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A1)

i	0	1	2	3
x_i	0	2	4	8
y_i	2	12	26	70

a) $p_n(x) := \sum_{i=0}^{n} y_i \cdot L_i^{(n)}(x)$ $\qquad L_i^{(n)}(x) := \prod_{j=0, j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$

$$L_0^{(2)}(x) = \prod_{\substack{j=0\\ j\neq 0}}^{2} \frac{x-x_j}{x_0-x_j} = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2)(x-4)}{(0-2)(0-4)} = \frac{1}{8}(x^2-6x+8)$$

$$L_1^{(2)}(x) = \prod_{\substack{j=0\\ j\neq 1}}^{2} \frac{x-x_j}{x_0-x_j} = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0)(x-4)}{(2-0)(2-4)} = -\frac{1}{4}(x^2-4x)$$

$$L_2^{(2)}(x) = \prod_{\substack{j=0\\ j\neq 2}}^{2} \frac{x-x_j}{x_0-x_j} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0)(x-2)}{(4-0)(4-2)} = \frac{1}{8}(x^2-2x)$$

$$p_2(x) = \sum_{i=0}^{2} y_i \cdot L_i^{(2)}(x) = 2\cdot L_0^{(2)}(x) + 12\cdot L_1^{(2)}(x) + 26\cdot L_2^{(2)}(x)$$
$$= 2\cdot\frac{1}{8}(x^2-6x+8) + 12\cdot\left(-\frac{1}{4}(x^2-4x)\right) + 26\cdot\frac{1}{8}(x^2-2x)$$
$$= \underline{\underline{\frac{1}{2}x^2+4x+2}}$$

b) $p_n(x) := \sum_{i=0}^{n} y[x_0,\dots,x_i]\cdot N_i(x)$ $\qquad N_i(x) := \prod_{j=0}^{i-1}(x-x_j)$

$$y[x_0] = y_0 = 2$$

$$y[x_0,x_1] = \frac{y[x_1]-y[x_0]}{x_1-x_0} = \frac{y_1-y_0}{x_1-x_0} = \frac{12-2}{2-0} = 5$$

$$y[x_1,x_2] = \frac{y[x_2]-y[x_1]}{x_2-x_1} = \frac{y_2-y_1}{x_2-x_1} = \frac{26-12}{4-2} = 7$$

$$y[x_0,x_1,x_2] = \frac{y[x_1,x_2]-y[x_0,x_1]}{x_2-x_0} = \frac{7-5}{4-0} = \frac{1}{2}$$

$$p_2(x) = y[x_0] + y[x_0,x_1](x-x_0) + y[x_0,x_1,x_2](x-x_0)(x-x_1)$$
$$= 2 + 5(x-0) + \frac{1}{2}(x-0)(x-2)$$

$$= 2 + 5x + \frac{1}{2}(x^2 - 2x)$$
$$= 2 + 5x + \frac{1}{2}x^2 - x$$
$$= \frac{1}{2}x^2 + 4x + 2$$

c) In Newtonscher Darstellung

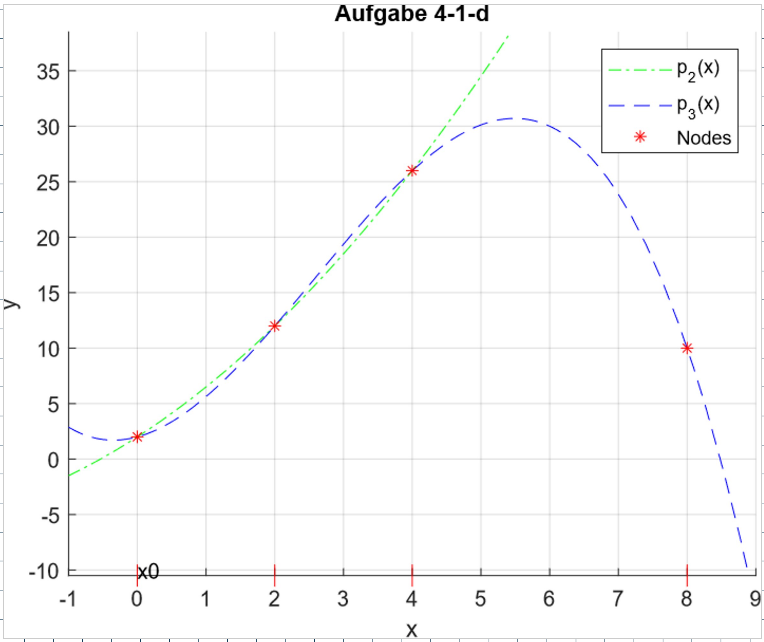
$$y[x_2, x_3] = \frac{y[x_3] - y[x_2]}{x_3 - x_2} = \frac{y_3 - y_2}{x_3 - x_2} = \frac{10 - 26}{8 - 4} = -4$$

$$y[x_1, x_2, x_3] = \frac{y[x_2, x_3] - y[x_1, x_2]}{x_3 - x_1} = \frac{-4 - 7}{8 - 2} = -\frac{11}{6}$$

$$y[x_0, x_1, x_2, x_3] = \frac{y[x_1, x_2, x_3] - y[x_0, x_1, x_2]}{x_3 - x_0} = \frac{-\frac{11}{6} - \frac{1}{2}}{8 - 0}$$
$$= \frac{-\frac{11}{6} - \frac{3}{6}}{8}$$
$$= \frac{-\frac{14}{6}}{8}$$
$$= -\frac{7}{24}$$

$$p_3(x) = p_2(x) + y[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)$$
$$= \frac{1}{2}x^2 + 4x + 2 - \frac{7}{24}(x - 0)(x - 2)(x - 4)$$
$$= \frac{1}{2}x^2 + 4x + 2 - \frac{7}{24}(x^3 - 6x^2 + 8x)$$
$$= -\frac{7}{24}x^3 + \frac{9}{4}x^2 + \frac{5}{3}x + 2$$

d) (Bonus)



A2] a)

Verwendung von Zeilensummennorm

A regulär: $\det(A) \neq 0$

$$\det(A) = \det\left(\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix}\right) = \frac{1}{2} \cdot -1 - \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{2} - \frac{1}{4} = \underline{\underline{-\frac{3}{4} \neq 0}}$$

$\Rightarrow$ A regulär

x berechnen

$$\left| \begin{array}{l} \frac{1}{2}x_1 + \frac{1}{2}x_2 = 1 \\ \frac{1}{2}x_1 - 1x_2 = 1 \end{array} \right.$$

$$\Leftrightarrow \left| \begin{array}{l} \frac{1}{2}x_1 + \frac{1}{2}x_2 = 1 \\ x_2 = \frac{1}{2}x_1 - 1 \end{array} \right.$$

$$\Leftrightarrow \left| \begin{array}{l} \frac{1}{2}x_1 + \frac{1}{2}x_1 - 1 = 1 \\ x_2 = \frac{1}{2}x_1 - 1 \end{array} \right.$$

$$\Leftrightarrow \left| \begin{array}{l} x_1 = 2 \\ x_2 = \frac{1}{2}x_1 - 1 \end{array} \right.$$

$\Leftrightarrow \left| \begin{array}{l} x_1 = 2 \\ x_2 = \frac{1}{2} \cdot 2 - 1 = 0 \end{array} \right.$

$x_1 = 2 \quad x_2 = 0$

$$\Delta A = A \cdot \pm 1\% = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} \cdot \pm 0{,}01 = \pm \begin{pmatrix} 0{,}005 & 0{,}005 \\ 0{,}005 & -0{,}01 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det(A)} \cdot \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{-\frac{3}{4}} \cdot \begin{pmatrix} -1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

$$\|A^{-1}\|_\infty = \max\left(\left|\tfrac{4}{3}\right| + \left|\tfrac{2}{3}\right|; \left|\tfrac{2}{3}\right| + \left|-\tfrac{2}{3}\right|\right) = \frac{4}{3} + \frac{2}{3} = 2$$

$$\|\Delta A\|_\infty = \max(|\pm 0{,}005| + |\pm 0{,}005|; |\pm 0{,}005| + |\pm 0{,}01|) = 0{,}005 + 0{,}01 = 0{,}015$$

$$\|A\|_\infty = \max\left(\left|\tfrac{1}{2}\right| + \left|\tfrac{1}{2}\right|; \left|\tfrac{1}{2}\right| + |-1|\right) = \frac{1}{2} + 1 = \frac{3}{2} = 1{,}5$$

$\|A^{-1}\|_\infty \cdot \|\Delta A\|_\infty < 1$:

$$\|A^{-1}\|_\infty \cdot \|\Delta A\|_\infty = 2 \cdot 0{,}015 = 0{,}03 < 1 \checkmark$$

$$\Delta b = b \cdot \pm 3\% = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \pm 0{,}03 = \pm \begin{pmatrix} 0{,}03 \\ 0{,}03 \end{pmatrix}$$

$$\|b\|_\infty = \max(|1|; |1|) = 1$$

$$\|\Delta b\|_\infty = \max(|\pm 0{,}03|; |\pm 0{,}03|) = 0{,}03$$

$$\mathrm{cond}_\infty(A) = \frac{3}{2} \cdot 2 = 3$$

$$\frac{\|\Delta x\|_\infty}{\|x\|_\infty} \leq \frac{\mathrm{cond}_\infty(A)}{1 - \mathrm{cond}_\infty(A) \cdot \frac{\|\Delta A\|_\infty}{\|A\|_\infty}} \cdot \left(\frac{\|\Delta A\|_\infty}{\|A\|_\infty} + \frac{\|\Delta b\|_\infty}{\|b\|_\infty} \right)$$

$$= \frac{3}{1 - 3 \cdot \frac{0{,}015}{1{,}5}} \cdot \left(\frac{0{,}015}{1{,}5} + \frac{0{,}03}{1} \right)$$

$$= \frac{3}{1 - 3 \cdot 0{,}01} \cdot (0{,}01 + 0{,}03)$$

$$= \frac{3}{1 - 0{,}03} \cdot 0{,}04$$

$$= \frac{3}{0{,}97} \cdot 0{,}04$$

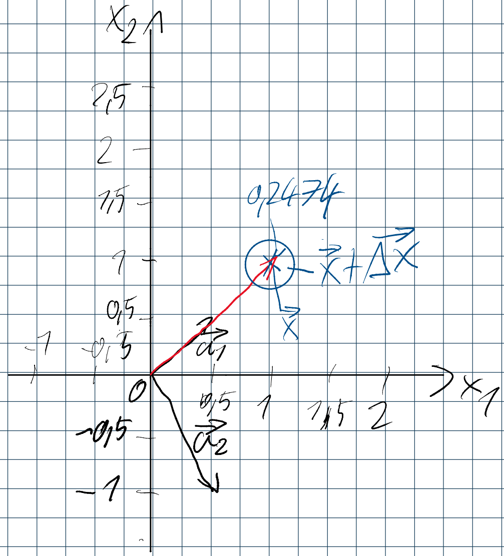
$$= \underline{\underline{\frac{12}{97}}}$$

b) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -1 \end{pmatrix} = (\vec{a}_1, \vec{a}_2)$ $\quad \vec{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \vec{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ $\quad \frac{\|\Delta x\|_\infty}{\|x\|_\infty} \leq \frac{12}{97}$

$$\Leftrightarrow \|\Delta x\|_\infty = \frac{\|\Delta x\|_\infty}{\|x\|_\infty} \cdot \|x\|_\infty \leq \frac{12}{97} \cdot \|x\|_\infty$$

$$= \frac{12}{97} \cdot 2$$

$$= \frac{24}{97} \approx 0{,}2474$$



$$\|x\|_\infty = \max(|2|; |0|) = 2$$