Giuseppe Peano's

Classic Mathematical Text

Arithmetices Principia, Nova Methodo Exposita

OR

The Principles of Arithmetic, Presented by a New Method

BOTH

in the original Latin

AND

in parallel English Translation

WITH

Modern Mathematical Notation

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This document is hosted at https://github.com/mdnahas/Peano_Book

Note from Original Translation

Below is Giuseppe Peano's *Arithmetices principia* as first published[1], i.e. as "*Arithmetices principia, nova methodo exposita*"[2] , which appeared translated to English[3] in 1967 as "*The principles of arithmetic, presented by a new method*"[4], as well as in 1973[5]. This present document[6] is the only (to my knowledge) side-by-side Latin-English translation of the Latin original. The mathematical notation (in the English, right, column) got updated to currently canonically-used or easy-to-decrypt symbols in the international and/or English mathematical community; which is also a feature currently unseen in any reprint.

Red text is mathematical commentary.

Gray text is irrelavent for modern mathematical notation.

Dashed lines (..................) indicate pages in the original treatise.

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[1] H. Kennedy, *Peano. Life and Works of Giuseppe Peano*, San Francisco: Peremptory Publications, 2002, p. 41.

[2] G. Peano, *Arithmetices principia, nova methodo exposita*, Bocca, Torino, 1889.

[3] These English translations listed are the only ones (to my knowledge) and all the English translations listed in:

I. Grattan-Guiness (ed.), *Landmark Writings in Western Mathematics 1640-1940*, Amsterdam: Elsevier, 2005, p. 614.

[4] G. Peano, (1889), "The principles of arithmetic presented by a new method" in: J. van Heijenoort (ed.), *From Frege to Gödel. A source book in mathematical logic. 1879-1931*, Cambridge: Harvard University Press, 1967, p. 83-97.

[5] G. Peano, *Selected works of Giuseppe Peano*, H. Kennedy (ed.), London: George Allen & Unwin, 1973, p. 101-134.

[6] Written by Vincent Verheyen. Last updated on 17/8/2015. I encourage you to use your reason for good. If you want my support, please contact me via http://vincentverheyen.com/contact. It is possible to contribute to the flourishing of knowledge, even when you have an intelligence like mine. Thank you and good luck studying.

I would like to thank Mauro Allegranzo and acknowledge his support of this work and his various comments during its creation.

I

ARITHMETICES PRINCIPIA

NOVA METHODO EXPOSITA

A

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[F, B, Labor et honor]

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THE PRINCIPLES OF ARITHMETIC

PRESENTED BY A NEW METHOD

By

GIUSEPPE PEANO

professor at the Royal Military Academy

teaching Analysis of the infinite at the Royal Turin Athenaeum.

[The seal of Fratres Bocca publishing, with an F, a B, and a belt saying "Work and honor"]

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II

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PRAEFATIO

Quaestiones, quae ad mathematicae fundamenta pertinent, etsi hisce temporibus a multis tractatae, satisfacienti solutione et adhuc carent. Hic difficultas maxime ex sermonis ambiguitate oritur.

Quare summi interest verba ipsa, quibus utimur attente perpendere. Hoc examen mihi proposui, atque mei studii resultatus, et arithmeticae applicationes in hoc scripto expono.

Ideas omnes quae in arithmeticae principiis occurrunt, signis indicavi, ita ut quaelibet propositio his tantum signis enuncietur.

Signa aut ad logicam pertinent, aut proprie ad arithmeticam. Logicae signa quae hic occurrunt, sunt numero ad decem, quamvis non omnia necessaria. Horum signorum usus et proprietas nonnullae in priore parte communi sermone explicantur. Ipsorum theoriam fusius hic exponere nolui. Arithmeticae signa, ubi occurrunt, explicantur.

His notationibus quaelibet propositio formam assumit atque praecisionem, qua in algebra aequationes gaudent, et a propositionibus ita scriptis aliea deducuntur, idque processis qui aequationum resolutioni assimilantur. Hoc caput totius scripti.

Sique, confectis signis quibus arithmeticae propositiones scribere possim, in earum tractatione usus sum methodo, quam quia et in aliis studiis sequenda foret, breviter exponam.

Ex arithmeticae signis quae caeteris, una cum logicae signis exprimere licet,

PREFACE

Questions pertaining to the foundations of mathematics, although treated by many these days, still lack a satisfactory solution. The difficulty arises principally from the ambiguity of ordinary language.

For this reason it is of the greatest concern to consider attentively the words we use. I resolved to do this, and am presenting in this paper the results of my study with applications to arithmetic.

I have indicated by symbols all the ideas which occur in the fundamentals of arithmetic, so that every proposition is stated with just these symbols.

The symbols pertain either to logic or to arithmetic. The symbols of logic that occur here are about ten in number, although not all are necessary. The use of these symbols and several of their properties are explained in ordinary language in the first part. I did not wish to present their theory more fully here. The symbols of arithmetic are explained as they occur.

With this notation every proposition assumes the form and precision equations enjoy in algebra, and from propositions so written others may be deduced, by a process which resembles the solution of algebraic equations. That is the chief reason for writing this paper.

Having made up the symbols with which I can write arithmetical propositions, in treating them I have used a method which, because it is to be followed in later studies, I shall present briefly.

Those arithmetical symbols which may be expressed by using others along with ideas significant quas definire possumus. Ita omnia definivi signa, si quatuor excipias, quae in explicationibus §1 continentur. Si, ut puto, haec ulterius reduci nequeunt, ideas ipsis expressas, ideis quae prius notae supponuntur, definire non licet.

symbols of logic represent the ideas we can define. Thus I have defined every symbol, if you except the four which are contained in the explanations of §1. If, as I believe, these cannot be reduced further, then the ideas expressed by them may not be defined by ideas already supposed to be known.

IV

Propositiones, quae logicae operationibus a caeteris deducuntur, sunt *theoremata*; quae vero non, *axiomata* vocavi. Axiomata hic sunt novem (§1), et signorum, quae definitione carent, proprietates fundamentales exprimunt.

Propositions that are deduced from others by the operations of logic are *theorems*; those for which this is not true I have called *axioms*. There are nine axioms here (See §1), and they express fundamental properties of the undefined symbols.

In §1-6 numerorum proprietates communes demonstravi; brevitatis causa, demonstrationes praecedentibus similes omisi; demonstrationum communem formam immutare oportet ut logicae signis exprimantur; haec transformatio interdum difficilior est, tamen inde demonstrationis natura clarissime patet.

In §1-6 I have proved the ordinary properties of numbers; for the sake of brevity, I have omitted proofs which are similar to preceding ones. The ordinary form of proofs has had to be altered in order that they may be expressed with the symbols of logic. This transformation is sometimes rather difficult but the nature of the proof then becomes quite clear.

In sequentibus § varia tractavi, ut huius methodi potentia magis videatur.

In the following sections I have treated various things so that the power of the method is better seen.

In §7 nonnulla theoremata, quae ad numerorum theoriam pertinent, continentur. In §8 et 9 rationalium et irrationalium definitiones inveniuntur.

In §7 are several theorems pertaining to the theory of numbers. In §8 and 9 are found the definitions of rationals and irrationals.

Denique, in §10, theoremata exposui nonnulla, quae nova esse puto, ad entium theoriam pertinentia, quae cl.$^{\text{mus}}$ Cantor *Punktmenge (ensemble de points)* vocavit.

Finally, in §10 I have given several theorems, which I believe to be new, pertaining to the theory of those entities which Professor Cantor has called *Punktmenge (ensemble de points)*.

In hoc scripto aliorum studiis usus sum. Logicae notationes et propositiones quae in num. II, III et IV continentur, si nonnullas excipias, ad multorum opera, inter quae Boole praecipue, referenda sunt.[1]

In this paper I have used the research of others. The notations and propositions of logic which are contained in numbers II, III, and IV, with some exceptions, represent the work of many, among them Boole especially.[1]

V

Signum ϵ, quod cum signo Ↄ confundere non licet, inversionis in logica appli-

The symbol $\in$, which must not be confused with the symbol $\subset$, applications of cationes, et paucas alias institui conventiones, ut ad exprimendam quamlibet propositionem pervenirem.

In arithmeticae demonstrationibus usus sum libro: H. Grassmann, *Lehrbuch der Arithmetik*, Berlin 1861.

Utilius quoque mihi fuit recens scriptum: R. Dedekind, *Was sind und was sollen die Zahlen*; Braunschweig, 1888, in quo quaestiones, quae ad numerorum fundamenta pertinent, acute examinantur.

Hic meus libellus ut novae methodi specimen habendus est. Hisce notationibus innumeras alias propositiones, ut quae ad rationales et irrationales pertinent, enunciare et demonstrare possumus. Sed, ut aliae theoriae tractentur, nova signa, quae nova indicant entia, instituere necesse est. Puto vero his tantum logicae signis propositiones cuiuslibet scientiae exprimi posse, dummodo adiungantur signa quae entia huius scientiae representant.

the inverse in logic, and a few other conventions, I have adopted so that I could express any proposition whatever.

In the proofs of arithmetic I used the book H. Grassmann, *Lehrbuch der Arithmetik* (Berlin 1861).

Also quite useful to me was the recent work by R. Dedekind, *Was sind und was sollen die Zahlen* (Braunschweig, 1888), in which questions pertaining to the foundations of numbers are acutely examined.

My booklet should be taken as a sample of this new method. With these notations we can state and prove innumerable other propositions, such as those which pertain to rationals and irrationals. But in order to treat other theories, it is necessary to adopt new symbols to indicate new entities. I believe, however, that with only these symbols of logic the propositions of any science can be expressed, so long as the symbols which represent the entities of the science are added.

COMMENTARY: Peano's mathematics has been translated into modern notation and language. This is a convenience for the casual reader. We caution expert readers that time has passed and that Peano's usage could differ in subtle ways from modern concepts.

Signorum tabula

Logicam signa

Signum	Significatio	Pag.
P	propositio	VII

Table of symbols

Symbols of logic

Symbol	Meaning	Pages
PROP *or* P#	proposition	VII

K	classis	X
$\cap$	et	VII, X
$\cup$	vel	VIII, X, XI
$-$	non	VIII, X
Λ	absurdum *aut* nihil	VIII, XI
Ɔ	deducitur *aut* continetur	VIII, XI
$=$	est aequalis	VIII
ϵ	est	X
[ ]	inversionis signum	XI
∍	qui *vel* $[\epsilon]$	XII
Th	Theorema	XVI
Hp	Hypothesis	
Ts	Thesis	
L	Logica	

SET	set	X
$\wedge$	and	VII, X
$\vee$	or	VIII, X, XI
$\neg$	not	VIII, X
$\perp$	false *or*	VIII, XI
$\varnothing$	nothing	
$\rightarrow$	one deduces *or*	VIII, XI
$\subset$	is contained in	
$=$	equals	VIII
$\in$	is a	X
$\{x \| \ldots\}$	notation of the inverse	XI
$\ni$	such that *or* $[\in]$	XII
Th	Theorem	XVI
Hp	Hypothesis	
Ts	Thesis	
L#	Logic	

Arithmeticae signa

Signum	Significatio	Pag.

Signa 1, 2, ..., $=$, $>$, $<$, $+$, $-$, $\times$ vulgarem habent significationem. Divisionis signum est $/$.

N	numerus integer positivus	1
R	num. rationalis positivus	12
Q	quantitas, *sive* numerus realis positivus	16
Np	numerus primus	9
M	maximus	6
Ɯ	minimus	6
T	terminus, *vel* limes summus	15
D	dividit	9

Symbols of arithmetic

Symbol	Meaning	Pag.

The symbols 1, 2, ..., $=$, $>$, $<$, $+$, $-$, $\times$ have their usual meaning. The symbol of division is $/$.

$\mathbb{N}$	positive integers	1
$\mathbb{Q}^+$	positive rational numbers	12
$\mathbb{R}^+$	quantity *or* positive real numbers	16
$\mathbb{P}$	prime number	9
max()	maximum	6
min()	minimum	6
sup()	boundary *or* upper limit	15
$\|$	divides	9

ꓷ	est multiplex	9
π	est primus cum	6

Signa composita

$- <$	non est minor
$= \cup >$	est aequalis aut maior
϶ D	divisor
M ϶ D	maximus divisor

ꓷ	is divisible	9
π	is prime with	6

Composite symbols

$\not<$	is not less than
$\geq$	is equal to or greater than
϶ D	is a divisor
M ϶ D	is the greatest divisor

VII

Logicae notationes.

I. De punctuatione.

Litteris $a, b, \ldots x, y, \ldots x', y', \ldots$ entia indicamus indeterminata quaecumque. Entia vero determinata signis, sive litteris P, K , N , ... indicamus.

Signa plerumque in eadem linea scribemus. Ut ordo pateat quo ea coniungere oporteat, *parenthesibus* ut in algebra, sive *punctis* . : ∴ :: etc. utimur.

Ut formula punctis divisa, intelligatur, primum signa quae nullo puncto seperantur colligenda sunt, postea quae uno puncto, deinde quae duobus punctis, etc.

Ex. g. sint $a, b, c, \ldots$ signa quaecumque. Tunc $ab \;.\; cd$ significat $(ab)(cd)$; et $ab \;.\; cd : ef \;.\; gh \therefore k$ significat $(((ab)(cd))((ef)(gh)))k$.

Punctuationis signa omittere licet si formulae quae diversa punctuatione existerent eundem habeant sensum; vel si una tantum formula, et ipsa quam scribere volumus, sensum habeat.

Ut ambiguitatis periculum absit, aritmeticae operationum signis . : nunquam

Notations of logic.

I. Punctuation.

By the letters $a, b, \ldots A, B, \ldots x, y, \ldots x', y', \ldots$ we indicate any variables. Constants are, however, indicated by the symbols, or rather by the letters, $\mathbb{P}$, **SET**, $\mathbb{N}, \ldots$

Generally we write symbols on the same line. So that it will be clear how they are to be joined, we use *parentheses*, as in algebra, or rather *points* . : ∴ :: etc.

So that a formula divided by points may be understood, first the symbols which are not separated by points are taken together, then those separated by one point, then those by two points, etc.

For example, let $a, b, c, \ldots$ be any symbols. Then $ab \;.\; cd$ means $(ab)(cd)$; and $ab \;.\; cd : ef \;.\; gh \therefore k$ means $\{[(ab)(cd)][(ef)(gh)]\}k$.

The symbols of punctuation may be omitted if formulas having different punctuation have the same meaning, or if just one formula, that being the one we wish to write, has meaning.

To avoid the danger of ambiguity, we never use . : as symbols of arithmetical utimur.

Parenthesum figura una est ( ); si in eadem formula, parentheses et puncta occurant, primum quae parenthesibus continentur, colligantur.

operations.

The shape of one parentheses is ( ); if parentheses and points occur in the same formula, whatever is contained in parentheses is to be gathered first.

II. De propositionibus.

Signo P significatur *propositio.*

Signum $\cap$ legitur *et.* Sint a, b propositiones; tunc $a \cap b$ est simultanea affirmatio propositionum a, b. Brevitatis causa, loco $a \cap b$ vulgo scribemus $a\ b$.

II. Propositions.

The symbol **PROP** means *proposition.*

The symbol $\wedge$ is read *and.* Let a, b, be propositions; then $a \wedge b$ is the simultaneous affirmation of the propositions a, b. For the sake of brevity, instead of $a \wedge b$, we ordinarily write ab.

VIII

Signum $-$ legitur *non.* Sit a quaedam P; tunc $-a$ est negatio propositionis a.

Signo $\cup$ legitur *vel.* Sint a, b propositiones; tunc $a \cup b$ idem est ac $- : -a \,.\, -b$.

[Signo V significatur *verum*, sive *identitas*; sed hoc signo numquam utimur].

Signum Λ significat *falsum*, sive *absurdum.*

[Signum C significat *est consequentia*; ita b C a legitur b *est consequentia propositionis* a. Sed hoc signo nunquam utimur].

Signum Ɔ significat *deducitur*; ita a Ɔ b significat quod b C a. Si propositiones a, b entia indeterminata continent x, y,..., scilicet sunt inter ipsa entia conditiones, tunc a Ɔ$_{x,y,\ldots}$ b significat: quaecumque sunt $x, y, \ldots$, a propositione a deducitur b. Si vero ambiguitatis periculum absit, loco Ɔ$_{x,y,\ldots}$, scribemus solum Ɔ.

Signum $=$ significat *est aequalis.* Sint a, b propositiones; tunc $a = b$ idem significat quod a Ɔ b . b Ɔ a; propositio $a =_{x,y,\ldots} b$ idem significat quod a Ɔ$_{x,y,\ldots}$ b . b Ɔ$_{x,y,\ldots}$ a.

The symbol $\neg$ is read *not.* Let a be a **PROP**; then $\neg a$ is the negation of the proposition a.

The symbol $\vee$ is read *or.* Let a, b be propositions; then $a \vee b$ is the same as $\neg[(\neg a) \wedge (\neg b)]$.

The symbol $\top$ means *true*, or *identity*, but we never use this symbol.

The symbol $\perp$ means *false*, or *absurd.*

The symbol $\leftarrow$ means *is a consequence of.* Thus $b \leftarrow a$ is read b *is a consequence of the proposition* a. But we never use this symbol.

The symbol $\rightarrow$ means *one deduces*; thus $a \rightarrow b$ means the same as $b \leftarrow a$. If the propositions a, b contain the variables $x, y, \ldots$, that is, express conditions on these objects, then $a \xrightarrow[\forall x,y,\ldots]{} b$ means: whatever the x, y,..., from propositions a one deduces b. If indeed there is no danger of ambiguity, instead of $\xrightarrow[\forall x,y,\ldots]{}$, we write only $\rightarrow$.

The symbol $=$ means *equals.* Let a, b be propositions; then $a = b$ means the same as $(a \rightarrow b) \wedge (b \rightarrow a)$; proposition $a \underset{\forall x,y,\ldots}{=} b$ means the same as $(a \xrightarrow[\forall x,y,\ldots]{} b) \wedge (b \xrightarrow[\forall x,y,\ldots]{} a)$.

III. Logicae propositiones.

Sint $a, b, c, \ldots$ propositiones. Tunc erit:

1. $a \supset a$
2. $a \supset b \;.\; b \supset c \;:\; \supset \;:\; a \supset c$
3. $a = b \;.\; = \;:\; a \supset b \;.\; b \supset a.$
4. $a = a$
5. $a = b \;.\; = \;.\; b = a$
6. $a = b \;.\; b \supset c \;:\; \supset \;.\; a \supset c$
7. $a \supset b \;.\; b = c \;:\; \supset \;.\; a \supset c$
8. $a = b \;.\; b = c \;:\; \supset \;.\; a = c$
9. $a = b \;.\; \supset \;.\; a \supset b$
10. $a = b \;.\; \supset \;.\; b \supset a$

11. $ab \supset a$
12. $ab = ba$
13. $a\,(bc) = (ab)\,c = abc$

IX

14. $aa = a$
15. $a = b \;.\; \supset \;.\; ac = bc$
16. $a \supset b \;.\; \supset \;.\; ac \supset bc$
17. $a \supset b \;.\; c \supset d \;:\; \supset \;.\; ac \supset bd$
18. $a \supset b \;.\; a \supset c \;:\; = \;.\; a \supset bc$
19. $a = b \;.\; c = d \;:\; \supset \;.\; ac = bd$

20. $-(-a) = a$
21. $a = b \;.\; = \;.\; -a = -b.$
22. $a \supset b \;.\; = \;.\; -b \supset -a =$

III. Propositions of logic.

Let $a, b, c, \ldots$ be propositions. We have:

1. $a \to a$
2. $[(a \to b) \wedge (b \to c)] \to (a \to c)$
3. $(a = b) = [(a \to b) \wedge (b \to a)]$
4. $a = a$
5. $(a = b) = (b = a)$
6. $[(a = b) \wedge (b \to c)] \to (a \to c)$
7. $[(a \to b) \wedge (b = c)] \to (a \to c)$
8. $[(a = b) \wedge (b = c)] \to (a = c)$
9. $(a = b) \to (a \to b)$
10. $(a = b) \to (b \to a)$

11. $(a \wedge b) \to a$
12. $(a \wedge b) = (b \wedge a)$
13. $(a \wedge (b \wedge c)) = ((a \wedge b) \wedge c) = (a \wedge b \wedge c)$

14. $(a \wedge a) = a$
15. $(a = b) \to [(a \wedge c) = (b \wedge c)]$
16. $(a \to b) \to [(a \wedge c) \to (b \wedge c)]$
17. $[(a \to b) \wedge (c \to d)] \to [(a \wedge c) \to (b \wedge d)]$
18. $[(a \to b) \wedge (a \to c)] = [(a \to (b \wedge c)]$
19. $[(a = b) \wedge (c = d)] \to [(a \wedge c) = (b \wedge d)]$

20. $\neg(\neg a) = a$
21. $(a = b) = [(\neg a) = (\neg b)]$
22. $(a \to b) = [(\neg b) \to (\neg a)]$

23. $a \cup b \;.\; = \therefore - : -a \;.\; -b$

24. $-(ab) = (-a) \cup (-b)$

25. $-(a \cup b) = (-a)\ (-b)$

26. $a \supset .\ a \cup b$

27. $a \cup b = b \cup a$

28. $a \cup (b \cup c) = (a \cup b) \cup c = a \cup b \cup c$

29. $a \cup a = a$

30. $a\ (b \cup c) = ab \cup ac$

31. $a = b \;.\supset.\; a \cup c = b \cup c$

32. $a \supset b \;.\supset.\; a \cup c \supset b \cup c$

33. $a \supset b \;.\; c \supset d : \supset : a \cup c \;.\supset.\; b \cup d$

34. $b \supset a \;.\; c \supset a : = .\ b \cup c \supset a$

35. $a - a = \Lambda$

36. $a\ \Lambda = \Lambda$

37. $a \cup \Lambda = \mathrm{a}$

38. $a \supset \Lambda \;.=.\; a = \Lambda$

39. $a \supset b \;.=.\; a - b = \Lambda$

40. $\Lambda \supset a$

41. $a \cup b = \Lambda \;. = :\; a = \Lambda \;.\; b = \Lambda$

42. $a \supset .\ b \supset c : = : ab \supset c$

43. $a \supset .\ b = c : = .\ ab = ac$

23. $(a \vee b) = \neg[(\neg a) \wedge (\neg b)]$

24. $[\neg(a \wedge b)] = [(\neg a) \vee (\neg b)]$

25. $[\neg(a \vee b)] = [(\neg a) \wedge (\neg b)]$

26. $a \to (a \vee b)$

27. $(a \vee b) = (b \vee a)$

28. $[a \vee (b \vee c)] = [(a \vee b) \vee c] = (a \vee b \vee c)$

29. $(a \vee a) = a$

30. $[a \wedge (b \vee c)] = [(a \wedge b) \vee (a \wedge c)]$

31. $(a = b) \to [(a \vee c) = (b \vee c)]$

32. $(a \to b) \to [(a \vee c) \to (b \vee c)]$

COMMENTARY: Proposition #33 appears to be false when $a = b = d = \top$ and $c = \bot$. It does hold if the first "and" is changed to an "or". Compare with proposition #17.

33. $[(a \to b) \wedge (c \to d)] \to [(a \vee c) \to (b \vee d)]$

34. $[(b \to a) \wedge (c \to a)] = [(b \vee c) \to a]$

35. $[a \wedge \neg a] = \bot$

36. $(a \wedge \bot) = \bot$

37. $(a \vee \bot) = a$

38. $(a \to \bot) = (a = \bot)$

39. $(a \to b) = [(a \wedge \neg b) = \bot]$

40. $\bot \to a$

41. $[(a \vee b) = \bot] = [(a = \bot) \wedge (b = \bot)]$

42. $[a \to (b \to c)] = [(a \wedge b) \to c]$

43. $[a \to (b = c)] = [(a \wedge b) = (a \wedge c)]$

Sit α quoddam relationis signum (ex. gr. $=$, Ɔ), ita ut $a\ \alpha\ b$ sit quaedam propositio. Tunc loco $-\ .\ a\ \alpha\ b$ scribemus $a\ -\ \alpha\ b$; scilicet:

$$a - = b\ .\ =\ :\ -\ .\ a = b$$

$$a - \text{Ɔ}\ b\ .\ =\ :\ -\ .\ a\ \text{Ɔ}\ b$$

Ita signum $- =$ significat *non est aequalis*. Si propositio a indeterminatum continet x, $a - =_x \Lambda$ significat: sunt x quae conditioni a satisfaciunt. Signum $- \text{Ɔ}$ significat *non deducitur*.

Similter, si α et β sunt relationis signa, loco $a\ \alpha\ b$, et $a\ \alpha\ b\ .\ \cup\ .\ a\ \beta\ b$ scribere possumus $a\ .\ \alpha\ \beta\ .\ b$ et $a\ .\ \alpha \cup \beta\ .\ b$. Ita, si a et b sunt propositiones, formula $a\ .\ \text{Ɔ} - =\ .\ b$ dicit: ab a *deducitur* b, sed non vice versa.

$$a\ .\ \text{Ɔ} - =\ .\ b\ :\ =\ :\ a\ \text{Ɔ}\ b\ .\ b - \text{Ɔ}\ a$$

Formulae:

$$a\ \text{Ɔ}\ b\ .\ b\ \text{Ɔ}\ c\ .\ a - \text{Ɔ}\ c\ :\ = \Lambda$$

$$a = b\ .\ b = c\ .\ a - = c\ :\ = \Lambda$$

$$a\ \text{Ɔ}\ b\ .\ b\ \text{Ɔ} - = c\ :\ \text{Ɔ}\ .\ a\ \text{Ɔ} - = c$$

$$a\ \text{Ɔ} - = b\ .\ b\ \text{Ɔ}\ c\ :\ \text{Ɔ}\ .\ a\ \text{Ɔ} - = c$$

Sed his notationibus raro utimur.

IV. De classibus.

Signo K significatur *classis*, sive entium aggregatio.

Signum ϵ significat *est*. Ita $a\ \epsilon\ b$ legitur *a est quoddam b*; $a\ \epsilon$ K significat *a est quaedam classis*; $a\ \epsilon$ P significat *a est quaedam propositio*.

Loco $-(a\ \epsilon\ b)$ scribemus $a - \epsilon\ b$; signum $- \epsilon$ significat *non est*; scilicet:

44. $\quad a - \epsilon\ b\ .\ =\ :\ -\ .\ a\ \epsilon\ b$

Let α be the symbol of some relation (e.g., $=$, $\to$) so that $a\ \alpha\ b$ is a proposition. Then instead of $\neg(a\ \alpha\ b)$, we write $a \not\alpha\, b$. Thus:

$$(a \neq b) = \neg(a = b)$$

$$(a \not\to b) = \neg(a \to b)$$

COMMENTARY: Peano's $=_x$ operator contains a "for all x", while the $- =_x$ operator contains an "exists x".

Thus the symbol $\neq$ means *is not equal to*. If the proposition a contains the variable $x, a \underset{\exists x}{\neq} \bot$ means: there is an x which satisfies condition a. The symbol $\not\to$ means *one does not deduce*.

Similarly, if α and β are symbols of relations, instead of $(a\ \alpha\ b) \wedge (a\ \beta\ b)$ and $(a\ \alpha\ b) \vee (a\ \beta\ b)$, we may write $a\ (\alpha \wedge \beta)\ b$ and $a\ (\alpha \vee \beta)\ b$. Thus, if a and b are propositions, the formula $a\ (\to \wedge \neq)\ b$ says: from a *one deduces* b, but not vice versa.

$$[a\ (\to \wedge \neq)\ b] = [(a \to b) \wedge (b \not\to a)]$$

Formulas:

$$[(a \to b) \wedge (b \to c) \wedge (a \not\to c)] = \bot$$

$$[(a = b) \wedge (b = c) \wedge (a \neq c)] = \bot$$

$$\Big((a \to b) \wedge [b\ (\to \wedge \neq)\ c]\Big) \to [a\ (\to \wedge \neq)\ c]$$

$$\Big([a\ (\to \wedge \neq)\ b] \wedge (b \to c)\Big) \to [a\ (\to \wedge \neq)\ c]$$

But we shall rarely use these notations.

IV. Sets.

The symbol **SET** means a *set*, or aggregate of entities.

The symbol $\in$ means *is*. Thus $a \in B$ is read *a is (an element of) B*; $A \in$ **SET** means *A is a set*; $a \in$ **PROP** means *a is a proposition*.

Instead of $\neg(a \in b)$ we shall write $a \notin b$. The symbol $\notin$ means *is not*; thus:

44. $\quad (a \notin b) = [\neg(a \in b)]$

Signum a, b, $c \mathrel{\epsilon} m$ significat: a, b et c sunt m; scilicet:

45. $a,\ b,\ c \mathrel{\epsilon} m\ .\ =:\ a \mathrel{\epsilon} m\ .\ b \mathrel{\epsilon} m\ .\ c \mathrel{\epsilon} m$

Sit a classis; tunc $-a$ significatur classis indiviuis constituta quae non sunt a.

46. $a \mathrel{\epsilon} \mathrm{K}\ .\ \supset :\ x \mathrel{\epsilon} -\ a\ .\ =\ .\ x - \epsilon\ a$

Sint a, b classes; $a \cap b$, sive a b, est classis individuis constituta

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quae eodem tempore sunt a et b; $a \cup b$ est classis individuis constituta qui sunt a vel b.

47. $a, b \mathrel{\epsilon} \mathrm{K}\ .\ \supset \therefore\ a\ x\ \epsilon.\ a\ b :\ =\ :\ x \mathrel{\epsilon} a\ .\ x \mathrel{\epsilon} b$

48. $a, b \mathrel{\epsilon} \mathrm{K}\ .\ \supset \therefore\ a \cup x\ \epsilon.\ a \cup b :\ =\ :\ x \mathrel{\epsilon} a\ .\ \cup\ .\ x \mathrel{\epsilon} b$

Signum Λ indicat classem quae nullum continet individuum. Ita:

49. $a \mathrel{\epsilon} \mathrm{K}\ .\ \supset \therefore\ a = \Lambda :\ =\ :\ x \mathrel{\epsilon} a\ .\ =_x \Lambda$

[Signo V, quod classem ex omnibus individuis constitutam, de quibus quaestio est, indieat, non utimur].

Signum $\supset$ significat *continetur*. Ita $a \supset b$ significat *classis a continetur in classi b.*

50. $a, b \mathrel{\epsilon} \mathrm{K}\ .\ \supset \therefore\ a \supset b :\ =\ :\ x \mathrel{\epsilon} a\ .\ \supset_x .\ x \mathrel{\epsilon} b$

[Formula $b \subset a$ significare potest *classis b continet classem a*; at signo $\subset$ non utiumur].

Hic signa Λ et $\supset$ significationem habent quae paullo a praecedenti differt; sed nulla orietur ambiguitas. Nam si de propositionibus agatur, haec signa legantur *absurdum* et *deducitur*; si vero de classibus, *nihil* et *continetur*.

The notation $a, b, c \in M$ means: a, b, and c are in M; thus:

45. $[a, b, c \in M] = [(a \in M) \wedge (b \in M) \wedge (c \in M)]$

Let A be a set. Then $\overline{A}$ means that set made up of individuals that are not in A.

46. $A \in \mathbf{SET} \to [(x \in \overline{A}) = (x \notin A)]$

Let A, B be sets. Then $A \cap B$, or AB, is the set composed of individuals

which are at the same time in A and B; $A \cup B$ is the set composed of individuals which are in A or B.

47. $(A, B \in \mathbf{SET}) \to [(x \in (A \cap B)] = [(x \in A) \wedge (x \in B)]$

48. $(A, B \in \mathbf{SET}) \to [(x \in (A \cup B)] = [(x \in A) \vee (x \in B)]$

The symbol $\varnothing$ indicates the set which contains no individuals. Thus:

49. $A \in \mathbf{SET} \to (A = \varnothing) = [(x \in A) \underset{\forall x}{=} \perp]$

[We shall not use the symbol $\mathbb{U}$, which indicates the set composed of all individuals being considered].

The symbol $\subset$ means *is contained*. Thus $A \subset B$ means *the set A is contained in the set B.*

50. $(A, B \in \mathbf{SET}) \to (A \subset B) = [(x \in A) \underset{\forall x}{\longrightarrow} (x \in B)]$

[The formula $B \supset A$ could mean *the set B contains the set A*, but we shall not use the symbol $\supset$].

COMMENTARY: The following paragraph describes ambiguity in Peano's notation. In translating the expressions to modern notation, the ambiguity has been removed. (Except for the equal sign, which is shared by sets and propositions.)

The symbols $(\perp/\varnothing)$ and $(\to\ /\ \subset)$ have meanings here which are slightly different from the preceding, but no ambiguity will arise, for if propositions are being considered, the symbols are read *absurd* $(\perp)$ and *one deduces* $(\to)$, but if sets are being considered, they are read *empty* $(\varnothing)$ and *is contained* $(\subset)$.

Formula $a = b$, si a et b sint classes, significat $a \supset b \,.\, b \supset a$. Itaque

51. $a, b \,\epsilon\, \mathrm{K} \,.\, \supset \therefore a = b : = : x \,\epsilon\, a \,.\, =_x .\ x \,\epsilon\, b$

Propositiones 1...41 quoque subsistunt, si a, b... classes indicant; praeterea est:

52. $a \,\epsilon\, b \,.\, \supset .\ b \,\epsilon\, \mathrm{K}$

53. $a \,\epsilon\, b \,.\, \supset .\ b - = \Lambda$

54. $a \,\epsilon\, b \,.\ b = c : \supset .\ a \,\epsilon\, c$

55. $a \,\epsilon\, b \,.\ b \supset c : \supset .\ a \,\epsilon\, c$

Sit s classis, et k classis quae in s contineatur; tunc dicimus k esse individuum classis s, si k ex uno tantum constat individuo. Itaque:

56. $s \,\epsilon\, \mathrm{K} \,.\, k \supset s : \supset :: k \,\epsilon\, s \,.\, = \therefore k - = \Lambda : x, y \,\epsilon\, k \,.\, \supset_{x,y} .\ x = y$

V. De inversione.

Inversionis signum est [ ], eiusque usum in sequenti numero explicabimus. Hic tantum casus particulares exponimus.

1. Sit a propositio, indeterminatum continens x; tunc scriptura $[x] \,\epsilon\, a$, quae legitur *ea x quibus a*, sive *solutiones*, vel *radices* conditionis a, classem significat individuis constitutam, quae conditioni a satisfaciunt. Itaque:

57. $a \,\epsilon\, \mathrm{P} \,.\, \supset : [x\,\epsilon]\ a \,.\, \epsilon\, \mathrm{K}$

58. $a \,\epsilon\, \mathrm{K} \,.\, \supset \therefore [x\,\epsilon] \,.\, x \,\epsilon\, a : = a$

59. $a \,\epsilon\, \mathrm{P} \,.\, \supset \therefore x \,\epsilon .\ [x\,\epsilon]\ a : = a$

Sint α, β propositiones indeterminatum continentes x; erit:

60. $[x\,\epsilon]\ (\alpha\ \beta) = ([x\,\epsilon]\ \alpha)\ ([x\,\epsilon]\ \beta)$

The formula $A = B$, if A and B are sets, means $(A \subset B) \wedge (B \subset A)$. Thus

51. $(A, B \in \mathbf{SET}) \to (A = B) = [(x \in A) \underset{\forall x}{=} (x \in B)]$

Propositions 1-41 also hold if $a, b, \ldots$ indicate sets. In addition, we have:

52. $(a \in B) \to B \in \mathbf{SET}$

53. $(a \in B) \to (B \neq \varnothing)$

54. $[(a \in B) \wedge (B = C)] \to (a \in C)$

55. $[(a \in B) \wedge (B \subset C)] \to (a \in C)$

COMMENTARY: Below, Peano defines subsets that each contain a single element.

Let A be a set, and B be a set which is contained in A; then we say that B is an individual of the set A, if B consists of only one individual. That is:

56. $[A \in \mathbf{SET} \wedge (B \subset A)] \to \Big[(B \in A) = \Big((B \neq \varnothing) \wedge [((x, y) \in B) \xrightarrow[\forall x,y]{} (x = y)]\Big)\Big]$

V. The inverse.

COMMENTARY: By inversion, Peano means going backwards from a proposition to a set. It is not a mathematical inverse; it is "set-builder notation".

The notation of the inverse is $\{x|\ldots\}$, and we shall explain its use in the following section. Here we give some particular examples.

1. Let a be a proposition containing the variable x; then the expression $\{x|a\}$, which is read *those x such that a*, or *solutions*, or *roots* of the condition a, indicates the set consisting of individuals which satisfy the condition a. That is:

57. $a \in \mathbf{PROP} \to (\{x|a\} \in \mathbf{SET})$

58. $A \in \mathbf{SET} \to \{x|x \in A\} = A$

59. $a \in \mathbf{PROP} \to (x \in \{x|a\}) = a$

Let α, β be propositions containing the variable x. We will have:

60. $\{x|\alpha \wedge \beta\} = \{x|\alpha\} \cap \{x|\beta\}$

61. $[x\ \epsilon] - \alpha = - [x\ \epsilon]\ \alpha$

62. $[x\ \epsilon]\ (\alpha \cup \beta) = [x\ \epsilon]\ \alpha \cup [x\ \epsilon]\ \beta$

63. $\alpha \supset_x \beta\ .=.\ [x\ \epsilon]\ \alpha \supset [x\ \epsilon]\ \beta$

64. $\alpha =_x \beta\ .=.\ [x\ \epsilon]\ \alpha = [x\ \epsilon]\ \beta$

2. Sint x, y entia quacumque; system ex ente x et ex ente y compositum ut novum ens consideramus, et signo (x, y) indicamus; similiterque si entium numerus maior fit. Sit α propositio indeterminata continens x, y; tunc $[(x, y)\ \epsilon]$ α significat classem entibus (x, y) constitutam, quae conditioni α satisfaciunt. Erit:

65. $\alpha \supset_{x,y} \beta\ .=.\ [(x, y)\ \epsilon]\ \alpha \supset [(x, y)\ \epsilon]\ \beta$

66. $[(x, y)\ \epsilon]\ \alpha - = \Lambda\ .=\therefore [x\ \epsilon]\ .\ [y\ \epsilon]\ \alpha - = \Lambda : - = \Lambda$

3. Sit $x\ \alpha\ y$ relatio inter indeterminata x et y (ex. g. in logica relationes $x = y$, $x - = y$, $x \supset y$; in arithmetica $x < y$, $x > y$, etc). Tunc signo $[\epsilon\ \alpha]$ y ea x indicamus, quae relationi $x\ \alpha\ y$ satisfaciunt. Commoditatis causa, loco $[\epsilon]$, signo $\ni$ utimur. Ita $\ni \alpha\ y\ .=:\ [x\ \epsilon]\ .\ x\ \alpha\ y$, et signum $\ni$ legitur *qui*, vel *quae*. Ex. gr. sit y numerus; tunc $\ni < y$ classem indicat numeris x compositam qui conditioni $x < y$ satisfaciunt, scilicet, *qui sunt minores* y, vel simpliciter *minores* y. Similiter, quum signum D significet *dividit*, vel *est divisor*, formula $\ni$D significat *qui dividunt* vel divisores. Deducitur $x\ \epsilon \ni \alpha\ y = x\ \alpha\ y$.

4. Sit α formula indeterminate continens x. Tunc scriptura $x'[x]\alpha$, quae legitur x' *loco* x *in* α *substituto*, formulam indicat quae obtinetur si in α, loco x, x' legimus. Deducitur $x[x]\alpha = \alpha$.

5. Sit α formula, quae indeterminata $x, y, ...$ continet. Tunc

$$(x', y', ...)\ [x, y, ...]\ \alpha,$$

61. $\{x|\neg\alpha\} = \overline{\{x|\alpha\}}$

62. $\{x|\alpha \vee \beta\} = \{x|\alpha\} \cup \{x|\beta\}$

63. $\alpha \underset{\forall x}{\longrightarrow} \beta = \{x|\alpha\} \subset \{x|\beta\}$

64. $(\alpha \underset{\forall x}{=} \beta) = (\{x|\alpha\} = \{x|\beta\})$

2. Let x, y be any entities. We shall consider the system composed of the entity x and the entity y as a new entity, and indicate it by the notation (x, y); and similarly if the number of entities becomes larger. Let α be a proposition containing the variables x, y; then $\{(x, y)|\alpha\}$ indicates the set of entities (x, y) which satisfy the condition α. We have:

65. $\alpha \underset{\forall x,y}{\longrightarrow} \beta = \{(x, y)|\alpha\} \subset \{(x, y)|\beta\}$

66. $\Big(\{(x, y)|\alpha\} \neq \varnothing\Big) = \Big(\{x|\{y|\alpha\} \neq \varnothing\}\} \neq \varnothing\Big)$

3. Let $x\ \alpha\ y$ be a relation between the variables x and y (e.g., in logic, the relations $x = y$, $x \neq y$, $x \to y$; in arithmetic, $x < y$, $x > y$, ...). Then the notation $[\in \alpha]\ y$ denotes the x that satisfy the relation $x\,\alpha\,y$. For the sake of convenience, we use the symbol $\ni$ instead of the notation $[\in]$. Thus, $\ni \alpha\, y = \{x|x\,\alpha\,y\}$, and the symbol $\ni$ is read *the objects that*. For example, let y be a number; then $\ni < y$ denotes the set formed by the numbers x that satisfy the condition $x < y$, that is, *the objects that are smaller than* y, or simply *the objects smaller than* y. Similarly, if the symbol $|$ means *divides* or *is a divisor of*, the formula $\ni |$ means *the objects that divide* or *the divisors*. It follows that $x \in (\ni \alpha\, y) = x\,\alpha\,y$.

4. Let α be a formula containing the variable x. Then the expression $\alpha[x := x']$, which is read x' *being substituted for* x *in* α, denotes the formula obtained if, in α, we read x' for x. It follows that $\alpha[x := x] = \alpha$.

5. Let α be a formula that contains the variables $x, y, ...$ Then

$$\alpha[x := x', y := y', \ldots],$$

quae legitur $x'y', \ldots$ *loco* $x, y, \ldots$ *in* α *substitutis*, formulam indicat quae obtinetur si in α loco $x, y, \ldots$, litterae $x'y', \ldots$ scribantur. Deducitur $(x, y)\ [x, y]\ \alpha = \alpha$.

VI. De functionibus.

Logicae notationes quae praecedunt exprimendae cuilibet arithmeticae propositioni sufficiunt, iisdemque tantum utimur. Hic notationes alias nonnullas breviter explicamus, quae utiles fieri possunt.

Sit s quaedam classis; supponimus aequalitatem inter entia systematis s definitam, quae conditionibus satisfaciat:

$$a = a$$

$$a = b\ .=.\ b = a$$

$$a = b\ .\ b = c : \supset .\ a = c$$

Sit φ signum, sive signorum aggregatus, ita ut si x est ens classis s, scriptura $\varphi\ x$ novum indicet ens; supponimus quoque aequalitatem inter entia $\varphi\ x$ definitam; et si x et y sunt entia classis s, et est $x = y$, supponimus deduci posse $\varphi\ x = \varphi\ y$. Tunc signum φ dicitur esse *functionis praesignum in classi* s, et scribemus $\varphi \ni F'\ s$.

$$s \epsilon \mathrm{K}\ . \supset :: \varphi\ F'\ s\ . = \therefore x, y\ \epsilon\ s\ .\ x = y : \supset_{x,y} .\ \varphi x = \varphi y$$

Verum si, cum sit x quodlibet ens classis s, scriptura $x\varphi$ novum indicet ens, et,

which is read $x', y', \ldots$ *being substituted for* $x, y, \ldots$ in α, denotes the formula obtained if, in α, the letters $x', y', \ldots$ are written for $x, y, \ldots$ It follows that $\alpha[x := x, y := y] = \alpha$.

VI. Functions.

COMMENTARY: Peano has a very structural or syntactical version of a function. There is no parameter. A presymbol function ("functionis praesignum") is text like "2 + " where if you append a number to the end, you have a valid expression. A postsymbol function would be "+ 2".

The symbols of logic introduced above suffice to express any proposition of arithmetic, and we shall only use these. We explain here briefly some other symbols that may be useful.

Let S be a set; we assume that equality is defined between the elements of the system S so as to satisfy the conditions:

$$a = a.$$

$$(a = b) = (b = a)$$

$$[(a = b) \wedge (b = c)] \rightarrow a = c$$

Let φ be a symbol or an aggregate of symbols such that, if x is an element of the set S, the expression φx denotes a new object; we assume also that equality is defined between the objects φx; further, if x and y are elements of the set S and if $x = y$, we assume it is possible to deduce $\varphi x = \varphi y$. Then the symbol φ is said to be a *function presymbol in the set* S, and we write $\varphi \ni \mathbf{FUNC}'\ S$.

COMMENTARY: In the description above, Peano writes $\varphi \ni \mathbf{FUNC}'\ S$. However, in the equation below, he drops the $\ni$ and writes $\varphi\ \mathbf{FUNC}'\ S$. He repeats this occurance with function postsymbols further on.

$$S \in \mathbf{SET} \rightarrow \varphi\ \mathbf{FUNC}'\ S = \left([(x, y \in S) \wedge x = y] \xrightarrow[\forall x,y]{} (\varphi x = \varphi y) \right)$$

If, x being any element of the set S, the expression $x\varphi$ denotes a new object and ex, $x = y$ deducitur $x\varphi = y\varphi$, tunc dicimus φ esse *functionis postsignum in classi* s et scribemus $\varphi \,\epsilon\, s\,'F$.

$$s \,\epsilon\, \mathrm{K} \,.\, \supset :: \varphi \; s\,'F \,.=\therefore x, y \,\epsilon\, s \,.\, x = y : \supset_{x,y} .\, x\varphi = y\varphi$$

Exempla. Sit a numerus; tunc $a\,+$ est functionis praesignum in numerorum classe, et $+\,a$ est functionis postsignum; quicumque enim est numerus x, formulae $a + x$ et $x + a$ novos indicant numeros, et ex $x = y$ deducitur $a + x = a + y$, et $x + a = y + a$. Itaque

$$a \,\epsilon\, \mathrm{N} \,.\supset : a + .\,\epsilon\,.\, F'\ \mathrm{N}$$

$$a \,\epsilon\, \mathrm{N} \,.\supset : +\, a \,.\,\epsilon\,.\, N\,'F$$

Sit φ functionis praesignum in classe s. Tunc $[\varphi]y$ classem significat iis x constitutam, quae conditioni $\varphi x = y$ satisfaciunt; scilicet:

Def. $\quad s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, F'\, s : \supset : [\varphi]y \,.=.\, [x\,\epsilon]\ (\varphi x = y)$

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Classis $[\varphi]y$ vel unum vel plura, vel etiam nullum individuum continere potest. Erit:

$$s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, F'\, s : \supset : y = \varphi\, x \,.=.\, x \,\epsilon\, [\varphi]y$$

Si vero φy uno tantum constat individuo, erit $y = \varphi x \,.=.\, x = [\varphi]y$

Sit φ functions postsignum; similiter ponimus:

$$s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, s\,'F : \supset \therefore y\,[\varphi] = [x\,\epsilon]\ (x\varphi = y).$$

Signum [ ] dicitur *inversionis signum*, eiusque usus nonullos in logica iam exposuimus. Nam si α est propositio indeterminatum continens x, atque A est classis individuis x composita quae conditioni α satisfaciunt, erit $x \,\epsilon\, a \,.=\alpha$, tunc $a = [x\epsilon]\ \alpha$, ut in V, i.

$x\varphi = y\varphi$ follows from $x = y$, then we say that φ is a *function postsymbol in the set* S, and we write $\varphi \in S\,'\mathbf{FUNC}$.

$$S \in \mathbf{SET} \to \varphi\ S\,'\mathbf{FUNC} = \Big([(x, y \in S) \wedge x = y] \xrightarrow[\forall x,y]{} x\varphi = y\varphi\Big)$$

Examples. Let a be a number; then $a\,+$ is a function presymbol in the set of numbers, and $+\,a$ is a function postsymbol; for any number x, formulas $a + x$ and $x + a$ denote new numbers; $a + x = a + y$ and $x + a = y + a$ follow from $x = y$. Thus

$$a \in \mathbb{N} \to [(a\,+) \in (\mathbf{FUNC}'\ \mathbb{N})]$$

$$a \in \mathbb{N} \to [(+\,a) \in (\mathbb{N}\,'\mathbf{FUNC})]$$

Let φ be a function presymbol in the set S. Then $[\varphi]y$ denotes the set composed of x that satisfy the condition $\varphi x = y$; that is,

Def. $\quad S \in \mathbf{SET} \wedge \varphi \in \mathbf{FUNC}'\ S \to [\varphi]y = \{x | \varphi x = y\}$

The set $[\varphi]y$ may contain one or several individuals, or none at all. We have:

$$S \in \mathbf{SET} \wedge \varphi \in \mathbf{FUNC}'\ S \to (y = \varphi x) = (x \in [\varphi]y)$$

But if φy consists of just one individual, we have $(y = \varphi x) = (x = [\varphi]y)$

Let φ be a function postsymbol; we write similarly:

$$S \in \mathbf{SET} \wedge \varphi \in S\,'\mathbf{FUNC} \to y[\varphi] = \{x | x\varphi = y\}.$$

The notation $\{x| \ldots\}$ is called *notation of the inverse*, and we have already presented some of its uses in logic. If α is a proposition containing the variable x and A is a set composed of the individuals x that satisfy the condition α, we have $(x \in A) = \alpha$, and then $A = \{x|\alpha\}$, as in V, 1.

COMMENTARY: In the following, Peano relies on the syntax of his substitution operator to create a presymbol function. The modern substitution operator has a different syntax and therefore doesn't work here.

Sit α formula indeterminate continens x, sitque φ functionis praesignum, quod litterae x praepositum, formulam α gignat; scilicet sit $\alpha = \varphi\ x$; tunc erit $\varphi = \alpha[x]$, et si x' est novum ens, erit $\varphi x' = \alpha[x]x'$, scilicet, si α est formula indeterminatum continens x, tunc $\alpha[x]x'$ significat id quod obtinetur si in α, loco x, x' ponatur.

Let α be a formula containing the variable x and let φ be a function presymbol that yields the formula α when written before the letter x; that is, let $\alpha = \varphi x$; then we have $\varphi = \alpha[x :=?]$, and if x' is a new object, we have $\varphi x' = \alpha[x := x']$, that is, if α is a formula containing the variable x, then $\alpha[x := x']$ means what is obtained when, in α, we put x' for x.

COMMENTARY: Below, Peano attempts to do the same with function postsymbols. However, it is unclear how it works, since the substitution operator is not symmetric.

Similiter, sit α formula indeterminate continens x, sitque φ functionis postsignum, ut $x\varphi = \alpha$; deducitur $\varphi = [x]\alpha$; tunc, si x' est novum ens, erit $x'\varphi = x'[x]\alpha$, scilicet $x'[x]\alpha$ rursum indicat id quod obtinetur si in α, loco x, x' legatur, ut in V, 4.

Similarly, let α be a formula containing the variable x and let φ be a function postsymbol, such that $x\varphi = \alpha$; it follows that $\varphi = ?[x := \alpha]$. Then, if x' is a new object, have $x'\varphi = x'[x := \alpha]$; that is, $x'[x := \alpha]$ again denotes what is obtained, when, in α, read x' for x, as in V, 4..

Alios quoque usus in logica signum [] habere potest, quos breviter esponimus, quum ipsis non utamur. Sint a et b duae classes; tunc $[a\cap]b$ sive $b[\cap a]$ classes indicat x, quae conditioni $b = a \cap x$, sive $b = x \cap a$ satisfaciunt. Si b in a non continetur, nulla classis huic conditioni satifacit; si b in a continetur, signum $b[\cap a]$ omnes indicat classes quae b continent atque in $b \cup -a$ continentur.

In Arithmetica, sint a, b numeri; tunc $[b + a]$ sive $[a+]b$ numerum indicat x, qui conditioni $b = x + a$, sive $b = a + x$ satisfacit, nempe $b - a$. Similiter erit $b[\times a] = [a\times]b = b/a$. Et in analysi hoc signum usuvenire potest; itaque

$y = \sin\ x\ . = .\ x\ \epsilon\ [\sin]\ y$ (loco $x = \text{arc}\ \sin\ y$).

$dF(x) = f(x)dx\ . = .\ F\ (x)\ \epsilon\ [d]\ f(x)dx$ (loco $F(x) = \int f(x)dx$).

The symbol [] can have other uses in logic, which we present only briefly, since we shall not use it in these ways. Let A and B be two sets; then $[A\cap]B$ or $B[\cap A]$ denotes the sets X that satisfy the condition $B = A \cap X$, or $B = X \cap A$. If B is not contained in A, no set satisfies this condition; if B is contained in A, the notation $B[\cap A]$ denotes all sets that contain B and are contained in $B \cup \overline{A}$.

In arithmetic, let a and b be numbers; then $[b + a]$ or $[a+]b$ denotes the number x that satisfies the condition $b = x + a$, or $b = a + x$, that is $b - a$. Similarly we have $b[\times a] = [a\times]b = b/a$. This notation can even find a use in analysis; thus

$(y = \sin x) = (x \in [\sin]y)$ (instead of $x = \arcsin y$).

$\Big(dF(x) = f(x)\mathrm{d}x\Big) = \Big(F(x) \in [d]f(x)\mathrm{d}x\Big)$ (instead of $F(x) = \int f(x)dx$).

Sit rursum φ functionis praesignum in classi s, sitque k classis

Let φ again be a function presymbol in a set S and let C be a set

in s contenta; tunc φk classem indicat omnibus φx compositam, ubi x sunt entia classis k; scilicet

contained in S; then φC denotes the set consisting of all φx, where the x are the elements of set C; that is

Def. $s \,\epsilon\, \mathrm{K} \,.\, k \,\epsilon\, \mathrm{K} \,.\, k \supset s \,.\, \varphi \,\epsilon\, F'\, s : \supset .\, \varphi\, k = [y\epsilon]\, (k \,.\, [\varphi]y : - = \Lambda)$

Sive $s \,\epsilon\, \mathrm{K} \,.\, k \,\epsilon\, \mathrm{K} \,.\, k \supset s \,.\, \varphi \,\epsilon\, F'\, s : \supset .\, \varphi\, k = [y\epsilon]\, ([x\, \epsilon] : x \,\epsilon\, k \,.\, [\varphi]x = y \therefore - = \Lambda)$

Def. $s \,\epsilon\, \mathrm{K} \,.\, k \,\epsilon\, \mathrm{K} \,.\, k \supset s \,.\, \varphi \,\epsilon\, s\,'F : \supset .\, k\, \varphi = [y\epsilon]\, (k \,.\, y\, [\varphi] : - = \Lambda)$

Itaque, si $\varphi \,\epsilon\, F's$, tunc $\varphi\, s$ classem indicat omnibus $\varphi\, x$ constitutam, ubi x sint entia classis s. Erit:

$$s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, F's \,.\, y \,\epsilon\, \varphi s : \supset : \varphi[\varphi]y = y$$

$$s \,\epsilon\, \mathrm{K} \,.\, a, b \,\epsilon\, \mathrm{K} \,.\, a \supset s \,.\, b \supset s \,.\, \varphi \,\epsilon\, F's : \supset .\, \varphi(a \cup b) = (\varphi a) \cup (\varphi b)$$

$$s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, F's : \supset .\, \varphi\, \Lambda = \Lambda$$

$$s \,\epsilon\, \mathrm{K} \,.\, a, b \,\epsilon\, \mathrm{K} \,.\, b \supset s \,.\, a \supset b \,.\, \varphi \,\epsilon\, F's : \supset .\, \varphi a \supset \varphi b$$

$$s \,\epsilon\, \mathrm{K} \,.\, a, b \,\epsilon\, \mathrm{K} \,.\, a \supset s \,.\, b \supset s \,.\, \varphi \,\epsilon\, F's : \supset .\, \varphi(ab) \supset (\varphi a)(\varphi b)$$

Sit a quaedam classis; tunc $a \cap \mathrm{K}$, sive $\mathrm{K} \cap a$, sive $\mathrm{K}\, a$, classes omnes indicat formae $a \cap x$, sive $x \cap a$, xa, ubi x est classis quacumque; scilicet $\mathrm{K}\, a$ indicat classes quae in a continentur. Formula $x \,\epsilon\, \mathrm{K}\, a$ idem significat quod $x \,\epsilon\, \mathrm{K} \,.\, x \supset a$. Hac conventione quandoque utimur; ita $\mathrm{K}\, N$ significat *numerorum classem.*

Similiter, si a est classis, $\mathrm{K} \cup a$ indicat classes quae a continent. Sit a numerus; tunc $a + N$, sive $N + a$, *numeros* indicat *numero a maiores*; $a \times N$, sive $N \times a$, sive Na indicat *multiplices numeri a*; a^N indicat *potestas numeri a*; N^2, N^3, ... indicat *numeros quadratos*, vel *numeros cubos*,...

Functional signorum aequalitatem, productum, potestas, ita definire licet:

Def. $s \,\epsilon\, \mathrm{K} \,.\, \varphi, \psi \,\epsilon\, F's : \supset \therefore \varphi = \psi : = : x \,\epsilon\, s \,.\, \supset .\, \varphi x = \psi x$

Def. $s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, F's \,.\, \psi \,\epsilon\, F'\varphi s \,.\, x \,\epsilon\, s : \supset .\, \psi\varphi x = \psi(\varphi x)$

Def. $S \in \mathbf{SET} \wedge K \in \mathbf{SET} \wedge K \subset S \wedge \varphi \in \mathbf{FUNC}'\, S \rightarrow \varphi K = \{y | K \cap [\varphi]y \neq \varnothing\}$

Or $S \in \mathbf{SET} \wedge K \in \mathbf{SET} \wedge K \subset S \wedge \varphi \in \mathbf{FUNC}'\, s \rightarrow \varphi K = \{y | \{x | x \in K \wedge [\varphi]x = y\} \neq \varnothing\}$

Def. $S \in \mathbf{SET} \wedge K \in \mathbf{SET} \wedge K \subset S \wedge \varphi \in S\,'\mathbf{FUNC} \rightarrow K\varphi = \{y | (K \cap y[\varphi]) \neq \varnothing\}$

Thus, if $\varphi \in \mathbf{FUNC}'\, S$, then φS denotes the set composed of all φx, where the x are elements of the set S. We have:

$$S \in \mathbf{SET} \wedge \varphi \in \mathbf{FUNC}'\, S \wedge y \in \varphi S \rightarrow \varphi[\varphi]y = y$$

$$S \in \mathbf{SET} \wedge (A, B \in \mathbf{SET}) \wedge A \subset S \wedge B \subset S \wedge \varphi \in \mathbf{FUNC}'\, S \rightarrow \varphi(A \cup B) = [(\varphi A) \cup (\varphi B)]$$

$$S \in \mathbf{SET} \wedge \varphi \in \mathbf{FUNC}'\, S \rightarrow \varphi\varnothing = \varnothing$$

$$S \in \mathbf{SET} \wedge (A, B \in \mathbf{SET}) \wedge B \subset S \wedge A \subset B \wedge \varphi \in \mathbf{FUNC}'\, S \rightarrow \varphi A \subset \varphi B$$

$$S \in \mathbf{SET} \wedge (A, B \in \mathbf{SET}) \wedge A \subset S \wedge B \subset S \wedge \varphi \in \mathbf{FUNC}'\, S \rightarrow \varphi(A \cap B) \subset [(\varphi A) \cap (\varphi B)]$$

Let A be a set; then $A \cap \mathbf{SET}$, or $\mathbf{SET} \cap A$, or $\mathbf{SET}A$, denotes all sets of the form $A \cap X$, or $X \cap A$, XA, where X is any set; that is $\mathbf{SET}A$ denotes the sets that are contained in A. The formula $X \in \mathbf{SET}A$ means the same as $X \in \mathbf{SET} \wedge X \subset A$. We shall sometimes use this convention; thus $\mathbf{SET}\mathbb{N}$ means *a set of numbers.*

COMMENTARY: Peano's natural numbers start at 1. This is why in the following text, $a + \mathbb{N}$ is the set of numbers "greater than a", rather than "greater than or equal to a".

Similarly, if A is a set, $\mathbf{SET} \cup A$ indicates the sets that contain A. Let a be a number; then $a + \mathbb{N}$, or $\mathbb{N} + a$, denotes *the numbers greater than the number a*; $a \times \mathbb{N}$, or $\mathbb{N} \times a$, or $\mathbb{N}a$ denotes *the multiples of the number a*; $a^{\mathbb{N}}$ denotes *the powers of the number a*; $\mathbb{N}^2$, $\mathbb{N}^3$, ... denote *the squares*, *the cubes*,...

Equality, product and powers can be defined thus for function symbols:

Def. $S \in \mathbf{SET} \wedge (\varphi, \psi \in \mathbf{FUNC}'\, S) \rightarrow (\varphi = \psi) = (x \in S \rightarrow \varphi x = \psi x)$

Def. $S \in \mathbf{SET} \wedge \varphi \in \mathbf{FUNC}'\, S \wedge \psi \in \mathbf{FUNC}'\, \varphi S \wedge x \in S \rightarrow \psi\varphi x = \psi(\varphi x)$

Itaque, in definitionis hypothesi, erit $\psi\varphi$ novum functionis praesignum; idque *productum signorum* ψ *et* φ vocatur.

Similiterque, si φ, ψ sunt functionis postsigna.

Haec valet propositio:

$$s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, F's \,.\, \varphi s \supset s :\supset: \varphi\varphi s \supset s \,.\, \varphi\varphi\varphi s \supset s. \text{ etc.}$$

Funcitones $\varphi\varphi, \varphi\varphi\varphi, \ldots$ *iteraiae* vocantur, et communiter signis $\varphi^2, \varphi^3, \ldots$ indicantur, ut operationis φ potestates.

Thus, if we assume this definition, we have the new function presymbol $\psi\varphi$; it is called the *product of the symbols* ψ *and* φ.

Similarly if φ, ψ are function postsymbols.

The following proposition holds:

$$S \in \mathbf{SET} \wedge \varphi \in \mathbf{FUNC}'\, S \wedge \varphi S \subset S \rightarrow (\varphi\varphi S \subset S \wedge \varphi\varphi\varphi S \subset S \wedge \ldots)$$

The functions $\varphi\varphi, \varphi\varphi\varphi, \ldots$ are said to be *iterated* and are generally denoted by the symbols $\varphi^2, \varphi^3, \ldots$ as powers of the operation φ.

XVI

Si vero φ est functionis postsignum, ha faciliori notatione, absque ambiguitate, uti licet:

Def. $\quad s \,\epsilon\, \mathrm{K} \,.\, \varphi \,\epsilon\, s'F \,.\, s\varphi \supset s :\supset: \varphi 1 = \varphi \,.\, \varphi 2 = \varphi\varphi \,.\, \varphi 3 = \varphi\varphi\varphi.$ etc.

In definitionis hypothesi, si $m, n \,\epsilon\, N$, erit $\varphi\ (m+n) = (\varphi m)(\varphi n)$; $(\varphi m)n = \varphi(mn)$

Si hac definitione in Arithmetica utimur, haec invenimus. Numerum qui sequitur numerum a signo faciliori $a+$ indicare possumus; tunc $a+1, a+2, \ldots$ et, si b est numerus, $a+b$, sensum habent $a+, a++, \ldots$ quod a definitione in §1 patet. Propositionem 6 in §1 scribere possumus $N+ \supset N$. Si a, b, c sunt numeri, tunc $a :+b \,.\, c$ significat $a+bc$, et $a : \times b \,.\, c$ significat ab^c.

Multi aliis proprietatibus gaudent functionem signa, praesertim si conditioni satisfaciunt: $\varphi x = \varphi y \,.\, \supset .\ x = y$. Functionis signum quod huic conditioni satisfacit vocatur a clarrissimo Dedekind *simile* (ähnliche Abbildung).

Sed his exponendis locus deest.

But if φ is a function postsymbol, we can use the following more convenient notation without ambiguity:

Def. $\quad S \in \mathbf{SET} \wedge \varphi \in S\,'\mathbf{FUNC} \wedge S\varphi \subset S \rightarrow (\varphi 1 = \varphi \wedge \varphi 2 = \varphi\varphi \wedge \varphi 3 = \varphi\varphi\varphi \wedge \ldots)$

Assuming this definition, if $m, n \in \mathbb{N}$, we have $\varphi(m+n) = (\varphi m)(\varphi n)$; $(\varphi m)n = \varphi(mn)$

If we use this definition in arithmetic, we obtain the following. We can denote the number that follows the number a by the more convenient notation $a+$; then $a+1$, $a+2$,..., and, if b is a number, $a+b$, have the meaning of $a+$, $a++$,..., which is clear from the definition in §1 below. Proposition 6 in §1 can be written $\mathbb{N}+ \subset \mathbb{N}$. If a, b, c are numbers, then $a :+b \,.\, c$ means $a+bc$, and $a : \times b \,.\, c$ means ab^c.

Function symbols possess many other properties, especially if they satisfy the condition: $(\varphi x = \varphi y) \rightarrow (x = y)$. A function symbol that satisfies this condition is called *equivalent*[2] by Dedekind (ähnliche Abbildung).

But we lack the space to present these properties.

Declarationes.

Defenitio, vel breviter *Def.* est propositio formam habens $x = a$, sive $\alpha \supset .\ x = a$, ubi α est signorum aggregatus sensum habens notum; x est signum, vel

Remarks.

A *Definition*, or *Def.* for short, is a proposition of the form $x = a$ or $\alpha \rightarrow (x = a)$, where a is an aggregate of symbols having a known meaning, x is a symbol or an signorum aggregatus significatione adhuc carnes; α vero est conditio sub qua definitio datur.

Theorema, (Theor. vel Th) est propositio quae demonstratur. Si theorema formam habet $\alpha \supset \beta$, ubi α et β sunt propositiones, tunc α dicitur *Hypothesis* (Hyp. vel breviter Hp.), β vero *Thesis* (Thes. vel Ts.). Hyp. ac Ts. a Theorematis forma pendent; nam si loco $\alpha \supset \beta$ scribemus $-\beta \supset -\alpha$, erit $-\beta$ Hp, et $-\alpha$ Ts.; si vero scribemus $\alpha - \beta = \Lambda$, Hp. ac Ts. absunt.

In quolibet § signum P quod quidam numerus sequatur, propositionem indicat eiusdem § hoc numero signatam. Logicae propositiones indicantur signo L et propositiones numero.

Formulae quae in una linea non continentur, in altera linea, nullo interposito signo, sequuntur.

aggregate of symbols, hitherto without meaning, and α is the condition under which the definition is given.

A *theorem* (Theor. or Th.) is a proposition that is proved. If a theorem has the form $\alpha \supset \beta$, where α and β are propositions, then α is called the *hypothesis* (Hyp., or even shorter, Hp.) and β the *thesis* (Thes. or Ts.). Hyp. and Ts. depend on the form of the theorem; in fact, if we write $\alpha \supset \beta$ instead of $-\beta \supset -\alpha$, then $-\beta$ is the Hp., and $-\alpha$ the Ts.; if we write $\alpha - \beta = \Lambda$, Hp. and Ts. do not exist.

In any § below, the symbol P followed by a number denotes the proposition indicated by that number in the same §. Propositions of logic are indicated by the symbol L and the number of the proposition.

Formulas that do not fit on one line are continued on the next line without any intervening symbol.

ARITHMETICES PRINCIPIA.

§1. De numeris et de additione.

Explicationes.

Signo	N	significatur	*numerus (integer positivus).*
≫	1	≫	*unitas.*
≫	$a+1$	≫	*sequens* a, sive a *plus* 1.
≫	$=$	≫	*est aequalis.* Hoc ut novum signum considerandum est, etsi logicae signi figuram habeat.

Axiomata.

1. $1 \,\epsilon\, \mathrm{N}\,.$
2. $a \,\epsilon\, \mathrm{N} \,.\supset.\, a = a.$
3. $a, b, c \,\epsilon\, \mathrm{N} \,.\supset:\, a = b \,.=.\, b = a.$
4. $a, b \,\epsilon\, \mathrm{N} \,.\supset\therefore\, a = b \,.\, b = c :\supset.\, a = c.$

THE PRINCIPLES OF ARITHMETIC.

§1. Numbers and addition.

Explanations.

COMMENTARY: Peano starts his natural numbers at 1; most modern versions start at 0. Also, while Peano calls it "successor" ("sequens"), he does not use the modern convention of using "S" for it.

The symbol	$\mathbb{N}$	means	*number (positive integer).*
”	1	”	*unity.*
”	$a+1$	”	*the successor of* a, or a *plus* 1.
”	$=$	”	*is equal to.* This must be considered as a new symbol, although it has the appearance of a symbol of logic.

Axioms.

1. $1 \in \mathbb{N}.$
2. $a \in \mathbb{N} \to a = a.$
3. $a, b, c \in \mathbb{N} \to [(a = b) = (b = a)].$
4. $a, b \in \mathbb{N} \to [(a = b \wedge b = c) \to a = c].$

5. $a = b \,.\, b \;\epsilon\; \mathrm{N} : \supset . \, a \;\epsilon\; \mathrm{N} \,.$

6. $a \;\epsilon\; \mathrm{N} \,. \supset . \, a + 1 \;\epsilon\; \mathrm{N} \,.$

7. $a, b \;\epsilon\; \mathrm{N} \,. \supset : a = b \,. = . \, a + 1 = b + 1.$

8. $a \;\epsilon\; \mathrm{N} \,. \supset . \, a + 1 \mathbin{-}= 1.$

9. $k \;\epsilon\; \mathrm{K} \;\therefore 1 \;\epsilon\; k \therefore x \;\epsilon\; \mathrm{N} \,.\, x \;\epsilon\; k : \supset_x . \, x + 1 \;\epsilon\; k :: \supset . \, \mathrm{N} \supset k.$

Definitiones.

10. $2 = 1 + 1$; $3 = 2 + 1$; $4 = 3 + 1$; etc.

5. $(a = b \land b \in \mathbb{N}) \to a \in \mathbb{N}.$

6. $a \in \mathbb{N} \to (a + 1 \in \mathbb{N}).$

7. $a, b \in \mathbb{N} \to [(a = b) = (a + 1 = b + 1)].$

8. $a \in \mathbb{N} \to (a + 1 \neq 1).$

9. $\Big(A \in \mathbf{SET} \land 1 \in A \land [(x \in \mathbb{N} \land x \in A) \underset{\forall x}{\longrightarrow} (x + 1 \in A)]\Big) \to \mathbb{N} \subset A.$

Definitions.

10. $2 = 1 + 1$; $3 = 2 + 1$; $4 = 3 + 1$; etc.

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Theoremata.

11. $2 \;\epsilon\; \mathrm{N} \,.$

Demonstratio:

$\mathrm{P}1 \,. \supset :$	$1 \;\epsilon\; N$	(1)

Theorems.

COMMENTARY: Below is Peano's first proof. The proof style taught in American high schools has two columns: the left column has mathematical statements and the right column has the rationale for each statement. In this first proof, Peano's style is similar except he puts the rationale in the left column with a $\to$ and the statement in the right column. Essentially, the rationale implies the statement.

COMMENTARY: Peano's proof notation uses "P1" and "P6" to refer to propositions #1 and #6 proved earlier. The "$[a := 1]$" after "(P6)" is notation for substituting 1 for a in proposition #6. The notations "(1)" and "(2)" refer to the results of steps 1 and 2 of the proof. Modus ponens is indicated by anding the antecedent with the implication, as in "$(1) \land (2)$".

11. $2 \in \mathbb{N}.$

Proof:

$\mathrm{P}1 \to$	$1 \in \mathbb{N}$	(1)

$1[a](\mathrm{P6}) . \supset :$	$1 \,\epsilon\, \mathrm{N} . \supset . 1+1 \,\epsilon\, N$	(2)
$(1)(2) . \supset :$	$1+1 \,\epsilon\, N$	(3)
$\mathrm{P10} . \supset :$	$2 = 1+1$	(4)
$(4) .(3) .(2, 1+1)[a, b](\mathrm{P5}) : \supset :$	$2 \,\epsilon\, N$	(Theor.)

Nota. - Huius facillimae demonstrationis gradus omnes ecplicite scripsimus. Brevitatis causa ipsam ita scribemus:

$$\mathrm{P1} . 1[a](\mathrm{P6}) : \supset : 1+1 \,\epsilon\, \mathrm{N} . \mathrm{P10} .(2, 1+1)[a, b](\mathrm{P5}) : \supset : \mathrm{Th.}$$

vel

$$\mathrm{P1} . \mathrm{P6} : \supset : 1+1 \,\epsilon\, \mathrm{N} . \mathrm{P10} . \mathrm{P5} : \supset : \mathrm{Th.}$$

12. $3, 4, \ldots \,\epsilon\, \mathrm{N}$.

13. $a, b, c, d \,\epsilon\, \mathrm{N} . a = b . b = c . c = d : \supset : a = d.$

Dem. $\mathrm{Hyp.P4} : \supset : a, c, d \,\epsilon\, \mathrm{N} . a = c . c = d . \mathrm{P4} : \supset :$ Thes.

14. $a, b, c \,\epsilon\, \mathrm{N} . a = b . b = c . a - = c := \Lambda.$

Dem. $\mathrm{P4} . \mathrm{L39} : \supset .$ Theor.

15. $a, b, c \,\epsilon\, \mathrm{N} . a = b . b - = c : \supset . a - = c.$

16. $a, b \,\epsilon\, \mathrm{N} . a+1 = b+1 : \supset . a = b.$

17. $a, b \,\epsilon\, \mathrm{N} . \supset : a - = b . = . a+1 - = b+1.$

Dem. $\mathrm{P7} . \mathrm{L21} : \supset .$ Theor.

$(\mathrm{P6})[a := 1] \rightarrow$	$1 \in \mathbb{N} \rightarrow (1+1 \in \mathbb{N})$	(2)
$(1) \wedge (2) \rightarrow$	$1+1 \in \mathbb{N}$	(3)
$\mathrm{P10} \rightarrow$	$2 = 1+1$	(4)
$(4) \wedge (3) \wedge (\mathrm{P5})[a := 2, b := 1+1] \rightarrow$	$2 \in \mathbb{N}$	(Theor.)

Note. - We have explicitly written every step of this very easy proof. For the sake of brevity, we shall write it as follows:

COMMENTARY: Since each statement in Peano's proofs is an implication, they can be combined using the axioms of logic. Peano shortens proofs by combining many steps together.

$$\Big(\mathrm{P1} \wedge (\mathrm{P6})[a := 1]\Big) \rightarrow \Big((1+1 \in \mathbb{N}) \wedge \mathrm{P10} \wedge (\mathrm{P5})[a := 2, b := 1+1]\Big) \rightarrow \mathrm{Th.}$$

or

$$[\mathrm{P1} \wedge \mathrm{P6}] \rightarrow [(1+1 \in \mathbb{N}) \wedge \mathrm{P10} \wedge \mathrm{P5}] \rightarrow \mathrm{Th.}$$

12. $3, 4, \ldots \in \mathbb{N}.$

COMMENTARY: Below, L# refers to a proposition of logic from the first half of this book. The text "theorem", "Theor" or "Th" refers to what is to be proven. If the theorem is an implication, "hypothesis" or "Hyp" refers to its antecedent and "thesis", "Thes" or "Ts" refers to its consequent.

13. $(a, b, c, d \in \mathbb{N} \wedge a = b \wedge b = c \wedge c = d) \rightarrow a = d.$

Proof $(\mathrm{Hyp} \wedge \mathrm{P4}) \rightarrow [(a, c, d \in \mathbb{N}) \wedge a = c \wedge c = d \wedge \mathrm{P4}] \rightarrow$ Thes.

14. $(a, b, c \in \mathbb{N} \wedge a = b \wedge b = c \wedge a \neq c) = \perp.$

Proof $(\mathrm{P4} \wedge \mathrm{L39}) \rightarrow$ Theor.

15. $(a, b, c \in \mathbb{N} \wedge a = b \wedge b \neq c) \rightarrow a \neq c.$

16. $[a, b \in \mathbb{N} \wedge (a+1 = b+1)] \rightarrow a = b.$

17. $(a, b \in \mathbb{N}) \rightarrow [(a \neq b) = (a+1 \neq b+1)].$

Proof $(\mathrm{P7} \wedge \mathrm{L21}) \rightarrow$ Theor.

Definitio.

18. $a, b \,\epsilon\, \mathrm{N} \,.\supset .\, a + (b+1) = (a+b)+1$

Nota. - Hanc definitionem ita legere oportet: si a et b sunt numeri, et $(a+b)+1$ sensum habet (scilicet si $a+b$ est numerus), sed $a+(b+1)$ nondum definitus est, tunc $a+(b+1)$ significat numerum qui $a+b$ sequitur.

Ab hac definitione, et a praecedentibus deducitur:

$$a \,\epsilon\, \mathrm{N} \,.\supset \therefore a+2 = a+(1+1) = (a+1)+1$$

$$a \,\epsilon\, \mathrm{N} \,.\supset \therefore a+3 = a+(2+1) = (a+2)+1, \text{ etc.}$$

Definition.

18. $(a, b \in \mathbb{N}) \to [a + (b+1) = (a+b)+1]$

Note. - This definition should be read: if a and b are numbers, and $(a+b)+1$ has meaning (that is, if $a+b$ is a number), but $a+(b+1)$ has not yet been defined, then $a+(b+1)$ indicates the number that follows $a+b$.

From this definition, and the preceding, we deduce that:

$$a \in \mathbb{N} \to a+2 = a+(1+1) = (a+1)+1$$

$$a \in \mathrm{N} \to a+3 = a+(2+1) = (a+2)+1, \text{ etc.}$$

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Theoremata.

19. $a, b \,\epsilon\, \mathrm{N} \,.\supset .\, a+b \,\epsilon\, N.$

Dem. $a \,\epsilon\, \mathrm{N} \,.\, \mathrm{P6} :\supset: a+1 \,\epsilon\, \mathrm{N} :\supset: 1 \,\epsilon\, [b\,\epsilon]\ \mathrm{Ts}.$ (1)

$a \,\epsilon\, \mathrm{N} \,.\supset :: b \,\epsilon\, \mathrm{N} \,.\, b \,\epsilon\, [b\,\epsilon]\ \mathrm{Ts} :\supset: a+b \,\epsilon\, \mathrm{N} \,.\, \mathrm{P6}$

$:\supset: (a+b)+1 \,\epsilon\, \mathrm{N} \,.\, \mathrm{P18} :\supset: a+(b+1) \,\epsilon\, \mathrm{N} :\supset: (b+1) \,\epsilon\, [b\,\epsilon]$ Ts. (2)

$a \,\epsilon\, \mathrm{N} \,.(1)\,.(2)\,.\supset :: 1 \,\epsilon\, [b\,\epsilon]\ \mathrm{Ts} \therefore b \,\epsilon\, \mathrm{N} \,.\, b \,\epsilon\, [b\,\epsilon]\ \mathrm{Ts}$

$:\supset: b+1 \,\epsilon\, [b\,\epsilon]\ \mathrm{Ts} \therefore ([b\,\epsilon]\ \mathrm{Ts}\,)[k]\ \mathrm{P9} :: \supset : \mathrm{N} \supset [b\,\epsilon]\ \mathrm{Ts} \,.\, (\mathrm{L50})$

$:: \supset : b \,\epsilon\, \mathrm{N} \,.\supset \mathrm{Ts}.$ (3)

$(3)\,.\, (\mathrm{L42}) :\supset: a, b \,\epsilon\, \mathrm{N} \,.\supset .$ Thesis. (Theor.)

20. *Def.* $a+b+c = (a+b)+c.$

21. $a, b, c \,\epsilon\, \mathrm{N} \,.\supset .\, a+b+c \,\epsilon\, N.$

22. $a, b, c \,\epsilon\, \mathrm{N} \,.\supset : a = b \,.=.\, a+c = b+c.$

Dem. $a, b \,\epsilon\, \mathrm{N} \,.\, \mathrm{P7} :\supset .\, 1 \,\epsilon\, [c\,\epsilon]\mathrm{Ts}.$ (1)

$a, b \,\epsilon\, \mathrm{N} \,.\supset :: c \,\epsilon\, \mathrm{N} \,.\, c \,\epsilon\, [c\,\epsilon]\ \mathrm{Ts} \therefore \supset \therefore a = b \,.=.\, a+c = b+c :$

$a+c, b+c \,\epsilon\, \mathrm{N} : a+c = b+c \,.=.\, a+c+1 = b+c+1 \therefore \supset \therefore a =$

Theorems.

19. $a, b \in \mathbb{N} \to a+b \in \mathbb{N}.$

Proof $(a \in \mathbb{N} \wedge \mathrm{P6}) \to a+1 \in \mathbb{N} \to 1 \in \{b|\mathrm{Ts}\}.$ (1)

$a \in \mathbb{N} \to \Big((b \in \mathbb{N} \wedge b \in \{b|\mathrm{Ts}\}) \to (a+b \in \mathbb{N} \wedge \mathrm{P6}) \to$

$[(a+b)+1 \in \mathbb{N} \wedge \mathrm{P18}] \to a+(b+1) \in \mathbb{N} \to (b+1) \in \{b|\mathrm{Ts}\}\Big).$ (2)

$[a \in \mathbb{N} \wedge (1) \wedge (2)] \to \Big(1 \in \{b|\mathrm{Ts}\} \wedge [(b \in \mathbb{N} \wedge b \in \{b|\mathrm{Ts}\}) \to b+1 \in$

$\{b|\mathrm{Ts}\}] \wedge \mathrm{P9}[A := \{b|\mathrm{Ts}\}]\Big) \to [\mathbb{N} \subset \{b|\mathrm{Ts}\} \wedge \mathrm{L50}] \to (b \in \mathbb{N} \to \mathrm{Ts}).$

(3)

$[(3) \wedge\ (\mathrm{L42})\ ] \to [(a, b \in \mathbb{N}) \to$ Thesis $].$ (Theor.)

20. *Def.* $a+b+c = (a+b)+c.$

21. $a, b, c \in \mathbb{N} \to a+b+c \in \mathbb{N}.$

22. $a, b, c \in \mathbb{N} \to (a = b) = (a+c = b+c).$

Proof $[(a, b \in \mathbb{N}) \wedge \mathrm{P7}] \to 1 \in \{c|\mathrm{Ts}\}.$ (1)

$a, b \in \mathbb{N} \to \Big[(c \in \mathbb{N} \wedge c \in \{c|\mathrm{Ts}\}) \to \Big([(a = b) = (a+c =$

$b+c)] \wedge (a+c, b+c \in \mathbb{N}) \wedge [(a+c = b+c) = (a+c+1 = b+c+1)]\Big) \to$

$b \,.=.\, a + (c+1) = b + (c+1) \therefore \text{Ɔ} \therefore (c+1) \;\epsilon\; [c\;\epsilon]\mathrm{Ts}.$ (2)

$a, b \;\epsilon\; \mathrm{N} \,.(1)\,.(2) : \text{Ɔ} :: 1 \;\epsilon\; [c\;\epsilon]\; \mathrm{Ts} \;\therefore c \;\epsilon\; [c\;\epsilon]\; \mathrm{Ts} \,.\text{Ɔ}.(c+1) \;\epsilon\; [c\;\epsilon]$

$\mathrm{Ts} :: \text{Ɔ} :: c \;\epsilon\; \mathrm{N} \,.\text{Ɔ}.\, \mathrm{Ts}.$ (3)

23. $a, b, c \;\epsilon\; \mathrm{N} \,.\text{Ɔ}.\, a + (b+c) = a + b + c.$

Dem. $a, b \;\epsilon\; \mathrm{N} \,.\, \mathrm{P18} \,.\, \mathrm{P20} : \text{Ɔ} .\, 1 \;\epsilon\; [c\;\epsilon]\; \mathrm{Ts}.$ (1)

$a, b \;\epsilon\; \mathrm{N} \,.\text{Ɔ} \therefore c \;\epsilon\; \mathrm{N} \,.\, c \;\epsilon\; [c\;\epsilon]\; \mathrm{Ts} : \text{Ɔ} : a + (b+c) = a + b + c \,.\, \mathrm{P7}$

$: \text{Ɔ} : a + (b+c) + 1 = a + b + c + 1 \,.\, \mathrm{P18}$

$: \text{Ɔ} : a + (b + (c+1)) = a + b + (c+1) : \text{Ɔ} .\, c + 1 \;\epsilon\; [c\;\epsilon]\; \mathrm{Ts}.$ (2)

$(1)(2)\ (\mathrm{P9}) \,.\text{Ɔ}.\, \mathrm{Theor.}$

24. $a \;\epsilon\; \mathrm{N} \,.\text{Ɔ}.\, 1 + a = a + 1.$

Dem. $\mathrm{P2} \,.\text{Ɔ}.\, 1 \;\epsilon\; [a\;\epsilon]\; \mathrm{Ts}.$ (1)

$a \;\epsilon\; \mathrm{N} \,.\, a \;\epsilon\; [a\;\epsilon]\; \mathrm{Ts}$

$: \text{Ɔ} : 1 + 1 = a + 1 : \text{Ɔ} : 1 + (a+1) = (a+1) + 1 : \text{Ɔ} : (a+1) \;\epsilon\; [a\;\epsilon]$

$\mathrm{Ts}.$ (2)

$(1)(2) \,.\text{Ɔ}.\, \mathrm{Theor.}$

24'. $a, b \;\epsilon\; \mathrm{N} \,.\text{Ɔ}.\, 1 + a + b = a + 1 + b.$

Dem. $\mathrm{Hyp.}\ \mathrm{P24} : \text{Ɔ} : 1 + a = a + 1 \,.\, \mathrm{P22} : \text{Ɔ} .\, \mathrm{Thesis.}$

$\Big((a = b) = [a + (c+1) = b + (c+1)]\Big) \to (c+1) \in \{c|\mathrm{Ts}\}\Big].$ (2)

$[(a, b \in \mathbb{N}) \wedge (1) \wedge (2)] \to \Big(1 \in \{c|\mathrm{Ts}\} \wedge [c \in \{c|\mathrm{Ts}\} \to (c+1) \in$

$\{c|\mathrm{Ts}\}]\Big) \to [(c \in \mathbb{N}) \to \mathrm{Ts}].$ (3)

23. $a, b, c \in \mathbb{N} \to a + (b+c) = a + b + c.$

Proof $[(a, b \in \mathbb{N}) \wedge \mathrm{P18} \wedge \mathrm{P20}] \to 1 \in \{c|\mathrm{Ts}\}.$ (1)

$a, b \in \mathbb{N} \to \Big[(c \in \mathbb{N} \wedge c \in \{c|\mathrm{Ts}\}) \to [a + (b+c) =$

$a + b + c \wedge \mathrm{P7}] \to [a + (b+c) + 1 = a + b + c + 1 \wedge \mathrm{P18}] \to$

$\Big(a + [b + (c+1)] = a + b + (c+1)\Big) \to c + 1 \in \{c|\mathrm{Ts}\}\Big].$ (2)

$[(1) \wedge (2) \wedge \mathrm{P9}] \to \mathrm{Theor.}$

24. $a \in \mathbb{N} \to 1 + a = a + 1.$

Proof $\mathrm{P2} \to 1 \in \{a|\mathrm{Ts}\}.$ (1)

$(a \in \mathbb{N} \wedge a \in \{a|\mathrm{Ts}\}) \to (1 + 1 = a + 1) \to [1 + (a+1) =$

$(a+1) + 1] \to [(a+1) \in \{a|\mathrm{Ts}\}].$ (2)

$[(1) \wedge (2)] \to \mathrm{Theor.}$

24'. $a, b \in \mathbb{N} \to 1 + a + b = a + 1 + b.$

Proof $(\mathrm{Hyp.} \wedge \mathrm{P24}) \to [(1 + a = a + 1) \wedge \mathrm{P22}] \to \mathrm{Thesis.}$

25. $a, b \;\epsilon\; \mathrm{N} \,.\text{Ɔ}.\, a + b = b + a.$

Dem. $a \;\epsilon\; \mathrm{N} \,.\, \mathrm{P24} : \text{Ɔ} : 1 \;\epsilon\; [b\;\epsilon]\; \mathrm{Ts}.$ (1)

$a \;\epsilon\; \mathrm{N} \,.\text{Ɔ} \therefore b \;\epsilon\; \mathrm{N} \,.\, b \;\epsilon\; [b\;\epsilon]\; \mathrm{Ts} : \text{Ɔ} : a + b = b + a \,.\, \mathrm{P7}$ (2)

$: \text{Ɔ} : (a+b)+1 = (b+a)+1 \,.(a+b)+1 = a+(b+1) \,.(b+a)+1 =$

$1+(b+a) \,.\, 1+(b+a) = (1+b)+a \,.(1+b)+a = (b+1)+a :$

$\text{Ɔ} : a + (b+1) = (b+1) + a : \text{Ɔ} : (b+1) \;\epsilon\; [b\;\epsilon]\; \mathrm{Ts}.$

$(1)(2) \,.\text{Ɔ}.\, \mathrm{Theor.}$

26. $a, b, c \;\epsilon\; \mathrm{N} \,.\text{Ɔ} : a = b \,.=.\, c + a = c + b.$

27. $a, b, c \;\epsilon\; \mathrm{N} \,.\text{Ɔ} : a + b + c = a + c + b.$

25. $a, b \in \mathbb{N} \to a + b = b + a.$

Proof $a \in \mathbb{N} \wedge \mathrm{P24} \to 1 \in \{b|Ts\}.$ (1)

$a \in \mathbb{N} \to \Big((b \in \mathbb{N} \wedge b \in \{b|\mathrm{Ts}\}) \to (a + b = b + a \wedge \mathrm{P7}) \to$ (2)

$[(a+b)+1 = (b+a)+1 \wedge (a+b)+1 = a+(b+1) \wedge (b+a)+1 =$

$1+(b+a) \wedge 1+(b+a) = (1+b)+a \,.(1+b)+a = (b+1)+a] \to$

$[a + (b+1) = (b+1) + a] \to [(b+1) \in \{b|\mathrm{Ts}\}]\Big).$

$(1)(2) \to \mathrm{Theor.}$

26. $a, b, c \in \mathbb{N} \to (a = b) = (c + a = c + b).$

27. $a, b, c \in \mathbb{N} \to a + b + c = a + c + b.$

28. $a, b, c, d \,\epsilon\, \mathrm{N} \,.\, a = b \,.\, c = d : \supset . \, a + c = b + d.$

§2. De substractione.

Explicationes.

Signo	$-$	legitur	*minus.*
≫	$<$	≫	*est minor.*
≫	$>$	≫	*est maior.*

Definitiones.

1. $a, b \,\epsilon\, \mathrm{N} \,.\supset : b - a = \mathrm{N}\,[x\,\epsilon](x + a = b).$

2. $a, b \,\epsilon\, \mathrm{N} \,.\supset : a < b \,.=.\, b - a \,-\!= \Lambda.$

3. $a, b \,\epsilon\, \mathrm{N} \,.\supset : b > a \,.=.\, a < b.$

$a{+}b{-}c = (a{+}b){-}c; a{-}b{+}c = (a{-}b){+}c; a{-}b{-}c = (a{-}b){-}c.$

Theoremata.

4. $a, b, a', b' \,\epsilon\, \mathrm{N} \,.\, a = a' \,.\, b = b' : \supset : b - a = b' - a'.$

Dem. Hyp $.\supset : x + a = b \,.=.\, x + a' = b' : \supset .$ Thesis.

5. $a, b \,\epsilon\, \mathrm{N} \,.\supset : a < b \,.=.\, b - a \,\epsilon\, N.$

Dem. $a, b \,\epsilon\, \mathrm{N} : \supset \therefore x, y \,\epsilon\, b{-}a \,.\supset_{x,y} : x, y \,\epsilon\, \mathrm{N} \,.\, x{+}a = b \,.\, y{+}a = b \,.$ (1)

§1 P22 $: \supset : x = y.$

$a, b \,\epsilon\, \mathrm{N} \,.\, a < b \,.$ P2 (2)

$.(1) : \supset \therefore b - a \,-\!= \Lambda : x, y \,\epsilon\, b - a \,.\supset .\, x = y : (N, b - a)[s, k]$

(L56) $\therefore \supset \therefore b - a \,\epsilon\, N.$

28. $(a, b, c, d \in \mathbb{N} \wedge a = b \wedge c = d) \to a + c = b + d.$

§2. Subtraction.

Explanations.

The symbol	$-$	is read	*minus.*
”	$<$	”	*is less than.*
”	$>$	”	*is greater than.*

Definitions.

1. $a, b \in \mathbb{N} \to b - a = \mathbb{N} \cap \{x | x + a = b\}.$

2. $a, b \in \mathbb{N} \to (a < b) = (b - a \neq \bot).$

3. $a, b \in \mathbb{N} \to (b > a) = (a < b).$

$a{+}b{-}c = (a{+}b){-}c; a{-}b{+}c = (a{-}b){+}c; a{-}b{-}c = (a{-}b){-}c.$

Theorems.

4. To do....

$a, b \,\epsilon\, \mathrm{N} \,.\, b - a \,\epsilon\, \mathrm{N} \,.\, (\mathrm{L56}) : \supset : b - a \,\text{-}\!= \Lambda : \supset : a < b.$ (3)

$(2)(3) \,.\supset.$ Theor.

6. $a, b \,\epsilon\, \mathrm{N} \,.\, a < b : \supset . \, b - a + a = b.$

Dem. Hyp . P5 . P1 $: \supset : b - a \,\epsilon\, \mathrm{N} \,.(b - a) \,\epsilon\, [x \,\epsilon](x + a = b) : \supset :$

Thes.

7. $a, b, c \,\epsilon\, \mathrm{N} \,. \supset : c = b - a \,. = .\, c + a = b.$

Dem. Hyp . §1 P22 . P6

$: \supset : c = b - a \,. = .\, c + a = b - a + a \,. = .\, c + a = b.$

8. $a, b \,\epsilon\, \mathrm{N} \,. \supset .\, a + b - a = b.$

Dem. $(a + b, b)[b, c]$ P7 $. \supset .$ Theor.

9. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a < b : \supset : c + (b - a) = c + b - a.$

Dem. Hyp . P6 $: \supset : (b - a) + a = b : \supset : c + (b - a) + a = c + b \,.$

P7 $: \supset :$ Thesis.

10. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a > b + c : \supset .\, a - (b + c) = a - b - c.$

11. $a, b, c \,\epsilon\, \mathrm{N} \,.\, b > c \,.\, a > b - c : \supset .\, a - (b - c) = a + c - b.$

12. $a, b, a', b' \,\epsilon\, \mathrm{N} \,.\, a = a' \,.\, b = b' : \supset : a < b \,. = .\, a' < b'.$

Dem. Hyp $. \supset .\, b - a = b' - a' \,. \supset .\, b - a \,\epsilon\, \mathrm{N} = b' - a' \,\epsilon\, \mathrm{N} \,. \supset .$ Thes.

13. $a, b \,\epsilon\, \mathrm{N} \,. \supset .\, a < a + b.$

Dem. Hyp $. P8 : \supset : a + b - a = b : \supset .\, a + b - a \,\epsilon\, \mathrm{N}$. P5 $: \supset :$

Thesis.

14. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a < b \,.\, b < c : \supset .\, a < c.$

Dem. Hyp $. \supset : b - a \,\epsilon\, \mathrm{N} \,.\, c - b \,\epsilon : \supset : (b - a) + (c - b) \,\epsilon\, \mathrm{N} : \supset :$

$c - a \,\epsilon\, \mathrm{N} : \supset .$ Thesis.

15. $a, b, c \,\epsilon\, \mathrm{N} \,. \supset : a < b \,. = .\, a + c < b + c.$

Dem. Hyp $. \supset : a < b \,. = .\, b - a \,\epsilon\, \mathrm{N} \,. = .(b + c) - (a + c) \,\epsilon\, \mathrm{N} \,. =$

$.\, a + c < b + c.$

16. $a, b, a', b' \,\epsilon\, \mathrm{N} \,.\, a < b \,.\, a' < b' : \supset .\, a + a' < b + b'.$

Dem. Hyp $. \supset : a + a' < b + a' . b + a' < b + b' : \supset .$ Thesis.

17. $a, b, c \,\epsilon\, \mathrm{N} . a < b < c : \supset . c - a > c - b.$

Dem. Hyp $. \supset b - a \,\epsilon\, \mathrm{N} . c - b \,\epsilon\, \mathrm{N} . (c - b) + (b - a) = c - a : \supset .$

Thesis.

18. $a \,\epsilon\, \mathrm{N} . \supset : a = 1 . \cup . a > 1.$

Dem. $1 \,\epsilon\, [a \,\epsilon]$ Thesis.

$a \,\epsilon\, \mathrm{N}$. P13 $: \supset : a + 1 > 1 : \supset : a + 1 \,\epsilon\, [a \,\epsilon]$ Thesis.

$(1)(2) . \supset .$ Theor.

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19. $a, b \,\epsilon\, \mathrm{N} . \supset . a + b\,{-} = b.$

Dem. $a \,\epsilon\, \mathrm{N}$. §1 P8 $: \supset : a + 1\,{-} = 1 : \supset : 1 \,\epsilon\, [b \,\epsilon]$ Thesis. (1)

$a \,\epsilon\, \mathrm{N} . b \,\epsilon\, \mathrm{N} . b \,\epsilon\, [b \,\epsilon]$ Ts $: \supset : a + b\,{-} = b .$ §1 P17 (2)

$: \supset : a + (b + 1)\,{-} = b + 1 : \supset : b + 1 \,\epsilon\, [b \,\epsilon]$ Ts.

$(1)(2) . \supset .$ Theor.

20. $a, b \,\epsilon\, \mathrm{N} . a < b . a = b := \Lambda.$

Dem. Hyp $: \supset : b - a \,\epsilon\, \mathrm{N} . (b - a) + a = a .$ P19 $: \supset : \Lambda.$

21. $a, b \,\epsilon\, \mathrm{N} . a > b . a = b := \Lambda.$

22. $a, b \,\epsilon\, \mathrm{N} . a > b . a < b := \Lambda.$

23. $a, b \,\epsilon\, \mathrm{N} : \supset : a < b . \cup . a = b . \cup . a > b.$

Dem $a \,\epsilon\, \mathrm{N}$. P18 $: \supset . 1 \,\epsilon\, [b \,\epsilon]$ Ts. (1)

$a, b \,\epsilon\, \mathrm{N} . a < b : \supset . a < b + 1.$ (2)

$a, b \,\epsilon\, \mathrm{N} . a = b : \supset . a < b + 1.$ (3)

$a, b \,\epsilon\, \mathrm{N} . a > b : \supset : a - b \,\epsilon\, \mathrm{N}$. P18 (4)

$: \supset : a - b = 1 . \cup . a - b > 1.$

$a, b \,\epsilon\, \mathrm{N} . a - b = 1 : \supset . a = b + 1.$ (5)

$a, b \,\epsilon\, \mathrm{N} . a - b > 1 : \supset . a > b + 1.$ (6)

$a, b \,\epsilon\, \mathrm{N} . a > b . (4)(5)(6) : \supset : a = b + 1 . \cup . a > b + 1 .$ (7)

$a, b \,\epsilon\, \mathrm{N} : a > b . \cup . a = b . \cup . a > b : (2)(3)(7) \therefore \supset \therefore a <$ (8)

$b+1 . \cup . a = b+1 . \cup . a > b+1.$

$a, b \;\epsilon\; \mathrm{N} . b \;\epsilon\; [b\;\epsilon]\; \mathrm{Ts} . (8) : \supset : b+1 \;\epsilon\; [b\;\epsilon]\; \mathrm{Ts}.$ (9)

$(1)(9) . \supset .$ Theor.

§3. De maximis et minimis.

Explicationes.

Sit $a \;\epsilon\; \mathrm{K}\; N$, hoc est sit a quaedam numerorum classis; tunc Ma legatur *maximus inter* a, et $\text{W}\, a$ legatur *minimus inter* a.

Definitiones.

1. $a \;\epsilon\; \mathrm{K}\;\; \mathrm{N} . \supset : \mathrm{M}\; a = [x\;\epsilon](x \;\epsilon\; a \therefore a . \epsilon > x := \Lambda).$

2. $a \;\epsilon\; \mathrm{K}\;\; \mathrm{N} . \supset : \text{W}\, a = [x\;\epsilon](x \;\epsilon\; a \therefore a . \epsilon < x := \Lambda).$

§3. Maxima and minima.

Explanations.

Let $A \;\epsilon\; \textbf{SET}\mathbb{N}$, that is, let A be a set of numbers; then $\max(A)$ is read *greatest among* A, and $\min(A)$ is read *least among* A.

Definitions.

1. $A \in \textbf{SET}\mathbb{N} \rightarrow \max(A) = \left\{ x \middle| x \in A \wedge [(A \cap \{z|z > x\}) = \bot] \right\}.$

2. $A \in \textbf{SET}\mathbb{N} \rightarrow \min(A) = \left\{ x \middle| x \in A \wedge [(A \cap \{z|z < x\}) = \bot] \right\}.$

Theoremata.

3. $n \;\epsilon\; \mathrm{N} . a \;\epsilon\; \mathrm{K}\;\; \mathrm{N} . a - = \Lambda . a \ni > n = \Lambda : \supset . Ma \;\epsilon\; N.$

Dem. $a \;\epsilon\; \mathrm{K}\;\; \mathrm{N} . a - = \Lambda . a \ni > 1 = \Lambda : \supset : a = 1 : \supset . Ma = 1 :$ (1)

$\supset . Ma \;\epsilon\; N.$

$(1) \supset : 1 \;\epsilon\; [n\;\epsilon]\; (\mathrm{Hp} \supset \mathrm{Ts}).$ (2)

$n \;\epsilon\; \mathrm{N} . a \;\epsilon\; \mathrm{K}\;\; \mathrm{N} . a \ni > n+1 = \Lambda . n+1 \;\epsilon\; a : \supset : n+1 =$ (3)

$Ma : \supset : Ma \;\epsilon\; N.$

$n \;\epsilon\; \mathrm{N} . a \;\epsilon\; \mathrm{K}\;\; \mathrm{N} . a \ni > n+1 = \Lambda . n+1 - \epsilon\; a : \supset :$ (4)

$a \ni > n = \Lambda.$

$n \;\epsilon\; [n\;\epsilon]\; (\mathrm{Hp} \supset \mathrm{Ts})$ (5)

Theorems.

3. To do....

$. a \,\epsilon\, \mathrm{K\ N} \,.\, a \ni > n+1 = \Lambda \,.\, n+1 - \epsilon\, a : \supset : M a \,\epsilon\, N.$

$n \,\epsilon\, [n\,\epsilon]\ (\mathrm{Hp} \supset \mathrm{Ts})\ .(6) : \supset .(n+1) \,\epsilon\, [n\,\epsilon]\ (\mathrm{Hp} \supset \mathrm{Ts}).$ (7)

$(2)(7) \,.\, \S 1\ \mathrm{P9} : \supset : n \,\epsilon\, \mathrm{N} \,.\supset .\ \mathrm{Hp} \supset \mathrm{Ts}.$ (Th.)

4. $a \,\epsilon\, \mathrm{K\ N} \,.\, a - = \Lambda : \supset . \mathcal{W} a \,\epsilon\, N.$

5. $a \,\epsilon\, \mathrm{K\ N} \,.\supset : \mathcal{W} a = M[x\,\epsilon](a \ni < x = \Lambda).$

§4. De multiplicatione.

Definitiones.

1. $a \,\epsilon\, \mathrm{N} \,.\supset .\, a \times 1 = a.$
2. $a, b \,\epsilon\, \mathrm{N} \,.\supset .\, a \times (b+1) = a \times b + a.$

$ab = a \times b; ab + c = (ab) + c; abc = (ab)c.$

Theoremata.

3. $a, b \,\epsilon\, \mathrm{N} \,.\supset .\, ab \,\epsilon\, N.$

Dem. $a, b \,\epsilon\, \mathrm{N}\ .\ \mathrm{P1} : \supset : a \times 1 \,\epsilon\, \mathrm{N} : \supset . 1 \,\epsilon\, [b\,\epsilon]\ \mathrm{Ts}.$

$a, b \,\epsilon\, \mathrm{N} \,.\, b \,\epsilon\, [b\,\epsilon]\ \mathrm{Ts} : \supset : a \times b \,\epsilon\, \mathrm{N} \,.\S 1\ \mathrm{P19} : \supset : ab + a \,\epsilon\, \mathrm{N}\ .$ (2)

$\mathrm{P1} : \supset : a(b+1) \,\epsilon\, \mathrm{N} : \supset : b+1 \,\epsilon\, [b\,\epsilon]\ \mathrm{Ts}.$

$(1)(2) \,.\supset .$ Theor.

§4. Multiplication.

Definitions.

1. $a \in \mathbb{N} \to a \times 1 = a.$
2. $a, b \in \mathbb{N} \to a \times (b+1) = a \times b + a.$

$ab = a \times b; ab + c = (ab) + c; abc = (ab)c.$

Theorems.

3. To do....

4. $a, b, c \,\epsilon\, \mathrm{N} \,.\supset .(a+b)c = ac + bc.$

COMMENTARY: Euclid's *Elements* was *the* mathematical text from 300 BCE up until Peano's era. Being able to prove propositions from it was an indication that

Nota. Haec est prop. 5^a Euclidis elem. libri VII.

Dem. $a, b \,\epsilon\, \mathrm{N} \,.\, \mathrm{P1} : \supset : 1 \,\epsilon\, [c \,\epsilon]\, \mathrm{Ts}.$ (1)

$a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\epsilon\, [c \,\epsilon]\, \mathrm{Ts}$ (2)

$: \supset : (a+b)c = ac + bc \,.\, \S 1\ \mathrm{P22}$

$: \supset : (a+b)c + a + b = ac + bc + a + b \,.\, \mathrm{P2}$

$: \supset : (a+b)(c+1) = a(c+1) + b(c+1) : \supset :$

$c + 1 \,\epsilon\, [c \,\epsilon]\, \mathrm{Ts}.$

$(1)(2) \,.\supset.\, \mathrm{Theor.}$

5. $a \,\epsilon\, \mathrm{N} \,.\supset.\, 1 \times a = a \,.$

Dem. $1 \,\epsilon\, [a \,\epsilon]\, \mathrm{Ts}.$ (1)

$a \,\epsilon\, [a \,\epsilon]\, \mathrm{Ts} \,.\supset.\, 1 \times a = a \,.\supset.\, 1 \times a + 1 =$ (2)

$a + 1 \,.\times.\, 1 \times (a+1) = a + 1 \,.\supset.\, a + 1 \,\epsilon\, [a \,\epsilon]$

Ts.

$(1)(2) \,.\supset.\, \mathrm{Theor.}$

6. $a, b \,\epsilon\, \mathrm{N} \,.\supset.\, ba + a = (b+1)a.$

7. $a, b \,\epsilon\, \mathrm{N} \,.\supset.\, ab = ba.$ (Eucl. VII, 16)

Dem. $a \,\epsilon\, \mathrm{N} \,.\, \mathrm{P5} \,.\, \mathrm{P1}$ (1)

$: \supset .\, a \times 1 = a = 1 \times a : \supset : 1 \,\epsilon\, [b \,\epsilon]\, \mathrm{Ts}.$

$a, b \,\epsilon\, \mathrm{N} \,.\, b \,\epsilon\, [b \,\epsilon]\, \mathrm{Ts}$ (2)

$: \supset : ab = ba : \supset : ab + a = ba + a \,.\, \mathrm{P1}$

$.\mathrm{P6}\!: \supset : a(b+1) = (b+1)a : \supset : b + 1 \,\epsilon\, [b \,\epsilon]$

Ts.

$(1)(2) \,.\supset.\, \mathrm{Theor.}$

8. $a, b, c \,\epsilon\, \mathrm{N} \,.\supset.\, a(b+c) = ab + ac.$

Dem. $\mathrm{P4} \,.\, \mathrm{P7} : \supset.\, \mathrm{Theor.}$

Peano's axioms matched the prevailing notions of mathematics.

Note. This is Proposition #5 of Euclid's *Elements*, Book VII.

Dem. To do.... (1)

9. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a = b : \supset : ac = bc.$

Dem. $a, b \,\epsilon\, \mathrm{N} \,.\, a = b :: \supset :: 1 \,\epsilon\, [c\,\epsilon]\ \mathrm{Ts}\ \therefore c \,\epsilon\, [c\,\epsilon]\ \mathrm{Ts}$

$. \supset : ac = bc \,.\, a = b : \supset : ac + a = bc + b : \supset :$

$a(c+1) = b(c+1) : \supset : c + 1 \,\epsilon\, [c\,\epsilon]\ \mathrm{Ts}$

$:: \supset : c \,\epsilon\, \mathrm{N} \,. \supset .\ \mathrm{Ts}.$

10. $a, b, c \,\epsilon\, \mathrm{N} : a < b : \supset .(b-a)c = bc - ac.$ (Eucl. VII, 7)

Dem. $\mathrm{Hyp} \,. \supset : b - a \,\epsilon\, \mathrm{N} \,.(b-a) + a = b : \supset :$

$(b-a)c + ac = bc : \supset : (b-a)c = bc - ac.$

11. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a < b : \supset : ac < bc.$

Dem. $\mathrm{Hyp} \,. \supset : b - a \,\epsilon\, \mathrm{N} \,.\ \mathrm{P3} : \supset : (b-a)c \,\epsilon\, \mathrm{N}\ .$

$\mathrm{P10} : \supset : bc - ac \,\epsilon\, \mathrm{N} : \supset$ Thesis.

12. $a, b, c \,\epsilon\, \mathrm{N} \,. \supset \therefore a < b \,. = .\, ac < bc : a = b \,. =$

$.\, ac = bc : a > b \,. = .\, ac > bc.$

13. $a, b, a', b' \,\epsilon\, \mathrm{N} \,.\, a < a' \,.\, b < b' : \supset : ab < a'b'.$

14. $a, b \,\epsilon\, \mathrm{N} : \supset .\, ab \,. > \cup = .\, a.$

15. $a, b, c \,\epsilon\, \mathrm{N} \,. \supset .\, a(bc) = abc.$

Dem. $a, b \,\epsilon\, \mathrm{N} \,.\ \mathrm{P1} : \supset : 1 \,\epsilon\, [c\,\epsilon]\ \mathrm{Ts}.$ (1)

$a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\epsilon\, [c\,\epsilon]\ \mathrm{Ts}$ (2)

$: \supset : a(bc) = abc : \supset : a(bc) + ab = abc + ab :$

$\supset : a(bc + b) = ab(c+1) : \supset : a(b(c+1)) =$

$ab(c+1) : \supset : c + 1 \,\epsilon\, [c\,\epsilon]\ \mathrm{Ts}.$

$(1)(2) \,. \supset .$ Theor.

§5. De potestatibus.

§5. Powers.

Definitiones.

1. $a \epsilon \mathrm{N} . \supset . a^1 = a.$
2. $a, b \epsilon \mathrm{N} . \supset . a^{b+1} = a^b a.$

Definitions.

1. $a \in \mathbb{N} \to a^1 = a.$
2. $a, b \in \mathbb{N} \to a^{b+1} = a^b a.$

Theoremata.

3. $a, b \epsilon \mathrm{N} . \supset . a^b \epsilon N.$

Dem. $a \epsilon \mathrm{N} . \mathrm{P1} : \supset . 1 \epsilon [b \epsilon] \mathrm{Ts}.$ (1)

$a, b \epsilon \mathrm{N} . b \epsilon [b \epsilon] \mathrm{Ts}$ (2)

$: \supset : a^b \epsilon \mathrm{N} . \S 4\mathrm{P3} : \supset : a^b a \epsilon \mathrm{N} .$

$\mathrm{P1} : \supset : a^{b+1} \epsilon \mathrm{N} : \supset : b+1 \epsilon [b \epsilon] \mathrm{Ts}.$

$(1)(2) . \supset . .$ Theor.

4. $a \epsilon \mathrm{N} . \supset . 1^a = 1.$
5. $a, b, c \epsilon \mathrm{N} . \supset . a^{b+c} = a^b a^c.$
6. $a, b, c \epsilon \mathrm{N} . \supset . (ab)^c = a^c b^c.$
7. $a, b, c \epsilon \mathrm{N} . \supset . (a^b)^c = a^{bc}.$
8. $a, b, c \epsilon \mathrm{N} . \supset \therefore a < b . = . a^c < b^c : a = b . = . a^c = b^c : a > b . = . a^c > b^c.$
9. $a, b, c \epsilon \mathrm{N} . a > 1 . \supset \therefore b < c . = . a^b < a^c : b = c . = . a^b = a^c : b > c . = . a^b > a^c.$

Theorems.

3. To do....

§6. De divisione.

Explicationes.

Signum	/	legatur	*divisus per.*
»	D	»	*dividit*, sive *est divisor.*

§6. Division.

Explanations.

The symbol	/	is read	*divided by.*
”	|	”	*divides*, or *is a divisor of.*

≫	ᗡ	≫	*est multiplex.*
≫	Np	≫	*numerus primus.*
≫	π	≫	*est primus cum.*

”	ᗡ	”	*is a multiple of.*
”	$\mathbb{P}$	”	*prime number.*
”	π	”	*is prime with.*

Definitiones.

1. $a, b \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} : b/a = \mathrm{N}\,[x\,\epsilon](xa = b).$
2. $a, b \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} : a \;\mathrm{D}\; b \,. = .\, b/a \,- = \Lambda.$
3. $a, b \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} : b \,\text{ᗡ}\, a \,. = .\, a \;\mathrm{D}\; b.$
4. $\mathrm{Np} = \mathrm{N}\,[x\,\epsilon](\ni \mathrm{D}\; x \,.\, \ni > 1 \,.\, \ni < x := \Lambda).$
5. $a, b \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} :: a \,\pi\, b \therefore = \therefore \ni \mathrm{D}\; a \,.\, \ni \mathrm{D}\; b \,.\, \ni > 1 := \Lambda.$
6. $a, b \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} \therefore \ni \mathrm{D}\,(a, b) :=: \ni \mathrm{D}\; a \,.\cap.\, \ni \mathrm{D}\; b.$
7. $a, b \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} \therefore \ni \text{ᗡ}(a, b) :=: \ni \text{ᗡ}\, a \,.\cap.\, \ni \text{ᗡ}\, b.$

$ab/c = (ab)/c; a/b/c = (a/b)/c; a/b \times c = (a/b)c.$

Definitions.

1. $a, b \in \mathbb{N} \to b/a = \mathbb{N} \cap \{x | xa = b\}.$
2. $a, b \in \mathbb{N} \to a|b = (b/a \neq \varnothing).$
3. $a, b \in \mathbb{N} \to b \,\text{ᗡ}\, a = (a|b).$
4. $\mathbb{P} = \mathbb{N} \cap \Big\{x \Big| \Big(\{z|(z|x)\} \cap \{z|z > 1\} \cap \{z|z < x\}\Big) = \varnothing\Big\}.$
5. $a, b \in \mathbb{N} \to a \,\pi\, b = \Big[\Big(\{z|(z|a)\} \cap \{z|(z|b)\} \cap \{z|z > 1\}\Big) = \varnothing\Big].$
6. $a, b \in \mathbb{N} \to \ni |\,(a, b) = \Big(\{z|(z|a)\} \cap \{z|(z|b)\}\Big).$
7. $a, b \in \mathbb{N} \to \ni \text{ᗡ}(a, b) = \Big(\{z|z \,\text{ᗡ}\, a\} \cap \{z|z \,\text{ᗡ}\, b\}\Big).$

$ab/c = (ab)/c; a/b/c = (a/b)/c; a/b \times c = (a/b)c.$

Theoremata.

Nota. Haec theoremata ut in substractione demonstrantur.

8. $a, b, a', b' \,\epsilon\, \mathrm{N} \,.\, a = a' \,.\, b = b' : \text{Ↄ} \,.\, a/b = a'/b'.$
9. $a, b, a', b' \,\epsilon\, \mathrm{N} \,.\, a = a' \,.\, b = b' : \text{Ↄ} : a \;\mathrm{D}\; b \,. = .\, a' \;\mathrm{D}\; b'.$
10. $a, b, c \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} : ac = b \,. = .\, c = b/a.$
11. $a, b \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} : a \;\mathrm{D}\; b \,. = .\, b/a \,\epsilon\, N.$
12. $a \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} \,.\, a/1 = a.$
13. $a \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} \,.\, a/a = 1.$
14. $a \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} \,.\, 1 \;\mathrm{D}\; a.$
15. $a \,\epsilon\, \mathrm{N} \,.\, \text{Ↄ} \,.\, a \;\mathrm{D}\; a.$

Theorems.

Note. These theorems are proved as for subtraction.

8. To do....

16. $a, b \,\epsilon\, \mathrm{N} \,.\, ab/b = a.$

17. $a, b \,\epsilon\, \mathrm{N} \,.\, a \;\mathrm{D}\; b : \supset . \, a(b/a) = b.$

18. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \;\mathrm{D}\; b : \supset . \, a(b/c) = ab/c.$

19. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a \,\text{Ɑ}\, bc : \supset : a/(bc) = a/b/c.$

20. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a \,\text{Ɑ}\, b \,.\, b \,\text{Ɑ}\, c : \supset . \, a/(b/c) = a/b \times c.$

21. $a, m, n \,\epsilon\, \mathrm{N} \,.\, m > n : \supset . \, a^m/a^n = a^{m-n}.$

22. $a, b \,\epsilon\, \mathrm{N} \,. \supset . \, a \;\mathrm{D}\; ab.$

23. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a \;\mathrm{D}\; b \,.\, b \;\mathrm{D}\; c : \supset . \, a \;\mathrm{D}\; c.$

24. $a, b, c \,\epsilon\, \mathrm{N} \,.\, a \;\mathrm{D}\; b \;\mathrm{D}\; c : \supset . \, c/a \,\text{Ɑ}\, c/b.$

25. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \;\mathrm{D}\; a \,.\, c \;\mathrm{D}\; b : \supset . (a+b)/c =$

$a/c + b/c.$

26. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \;\mathrm{D}\; a \,.\, c \;\mathrm{D}\; b \,.\, a > b : \supset :$

$(a-b)/c = a/c - b/c.$

27. $a, b, c, \epsilon\, \mathrm{N} \,.\, c \;\mathrm{D}\; a \,.\, c \;\mathrm{D}\; b : \supset . \, c \;\mathrm{D}\; a + b.$

28. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \;\mathrm{D}\; a \,.\, c \;\mathrm{D}\; b \,.\, a > b : \supset . \, c \;\mathrm{D}\; a - b.$

29. $a, b, c, m, n \,\epsilon\, \mathrm{N} \,.\, c \;\mathrm{D}\; a \,.\, c \;\mathrm{D}\; b :$

$\supset . \, c \;\mathrm{D}\; ma + nb.$

30. $a, b, c, m, n \,\epsilon\, \mathrm{N} \,.\, c \;\mathrm{D}\; a \,.\, c \;\mathrm{D}\; b \,.\, ma > nb :$

$\supset . \, c \;\mathrm{D}\; ma - nb.$

31. $a, b \,\epsilon\, \mathrm{N} \,.\, a \;\mathrm{D}\; b : \supset : a \,. < \cup = .\, b.$

Dem. Hyp . P11 . P17 . §4 P14 $: \supset : b/a \,\epsilon$

$\mathrm{N} \,.\, a(b/a) = b \,.\, a < \cup = a(b/a) : \supset .$ Thesis.

32. $a, b \,\epsilon\, \mathrm{N} \,.\, a \;\mathrm{D}\; b \,.\, b \;\mathrm{D}\; a : \supset . \, a = b.$

33. $a \,\epsilon\, \mathrm{N} \,. \supset . \, M \ni \mathrm{D}\; a = a.$

34. $a, b \,\epsilon\, \mathrm{N} \,.\, a > b : \supset . \ni \mathrm{D}\,(a, b) = \ni \mathrm{D}\,(b, a-b).$

Dem. Hyp. P28 (1)

$: \supset \therefore x \;\mathrm{D}\; a \,.\, x \;\mathrm{D}\; b : \supset : x \;\mathrm{D}\; b \,.\, x \;\mathrm{D}\,(a-b)$

Hyp. P27 $: \supset \therefore x \mathrm{~D~} b \,.\, x \mathrm{~D} \,(a-b) : \supset :$ (2)

$x \mathrm{~D~} b \,.\, x \mathrm{~D} \,(b+(a-b)) : \supset : x \mathrm{~D~} b \,.\, x \mathrm{~D~} a.$

$(1)(2) \supset :$ Hyp. (Th.)

$\supset \therefore x \mathrm{~D~} a \,.\, x \mathrm{~D~} b :=: x \mathrm{~D~} b \,.\, xD(a-b).$

35. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset : M \ni \mathrm{D}\,(a,b) \,\epsilon\, N.$

Dem. $1 \mathrm{~D~} a \,.\, 1 \mathrm{~D~} b : \supset : \ni \mathrm{D}\,(a,b) -\!= \Lambda.$ (1)

$\ni \mathrm{D}\,(a,b) \,.\, \ni > a := \Lambda.$ (2)

$(1)(2) \,.\, \S 3$ P3 $: \supset .$ Th.

36. $a, b \,\epsilon\, \mathrm{N} \,.\, \supset .\, \ni \mathrm{D}\,(a,b) = \ni \mathrm{D~} M \ni \mathrm{D}\,(a,b).$ (Eucl. VII, 2)

Dem. $k = \mathrm{N}\,[c\,\epsilon]$ (Hp. $a < c \,.\, b < c : \supset .$ Ts.). (1)

$a \,\epsilon\, \mathrm{N} \,.\, b \,\epsilon\, \mathrm{N} \,.\, a < 1 \,.\, b < 1 := \Lambda.$ (2)

$(1)(2) \,.\, \supset .\, 1 \,\epsilon\, \mathrm{K}\,.$ (3)

$a, b \,\epsilon\, \mathrm{N} \,.\, a < c+1 \,.\, b < c+1 :$ (4)

$\supset \therefore a < c \,.\, b < c : \cup : a = c \,.\, b < c : \cup :$

$a < c \,.\, b = c : \cup : a = c \,.\, b = c.$

$c \,\epsilon\, k \,.\, a, b \,\epsilon\, \mathrm{N} \,.\, a < c \,.\, b < c : \supset :$ Ts. (5)

$c \,\epsilon\, k \,.\, a, b \,\epsilon\, \mathrm{N} \,.\, a = c \,.\, b < c : \supset : c \,\epsilon$ (6)

$k \,.\, b < c/pa - b < c \,.\, \ni \mathrm{D}\,(a,b) = \ni \mathrm{D}\,(b, a-$

$b) : \supset : \ni \mathrm{D}\,(b, a-b) = \ni \mathrm{D~} m \ni \mathrm{D}\,(b, a-b) :$

$\supset : \ni \mathrm{D}\,(a,b) = \ni \mathrm{D~} M \ni \mathrm{D}\,(a,b) : \supset :$ Ts.

$(a,b)[b,a](6) \supset .\, c \,\epsilon\, k \,.\, a, b \,\epsilon\, \mathrm{N} \,.\, a < c \,.\, b = c :$ (7)

$\supset :$ Ts.

$c \,\epsilon\, k \,.\, a, b \,\epsilon\, \mathrm{N} \,.\, a = c \,.\, b = c : \supset : \ni \mathrm{D}\,(a,b) =$ (8)

$\ni \mathrm{D~} c = \ni \mathrm{D~} M \ni \mathrm{D~} c = \ni \mathrm{D~} M \ni \mathrm{D}\,(a,b) : \supset :$

Ts.

$(4)(5)(6)(7)(8) \,.\, \supset .\, c \,\epsilon\, k \,.\, a, b \,\epsilon$ (9)

$\mathrm{N} \,.\, a < c+1 \,.\, b < c+1 : \supset :$ Ts.

$(9) \supset .\, c \,\epsilon\, k \,.\, \supset .\, (c+1) \,\epsilon\, k.$ (10)

$(1)(10) . \supset \therefore c \in \mathrm{N} . \mathrm{Hp}.\ a < c . b < c : \supset : \mathrm{Ts}.$ (11)

$(a+b)[c](11) . \supset : \mathrm{Hp}. \supset . \mathrm{Ts}.$ (Th.)

37. $a, b, m \in \mathrm{N} . \supset . M \ni \mathrm{D}\,(am, bm) = m \times M \ni \mathrm{D}\,(a, b).$

§7. Theoremata varia.

1. $a, b \in \mathrm{N} . a^2 + b^2 \,\text{Ɑ}\, 7 : \supset : a \,\text{Ɑ}\, 7 . b \,\text{Ɑ}\, 7.$
2. $x \in \mathrm{N} . \supset . x(x+1) \,\text{Ɑ}\, 2.$
3. $x \in \mathrm{N} . \supset . x(x+1)(x+2) \,\text{Ɑ}\, 6.$
4. $x \in \mathrm{N} . \supset . x(x+1)(2x+1) \,\text{Ɑ}\, 6.$
5. $x \in \mathrm{N} . \supset : x . \pi . x+1.$
6. $x \in \mathrm{N} . \supset : 2x-1 . \pi . 2x+1.$
7. $x \in \mathrm{N} . \supset . (2x+1)^2 - 1 \,\text{Ɑ}\, 8.$
8. $a \in \mathrm{N} . a > 1 : \supset \therefore \mathrm{Np} . \ni > 1 . \ni \mathrm{D}\, a : - = \Lambda.$ (Eucl. VII, 31)
9. $a, b \in \mathrm{N} \therefore b^2 > a \therefore \ni \mathrm{D}\, a . \ni > 1 . \ni < b := \Lambda :: \supset . a \in \mathrm{Np}.$
10. $a, b \in \mathrm{N} . a \in \mathrm{Np} . a - \mathrm{D}\, b : \supset : a \,\pi\, b.$ (Eucl. VII, 29)
11. $a, b, c \in \mathrm{N} . a \,\mathrm{D}\, bc . a \,\pi\, b : \supset . a \,\mathrm{D}\, c.$
12. $a, b \in \mathrm{N} . m = M \ni \mathrm{D}\,(a, b) : \supset : a/m \,\pi\, . b/m.$
13. $a \in \mathrm{Np} . b, c \in \mathrm{N} . a \,\mathrm{D}\, bc : \supset : a \,\mathrm{D}\, b . \cup . a \,\mathrm{D}\, c.$ (Eucl. VII, 30)
14. $a \in \mathrm{Np} . b, n \in \mathrm{N} : \supset : a \,\mathrm{D}\, b^n . = . a \,\mathrm{D}\, b.$ (Eucl. IX, 12)
15. $a, b, c \in \mathrm{N} . a \,\pi\, b . c \,\mathrm{D}\, a : \supset : c \,\pi\, b.$ (Eucl. VII, 23)
16. $a, b, c \in \mathrm{N} . \supset \therefore a \,\pi\, b . a \,\pi\, c :=: a \,\pi\, bc.$ (Eucl. VII, 24)

§7. Various theorems.

1. To do....

17. $a, b, c \,\epsilon\, \mathrm{N} \,.\, b \,\pi\, c \,.\, b \,\mathrm{D}\, a \,.\, c \,\mathrm{D}\, a : \reflectbox{C} \,.\, bc \,\mathrm{D}\, a.$

18. $a, b, c \,\epsilon\, \mathrm{N} \quad pa \,\pi\, b : \reflectbox{C} : \ni \mathrm{D}\,(ac, b) = \ni \mathrm{D}\,(c, b).$

19. $a, b \,\epsilon\, \mathrm{N} \,.\, \reflectbox{C} \,.\, \reflectbox{M} \ni \reflectbox{D}(a, b) \,\epsilon\, N.$

20. $a, b \,\epsilon\, \mathrm{N} \,.\, \reflectbox{C} \,.\, \reflectbox{M} \ni \reflectbox{D}(a, b) = ab/M \ni \mathrm{D}\,(a, b).$ (Eucl. VII, 34)

21. $a, b, c \,\epsilon\, \mathrm{N} \,.\, c \,\reflectbox{D}\, a \,.\, c \,\reflectbox{D}\, b : \reflectbox{C} : c \,\reflectbox{D}\, \reflectbox{M} \ni \mathrm{D}\,(a, b).$ (Eucl. VII, 35)

22. $x \,\epsilon\, \mathrm{N} \,.\, x < 41 : \reflectbox{C} \,.\, 41 - x + x^2 \,\epsilon\, \mathrm{Np}\,.$

23. $M \,.\, \mathrm{Np} := \Lambda.$ (Eucl. IX, 20)

COMMENTARY: Fermat's Little Theorem

23. $n \,\epsilon\, \mathrm{Np} \,.\, a \,\epsilon\, \mathrm{N} \,.\, a \,\text{-}\reflectbox{D}\, n : \reflectbox{C} \,.\, a^{n-1} - 1 \,\reflectbox{D}\, n.$ (Fermat)

§8. Numerorum rationes.

§8. Rational numbers.

Explicationes.

Si $p, q \,\epsilon\, \mathrm{N}$, tunc $\frac{p}{q}$ legitur *ratio numeri p numero q*.

Signum R legitur *duorum numerorum ratio*, et indicat numeros rationales positivos.

Explanations.

COMMENTARY: Peano uses R for rational numbers and Q for reals, which is the opposite of the modern usage.

If $p, q \,\epsilon\, \mathbb{N}$, then $\frac{p}{q}$ is read *the ratio of the number p* to the number q.

The symbol $\mathbb{Q}^+$ is read *ratio of two numbers*, and indicates the positive rational numbers.

Definitiones.

1. $m, p, q \,\epsilon\, \mathrm{N} \,.\, \reflectbox{C} \,.\, m\frac{p}{q} = mp/q.$

2. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\, \reflectbox{C} :: \frac{p}{q} = \frac{p'}{q'} \,.=\, \therefore x \,\epsilon$
$\mathrm{N} \,.\, x\frac{p}{q}, x\frac{p'}{q'} \,\epsilon\, \mathrm{N} : \reflectbox{C}_x \,.\, x\frac{p}{q} = x\frac{p'}{q'}.$

3. $R =:: [x\,\epsilon] \therefore p, q \,\epsilon\, \mathrm{N} \,.\, \frac{p}{q} = x : \text{-} = \Lambda.$

Definitions.

1. $m, p, q \in \mathbb{N} \to m\frac{p}{q} = mp/q.$

2. $p, q, p', q' \in \mathbb{N} \to (\frac{p}{q} = \frac{p'}{q'}) = [x \in$
$\mathbb{N} \wedge (x\frac{p}{q}, x\frac{p'}{q'} \in \mathbb{N}) \underset{\forall x}{\longrightarrow} (x\frac{p}{q} = x\frac{p'}{q'})].$

3. $\mathbb{Q}^+ = \{x | [(p, q \in \mathbb{N}) \wedge \frac{p}{q} = x] \neq \bot\}.$

4. $p \,\epsilon\, \mathrm{N} \,.\supset.\, \frac{p}{1} = p.$

4. $p \in \mathbb{N} \to \frac{p}{1} = p.$

Theoremata.

5. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\supset::\, \frac{p}{q} = \frac{p'}{q'} \,.=.\, pq' = p'q.$ (Eucl. VII, 19)

Dem. Hp. $\frac{p}{q} = \frac{p'}{q'} :\supset\therefore qq', qq'\frac{p}{q}, qq'\frac{p'}{q'} \,\epsilon\, \mathrm{N} \,.\, \mathrm{P2}$ (1)

$\therefore\supset\therefore qq'\frac{p}{q} = qq'\frac{p'}{q'} \,.\, qq'\frac{p}{q} = pq' \,.\, qq'\frac{p'}{q'} =$

$p'q \therefore\supset\therefore pq' = p'q.$

Hp. $pq' = p'q \therefore\supset\therefore x \,\epsilon\, \mathrm{N} \,.\, x\frac{p}{q}, x\frac{p'}{q'} \,\epsilon\, \mathrm{N} :\supset_x :$ (2)

$xp'q' = xp'q :\supset: (x\frac{p}{q})qq' = (x\frac{p'}{q'})qq' :\supset:$

$x\frac{p}{q} = x\frac{p'}{q'}.$

$(1)(2) \,.\supset.\, \mathrm{Th}.$

6. $m, p, q \,\epsilon\, \mathrm{N} \,.\supset.\, \frac{p}{q} = \frac{mp}{mq}.$ (Eucl. VII, 17)

7. $p, q \,\epsilon\, \mathrm{N} \,.\, m \,\epsilon\, \mathrm{N} \,.\, m \,\mathrm{D}\, p \,.\, m \,\mathrm{D}\, q :\supset.\, \frac{p}{q} =$

$\frac{p/m}{q/m}.$

8. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\, p \,\pi\, q \,.\, p' \,\pi\, q' \,.\, \frac{p}{q} = \frac{p'}{q'} :\supset: p =$

$p' \,.\, q = q'.$

9. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\, p' \,\pi\, q' \,.\, \frac{p}{q} = \frac{p'}{q'} :\supset: p'/p =$

$q'/q = M \ni \mathrm{D}\,(p, q).$

10. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\, \frac{p}{q} = \frac{p'}{q'} \,.\, p \,\pi\, q \,.\, q' < q :=\Lambda.$ (Eucl. VII, 21)

11. $p, q, p', q' \,\epsilon\, \mathrm{N} :\supset: \frac{p}{q} = \frac{p'}{q'} \,.=.\, \frac{p}{p'} = \frac{q}{q'} \,.=$ (Eucl. VII, 13)

$.\, \frac{q}{p} = \frac{q'}{p'}.$

12. $p, q \,\epsilon\, \mathrm{N} \,.\supset::\, [m\,\epsilon] : m \,\epsilon\, \mathrm{N} \,.\, m\frac{p}{q} \,\epsilon\, \mathrm{N} \,\therefore - =$

$\Lambda.$

12'. $a \,\epsilon\, \mathrm{R} \,.\supset::\, [m\,\epsilon] : m \,\epsilon\, \mathrm{N} \,.\, ma \,\epsilon\, \mathrm{N} \,\therefore - = \Lambda.$

Theorems.

5. To do....

13. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\supset::\, [(r, s, l)\,\epsilon] : r, s, t \,\epsilon$

$\mathrm{N} \,.\, \frac{p}{q} = \frac{r}{t} \,.\, \frac{p'}{q'} = \frac{s}{t} \therefore - = \Lambda.$

13'. $a, b \,\epsilon\, \mathrm{R} \,.\, \text{Ɔ} :: [(r, s, t) \,\epsilon] : r, s, t \,\epsilon\, \mathrm{N} \,.\, a =$
$\frac{r}{t} \,.\, b = \frac{s}{t} \therefore \mathbf{-} = \Lambda.$

14. $a, b, c \,\epsilon\, \mathrm{R} \,.\, \text{Ɔ} :: [(m, n, p, q) \,\epsilon] : m, n, p, q \,\epsilon$
$\mathrm{N} \,.\, a = \frac{m}{q} \,.\, b = \frac{n}{q} \,.\, c = \frac{p}{q} \therefore \mathbf{-} = \Lambda.$

15. $p, q, r \,\epsilon\, \mathrm{N} \,.\, a = \frac{p}{r} \,.\, b = \frac{q}{r} : \text{Ɔ} : a = b \,.=.\, p =$
$q.$

16. $m \,\epsilon\, \mathrm{N} \,.\, a, b \,\epsilon\, \mathrm{R} \,.\, a = b \,.\, ma \,\epsilon\, \mathrm{N} : \text{Ɔ} \,.\, mb \,\epsilon\, N.$

17. $a, b, c \,\epsilon\, \mathrm{R} \,.\, \text{Ɔ} \therefore a = a.$
$\text{Ɔ} \therefore a = b \,.=.\, b = a.$
$\text{Ɔ} \therefore a = b \,.\, b = c : \text{Ɔ} \,.\, a = c.$

18. $N \,\text{Ɔ}\, \mathrm{R}\,.$

Definitiones.

19. $a, b \,\epsilon\, \mathrm{R} \,.\, \text{Ɔ} :: a < b \,.=\therefore x \,\epsilon\, \mathrm{N} \,.\, xa, xb \,\epsilon\, \mathrm{N} :$
$\text{Ɔ} \,.\, xa < xb.$

20. $a, b \,\epsilon\, \mathrm{R} \,.\, \text{Ɔ} : b > a \,.=.\, a < b.$

Theoremata.

21. $p, q, r \,\epsilon\, \mathrm{N} \,.\, a = \frac{p}{r} \,.\, b = \frac{q}{r} : \text{Ɔ} : a < b \,.=$
$.\, p < q.$

22. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\, \text{Ɔ} : \frac{p}{q} < \frac{p'}{q'} \,.=.\, pq' < p'q.$

23. $p, q, r \,\epsilon\, \mathrm{N} \,.\, a = \frac{r}{p} \,.\, b = \frac{r}{q} : \text{Ɔ} : a < b \,.=.\, p >$
$q.$

24. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\, \frac{p}{q} < \frac{p'}{q'} : \text{Ɔ} \,.\, \frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}.$

25. $a \,\epsilon\, \mathrm{R} \,.\, \text{Ɔ} \therefore ? \,.\, \backepsilon > a : \mathbf{-} = \Lambda.$

26. $a \,\epsilon\, \mathrm{R} \,.\, \text{Ɔ} \therefore \mathrm{R} \,.\, \backepsilon < a : \mathbf{-} = \Lambda.$

27. $a, b \,\epsilon\, \mathrm{R} \,.\, a < b : \text{Ɔ} \therefore \mathrm{R} \,.\, \backepsilon > a \,.\, \backepsilon < b : \mathbf{-} =$

Definitions.

19. $a, b \in \mathbb{Q}^+ \rightarrow a < b = ([x \in \mathbb{N} \wedge (xa, xb \in \mathbb{N})] \rightarrow xa <$
$xb).$

20. $a, b \in \mathbb{Q}^+ \rightarrow b > a = a < b.$

Theorems.

21. To do....

$\Lambda.$

28. $a, b \epsilon \mathrm{R} : \supset \therefore a < b \,.\, a = b := \Lambda.$

$\supset \therefore a > b \,.\, a = b := \Lambda.$

$\supset \therefore a < b \,.\, a > b := \Lambda.$

$\supset \therefore a \mathbin{-} < b \,.\, a \mathbin{-} = b \,.\, a \mathbin{-} > b := \Lambda.$

29. $a, b, c \,\epsilon\, \mathrm{R} : \supset \therefore a < \cup = b \,.\, b < c : \supset : a < c.$

$\supset \therefore a < b \,.\, b < \cup = c : \supset : a < c.$

Definitiones.

30. $a, b \epsilon \mathrm{R} \,.\supset.\, a + b = [c \,\epsilon](c \,\epsilon\, \mathrm{R} \therefore x \,\epsilon\, \mathrm{N} \,.\, xa, xb, xc \,\epsilon$ $\mathrm{N} : \supset_x .\, xa + xb = xc).$

31. $a, b \epsilon \mathrm{R} \,.\supset :: b - a = \therefore [x \,\epsilon](x \,\epsilon\, \mathrm{R} \,.\, a + x = b).$

32. $a, b \epsilon \mathrm{R} \,.\supset.\, ab = [c \,\epsilon](c \,\epsilon\, \mathrm{R} \therefore x \,\epsilon\, \mathrm{N} \,.\, xa, (xa)b, xc \,\epsilon$ $\mathrm{N} : \supset_x .(xa)b = xc).$

33. $a, b \epsilon \mathrm{R} \,.\supset.\, b/a = [x \,\epsilon](x \,\epsilon\, \mathrm{R} \,.\, ax = b).$

Theoremata.

34. $p, q, r \,\epsilon\, \mathrm{N} \,.\supset \frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}.$

35. $a, b \,\epsilon\, \mathrm{R} \,.\supset.\, a + b \,\epsilon\, \mathrm{R}\,.$

36. $p, q, r \,\epsilon\, \mathrm{N} \,.\, p < q : \supset .\, \frac{q}{r} - \frac{p}{r} = \frac{q-p}{r}.$

37. $a, b \,\epsilon\, \mathrm{R} \,.\, a < b : \supset .\, b - a \,\epsilon\, \mathrm{R}\,.$

38. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\supset.\, \frac{p}{q}\frac{p'}{q'} = \frac{pp'}{qq'}.$

39. $a, b \,\epsilon\, \mathrm{R} \,.\supset.\, ab \,\epsilon\, \mathrm{R}\,.$

40. $p, q, p', q' \,\epsilon\, \mathrm{N} \,.\supset.\, \frac{p}{q}/\frac{p'}{q'} = \frac{pq'}{p'q}.$

41. $a, b \,\epsilon\, \mathrm{R} \,.\supset.\, b/a \,\epsilon\, \mathrm{R}\,.$

42. $p, q \,\epsilon\, \mathrm{N} \,.\supset.\, \frac{p}{q} = \frac{p}{q}.$

Definitions.

30. $a, b \in \mathbb{Q}^+ \to a + b = \Big\{c \Big| c \in \mathbb{Q}^+ \wedge \Big([x \in \mathbb{N} \wedge (xa, xb, xc \in \mathbb{N})] \xrightarrow[\forall x]{} xa + xb = xc\Big)\Big\}.$

31. $a, b \in \mathbb{Q}^+ \to b - a = \{x | x \in \mathbb{Q}^+ \wedge a + x = b\}.$

32. $a, b \in \mathbb{Q}^+ \to ab = \Big\{c \Big| c \in \mathbb{Q}^+ \wedge \Big[\Big(x \in \mathbb{N} \wedge [xa, (xa)b, xc \in \mathbb{N}]\Big) \xrightarrow[\forall x]{} (xa)b = xc\Big]\Big\}.$

33. $a, b \in \mathbb{Q}^+ \to b/a = \{x | x \in \mathbb{Q}^+ \wedge ax = b\}.$

Theorems.

34. To do....

§9. Rationalum systemata. Irrationales.

Explicatio.

Si $a \epsilon \mathrm{K}\ \mathrm{R}$, signum $\mathrm{T}\, a$ legitur *terminus summus*, vel *limes summus classis a*. Supra hoc novum ens relationes ac operationes tantum definimus.

Definitiones.

1. $a \epsilon \mathrm{K}\ \mathrm{R} \,.\, x \epsilon \mathrm{R} : \supset :: x < \mathrm{T}\, a \,.=\, \therefore a \,.\, \ni > x : - =$ $\Lambda.$

2. $a \epsilon \mathrm{K}\ \mathrm{R} \,.\, x \epsilon \mathrm{R} : \supset ::: x = \mathrm{T}\, a \,.=:.: a \,.\, \ni > x :=$ $\Lambda :: u \epsilon \mathrm{R} \,.\, u < x : \supset_x \therefore a \,.\, \ni > u : - = \Lambda.$

3. $a \epsilon \mathrm{K}\ \mathrm{R} \,.\, x \epsilon \mathrm{R} : \supset \therefore x > \mathrm{T}\, a \,.=:$ $x - < \mathrm{T}\, a \,.\, x - = \mathrm{T}\, a.$

§9. The system of rationals. Irrationals.

Explanation.

If $A \in \mathbf{SET}\mathbb{Q}^+$, the symbol $\sup(A)$ is read *upper boundary*, or *upper limit of the set A*. We shall define only a few relations and operations on this new entity.

Definitions.

1. $(A \in \mathbf{SET}\mathbb{Q}^+ \wedge x \in \mathbb{Q}^+) \to x < \sup(A) =$ $[(A \cap \{z|z > x\}) \neq \varnothing].$

2. $(A \in \mathbf{SET}\mathbb{Q}^+ \wedge x \in \mathbb{Q}^+) \to (x = \sup(A)) =$ $\Big[[(A \cap \{z|z > x\}) = \varnothing] \wedge \Big((u \in \mathbb{Q}^+ \wedge u < x) \underset{\forall u}{\longrightarrow}$ $[(A \cap \{z|z > u\}) \neq \varnothing]\Big)\Big].$

3. $(A \in \mathbf{SET}\mathbb{Q}^+ \wedge x \in \mathbb{Q}^+) \to (x > \sup(A)) = [(x \not< $ $\sup(A)) \wedge (x \neq \sup(A))].$

Theorema.

4. $x \epsilon \mathrm{R} \,.\supset :: x = \therefore \mathrm{T} : \mathrm{R} \,.\, \ni < x.$

Explicatio.

Signum Q legitur *quantitas*, numerosque indicat reales positivos, rationales aut irrationales, 0 et ∞ exceptis.

Definitiones.

Theorem.

4. $x \in \mathbb{Q}^+ \to x = \sup(\mathbb{Q}^+ \cap \{z|z < x\}).$

Explanation.

The symbol $\mathbb{R}^+$ is read *quantity*, and indicates the positive real numbers, rational or irrational, with the exception of 0 and ∞.

Definitions.

5. $\mathrm{Q} = [x\,\epsilon](a\,\epsilon\,\mathrm{K}\;\mathrm{R} : a\,- = \Lambda : \mathrm{R}\,\epsilon > \mathrm{T}\,a\,.\,- = \Lambda : \mathrm{T}\,a = x \therefore - = \Lambda)$.

6. $a, b\,\epsilon\,\mathrm{Q}\,.\supset :: a = b\,. = \therefore \mathrm{R}\,.\ni < a :=: \mathrm{R}\,.\ni < b$.

7. $a, b\,\epsilon\,\mathrm{Q}\,.\supset :: a < b\,. = \therefore \mathrm{R}\,.\ni > a\,.\ni < b : - = \Lambda$.

8. $a, b\,\epsilon\,\mathrm{Q}\,.\supset : b > a\,. = .\,a < b$.

Theoremata.

9. $a\,\epsilon\,\mathrm{Q}\,.\supset \therefore \mathrm{R}\,.\ni < a : - = \Lambda$.

10. $a\,\epsilon\,\mathrm{Q}\,.\supset \therefore \mathrm{R}\,.\ni > a : - = \Lambda$.

11. $\mathrm{R} \supset \mathrm{Q}$.

Subsistunt quoque propositiones quae a P17, 28, 29 in §8 obtinentur, si loco R legatur Q.

Definitiones.

12. $a, b\,\epsilon\,Q\,.\supset.\,a + b = \mathrm{T}\,[z\,\epsilon]([(x, y)\,\epsilon] : x, y\,\epsilon\,\mathrm{R}\,.\,x < a\,.\,y < b\,.\,x + y = z \therefore - = \Lambda)$.

13. $a, b\,\epsilon\,Q\,.\supset.\,ab = \mathrm{T}\,[z\,\epsilon]([(x, y)\,\epsilon] : x, y\,\epsilon\,\mathrm{R}\,.\,x < a\,.\,y < b\,.\,xy = z \therefore - = \Lambda)$.

Ut valeant hae definitiones, demonstrandum est subsistere propositiones 12 et 13, si $a, b\,\epsilon\,\mathrm{R}$.

Substractionem et divisionem ut operationes inversas additiones et multiplicationis definire licet, illarumque proprietas demonstrare.

5. $\mathbb{R}^+ = \{x|[A \in \mathbf{SET}\mathbb{Q}^+ \wedge A \neq \varnothing \wedge (\mathbb{Q}^+ \cap \{z|z > \sup(A)\}) = \varnothing \wedge \sup(A) = x] \neq \bot\}$.

6. $a, b \in \mathbb{R}^+ \to (a = b) = [(\mathbb{Q}^+ \cap \{z|z < a\}) = (\mathbb{Q}^+ \cap \{z|z < b\})]$

7. $a, b \in \mathbb{R}^+ \to a < b = [(\mathbb{Q}^+ \cap \{z|z > a\} \cap \{z|z < b\}) \neq \varnothing]$.

8. $a, b \in \mathbb{R}^+ \to b > a = a < b$.

Theorems.

9. $a \in \mathbb{R}^+ \to (\mathbb{Q}^+ \cap \{z|z < a\}) \neq \varnothing$.

10. $a \in \mathbb{R}^+ \to (\mathbb{Q}^+ \cap \{z|z > a\}) \neq \varnothing$.

11. $\mathbb{Q}^+ \subset \mathbb{R}^+$.

The propositions obtained from P17, 28, 29 in §8 also hold, by reading $\mathbb{R}^+$ for $\mathbb{Q}^+$.

Definitions.

12. $a, b \in \mathbb{R}^+ \to a + b = \sup\left(\left\{z \middle| \{(x, y)|(x, y \in \mathbb{Q}^+) \wedge x < a \wedge y < b \wedge x + y = z\} \neq \varnothing\right\}\right)$.

13. $a, b \in \mathbb{R}^+ \to ab = \sup\left(\left\{z \middle| \{(x, y)|(x, y \in \mathbb{Q}^+) \wedge x < a \wedge y < b \wedge xy = z\} \neq \varnothing\right\}\right)$.

In order for these definitions to have meaning, it must be proved that propositions 12 and 13 hold, if $a, b\,\epsilon\,\mathbb{Q}^+$.

Subtraction and division could be defined as the inverse operations to addition and multiplication, and their properties could be proved.

§10. Quantitatum systemata.

Explicationes.

Si $a \,\epsilon\, \mathrm{K\ Q}$, signa $\mathbf{I}\, a$, $\mathbf{E}\, a$, $\mathbf{L}\, a$ leguntur: *interior, exterior, limes classis a.*

§10. System of quantities.

Explanations.

If $A \in \mathbf{SET}\mathbb{R}^+$, the symbols $\mathrm{int}(A)$, $\mathrm{ext}(A)$, $\mathrm{bd}(A)$ are read: *interior, exterior, limit of the set A.*

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Definitiones.

1. $a \,\epsilon\, \mathrm{K\ Q} \,.\supset\, \mathbf{I}\, a = \mathrm{Q}\,[x\,\epsilon]([(u,v)\,\epsilon] :: u,v\,\epsilon\; \mathrm{Q} \therefore u < x < v \therefore \ni > u \,.\, \ni < v : \supset : a : . : \mathbf{-} = \Lambda)$.
2. $a \,\epsilon\, \mathrm{K\ Q} \,.\supset .\, \mathbf{E}\, a = \mathbf{I}\,(\mathbf{-}\,a)$.
3. $a \,\epsilon\, \mathrm{K\ Q} \,.\supset .\, \mathbf{L}\, a = (\mathbf{-}\,\mathbf{I}\, a)(\mathbf{-}\,\mathbf{E}\, a)$.

Theoremata.

4. $a \,\epsilon\, \mathrm{K\ Q} \,.\, x,u,v\,\epsilon\; Q \,.\, u < x < v \,.(\ni > u \,.\, \ni < v : \supset a) : \supset .\, x \,\epsilon\; \mathbf{I}\, a$.
5. $a \,\epsilon\, \mathrm{K\ Q} \,.\, x \,\epsilon\, \mathbf{I}\, a : \supset : [(u,v)\,\epsilon](u,v\,\epsilon\; \mathrm{Q} \therefore u < x < v \therefore \ni > u \,.\, \ni < v : \supset : a)\, \mathbf{-} = \Lambda$.

Dem. P1 = (P4)(P5).

6. $a \,\epsilon\, \mathrm{K\ Q} \,.\, u,v\,\epsilon\; \mathrm{Q} \,.(\ni > u \,.\, \ni < v : \supset a) \therefore \supset \therefore \ni > u \,.\, \ni < v : \supset \mathbf{I}\, a$.

Dem. P6 = P4.

Definitions.

1. $A \in \mathbf{SET}\mathbb{R}^+ \to \mathrm{int}(A) = \mathbb{R}^+ \cap \left\{x \middle| \{(u,v) | (u,v \in \mathbb{R}^+) \wedge (u < x < v) \wedge (\{z | z > u\} \cap \{z | z < v\}) \subset A\} \neq \varnothing \right\}$.
2. $A \in \mathbf{SET}\mathbb{R}^+ \to \mathrm{ext}(A) = \mathrm{int}\left(\overline{A}\right)$.
3. $A \in \mathbf{SET}\mathbb{R}^+ \to \mathrm{bd}(A) = \overline{\mathrm{int}(A)} \cap \overline{\mathrm{ext}(A)}$.

Theorems.

4. To do....

7. $a \in \mathrm{K\ Q} . \supset . \mathbf{I}\, a \supset a.$

8. $a \in \mathrm{K\ Q} . \supset . \mathbf{II}\, a = \mathbf{I}\, a.$

Dem. Hp. $(\mathbf{I}\, a)[a]$ P7 $: \supset . \mathbf{II}\, a \supset \mathbf{I}\, a$ (1)

Hp. (2)

$x, u, v \in \mathrm{Q} . u < x < v . (\ni > u . \ni < v : \supset a) .$

P6 $: \supset : u, v \in \mathrm{Q} . u < x < v . (\ni > u . \ni < v :$

$\supset \mathbf{I}\, a)$

Hp. $x \in \mathbf{I}\, a . (2) : \supset : x \in \mathbf{II}\, a$ (3)

Hp. $(3) : \supset : \mathbf{I}\, a\, \mathbf{II}\, a$ (4)

Hp. $(1) . (4) : \supset :$ Ts. (Theor.)

9. $a, b \in \mathrm{K\ Q} . a \supset b : \supset . \mathbf{I}\, a \supset \mathbf{I}\, b$

Dem. Hp. $x, u, v \in \mathrm{Q} . u < x < v . (\ni > u . \ni < v :$ (1)

$\supset a) : \supset \therefore \ni > u . \ni < v : \supset b$

Hp. $x \in \mathbf{I}\, a : \supset : x \in \mathbf{I}\, b$ (Theor.)

10. $a, b \in \mathrm{K\ Q} : \supset : \mathbf{I}\, (ab) \supset \mathbf{I}\, a$

Dem. $(ab, a)[a, b]$ P9 $. = .$ P10

11. $a, b \in \mathrm{K\ Q} . \supset . \mathbf{I}\, (ab) \supset (\mathbf{I}\, a)(\mathbf{I}\, b)$

Dem. P11 $=:$ P10 $. \cap . (b, a)[a, b]$ P10

12. $a, b \in \mathrm{K\ Q} . \supset . \mathbf{I}\, a \supset \mathbf{I}\, (a \cup b)$

13. $a, b \in \mathrm{K\ Q} . \supset . \mathbf{I}\, a \cup \mathbf{I}\, b \supset \mathbf{I}\, (a \cup b)$

14. $a, b \in \mathrm{K\ Q} . \supset . \mathbf{I}\, (ab) = (\mathbf{I}\, a)(\mathbf{I}\, b)$

Dem. Hp. P11 $: \supset . \mathbf{I}\, (ab) \supset (\mathbf{I}\, a)(\mathbf{I}\, b)$ (1)

Hp. $x \in \mathrm{Q} . u, v \in$ (2)

$\mathrm{Q} . u < x < v . (\ni > u . \ni < v : \supset a) . u', v' \in$

$\mathrm{Q} . u' < x < v' . (\ni > u' . \ni < v' : \supset b) . u'' =$

$M(u \cup u') . v'' = \mathrm{W}(v, v') : \supset : u'', v'' \in$

$Q . u'' < x < v'' . (\ni > u'' . \ni > v'' : \supset : ab)$

Hp. $x \,\epsilon\, \mathbf{I}\, a . x \,\epsilon\, \mathbf{I}\, b . (2) : \supset . x \,\epsilon\, \mathbf{I}\,(ab)$ (3)

Hp. $(3) : \supset : (\mathbf{I}\, a)(\mathbf{I}\, b) \supset \mathbf{I}\,(ab)$ (4)

Hp. $(1) . (4) : \supset .$ Ts.

15. $a \,\epsilon\, \mathrm{K\ Q} . \supset . \mathbf{E}\, a \supset -a$

Dem. P15 $= (-a)[a]$ P7

16. $a \,\epsilon\, \mathrm{K\ Q} . \supset \therefore \mathbf{I}\, a . \mathbf{E}\, a := \Lambda$

Dem. Hp. P7 . P15 $: \supset \therefore \mathbf{I}\, a . \mathbf{E}\, a : \supset : a - a := \Lambda$

17. $a \,\epsilon\, \mathrm{K\ Q} . \supset . \mathbf{I}\ \mathbf{E}\, a = \mathbf{E}\, a$

Dem. P17 $= (-a)[a]$ P8

18. $a, b \,\epsilon\, \mathrm{K\ Q} . b \supset a : \supset . \mathbf{E}\, a \supset \mathbf{E}\, b$

Dem. P18 $= (-a, -b)[a, b]$ P9

19. $a, b \,\epsilon\, \mathrm{K\ Q} . \supset : \mathbf{E}\, a \cup \mathbf{E}\, b . \supset \mathbf{E}\,(ab)$

20. $a, b \,\epsilon\, \mathrm{K\ Q} . \supset . \mathbf{E}\,(a \cup b) = (\mathbf{E}\, a)(\mathbf{E}\, b)$

Dem. P20 $= (-a, -b)[a, b]$ P14

21. $a \,\epsilon\, \mathrm{K\ Q} . \supset . \mathbf{L}\,(-a) = \mathbf{L}\, a$

22. $a \,\epsilon\, \mathrm{K\ Q} . \supset \therefore \mathbf{I}\, a . \mathbf{L}\, a := \Lambda$

$\supset \therefore \mathbf{E}\, a . \mathbf{L}\, a := \Lambda$

$\supset \therefore -\mathbf{I}\, a . -\mathbf{E}\, a . -\mathbf{L}\, a := \Lambda$

Dem. P22 = P3

23. $a \,\epsilon\, \mathrm{K\ Q} . \supset : a \supset . \mathbf{I}\, a \cup \mathbf{L}\, a$

24. $a \,\epsilon\, \mathrm{K\ Q} . \supset . \mathbf{I}\,(a\, \mathbf{L}\, a) = \Lambda$

Dem. Hp. P14 . P7 . P22 $: \supset : \mathbf{I}\,(a\, \mathbf{L}\, a) . =$

$. \mathbf{I}\, a\, \mathbf{I}\ \mathbf{L}\, a . \supset . \mathbf{I}\, a\, \mathbf{L}\, a . = . \Lambda$

25. $a, b \,\epsilon\, \mathrm{K\ Q} . a \supset b : \supset : \mathbf{L}\, a . \supset . \mathbf{I}\, b \cup \mathbf{L}\, b$

Dem. Hp. P18 $: \supset : \mathbf{E}\, b \supset \mathbf{E}\, a : \supset :$

$\mathbf{I}\, a \cup \mathbf{L}\, a . \supset . \mathbf{I}\, b \cup \mathbf{L}\, b : \supset .$ Ts.

26. $a, b \,\epsilon\, \mathrm{K\ Q} . \supset :$

$\mathbf{L}\,(ab) \supset .\ \mathbf{I}\ a\ \mathbf{L}\ b \cup \mathbf{I}\ b\ \mathbf{L}\ a \cup \mathbf{L}\ a\ \mathbf{L}\ b$

Dem. Hp. $\supset : ab \supset a\ .\ ab \supset b\ .$ P25

$: \supset :\ \mathbf{L}\,(ab) \supset \mathbf{I}\ a \cup \mathbf{L}\ a\ .\ \mathbf{L}\,(ab) \supset \mathbf{I}\ b \cup \mathbf{L}\ b :$

$\supset :\ \mathbf{L}\,(ab) \supset (\mathbf{I}\ a \cup \mathbf{L}\ a)(\mathbf{I}\ b \cup$

$\mathbf{L}\ b)\ .\ \mathbf{L}\,(ab)(\mathbf{I}\ a)(\mathbf{I}\ b) = \mathbf{L}\,(ab)\ \mathbf{I}\,(ab) = \Lambda :$

$\supset :$ Ts.

26’ $a, b \epsilon$ K Q $. \supset .\ \mathbf{L}\,(ab) \supset \mathbf{L}\ a \cup \mathbf{L}\ b$

27. $a, b \epsilon$ K Q $. \supset :\ \mathbf{L}\,(a \cup b) =$

$\mathbf{L}\ a\ \mathbf{E}\ b \cup \mathbf{L}\ b\ \mathbf{E}\ a \cup \mathbf{L}\ a\ \mathbf{L}\ b$

Dem. P27 $= (-a, -b)[a, b]$ P26

27’. $a, b \epsilon$ K Q $. \supset :\ \mathbf{L}\,(a \cup b) \supset \mathbf{L}\ a \cup \mathbf{L}\ b$

28. $a \epsilon$ K Q $. \supset .\ \mathbf{L}\ \mathbf{I}\ a \supset \mathbf{L}\ a$

Dem. Hp. P7 $: \supset :\ \mathbf{I}\ a \supset a\ .$ P25 (1)

$: \supset :\ \mathbf{L}\ \mathbf{I}\ a \supset \mathbf{I}\ a \cup \mathbf{L}\ a$

Hp. P8 . P22 (2)

$: \supset .\ \mathbf{L}\ \mathbf{I}\ a\ \mathbf{I}\ a = \mathbf{L}\ \mathbf{I}\ a\ \mathbf{II}\ a = \Lambda$

(1)(2) $. \supset .$ Theor.

28’. $a \epsilon$ K Q $. \supset \mathbf{L}\ \mathbf{E}\ a \supset \mathbf{L}\ a$

29. $a \epsilon$ K Q $. \supset .\ \mathbf{LL}\ a \supset \mathbf{L}\ \mathbf{I}\ a \cup \mathbf{L}\ \mathbf{E}\ a$

Dem. Hp. $\supset :\ \mathbf{LL}\ a = \mathbf{L}\,(\mathbf{I}\ a \cup \mathbf{E}\ a)\ .$ P27’ $: \supset .$ Ts.

29’. $a \epsilon$ K Q $. \supset .\ \mathbf{LL}\ a \supset \mathbf{L}\ a$

Dem. P29 . P28 . P28’ $: \supset .$ Theor.

30. $a \epsilon$ K Q $. \supset .\ \mathbf{L}\ a = \mathbf{I}\ \mathbf{L}\ a \cup \mathbf{LL}\ a$

Dem. Hp. P23 $: \supset .\ \mathbf{L}\ a \supset \mathbf{I}\ \mathbf{L}\ a \cup \mathbf{LL}\ a$ (1)

Hp. P7 $: \supset .\ \mathbf{I}\ \mathbf{L}\ a \supset \mathbf{L}\ a$ (2)

Hp. P29’ $: \supset .\ \mathbf{LL}\ a \supset \mathbf{L}\ a$ (3)

(1)(2)(3) $. \supset .$ Theor.

31. $a \epsilon$ K Q $. \supset . \mathbf{L} \ \mathbf{I} \ \mathbf{L} \ a \supset \mathbf{LL} \ a$

Dem. P31 $= (\mathbf{L} \ a)[a]$ P28

32. $a \epsilon$ K Q $. \supset . \mathbf{I} \ \mathbf{LL} \ a = \Lambda$

Dem. Hp. P29' $: \supset : \mathbf{LL} \ a = \mathbf{L} \ a \ \mathbf{LL} \ a . (\mathbf{L} \ a)[a]$

P24 $: \supset$ Ts.

33. $a \epsilon$ K Q $. \supset : \mathbf{I} \ \mathbf{L} \ \mathbf{I} \ \mathbf{L} \ a = \Lambda$

Dem. P31 . P32 $: \supset . P33$

34. $a \epsilon$ K Q $. \supset . \mathbf{LL} \ \mathbf{L} \ a = \mathbf{LL} \ a$

Dem. $(\mathbf{L} \ a)[a]$ P30 $/p$ P32 $: \supset .$ Theor.

35. $a, b \epsilon$ K Q $. \supset . \mathbf{I} \ a \ \mathbf{L} \ b \supset \mathbf{L} \, (ab)$

Dem. Hp. P14 (1)

$: \supset . \mathbf{I} \ a \ \mathbf{L} \ b \ \mathbf{I} \, (ab) = \mathbf{I} \ a \ \mathbf{I} \ b \ \mathbf{L} \ b = \Lambda$

Hp. P2 . P14 (2)

$: \supset . \mathbf{I} \ a \ \mathbf{L} \ b \ \mathbf{E} \, (ab) = \mathbf{I} \ a \ \mathbf{L} \ b \ \mathbf{I} \, (- a \cup - b) =$

$\mathbf{I} \, (a - b) \ \mathbf{L} \ b = \mathbf{I} \ a \ \mathbf{E} \ b \ \mathbf{L} \ b = \Lambda$

(1)(2) $\supset$ Theor.

36. $a, b \epsilon$ K Q $. \supset . \mathbf{I} \ a \ \mathbf{L} \ b \cup \mathbf{I} \ b \ \mathbf{L} \ a \supset \mathbf{L} \ ab.$ (Vide P26)

Dem. P36 =: P35 $.(b, a)[a, b]$ P35

37. $a, b \epsilon$ K Q $. \supset . \mathbf{E} \ a \ \mathbf{L} \ b \cup \mathbf{E} \ b \ \mathbf{L} \ a \cup \mathbf{L} \, (a \cup b).$ (Vide P27)

Dem. P37 $= (- a, - b)[a, b]$ P36

38. $a, b \epsilon$ K Q $. \supset . \mathbf{I} \, (a \cup b) \supset \mathbf{I} \ a \cup \mathbf{I} \ b \cup \mathbf{L} \ a \ \mathbf{L} \ b$ (Vide P13)

Dem. Hp. (1)

$\supset . \mathbf{I} \, (a \cup b) \supset (\mathbf{I} \ a \cup \mathbf{L} \ a \cup \mathbf{E} \ a)(\mathbf{I} \ b \cup \mathbf{L} \ b \cup \mathbf{E} \ b)$

Hp. P20 . P16 (2)

$: \supset . \mathbf{I} \, (a \cup b) \ \mathbf{E} \ a \ \mathbf{E} \ b = \mathbf{I} \, (a \cup b) \ \mathbf{E} \, (a \cup b) = \Lambda$

Hp. P37: $\supset$: $\mathbf{I}\,(a \cup b)(\,\mathbf{E}\ a\ \mathbf{L}\ b \cup$ (3)

$\mathbf{E}\ b\ \mathbf{L}\ a)\,.\supset.\ \mathbf{I}\,(a \cup b)\ \mathbf{L}\,(a \cup b)\,. = \Lambda$

(1)(2)(3) $.\supset.$ Theor.

38'. $a, b\ \epsilon$ K Q $.\supset.\ \mathbf{E}\,(ab) \supset \mathbf{E}\ a \cup \mathbf{E}\ b \cup \mathbf{L}\ a\ \mathbf{L}\ b$ (Vide P19)

39. $a\ \epsilon$ K Q $.\supset.\ \mathbf{I}\ \mathbf{L}\ a\ \mathbf{L}\ \mathbf{I}\ a = \Lambda$

Dem. Hp. P36 : $\supset$: $\mathbf{I}\ \mathbf{L}\ a\ \mathbf{L}\ \mathbf{I}\ a \supset \mathbf{L}\,(\,\mathbf{L}\ a\ \mathbf{I}\ a) = \Lambda$

40. $a\ \epsilon$ K Q $.\supset.\ \mathbf{L}\ \mathbf{I}\ a \supset \mathbf{LL}\ a$

Dem. Hp. P28 . P30 . P39 : $\supset$ Theor.

40'. $a\ \epsilon$ K Q $.\supset.\ \mathbf{L}\ \mathbf{E}\ a \supset \mathbf{LL}\ a$

41. $a\ \epsilon$ K Q $.\supset \mathbf{LL}\ a = \mathbf{L}\ \mathbf{I}\ a \cup \mathbf{L}\ \mathbf{E}\ a$

Dem. P29 . P40 . P40' : $\supset$. Theor.

42. $a\ \epsilon$ K Q $.\supset.\ \mathbf{I}\ \mathbf{L}\ \mathbf{I}\ a = \Lambda$

$\supset.\ \mathbf{I}\ \mathbf{L}\ \mathbf{E}\ a = \Lambda$

$\supset.\ \mathbf{LL}\ \mathbf{I}\ a = \mathbf{L}\ \mathbf{I}\ a$

$\supset.\ \mathbf{LL}\ \mathbf{E}\ a = \mathbf{L}\ \mathbf{E}\ a$

43. $a, b\ \epsilon$ K Q $.\supset.\ \mathbf{I}\,(\,\mathbf{I}\ a \cup \mathbf{I}\ b) = \mathbf{I}\ a \cup \mathbf{I}\ b$

Dem. Hp. P7 : $\supset.\ \mathbf{I}\,(\,\mathbf{I}\ a \cup \mathbf{I}\ b) \supset \mathbf{I}\ a \cup \mathbf{I}\ b$ (1)

Hp. P8 . P13 (2)

: $\supset$: $\mathbf{I}\ a \cup \mathbf{I}\ b\,. = .\ \mathbf{II}\ a \cup \mathbf{II}\ b\,.\supset.\ \mathbf{I}\,(\,\mathbf{I}\ a \cup \mathbf{I}\ b)$

(1)(2) $\supset$ Theor.

44. $a, b\ \epsilon$ K Q $.\supset.\ \mathbf{I}\,(\,\mathbf{LL}\ a \cup \mathbf{LL}\ b) = \Lambda$

Dem. Hp. P38 . P32 . P34 (1)

: $\supset.\ \mathbf{I}\,(\,\mathbf{LL}\ a \cup \mathbf{LL}\ b) \supset \mathbf{LL}\ a\ \mathbf{LL}\ b \supset \mathbf{LL}\ a$

Hp. (1) . P8

: $\supset.\ \mathbf{I}\,(\,\mathbf{LL}\ a \cup \mathbf{LL}\ b) \cup \mathbf{I}\ \mathbf{LL}\ a = \Lambda$

45. $a\ \epsilon$ K Q $.\supset.\ \mathbf{I}\,(\,\mathbf{I}\ a \cup \mathbf{E}\ a) = \mathbf{I}\ a \cup \mathbf{E}\ a$

Dem. P8 . P17 $.(-a)[b]$ P43 : $\supset$. Theor.

45'. $a\ \epsilon$ K Q $.\supset.\ \mathbf{E}\ \mathbf{L}\ a = \mathbf{I}\ a \cup \mathbf{E}\ a$

46. $a \epsilon \mathrm{K\ Q} \,.\!\supset\!.\ \mathbf{E\ I}\, a = -(\mathbf{I}\, a \cup \mathbf{L\ I}\, a)$

46'. $a \epsilon \mathrm{K\ Q} \,.\!\supset\!.\ \mathbf{EE}\, a = -(\mathbf{E}\, a \cup \mathbf{L\ E}\, a)$ 46'.

END

Endnotes

1 **Giuseppe Peano's footnote (original):**

Boole:

The mathematical analysis of logic ..., Cambridge, 1847.

The calculus of logic, Camb. and Dublin Math. Journal, 1848.

An investigation of the laws of thought ..., London, 1854.

E. Schröder:

Der Operationskreis des Logikkalculus, Leipzig, 1877.

Ipse iam nonnulla quae ad logicam pertinent tractavit in praecedenti opera.

Lehrbuch der Arithmetik und Algebra ..., Leipzig, 1873.

Boole e Schröder theorias brevissime exposui in meo libro *Calcolo geometrico ...*, Torino, 1888.

Vide:

C. S. Pierce, *On the Algebra of logic*; American Journal, III, 15; VII, 180.

Jevons, *The principles of science*, London, 1883.

Mc.Coll., *The calculus of equivalent statements*, Proceedings of the London Math. Society, 1878, Vol. IX, 9. Vol X, 16.

1 **Giuseppe Peano's footnote (translated):**

Boole:

The mathematical analysis of logic ... (Cambridge, 1847.)

'The calculus of logic,' *Camb. and Dublin Math. J.*, 3 (1848), 193-98.

An investigation of the laws of thought ... (London, 1854).

E. Schröder:

Der Operationskreis des Logikkalculus (Leipzig, 1877).

He had already treated several matters pertaining to logic in a preceding work.

Lehrbuch der Arithmetik und Algebra ... (Leipzig, 1873).

I gave a very brief presentation of the theories of Boole and Schröder in my book *Calcolo geometric* etc. (Torino, 1888).

Cf:

C. S. Pierce, 'On the Algebra of logic,' *American J. Math.*, 3 (1880), 15-57; 7 (1885), 180-202.

Jevons, *The principles of science* (London, 1883).

Mc.Coll., 'The calculus of equivalent statements,' *Proc. London Math. Soc.*, 9 (1878), 9-20; 10 (1878), 16-28.

2 The 2 other translation, mentioned at the beginning of this current document, translated "ähnlich" literally to "similar", instead of "equivalent". However, additional information can be found in a footnote of the first translation:

> *"Today "similar" has another meaning and instead we would say "equivalent"."*

G. Peano, (1889), "The principles of arithmetic presented by a new method" in: J. van Heijenoort (ed.), *From Frege to Gödel. A source book in mathematical logic. 1879-1931*, Cambridge: Harvard University Press, 1967, p. 93.