

Formal Methods in Software Engineering

Theory of Computation — Spring 2025

Konstantin Chukharev

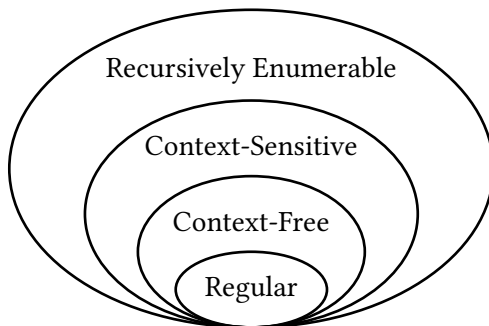
§1 Languages

Formal Languages

Definition 1 (Formal language): A set of strings over an alphabet Σ , closed under concatenation.

Formal languages are classified by *Chomsky hierarchy*:

- Type 0: Recursively Enumerable
- Type 1: Context-Sensitive
- Type 2: Context-Free
- Type 3: Regular



Examples:

- $L = \{a^n \mid n \geq 0\}$
- $L = \{a^n b^n \mid n \geq 0\}$
- $L = \{a^n b^n c^n \mid n \geq 0\}$
- $L = \{\langle M, w \rangle \mid M \text{ is a TM that halts on input } w\}$



Noam Chomsky

Decision Problems as Languages

Definition 2 (Decision problem): A *decision problem* is a question with a “yes” or “no” answer.

Formally, set of inputs for which the problem has an answer “yes” corresponds to a subset $L \subseteq \Sigma^*$, where Σ is an alphabet.

Example: SAT Problem as a language:

$$\text{SAT} = \{\varphi \mid \varphi \text{ is a satisfiable Boolean formula}\}$$

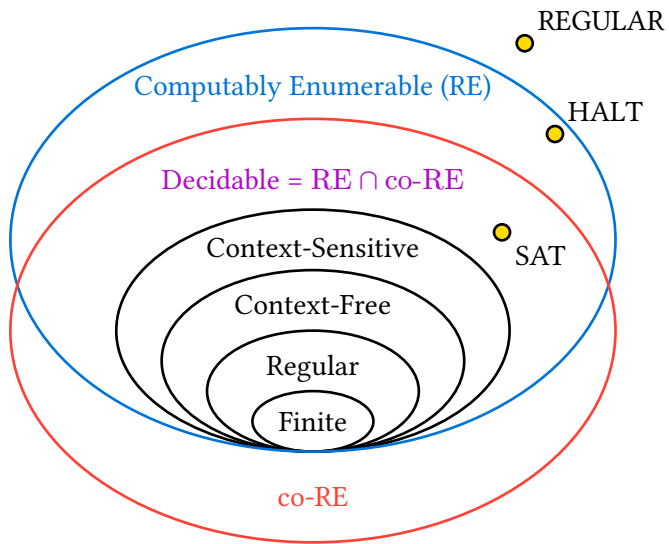
Example: Validity Problem as a language:

$$\text{VALID} = \{\varphi \mid \varphi \text{ is a valid logical formula (tautology)}\}$$

Example: Halting Problem as a language:

$$\text{HALT} = \{\langle M, w \rangle \mid \text{Turing machine } M \text{ halts on input } w\}$$

Language Classes



§2 Machines

Finite Automata

Definition 3: Deterministic Finite Automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where:

- Q is a *finite* set of states,
- Σ is an *alphabet* (finite set of input symbols),
- $\delta : Q \times \Sigma \longrightarrow Q$ is a *transition function*,
- $q_0 \in Q$ is the *start* state,
- $F \subseteq Q$ is a set of *accepting* states.

DFA recognizes *regular* languages (Type 3).

Example: Automaton \mathcal{A} recognizing strings with an even number of 0s, $\mathcal{L}(\mathcal{A}) = \{0^n \mid n \text{ is even}\}$.

	0	1
q_0	q_1	q_1
q_1	q_0	q_1



Turing Machines

Informally, a Turing machine is a *finite-state* machine with an *infinite tape* and a *head* that can read and write symbols. Initially, the tape contains the *input* string, the rest are blanks, and the machine is in the *start* state. At each step, the machine reads the symbol under the head, changes the state, writes a new symbol, and moves the head left or right. When the machine reaches the *accept* or *reject* state, it immediately halts.

Note: If the machine never reaches the *accept* or *reject* state, it *loops* forever.



TM Formal Definition

Definition 4: Turing Machine (TM) is a 7-tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$ where:

- Γ is a *tape alphabet* (including blank symbol $\square \in \Gamma$),
- $\Sigma \subseteq \Gamma$ is a *input alphabet*,
- $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\}$ is a transition function,
- q_{acc} and q_{rej} are the *accept* and *reject* states.

TM recognizes *recursively enumerable* languages (Type 0).

TM Language

Definition 5: The language *recognized* by a TM M , denoted $\mathcal{L}(M)$, is the set of strings $w \in \Sigma^*$ that M accepts, that is, for which M halts in the *accept* state.

- For any $w \in \mathcal{L}(M)$, M accepts w .
- For any $w \notin \mathcal{L}(M)$, M does not accept w , that is, M either *rejects* w or *loops forever* on w .

Definition 6: A TM is a *decider* if it halts on all inputs.

TM Configuration

Definition 7: A *configuration* of a TM is a *string* $(u; q; v)$ where $u, v \in \Gamma^*$, $q \in Q$, meaning:

- Tape contents: uv followed by the blanks.
- Current state is q .
- Head position: at the first symbol of v .

For example, configuration $(u; q; av)$, where $a \in \Gamma$, is represented as follows:



TM Computation

Definition 8 (Computation): The process of *computation* by a TM on input $w \in \Sigma^*$ is a *sequence* of configurations C_1, C_2, \dots, C_n .

- $C_1 = (\triangleright; q_0; w)$ is the *start* configuration with input $w \in \Sigma^*$.
- $C_i \Rightarrow C_{i+1}$ for each i .
- C_n is a *final* configuration.

Configuration C_1 *yields* C_2 , denoted $C_1 \Rightarrow C_2$, if TM can move from C_1 to C_2 in *one* step.

- See the formal definition on the next slide.

The relation \Rightarrow^* is the *reflexive* and *transitive* closure of \Rightarrow .

- $C_1 \Rightarrow^* C_2$ denotes “yields in *some* number of steps”.

TM Yields Relation

Definition 9 (Yields): Let $u, v \in \Gamma^*$, $a, b, c \in \Gamma$, $q_i, q_j \in Q$.

- Move left: $(ua; q_i; bv) \Rightarrow (u; q_j; acv)$ if $\delta(q_i, b) = (q_j, c, L)$ (overwrite b with c , move left)
- Move right: $(u; q_i; bav) \Rightarrow (uc; q_j; av)$ if $\delta(q_i, b) = (q_j, c, R)$ (overwrite b with c , move right)



Special case for the left end:

- $(\triangleright; q_i; bv) \Rightarrow (\triangleright; q_j; cv)$ if $\delta(q_i, b) = (q_j, c, L)$ (overwrite b with c , do not move).

Recognizing vs Deciding

There are *two* types of Turing machines:

1. Total TM: always halts. Also called *decider*.
2. General TM: may loop forever. Also called *recognizer*.

Definition 10 (Recognition): A TM *recognizes* a language L , if it halts and accepts all words $w \in L$, but no others. A language recognized by a TM is called *semi-decidable* or *recursively enumerable* or *recursively computable* or *Turing-recognizable*. The set of all recognizable languages is denoted by **RE**.

Definition 11 (Decision): A TM *decides* a language L , if it halts and accepts all words $w \in L$, and halts and rejects any other word $w \notin L$. A language decided by a TM is called *decidable* or *recursive* or *computable*. The set of all decidable languages is denoted by **R**.

MIU. MU?

Definition 12 (MIU system): The *MIU system* is a “formal system” consisting of:

- an alphabet $\Sigma = \{M, I, U\}$,
- a single axiom: MI,
- a set of inference rules:

Rule	Description	Example
$xI \vdash xIU$	add U to the end of any string ending with I	MI to MIU
$Mx \vdash Mxx$	double the string after M	MIU to MIUIU
$xIIIy \vdash xUy$	replace any III with U	MUIIIU to MUUU
$xUUy \vdash xy$	remove any UU	MUUU to MU

Question: Is MU a theorem of the MIU system?

§3 Complexity

P and NP

Definition 13: Class P consists of problems that can be *solved* in *polynomial time*.

Equivalently, $L \in P$ iff L is *decidable* in polynomial time by a *deterministic* TM.

Examples: Shortest path, primality testing (AKS algorithm), linear programming.

Definition 14: Class NP consists of problems where a *certificate* of a solution (“yes” answer) can be *verified* in polynomial time.

Equivalently, $L \in \text{NP}$ iff L is *decidable* in polynomial time by a *non-deterministic* TM.

Equivalently, $L \in \text{NP}$ iff L is *recognizable* in polynomial time by a *deterministic* TM.

Examples: SAT, graph coloring, graph isomorphism, subset sum, knapsack, vertex cover, clique.

NP-Hard and NP-Complete

Definition 15: A problem H is *NP-hard* if every problem $L \in \text{NP}$ is polynomial-time *reducible* to H .

Examples: Halting problem (undecidable), Traveling Salesman Problem (TSP).

Definition 16: A problem H is *NP-complete* if:

1. $H \in \text{NP}$
2. H is NP-hard

Examples: SAT, 3-SAT, Hamiltonian path... Actually, almost all NP problems are NP-complete!

Theorem 1 (Cook–Levin): SAT is NP-complete.

co-NP

Definition 17: Complexity class co-NP contains problems where “no” instances can be *verified* in *polynomial time*.

Equivalently, $L \in \text{co-NP}$ iff the complement of L is in NP:

$$\text{co-NP} = \{L \mid \overline{L} \in \text{NP}\}$$

Open question: $\text{NP} \stackrel{?}{=} \text{co-NP}$? Implies $\text{P} \neq \text{NP}$ if false.

Examples:

- **VALID:** Check if a Boolean formula is always true (tautology).
- **UNSAT:** Check if a formula has no satisfying assignment.

Computational Hierarchy

$$P \subseteq NP \subseteq PSPACE \subseteq EXP \subset R \subset RE$$

- **RE**
Languages *accepted* (*recognized*) by any TM.
- **R** = $RE \cap \text{co-RE}$
Languages *decided* by any TM (always halt).
- **EXP**
Languages *decided* by a *deterministic* TM in *exponential time*.
- **PSPACE**
Languages *decided* by a *deterministic* TM in *polynomial space*.
- **NP**
Languages *accepted* (*recognized*) by any TM, or *decided* by a *non-deterministic* TM, in *polynomial time*.
- **P**
Languages *decided* by a *deterministic* TM in *polynomial time*.



Complexity Zoo

TODO

See also: https://complexityzoo.net/Petting_Zoo

§4 Computability

Computable Functions

Definition 18 (Church–Turing thesis): *Computable functions* are exactly the functions that can be calculated using a mechanical (that is, automatic) calculation device given unlimited amounts of time and storage space.

“Every model of computation that has ever been imagined can compute only computable functions, and all computable functions can be computed by any of several models of computation that are apparently very different, such as Turing machines, register machines, lambda calculus and general recursive functions.”

Definition 19 (Computable function): A partial function $f : \mathbb{N}^k \hookrightarrow \mathbb{N}$ is *computable* (“can be calculated”) if there exists a computer program with the following properties:

- If $f(x)$ is defined, then the program terminates on the input x with the value $f(x)$ stored in memory.
- If $f(x)$ is undefined, then the program never terminates on the input x .

Effective Procedures

Definition 20 (Effective procedure): An *effective procedure* is a finite, deterministic, mechanical algorithm that guarantees to terminate and produce the correct answer in a finite number of steps.

An algorithm (set of instructions) is called an *effective procedure* if it:

- Consists of *exact*, finite steps.
- Always *terminates* in finite time.
- Produces the *correct* answer for given inputs.
- Requires no external assistance to execute.
- Can be performed *manually*, with pencil and paper.

Definition 21: A function is *computable* if there exists an effective procedure that computes it.

Examples of Computable Functions

Examples:

- The function $f(x) = x^2$ is computable.
- The function $f(x) = x!$ is computable.
- The function $f(n) = \text{"}n\text{-th prime number"}$ is computable.
- The function $f(n) = \text{"the } n\text{-th digit of } \pi \text{"}$ is computable.
- The Ackermann function is computable.
- The function that answers the question "Does God exist?" is computable.
- If the Collatz conjecture is true, the stopping time (number of steps to reach 1) of any n is computable.

§5 Decidability

Decidable Sets

Definition 22 (Decidable set): Given a universal set \mathcal{U} , a set $S \subseteq \mathcal{U}$ is *decidable* (or *computable*, or *recursive*) if there exists a computable function $f : \mathcal{U} \rightarrow \{0, 1\}$ such that $f(x) = 1$ iff $x \in S$.

Examples:

- The set of all WFFs is decidable.
 - *We can check if a given string is well-formed by recursively verifying the syntax rules.*
- For a given finite set Γ of WFFs, the set $\{\alpha \mid \Gamma \models \alpha\}$ of all tautological consequences of Γ is decidable.
 - *We can decide $\Gamma \models \alpha$ using a truth table algorithm by enumerating all possible interpretations (at most $2^{|\Gamma|}$) and checking if each satisfies all formulas in Γ .*
- The set of all tautologies is decidable.
 - *It is the set of all tautological consequences of the empty set.*

Undecidable Sets

Definition 23 (Undecidable set): A set S is *undecidable* if it is not decidable.

Example: The existence of *undecidable* sets of expressions can be shown as follows.

An algorithm is completely determined by its *finite* description. Thus, there are only *countably many* effective procedures. But there are uncountably many sets of expressions. (Why? The set of expressions is countably infinite. Therefore, its power set is uncountable.) Hence, there are *more* sets of expressions than there are possible effective procedures.

§6 Undecidability

Halting Problem

Definition 24 (Halting problem \boxtimes): Given a program and an input, determine whether the program *halts* (stops after a finite time) on that input or *loops* forever.

Theorem 2 (Turing): The halting problem is undecidable.

Proof sketch: Suppose there exists a procedure H that decides the halting problem. We can construct a program P that takes itself as input and runs H on it. If H says that P halts, then P enters an infinite loop. If H says that P does not halt, then P halts. This leads to a contradiction, proving that H cannot exist. \square

Halting Problem Pseudocode

```
def halts(P, x) -> bool:
    """
    Returns True if program P halts on input x.
    Returns False if P loops forever.
    """

def self_halts(P):
    if halts(P, P):
        while True: # loop forever
    else:
        return # halt
```

Observe that `halts(self_halts, self_halts)` cannot return neither `True` nor `False`. **Contradiction!**

Thus, the halts *does not exist* (cannot be implemented), and thus the halting problem is *undecidable*.

Post Correspondence Problem

Definition 25 (Post correspondence problem \boxtimes): Given two finite lists a_1, \dots, a_n and b_1, \dots, b_n of strings (over the alphabet with at least two symbols), determine whether there exists a sequence of indices i_1, \dots, i_k , such that $a_{i_1} \dots a_{i_k} = b_{i_1} \dots b_{i_k}$.

Example: Let $A = [a, ab, bba]$, $B = [baa, aa, bb]$. A solution is $(3, 2, 3, 1)$:

$$a_3 a_2 a_3 a_1 = bba \cdot ab \cdot bba \cdot a = bbaabbbbaa = bb \cdot aa \cdot bb \cdot baa = b_3 b_2 b_3 b_1$$

An alternative formulation of PCP is a collection of *dominoes*, each with a *top* and a *bottom* half, with an unlimited supply of each block, and the goal is to find a sequence of blocks such that the string formed by the *top* halves is equal to the string formed by the *bottom* halves.

bba	ab	bba	a
bb	aa	bb	baa
$i_1 = 3$	$i_2 = 2$	$i_3 = 3$	$i_4 = 1$

§7 Semi-decidability

Semi-decidability

Suppose we want to determine $\Sigma \models \alpha$ where Σ is infinite. In general, it is undecidable.

However, it is possible to obtain a weaker result.

Definition 26 (Semi-decidable set): A set S is *computably enumerable* if there is an *enumeration procedure* which lists, in some order, every member of S : $s_1, s_2, s_3 \dots$

Equivalently (see Theorem 3), a set S is *semi-decidable* if there is an algorithm such that the set of inputs for which the algorithm halts is exactly S .

Note that if S is infinite, the enumeration procedure will *never* finish, but every member of S will be listed *eventually*, after some finite amount of time.

Some properties:

- Decidable sets are closed under union, intersection, Cartesian product, and complement.
- Semi-decidable sets are closed under union, intersection, and Cartesian product.

Enumerability and Semi-decidability

Theorem 3: A set S is computably enumerable iff it is semi-decidable.

Proof (\Rightarrow): *If S is computably enumerable, then it is semi-decidable.*

Since S is computably enumerable, we can check if $\alpha \in S$ by enumerating all members of S and checking if α is among them. If it is, we answer “yes”; otherwise, we continue enumerating. Thus, if $\alpha \in S$, the procedure produces “yes”. If $\alpha \notin S$, the procedure runs forever. □

Enumerability and Semi-decidability [2]

Proof (\Leftarrow): *If S is semi-decidable, then it is computably enumerable.*

Suppose we have a procedure P which, given α , terminates and produces “yes” iff $\alpha \in S$. To show that S is computably enumerable, we can proceed as follows.

1. Construct a systematic enumeration of *all* expressions (for example, by listing all strings over the alphabet in length-lexicographical order): $\beta_1, \beta_2, \beta_3, \dots$
2. Break the procedure P into a finite number of “steps” (for example, by program instructions).
3. Run the procedure on each expression in turn, for an increasing number of steps (see dovetailing):
 - Run P on β_1 for 1 step.
 - Run P on β_1 for 2 steps, then on β_2 for 2 steps.
 - ...
 - Run P on each of β_1, \dots, β_n for n steps each.
 - ...
4. If P produces “yes” for some β_i , output (yield) β_i and continue enumerating.

This procedure will eventually list all members of S , thus S is computably enumerable. □

Dual Enumerability and Decidability

Theorem 4: A set is decidable iff both it and its complement are semi-decidable.

Proof (\Rightarrow): *If A is decidable, then both A and its complement \bar{A} are effectively enumerable.*

Since A is decidable, there exists an effective procedure P that halts on all inputs and returns “yes” if $\alpha \in A$ and “no” otherwise.

To enumerate A :

- Systematically generate all expressions $\alpha_1, \alpha_2, \alpha_3, \dots$
- For each α_i , run P . If P outputs “yes”, yield α_i . Otherwise, continue.

Similarly, enumerate \bar{A} by yielding α_i when P outputs “no”.

Both enumerations are effective, since P always halts, so A and its complement are semi-decidable. □

Dual Enumerability and Decidability [2]

Proof (\Leftarrow): *If both A and its complement \overline{A} are effectively enumerable, then A is decidable.*

Let E be an enumerator for A and \overline{E} an enumerator for \overline{A} .

To decide if $\alpha \in A$, *interleave* the execution of E and \overline{E} , that is, for $n = 1, 2, 3, \dots$

- Run E for n steps and if it produces α , *halt* and output “yes”.
- Run \overline{E} for n steps and if it produces α , *halt* and output “no”.

Since α is either in A or in \overline{A} , one of the enumerators will eventually produce α . The interleaving with increasing number of steps ensures fair scheduling without starvation.

Remark: The “dovetailing” technique (alternating between enumerators with increasing step) avoids infinite waiting while maintaining finite memory requirements. The alternative is to run both enumerators *simultaneously*, in parallel, using, for example, two computers. □

Enumerability of Tautological Consequences

Theorem 5: If Σ is an effectively enumerable set of WFFs, then the set $\{\alpha \mid \Sigma \models \alpha\}$ of tautological consequences of Σ is effectively enumerable.

Proof: Consider an enumeration of the elements of Σ : $\sigma_1, \sigma_2, \sigma_3, \dots$

By the compactness theorem, $\Sigma \models \alpha$ iff $\{\sigma_1, \dots, \sigma_n\} \models \alpha$ for some n .

Hence, it is sufficient to successively test (using truth tables)

$$\emptyset \models \alpha,$$

$$\{\sigma_1\} \models \alpha,$$

$$\{\sigma_1, \sigma_2\} \models \alpha,$$

and so on. If any of these tests succeeds (each is decidable), then $\Sigma \models \alpha$.

This demonstrates that there is an effective procedure that, given any WFF α , will output “yes” iff α is a tautological consequence of Σ . Thus, the set of tautological consequences of Σ is effectively enumerable. \square

§8 Universal Machines

Universal Turing Machine

A *universal Turing machine* is a Turing machine that is capable of computing any computable sequence. [1]

Definition 27: A *universal Turing machine* U_{TM} is a Turing machine that can simulate any other TM.

High-level description of a universal Turing machine U_{TM} :

- Given an input $\langle M, w \rangle$, where M is a TM and $w \in \Sigma^*$:
 - Run (simulate a computation of) M on w .
 - If M halts and accepts w , U_{TM} accepts $\langle M, w \rangle$.
 - If M halts and rejects w , U_{TM} rejects $\langle M, w \rangle$.
 - *Implicitly*, if M loops on w , U_{TM} loops on $\langle M, w \rangle$.

Definition 28: The *language of a universal Turing machine* U_{TM} is the set A_{TM} of all pairs (M, w) such that M is a TM and M accepts w .

$$A_{\text{TM}} = \mathcal{L}(U_{\text{TM}}) = \{\langle M, w \rangle \mid M \text{ is a TM and } w \in \mathcal{L}(M)\}$$

Diagonalization Language

Consider all possible Turing machines, listed in some order, and all strings that are valid TM descriptions:

$$\langle M_0 \rangle, \langle M_1 \rangle, \dots$$

Definition 29: Construct the *diagonalization language* L_Δ of all TMs that do not accept their own description:

$$L_\Delta = \mathcal{L}(M_\Delta) = \{\langle M \rangle \mid M \text{ is a TM and } \langle M \rangle \notin \mathcal{L}(M)\}$$

Note: M_Δ is *not* listed in the table, since its behavior differs from each other M_i at least on input $\langle M_i \rangle$.

	$\langle M_0 \rangle$	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$...
M_0	Acc	No	No	Acc	No	...
M_1	Acc	Acc	Acc	Acc	Acc	...
M_2	Acc	Acc	No	No	No	...
M_3	No	Acc	Acc	Acc	Acc	...
M_4	No	Acc	No	No	No	...
...
M_Δ	No	No	Acc	No	Acc	...

Diagonalization Language is not Recognizable

$$L_{\Delta} = \{\langle M \rangle \mid \langle M \rangle \notin \mathcal{L}(M)\}$$

Theorem 6: $L_{\Delta} \notin \text{RE}$.

Proof: Suppose L_{Δ} is recognizable. Then there exists a recognizer R such that $\mathcal{L}(R) = L_{\Delta}$.

It is the case that either $\langle R \rangle \notin \mathcal{L}(R)$ or $\langle R \rangle \in \mathcal{L}(R)$.

1. $\langle R \rangle \notin \mathcal{L}(R)$. Thus, $\langle R \rangle \in L_{\Delta}$. Since $\mathcal{L}(R) = L_{\Delta}$, $\langle R \rangle \in \mathcal{L}(R)$. Contradiction.
2. $\langle R \rangle \in \mathcal{L}(R)$. Thus, $\langle R \rangle \notin L_{\Delta}$. Since $\mathcal{L}(R) = L_{\Delta}$, $\langle R \rangle \in \mathcal{L}(R)$. Contradiction.

In either case, we reach a contradiction. Therefore, the initial assumption that L_{Δ} is recognizable must be false. Thus, L_{Δ} is not recognizable. □

Universal Language

$$A_{\text{TM}} = \mathcal{L}(U_{\text{TM}}) = \{\langle M, w \rangle \mid M \text{ is a TM and } w \in \mathcal{L}(M)\}$$

Theorem 7: $A_{\text{TM}} \in \text{RE}$.

Proof: U_{TM} is a TM that recognizes A_{TM} . □

Theorem 8: $\overline{A}_{\text{TM}} \notin \text{RE}$

Proof: $L_{\Delta} \leq_M \overline{A}_{\text{TM}}$. Build a recognizer (impossible) for L_{Δ} using a (hypothetical) recognizer for \overline{A}_{TM} . □

Theorem 9: $A_{\text{TM}} \notin \text{R}$.

Proof: R is closed under complement. A language A is decidable iff it is both recognizable ($A \in \text{RE}$) and co-recognizable ($\overline{A} \in \text{RE}$). We know that $\overline{A}_{\text{TM}} \notin \text{RE}$, thus A_{TM} cannot be decidable. □

§9 Reductions

Mapping Reductions

TODO

Extremely Hard Problem

Regular languages are decidable. Some Turing machines accept regular languages and some do not.

Definition 30: Let **REGULAR** be the language of all TMs that accept regular languages.

$$\text{REGULAR}_{\text{TM}} = \{\langle M \rangle \mid \mathcal{L}(M) \text{ is regular}\}$$

This language is *neither* recognizable nor co-recognizable. (See theorems on the next slides.)

- *No computer program can confirm that a given Turing machine has a regular language.*
- *No computer program can confirm that a given Turing machine has a non-regular language.*
- *This problem is beyond the limits of what computers can ever do.*

REGULAR is not Recognizable

Theorem 10: $\text{REGULAR}_{\text{TM}} \notin \text{RE}$.

Proof: $L_{\Delta} \leq_M \text{REGULAR}_{\text{TM}}$.



REGULAR is not even co-Recognizable

Theorem 11: $\text{REGULAR}_{\text{TM}} \notin \text{co-RE}$

Proof: $\overline{L}_{\Delta} \leq_M \text{REGULAR}_{\text{TM}}$.

□

TODO

- ☐ Decidable language outside of NP
- ☒ Universal TM
- ☐ Encodings
- ☐ Programs
- ☐ Rice's theorem

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