Formal Methods in Software Engineering

Normal Forms — Spring 2025

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§1 Normal Forms

Normal Forms in Propositional Logic

Definition 1 (Normal form): A *normal form* is a standardized syntactic representation of logical formulas with a *restricted* structure.

Normal forms enable efficient reasoning, simplification, and decision procedures, making them essential in automated theorem proving, model checking, and logic synthesis.

There are several *normal forms* commonly used in propositional logic:

- Negation normal form (NNF)
- Conjunctive normal form (CNF)
- Disjunctive normal form (DNF)
- Algebraic normal form (ANF)
- Binary decision diagram (BDD)

Each normal form has its own advantages and disadvantages, and is used in different contexts.

Every propositional formula can be converted to an *equivalent* formula in any of these normal forms.

Negation Normal Form

Definition 2 (Negation Normal Form (NNF)): A formula is in *negation normal form* if the negation operator ($\neg$) is only applied to variables, and the only allowed logical connectives are $\land$ and $\lor$.

Example: The formula $(p \land q) \lor (\neg p \land \neg q)$ is in NNF.

Example: The formula $\neg(p \land q) \lor (\neg p \land \neg q)$ is *not* in NNF due to $\neg(\ldots)$.

Grammar for NNF formulas:

$$\begin{aligned} \langle \text{Atom} \rangle &::= \top \mid \bot \mid \langle \text{Variable} \rangle \\ \langle \text{Literal} \rangle &::= \langle \text{Atom} \rangle \mid \neg \langle \text{Atom} \rangle \\ \langle \text{Formula} \rangle &::= \langle \text{Literal} \rangle \mid \langle \text{Formula} \rangle \land \langle \text{Formula} \rangle \mid \langle \text{Formula} \rangle \lor \langle \text{Formula} \rangle \end{aligned}$$

Literals

Definition 3 (Literal): A *literal* is a propositional variable or its negation.

- p is a *positive literal.*
- $\neg p$ is a *negative literal.*

Definition 4 (Complement): The *complement* of a literal p is denoted by $\overline{p}$.

$$\overline{p} = \begin{cases} \neg p \text{ if } p \text{ is positive} \\ p \text{ if } p \text{ is negative} \end{cases}$$

Note: *complementary* literals p and $\overline{p}$ are each other's completement.

NNF Transformation

Any propositional formula can be converted to NNF by the repeated application of the following rewriting rules ($\Longrightarrow$) to the formula and its sub-formulas, to completion (until none apply):

Description	**Rewrite rule**
Eliminate implications	$(A \to B) \Longrightarrow (\neg A \vee B)$
Eliminate bi-implications	$(A \leftrightarrow B) \Longrightarrow (\neg A \vee B) \wedge (A \vee \neg B)$
Push negation inside conjunctions	$\neg(A \wedge B) \Longrightarrow (\neg A \vee \neg B)$
Push negation inside disjunctions	$\neg(A \vee B) \Longrightarrow \neg A \wedge \neg B$
Eliminate double negations	$\neg\neg A \Longrightarrow A$

Theorem 1: Every well-formed formula not containing $\leftrightarrow$ can be converted to an *equivalent* NNF with a *linear increase* in the size[1] of the formula.

[1]For example, number of variable occurences, or number of sub-formulas.

Exponential Blowup of NNF

The NNF of formulas containing $\leftrightarrow$ can grow *exponentially* in size.

Example: Let's convert the following formula to NNF…

$$
\begin{aligned}
F &= a \leftrightarrow (b \leftrightarrow (c \leftrightarrow d)) \Longrightarrow \\
&= a \leftrightarrow (b \leftrightarrow ((c \rightarrow d) \land (d \rightarrow c))) \Longrightarrow \\
&= a \leftrightarrow ((b \rightarrow ((c \rightarrow d) \land (d \rightarrow c))) \land (((c \rightarrow d) \land (d \rightarrow c)) \rightarrow b)) \Longrightarrow \\
&= a \leftrightarrow ((b \lor (\ldots)) \land (\neg(\ldots) \lor b)) \Longrightarrow \\
&= (\neg a \lor (\ldots)) \land (a \lor \neg(\ldots)) \Longrightarrow \\
&= (\neg a \lor ((b \lor (\ldots)) \land (\neg(\ldots) \lor b))) \land \\
&\quad\; (a \lor \neg((b \lor (\ldots)) \land (\neg(\ldots) \lor b)))
\end{aligned}
$$

The original F contains only 4 variable occurences, while the NNF of F contains 16 variable occurences.

Disjunctive Normal Form

Definition 5 (Disjunctive Normal Form (DNF)): A formula is said to be in *disjunctive normal form* if it is a disjunction of *cubes* (conjunctions of literals).

$$A = \bigvee_i \bigwedge_j p_{ij}$$

Example: $A = (p \land q) \lor (\neg p \land q \land r) \lor \neg q$

Grammar for DNF formulas:

$$\begin{aligned}
\langle \text{Atom} \rangle &::= \top \mid \bot \mid \langle \text{Variable} \rangle \\
\langle \text{Literal} \rangle &::= \langle \text{Atom} \rangle \mid \neg \langle \text{Atom} \rangle \\
\langle \text{Cube} \rangle &::= \langle \text{Literal} \rangle \mid \langle \text{Literal} \rangle \land \langle \text{Cube} \rangle \\
\langle \text{Formula} \rangle &::= \langle \text{Cube} \rangle \mid \langle \text{Cube} \rangle \lor \langle \text{Formula} \rangle
\end{aligned}$$

Cubes and Clauses

Definition 6 (Cube): A *cube* is a conjunction of literals.

Definition 7 (Clause): A *clause* is a disjunction of literals.

- An *empty clause* is a clause with no literals, commonly denoted by $\square$.
- A *unit clause* is a clause with a single literal, that is, just a literal itself.
- A *Horn clause* is a clause with at most one positive literal.

Note: $\square$ is *false in every interpretation*, that is, unsatisfiable.

Conjunctive Normal Form

Definition 8 (Conjunctive Normal Form (CNF)): A formula is said to be in *conjunctive normal form* if it is a conjunction of *clauses*.

$$A = \bigwedge_i \bigvee_j p_{ij}$$

Example: $A = (\neg p \vee q) \wedge (\neg p \vee q \vee r) \wedge \neg q$

Satisfiability on CNF

An interpretation ν satisfies a clause $C = p_1 \vee \ldots \vee p_n$ if it satisfies some (at least one) literal p_k in C.

An interpretation ν satisfies a CNF formula $A = C_1 \wedge \ldots \wedge C_n$ if it satisfies every clause C_i in A.

A CNF formula A is *satisfiable* if there exists an interpretation ν that satisfies A.

The **SAT problem** is about determining whether a given CNF formula is satisfiable.

CNF Transformation

Any propositional formula can be converted to CNF by the repeated application of these rewriting rules:

- Any NNF transformation rules.
- Distribute $\lor$ over $\land$ (another source of exponential blowup):
  - $A \lor (B \land C) \implies (A \lor B) \land (A \lor C)$
  - $(A \land B) \lor C \implies (A \lor C) \land (B \lor C)$
- Normalize nested $\land$ and $\lor$ operators:
  - $A \land (B \land C) \implies (A \land B \land C)$
  - $A \lor (B \lor C) \implies (A \lor B \lor C)$

Theorem 2: Every well-formed formula α can be converted to an *equivalent* CNF α' with a *potentially exponential increase* in the size of the formula.

Exponential Blowup of CNF

Distributive law is the main source of the exponential blowup in CNF conversion:

$$
n \text{ cubes} \left\{ \begin{array}{l} (x_1 \land y_1) \lor \\ (x_2 \land y_2) \lor \\ \dots \\ (x_n \land y_n) \lor \end{array} \right. \xrightarrow{\text{CNF}} \left. \begin{array}{l} (x_1 \lor x_2 \lor \dots \lor x_n) \land \\ (y_1 \lor x_2 \lor \dots \lor x_n) \land \\ \dots \\ (x_1 \lor y_2 \lor \dots \lor y_n) \land \\ (y_1 \lor y_2 \lor \dots \lor y_n) \end{array} \right\} 2^n \text{ clauses}
$$

Is there a way to avoid the exponential blowup? *Yes!*

Tseitin Transformation

A space-efficient way to convert a formula to CNF is the *Tseitin transformation*, which is based on so-called "*naming*" or "*definition introduction*", allowing to replace subformulas with the "*fresh*" (new) variables.

1. Take a subformula A of a formula F.
2. Introduce a new propositional variable n.
3. Add a *definition* for n, that is, a formula stating that n is equivalent to A.
4. Replace A with n in F.

Overall, construct $S := F[n/A] \land (n \leftrightarrow A)$

$$
\begin{aligned}
F &= p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow \overbrace{(p_5 \leftrightarrow p_6)}^{A}))) \Longrightarrow \\
S &= p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow n))) \land \\
&\qquad n \leftrightarrow (p_5 \leftrightarrow p_6)
\end{aligned}
$$

Note: The resulting formula is, in general, **not equivalent** to the original one, but it is *equisatisfiable*, i.e., it is satisfiable iff the original formula is satisfiable.

Equisatisfiability

Definition 9 (Equisatisfiability): Two formulas A and B are *equisatisfiable* if A is satisfiable *if and only if* B is satisfiable.

The set S of clauses obtained by the Tseitin transformation is *equisatisfiable* with the original formula F.

- Every model of S is a model of F.
- Every model of F can be extended to a model of S by assigning the values of fresh variables according to their definitions.

Avoiding the Exponential Blowup

Example: $F = p_1 \leftrightarrow (p_2 \leftrightarrow (p_3 \leftrightarrow (p_4 \leftrightarrow (p_5 \leftrightarrow p_6))))$

Applying the Tseitin transformation gives us:

$$
\begin{aligned}
S = p_1 &\leftrightarrow (p_2 \leftrightarrow n_3) \land \\
n_3 &\leftrightarrow (p_3 \leftrightarrow n_4) \land \\
n_4 &\leftrightarrow (p_4 \leftrightarrow n_5) \land \\
n_5 &\leftrightarrow (p_5 \leftrightarrow p_6)
\end{aligned}
$$

The equivalent CNF of F consists of $2^5 = 32$ clauses, and grows exponentially with number of variables.

The equisatisfiable CNF of F consists of 16 clauses, yet introduces 3 fresh variables, and grows linearly with the number of variables.

Clausal Form

Definition 10 (Clausal form): A *clausal form* of a formula F is a set S_F of clauses which is satisfiable iff F is satisfiable.

A clausal form of a *set* of formulas S is a set S' of clauses which is satisfiable iff S is satisfiable.

Even stronger requirement:
- F and S_F have the same models in the language of F.
- S and S' have the same models in the language of S.

The main advantage of the clausal form over the equivalent CNF is that we can convert any formula into a set of clauses in *almost linear time*.

1. If F is a formula which has the form $C_1 \wedge \ldots \wedge C_n$, where $n > 0$ and each C_i is a clause, then its clausal form is $S \stackrel{\text{def}}{=} \{C_1, \ldots, C_n\}$.
2. Otherwise, apply Tseitin transformation: introduce a name for each subformula A of F such that A is not a literal and use this name instead of a subformula A.

TODO

- [ ] Exercises
- [ ] Example: convert formula to clausal form
- [ ] DNF vs CNF satisfiability