

Formal Methods in Software Engineering

Boolean Satisfiability — Spring 2025

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§1 Boolean Satisfiability

Boolean Satisfiability Problem (SAT)

SAT is the classical NP-complete problem of determining whether a given Boolean formula is *satisfiable*, that is, whether there exists an assignment of truth values to its variables that makes the formula true.

$$\exists X. f(X) = 1$$

SAT is a *decision* problem — the answer is either “yes” or “no”. In practice, we are mainly interested in *finding* the actual satisfying assignment if it exists — this search problem is called *functional* SAT.

Cook–Levin Theorem

Historically, SAT was the first problem proven to be NP-complete, independently by Stephen Cook [1] and Leonid Levin [2] in 1971.

Theorem 1 (Cook–Levin): SAT is NP-complete.

That is, SAT is in NP, and *any* problem in NP can be *reduced* to SAT in polynomial time.

Cook's original proof was based on the concept of *Turing reductions* (now called *Cook reductions*).

Richard Karp [3] later refined this with a stronger concept of polynomial-time *many-one reductions* (now known as *Karp reductions*), establishing the modern framework for NP-completeness.



Stephen Cook



Leonid Levin



Richard Karp

Karp Reduction

Definition 1 (Many-one reduction_□): A polynomial-time *many-one reduction* from problem A to B , denoted $A \leq_p B$, is a polynomial-time computable function f such that for every instance x of A , x is a “yes” instance of A if and only if $f(x)$ is a “yes” instance of B .

A reduction of this kind is also called a *polynomial transformation* or *Karp reduction*.

Proving the Cook–Levin Theorem

Proof ($SAT \in NP$): A satisfying assignment serves as a *certificate* verifiable in linear time. □

Proof of Theorem 1: *Any problem in NP can be reduced to SAT in polynomial time.*

A problem L is in NP if there exists a polynomial-time *verifier* (Turing machine) $V(x, c)$ that checks whether c is a valid certificate for $x \in L$.

A *Karp reduction* from L to SAT is a polynomial-time computable function mapping instances x of L to propositional formulas φ_x , such that φ_x is satisfiable iff $x \in L$.

Construct a formula φ_x encoding the computation of $V(x, c)$ for input x and certificate c .

- Variables encode the Turing machine state, tape cells, and head position at each step t .
- Clauses enforce the initial configuration (fixed input x , free variables for certificate c), valid transitions (per V 's rules), and acceptance at step $T \in \mathcal{O}(p(|x|))$.

A satisfying assignment to φ_x corresponds to a valid certificate c causing $V(x, c)$ to accept, thus $x \in L$. □

This foundational result shows that SAT is a “universal” problem for NP.

Solving General Search Problems with SAT

SAT solvers are powerful tools for solving general search problems. Given a problem, we can encode it as a SAT instance and use a SAT solver to find a solution.

To model a search problem as a SAT instance, the general approach is as follows:

1. Define propositional variables to represent the problem's *state*.
2. Encode the problem's *constraints* using propositional formulas.
3. Translate the formulas into a *clausal form*.
4. Run a SAT solver to *find* a satisfying assignment or *prove* its non-existence.

Example: Graph Coloring

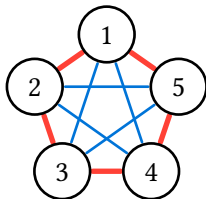
Graph $G = (V, E)$ consists of a set V of vertices and a set E of edges, where each edge is an unordered pair of vertices. A complete graph on n vertices, denoted K_n , is a graph with $|V| = n$ such that E contains all possible pairs of vertices. In total, K_n has $\frac{n(n-1)}{2}$ edges.

Given a graph, color its vertices such that no two adjacent vertices have the same color.

Given a complete graph K_n , color its edges using k colors without creating a monochromatic triangle.

What is the largest complete graph for which this is possible for a given number of colors?¹

- For $k = 1$, the answer is $n = 2$.
 - The graph K_2 has only one edge, which can be colored with a single color.
- For $k = 2$, the answer is $n = 5$.
 - See the example of 2-colored K_5 on the right.
- For $k = 3$, the answer is $n = 16$.
 - This is the work for a SAT solver. See the next slides.



¹For a more general case, see [Ramsey's theorem](#)

Modelling and Solving the Graph Coloring Example

1. Describe states using propositional variables.

- A simple (*one-hot*, or *direct*) encoding uses three (as $k = 3$) variables for each edge: e_1 , e_2 , and e_3 . There are $2^3 = 8$ possible combinations of values of three variables, but only 3 of them are *valid*.

2. Describe constraints using propositional formulas.

- Each edge must be colored with exactly one color. For each edge e :

$$(e_1 \vee e_2 \vee e_3) \wedge \neg(e_1 \wedge e_2) \wedge \neg(e_1 \wedge e_3) \wedge \neg(e_2 \wedge e_3)$$

- No monochromatic triangles are allowed. For each triangle (e, f, g) :

$$\neg(e_1 \wedge f_1 \wedge g_1) \wedge \neg(e_2 \wedge f_2 \wedge g_2) \wedge \neg(e_3 \wedge f_3 \wedge g_3)$$

3. Translate the formula into a clausal form.

4. Run a SAT solver to find a satisfying assignment or prove its non-existence.

Now, perform this process for increasing values of n , and find the largest n for which the formula is satisfiable. The answer is $n = 16$ for $k = 3$.

Code for Graph Coloring Example

```
n = 17
k = 3
m = n * (n - 1) // 2

edges = {}
for u in range(1, n + 1):
    for v in range(u + 1, n + 1):
        edges[(u, v)] = len(edges) + 1

def color(e, c):
    return (e - 1) * k + c

clauses = []
for e in range(1, m + 1):
    # At least one color is assigned to edge e
    clauses.append([
        color(e, c) for c in range(1, k + 1)
    ])
    # At most one color is assigned to edge e
    for c1 in range(1, k + 1):
        for c2 in range(c1 + 1, k + 1):
            clauses.append([
```

```
                -color(e, c1), -color(e, c2)
            ])
# No mono-chromatic triangles
for v1 in range(1, n + 1):
    for v2 in range(v1 + 1, n + 1):
        for v3 in range(v2 + 1, n + 1):
            e12 = edges[(v1, v2)]
            e23 = edges[(v2, v3)]
            e13 = edges[(v1, v3)]
            for c in range(1, k + 1):
                clauses.append([
                    -color(e12, c),
                    -color(e23, c),
                    -color(e13, c)
                ])

print(f"p cnf {color(len(edges), k)} {len(clauses)}")
for clause in clauses:
    print(" ".join(map(str, clause)) + " 0")
```

TODO

- ☐ Encodings
- ☐ SAT Solvers
- ☐ Applications
- ☐ Exercises

Bibliography

- [1] S. A. Cook, “The complexity of theorem-proving procedures,” in *Proceedings of the Third Annual ACM Symposium on Theory of Computing*, 1971, pp. 151–158. doi: [10.1145/800157.805047](https://doi.org/10.1145/800157.805047).
- [2] L. A. Levin, “Universal sequential search problems,” *Problemy Peredachi Informatsii*, vol. 9, no. 3, pp. 115–116, 1973, [Online]. Available: <http://mi.mathnet.ru/ppi914>
- [3] R. M. Karp, “Reducibility among Combinatorial Problems,” *Complexity of Computer Computations*. Springer, pp. 85–103, 1972. doi: [10.1007/978-1-4684-2001-2_9](https://doi.org/10.1007/978-1-4684-2001-2_9).