Formal Methods in Software Engineering

Satisfiability Modulo Theories — Spring 2025

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§1 First-Order Theories

Motivation

Consider the signature $\Sigma = \langle \Sigma^S, \Sigma^F \rangle$ for a fragment of number theory:

- $\Sigma^S = \{\texttt{Nat}\}$, $\Sigma^F = \{0, 1, +, <\}$
- $\text{rank}(0) = \text{rank}(1) = \langle \texttt{Nat} \rangle$
- $\text{rank}(+) = \langle \texttt{Nat}, \texttt{Nat}, \texttt{Nat} \rangle$
- $\text{rank}(<) = \langle \texttt{Nat}, \texttt{Nat}, \texttt{Bool} \rangle$

1. Consider the Σ-sentence: $\forall x : \texttt{Nat}.\, \neg(x < x)$
   - Is it *valid*, that is, true under *all* interpretations?
   - No, e.g., if we interpret $<$ as *equals* or *divides*.

2. Consider the Σ-sentence: $\neg\exists x : \texttt{Nat}.\, (x < 0)$
   - Is it *valid*?
   - No, e.g., if we interpret `Nat` as the set of *all* integers.

3. Consider the Σ-sentence: $\forall x : \texttt{Nat}.\forall y : \texttt{Nat}.\forall z : \texttt{Nat}.\, (x < y) \land (y < z) \rightarrow (x < z)$
   - Is it *valid*?
   - No, e.g., if we interpret $<$ as the *successor* relation.

Motivation [2]

In practice, we often *do not care* about satisfiability or validity in *general*, but rather with respect to a *limited class* of interpretations.

A practical reason:

- When reasoning in a particular application domain, we typically have *specific* data types/structures in mind (e.g., integers, strings, lists, arrays, finite sets, …).
- More generally, we are typically *not* interested in *arbitrary* interpretations, but rather in *specific* ones.

Theories formalize this domain-specific reasoning: we talk about satisfiability and validity *with respect to a theory* or "*modulo a theory*".

A computational reason:

- The validity problem for FOL is *undecidable* in general.
- However, the validity problem for many *restricted* theories, is *decidable*.

First-Order Theories

Hereinafter, we assume that we have an infinite set of variables X.

Definition 1 (Theory): A first-order *theory* $\mathcal{T}$ is a pair[1] $\langle \Sigma, \boldsymbol{M} \rangle$, where

- $\Sigma = \langle \Sigma^S, \Sigma^F \rangle$ is a first-order signature,
- $\boldsymbol{M}$ is a class[2] of Σ-interpretations over X that is *closed under variable re-assignment*.

Definition 2: $\boldsymbol{M}$ is *closed under variable re-assignment* if every Σ-interpretation that differs from one in $\boldsymbol{M}$ in the way it interprets the variables in X is also in $\boldsymbol{M}$.

A theory limits the interpretations of Σ-formulas to those from $\boldsymbol{M}$.

[1] Here, we use **bold** style for $\boldsymbol{M}$ to denote that it is *not a single* model, but a *collection* of them.

[2] *Class* is a generalization of a set.

Theory Examples

Example: Theory of Real Arithmetic $\mathcal{T}_{\mathrm{RA}} = \langle \Sigma_{\mathrm{RA}}, \boldsymbol{M}_{\mathrm{RA}} \rangle$:

- $\Sigma^S_{\mathrm{RA}} = \{\texttt{Real}\}$
- $\Sigma^F_{\mathrm{RA}} = \{+, -, \times, \leq\} \cup \{q \mid q \text{ is a decimal numeral}\}$
- All $\mathcal{I} \in \boldsymbol{M}_{\mathrm{RA}}$ interpret `Real` as the set of *real numbers* $\mathbb{R}$, each q as the *decimal number* that it denotes, and the function symbols in the usual way.

Example: Theory of Ternary Strings $\mathcal{T}_{\mathrm{TS}} = \langle \Sigma_{\mathrm{TS}}, \boldsymbol{M}_{\mathrm{TS}} \rangle$:

- $\Sigma^S_{\mathrm{TS}} = \{\texttt{String}\}$
- $\Sigma^F_{\mathrm{TS}} = \{\cdot, <\} \cup \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$
- All $\mathcal{I} \in \boldsymbol{M}_{\mathrm{TS}}$ interpret `String` as the set $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}^*$ of all finite strings over the characters $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, symbol $\cdot$ as string concatenation (e.g., $\mathrm{a} \cdot \mathrm{b} = \mathrm{ab}$), and $<$ as lexicographic order.

$\mathcal{T}$-interpretations

Definition 3 (Reduct): Let Σ and Ω be two signatures over variables X, where $\Omega \supseteq \Sigma$, that is, $\Omega^S \supseteq \Sigma^S$ and $\Omega^F \supseteq \Sigma^F$.

Let $\mathcal{I}$ be an Ω-interpretation over X.

The *reduct* $\mathcal{I}^\Sigma$ of $\mathcal{I}$ to Σ is a Σ-interpretation obtained from $\mathcal{I}$ by resticting it to the symbols in Σ.

Definition 4 ($\mathcal{T}$-interpretation): Given a theory $\mathcal{T} = \langle \Sigma, \boldsymbol{M} \rangle$, a *$\mathcal{T}$-interpretation* is any Ω-interpretation $\mathcal{I}$ for some signature $\Omega \supseteq \Sigma$ such that $\mathcal{I}^\Sigma \in \boldsymbol{M}$.

Note: This definition allows us to consider the satisfiability in a theory $\mathcal{T} = \langle \Sigma, \boldsymbol{M} \rangle$ of formulas that contain sorts or function symbols not in Σ. These symbols are usually called *uninterpreted* (in $\mathcal{T}$).

$\mathcal{T}$-interpretations [2]

Example: Consider again the theory of real arithmetic $\mathcal{T}_{\mathrm{RA}} = \langle \Sigma_{\mathrm{RA}}, \boldsymbol{M}_{\mathrm{RA}} \rangle$.

All $\mathcal{I} \in \boldsymbol{M}_{\mathrm{RA}}$ interpret `Real` as $\mathbb{R}$ and function symbols as usual.

Which of the following interpretations are $\mathcal{T}_{\mathrm{RA}}$-interpretations?

1. $\texttt{Real}^{\mathcal{I}_1} = \mathbb{Q}$, symbols in Σ^F_{RA} interpreted as usual. ✗
2. $\texttt{Real}^{\mathcal{I}_2} = \mathbb{R}$, symbols in Σ^F_{RA} interpreted as usual, and $\texttt{String}^{\mathcal{I}_2} = \{0.5, 1.3\}$. ✓
3. $\texttt{Real}^{\mathcal{I}_3} = \mathbb{R}$, symbols in Σ^F_{RA} interpreted as usual, and $\log^{\mathcal{I}_3}$ is the successor function. ✓

$\mathcal{T}$-satisfiability, $\mathcal{T}$-entailment, $\mathcal{T}$-validity

Definition 5 ($\mathcal{T}$-satisfiability): A Σ-formula α is *satisfiable in $\mathcal{T}$*, or *$\mathcal{T}$-satisfiable*, if it is satisfied by *some* $\mathcal{T}$-interpretation $\mathcal{I}$.

Definition 6 ($\mathcal{T}$-entailment): A set Γ of formulas *$\mathcal{T}$-entails* a formula α, if every $\mathcal{T}$-interpretation that satisfies all formulas in Γ also satisfies α.

Definition 7 ($\mathcal{T}$-validity): A formula α is *$\mathcal{T}$-valid*, if it is satisfied by *all* $\mathcal{T}$-interpretations.

Note: A formula α is *$\mathcal{T}$-valid* iff $\emptyset \vDash \alpha$.

Example: Which of the following Σ_{RA}-formulas is satisfiable or valid in $\mathcal{T}_{\mathrm{RA}}$?

1. $(x_0 + x_1 \leq 0.5) \land (x_0 - x_1 \leq 2)$ — *satisfiable*, *falsifiable*
2. $\forall x_0 .\, (x_0 + x_1 \leq 1.7) \to (x_1 \leq 1.7 - x_0)$ — *satisfiable*, *valid*
3. $\forall x_0 . \forall x_1 .\, (x_0 + x_1 \leq 1)$ — *unsatisfiable*, *falsifiable*

FOL vs Theory

For every signature Σ, entailment and validity in “pure” FOL can be seen as entailment and validity in the theory $\mathcal{T}_{\text{FOL}} = \langle \Sigma, \boldsymbol{M}_{\text{FOL}} \rangle$ where $\boldsymbol{M}_{\text{FOL}}$ is the class of *all possible* Σ-interpretations.

- Pure first-order logic = reasoning over *all* possible interpretations.
- Reasoning modulo a theory = *restricting* interpretations with some domain constraints.
- Theories make automated reasoning *feasible* in many domains.

Axiomatization

Definition 8 (Axiomatic theory): A first-order *axiomatic theory* $\mathcal{T}$ is defined by a signature Σ and a set $\mathcal{A}$ of Σ-sentences, or *axioms*.

Definition 9 ($\mathcal{T}$-validity in axiomatic theory): An Ω-formula α is *valid* in an axiomatic theory $\mathcal{T}$ if it is entailed by the axioms of $\mathcal{T}$, that is, every Ω-interpretation $\mathcal{I}$ that satisfies $\mathcal{A}$ also satisfies α.

Note: Axiomatic theories are a *special case* of the general definition (via $\boldsymbol{M}$) of theories.

- Given an axiomatic theory $\mathcal{T}'$ defined by Σ and $\mathcal{A}$, we can define a theory $\mathcal{T} = \langle \Sigma, \boldsymbol{M} \rangle$ where $\boldsymbol{M}$ is the class of all Σ-interpretations that satisfy all axioms in $\mathcal{A}$.
- It is not hard to show that a formula α is valid in $\mathcal{T}$ *iff* it is valid in $\mathcal{T}'$.

Note: Not all theories are first-order axiomatizable.

Non-Axiomatizable Theories

Note: Not all theories are first-order axiomatizable.

Example: Consider the theory $\mathcal{T}_{\mathtt{Nat}}$ of the natural numbers, with signature Σ with $\Sigma^S = \{\mathtt{Nat}\}$, $\Sigma^F = \{0, S, +, <\}$, and $M = \{\mathcal{J}\}$ where $\mathtt{Nat}^{\mathcal{J}} = \mathbb{N}$ and Σ^F is interpreted as usual.

Any set of axioms (for example, *Peano axioms*) for this theory is satisfied by *non-standard models*, e.g., interpretations $\mathcal{J}'$ where $\mathtt{Nat}^{\mathcal{J}'}$ includes other chains of elements besides the natural numbers.

However, these models *falsify* formulas that are *valid* in $\mathcal{T}_{\mathtt{Nat}}$.

For example, "every number is either zero or a successor": $\forall x.\, (x \doteq 0) \lor \exists y.\, (x \doteq S(y))$.

- `true` in the *standard* model, i.e. $\mathtt{Nat}^{\mathcal{J}} = \mathbb{N} = \{0, 1 := S(0), 2 := S(1), ...\}$.
- `false` in *non-standard* models, e.g., $\mathtt{Nat}^{\mathcal{J}'} = \{0, 1, 2, ...\} \cup \{\omega, \omega + 1, ...\}$
  - Intuitively, ω is "an infinite element".
  - The successor function still applies: $S(\omega) = \omega + 1$, $S(\omega + 1) = \omega + 2$, etc.
  - Even the addition and multiplication still works: $\omega + 3 = S(S(S(\omega)))$, $\omega \times 2 = \omega + \omega$.
  - But ω is larger than all standard numbers: $\omega > 0, \omega > 1, ...$

Peano Arithmetic

Definition 10: *Peano arithmetic* $\mathcal{T}_{\mathrm{PA}}$, or *first-order arithmetic*, is the axiomatic theory of natural numbers with signature $\Sigma_{\mathrm{PA}}^F = \{0, S, +, \times, =\}$ and *Peano axioms*:

1. $\forall x.\, (S(x) \neq 0)$ (zero)
2. $\forall x. \forall y.\, (S(x) = S(y)) \to (x = y)$ (successor)
3. $F[0] \land (\forall x.\, F[x] \to F[x + 1]) \to \forall x.\, F[x]$ (induction)
4. $\forall x.\, (x + 0 = x)$ (plus zero)
5. $\forall x. \forall y.\, (x + S(y) = S(x + y))$ (plus successor)
6. $\forall x.\, (x \times 0 = 0)$ (times zero)
7. $\forall x. \forall y.\, (x \times S(y) = (x \times y) + x)$ (times successor)

Axiom (induction) is the *induction axiom schema*. It stands for an *infinite* set of axioms, one for each Σ_{PA}-formula F with one free variable. The notation $F[\alpha]$ means that F contains α as a sub-formula.

The *intended interpretation* (*standard models*) of $\mathcal{T}_{\mathrm{PA}}$ have the domain $\mathbb{N}$ and the usual interpretations of the function symbols as $0_{\mathbb{N}}$, $S_{\mathbb{N}}$, $+_{\mathbb{N}}$, and $\times_{\mathbb{N}}$.

Presburger Arithmetic

Note: Satisfiability and validity in $\mathcal{T}_{\mathrm{PA}}$ is undecidable. Therefore, we need a more restricted theory of arithmetic that does not include multiplication.

Definition 11: *Presburger arithmetic* $\mathcal{T}_{\mathbb{N}}$ is the axiomatic theory of natural numbers with signature $\Sigma_{\mathbb{N}}^{F} = \{0, S, +, =\}$ and the *subset* of *Peano axioms*:

1. $\forall x. \, (S(x) \neq 0)$ (zero)
2. $\forall x. \forall y. \, (S(x) = S(y)) \rightarrow (x = y)$ (successor)
3. $F[0] \land (\forall x. \, F[x] \rightarrow F[x + 1]) \rightarrow \forall x. \, F[x]$ (induction)
4. $\forall x. \, (x + 0 = x)$ (plus zero)
5. $\forall x. \forall y. \, (x + S(y) = S(x + y))$ (plus successor)

Note: Presburger arithmetic is decidable.

Completeness of Theories

Definition 12: A Σ-theory $\mathcal{T}$ is *complete* if for every Σ-sentence α, either α or $\neg\alpha$ is valid in $\mathcal{T}$.

Note: In a complete Σ-theory, every Σ-sentence is either valid or unsatisfiable.

Example: Any theory $\mathcal{T} = \langle \Sigma, \boldsymbol{M} \rangle$ where all interpretations in $\boldsymbol{M}$ only differ in how they interpret the variables (e.g., $\mathcal{T}_{\mathrm{RA}}$) is *complete*.

Example: The axiomatic (mono-sorted) theory of *monoids* with $\Sigma^F = \{ \cdot , \varepsilon \}$ and axioms

$$\forall x. \forall y. \forall z. \, (x \cdot y) \cdot z \doteq x \cdot (y \cdot z) \qquad \forall x. \, (x \cdot \varepsilon \doteq x) \qquad \forall x. \, (\varepsilon \cdot x \doteq x)$$

is *incomplete*. For example, the sentence $\forall x. \forall y. \, (x \cdot y \doteq y \cdot x)$ is `true` in some monoids (e.g. the addition of integers *is* commutative) but `false` in others (e.g. the concatenation of strings *is not* commutative).

Completeness of Theories [2]

Example: The axiomatic (mono-sorted) theory of *dense linear orders without endpoints* with $\Sigma^F = \{\prec\}$ and the following axioms is *complete*.

$$\forall x.\forall y.(x \prec y) \rightarrow \exists z.\,((x \prec z) \wedge (z \prec y)) \quad \text{(dense)}$$

$$\forall x.\forall y.\,((x \prec y) \vee (y \prec x) \vee (x \doteq y)) \quad \text{(linear)}$$

$$\forall x.\,\neg(x \prec x) \quad \forall x.\forall y.\forall z.\,((x \prec y) \wedge (y \prec z) \rightarrow (x \prec z)) \quad \text{(orders)}$$

$$\forall x.\exists y.\,(y \prec x) \quad \forall x.\exists y.\,(x \prec y) \quad \text{(without endpoints)}$$

Decidability

Recall that a set A is *decidable* if there exists a *terminating* procedure that, given an input element a, returns (after *finite* time) either "yes" if $a \in A$ or "no" if $a \notin A$.

Definition 13: A theory $\mathcal{T} = \langle \Sigma, \boldsymbol{M} \rangle$ is *decidable* if the set of all *$\mathcal{T}$-valid* Σ-formulas is decidable.

Definition 14: A *fragment* of $\mathcal{T}$ is a *syntactically-restricted subset* of $\mathcal{T}$-valid Σ-formulas.

Example: The *quantifier-free* fragment of $\mathcal{T}$ is the set of all $\mathcal{T}$-valid Σ-formulas *without quantifiers*.

Example: The *linear* fragment of $\mathcal{T}_{\mathrm{RA}}$ is the set of all $\mathcal{T}$-valid Σ_{RA}-formulas *without multiplication* ($\times$).

Axiomatizability

Definition 15: A theory $\mathcal{T} = \langle \Sigma, M \rangle$ is *recursively axiomatizable* if M is the class of all interpretations satisfying a *decidable set* of first-order axioms $\mathcal{A}$.

Theorem 1 (Lemma): Every recursively axiomatizable theory $\mathcal{T}$ admits a procedure $E_{\mathcal{T}}$ that *enumerates* all $\mathcal{T}$-valid formulas.

Theorem 2: For every *complete* and *recursively axiomatizable* theory $\mathcal{T}$, validity in $\mathcal{T}$ is decidable.

Proof: Given a formula α, use $E_{\mathcal{T}}$ to enumerate all valid formulas. Since $\mathcal{T}$ is complete, either α or $\neg\alpha$ will eventually (after *finite* time) be produced by $E_{\mathcal{T}}$. $\square$

§2 Introduction to SMT

Common Theories in SMT

Satisfiability Modulo Theories (SMT) traditionally focuses on theories with *decidable quantifier-free fragments*.

SMT is concerned with (un)satisfiability, but recall that a formula α is *$\mathcal{T}$-valid* iff $\neg\alpha$ is *$\mathcal{T}$-unsatisfiable*.

Checking the (un)satisfiability of quantifier-free formulas in main background theories *efficiently* has a large number of applications in:

- hardware and software verification
- model checking
- symbolic execution
- compiler validation
- type checking
- planning and scheduling
- software synthesis
- cyber-security
- verifiable machine learning
- analysis of biological systems

Further, we are going to study:
- A few of those *theories* and their *decision procedures*.
- *Proof systems* to reason *modulo theories* automatically.

From Quantifier-Free Formulas to Conjunctions of Literals

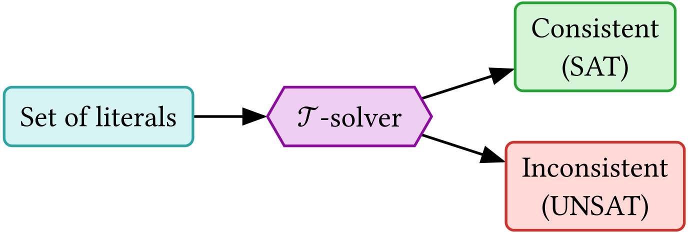
Theorem 3: The satisfiability of *quantifier-free* formulas in a theory $\mathcal{T}$ is *decidable* iff the satisfiability in $\mathcal{T}$ of *conjunctions of literals* is decidable.

Here, *literal* is an atom or its negation. For example: $(a \doteq b)$, $\neg(a + 1 < b)$, $(f(b) \doteq g(f(a)))$.

Proof: A quantifier-free formula can be transformed into disjunctive normal form (DNF), and its satisfiability reduces to checking satisfiability of conjunctions of literals. Conversely, a conjunction of literals is a special case of a quantifier-free formula. Thus, the two satisfiability problems are equivalent. $\square$

Theory Solvers

Definition 16 ($\mathcal{T}$-solver): A *theory solver*, or *$\mathcal{T}$-solver*, is a specialized decision procedure for the satisfiability of conjunctions of literals in a theory $\mathcal{T}$.



Theory of Uninterpreted Functions

Definition 17: Given a signature Σ, the most general theory consists of the class of *all* Σ-interpretations. In fact, this is a *family* of theories parameterized by the signature Σ.

It is known as the theory of *equality with uninterpreted functions* $\mathcal{T}_{\text{EUF}}$, or the *empty theory*, since it contains no *sentences*.

Example: $(a \doteq b) \land (f(a) \doteq b) \land \neg(g(a) \doteq g(f(a)))$ Is this formula satisfiable in $\mathcal{T}_{\text{EUF}}$?

Both validity and satisfiability are undecidable in $\mathcal{T}_{\text{EUF}}$.

- Validity in $\mathcal{T}_{\text{EUF}}$ is *semi-decidable* — this is just a validity in FOL.
- Since a formula α is $\mathcal{T}$-satisfiable iff $\neg\alpha$ is not $\mathcal{T}$-valid, $\mathcal{T}_{\text{EUF}}$-satisfiability is *co-recognizable*.

However, the satisfiability of *conjunctions of $\mathcal{T}_{\text{EUF}}$-literals* is *decidable*, in polynomial time, using the *congruence closure* algorithm.

Theory of Real Arithmetic

Definition 18: The theory of *real arithmetic* $\mathcal{T}_{\mathrm{RA}}$ is a theory of inequalities over the real numbers.

- $\Sigma^S = \{\texttt{Real}\}$
- $\Sigma^F = \{+, -, \times, <\} \cup \{q \mid q \text{ is a decimal numeral}\}$
- $\boldsymbol{M}$ is the class of interpretations that interpret `Real` as the set of *real numbers* $\mathbb{R}$, and the function symbols in the usual way.

Satisfiability in the full $\mathcal{T}_{\mathrm{RA}}$ is *decidable* (in worst-case doubly-exponential time).

Restricted fragments of $\mathcal{T}_{\mathrm{RA}}$ can be decided more efficiently.

Example: Quantifier-free linear real arithmetic (`QF_LRA`) is the theory of *linear* inequalities over the reals, where $\times$ can only be used in the form of *multiplication by constants* (decimal numerals).

The satisfiability of conjunctions of literals in `QF_LRA` is *decidable* in *polynomial time*.

Theory of Integer Arithmetic

Definition 19: The theory of *integer arithmetic* $\mathcal{T}_{\text{IA}}$ is a theory of inequalities over the integers.

- $\Sigma^S = \{\texttt{Int}\}$
- $\Sigma^F = \{+, -, \times, <\} \cup \{n \mid n \text{ is an integer numeral}\}$
- M is the class of interpretations that interpret `Int` as the set of *integers* $\mathbb{Z}$, and the function symbols in the usual way.

Satisfiability in $\mathcal{T}_{\text{IA}}$ is *not even semi-decidable*!

Satisfiability of quantifier-free Σ-formulas in $\mathcal{T}_{\text{IA}}$ is *undecidable* as well.

Linear integer arithmetic (`LIA`, also known as *Presburger arithmetic*) is decidable, but not efficiently (in worst-case triply-exponential time). Its quantifier-free fragment (`QF_LIA`) is NP-complete.

Theory of Arrays with Extensionality

Definition 20: The theory of *arrays* $\mathcal{T}_{\mathrm{AX}}$ is useful for modelling RAM or array data structures.

- $\Sigma^S = \{\mathtt{A}, \mathtt{I}, \mathtt{E}\}$ (arrays, indices, elements)
- $\Sigma^F = \{\mathrm{read}, \mathrm{write}\}$, where $\mathrm{rank}(\mathrm{read}) = \langle \mathtt{A}, \mathtt{I}, \mathtt{E} \rangle$ and $\mathrm{rank}(\mathrm{write}) = \langle \mathtt{A}, \mathtt{I}, \mathtt{E}, \mathtt{A} \rangle$

Let a be a variable of sort $\mathtt{A}$, variable i of sort $\mathtt{I}$, and variable v of sort $\mathtt{E}$.

- $\mathrm{read}(a, i)$ denotes the value stored in array a at index i.
- $\mathrm{write}(a, i, v)$ denotes the array that stores value v at index i and is otherwise identical to a.

Example: $\mathrm{read}(\mathrm{write}(a, i, v), i) \doteq_{\mathtt{E}} v$

- Is this formula *intuitively* valid/satisfiable/unsatisfiable in $\mathcal{T}_{\mathrm{A}}$?

Example: $\forall i.\ (\mathrm{read}(a, i) \doteq_{\mathtt{E}} \mathrm{read}(a', i)) \rightarrow (a \doteq_{\mathtt{A}} a')$

- Is this formula *intuitively* valid/satisfiable/unsatisfiable in $\mathcal{T}_{\mathrm{A}}$?

Theory of Arrays with Extensionality [2]

Definition 21: The theory of arrays $\mathcal{T}_{\mathrm{AX}} = \langle \Sigma, M \rangle$ is finitely axiomatizable.

M is the class of interpretations that satisfy the following axioms:

1. $\forall a. \forall i. \forall v. \left(\mathrm{read}(\mathrm{write}(a, i, v), i) \doteq_{\mathsf{E}} v\right)$
2. $\forall a. \forall i. \forall j. \forall v. \neg(i \doteq_{\mathsf{I}} j) \to \left(\mathrm{read}(\mathrm{write}(a, i, v), j) \doteq_{\mathsf{E}} \mathrm{read}(a, j)\right)$
3. $\forall a. \forall b. \left(\forall i. \left(\mathrm{read}(a, i) \doteq_{\mathsf{E}} \mathrm{read}(b, i)\right)\right) \to (a \doteq_{\mathsf{A}} b)$

Note: The last axiom is called *extensionality* axiom. It states that two arrays are equal if they have the same values at all indices. It can be omitted to obtain a theory of arrays *without extensionality* $\mathcal{T}_{\mathrm{A}}$.

Validity and satisfiability in $\mathcal{T}_{\mathrm{AX}}$ is *undecidable*.

There are several *decidable fragments* of $\mathcal{T}_{\mathrm{A}}$.

Survey of Decidability and Complexity

Theory	Description	Full	QF	Full complexity	QFC complexity
PL	Propositional Logic	—	yes	NP-complete	$\Theta(n)$
$\mathcal{T}_{\mathrm{EUF}}$	Equality	no	yes	undecidable	$\mathcal{O}(n \log n)$
$\mathcal{T}_{\mathrm{PA}}$	Peano Arithmetic	no	no	undecidable	undecidable
$\mathcal{T}_{\mathbb{N}}$	Presburger Arithmetic	yes	yes	$\Omega(2^{2^n}), \mathcal{O}\left(2^{2^{2^{kn}}}\right)$	NP-complete
$\mathcal{T}_{\mathbb{Z}}$	Linear Integers (LIA)	yes	yes	$\Omega(2^{2^n}), \mathcal{O}\left(2^{2^{2^{kn}}}\right)$	NP-complete
$\mathcal{T}_{\mathbb{R}}$	Reals	yes	yes	$\mathcal{O}\left(2^{2^{kn}}\right)$	$\mathcal{O}\left(2^{2^{kn}}\right)$
$\mathcal{T}_{\mathbb{Q}}$	Linear Rationals	yes	yes	$\Omega(2^n), \mathcal{O}\left(2^{2^{kn}}\right)$	PTIME
$\mathcal{T}_{\mathrm{RDS}}$	Recursive Data Structures	no	yes	undecidable	$\mathcal{O}(n \log n)$
$\mathcal{T}_{\mathrm{ARDS}}$	Acyclic RDS	yes	yes	not elementary recursive	$\Theta(n)$
$\mathcal{T}_{\mathrm{A}}$	Arrays	no	yes	undecidable	NP-complete
$\mathcal{T}_{\mathrm{AX}}$	Arrays with Extensionality	no	yes	undecidable	NP-complete

Survey of Decidability and Complexity [2]

Legend for the table:

- **"Full"** denotes the decidability of a complete theory *with* quantifiers.
- **"QF"** denotes the decidability of a *quantifier-free* theory.
- **"Full complexity"** denotes the complexity of the satisfiability in a complete theory *with quantifiers*.
- **"QFC complexity"** denotes the complexity of the satisfiability in a *quantifier-free conjunctive* fragment.
- For complexities, n is the size of the input formula, k is some positive integer.
- "*Not elementary recursive*" means the runtime cannot be bounded by a fixed-height stack of exponentials.

§3 Difference Logic

Difference Logic

Definition 22: *Difference logic* (DL) is a fragment of linear integer arithmetic consisting of conjunctions of literals of the very restricted form:

$$x - y \bowtie c$$

where x and y are integer variables, c is a numeral, and $\bowtie \in \{=, <, \leq, >, \geq\}$.

A solver for difference logic consists of three steps:

1. Literals normalization.
2. Conversion to a graph.
3. Cycle detection.

Decision Procedure for DL

Step 1: Rewrite each literal using $\leq$ by applying the following rules:

1. $(x - y = c) \longrightarrow (x - y \leq c) \wedge (x - y \geq c)$
2. $(x - y \geq c) \longrightarrow (y - x \leq -c)$
3. $(x - y > c) \longrightarrow (y - x < -c)$
4. $(x - y < c) \longrightarrow (x - y \leq c - 1)$

Step 2: Construct a weighted directed graph G with a vertex for each variable and an edge $x \xrightarrow{c} y$ for each literal $(x - y \leq c)$.

Step 3: Check for *negative cycles* in G.

- Use, for example, the Bellman-Ford algorithm.
- If G contains a negative cycle, the set of literals is *inconsistent* (UNSAT).
- Otherwise, the set of literals is *consistent* (SAT).

Difference Logic Example

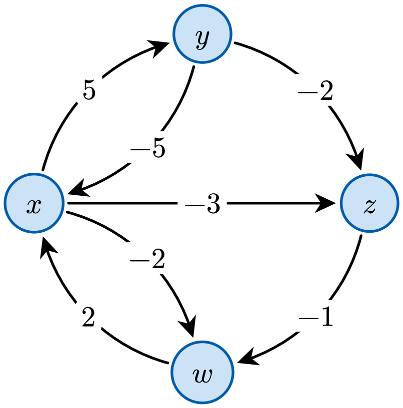
Consider the following set of difference logic literals:

$$(x - y = 5) \wedge (z - y \geq 2) \wedge (z - x > 2) \wedge (w - x = 2) \wedge (z - w < 0)$$

Normalize the literals:

- $(x - y = 5) \implies (x - y \leq 5) \wedge (y - x \leq -5)$
- $(z - y \geq 2) \implies (y - z \leq -2)$
- $(z - x > 2) \implies (x - z \leq -3)$
- $(w - x = 2) \implies (w - x \leq 2) \wedge (x - w \leq -2)$
- $(z - w < 0) \implies (z - w \leq -1)$

UNSAT because of the negative cycle: $x \xrightarrow{-3} z \xrightarrow{-1} w \xrightarrow{2} x$.



§4 Equality

Theory of Equality with Uninterpreted Functions

Definition 23: The theory of equality with uninterpreted functions $\mathcal{T}_{\text{EUF}}$ is defined by the signature $\Sigma^F = \{\doteq, f, g, h, \ldots\}$ (*interpreted* equality and *uninterpreted* functions) and the following axioms:

1. $\forall x.\, x \doteq x$ (reflexivity)
2. $\forall x. \forall y.\, (x \doteq y) \to (y \doteq x)$ (symmetry)
3. $\forall x. \forall y. \forall z.\, (x \doteq y) \land (y \doteq z) \to (x \doteq z)$ (transitivity)
4. $\forall \boldsymbol{x}. \forall \boldsymbol{y}.\, (\boldsymbol{x} = \boldsymbol{y}) \to (f(\boldsymbol{x}) \doteq f(\boldsymbol{y}))$ (function congruence)

Flattening

Definition 24: A literal is *flat* if it is of the form:

- $x \doteq y$
- $\neg(x \doteq y)$
- $x \doteq f(\boldsymbol{z})$

where x and y are variables, f is a function symbol, and $\boldsymbol{z}$ is a tuple of 0 or more variables.

Note: Any set of literals can be converted to an equisatisfiable set of *flat* literals by introducing *new* variables and equating non-equational atoms to `true`.

Example: Consider the set of literals: $\{x + y > 0, y \doteq f(g(z))\}$.

We can convert it to an equisatisfiable set of flat literals by introducing fresh variables v_i:

$$\{\, v_1 \doteq v_2 > v_3, \quad v_1 \doteq \texttt{true}, \quad v_2 \doteq x + y, \quad v_3 \doteq 0, \quad y \doteq f(v_4), \quad v_4 \doteq g(z) \,\}$$

Hereinafter, we will assume that all literals are *flat*.

Notation and Assumptions

- We abbreviate $\neg(s \doteq t)$ with $s \not\doteq t$.
- For tuples $\boldsymbol{u} = \langle u_1, \ldots, u_n \rangle$ and $\boldsymbol{v} = \langle v_1, \ldots, v_n \rangle$, we abbreviate $(u_1 \doteq v_1) \land \ldots \land (u_n \doteq v_n)$ with $\boldsymbol{u} = \boldsymbol{v}$.
- Γ is used to refer to the "current" proof state in rule premises.
- $\Gamma, s \doteq t$ is an abbreviation for $\Gamma \cup \{s \doteq t\}$.
- If applying a rule R does not change Γ, then R *is not applicable* to Γ, that is, Γ is *irreducible* w.r.t. R.

Satisfiability Proof System for `QF_UF`

Let `QF_UF` be the quantifier-free fragment of FOL over some signature Σ.

Below is a simple *satisfiability proof system* R_{UF} for `QF_UF`:

$$\textsc{Refl}\ \frac{x \text{ occurs in } \Gamma}{\Gamma := \Gamma, x \doteq x} \qquad \textsc{Cong}\ \frac{x \doteq f(\boldsymbol{u}) \in \Gamma \quad y \doteq f(\boldsymbol{v}) \in \Gamma \quad \boldsymbol{u} = \boldsymbol{v} \in \Gamma}{\Gamma := \Gamma, x \doteq y}$$

$$\textsc{Symm}\ \frac{x \not\doteq y \in \Gamma}{\Gamma := \Gamma, y \doteq x} \qquad \textsc{Contr}\ \frac{x \doteq y \in \Gamma \quad x \not\doteq y \in \Gamma}{\text{UNSAT}}$$

$$\textsc{Trans}\ \frac{x \not\doteq y \in \Gamma \quad y \doteq z \in \Gamma}{\Gamma := \Gamma, x \doteq z} \qquad \textbf{SAT}\ \frac{\text{No other rules apply}}{\text{SAT}}$$

Is R_{UF} *sound*?

Is R_{UF} *terminating*?

Example Derivation in R_{UF}

$$\textsc{Refl}\ \frac{x \text{ occurs in } \Gamma}{\Gamma := \Gamma, x \doteq x} \qquad \textsc{Trans}\ \frac{x \not\doteq y \in \Gamma \qquad y \doteq z \in \Gamma}{\Gamma := \Gamma, x \doteq z} \qquad \textsc{Contr}\ \frac{x \doteq y \in \Gamma \qquad x \not\doteq y \in \Gamma}{\text{UNSAT}}$$

$$\textsc{Symm}\ \frac{x \not\doteq y \in \Gamma}{\Gamma := \Gamma, y \doteq x} \qquad \textsc{Cong}\ \frac{x \doteq f(\boldsymbol{u}) \in \Gamma \qquad y \doteq f(\boldsymbol{v}) \in \Gamma \qquad \boldsymbol{u} = \boldsymbol{v} \in \Gamma}{\Gamma := \Gamma, x \doteq y} \qquad \textbf{SAT}\ \frac{\text{No other rules apply}}{\text{SAT}}$$

Example: Determine the satisfiability of the following set of literals: $a \doteq f(f(a))$, $a \doteq f(f(f(a)))$, $g(a, f(a)) \not\doteq g(f(a), a)$. Flatten the literals and construct the following proof:

$$\dfrac{a \doteq f(a_1), a_1 \doteq f(a), a \doteq f(a_2), a_2 \doteq f(a_1), a_3 \not\doteq a_4, a_3 \doteq g(a, a_1), a_4 \doteq g(a_1, a)}{\dfrac{a_1 \doteq a_1}{\dfrac{a \doteq a_2}{\dfrac{a_1 \doteq a}{\dfrac{a \doteq a_1}{\dfrac{a_3 \doteq a_4}{\text{UNSAT}}\ \textsc{Contr} \text{ applied to } a_3 \doteq a_4, a_3 \not\doteq a_4}\ \textsc{Cong} \text{ applied to } a_3 \doteq g(a, a_1), a_4 \doteq g(a_1, a), a \doteq a_1, a_1 \doteq a}\ \textsc{Symm}}\ \textsc{Cong} \text{ applied to } a_1 \doteq f(a), a \doteq f(a_2), a \doteq a_2}\ \textsc{Cong} \text{ applied to } a \doteq f(a_1), a_2 \doteq f(a_1), a_1 \doteq a_1}\ \textsc{Refl}$$

Soundness of R_{UF}

Theorem 4 (Refutation soundness): A literal set Γ_0 is unsatisfiable if R_{UF} derives UNSAT from it.

Proof: All rules except SAT are satisfiability-preserving.

If a derivation from Γ_0 ends with UNSAT, then Γ_0 must be unsatisfiable. □

Theorem 5 (Solution soundness): A literal set Γ_0 is satisfiable if R_{UF} derives SAT from it.

Proof: Let Γ be a proof state to which SAT applies. From Γ, we can construct an interpretation $\mathcal{I}$ that satisfies Γ_0. Let $s \sim t$ iff $(s \doteq t) \in \Gamma$. One can show that $\sim$ is an equivalence relation.

Let the domain of $\mathcal{I}$ be the equivalence classes $E_1, \ldots, E_k$ of $\sim$.

- For every variable or a constant t, let $t^{\mathcal{I}} = E_i$ if $t \in E_i$ for some i. Otherwise, let $t^{\mathcal{I}} = E_1$.
- For every unary function symbol f, and equivalence class E_i, let $f^{\mathcal{I}}$ be such that $f^{\mathcal{I}}(E_i) = E_j$ if $f(t) \in E_j$ for some $t \in E_i$. Otherwise, let $f^{\mathcal{I}}(E_i) = E_1$. Define $f^{\mathcal{I}}$ for non-unary f similarly.

We can show that $\mathcal{I} \vDash \Gamma$. This means that $\mathcal{I}$ models Γ_0 as well since $\Gamma_0 \subseteq \Gamma$. □

Termination in R_{UF}

Theorem 6: Every derivation strategy for R_{UF} terminates.

Proof: R_{UF} adds to the current state Γ only equalities between variables of Γ_0.

So, at some point it will run out of new equalities to add. $\square$

Completeness of R_{UF}

Theorem 7 (Refutation completeness): Every derivation strategy applied to an unsatisfiable state Γ_0 ends with UNSAT.

Proof: Let Γ_0 be an unsatisfiable state. Suppose there was a derivation from Γ_0 that did not end with UNSAT. Then, by the termination theorem, it would have to end with SAT. But then R_{UF} would be not be solution sound. $\square$

Theorem 8 (Solution completeness): Every derivation strategy applied to a satisfiable state Γ_0 ends with SAT.

Proof: Let Γ_0 be a satisfiable state. Suppose there was a derivation from Γ_0 that did not end with SAT. Then, by the termination theorem, it would have to end with UNSAT. But then R_{UF} would be not be refutation sound. $\square$

§5 Arrays

Theory of Arrays

Definition 25: The theory of *arrays* $\mathcal{T}_{\mathrm{AX}}$ is defined by the signature $\Sigma^S = \{\mathtt{A}, \mathtt{I}, \mathtt{E}\}$ (arrays, indices, elements), $\Sigma^F = \{\mathrm{read}, \mathrm{write}\}$ and the following axioms:

1. $\forall a. \forall i. \forall v. \left(\mathrm{read}(\mathrm{write}(a, i, v), i) \doteq_{\mathtt{E}} v\right)$
2. $\forall a. \forall i. \forall j. \forall v. \neg(i \doteq_{\mathtt{I}} j) \rightarrow \left(\mathrm{read}(\mathrm{write}(a, i, v), j) \doteq_{\mathtt{E}} \mathrm{read}(a, j)\right)$
3. $\forall a. \forall b. \left(\forall i. \left(\mathrm{read}(a, i) \doteq_{\mathtt{E}} \mathrm{read}(b, i)\right)\right) \rightarrow (a \doteq_{\mathtt{A}} b)$

Example

```c
void ReadBlock(int data[], int x, int len) {
  int i = 0;
  int next = data[0];
  for (; i < next && i < len; i = i + 1) {
    if (data[i] == x)
      break;
    else
      Process(data[i]);
  }
  assert(i < len);
}
```

One pass through this code can be translated into the following $\mathcal{T}_A$ formula:

$$(\mathrm{i} \doteq 0) \land (\mathrm{next} \doteq \mathrm{read}(\mathrm{data}, 0)) \land (\mathrm{i} < \mathrm{next}) \land$$
$$\land\ (\mathrm{i} < \mathrm{len}) \land (\mathrm{read}(\mathrm{data}, \mathrm{i}) \doteq x) \land \neg(\mathrm{i} < \mathrm{len})$$

Satisfiability Proof System for `QF_AX`

The satisfiability proof system R_{AX} for $\mathcal{T}_{\text{AX}}$ *extends* the proof system R_{UF} for $\mathcal{T}_{\text{UF}}$ with the following rules:

$$\textbf{RIntro1}\ \frac{b \doteq \text{write}(a, i, v) \in \Gamma}{\Gamma := \Gamma, v \doteq \text{read}(b, i)}$$

$$\textbf{RIntro2}\ \frac{b \doteq \text{write}(a, i, v) \in \Gamma \qquad u \doteq \text{read}(x, j) \in \Gamma \qquad x \in \{a, b\}}{\Gamma := \Gamma, i \doteq j \qquad \Gamma := \Gamma, i \not\doteq j, u \doteq \text{read}(a, j), u \doteq \text{read}(b, j)}$$

$$\textbf{Ext}\ \frac{a \not\doteq b \in \Gamma \qquad a \text{ and } b \text{ are arrays}}{\Gamma := \Gamma, u \not\doteq v, u \doteq \text{read}(a, k), v \doteq \text{read}(b, k)}$$

- **RIntro1**: After writing v at index i, the reading at the same index i gives us back the value v.
- **RIntro2**: After writing v in a at index i, the reading from a or b at index j *splits* in two cases: (1) i equals j, (2) a and b have the same value u at position j.
- **Ext**: If two arrays a and b are distinct, they must differ at some index k.

Example Derivation in R_{AX}

$$\textsc{RIntro1}\ \frac{b \doteq \mathrm{write}(a,i,v) \in \Gamma}{\Gamma := \Gamma, v \doteq \mathrm{read}(b,i)} \qquad \textsc{Ext}\ \frac{a \not\doteq b \in \Gamma \qquad a \text{ and } b \text{ are arrays}}{\Gamma := \Gamma, u \not\doteq v, u \doteq \mathrm{read}(a,k), v \doteq \mathrm{read}(b,k)}$$

$$\textsc{RIntro2}\ \frac{b \doteq \mathrm{write}(a,i,v) \in \Gamma \qquad u \doteq \mathrm{read}(x,j) \in \Gamma \qquad x \in \{a,b\}}{\Gamma := \Gamma, i \doteq j \qquad \Gamma := \Gamma, i \not\doteq j, u \doteq \mathrm{read}(a,j), u \doteq \mathrm{read}(b,j)}$$

Example: Determine the satisfiability of $\{\mathrm{write}(a_1, i, \mathrm{read}(a_1, i)) \doteq \mathrm{write}(a_2, i, \mathrm{read}(a_2, i)), a_1 \not\doteq a_2\}$.

First, flatten the literals:

$$\{\mathrm{write}(a_1, i, \mathrm{read}(a_1, i)) \doteq \mathrm{write}(a_2, i, \mathrm{read}(a_2, i))\} \longrightarrow$$

$$\longrightarrow \{a_1' \doteq a_2', a_1' \doteq \mathrm{write}(a_1, i, \mathrm{read}(a_2, i)), a_2' \doteq \mathrm{write}(a_2, i, \mathrm{read}(a_1, i)), a_1 \not\doteq a_2\} \longrightarrow$$

$$\longrightarrow \{a_1' \doteq a_2', a_1' \doteq \mathrm{write}(a_1, i, v_2), v_2 \doteq \mathrm{read}(a_2, i), a_2' \doteq \mathrm{write}(a_2, i, v_1), v_1 \doteq \mathrm{read}(a_1, i), a_1 \not\doteq a_2\}$$

Example Derivation in R_{AX} [2]

1. $a_1' \doteq a_2', a_1' \doteq \mathrm{write}(a_1, i, v_2), v_2 \doteq \mathrm{read}(a_2, i), a_2' \doteq \mathrm{write}(a_2, i, v_1), v_1 \doteq \mathrm{read}(a_1, i), a_1 \not\doteq a_2$
2. (by Refl) $a_1 \doteq a_1$
3. (by Refl) $a_2 \doteq a_2$
4. (by Ext) $u_1 \not\doteq u_2, u_1 \doteq \mathrm{read}(a_1, n), u_2 \doteq \mathrm{read}(a_2, n)$
5. (by RIntro2) split

---

6. $i \doteq n$
7. (by Cong) $v_1 \doteq u_1$
8. (by Symm) $u_1 \doteq v_1$
9. (by Cong) $v_2 \doteq u_2$
10. (by RIntro1) $v_2 \doteq \mathrm{read}(a_1', i)$
11. (by RIntro1) $v_1 \doteq \mathrm{read}(a_2', i)$
12. (by Refl) $i \doteq i$
13. (by Cong) $v_1 \doteq v_2$
14. (by Trans) $u_1 \doteq u_2$
15. (by Contr) UNSAT

---

6. $i \not\doteq n, u_1 \doteq \mathrm{read}(a_1', n)$
7. (by RIntro2) split

---

8. $i \doteq n$
9. (by Contr) UNSAT

---

8. $i \not\doteq n, u_2 \doteq \mathrm{read}(a_2', n)$
9. (by Relf) $n \doteq n$
10. (by Cong) $u_1 \doteq u_2$
11. (by Contr) UNSAT

§6 Arithmetic

Theory of Real Arithmetic

Definition 26: The theory of *real arithmetic* $\mathcal{T}_{\mathrm{RA}}$ is defined by the signature $\Sigma^S_{\mathrm{RA}} = \{\texttt{Real}\}$, $\Sigma^F_{\mathrm{RA}} = \{+, -, \times, \leq\} \cup \{q \mid q \text{ is a decimal numeral}\}$ and the class of interpretations $\boldsymbol{M}_{\mathrm{RA}}$ that interpret `Real` as the set of *real numbers* $\mathbb{R}$, and the function symbols in the usual way.

Quantifier-free linear real arithmetic (`QF_LRA`) is the theory of *linear inequalities* over the reals, where $\times$ can only be used in the form of *multiplication by constants (decimal numerals)*.

Linear Programming

Definition 27: A *linear program* (LP) consists of:

1. An $m \times n$ matrix $\boldsymbol{A}$, the *contraint matrix*.
2. An m-dimensional vector $\boldsymbol{b}$.
3. An n-dimensional vector $\boldsymbol{c}$, the *objective function*.

Let $\boldsymbol{x}$ be a vector of n variables.

Goal: Find a solution $\boldsymbol{x}$ that *maximizes* $\boldsymbol{c}^T\boldsymbol{x}$ subject to the linear constraints $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ (and[3] $\boldsymbol{x} \geq \boldsymbol{0}$).

Note: All **bold**-styled symbols denote *vectors* or *matrices*, e.g., $\boldsymbol{x}$, $\boldsymbol{A}$, $\boldsymbol{0}$.

[3] The constraint $\boldsymbol{x} \geq \boldsymbol{0}$ is introduced when LP is expressed in *standard form*, explained later in these slides.

Example and Terminology

Example: Maximize $2x_2 - x_1$ subject to:

$$x_1 + x_2 \leq 3$$
$$2x_1 - x_2 \leq -5$$

Here, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Find $\boldsymbol{x}$ that maximizes $\boldsymbol{c}^T\boldsymbol{x}$ subject to $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$.

Definition 28: An assignment of $\boldsymbol{x}$ is a *feasible solution* if it satisfies $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$.

- Is $\boldsymbol{x} = \langle 0, 0 \rangle$ a feasible solution? ✗
- Is $\boldsymbol{x} = \langle -2, 1 \rangle$ a feasible solution? ✓

Definition 29: For a given assignment $\boldsymbol{x}$, the value $\boldsymbol{c}^T\boldsymbol{x}$ is the *objective value*, or *cost*, of $\boldsymbol{x}$.

- What is the objective value of $\boldsymbol{x} = \langle -2, 1 \rangle$?

Example and Terminology [2]

Definition 30: An *optimal solution* is a feasible solution with a *maximal* objective value among all feasible solutions.

Definition 31: If a linear program has no feasible solutions, it is *infeasible*.

Definition 32: The linear program is *unbounded* if the objective value of the optimal solution is ∞.

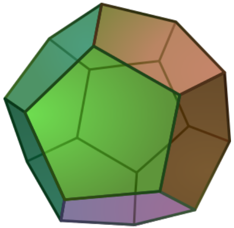
Geometric Interpretation

Definition 33: A *polytope* is a generalization of 3-dimensional polyhedra to higher dimensions.

Definition 34: A polytope P is *convex* if every point on the line segment connecting any two points in P is also within P.

Formally, for all $a, b \in \mathbb{R}^n \cap P$, and for all $\lambda \in [0; 1]$, it holds that $\lambda a + (1 - \lambda) b \in P$.

Note: For an $m \times n$ constraint matrix $\boldsymbol{A}$, the set of points $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$ forms a *convex polytope* in n-dimensional space.



LP goal: find a point $\boldsymbol{x}$ *inside the polytope* that maximizes $\boldsymbol{c}^T \boldsymbol{x}$ for a given $\boldsymbol{c}$.

Note: LP is *infeasible* iff the polytope is *empty*.

Note: LP is *unbounded* iff the polytope is *open* in the direction of the objective function.

Note: The *optimal solution* for a bounded LP lies on a *vertex* of the polytope.

Satisfiability as Linear Programming

Our goal is to use LP to check the satisfiability of *sets of linear $\mathcal{T}_{\mathrm{RA}}$-literals*.

Step 1: Convert equalities to inequalities.

- A linear $\mathcal{T}_{\mathrm{RA}}$-equality can be written to have the form $\boldsymbol{a}^T\boldsymbol{x} = \boldsymbol{b}$.
- We rewrite this further as $\boldsymbol{a}^T\boldsymbol{x} \geq \boldsymbol{b}$ and $\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b}$.
- And finally to $-\boldsymbol{a}^T\boldsymbol{x} \leq -\boldsymbol{b}$ and $\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b}$.

Step 2: Handle inequalities.

- A $\mathcal{T}_{\mathrm{RA}}$-literal of the form $\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b}$ is already in the desired form.
- A $\mathcal{T}_{\mathrm{RA}}$-literal of the form $\neg(\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b})$ is transformed as follows:

$$\neg(\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b}) \longrightarrow (\boldsymbol{a}^T\boldsymbol{x} > \boldsymbol{b}) \longrightarrow (-\boldsymbol{a}^T\boldsymbol{x} < -\boldsymbol{b}) \longrightarrow (-\boldsymbol{a}^T\boldsymbol{x} + y \leq -\boldsymbol{b}), (y > 0)$$

  where y is a fresh variable used for all negated inequalities.

  Example: $\neg(2x_1 - x_2 \leq 3)$ rewrites to $-2x_1 + x_2 + y \leq -3$, $y > 0$
- If there are no negated inequalities, add the inequality $y \leq 1$, where y is a fresh variable.

Satisfiability as Linear Programming [2]

- In either case, we end up with a set of the form $\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b} \cup \{y > 0\}$

Step 3: Check the satisfiability of $\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b} \cup \{y > 0\}$.

Encode it as LP: maximize y subject to $\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{b}$.

The final system is *satisfiable* iff the *optimal value* for y is *positive*.

Methods for Solving LP

- *Simplex* (Dantzig, 1947) — exponential time $\mathcal{O}(2^n)$
- *Ellipsoid* (Khachiyan, 1979) — polynomial time $\mathcal{O}(n^6)$
- *Projective* (Karmarkar, 1984) — polynomial time $\mathcal{O}(n^{3.5})$
- And many more tricky algorithms approaching $\mathcal{O}(n^{2.5})$

Note: Although the Simplex method is the *oldest* and the *least efficient in theory*, it can be implemented to be *quite efficient in practice*. It remains the most popular and we will focus on it next.

Standard Form

Any LP can be transformed to *standard form*:

$$
\begin{aligned}
\text{maximize} \sum_{j=1}^{n} c_j x_j \\
\text{such that} \sum_{j=1}^{m} a_{ij} x_j \leq b_i \text{ for } i = 1, \dots, m \\
x_j \geq 0 \text{ for } j = 1, \dots, n
\end{aligned}
$$

Example: Next, we are going to use the following running example LP:

$$
\text{maximize } 5x_1 + 4x_2 + 3x_3
$$

$$
\text{such that } \begin{cases} 2x_1 + 3x_2 + x_3 \leq 5 \\ 4x_1 + x_2 + 2x_3 \leq 11 \\ 3x_1 + 4x_2 + 2x_3 \leq 8 \\ x_1, x_2, x_3 \geq 0 \end{cases}
$$

Slack Variables

- Observe the first inequality: $2x_1 + 3x_2 + x_3 \le 5$
- Define a *new variable* to represent the *slack*:

$$x_4 = 5 - 2x_1 - 3x_2 - x_3, \quad x_4 \ge 0$$

- Do this for each constraint, so that everything becomes *equalities*.
- Define a new variable to represent the *objective value*: $z = 5x_1 + 4x_2 + 3x_3$

$$\boxed{\begin{aligned} &\max 5x_1 + 4x_2 + 3x_3 \\ &\text{s.t.} \begin{cases} 2x_1 + 3x_2 + x_3 \le 5 \\ 4x_1 + x_2 + 2x_3 \le 11 \\ 3x_1 + 4x_2 + 2x_3 \le 8 \\ x_1, x_2, x_3 \ge 0 \end{cases} \end{aligned}} \longrightarrow \boxed{\begin{aligned} &\max z \\ &\text{s.t.} \begin{cases} x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ x_5 = 11 - 4x_1 - x_2 - 2x_3 \\ x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \\ z = 5x_1 + 4x_2 + 3x_3 \\ x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{cases} \end{aligned}}$$

Note: Optimal solution remains optimal for the new problem.

The Simplex Strategy

- Start with a feasible solution.
  - For our example, assign 0 to all variables.

    $x_1 \mapsto 0, x_2 \mapsto 0, x_3 \mapsto 0$
  - Assign the introduced variables their computed values.

    $x_4 \mapsto 5, x_5 \mapsto 11, x_6 \mapsto 8, z \mapsto 0$
- Iteratively improve the objective value.
  - Go from $\boldsymbol{x}$ to $\boldsymbol{x}'$ only if $z(\boldsymbol{x}) \leq z(\boldsymbol{x}')$.

What can we improve here?

$$\begin{cases} x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ x_5 = 11 - 4x_1 - x_2 - 2x_3 \\ x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \\ z = 5x_1 + 4x_2 + 3x_3 \end{cases}$$

One option is to make x_1 larger, leaving x_2 and x_3 unchanged:

- $x_1 = 1 \quad \longrightarrow \quad x_4 = 3, x_5 = 7, x_6 = 1, z = 5$ ✓
- $x_1 = 2 \quad \longrightarrow \quad x_4 = 1, x_5 = 3, x_6 = 2, z = 10$ ✓
- $x_1 = 3 \quad \longrightarrow \quad x_4 = -1, \ldots$ ✗ *no longer feasible!*

The Simplex Strategy [2]

We can't increase x_1 *too much*. Let's increase it as much as possible, *without compromising feasibility*.

$$x_1 \mapsto 0, x_2 \mapsto 0, x_3 \mapsto 0$$

$$\begin{cases} x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ x_5 = 11 - 4x_1 - x_2 - 2x_3 \\ x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \\ z = 5x_1 + 4x_2 + 3x_3 \end{cases} \longrightarrow \begin{cases} x_1 \leq \frac{5}{2} \\ x_1 \leq \frac{11}{4} \\ x_1 \leq \frac{8}{3} \end{cases}$$

Select the *tightest bound*, $x_1 \leq \frac{5}{2}$.

- New assignment: $x_1 \mapsto \frac{5}{2}, x_2 \mapsto x_3 \mapsto x_4 \mapsto 0, x_5 \mapsto 1, x_6 \mapsto \frac{1}{2}, z \mapsto \frac{25}{2}$
- This indeed improves the objective value z.

The Simplex Strategy [3]

Current assignment:

- $x_1 \mapsto \frac{5}{2}, x_2 \mapsto x_3 \mapsto x_4 \mapsto 0, x_5 \mapsto 1, x_6 \mapsto \frac{1}{2}, z \mapsto \frac{25}{2}$

How do we continue?

For the first iteration we had:

- A *feasible solution*.
- An *equation system* where the variables with positive values are expressed in terms of variables with 0 value.

Does the current *equation system* satisfy this property? *No* ✗

$$\begin{cases} x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ x_5 = 11 - 4x_1 - x_2 - 2x_3 \\ x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \\ z = 5x_1 + 4x_2 + 3x_3 \end{cases}$$

The Simplex Strategy [4]

What should we change?

- Initially, x_1 was 0 and x_4 was positive.
- Now, x_1 is positive and x_4 is 0.

Isolate x_1 and *eliminate* it from right-hand-side:

- $x_4 = 5 - 2x_1 - 3x_2 - x_3 \quad \longrightarrow \quad x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4$

$$x_1 \mapsto \frac{5}{2}, x_2 \mapsto x_3 \mapsto x_4 \mapsto 0$$

$$\begin{cases} x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ x_5 = 11 - 4x_1 - x_2 - 2x_3 \\ x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \\ z = 5x_1 + 4x_2 + 3x_3 \end{cases}$$

$$\begin{cases} x_4 = 5 - 2x_1 - 3x_2 - x_3 \\ x_5 = 11 - 4x_1 - x_2 - 2x_3 \\ x_6 = 8 - 3x_1 - 4x_2 - 2x_3 \\ z = 5x_1 + 4x_2 + 3x_3 \end{cases} \longrightarrow \begin{cases} x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 = 1 + 5x_2 + \qquad + 2x_4 \\ x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \end{cases}$$

The Simplex Strategy [5]

How can we improve z further?

- **Option 1**: decrease x_2 or x_4, but we can't since $x_2, x_4 \geq 0$.
- **Option 2**: increase x_3. *By how much?*

x_3's bounds: $x_3 \leq 5$, $x_3 \leq \infty$, $x_3 \leq 1$.

We increase x_3 to its tightest bound 1.

- New assignment: $x_1 \mapsto 2, x_2 \mapsto 0, x_3 \mapsto 1, x_4 \mapsto 0, x_5 \mapsto 0, x_6 \mapsto 0$.
- This gives $z = 13$, which is again an improvement.

$$x_1 \mapsto \frac{5}{2}, x_2 \mapsto 0, x_3 \mapsto 0, x_4 \mapsto 0$$

$$\begin{cases} x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 = 1 + 5x_2 + \phantom{\frac{1}{2}x_3} + 2x_4 \\ x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \end{cases}$$

As before, we switch x_6 and x_3, and *eliminate* x_3 from the right-hand-side:

$$\boxed{\begin{cases} x_1 = \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 = 1 + 5x_2 + \phantom{\frac{1}{2}x_3} + 2x_4 \\ x_6 = \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ z = \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4 \end{cases}} \longrightarrow \boxed{\begin{cases} x_1 = 2 - 2x_2 - 2x_4 + x_6 \\ x_5 = 1 + 5x_2 + 2x_4 \\ x_3 = 1 + x_2 + 3x_4 - 2x_6 \\ z = 13 - 3x_2 - x_4 - x_6 \end{cases}}$$

The Simplex Strategy [6]

Can we improve z again?

- No, because $x_2, x_4, x_6 \geq 0$, and all *appear with negative signs* in the objective function.

So, we are done, and the optimal value of z is 13.

The optimal solution is then $x_1 \mapsto 2, x_2 \mapsto 0, x_3 \mapsto 1$.

$x_1 \mapsto 2, x_2 \mapsto 0, x_3 \mapsto 1,$

$x_4 \mapsto 0, x_6 \mapsto 0$

$$\begin{cases} x_1 = 2 - 2x_2 - 2x_4 + x_6 \\ x_5 = 1 + 5x_2 + 2x_4 \\ x_3 = 1 + x_2 + 3x_4 - 2x_6 \\ z = 13 - 3x_2 - x_4 - x_6 \end{cases}$$

The Simplex Algorithm

$$\text{maximize} \sum_{j=1}^{n} c_j x_j$$

$$\text{such that} \sum_{j=1}^{m} a_{ij} x_j \leq b_i \text{ for } i = 1, ..., m$$

$$x_j \geq 0 \text{ for } j = 1, ..., n$$

1. Introduce slack variables $x_{n+1}, ..., x_{n+m}$.
2. Set $x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, ..., m$.
3. Start with initial, *feasible* solution. (commonly, $x_1 \mapsto 0, ..., x_n \mapsto 0$)
4. While some summands in the current objective function have *positive coefficients*, update the feasible solution to improve the objective value. Otherwise, stop.
5. Update the equations to *maintain the invariant* that all right-hand-side values have value 0.
6. Go to 4.

§7 CDCL($\mathcal{T}$)

CDCL($\mathcal{T}$) Architecture

$$\text{CDCL}(\mathcal{T}) = \text{CDCL}(X) + \mathcal{T}\text{-solver}$$

CDCL(X):

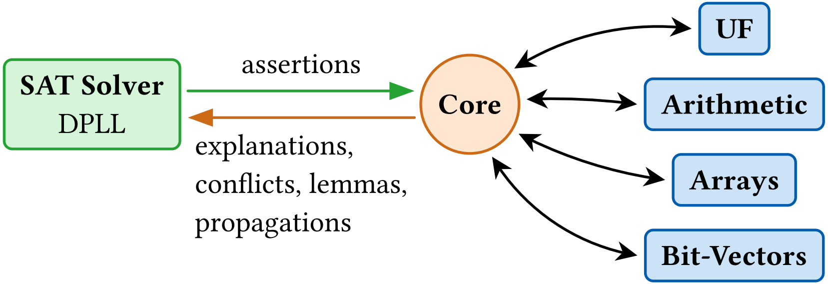
- Very *similar to a SAT solver*, enumerates Boolean models.
- Not allowed: pure literal rule (and other SAT specific heuristics).
- Required: incremental addition of clauses.
- Desirable: partial model detection.

$\mathcal{T}$-solver:

- Checks the $\mathcal{T}$-satisfiability of conjunctions of literals.
- Computes *theory propagations*.
- Produces *explanations* of $\mathcal{T}$-unsatisfiability/propagation.
- Must be *incremental* and *backtrackable*.

Typical SMT Solver Architecture

SAT Solver:
- Only sees *Boolean skeleton* of a problem.
- Builds *partial model* by assigning truth values to literals
- Sends these literals to the core as *assertions*



Theory Solvers:
- Check $\mathcal{T}$-satisfiability of sets of theory literals
- Incremental
- Backtrackable
- Conflict generation
- Theory propagation

Core:
- Sends each assertion to the appropriate theory
- Sends deduced literals to other theories/SAT solver
- Handles *theory combination*

§8 Combining Theories

Motivation

TODO

TODO

- [x] theory of arrays $\mathcal{T}_{\mathrm{A}}$
- [x] satisfiability proof system for $\mathcal{T}_{\mathrm{A}}$
- [x] example of derivation in R_{AX}
- [ ] soundness, termination, completeness of R_{AX}
- [ ] RDS solver
- [ ] Bit-vector solver
- [ ] String solver
- [x] LRA
- [x] Linear programming
- [ ] Simplex algorithm
- [ ] Combination of theories