# Formal Methods in Software Engineering

**Boolean Satisfiability** — Spring 2025

Konstantin Chukharev

# §1 Boolean Satisfiability

# Boolean Satisfiability Problem (SAT)

SAT is the classical NP-complete problem of determining whether a given Boolean formula is *satisfiable*, that is, whether there exists an assignment of truth values to its variables that makes the formula true.

$$\exists X . f(X) = 1$$

SAT is a *decision* problem, which means that the answer is either "yes" or "no". However, in practice, we are mainly interested in *finding* the actual satisfying assignment if it exists — this is a *functional* SAT problem.

Historically, SAT was the first problem proven to be NP-complete, independently by Stephen Cook [1] and Leonid Levin [2] in 1971.

Stephen Cook

Leonid Levin

# Cook–Levin Theorem

**Theorem 1** (Cook–Levin): SAT is NP-complete.

That is, SAT is in NP, and *any* problem in NP can be *reduced* to SAT in polynomial time.

The proof is due to Richard Karp [3], who introduced the concept of *polynomial-time many-one reductions*, also known as *Karp reductions*. The earlier proof by Cook was based on a weaker type of reduction called *Turing reduction* or *Cook reduction*.

**Definition 1** (Karp's many-one reduction): A polynomial-time *many-one reduction* from a problem $A$ to a problem $B$ is a polynomial-time computable function $f$ such that for every instance $x$ of $A$, $x$ is a "yes" instance of $A$ if and only if $f(x)$ is a "yes" instance of $B$. A reduction of this kind is denoted as $A \leq_p B$ and called a *polynomial transformation* or *Karp reduction*.

# Cook–Levin Theorem [2]

*Proof sketch*: A problem $L$ is in NP if there exists a polynomial-time verifier (Turing machine) $V(x, c)$ that verifies whether a certificate $c$ is a valid proof that $x \in L$.

A *Karp reduction* from $L$ to SAT is a polynomial-time computable function $f$ mapping instances $x$ of $L$ to propositional formulas $\varphi_x$, such that $x \in L$ iff $\varphi_x$ is satisfiable.

For input $x$, simulate $V(x, c)$ computation (with certificate $c$) as a *Turing machine* run. Encode its execution over $T = \mathcal{O}(p(|x|))$ steps into a propositional formula $\varphi_x$:

- Variables represent the machine's state, tape cells, and head position at each step $t$.
- Clauses enforce the initial configuration (input $x$ and empty certificate $c$), valid transitions between steps (per $V$'s rules), and the acceptance at step $T$.

A satisfying assignment to $\varphi_x$ corresponds to a valid certificate $c$ causing $V(x, c)$ to accept.

The encoding $x \mapsto \varphi_x$ is computable in polynomial time. Since $L \in \mathrm{NP}$ was arbitrary, *all NP problems can be reduced to SAT*, proving SAT is **NP-hard**. As SAT is also in **NP**, it is **NP-complete**. $\square$

<u>This foundational result shows that SAT is a "universal" problem for NP.</u>

# Solving General Search Problems with SAT

Modelling and solving general search problems:

1. Define a finite set of possible *states*.
2. Describe states using propositional *variables*.
3. Describe *legal* and *illegal* states using propositional *formulas*.
4. Construct a propositional *formula* describing the desired state.
5. Translate the formula into an *equisatisfiable* CNF formula.
6. If the formula is *satisfiable*, the satisfying assignment corresponds to the desired state.
7. If the formula is *unsatisfiable*, the desired state does not exist.

# Example: Graph Coloring

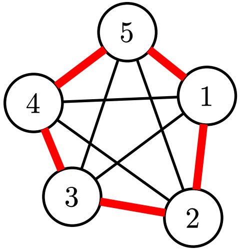
Recall that a graph $G = (V, E)$ consists of a set $V$ of vertices and a set $E$ of edges, where each edge is an unordered pair of vertices.

A complete graph on $n$ vertices, denoted $K_n$, is a graph with $|V| = n$ such that $E$ contains all possible pairs of vertices. In total, $K_n$ has $\frac{n(n-1)}{2}$ edges.

Given a graph, color its vertices such that no two adjacent vertices have the same color.

Given a complete graph $K_n$, color its edges using $k$ colors without creating a monochromatic triangle. What is the largest complete graph for which this is possible for a given number of colors?

- For $k = 1$, the answer is $n = 2$.
  - The graph $K_2$ has only one edge, which can be colored with a single color.
- For $k = 2$, the answer is $n = 5$.
  - See the example of 2-colored $K_5$ on the right.
- For $k = 3$, the answer is $n = 16$.
  - This is the work for a SAT solver. See the next slides.



# Modelling and Solving the Graph Coloring Example

1. *Define a finite set of possible states.*
   - Each possible edge coloring is a state. There are $3^{|E|}$ possible states.
2. *Describe states using propositional variables.*
   - A simple (*one-hot*, or *direct*) encoding uses three variables for each edge: $e_1$, $e_2$, and $e_3$. There are 8 possible combinations of values of three variables, which given a state space of $8^{|E|}$. This is larger than necessary, but keeps the encoding simple.
3. *Describe legal and illegal states using propositional formulas.*
   - For each edge $e \in E$, the formula $e_1 + e_2 + e_3 = 1$ (so called "cardinality constraint") ensures that each edge is colored with exactly one color. This reduces the state space to $3^{|E|}$.
4. *Construct a propositional formula describing the desired state.*
   - The desired state is one in which there are no monochromatic triangles. For each triangle $(e, f, g)$, we explicitly forbid it from being colored with the same color:

$$\neg((e_1 \leftrightarrow f_1) \wedge (f_1 \leftrightarrow g_1) \wedge (e_2 \leftrightarrow f_2) \wedge (f_2 \leftrightarrow g_2) \wedge (e_3 \leftrightarrow f_3) \wedge (f_3 \leftrightarrow g_3))$$

# Modelling and Solving the Graph Coloring Example [2]

5. *Translate the formula into an equisatisfiable CNF formula.*
    - This can be done using the Tseitin transformations.

6. *If the formula is satisfiable, the satisfying assignment corresponds to the desired state.*
    - The satisfying assignment corresponds to a valid edge coloring. Among variables $e_1$, $e_2$, and $e_3$, the single one with the value of 1 corresponds to the color of the edge.

7. *If the formula is unsatisfiable, the desired state does not exist.*
    - If the formula is unsatisfiable, there is no valid edge coloring.

Now, run a SAT solver for increasing values of $n$, and find the largest $n$ for which the formula is satisfiable. The answer is $n = 16$ for $k = 3$.

# TODO

- [ ] Encodings
- [ ] SAT Solvers
- [ ] Applications
- [ ] Exercises

# Bibliography

[1] S. A. Cook, “The complexity of theorem-proving procedures,” in *Proceedings of the Third Annual ACM Symposium on Theory of Computing*, 1971, pp. 151–158. doi: 10.1145/800157.805047.

[2] L. A. Levin, “Universal sequential search problems,” *Problemy Peredachi Informatsii*, vol. 9, no. 3, pp. 115–116, 1973, [Online]. Available: http://mi.mathnet.ru/ppi914

[3] R. M. Karp, “Reducibility among Combinatorial Problems,” *Complexity of Computer Computations.* Springer, pp. 85–103, 1972. doi: 10.1007/978-1-4684-2001-2_9.