - (-1)^n$$

## Fibonacci Numbers

*Example:* Find the closed formula for Fibonacci numbers.

- The recurrence relation is  $F_n = F_{n-1} + F_{n-2}$ .
- The characteristic equation is  $r^2 - r - 1 = 0$ .
- The roots are  $r_1 = (1 + \sqrt{5})/2$  and  $r_2 = (1 - \sqrt{5})/2$ .
- Therefore, the solution is  $F_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$  for some constants  $\alpha_1$  and  $\alpha_2$ .
- Using the initial conditions  $F_0 = 0$  and  $F_1 = 1$ , we get

$$\begin{cases} F_0 = \alpha_1 + \alpha_2 = 0 \\ F_1 = \alpha_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right) = 1 \end{cases}$$

- Solving these two equations gives  $\alpha_1 = 1/\sqrt{5}$  and  $\alpha_2 = -1/\sqrt{5}$ .
- Hence, the *closed formula* (also known as Binet's formula) for Fibonacci numbers is

$$F_n = \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^n}_{\varphi} - \frac{1}{\sqrt{5}} \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}_{\psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$



## Single (Repeated) Root Case

**Theorem 22:** Let  $c_1$  and  $c_2$  be real numbers. Suppose that  $r^2 - c_1 r - c_2 = 0$  has a *single (repeated)* root  $r_0$ . A sequence  $(a_n)$  is a solution of the recurrence relation  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$  if and only if  $a_n = (\alpha_1 + \alpha_2 n) r_0^n$  for  $n = 0, 1, 2, \dots$ , where  $\alpha_1$  and  $\alpha_2$  are constants.

*Example:* Solve  $a_n = 6a_{n-1} - 9a_{n-2}$  with  $a_0 = 1$  and  $a_1 = 6$ .

The characteristic equation is  $r^2 - 6r + 9 = 0$  with a single (repeated) root  $r_0 = 3$ . Hence, the solutions is of the form  $a_n = (\alpha_1 + \alpha_2 n) 3^n$ .

$$\begin{cases} a_0 = 1 = \alpha_1 \\ a_1 = 6 = (\alpha_1 + \alpha_2) \cdot 3 \end{cases} \implies \begin{cases} \alpha_1 = 1 \\ \alpha_2 = 1 \end{cases}$$

Thus, the *solution* is  $a_n = (1 + n) 3^n$ .

## General Case

**Theorem 23:** Let  $c_1, \dots, c_k$  be real numbers. Suppose that the recurrence  $a_n = c_1 a_{n-1} + \dots + c_k a_{n-k}$  has  $t$  distinct characteristic roots  $r_1, \dots, r_t$  with multiplicities  $m_1, \dots, m_t$ , respectively, so that  $m_i \geq 1$  for  $i = 1, \dots, t$  and  $m_1 + \dots + m_t = k$ .

Then the general of the recurrence relation is given by

$$a_n = \sum_{i=1}^t \left( r_i^n \cdot \sum_{j=0}^{s_i-1} \alpha_{i,j} x^j \right)$$

where  $\alpha_{i,j}$  are constants for  $1 \leq i \leq t$  and  $0 \leq j \leq m_i - 1$ .

*Example:* Find generic solution for  $a_n = 7a_{n-1} - 16a_{n-2} + 12a_{n-3}$ .

The characteristic equation  $r^3 - 7r^2 + 16r - 12 = 0$  has  $t = 2$  distinct roots:  $r_1 = 2$  repeated  $m_1 = 2$  times, and  $r_2 = 3$  with multiplicity  $m_2 = 1$ . Hence, the solution is of the form

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n) 2^n + \alpha_{2,0} 3^n$$

## Complex Case

*Example:* Solve the recurrence  $a_n = 6a_{n-1} - 16a_{n-2} + 25a_{n-3} - 20a_{n-4} + 8a_{n-5}$  with the initial conditions  $a_0 = 2, a_1 = 2, a_2 = 0, a_3 = -4, a_4 = -8$ .

Characteristic equation is  $r^5 - 6r^4 + 16r^3 - 25r^2 + 20r - 8 = 0$ .

Characteristic roots are:  $r_1 = (1 - i)$ , repeated twice,  $r_2 = (1 + i)$ , also repeated twice, and  $r_3 = 2$ .

General solution has the form:

$$a_n = (\alpha_{1,0} + \alpha_{1,1}n)(1 - i)^n + (\alpha_{2,0} + \alpha_{2,1}n)(1 + i)^n + \alpha_{3,0} 2^n$$

To find the constants, we can use the initial conditions:

$$\begin{cases} a_0 = 2 &= \alpha_{1,0} + \alpha_{1,1} &+ \alpha_{2,0} + \alpha_{2,1} &+ \alpha_{3,0} \\ a_1 = 2 &= (\alpha_{1,0} + \alpha_{1,1})(1 - i) &+ (\alpha_{2,0} + \alpha_{2,1})(1 + i) &+ 2\alpha_{3,0} \\ a_2 = 0 &= (\alpha_{1,0} + 2\alpha_{1,1})(1 - i)^2 &+ (\alpha_{2,0} + 2\alpha_{2,1})(1 + i)^2 &+ 4\alpha_{3,0} \\ a_3 = -4 &= (\alpha_{1,0} + 3\alpha_{1,1})(1 - i)^3 &+ (\alpha_{2,0} + 3\alpha_{2,1})(1 + i)^3 &+ 8\alpha_{3,0} \\ a_4 = -8 &= (\alpha_{1,0} + 4\alpha_{1,1})(1 - i)^4 &+ (\alpha_{2,0} + 4\alpha_{2,1})(1 + i)^4 &+ 16\alpha_{3,0} \end{cases} \implies \begin{cases} \alpha_{1,0} = 1 \\ \alpha_{1,1} = 0 \\ \alpha_{2,0} = 1 \\ \alpha_{2,1} = 0 \\ \alpha_{3,0} = 0 \end{cases}$$

## Linear Non-Homogeneous Recurrence Relations

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$$

*Example:*  $a_n = 3a_{n-1} + 2n$  is non-homogeneous.

**Definition 26:** An *associated homogeneous recurrence relation* is the relation without the term  $F(n)$ .

## Solving Non-Homogeneous Recurrence Relations

**Theorem 24:** If  $(a_n^{(p)})$  is a *particular* solution of the non-homogeneous linear recurrence relation  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n)$ , then *every solution* is of the form  $(a_n^{(p)} + a_n^{(h)})$ , where  $(a_n^{(h)})$  is a solution of the associated homogeneous recurrence relation.

*Example:* Find all solutions of the recurrence relation  $a_n = 3a_{n-1} + 2n$ . What is the solution with  $a_1 = 3$ ?

- First, solve the associated homogeneous recurrence relation  $a_n = 3a_{n-1}$ .
- It has a general solution  $a_n^{(h)} = \alpha 3^n$ , where  $\alpha$  is a constant.
- To find a particular solution, observe that  $F(n) = 2n$  is a polynomial in  $n$  of degree 1, so a reasonable trial solution is a linear function in  $n$ , for example,  $p_n = cn + d$ , where  $c$  and  $d$  are constants.
- Thus, the equation  $a_n = 3a_{n-1} + 2n$  becomes  $cn + d = 3(c(n-1) + d) + 2n$ .
- Simplify and reorder:  $(2 + 2c)n + (2d - 3c) = 0$ .

$$\begin{cases} 2 + 2c = 0 \\ 2d - 3c = 0 \end{cases} \implies \begin{cases} c = -1 \\ d = -3/2 \end{cases}$$

- Thus,  $a_n^{(p)} = -n - 3/2$  is a *particular* solution.

## Solving Non-Homogeneous Recurrence Relations [2]

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- By Theorem 24, all solutions are of the form

$$a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n,$$

where  $\alpha$  is a constant.

- To find the solution with  $a_1 = 3$ , let  $n = 1$  in the formula:  $3 = -1 - 3/2 + 3\alpha$ , thus  $\alpha = 11/6$ .
- The *solution* is  $a_n = -n - 3/2 + (11/6)3^n$ .

# Annihilators

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# Operators

**Definition 27:** *Operators* are higher-order functions that transform functions into other functions.

For example, differential and integral operators  $\frac{d}{dx}$  and  $\int dx$  are core operators in calculus.

In combinatorics, we are interested in the following three operators:

- *Sum*:  $(f + g)(n) := f(n) + g(n)$
- *Scale*:  $(\alpha \cdot f)(n) := \alpha \cdot f(n)$
- *Shift*:  $(\mathbf{E} f)(n) := f(n + 1)$

*Examples:*

- Scale and Shift operators are *linear*:  $\mathbf{E}(f - 3(g - h)) = \mathbf{E} f + (-3) \mathbf{E} g + 3 \mathbf{E} h$
- Operators are *composable*:  $(\mathbf{E} - 2)f := \mathbf{E} f + (-2)f$
- $\mathbf{E}^2 f = \mathbf{E}(\mathbf{E} f)$
- $\mathbf{E}^k f(n) = f(n + k)$
- $(\mathbf{E} - 2)^2 = (\mathbf{E} - 2)(\mathbf{E} - 2)$
- $(\mathbf{E} - 1)(\mathbf{E} - 2) = \mathbf{E}^2 - 3 \mathbf{E} + 2$



## Applying Operators

*Examples:* Below are the results of applying different operators to  $f(n) = 2^n$ :

$$2f(n) = 2 \cdot 2^n = 2^{n+1}$$

$$3f(n) = 3 \cdot 2^n$$

$$\mathbf{E} f(n) = 2^{n+1}$$

$$\mathbf{E}^2 f(n) = 2^{n+2}$$

$$(\mathbf{E} - 2)f(n) = \mathbf{E} f(n) - 2f(n) = 2^{n+1} - 2^{n+1} = 0$$

$$(\mathbf{E}^2 - 1)f(n) = \mathbf{E}^2 f(n) - f(n) = 2^{n+2} - 2^n = 3 \cdot 2^n$$

## Compound Operators

The compound operators can be seen as polynomials in “variable”  $\mathbf{E}$ .

*Example:* The compound operators  $\mathbf{E}^2 - 3\mathbf{E} + 2$  and  $(\mathbf{E} - 1)(\mathbf{E} - 2)$  are equivalent:

$$\text{Let } g(n) := (\mathbf{E} - 2)f(n) = f(n + 1) - 2f(n)$$

$$\begin{aligned}\text{Then } (\mathbf{E} - 1)(\mathbf{E} - 2)f(n) &= (\mathbf{E} - 1)g(n) \\ &= g(n + 1) - g(n) \\ &= [f(n + 2) - 2f(n + 1)] - [f(n + 1) - 2f(n)] \\ &= f(n + 2) - 3f(n + 1) + 2f(n) \\ &= (\mathbf{E}^2 - 3\mathbf{E} + 2)f(n) \quad \checkmark\end{aligned}$$

## Operators Summary

Operator	Definition
addition	$(f + g)(n) := f(n) + g(n)$
subtraction	$(f - g)(n) := f(n) - g(n)$
multiplication	$(\alpha \cdot f)(n) := \alpha \cdot f(n)$
shift	$\mathbf{E} f(n) := f(n + 1)$
k-fold shift	$\mathbf{E}^k f(n) := f(n + k)$
composition	$(\mathbf{X} + \mathbf{Y})f := \mathbf{X} f + \mathbf{Y} f$
	$(\mathbf{X} - \mathbf{Y})f := \mathbf{X} f - \mathbf{Y} f$
	$\mathbf{X} \mathbf{Y} f := \mathbf{X}(\mathbf{Y} f) = \mathbf{Y}(\mathbf{X} f)$
distribution	$\mathbf{X}(f + g) = \mathbf{X} f + \mathbf{X} g$

## Annihilators

**Definition 28:** An *annihilator* of a function  $f$  is any non-trivial operator that transforms  $f$  into zero.

TODO: examples!

## Annihilators Summary

Operator	Functions annihilated
$\mathbf{E} - 1$	$\alpha$
$\mathbf{E} - a$	$\alpha a^n$
$(\mathbf{E} - a)(\mathbf{E} - b)$	$\alpha a^n + \beta b^n$ [if $a \neq b$ ]
$(\mathbf{E} - a_0)(\mathbf{E} - a_1) \dots (\mathbf{E} - a_k)$	$\sum_{i=0}^k \alpha_i a_i^n$ [if $a_i$ are distinct]
$(\mathbf{E} - 1)^2$	$\alpha n + \beta$
$(\mathbf{E} - a)^2$	$(\alpha n + \beta) a^n$
$(\mathbf{E} - a)^2(\mathbf{E} - b)$	$(\alpha n + \beta) a^n + \gamma b^n$ [if $a \neq b$ ]
$(\mathbf{E} - a)^d$	$(\sum_{i=0}^{d-1} \alpha_i n^i) a^n$

## Properties of Annihilators

**Theorem 25:** If  $X$  annihilates  $f$ , then  $X$  also annihilates  $\alpha f$  for any constant  $\alpha$ .

**Theorem 26:** If  $X$  annihilates both  $f$  and  $g$ , then  $X$  also annihilates  $f \pm g$ .

**Theorem 27:** If  $X$  annihilates  $f$ , then  $X$  also annihilates  $\mathbf{E} f$ .

**Theorem 28:** If  $X$  annihilates  $f$  and  $Y$  annihilates  $g$ , then  $XY$  annihilates  $f \pm g$ .

## Annihilating Recurrences

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1. Write the recurrence in the *operator form*.
2. Find the *annihilator* for the recurrence.
3. *Factor* the annihilator, if necessary.
4. Find the *generic solution* from the annihilator.
5. Solve for coefficients using the *initial conditions*.

*Example:*  $r(n) = 5r(n-1)$  with  $r(0) = 3$ .

1.  $r(n+1) - 5r(n) = 0$   
 $(\mathbf{E} - 5)r(n) = 0$
2.  $(\mathbf{E} - 5)$  annihilates  $r(n)$ .
3.  $(\mathbf{E} - 5)$  is already factored.
4.  $r(n) = \alpha 5^n$  is a generic solution.
5.  $r(0) = \alpha = 3 \implies \alpha = 3$

Thus,  $r(n) = 3 \cdot 5^n$ .

## Annihilating Recurrences [2]

*Example:*  $T(n) = 2T(n-1) + 1$  with  $T(0) = 0$

1.  $T(n+1) - 2T(n) = 1$   
 $(\mathbf{E}-2)T(n) = 1$
2.  $(\mathbf{E}-2)$  does *not* annihilate  $T(n)$ : the residue is 1.  
 $(\mathbf{E}-1)$  annihilates the residue 1.  
Thus,  $(\mathbf{E}-1)(\mathbf{E}-2)$  annihilates  $T(n)$ .
3.  $(\mathbf{E}-1)(\mathbf{E}-2)$  is already factored.
4.  $T(n) = \alpha 2^n + \beta$  is a generic solution.
5. Find the coefficients  $\alpha, \beta$  using  $T(0) = 0$  and  $T(1) = 2T(0) + 1 = 1$ :

$$\begin{cases} T(0) = 0 = \alpha \cdot 2^0 + \beta \\ T(1) = 1 = \alpha \cdot 2^1 + \beta \end{cases} \implies \begin{cases} \alpha = 1 \\ \beta = -1 \end{cases}$$

Thus,  $T(n) = 2^n - 1$ .



## Annihilating Recurrences [3]

*Example:*  $T(n) = T(n-1) + 2T(n-2) + 2^n - n^2$

1. Operator form:

$$(\mathbf{E}^2 - \mathbf{E} - 2)T(n) = \mathbf{E}^2(2^n - n^2)$$

2. Annihilator:

$$(\mathbf{E}^2 - \mathbf{E} - 2)(\mathbf{E} - 2)(\mathbf{E} - 1)^3$$

3. Factorization:

$$(\mathbf{E} + 1)(\mathbf{E} - 2)^2(\mathbf{E} - 1)^3$$

4. Generic solution:

$$T(n) = \alpha(-1)^n + (\beta n + \gamma)2^n + \delta n^2 + \varepsilon n + \zeta$$

5. There are no initial conditions. We can only provide an asymptotic bound.

Thus,  $T(n) \in \Theta(n2^n)$

# ***Asymptotic Analysis***

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## Asymptotics 101

**Definition 29** (*Big-O notation*): The notation  $f \in O(g)$  means that the function  $f(n)$  is *asymptotically bounded from above* by the function  $g(n)$ , up to a constant factor.

$$f(n) \in O(g(n)) \iff \exists c > 0. \exists n_0. \forall n > n_0 : |f(n)| \leq c \cdot g(n)$$

**Definition 30** (*Small-o notation*): The notation  $f \in o(g)$  means that the function  $f(n)$  is *asymptotically dominated* by  $g(n)$ , up to a constant factor.

$$f(n) \in o(g(n)) \iff \forall c > 0. \exists n_0. \forall n > n_0 : |f(n)| \leq c \cdot g(n)$$

**Note:** The difference is only in the  $\exists c$  and  $\forall c$  quantifier.

**Note:** Flip  $\leq$  to  $\geq$  in the above definitions to obtain the dual notations:  $f \in \Omega(g)$  and  $f \in \omega(g)$ .

**Definition 31** (*Theta notation*):  $f \in \Theta(g)$  iff  $f \in O(g)$  and  $g \in O(f)$ .

## Limits

Notation	Name	Description	Limit definition
$f \in o(g)$	Small Oh	$f$ is dominated by $g$	$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 0$
$f \in O(g)$	Big Oh	$f$ is bounded above by $g$	$\limsup_{n \rightarrow \infty} \frac{ f(n) }{g(n)} < \infty$
$f \sim g$	Equivalence	$f$ is asymptotically equal to $g$	$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$
$f \in \Omega(g)$	Big Omega	$f$ is bounded below by $g$	$\liminf_{n \rightarrow \infty} \frac{f(n)}{g(n)} > 0$
$f \in \omega(g)$	Small Omega	$f$ dominates $g$	$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = \infty$

## Asymptotic Equivalence

**Definition 32:** The notation  $f \sim g$  means that functions  $f(n)$  and  $g(n)$  are *asymptotically equivalent*.

$$f \sim g \iff \forall \varepsilon > 0. \exists n_0. \forall n > n_0 : \left| \frac{f(n)}{g(n)} - 1 \right| \leq \varepsilon \iff \lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$$

**Note:**  $f \sim g$  and  $g \sim f$  are equivalent, since  $\sim$  is an equivalence relation.

**Note:**  $f \sim g$  and  $f \in \Theta(g)$  are *different* notions!

## Some Properties of Asymptotics

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$$f \in O(g) \text{ and } f \in \Omega(g) \iff f \in \Theta(g)$$

$$f \in O(g) \iff g \in \Omega(f)$$

$$f \in o(g) \iff g \in \omega(f)$$

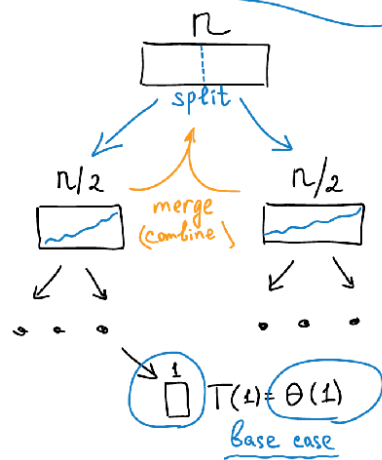
$$f \in o(g) \rightarrow f \in O(g)$$

$$f \in \omega(g) \rightarrow f \in \Omega(g)$$

$$f \sim g \rightarrow f \in \Theta(g)$$

## Divide-and-Conquer Algorithms Analysis

Divide-and-Conquer



$$T(n) = \underbrace{2 T(n/2)}_{\text{recursive work}} + \underbrace{\Theta(n)}_{\text{split/merge work}}$$

$$\left\{ \begin{aligned} \Sigma &= 2 \cdot 2 \cdot \underbrace{T(n/4)}_{\text{merge}} = n \end{aligned} \right.$$

$$\left\{ \begin{aligned} \Sigma &\approx 2^{i+1} T(n/2^{i+1}) = n \end{aligned} \right.$$

$$\left\{ \begin{aligned} \Sigma &= n \cdot \Theta(1) = n \end{aligned} \right.$$

$$\underbrace{\Theta(n)}_{\text{work on each level of recursion tree}} \cdot \underbrace{\Theta(\log n)}_{\text{height of recursion tree}} = \Theta(n \log n)$$

## Divide-and-Conquer Recurrence

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

- $T(n)$  is the *cost* of the recursive algorithm
- $a$  is the number of *parts* (*sub-problems*)
- $n / b$  is the *size* of each part
- $T\left(\frac{n}{b}\right)$  is the cost of each *sub-problem*
- $f(n)$  is the cost of *splitting* and *merging* the solutions of the subproblems

Hereinafter,  $c_{\text{crit}} = \log_b a$  is a *critical constant*.



## Master Theorem

The master theorem [3] applies to divide-and-conquer recurrences of the form

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

Case	Description	Condition	Bound
Case I	“merge” $\ll$ “recursion”	$f(n) \in O(n^c)$ where $c < c_{\text{crit}}$	$T(n) \in \Theta(n^{c_{\text{crit}}})$
Case II	“merge” $\approx$ “recursion”	$f(n) \in \Theta(n^{c_{\text{crit}}} \log^k n)$ where $k \geq 0$	$T(n) \in \Theta(n^{c_{\text{crit}}} \log^{k+1} n)$
Case III	“merge” $\gg$ “recursion”	$f(n) \in \Omega(n^c)$ where $c > c_{\text{crit}}$	$T(n) \in \Theta(f(n))$

**Note:** Case III also requires the *regularity condition* to hold:  $af(n/b) \leq kf(n)$  for some constant  $k < 1$  and all sufficiently large  $n$ .

**Note:** There is an *extended* Case II, with three sub-cases (IIa, IIb, IIc) for other values of  $k$ . See [wiki](#).

## Examples of Master Theorem Application

*Examples:* Determine the case of Master Theorem and the bound of  $T(n)$  for the following recurrences.

1.  $T(n) = 3T(n / 9) + \sqrt{n}$

2.  $T(n) = 2T(n / 4) + n^{0.51}$

3.  $T(n) = 5T(n / 25) + n^{0.49}$

4.  $T(n) = T(\lfloor \frac{n}{2} \rfloor) + T(\lceil \frac{n}{2} \rceil)$

5.  $T(n) = 3T(n / 9) + \frac{\sqrt{n}}{\log n}$

6.  $T(n) = 6T(n / 36) + \frac{\sqrt{n}}{\log^2 n}$

7.  $T(n) = 4T(n / 16) + \sqrt{\frac{n}{\log n}}$

## Akra–Bazzi Method

The Akra–Bazzi method [4] is a *generalization* of the master theorem to recurrences of the form

$$T(n) = f(n) + \sum_{i=1}^k a_i T\left(b_i n + \underbrace{h_i(n)}_{*}\right)$$

- $k$  is a constant
- $a_i > 0$
- $0 < b_i < 1$
- $h_i(n) \in O\left(\frac{n}{\log^2 n}\right)$  is a *small perturbation*

Bound of  $T(n)$  by Akra–Bazzi method:

$$T(n) \in \Theta\left(n^p \cdot \left(1 + \int_1^n \frac{f(x)}{x^{p+1}} dx\right)\right)$$

where  $p$  is the solution for the equation  $\sum_{i=1}^k a_i b_i^p = 1$

## Example of Akra–Bazzi Method Application

*Example:* Suppose the runtime of an algorithm is expressed by the following recurrence relation:

$$T(n) = \begin{cases} 1 & \text{for } 0 \leq n \leq 3 \\ n^2 + \frac{7}{4}T(\lfloor \frac{1}{2}n \rfloor) + T(\lceil \frac{3}{4}n \rceil) & \text{for } n > 3 \end{cases}$$

- Note that the Master Theorem *is not* applicable here, since there are *two* different recursive terms.
- Let's apply the Akra–Bazzi method. First, solve the equation  $\frac{7}{4}(\frac{1}{2})^p + (\frac{3}{4})^p = 1$ . This gives us  $p = 2$ .
- Next, use the formula from AB-method to obtain the bound:

$$\begin{aligned} T(x) &\in \Theta\left(x^p \left(1 + \int_1^x \frac{f(u)}{u^{p+1}} du\right)\right) = \\ &= \Theta\left(x^2 \left(1 + \int_1^x \frac{u^2}{u^3} du\right)\right) = \\ &= \Theta(x^2(1 + \ln x)) = \\ &= \Theta(x^2 \log x) \end{aligned}$$

# Advanced Topics

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## Gamma Function

**Definition 33:** *Gamma function*  $\Gamma(x)$  is the most common *extension* of the factorial function to real and complex numbers. It is defined for all complex numbers  $z \in \mathbb{C}$  (except non-positive integers) as

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

For positive integers  $z = n$ , it is defined as

$$\Gamma(n) = (n-1)!$$

**Motivation:** The factorial is defined for positive integers as  $n! = 1 \cdot 2 \cdot \dots \cdot n = (n-1)! \cdot n$ .

We want to *extend* this definition to *all real numbers* and capture its *recursive* nature.

Overall, we are looking for a *smooth* function  $\Gamma(x)$  such that:

- $\Gamma(n+1) = n!$  for all  $n \in \mathbb{N}$ , matching the factorial.
- $\Gamma(x+1) = x \cdot \Gamma(x)$ , satisfying a *recursive* property.
- $\Gamma(x)$  is defined for all *real* numbers  $x > 0$ .

## Definition of a Gamma Function

The main definition of a gamma function is known as *Euler integral of the second kind*:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

Gauss proposed a function  $\Gamma(x)$  defined by the *limit*

$$\Gamma(x) := \lim_{n \rightarrow \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)} = \lim_{n \rightarrow \infty} \frac{n! \cdot n^x}{\prod_{k=0}^n (x+k)} \quad \text{for } x > 0$$

## Integral Definition

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$$

Let's check that the integral definition is indeed a suitable definition of a gamma function.

$$\begin{aligned}\Gamma(z+1) &= \int_0^{\infty} t^z e^{-t} dt \\ &= [-t^z e^{-t}]_0^{\infty} + \int_0^{\infty} z t^{z-1} e^{-t} dt \\ &= \lim_{t \rightarrow \infty} (-t^z e^{-t}) - (-0^z e^{-0}) + z \int_0^{\infty} t^{z-1} e^{-t} dt\end{aligned}$$

Note that  $-t^z e^{-t} \rightarrow 0$  as  $t \rightarrow \infty$ , so:

$$\Gamma(z+1) = z \int_0^{\infty} t^{z-1} e^{-t} dt = z \cdot \Gamma(z)$$



## Limit Definition

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! \cdot n^x}{\prod_{k=0}^n (x+k)}$$

Let's check that the limit definition is indeed a suitable definition of a gamma function.

**Step 1.** Write  $\Gamma(x+1)$ .

$$\Gamma(x+1) = \lim_{n \rightarrow \infty} \frac{n! \cdot n^{x+1}}{\prod_{k=0}^n (x+1+k)} = \lim_{n \rightarrow \infty} \frac{n! \cdot n^{x+1}}{\prod_{k=1}^{n+1} (x+k)}$$

**Step 2.** Multiply both numerator and denominator by  $x$  and rearrange:

$$= \lim_{n \rightarrow \infty} \frac{n! \cdot n^{x+1}}{(x+1) \cdot \dots \cdot (x+n+1)} = \lim_{n \rightarrow \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)} \cdot \frac{n}{x+n+1} \cdot x$$

**Step 3.** Take the limit. As  $n \rightarrow \infty$ , the ratio  $\frac{n}{x+n+1}$  approaches 1.

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

## Equivalence of Definitions

Let's prove the equivalence of two definitions: integral and limit.

We claim:

$$\int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt \stackrel{?}{=} \lim_{n \rightarrow \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)}$$

Note that as  $n \rightarrow \infty$ , the integrand  $\left(1 - \frac{t}{n}\right)^n$  approaches  $e^{-t}$ , so this integral approximates  $\Gamma(x)$ .

Substitute  $u = \frac{t}{n}$ :

$$\int_0^n t^{x-1} \left(1 - \frac{t}{n}\right)^n dt = n^x \int_0^1 u^{x-1} (1-u)^n du = n^x \cdot B(x, n+1)$$

where  $B(x, n+1)$  is the Beta function:

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}$$

## Equivalence of Definitions [2]

Then:

$$I_n = n^x \cdot B(x, n+1) = n^x \cdot \frac{\Gamma(x) \cdot \Gamma(n+1)}{\Gamma(x+n+1)} = \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)}$$

Take the limit on both sides. Since  $\lim_{n \rightarrow \infty} I_n = \Gamma(x)$ , we have:

$$\Gamma(x) = \lim_{n \rightarrow \infty} \frac{n! \cdot n^x}{x \cdot (x+1) \cdot \dots \cdot (x+n)}$$

## Using the Gamma Function

$$n! = \Gamma(n + 1)$$

$$\binom{r}{k} = \frac{\Gamma(r + 1)}{\Gamma(k + 1) \cdot \Gamma(r - k + 1)}$$

$$\Gamma(n + 1) \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}$$

## Bibliography

- [1] G. Pólya, *Mathematics and Plausible Reasoning, Volume 1: Induction and Analogy in Mathematics*. 1954.
- [2] H. S. Wilf, *generatingfunctionology*, 3rd ed. 2006.
- [3] J. L. Bentley, D. Haken, and J. B. Saxe, “A General Method for Solving Divide-and-Conquer Recurrences,” *ACM SIGACT News*, vol. 12, no. 3, pp. 36–44, Sept. 1980, doi: [10.1145/1008861.1008865](https://doi.org/10.1145/1008861.1008865).
- [4] M. Akra and L. Bazzi, “On the Solution of Linear Recurrence Equations,” *Computational Optimization and Applications*, vol. 10, no. 2, pp. 195–210, May 1998, doi: [10.1023/A:1018373005182](https://doi.org/10.1023/A:1018373005182).