**𝓗omework #6** — **Automata Theory**

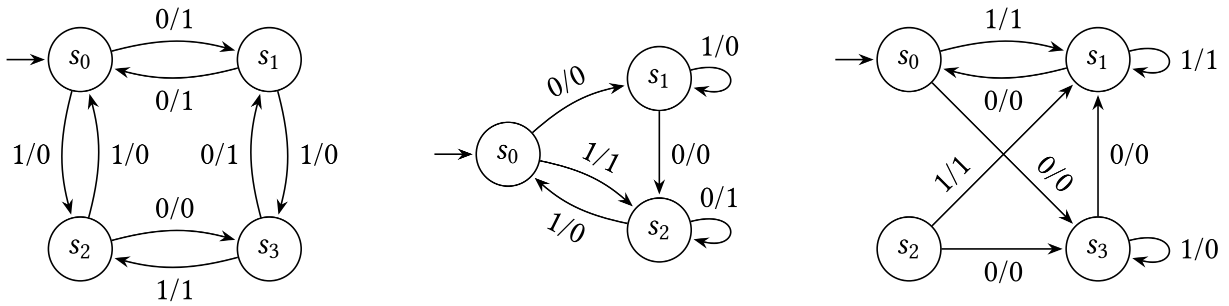
**𝓓iscrete 𝓜∀th** — **Spring 2025**

1. For each given regular expression $P$, construct a DFA (Deterministic Finite Automaton), and find the number of accepted word of length at most 5, i.e. the size of the set $\mathcal{L}' = \{w \in \mathcal{L}(P) \mid |w| \le 5\}$. For “any” (`.`) and “negative” (`[^.]`) matches, assume that the alphabet is $\Sigma = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$.
   (a) $P_1 =$ `ab*`
   (b) $P_2 =$ `a+b?c`
   (c) $P_3 =$ `[^cd]+c{3}`
   (d) $P_4 =$ `[^a](.|ddd)?`
   (e) $P_5 =$ `d(a|bc)*`
   (f) $P_6 =$ `((a|ab)[cd]){2}`

2. Describe the set of strings defined by each of these sets of productions in EBNF (extended Backus-Naur form).
   (a) $\langle string \rangle$ ::= $\langle L \rangle$+ $\langle D \rangle$? $\langle L \rangle$+
   $\langle L \rangle$ ::= `a` | `b` | `c`
   $\langle D \rangle$ ::= `0` | `1`
   (b) $\langle string \rangle$ ::= $\langle sign \rangle$? $\langle N \rangle$
   $\langle sign \rangle$ ::= ‘`+`’ | ‘`-`’
   $\langle N \rangle$ ::= $\langle D \rangle$ ($\langle D \rangle$ | `0`)*
   $\langle D \rangle$ ::= `1` | . . . | `9`
   (c) $\langle string \rangle$ ::= $\langle L \rangle$* ($\langle D \rangle$+)? $\langle L \rangle$*
   $\langle L \rangle$ ::= `x` | `y`
   $\langle D \rangle$ ::= `0` | `1`
   (d) $\langle string \rangle$ ::= $\langle C \rangle$ $\langle R \rangle$*
   $\langle C \rangle$ ::= `a` | . . . | `z` | `A` | . . . | `Z`
   $\langle D \rangle$ ::= `0` | . . . | `9`
   $\langle R \rangle$ ::= $\langle C \rangle$ | $\langle D \rangle$ | ‘`_`’

3. Let $\mathcal{G} = \langle V, T, S, P \rangle$ be the phrase-structure grammar with vocabulary $V = \{\mathtt{A}, \mathtt{S}\}$, terminal symbols $T = \{0, 1\}$, start symbol $S = \mathtt{S}$, and set of productions $P$: $\mathtt{S} \longrightarrow 1\mathtt{S}$, $\mathtt{S} \longrightarrow 00\mathtt{A}$, $\mathtt{A} \longrightarrow 0\mathtt{A}$, $\mathtt{A} \longrightarrow 0$.
   (a) Show that `111000` belongs to the language generated by $\mathcal{G}$.
   (b) Show that `11001` does not belong to the language generated by $\mathcal{G}$.
   (c) What is the language generated by $\mathcal{G}$?

4. Find the output generated from the input string `01110` for each of the following Mealy machines.



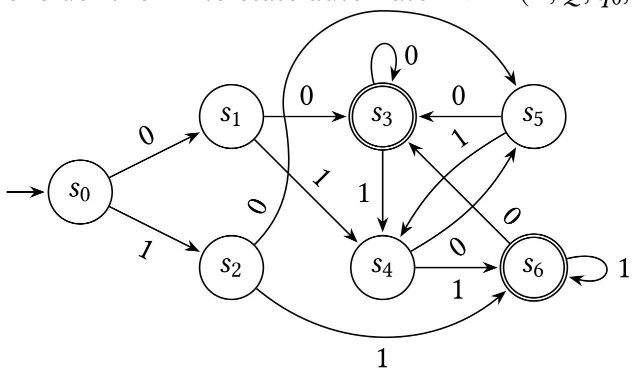

5. Construct a Moore machine for each of the following descriptions.
   (a) Determine the residue modulo 3 of the input treated as a binary number. For example, for input $\varepsilon$ (which corresponds to “value” 0) the residue is 0; `101` (5 in decimal) has residue 2; and `1010` (value 10) has residue 1.
   (b) Output the residue modulo 5 of the input from $\{0, 1, 2\}^*$ treated as a ternary (base 3) number.
   (c) Output $A$ if the binary input ends with `101`; output $B$ if it ends with `110`; otherwise output $C$.

6. Show that regular languages are *closed* under the following operations.
   (a) Union, that is, if $L_1$ and $L_2$ are regular languages, then $L_1 \cup L_2$ is also regular.
   (b) Concatenation, that is, if $L_1$ and $L_2$ are regular languages, then $L_1 \cdot L_2$ is also regular.
   (c) Kleene star, that is, if $L$ is a regular language, then $L^*$ is also regular.
   (d) Complement, that is, if $L$ is a regular language, then $\overline{L} = \Sigma^* - L$ is also regular.
   (e) Intersection, that is, if $L_1$ and $L_2$ are regular languages, then $L_1 \cap L_2$ is also regular.

7. Determine whether the following languages are regular or not. For non-regular languages, use Pumping lemma to prove that they are not regular. For each regular language, provide a regular expression and construct an $\varepsilon$-NFA.
   (a) $L_1 = \{w \in \{0,1\}^* \mid \text{length of } w \text{ is odd}\}$
   (b) $L_2 = \{0^n 1^n \mid n \in \mathbb{N}\}$
   (c) $L_3 = \{w \in \{0,1\}^* \mid w \text{ contains an even number of 1s}\}$
   (d) $L_4 = \{1^{n^2} \mid n \in \mathbb{N}\}$

8. Consider a finite-state automaton $M = (\Sigma, Q, q_0, F, \delta)$ and a non-negative integer $k$. Let $R_k$ be the relation on the set of states of $M$ such that $s\ R_k\ t$ if and only if for every input string $w \in \Sigma^*$ with $|w| \leq k$, $\delta(s,w)$ and $\delta(t,w)$ are both final states or both not final states. Furthermore, let $R^*$ be the relation on the set of states of $M$ such that $s\ R^*\ t$ if and only if for every input string $w \in \Sigma^*$, regardless of length, $\delta(s,w)$ and $\delta(t,w)$ are both final states or both not final states.
   (a) Show that for every nonnegative integer $k$, $R_k$ is an equivalence relation on $S$.
   Two states $s$ and $t$ are called $k$-equivalent if $s\ R_k\ t$.
   (b) Show that $R^*$ is an equivalence relation on $S$.
   Two states $s$ and $t$ are called $*$-equivalent if $s\ R^*\ t$.
   (c) Show that if two states $s$ and $t$ are $k$-equivalent ($k > 0$), then they are also $(k-1)$-equivalent.
   (d) Show that the equivalence classes of $R_k$ are a *refinement* of the equivalence classes of $R_{k-1}$.
   (e) Show that if two states $s$ and $t$ are $k$-equivalent for every non-negative integer $k$, then they are $*$-equivalent.
   (f) Show that all states in a given $R^*$-equivalence class are final or all are not final.
   (g) Show that if two states $s$ and $t$ are $*$-equivalent, then $\delta(s,a)$ and $\delta(t,a)$ are also $*$-equivalent for all $a \in \Sigma$.

9. Consider the finite-state automaton $M = (\Sigma, Q, q_0, F, \delta)$ depicted below.




   (a) Find the $k$-equivalence classes of $M$ for $k = 0, 1, 2, 3$.
   (b) Find the $*$-equivalence classes of $M$.
   (c) Construct the quotient automaton $\overline{M}$ of $M$.
   ▹ The quotient automaton $\overline{M}$ of the deterministic finite-state automaton $M = (\Sigma, S, s_0, F, \delta)$ is the finite state automaton $\overline{M} = (\Sigma, \overline{S}, [s_0]_{R^*}, \overline{F}, \overline{\delta})$, where the set of states $\overline{S}$ is the set of $R^*$-equivalence classes of $S$; the transition function $\overline{\delta}$ is defined by $\overline{\delta}([s]_{R^*}, a) = [\delta(s,a)]_{R^*}$ for all states $[s]_{R^*}$ of $\overline{M}$ and input symbols $a \in \Sigma$; and $\overline{F}$ is the set consiting of $R^*$-equivalence classes of final states of $M$.

10. Solve the following regex crosswords[1]. Fill each cell with a single ASCII character (an uppercase letter, a digit, a punctuation mark, or a space). Each row/column, when read left to right or top to bottom must match the regular expression(s) given for that row/column.

<table>
<tr><td></td><td>[^SPEAK]+</td><td>EP|IP|EF</td></tr>
<tr><td>HE|LL|O+</td><td></td><td></td></tr>
<tr><td>[PLEASE]+</td><td></td><td></td></tr>
</table>

<table>
<tr><td></td><td>(FI|A)+</td><td>(YE|OT)K</td><td>(.)[IF]+</td><td>[NODE]+</td><td>(FY|F|RG)+</td></tr>
<tr><td>(Y|F)(.)\2[DAF]\1</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>(U|O|I)*T[FRO]+</td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>[KANE]*[GIN]*</td><td></td><td></td><td></td><td></td><td></td></tr>
</table>

<table>
<tr><td></td><td>[^NRU](NO|ON)</td><td>(D|FU|UF)+</td><td>(FO|A|R)*</td><td>(N|A)*</td></tr>
<tr><td>[RUNT]*</td><td></td><td></td><td></td><td></td></tr>
<tr><td>O.*[HAT]</td><td></td><td></td><td></td><td></td></tr>
<tr><td>(.)*DO\1</td><td></td><td></td><td></td><td></td></tr>
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<table>
<tr><td></td><td>[^MCI]+</td><td>.A</td><td>(TM|BF)</td><td></td></tr>
<tr><td>(CAT|A-T)+</td><td></td><td></td><td></td><td>[^KI\sP]+</td></tr>
<tr><td>[MA\-\sE]+</td><td></td><td></td><td></td><td>(M|APS|EA)*</td></tr>
<tr><td></td><td>[AI][E\s]</td><td>[A\-Z]+</td><td>[\sT\-M]+</td><td></td></tr>
</table>

<table>
<tr><td></td><td>(.)\1(.)\2</td><td>[C\sOU]+</td><td>[^PU\sH]+</td><td></td></tr>
<tr><td>.[LUH]+</td><td></td><td></td><td></td><td>.*L+</td></tr>
<tr><td>(P|K)[^U]+</td><td></td><td></td><td></td><td>[PUF\s]*</td></tr>
<tr><td>.*C+[TIF]</td><td></td><td></td><td></td><td>[TIC]*</td></tr>
<tr><td>(NO|ONE|ION)*</td><td></td><td></td><td></td><td>[NOI\sE]+</td></tr>
<tr><td></td><td>[PIF]+</td><td>.*[OWE]*</td><td>(TN|LF|TF)*</td><td></td></tr>
</table>

<table>
<tr><td></td><td>.*E.*</td><td>[^P]I(IT|ME)</td><td>(EM|FE)(IT|IP)</td><td>(TS|PE|KE)*</td></tr>
<tr><td>(EP|ST)*</td><td></td><td></td><td></td><td></td></tr>
<tr><td>T[A-Z]*</td><td></td><td></td><td></td><td></td></tr>
<tr><td>.M.T</td><td></td><td></td><td></td><td></td></tr>
<tr><td>.*P.[S-X]+</td><td></td><td></td><td></td><td></td></tr>
</table>

[1] Credits: https://regexcrossword.com