

# Formal Methods in Software Engineering

**Propositional Logic** — Spring 2025

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# §1 Propositional Logic

## Motivation

- Boolean functions are at the core of logic-based reasoning.
- A Boolean function  $F(X_1, \dots, X_n)$  describes the output of a system based on its inputs.
- Boolean gates (AND, OR, NOT) form the building blocks of digital circuits.
- Propositional logic formalizes reasoning about Boolean functions and circuits.
- **Applications:**
  - Digital circuit design.
  - Verification and synthesis of hardware and software.
  - Expressing logical constraints in AI and optimization problems.
  - Automated reasoning and theorem proving.

# Boolean Circuits and Propositional Logic

**Boolean circuit** is a directed acyclic graph (DAG) of Boolean gates.

- Inputs: Propositional variables.
- Outputs: Logical expressions describing the circuit's behavior.

*“Can the output of a circuit ever be true?”*

- Propositional logic provides a formal framework to answer such questions.

**Real-world examples:**

- Error detection circuits.
- Arithmetic logic units (ALUs) in processors.
- Routing logic in network devices.

# What is Logic?

A formal logic is defined by its **syntax** and **semantics**.

## □ **Syntax**

- An **alphabet**  $\Sigma$  is a set of symbols.
- A finite sequence of symbols (from  $\Sigma$ ) is called an **expression** or **string** (over  $\Sigma$ ).
- A set of rules defines the **well-formed** expressions.

## □ **Semantics**

- Gives meaning to (well-formed) expressions.

# Syntax of Propositional Logic

## □ Alphabet

1. Logical connectives:  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ .
2. Propositional variables:  $A_1, A_2, \dots, A_n$ .
3. Parentheses for grouping:  $(, )$ .

## □ Well-Formed Formulas (WFFs)

Valid (**well-formed**) expressions are defined **inductively**:

1. A single propositional symbol (e.g.  $A$ ) is a WFF.
2. If  $\alpha$  and  $\beta$  are WFFs, so are:  $\neg\alpha$ ,  $(\alpha \wedge \beta)$ ,  $(\alpha \vee \beta)$ ,  $(\alpha \rightarrow \beta)$ ,  $(\alpha \leftrightarrow \beta)$ .
3. No other expressions are WFFs.

## Syntax of Propositional Logic [2]

### □ Conventions

- Large variety of propositional variables:  $A, B, C, \dots, p, q, r, \dots$
- Outer parentheses can be omitted:  $A \wedge B$  instead of  $(A \wedge B)$ .
- Operator precedence:  $\neg > \wedge > \vee > \rightarrow > \leftrightarrow$ .
- Left-to-right associativity for  $\wedge$  and  $\vee$ :  $A \wedge B \wedge C = (A \wedge B) \wedge C$ .
- Right-to-left associativity for  $\rightarrow$ :  $A \rightarrow B \rightarrow C = A \rightarrow (B \rightarrow C)$ .

## Semantics of Propositional Logic

- Each propositional variable is assigned a truth value:  $T$  (true) or  $F$  (false).
- More formally, *interpretation*  $\nu : V \rightarrow \{0, 1\}$  assigns truth values to all variables (atoms).
- Truth values of complex formulas are computed (evaluated) recursively:
  1.  $\llbracket p \rrbracket_\nu \triangleq \nu(p)$ , where  $p \in V$  is a propositional variable
  2.  $\llbracket \neg \alpha \rrbracket_\nu \triangleq 1 - \llbracket \alpha \rrbracket_\nu$
  3.  $\llbracket \alpha \wedge \beta \rrbracket_\nu \triangleq \min(\llbracket \alpha \rrbracket_\nu, \llbracket \beta \rrbracket_\nu)$
  4.  $\llbracket \alpha \vee \beta \rrbracket_\nu \triangleq \max(\llbracket \alpha \rrbracket_\nu, \llbracket \beta \rrbracket_\nu)$
  5.  $\llbracket \alpha \rightarrow \beta \rrbracket_\nu \triangleq (\llbracket \alpha \rrbracket_\nu \leq \llbracket \beta \rrbracket_\nu) = \max(1 - \llbracket \alpha \rrbracket_\nu, \llbracket \beta \rrbracket_\nu)$
  6.  $\llbracket \alpha \leftrightarrow \beta \rrbracket_\nu \triangleq (\llbracket \alpha \rrbracket_\nu = \llbracket \beta \rrbracket_\nu) = 1 - |\llbracket \alpha \rrbracket_\nu - \llbracket \beta \rrbracket_\nu|$



## §2 Foundations

## Truth Tables

$\alpha$	$\beta$	$\gamma$	$\alpha \wedge (\beta \vee \neg \gamma)$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

## Normal Forms

- **Conjunctive Normal Form (CNF):**

- A formula is in CNF if it is a conjunction of *clauses* (disjunctions of literals).

*Example:*  $(A \vee B) \wedge (\neg A \vee C) \wedge (B \vee \neg C)$  – CNF with 3 clauses.

- **Disjunctive Normal Form (DNF):**

- A formula is in DNF if it is a disjunction of *cubes* (conjunctions of literals).

*Example:*  $(\neg A \wedge B) \vee (B \wedge C) \vee (\neg A \wedge B \wedge \neg C)$  – DNF with 3 cubes.

- **Algebraic Normal Form (ANF):**

- A formula is in ANF if it is a sum of *products* of variables (or a constant 1).

*Example:*  $B \oplus AB \oplus ABC$  – ANF with 3 terms.

# Logical Laws and Tautologies

- **Associative and Commutative** laws for  $\wedge$ ,  $\vee$ ,  $\leftrightarrow$ :
  - ▶  $A \circ (B \circ C) \equiv (A \circ B) \circ C$
  - ▶  $A \circ B \equiv B \circ A$
- **Distributive laws:**
  - ▶  $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$
  - ▶  $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$
- **Negation:**
  - ▶  $\neg\neg A \equiv A$
- **De Morgan's laws:**
  - ▶  $\neg(A \wedge B) \equiv \neg A \vee \neg B$
  - ▶  $\neg(A \vee B) \equiv \neg A \wedge \neg B$

## Logical Laws and Tautologies [2]

- **Implication:**

- $(A \rightarrow B) \equiv (\neg A \vee B)$

- **Contraposition:**

- $(A \rightarrow B) \equiv (\neg B \rightarrow \neg A)$

- **Law of Excluded Middle:**

- $(A \vee \neg A) \equiv \top$

- **Contradiction:**

- $(A \wedge \neg A) \equiv \perp$

- **Exportation:**

- $((A \wedge B) \rightarrow C) \equiv (A \rightarrow (B \rightarrow C))$

## Completeness of Connectives

- All Boolean functions can be expressed using  $\{\neg, \wedge, \vee\}$  (so called “*standard Boolean basis*”).
- Even smaller sets are sufficient:
  - $\{\neg, \wedge\}$  – AIG (And-Inverter Graph), see also: AIGER format.
  - $\{\neg, \vee\}$
  - $\{\overline{\wedge}\}$  – NAND
  - $\{\overline{\vee}\}$  – NOR

## Incompleteness of Connectives

To prove that a set of connectives is incomplete, we find a property that is true for all WFFs expressed using those connectives, but that is not true for some Boolean function.

*Example:*  $\{\wedge, \rightarrow\}$  is not complete.

*Proof:* Let  $\alpha$  be a WFF which uses only these connectives. Let  $\nu$  be an interpretation such that  $\nu(A_i) = 1$  for all propositional variables  $A_i$ . Next, we prove by induction that  $\llbracket \alpha \rrbracket_\nu = 1$ .

- Base case:
  - $\llbracket A_i \rrbracket_\nu = \nu(A_i) = 1$
- Inductive step:
  - $\llbracket \beta \wedge \gamma \rrbracket_\nu = \min(\llbracket \beta \rrbracket_\nu, \llbracket \gamma \rrbracket_\nu) = 1$
  - $\llbracket \beta \rightarrow \gamma \rrbracket_\nu = \max(1 - \llbracket \beta \rrbracket_\nu, \llbracket \gamma \rrbracket_\nu) = 1$

Thus,  $\llbracket \alpha \rrbracket_\nu = 1$  for all WFFs  $\alpha$  built from  $\{\wedge, \rightarrow\}$ . However,  $\llbracket \neg A_1 \rrbracket_\nu = 0$ , so there is no such formula  $\alpha$  tautologically equivalent to  $\neg A_1$ . □

## §3 Semantical Aspects



# Validity, Satisfiability, Entailment

## □ Validity

- $\alpha$  is a **tautology** if  $\alpha$  is true under all truth assignments.

Formally,  $\alpha$  is **valid**, denoted “ $\models \alpha$ ”, iff  $\llbracket \alpha \rrbracket_\nu = 1$  for all interpretations  $\nu \in \{0, 1\}^V$ .

- $\alpha$  is a **contradiction** if  $\alpha$  is false under all truth assignments.

Formally,  $\alpha$  is **unsatisfiable** if  $\llbracket \alpha \rrbracket_\nu = 0$  for all interpretations  $\nu \in \{0, 1\}^V$ .

## □ Satisfiability

- $\alpha$  is **satisfiable (consistent)** if there exists an interpretation  $\nu \in \{0, 1\}^V$  where  $\llbracket \alpha \rrbracket_\nu = 1$ .

When  $\alpha$  is satisfiable by  $\nu$ , denoted  $\nu \models \alpha$ , this interpretation is called a **model** of  $\alpha$ .

- $\alpha$  is **falsifiable (invalid)** if there exists an interpretation  $\nu \in \{0, 1\}^V$  where  $\llbracket \alpha \rrbracket_\nu = 0$ .

## □ Entailment

- Let  $\Gamma$  be a set of WFFs. Then  $\Gamma$  **tautologically implies (semantically entails)**  $\alpha$ , denoted  $\Gamma \models \alpha$ , if every truth assignment that satisfies all formulas in  $\Gamma$  also satisfies  $\alpha$ .
- Formally,  $\Gamma \models \alpha$  iff for all interpretations  $\nu \in \{0, 1\}^V$  and formulas  $\beta \in \Gamma$ , if  $\nu \models \beta$ , then  $\nu \models \alpha$ .
- Note:  $\alpha \models \beta$ , where  $\alpha$  and  $\beta$  are WFFs, is just a shorthand for  $\{\alpha\} \models \beta$ .

## Implication vs Entailment

The **implication** operator ( $\rightarrow$ ) is a syntactic construct, while **entailment** ( $\models$ ) is a semantical relation.

They are related as follows:  $\alpha \rightarrow \beta$  is valid iff  $\alpha \models \beta$ .

*Example:*  $A \rightarrow (A \vee B)$  is valid (a tautology), and  $A \models A \vee B$

$A$	$B$	$A \vee B$	$A \rightarrow (A \vee B)$	$A \models A \vee B$
0	0	0	1	—
0	1	1	1	—
1	0	1	1	OK
1	1	1	1	OK

## Examples

- $A \vee B \wedge (\neg A \wedge \neg B)$  is satisfiable, but not valid.
- $A \vee B \wedge (\neg A \wedge \neg B) \wedge (A \leftrightarrow B)$  is unsatisfiable.
- $\{A \rightarrow B, A\} \models B$
- $\{A, \neg A\} \models A \wedge \neg A$
- $\neg(A \wedge B)$  is tautologically equivalent to  $\neg A \vee \neg B$ .

## Duality of SAT vs VALID

- **SAT:** Given a formula  $\alpha$ , determine if it is satisfiable.

$$\exists \nu. \llbracket \alpha \rrbracket_\nu$$

- **VALID:** Given a formula  $\alpha$ , determine if it is valid.

$$\forall \nu. \llbracket \alpha \rrbracket_\nu$$

- **Duality:**  $\alpha$  is valid iff  $\neg\alpha$  is unsatisfiable.
- Note: SAT is NP, but VALID is co-NP.

## Solving SAT using Truth Tables

### **Algorithm for satisfiability:**

To check whether  $\alpha$  is satisfiable, construct a truth table for  $\alpha$ . If there is a row where  $\alpha$  evaluates to true, then  $\alpha$  is satisfiable. Otherwise,  $\alpha$  is unsatisfiable.

### **Algorithm for semantical entailment (tautological implication):**

To check whether  $\{\alpha_1, \dots, \alpha_k\} \models \beta$ , check the satisfiability of  $(\alpha_1 \wedge \dots \wedge \alpha_k) \wedge (\neg\beta)$ . If it is unsatisfiable, then  $\{\alpha_1, \dots, \alpha_k\} \models \beta$ . Otherwise,  $\{\alpha_1, \dots, \alpha_k\} \not\models \beta$ .

## Compactness

Recall:

- A WFF  $\alpha$  is **satisfiable** if there exists an interpretation  $\nu$  such that  $\nu \models \alpha$ .
- Hereinafter, let  $\Gamma$  denote a *finite* set of WFFs, and  $\Sigma$  denote a *possibly infinite* set of WFFs.
- A set of WFFs  $\Sigma$  is **satisfiable** if there exists an interpretation  $\nu$  that satisfies all formulas in  $\Sigma$ .
- A set of WFFs  $\Sigma$  is **finitely satisfiable** if every finite subset of  $\Sigma$  is satisfiable.

**Theorem 1** (Compactness Theorem): A set of WFFs  $\Sigma$  is satisfiable iff it is finitely satisfiable.

*Proof* ( $\Rightarrow$ ): Suppose  $\Sigma$  is satisfiable, i.e. there exists an interpretation  $\nu$  that satisfies all formulas in  $\Sigma$ .

This direction is trivial: any subset of a satisfiable set is clearly satisfiable.

- For each finite subset  $\Sigma' \subseteq \Sigma$ ,  $\nu$  also satisfies all formulas in  $\Sigma'$ .
- Thus, every finite subset of  $\Sigma$  is satisfiable.

□

## Compactness [2]

*Proof* ( $\Leftarrow$ ): Suppose  $\Sigma$  is finitely satisfiable, i.e. every finite subset of  $\Sigma$  is satisfiable.

Construct a *maximal* finitely satisfiable set  $\Delta$  as follows:

- Let  $\alpha_1, \dots, \alpha_n, \dots$  be a fixed enumeration of all WFFs.
  - *This is possible since the set of all sequences of a countable set is countable.*
- Then, let:

$$\Delta_0 = \Sigma,$$
$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{\alpha_{n+1}\} & \text{if this is finitely satisfiable,} \\ \Delta_n \cup \{\neg\alpha_{n+1}\} & \text{otherwise.} \end{cases}$$

- *Note that each  $\Delta_n$  is finitely satisfiable by construction.*

## Compactness [3]

- Let  $\Delta = \bigcup_{n \in \mathbb{N}} \Delta_n$ . Note:
  1.  $\Sigma \subseteq \Delta$
  2.  $\alpha \in \Delta$  or  $\neg\alpha \in \Delta$  for any WFF  $\alpha$
  3.  $\Delta$  is finitely satisfiable by construction.

Now we need to show that  $\Delta$  is satisfiable (and thus  $\Sigma \subseteq \Delta$  is also satisfiable).

Define an interpretation  $\nu$  as follows: for each propositional variable  $p$ , let  $\nu(p) = 1$  iff  $p \in \Delta$ .

We claim that  $\nu \models \alpha$  iff  $\alpha \in \Delta$ . The proof is by induction on well-formed formulas.

- Base case:
  - Suppose  $\alpha \equiv p$  for some propositional variable  $p$ .
  - By definition,  $\llbracket p \rrbracket_\nu = \nu(p) = 1$ .
- Inductive step:
  - (*Note: we consider only two cases:  $\neg$  and  $\wedge$ , since they form a complete set of connectives.*)
  - Suppose  $\alpha \equiv \neg\beta$ .
    - $\llbracket \alpha \rrbracket_\nu = 1$  iff  $\llbracket \beta \rrbracket_\nu = 0$  iff  $\beta \notin \Delta$  iff  $\neg\beta \in \Delta$  iff  $\alpha \in \Delta$ .



## Compactness [4]

- ▶ Suppose  $\alpha \equiv \beta \wedge \gamma$ .
  - $\llbracket \alpha \rrbracket_\nu = 1$  iff both  $\llbracket \beta \rrbracket_\nu = 1$  and  $\llbracket \gamma \rrbracket_\nu = 1$  iff both  $\beta \in \Delta$  and  $\gamma \in \Delta$ .
  - If both  $\beta$  and  $\gamma$  are in  $\Delta$ , then  $\beta \wedge \gamma$  is in  $\Delta$ , thus  $\alpha \in \Delta$ .
    - Why? Because if  $\beta \wedge \gamma \notin \Delta$ , then  $\neg(\beta \wedge \gamma) \in \Delta$ . But then  $\{\beta, \gamma, \neg(\beta \wedge \gamma)\}$  is a finite subset of  $\Delta$  that is not satisfiable, which is a contradiction of  $\Delta$  being finitely satisfiable.
  - Similarly, if either  $\beta \notin \Delta$  or  $\gamma \notin \Delta$ , then  $\beta \wedge \gamma \notin \Delta$ , thus  $\alpha \notin \Delta$ .
    - Why? Again, suppose  $\beta \wedge \gamma \in \Delta$ . Since  $\beta \notin \Delta$  or  $\gamma \notin \Delta$ , at least one of  $\neg\beta$  or  $\neg\gamma$  is in  $\Delta$ . Wlog, assume  $\neg\beta \in \Delta$ . Then,  $\{\neg\beta, \beta \wedge \gamma\}$  is a finite subset of  $\Delta$  that is not satisfiable, which is a contradiction of  $\Delta$  being finitely satisfiable.
  - Thus,  $\llbracket \alpha \rrbracket_\nu = 1$  iff  $\alpha \in \Delta$ .

This shows that  $\llbracket \alpha \rrbracket_\nu = 1$  iff  $\alpha \in \Delta$ , thus  $\Delta$  is satisfiable by  $\nu$ . □

## Compactness [5]

**Corollary 1.1:** If  $\Sigma \models \alpha$ , then there is a finite  $\Sigma_0 \subseteq \Sigma$  such that  $\Sigma_0 \models \alpha$ .

*Proof:* Suppose that  $\Sigma_0 \not\models \alpha$  for every finite  $\Sigma_0 \subseteq \Sigma$ .

Then,  $\Sigma_0 \cup \{\neg\alpha\}$  is satisfiable for every finite  $\Sigma_0 \subseteq \Sigma$ , that is,  $\Sigma \cup \{\neg\alpha\}$  is finitely satisfiable.

Then, by the compactness theorem,  $\Sigma \cup \{\neg\alpha\}$  is satisfiable, thus  $\Sigma \not\models \alpha$ , which contradicts the theorem assumption that  $\Sigma \models \alpha$ . □

## §4 Proof Systems

# Natural Deduction

- **Natural deduction** is a proof system for propositional logic.
- **Axioms:**
  - **No axioms.**
- **Rules:**
  - **Introduction:**  $\wedge$ -introduction,  $\vee$ -introduction,  $\rightarrow$ -introduction,  $\neg$ -introduction.
  - **Elimination:**  $\wedge$ -elimination,  $\vee$ -elimination,  $\rightarrow$ -elimination,  $\neg$ -elimination.
  - **Reduction ad Absurdum**
  - **Law of Excluded Middle** (note: forbidden in *intuitionistic* logic)
- **Proofs** are constructed by applying rules to assumptions and previously derived formulas.

$$\underbrace{A_1, \dots, A_n \vdash A}_{\text{sequent}}$$

$$\frac{\Gamma_1 \vdash (\text{premise 1}) \quad \Gamma_2 \vdash (\text{premise 2}) \quad \dots}{\Gamma \vdash (\text{conclusion})} \text{ rule name}$$

## Inference Rules

$$\frac{}{\Gamma \vdash \varphi \vee \neg \varphi}$$
law of excluded middle

$$\frac{}{\Gamma, \varphi \vdash \varphi}$$
assumption

$$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \neg \alpha}{\Gamma \vdash \beta}$$
reduction ad absurdum

$$\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \alpha}$$
 $\wedge$ -elimination

$$\frac{\Gamma \vdash \alpha \wedge \beta}{\Gamma \vdash \beta}$$
 $\wedge$ -elimination

$$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \beta}{\Gamma \vdash \alpha \wedge \beta}$$
 $\wedge$ -introduction

$$\frac{\Gamma \vdash \alpha_1 \vee \alpha_2 \quad \Gamma, \alpha_1 \vdash \beta \quad \Gamma, \alpha_2 \vdash \beta}{\Gamma \vdash \beta}$$
 $\vee$ -elim

$$\frac{\Gamma \vdash \alpha}{\Gamma \vdash \alpha \vee \beta}$$
 $\vee$ -intro

$$\frac{\Gamma \vdash \beta}{\Gamma \vdash \alpha \vee \beta}$$
 $\vee$ -intro

$$\frac{\Gamma \vdash \alpha \quad \Gamma \vdash \alpha \rightarrow \beta}{\Gamma \vdash \beta}$$
 $\rightarrow$ -elimination

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \rightarrow \beta}$$
 $\rightarrow$ -introduction

## Example Derivation

Example:  $\underbrace{p \wedge q, r}_{\text{premises}} \vdash \underbrace{q \wedge r}_{\text{conclusion}}$

**Proof tree:**

$$\frac{\frac{\overline{p \wedge q}}{q} \wedge e \quad \frac{\overline{r}}{r}}{q \wedge r} \wedge i$$

**Linear proof (Fitch notation):**

1.  $p \wedge q$     **premise**
2.  $r$     **premise**
3.  $q$      **$\wedge e$  1**
4.  $q \wedge r$      **$\wedge i$  2,3**

## Exercises

1.  $\vdash (b \rightarrow c) \rightarrow ((\neg b \rightarrow \neg a) \rightarrow (a \rightarrow c))$
2.  $a \vee b \vdash b \vee a$
3.  $a \rightarrow c, b \rightarrow c, a \vee b \vdash c$
4.  $\neg a \vee b \vdash a \rightarrow b$
5.  $a \rightarrow b \vdash \neg a \vee b$
6.  $a \rightarrow b, a \rightarrow \neg b \vdash \neg a$
7.  $\neg p \rightarrow \perp \vdash p$  (with allowed  $\neg\neg$ E)
8.  $\vdash p \vee \neg p$
9.  $a \vee b, b \vee c, \neg b \vdash a \wedge c$
10.  $a \vee (b \rightarrow a) \vdash \neg a \rightarrow \neg b$
11.  $p \rightarrow \neg p \vdash \neg p$
12.  $a \rightarrow b, \neg b \vdash \neg a$
13.  $((a \rightarrow b) \rightarrow a) \rightarrow a$
14.  $\neg a \rightarrow \neg b \vdash b \rightarrow a$
15.  $\vdash (a \rightarrow b) \vee (b \rightarrow a)$

## Soundness and Completeness

- A formal system is **sound** if every provable formula is true in all models.

- ▶ **Weak soundness:** “every provable formula is a tautology”.

If  $\vdash \alpha$ , then  $\models \alpha$ .

- ▶ **Strong soundness:** “every derivable (from  $\Gamma$ ) formula is a logical consequence (of  $\Gamma$ )”.

If  $\Gamma \vdash \alpha$ , then  $\Gamma \models \alpha$ .

- A formal system is **complete** if every formula true in all models is provable.

- ▶ **Weak completeness:** “every tautology is provable”.

If  $\models \alpha$ , then  $\vdash \alpha$ .

- ▶ **Strong completeness:** “every logical consequence (of  $\Gamma$ ) is derivable (from  $\Gamma$ )”.

If  $\Gamma \models \alpha$ , then  $\Gamma \vdash \alpha$ .



# TODO

- ☒ Normal forms
- ☐ Canonical normal forms
- ☐ BDDs
- ☒ Natural deduction
- ☐ Sequent calculus
- ☐ Fitch notation
- ☐ Proof checkers
- ☐ Proof assistants
- ☐ Automatic theorem provers
- ☐ Abstract proof systems
- ☐ Intuitionistic logic
- ☒ Soundness and completeness
- ☐ Proof of soundness
- ☐ Proof of completeness