Formal Methods in Software Engineering

Boolean Satisfiability — Spring 2025

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§1 Boolean Satisfiability

Boolean Satisfiability Problem (SAT)

SAT is the classical NP-complete problem of determining whether a given Boolean formula is *satisfiable*, that is, whether there exists an assignment of truth values to its variables that makes the formula true.

$$\exists X . f(X) = 1$$

SAT is a *decision* problem — the answer is either "yes" or "no". In practice, we are mainly interested in *finding* the actual satisfying assignment if it exists — this search problem is called *functional* SAT.

Cook–Levin Theorem

Historically, SAT was the first problem proven to be NP-complete, independently by Stephen Cook [1] and Leonid Levin [2] in 1971.

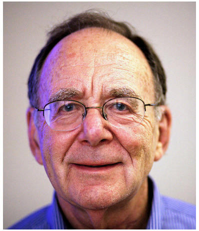
Theorem 1 (Cook–Levin): SAT is NP-complete.

That is, SAT is in NP, and *any* problem in NP can be *reduced* to SAT in polynomial time.

Cook's original proof was based on the concept of *Turing reductions* (now called *Cook reductions*). Richard Karp [3] later refined this with a stronger concept of polynomial-time *many-one reductions* (now known as *Karp reductions*), establishing the modern framework for NP-completeness.

Stephen Cook

Leonid Levin



Richard Karp

Karp Reduction

Definition 1 (Many-one reduction☒): A polynomial-time *many-one reduction* from problem A to B, denoted $A \leq_p B$, is a polynomial-time computable function f such that for every instance x of A, x is a "yes" instance of A if and only if $f(x)$ is a "yes" instance of B.

A reduction of this kind is also called a *polynomial transformation* or *Karp reduction*.

Proving the Cook–Levin Theorem

Proof (SAT ∈ NP): A satisfying assignment serves as a *certificate* verifiable in linear time. □

Proof of Theorem 1: *Any problem in NP can be reduced to SAT in polynomial time.*

A problem L is in NP if there exists a polynomial-time *verifier* (Turing machine) $V(x, c)$ that checks whether c is a valid certificate for $x \in L$.

A *Karp reduction* from L to SAT is a polynomial-time computable function mapping instances x of L to propositional formulas φ_x, such that φ_x is satisfiable iff $x \in L$.

Construct a formula φ_x encoding the computation of $V(x, c)$ for input x and certificate c.

- Variables encode the Turing machine state, tape cells, and head position at each step t.
- Clauses enforce the initial configuration (fixed input x, free variables for certificate c), valid transitions (per V's rules), and acceptance at step $T \in \mathcal{O}(p(|x|))$.

A satisfying assignment to φ_x corresponds to a valid certificate c causing $V(x, c)$ to accept, thus $x \in L$. □

This foundational result shows that SAT is a "universal" problem for NP.

Solving General Search Problems with SAT

SAT solvers are powerful tools for solving general search problems. Given a problem, we can encode it as a SAT instance and use a SAT solver to find a solution.

To model a search problem as a SAT instance, the general approach is as follows:

1. Define propositional variables to represent the problem's *state*.
2. Encode the problem's *constraints* using propositional formulas.
3. Translate the formulas into a *clausal form*.
4. Run a SAT solver to *find* a satisfying assignment or *prove* its non-existence.

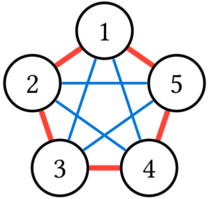
Example: Graph Coloring

Graph $G = (V, E)$ consists of a set V of vertices and a set E of edges, where each edge is an unordered pair of vertices. A complete graph on n vertices, denoted K_n, is a graph with $|V| = n$ such that E contains all possible pairs of vertices. In total, K_n has $\frac{n(n-1)}{2}$ edges.

Given a graph, color its vertices such that no two adjacent vertices have the same color.

Given a complete graph K_n, color its edges using k colors without creating a monochromatic triangle. What is the largest complete graph for which this is possible for a given number of colors?[1]

- For $k = 1$, the answer is $n = 2$.
  - The graph K_2 has only one edge, which can be colored with a single color.
- For $k = 2$, the answer is $n = 5$.
  - See the example of 2-colored K_5 on the right.
- For $k = 3$, the answer is $n = 16$.
  - This is the work for a SAT solver. See the next slides.



[1] For a more general case, see Ramsey's theorem

Modelling and Solving the Graph Coloring Example

1. *Describe states using propositional variables.*
   - A simple (*one-hot*, or *direct*) encoding uses three (as $k = 3$) variables for each edge: e_1, e_2, and e_3. There are $2^3 = 8$ possible combinations of values of three variables, but only 3 of them are *valid*.

2. *Describe constrains using propositional formulas.*
   - Each edge must be colored with exactly one color. For each edge e:

$$(e_1 \lor e_2 \lor e_3) \land \neg(e_1 \land e_2) \land \neg(e_1 \land e_3) \land \neg(e_2 \land e_3)$$

   - No monochromatic triangles are allowed. For each triangle (e, f, g):

$$\neg(e_1 \land f_1 \land g_1) \land \neg(e_2 \land f_2 \land g_2) \land \neg(e_3 \land f_3 \land g_3)$$

3. *Translate the formula into a clausal form.*

4. *Run a SAT solver to find a satisfying assignment or prove its non-existence.*

Now, perform this process for increasing values of n, and find the largest n for which the formula is satisfiable. The answer is $n = 16$ for $k = 3$.

Code for Graph Coloring Example

```
n = 17
k = 3
m = n * (n - 1) // 2

edges = {}
for u in range(1, n + 1):
    for v in range(u + 1, n + 1):
        edges[(u, v)] = len(edges) + 1

def color(e, c):
    return (e - 1) * k + c

clauses = []
for e in range(1, m + 1):
    # At least one color is assigned to edge e
    clauses.append([
      color(e, c) for c in range(1, k + 1)
    ])
    # At most one color is assigned to edge e
    for c1 in range(1, k + 1):
        for c2 in range(c1 + 1, k + 1):
```

```
            clauses.append([
              -color(e, c1), -color(e, c2)
            ])
# No mono-chromatic triangles
for v1 in range(1, n + 1):
    for v2 in range(v1 + 1, n + 1):
        for v3 in range(v2 + 1, n + 1):
            e12 = edges[(v1, v2)]
            e23 = edges[(v2, v3)]
            e13 = edges[(v1, v3)]
            for c in range(1, k + 1):
                clauses.append([
                  -color(e12, c),
                  -color(e23, c),
                  -color(e13, c)
                ])

print(f"p cnf {color(len(edges), k)}
{len(clauses)}")
for clause in clauses:
    print(" ".join(map(str, clause)) + " 0")
```

TODO

- □ Encodings
- □ SAT Solvers
- □ Applications
- □ Exercises

Bibliography

[1] S. A. Cook, “The complexity of theorem-proving procedures,” in *Proceedings of the Third Annual ACM Symposium on Theory of Computing*, 1971, pp. 151–158. doi: 10.1145/800157.805047.

[2] L. A. Levin, “Universal sequential search problems,” *Problemy Peredachi Informatsii*, vol. 9, no. 3, pp. 115–116, 1973, [Online]. Available: http://mi.mathnet.ru/ppi914

[3] R. M. Karp, “Reducibility among Combinatorial Problems,” *Complexity of Computer Computations.* Springer, pp. 85–103, 1972. doi: 10.1007/978-1-4684-2001-2_9.