Below is Giuseppe Peano's *Arithmetices principia* as first published[1], i.e. as "*Arithmetices principia, nova methodo exposita*"[2] , which appeared translated to English[3] in 1967 as "*The principles of arithmetic, presented by a new method*"[4], as well as in 1973[5]. This present document[6] is the only (to my knowledge) side-by-side Latin-English translation of the Latin original. The mathematical notation (in the English, right, column) got updated to currently canonically-used or easy-to-decrypt symbols in the international and/or English mathematical community; which is also a feature currently unseen in any reprint. In the middle of each dotted horizontal line is the number of the page below in the original Latin edition of 1889. Parts irrelevant for the mathematical-content and/or current notation got greyed.

I

Arithmetices principia
Nova methodo exposita
a
Ioseph Peano

in R. Academia militari professore
Analysin infinitorum in R. Taurinensi Athenæo docente.

Labor et honor

Augustae Taurinorum
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Romae — Via del Corse, 216-217.
Florentiae — Via Oerretani, 8.
1889

The principles of arithmetic
Presented by a new method
by
Giuseppe Peano

professor at the Royal Military Academy
teaching Analysis of the infinite at the Royal Turin Athenaeum.

Work and honor

At Turin
Published by Libreria Bocca
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II

Iuribus servatis

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III

Praefatio

Quaestiones, quae ad mathematicae fundamenta pertinent, etsi hisce temporibus a multis tractatae, satisfacienti solutione et adhuc carent. Hic difficultas maxime ex sermonis ambiguitate oritur.

Quare summi interest verba ipsa, quibus utimur attente perpendere. Hoc examen mihi proposui, atque mei studii resultatus, et arithmeticae applicationes in hoc scripto expono.

Ideas omnes quae in arithmeticae principiis occurrunt, signis indicavi, ita ut quaelibet propositio his tantum signis enuncietur.

Signa aut ad logicam pertinent, aut proprie ad arithmeticam. Logicae signa quae hic occurrunt, sunt numero ad decem, quamvis non omnia necessaria. Horum signorum usus et proprietas nonnullae in priore parte communi sermone explicantur. Ipsorum theoriam fusius hic exponere nolui. Arithmeticae signa, ubi occurrunt, explicantur.

His notationibus quaelibet propositio formam assumit atque praecisionem, qua in algebra aequationes gaudent, et a propositionibus ita scriptis aliea deducuntur, idque processis qui aequationum resolutioni assimilantur. Hoc caput totius scripti.

Sique, confectis signis quibus arithmeticae propositiones scribere possim, in earum tractatione usus sum methodo, quam quia et in aliis studiis sequenda foret, breviter exponam.

Ex arithmeticae signis quae caeteris, una cum logicae signis exprimere licet, ideas significant quas definire possumus. Ita omnia definivi signa, si quatuor excipias, quae in explicationibus §1 continentur. Si, ut puto, haec ulterius reduci nequeunt, ideas ipsis expressas, ideis quae prius notae supponuntur, definire non licet.

Preface

Questions pertaining tot the foundations of mathematics, although treated by many
still lack a satisfactory solution. The difficulty arises principally from the ambig
nary language.

For this reason it is of the greatest concern to consider attentively the words we use
to do this, and am presenting in this paper the results of my study with applicati
metic.

I have indicated by signs all the idea which occur in the fundamentals of arithmetic, s
proposition is stated with just these signs.

The signs pertain either to logic or to arithmetic. The signs of logic that occur her
ten in number, although not all are necessary. The use of these signs and seve
properties are explained in ordinary language in the first part. I did not wish to p
theory more fully here. The signs of arithmetic are explained as they occur.

With this notation every proposition assumes the form and precision equations enjoy
and from propositions so written others may be deduced, by a process which re
solution of algebraic equations. That is the chief reason for writing this paper.

Having made up the signs with which I can write arithmetical propositions, in trea
have used a method which, because it is to be followed in later studies, I shall pre

Those arithmetical signs which may be expressed by using others along with signs of
sent the ideas we can define. Thus I have defined every sign, if you except the fou
contained in the explanations of §1. If, as I believe, these cannot be reduced furth
ideas expressed by them may not be defined by ideas already supposed to be kno

IV

Propositiones, quae logicae operationibus a caeteris deducuntur, sunt *theoremata*; quae vero non, *axiomata* vocavi. Axiomata hic sunt novem (§1), et signorum, quae definitione carent, proprietates fundamentales exprimunt.

In §1-6 numerorum proprietates communes demonstravi; brevitatis causa, demonstrationes praecedentibus similes omisi; demonstrationum communem formam immutare oportet ut logicae signis exprimantur; haec transformatio interdum difficilior est, tamen inde demonstrationis natura clarissime patet.

In sequentibus § varia tractavi, ut huius methodi potentia magis videatur.

In §7 nonnulla theoremata, quae ad numerorum theoriam pertinent, continentur. In §8 et 9 rationalium et irrationalium definitiones inveniuntur.

Denique, in §10, theoremata exposui nonnulla, quae nova esse puto, ad entium theoriam pertinentia, quae cl.[mus] Cantor *Punktmenge (ensemble de points)* vocavit.

In hoc scripto aliorum studiis usus sum. Logicae notationes et propositiones quae in num. II, III et IV continentur, si nonnullas excipias, ad multorum opera, inter quae Boole praecipue, referenda sunt.[7]

Propositions which are deduced from others by the operations of logic are theorem
which this is not true I have called axioms. There are nine axioms here (§1), and t
fundamental properties of the undefined signs.

In §1-6 I have proved the ordinary properties of numbers; for the sake of brevity, I h
proofs which are similar to preceding ones. The ordinary form of proofs has had t
in order that they may be expressed with the signs of logic. This transformation is
rather difficult but the nature of the proof then becomes quite clear. In the follow
I have treated various things so that the power of the method is better seen.

In the following sections I have treated various things so that the power of the meth
seen.

In §7 are several theorems pertaining tot the theory of numbers. In §8 et 9 are foun
tions of rationals and irrationals.

Finally, in §10 I have given several theorems, which I believe to be new, pertaining t
of those entities which Professor Cantor has called *Punktmenge (ensemble de poi*

In this paper I have used the research of others. The notations and propositions of
are contained in numbers II, III, and IV, with some exceptions, represent the wo
among them Boole especially.[7]

V

Signum ϵ, quod cum signo Ɔ confundere non licet, inversionis in logica applicationes, et paucas alias institui conventiones, ut ad exprimendam quamlibet propositionem pervenirem.

In arithmeticae demonstrationibus usus sum libro: H. Grassmann, *Lehrbuch der Arithmetik*, Berlin 1861.

Utilius quoque mihi fuit recens scriptum: R. Dedekind, *Was sind und was sollen die Zahlen*; Braunschweig, 1888, in quo quaestiones, quae ad numerorum fundamenta pertinent, acute examinantur.

Hic meus libellus ut novae methodi specimen habendus est. Hisce notationibus innumeras alias propositiones, ut quae ad rationales et irrationales pertinent, enunciare et demonstrare possumus. Sed, ut aliae theoriae tractentur, nova signa, quae nova indicant entia, instituere necesse est. Puto vero his tantum logicae signis propositiones cuiuslibet scientiae exprimi posse, dummodo adiungantur signa quae entia huius scientiae representant.

The sign $\in$, which must not be confused with the sign $\subset$, applications of the inverse
a few other conventions, I have adopted so that I could express any proposition w

In the proofs of arithmetic I used the book H. Grassmann, *Lehrbuch der Arithmetik* (B

Also quite useful to me was the recent work by R. Dedekind, Was sind und wa
Zahlen (Braunschweig, 1888), in which questions pertaining to the foundations
are acutely examined.

My booklet should be taken as a sample of this new method. With these notations v
and prove innumerable other propositions, such as those which pertain to rationa
tionals. But in order to treat other theories, it is necessary to adopt new signs to i
entities. I believe, however, that with only these signs of logic the propositions of
can be expressed, so long as the signs which represent the entities of the science

VI

Signorum tabula / Table of signs

Signum	Significatio	Pag.	Sign	Meaning
Logicam signa			Signs of logic	
P	propositio	VII	proposition	proposition
K	classis	X	CLASS	class
$\cap$	et	VII, X	$\wedge$	and
$\cup$	vel	VIII, X, XI	$\vee$	or
$-$	non	VIII, X	$\neg$	not
Λ	absurdum *aut* nihil	VIII, XI	$\perp$	false *or*
			$\varnothing$	nothing
Ɔ	deducitur *aut* continetur	VIII, XI	$\rightarrow$	one deduces *or*
			$\subset$	is contained in
=	est aequalis	VIII	=	equals
ϵ	est	X	$\in$	is (an element of) *or*
			is a	is (a)
[ ]	inversionis signum	XI	*inversed*	sign of the inverse
∍	qui *vel* [ϵ]	XII	$\rightarrow$	such that *or*
			isn't	isn't
Th	Theorema	XVI	theorem	Theorem
Hp	Hypothesis		hypothesis	Hypothesis
Ts	Thesis		thesis	Thesis
L	Logica		logic	Logic
Arithmeticae signa			Signs of arithmetic	
Signa 1, 2, ..., =, >, <, +, -, × vulgarem habent significationem. Divisionis signum est /.			The signs 1, 2, ..., =, >, <, +, -, ∗ have their usual meaning. The sign	
N	numerus integer positivus	1	$\mathbb{N}$	positive integers
R	num. rationalis positivus	12	$\mathbb{R}$	postive rational numbers
Q	quantitas, *sive* numerus realis positivus	16	$\mathbb{Q}$	quantity *or* postive real numbers
Np	numerus primus	9	$\mathbb{P}$	prime number
M	maximus	6	M[aximum]	maximum
Ɯ	minimus	6	$\mathrm{m}_{\mathrm{inimum}}$	minimum
T	terminus, *vel* limes summus	15	Gr[eatest bound]	terminus *or* greatest bound
D	dividit	9	divides	divides
Ɑ	est multiplex	9	is divisible	is divisible
π	est primus cum	6	is $\mathbb{P}$rime with	is prime with
Signa composita			Composite signs	
$-<$	non est minor		$\not<$	is not less than
$=\cup>$	est aequalis aut maior		$\geq$	is equal to or greater than
∍ D	divisor		is a divisor	is a divisor
M ∍ D	maximus divisor		is the greatest divisor	is the greatest divisor

VII

Logicae notationes.

I. De punctuatione.

Litteris $a, b, ... x, y, ... x', y', ...$ entia indicamus indeterminata quaecumque. Entia vero dete
signis, sive litteris $P, K, N, ...$ indicamus.

Signa plerumque in eadem linea scribemus. Ut ordo pateat quo ea coniungere oporteat
thesibus ut in algebra, sive *punctis* . : .˙. :: etc. utimur.

Ut formula punctis divisa, intelligatur, primum signa quae nullo puncto seperantur co
sunt, postea quae uno puncto, deinde quae duobus punctis, etc.

Ex. g. sint $a, b, c, ...$ signa quaecumque. Tunc $ab \,.\, cd$ significat $(ab)(cd)$; et $ab \,.\, cd : ef$
significat $(((ab)(cd))((ef)(gh)))k$.

Punctuationis signa omittere licet si formulae quae diversa punctuatione existerent
habeant sensum; vel si una tantum formula, et ipsa quam scribere volumus, sensum

Ut ambiguitatis periculum absit, aritmeticae operationum signis . : nunquam utimur.

Parenthesum figura una est ( ); si in eadem formula, parentheses et puncta occurant,
quae parenthesibus continentur, colligantur.

II. De propositionibus.

Signo P significatur *propositio*.

Signum $\cap$ legitur *et*. Sint a, b propositiones; tunc $a \cap b$ est simultanea af
a, b. Brevitatis causa, loco $a \cap b$ vulgo scribemus $a\ b$.

Signum $-$ legitur *non*. Sit a quaedam P; tunc $-a$ est negatio propositioni

Signo $\cup$ legitur *vel*. Sint a, b propositiones; tunc $a \cup b$ idem est ac $- : -a$.

[Signo V significatur *verum*, sive *identitas*; sed hoc signo numquam utim

Signum Λ significat *falsum*, sive *absurdum*.

[Signum C significat *est consequentia*; ita b C a legitur b *est consequent*
hoc signo nunquam utimur].

Signum Ɔ significat *deducitur*; ita a Ɔ b significat quod b C a. Si propositi
minata continent x, y,..., scilicet sunt inter ipsa entia conditiones, tu
quaecumque sunt $x, y, ...$, a propositione a deducitur b. Si vero ambig
loco $Ɔ_{x,y,...}$, scribemus solum Ɔ.

Signum = significat *est aequalis*. Sint a, b propositiones; tunc $a = b$ idem s
Ɔ a; propositio $a =_{x,y,...} b$ idem significat quod $a\ Ɔ_{x,y,...} b \,.\, b\ Ɔ_{x,y,...} a$.

III. Logicae propositiones.

Sint $a, b, c, ...$ propositiones. Tunc erit:

1. a Ɔ a
2. a Ɔ b . b Ɔ c : Ɔ : a Ɔ c
3. $a = b$. = : a Ɔ b . b Ɔ a.
4. $a = a$
5. $a = b$. = . $b = a$
6. $a = b$. b Ɔ c : Ɔ . a Ɔ c
7. a Ɔ b . $b = c$: Ɔ . a Ɔ c
8. $a = b$. $b = c$: Ɔ . $a = c$
9. $a = b$. Ɔ . a Ɔ b
10. $a = b$. Ɔ . b Ɔ a

11. ab Ɔ a
12. $ab = ba$
13. $a\,(bc) = (ab)\,c = abc$

14. $aa = a$
15. $a = b$. Ɔ . $ac = bc$
16. a Ɔ b . Ɔ . ac Ɔ bc
17. a Ɔ b . c Ɔ d : Ɔ . ac Ɔ bd
18. a Ɔ b . a Ɔ c : = . a Ɔ bc
19. $a = b$. $c = d$: Ɔ . $ac = bd$

20. $-(-a) = a$
21. $a = b$. = . $-a = -b$.
22. a Ɔ b . = . $-b$ Ɔ $-a$ =

23. $a \cup b$. = .˙. $-$: $-a$. $-b$
24. $-(ab) = (-a) \cup (-b)$
25. $-(a \cup b) = (-a)\,(-b)$
26. a Ɔ . $a \cup b$
27. $a \cup b = b \cup a$
28. $a \cup (b \cup c) = (a \cup b) \cup c = a \cup b \cup c$
29. $a \cup a = a$
30. $a\,(b \cup c) = ab \cup ac$
31. $a = b$. Ɔ . $a \cup c = b \cup c$
32. a Ɔ b . Ɔ . $a \cup c$ Ɔ $b \cup c$
33. a Ɔ b . c Ɔ d : Ɔ : $a \cup c$. Ɔ . $b \cup d$
34. b Ɔ a . c Ɔ a : = . $b \cup c$ Ɔ a

35. $a - a = \Lambda$
36. $a\,\Lambda = \Lambda$
37. $a \cup \Lambda = \mathrm{a}$
38. a Ɔ Λ . = . $a = \Lambda$
39. a Ɔ b . = . $a - b = \Lambda$
40. Λ Ɔ a
41. $a \cup b = \Lambda$. = : $a = \Lambda$. $b = \Lambda$

42. a Ɔ . b Ɔ c : = : ab Ɔ c
43. a Ɔ . $b = c$: = . $ab = ac$

Sit α quoddam relationis signum (ex. gr. =, Ɔ), ita ut a
$-$. $a\ \alpha\ b$ scribemus $a - \alpha\ b$; scilicet:

$a - = b$. = : $-$. $a = b$

$a -$ Ɔ b . = : $-$. a Ɔ b

Ita signum $- =$ significat *non est aequalis*. Si propositio
significat: sunt x quae conditioni a satisfaciunt. Signu

Similter, si α et β sunt relationis signa, loco $a\ \alpha\ b$, et
. $\alpha\ \beta$. b et a . $\alpha \cup \beta$. b. Ita, si a et b sunt propositi
deducitur b, sed non vice versa.

a . Ɔ $-$ = . b : = : a Ɔ b . $b -$ Ɔ a

Formulae:

a Ɔ b . b Ɔ c . $a -$ Ɔ c : = Λ

$a = b$. $b = c$. $a - = c$: = Λ

a Ɔ b . b Ɔ $- = c$: Ɔ . a Ɔ $- = c$

a Ɔ $- = b$. b Ɔ c : Ɔ . a Ɔ $- = c$

Sed his notationibus raro utimur.

IV. De classibus.

Signo K significatur *classis*, sive entium

Signum $\in$ significat *est*. Ita $a \in b$ legitur a
$a \in$ P significat *a est quaedam propositi*

Loco $-(a \in b)$ scribemus $a - \in b$; signum

44. $a - \in b$. = : $-$. $a \in b$

Signum a, b, $c \in m$ significat: a, b et c su

45. a, b, $c \in m$. = : $a \in m$. $b \in m$. $c \in$

Sit a classis; tunc $-a$ significatur classis

46. $a \in$ K . Ɔ : $x \in - a$. = . $x - \in a$

Sint a, b classes; $a \cap b$, sive $a\ b$, est clas

quae eodem tempore sunt a et b; $a \cup b$ est

47. $a, b \in$ K . Ɔ .˙. $a\ x \in$. $a\ b$: = : $x \in a$
48. $a, b \in$ K . Ɔ .˙. $a \cup x \in$. $a \cup b$: = :

Signum Λ indicat classem quae nullum

49. $a \in$ K . Ɔ .˙. $a = \Lambda$: = : $x \in a$. $=_x \Lambda$

[Signo V, quod classem ex omnibus indiv
utimur].

Signum Ɔ significat *continetur*. Ita a Ɔ

50. $a, b \in$ K . Ɔ .˙. a Ɔ b : = : $x \in a$. $Ɔ_x$

[Formula b C a significare potest *classis*

Hic signa Λ et Ɔ significationem habent
ambiguitas. Nam si de propositionibus
vero de classibus, *nihil* et *continetur*.

Formula $a = b$, si a et b sint classes, sign

51. $a, b \in$ K . Ɔ .˙. $a = b$: = : $x \in a$. $=_x$

Propositiones 1...41 quoque subsistunt, si

52. $a \in \mathrm{b}$. Ɔ . $b \in$ K
53. $a \in \mathrm{b}$. Ɔ . $b - = \Lambda$
54. $a \in b$. $b = c$: Ɔ . $a \in c$
55. $a \in b$. b Ɔ c : Ɔ . $a \in c$

Sit s classis, et k classis quae in s contin
ex uno tantum constat individuo. Itaqu

56. $s \in$ K . k Ɔ s : Ɔ :: $k \in s$. = .˙. $k - =$

V.

Inversionis signum est [
particulares exponimus.

1. Sit a propositio, indete
quibus a, sive *solutiones*
quae conditioni a satisfa

57. $a \in$ P . Ɔ : $[x\epsilon]\ a$.
58. $a \in$ K . Ɔ .˙. $[x\epsilon]$. x

59. $\alpha \in \mathrm{P} \,.\, \supset \therefore x \in .\, [x\epsilon]\, \alpha : = \alpha$

Sint α, β propositiones indeterminatum continentes x; erit:

60. $[x\epsilon]\, (\alpha\, \beta) = ([x\epsilon]\, \alpha)\, ([x\epsilon]\, \beta)$

61. $[x\epsilon] - \alpha = -\, [x\epsilon]\, \alpha$

62. $[x\epsilon]\, (\alpha \cup \beta) = [x\epsilon]\, \alpha \cup [x\epsilon]\, \beta$

63. $\alpha \supset_x \beta \,.\, = .\, [x\epsilon]\, \alpha \supset [x\epsilon]\, \beta$

64. $\alpha =_x \beta \,.\, = .\, [x\epsilon]\, \alpha = [x\epsilon]\, \beta$

59. α is a $_{\text{proposition}} \rightarrow [(x \in \overline{x} \in \alpha) = \alpha]$

Let α, β be propositions containing the indeterminate x. We will have:

60. $[\overline{x} \in (\alpha \wedge \beta)] = [(\overline{x} \in \alpha) \wedge (\overline{x} \in \beta)]$

61. $(\overline{x} \notin \alpha) = [(\neg \overline{x}) \in \alpha]$

62. $[\overline{x} \notin (\alpha \cup \beta)] = [(\overline{x} \in \alpha) \cup (\overline{x} \in \beta)]$

63. $(\alpha\ (\underset{\forall x}{\longrightarrow} / \underset{\forall x}{\subset})\ \beta) = [(\overline{x} \in \alpha)\ (\rightarrow / \subset)\ (\overline{x} \in \beta)]$

64. $(\alpha \underset{\forall x}{=} \beta) = [(\overline{x} \in \alpha) = (\overline{x} \in \beta)]$

2\. Sint x, y entia quacumque; system ex ente x et ex ente y compositum ut novum ens consideramus, et signo (x, y) indicamus; similiterque si entium numerus maior fit. Sit α propositio indeterminata continens x, y; tunc $[(x, y)\epsilon]\, \alpha$ significat classem entibus (x, y) constitutam, quae conditioni α satisfaciunt. Erit:

65. $\alpha \supset_{x,y} \beta \,.\, = .\, [(x,y)\epsilon]\, \alpha \supset [(x,y)\epsilon]\, \beta$

66. $[(x,y)\epsilon]\, \alpha -= \Lambda \,.\, = \therefore [x\epsilon] \,.\, [y\epsilon]\, \alpha -= \Lambda : -= \Lambda$

2\. Let x, y be any entities. We shall consider the system composed of the entity x and the entity y as a new entity, and indicate it by the sign (x, y); and similarly if the number of entities becomes larger. Let α be a proposition containing the indeterminate x, y; then $[(x, y)\epsilon]\, \alpha$ indicates the class of entities (x, y) which satisfy the condition α. We have:

65\.

66\.

3\. Sit $x\ \alpha\ y$ relatio inter indeterminata x et y (ex. g. in logica relationes $x = y$, $x -= y$, $x \supset y$; in arithmetica $x < y$, $x > y$, etc). Tunc signo $[\epsilon\ \alpha]\ y$ ea x indicamus, quae relationi $x\ \alpha\ y$ satisfaciunt. Commoditatis causa, loco $[\epsilon]$, signo $\ni$ utimur. Ita $\ni \alpha\ y \,.\, = : [x\ \epsilon] \,.\, x\ \alpha\ y$, et signum $\ni$ legitur *qui*, vel *quae*. Ex. gr. sit y numerus; tunc $\ni < y$ classem indicat numeris x compositam qui conditioni $x < y$ satisfaciunt, scilicet, *qui sunt minores* y, vel simpliciter *minores* y. Similiter, quum signum D significet *dividit*, vel *est divisor*, formula $\ni$D significat *qui dividunt* vel divisores. Deducitur $x \in \ni \alpha\ y = x\ \alpha\ y$.

3\. Let $x\ \alpha\ y$ be a relation between the indeterminates x and y (eg. in logic, the relations $x = y$, $x \neq y$, $x \rightarrow y$; in arithmetic, $x < y$, $x > y$, ...). Then the sign $[\epsilon\ \alpha]\ y$ denotes the x that satisfy the relation $x\ \alpha\ y$. For the sake of convenience, we use the sign $\ni$ instead of the sign $[\epsilon]$. Thus, $\ni \alpha\ y \,.\, = : [x\ \epsilon] \,.\, x\ \alpha\ y$, and the sign $\ni$ is read *the objects that*. For example, let y be a number; then $\forall x < y$ denotes the class formed by the numbers x that satisfy the condition $x < y$, that is, *the objects that are smaller than* y, or simply *the objects smaller than* y. Similarly, if the sign D means *divides* or *is a divisor of*, the formula $\ni$D means *the objects that divide* or *the divisors*. It follows that $x \in \ni \alpha\ y = x\ \alpha\ y$.

4\. Sit α formula indeterminate continens x. Tunc scriptura $x'[x]\alpha$, quae legitur x' *loco x in α substituto*, formulam indicat quae obtinetur si in α, loco x, x' legimus. Deducitur $x[x]\alpha = \alpha$.

4\. Let α be a formula containing the indeterminate x. Then the expression $x'[x]\alpha$, which is read x' *being substituted for x in α*, denotes the formula obtained if, in α, we read x' for x. It follows that $x[x]\alpha = \alpha$.

5\. Sit α formula, quae indeterminata $x, y, \ldots$ continet. Tunc

$(x', y', \ldots)\ [x, y, \ldots]\ \alpha$,

5\. Let α be a formula that contains the indeterminates $x, y, \ldots$ Then

$(x', y', \ldots)\ [x, y, \ldots]\ \alpha$,

quae legitur $x'y', \ldots$ *loco* $x, y, \ldots$ *in α substitutis*, formulam indicat quae obtinetur si in α loco $x, y, \ldots$, litterae $x'y', \ldots$ scribantur. Deducitur $(x, y)\ [x, y]\ \alpha = \alpha$.

which is read $x', y', \ldots$ *being substituted for* $x, y, \ldots$ in α, denotes the formula obtained if, in α, the letters $x'_1, y', \ldots$ are written for $x, y, \ldots$ It follows that $(x, y)\ [x, y]\ \alpha = \alpha$.

VI. De functionibus.

Logicae notationes quae praecedunt exprimendae cuilibet arithmeticae propositioni sufficiunt, iisdemque tantum utimur. Hic notationes alias nonnullas breviter explicamus, quae utiles fieri possunt.

Sit s quaedam classis; supponimus aequalitatem inter entia systematis s definitam, quae conditionibus satisfaciat:

$a = a$

$a = b \,.\, = .\, b = a$

$a = b \,.\, b = c : \supset .\, a = c$

Sit φ signum, sive signorum aggregatus, ita ut si x est ens classis s, scriptura $\varphi\ x$ novum indicet ens; supponimus quoque aequalitatem inter entia $\varphi\ x$ definitam; et si x et y sunt entia classis s, et est $x = y$, supponimus deduci posse $\varphi\ x = \varphi\ y$. Tunc signum φ dicitur esse *functionis praesignum in classi* s, et scribemus $\varphi \ni F'\ s$.

$s \in K \,.\, \supset :: \varphi\ F'\ s \,.\, = \therefore x, y \in s \,.\, x = y : \supset_{x,y} .\, \varphi x = \varphi y$

Verum si, cum sit x quodlibet ens classis s, scriptura $x\varphi$ novum indicet ens, et, ex, $x = y$ deducitur $x\varphi = y\varphi$, tunc dicimus φ esse *functionis postsignum in classi* s et scribemus $\varphi \in s'$F.

$s \in K \,.\, \supset :: \varphi\ s\,'F \,.\, = \therefore x, y \in s \,.\, x = y : \supset_{x,y} .\, x\varphi = y\varphi$

Exempla. Sit a numerus; tunc $a +$ est functionis praesignum in numerorum classe, et $+ a$ est functionis postsignum; quicumque enim est numerus x, formulae $a + x$ et $x + a$ novos indicant numeros, et ex $x = y$ deducitur $a + x = a + y$, et $x + a = y + a$. Itaque

$a \in \mathrm{N} \,.\, \supset : a + .\, \in .\, F'\,\mathrm{N}$

$a \in \mathrm{N} \,.\, \supset : + a \,.\, \in .\, N\,'F$

Sit φ functionis praesignum in classe s. Tunc $[\varphi]y$ classem significat iis x constitutam, quae conditioni $\varphi x = y$ satisfaciunt; scilicet:

Def. $s \in \mathrm{K} \,.\, \varphi \in F'\ s : \supset : [\varphi]y \,.\, = .\, [x\epsilon]\ (\varphi x = y)$

VI. Functions.

The symbols of logic introduced above suffice to express any proposition of arithmeti
shall only these.We explain here briefly some other symbols that may be useful.

Let S be a class; we assume that equality is defined between the objects of the syste
satisfy the conditions:

$a = a.$

$(a = b) = (b = a)$

$[(a = b) \wedge (b = c)] \rightarrow a = c$

Let φ be a sign or an aggregate of signs such that, if x is an object of the class S, the
$\varphi\ x$ denotes a new object; we assume also that equality is defined between the ob
further, if x and y are objects of the class S and if $x = y$, we assume it is possible t
$x = \varphi\ y$. Then the sign φ is said to be a *function presign in the class* S, and we wr

...

If, x being any object of the class S, the expression $x\varphi$ denotes a new object and $x\varphi$ =
from $x = y$, then we say that φ is a *function postsign in the class* S, and we write

...

Examples. Let a be a number; then $a+$ is a function presign in the class of numbers,
function postsign; for any number x, formulas $a + x$ and $x + a$ denote new number
$a + x = a + y$ and $x + a = y + a$ follow from $x = y$. Thus

...

Let φ be a function presign in the class S. Then $[\varphi]y$ denotes the class composed of
satisfy the condition $\varphi x = y$; that is,

Def.

Classis $[\varphi]y$ vel unum vel plura, vel etiam nullum individuum continere potest. Erit:

$s \in \mathrm{K} \,.\, \varphi \in F'\ s : \supset : y = \varphi\ x \,.\, = .\, x \in [\varphi]y$

Si vero φy uno tantum constat individuo, erit $y = \varphi x \,.\, = .\, x = [\varphi]y$

Sit φ functions postsignum; similiter ponimus:

$s \in \mathrm{K} \,.\, \varphi \in s\,'F : \supset \therefore y\ |\varphi| = |x\epsilon|\ (x\varphi = y).$

Signum [ ] dicitur *inversionis signum*, eiusque usus nonullos in logica iam exposuimus. Nam si α est propositio indeterminatum continens x, atque A est classis individuis x composita quae conditioni α satisfaciunt, erit $x \in a \,.\, = \alpha$, tunc $a = [x\epsilon]\ \alpha$, ut in V, i.

Sit α formula indeterminate continens x, sitque φ functionis praesignum, quod litterae x praepositum, formulam α gignat; scilicet sit $\alpha = \varphi\ x$; tunc erit $\varphi = \alpha[x]$, et si x' est novum ens, erit $\varphi x' = \alpha[x]x'$, scilicet, si α est formula indeterminatum continens x, tunc $\alpha[x]x'$ significat id quod obtinetur si in α, loco x, x' ponatur.

Similiter, sit α formula indeterminate continens x, sitque φ functionis postsignum, ut $x\varphi = \alpha$; deducitur $\varphi = [x]\alpha$; tunc, si x' est novum ens, erit $x'\varphi = x'[x]\alpha$, scilicet $x'[x]\alpha$ rursum indicat id quod obtinetur si in α, loco x, x' legatur, ut in V, 4.

Alios quoque usus in logica signum [] habere potest, quos breviter esponimus, quum ipsis non utamur. Sint a et b duae classes; tunc $[a\cap]b$ sive $b[\cap a]$ classes indicat x, quae conditioni $b = a \cap x$, sive $b = x \cap a$ satisfaciunt. Si b in a non continetur, nulla classis huic conditioni satifacit; si b in a continetur, signum $b[\cap a]$ omnes indicat classes quae b continent atque in $b \cup -a$ continentur.

In Arithmetica, sint a, b numeri; tunc $[b + a]$ sive $[a+]b$ numerum indicat x, qui conditioni $b = x + a$, sive $b = a + x$ satisfacit, nempe $b - a$. Similiter erit $b[\times a] = [a \times]b = b/a$. Et in analysi hoc signum usuvenire potest; itaque

$y = sin\ x \,.\, = .\, x\epsilon\ [sin]\ y$	(loco $x = arc\ sin\ y$).
$dF(x) = f(x)dx \,.\, = .\, F\ (x)\epsilon\ [d]\ f(x)dx$	(loco $F(x) = \int f(x)dx$).

Sit rursum φ functionis praesignum in classi s, sitque k classis

The class $[\varphi]y$ may contain one or several individuals, or none at all. We have:

...

But if φy consists of just one individual, we have $y = \varphi x \,.\, = .\, x = [\varphi]y$

Let φ be a function postsign; we write similarly:

...

The sign [ ] is called *inversion sign*, and we have already presented some of its uses i
is a proposition containing the indeterminate x and A is a class composed of the i
x that satisfy the condition α, we have $x \in A \,.\, = \alpha$, and then $A = [x\epsilon]\ \alpha$, as in V, i.

Let α be a formula containing the indeterminate x and let φ be a function presign th
the formula α when written before the letter x; that is, let $\alpha = \varphi x$; then we have
and if x' is a new object, we have $\varphi x' = \alpha[x]x'$, that is, if α is a formula containin
indeterminate x, then $\alpha[x]x'$ means what is obtained when, in α, we put x' for x.

Similarly, let α be a formula containing the indeterminate x and let φ be a function
such that $x\varphi = \alpha$; it follows that $\varphi = [x]\alpha$. Then, if x' is a new object, have $x'\varphi = x$
is, $x'[x]\alpha$ again denotes what is obtained, when, in α, read x' for x, as in V, 4..

The sign [] can have other uses in logic, which we present only briefly, since we shall
in these ways. Let a and b be two classes; then $[a\cap]b$ sive $b[\cap a]$, or $b[\cap a]$ denotes
x that satisfy the condition $b = a \cap x$, or $b = x \cap a$. If b is not contained in a, no cla
this condition; if b is contained in a, the sign $b[\cap a]$ denotes all classes that conta
contained in $b \cup -a$.

In arithmetic, let a and b be numbers; then $[b + a]$ or $[a+]b$ denotes the number x th
the condition $b = x + a$, or $b = a + x$, that is $b - a$. Similarly we have $b[\times a] = [a \times]b$
sign can even find a use in analysis; thus

... ...

... ...

Let φ again be a function presign in a class S and let C be a class

in s contenta; tunc φk classem indicat omnibus φx compositam, ubi x sunt entia classis k; scilicet

Def. $s \in \mathrm{K} \,.\, k \in \mathrm{K} \,.\, k \supset s \,.\, \varphi \in F'\ s : \supset .\, \varphi\ k = [y\epsilon]\ (k \,.\, [\varphi]y : -= \Lambda)$

Sive $s \in \mathrm{K} \,.\, k \in \mathrm{K} \,.\, k \supset s \,.\, \varphi \in F'\ s : \supset .\, \varphi\ k = [y\epsilon]\ ([x\epsilon] : x \in k \,.\, [\varphi]x = y \therefore -= \Lambda)$

Def. $s \in \mathrm{K} \,.\, k \in \mathrm{K} \,.\, k \supset s \,.\, \varphi \in s\,'F : \supset .\, k\ \varphi = [y\epsilon]\ (k \,.\, y\ [\varphi] : -= \Lambda)$

Itaque, si $\varphi \in F's$, tunc $\varphi\ s$ classem indicat omnibus $\varphi\ x$ constitutam, ubi x sint entia classis s. Erit:

$s \in K \,.\, \varphi \in F's \,.\, y \in \varphi s : \supset : \varphi[\varphi]y = y$

$s \in K \,.\, a, b \in K \,.\, a \supset s \,.\, b \supset s \,.\, \varphi \in F's : \supset .\, \varphi(a \cup b) = (\varphi a) \cup (\varphi b)$

$s \in K \,.\, \varphi \in F's : \supset .\, \varphi\Lambda = \Lambda$

$s \in K \,.\, a, b \in K \,.\, b \supset s \,.\, a \supset b \,.\, \varphi \in F's : \supset .\, \varphi a \supset \varphi b$

$s \in K \,.\, a, b \in K \,.\, a \supset s \,.\, b \supset s \,.\, \varphi \in F's : \supset .\, \varphi(ab) \supset (\varphi a)(\varphi b)$

Sit a quaedam classis; tunc $a \cap K$, sive $K \cap a$, sive Ka, classes omnes indicat formae $a \cap x$, sive $x \cap a$, xa, ubi x est classis quacumque; scilicet Ka indicat classes quae in a continentur. Formula $x \in Ka$ idem significat quod $x \in K \,.\, x \supset a$. Hac conventione quandoque utimur; ita KN isgnificat *numerorum classem*.

Similiter, si a est classis, $K \cup a$ indicat classes quae a continent. Sit a numerus; tunc $a + N$, sive $N + a$, *numeros* indicat *numero a maiores*; $a \times N$, sive $N \times a$, sive Na indicat *multiplices numeri a*; a^N indicat *potestas numeri a*; $N^2, N^3, \ldots$ indicat *numeros quadratos*, vel *numeros cubos*,...

contained in S; then φC denotes the class consisting of all φx, where the x are the o
class C; that is

Def.

Or

Def.

Thus, if $\varphi \in F'S$, then $\varphi\ s$ denotes the class composed of all $\varphi\ x$, where the x are obj
class S. We have:

...

...

...

...

...

Let a be a class; then $a \cap K$, or $K \cap a$, or Ka, denotes all classes of the form $a \cap x$, or
where x is any class; that is Ka denotes the classes that are contained in a. The
$x \in Ka$ means the same as $x \in K \,.\, x \supset a$. We shall sometimes use this convention;
means *a class of numbers*.

Similarly, if A is a class, $K \cup A$ indicates the classes that contain A. Let a be a numb
$a + N$, or $N + a$, denotes *the numbers greater than the number a*; $a \times N$, or $N \times a$,
denotes *the multiples of the number a*; a^N denotes *the powers of the number a*; N
denote *the squares*, *the cubes*,...

Functional signorum aequalitatem, productum, potestas, ita definire licet:

Def. $s \in K \,.\, \varphi, \psi \in F's : \supset \therefore \varphi = \psi : = : x \in s \,.\, \supset .\, \varphi x = \psi x$

Def. $s \in K \,.\, \varphi \in F's \,.\, \psi \in F'\varphi s \,.\, x \in s : \supset .\, \psi\varphi x = \psi(\varphi x)$

Itaque, in definitionis hypothesi, erit $\psi\varphi$ novum functionis praesignum; idque *productum signorum ψ et φ* vocatur.

Similiterque, si φ, ψ sunt functionis postsigna.

Haec valet propositio:

$s \in K \,.\, \varphi \in F's \,.\, \varphi s \supset s : \supset : \varphi\varphi s \supset s \,.\, \varphi\varphi\varphi s \supset s.$ etc.

Funcitones $\varphi\varphi, \varphi\varphi\varphi, \ldots$ *iteraiae* vocantur, et communiter signis $\varphi^2, \varphi^3, \ldots$ indicantur, ut operationis φ potestates.

Equality, product and powers can be defined thus for function signs:

Def.

Def.

Thus, if we assume this definition, we have the new function presign $\psi\varphi$; it is called
of the signs ψ and φ.

Similarly if φ, ψ are function postsigns.

The following proposition holds:

...

The functions $\varphi\varphi, \varphi\varphi\varphi, \ldots$ are said to be *iterated* and are generally denoted by the si
as powers of the operation φ.

Si vero φ est functionis postsignum, ha faciliori notatione, absque ambiguitate, uti licet:

Def. $s \in K \,.\, \varphi \in s'F \,.\, s\varphi \supset s : \supset : \varphi 1 = \varphi \,.\, \varphi 2 = \varphi\varphi \,.\, \varphi 3 = \varphi\varphi\varphi.$ etc.

In definitionis hypothesi, si $m, n \in N$, erit $\varphi\ (m + n) = (\varphi m)(\varphi n)$; $(\varphi m)n = \varphi(mn)$

Si hac definitione in Arithmetica utimur, haec invenimus. Numerum qui sequitur numerum a signo faciliori $a+$ indicare possumus; tunc $a + 1, a + 2, \ldots$ et, si b est numerus, $a + b$, sensum habent $a+, a++, \ldots$ quod a definitione in §1 patet. Propositionem 6 in §1 scribere possumus $N+ \supset N$. Si a, b, c sunt numeri, tunc $a : +b \,.\, c$ significat $a + bc$, et $a : \times b \,.\, c$ significat ab^c.

Multi aliis proprietatibus gaudent functionem signa, praesertim si conditioni satisfaciunt: $\varphi x = \varphi y \,.\, \supset .\, x = y$. Functionis signum quod huic conditioni satisfacit vocatur a clarrissimo Dedekind *simile* (ähnliche Abbildung).

Sed his exponendis locus deest.

But if φ is a function postsign, we can use the following more convenient notation w
ambiguity:

Def.

Assuming this definition, if $m, n \in N$, we have $\varphi\ (m + n) = (\varphi m)(\varphi n)$; $(\varphi m)n = \varphi(mn)$

If we use this definition in arithmetice, we obtain the following. We can denote the n
follows the number a by the more convenient sign $a+$; then $a + 1$, $a + 2, \ldots$, and, if
number, $a + b$, have the meaning of $a+$, $a++, \ldots$, which is clear from the definitio
below. Proposition 6 in §1 can be written $N+ \supset N$. If a, b, c are numbers, then a :
means $a + bc$, and $a : \times b \,.\, c$ means ab^c.

Function signs possess many other properties, especially if they satisfy the conditio
$\supset .\, x = y$. A function sign that satisfies this condition is called *equivalent*[8] by De
(ähnliche Abbildung).

But we lack the space to present these properties.

Declarationes.

Defenitio, vel breviter *Def.* est propositio formam habens $x = a$, sive $\alpha \supset .\, x = a$, ubi α est signorum aggregatus sensum habens notum; x est signum, vel signorum aggregatus significatione adhuc carnes; α vero est conditio sub qua definitio datur.

Theorema, (Theor. vel Th) est propositio quae demonstratur. Si theorema formam habet $\alpha \supset \beta$, ubi α et β sunt propositiones, tunc α dicitur *Hypothesis* (Hyp. vel breviter Hp.), β vero *Thesis* (Thes. vel Ts.). Hyp. ac Ts. a Theorematis forma pendent; nam si loco $\alpha \supset \beta$ scribemus $-\beta \supset -\alpha$, erit $-\beta$ Hp, et $-\alpha$ Ts.; si vero scribemus $\alpha - \beta = \Lambda$, Hp. ac Ts. absunt.

In quolibet § signum P quod quidam numerus sequatur, propositionem indicat eiusdem § hoc numero signatam. Logicae propositiones indicantur signo L et propositiones numero.

Formulae quae in una linea non continentur, in altera linea, nullo interposito signo, sequuntur.

Remarks.

A *Definition*, or *Def.* for short, is a proposition of the form $x = a$ or α
aggregate of signs having a known meaning, x is a sign or an agg
without meaning, and α is the condition under which the definitio

A *theorem* (Theor. or Th.) is aproposition that is *proved*. If a theorem
α and β are propositions, then α is called the *hypothesis* (Hyp., or
the *thesis* (Thes. or Ts.). Hyp. and Ts. depend on the form of the t
$\alpha \supset \beta$ instead of $-\beta \supset -\alpha$, then $-\beta$ is the Hp, abd $-\alpha$ the Ts.; if we
Ts. do not exist.

In any § below, the sign P followed by a number denotes the propositi
number in the same §. Propositions of logic are indicated by the s
the proposition.

Formulas that do not fit on one line are continued on the next line wi

Arithmetices Principia.

§1. De numeris et de additione.

Explicationes.

Signo	N	significa
»	1	»
»	$a + 1$	»
»	=	»

Axi

1. $1 \in N$
2. $a \in \mathrm{N} \,.\, \supset .\, a = a$
3. $a, b, c \in \mathrm{N} \,.\, \supset : a = b \,.\, = .\, b$
4. $a, b \in \mathrm{N} \,.\, \supset \therefore a = b \,.\, b = c$:
5. $a = b \,.\, b \in \mathrm{N} : \supset .\, a \in N$
6. $a \in \mathrm{N} \,.\, \supset .\, a + 1 \in N$
7. $a, b \in \mathrm{N} \,.\, \supset : a = b \,.\, = .\, a +$
8. $a \in \mathrm{N} \,.\, \supset .\, a + 1 -= 1$
9. $k \in K \therefore 1 \in k \therefore x \in \mathrm{N} \,.\, x \in$

10. $2 = 1 +$

$\therefore \supset \therefore b - a \in N.$

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$a, b \in \mathrm{N} . b - a \in \mathrm{N} . (\mathrm{L}56) : \supset : b - a -= \Lambda : \supset : a < b.$ (3)

$(2)(3) . \supset .$ Theor.

6. $a, b \in \mathrm{N} . a < b : \supset . b - a + a = b.$

Dem. Hyp . P5 . P1 $: \supset : b - a \in \mathrm{N} . (b - a) \in [x \in](x + a = b) : \supset :$ Thes.

7. $a, b, c \in \mathrm{N} . \supset : c = b - a . = . c + a = b.$

Dem. Hyp . §1 P22 . P6 $: \supset : c = b - a . = . c + a = b - a + a . = . c + a = b.$

8. $a, b \in \mathrm{N} . \supset . a + b - a = b.$

Dem. $(a + b, b)[b, c]$ P7 $. \supset .$ Theor.

9. $a, b, c \in \mathrm{N} . a < b : \supset : c + (b - a) = c + b - a.$

Dem. Hyp . P6 $: \supset : (b - a) + a = b : \supset : c + (b - a) + a = c + b$. P7 $: \supset :$ Thesis.

10. $a, b, c \in \mathrm{N} . a > b + c : \supset . a - (b + c) = a - b - c.$

11. $a, b, c \in \mathrm{N} . b > c . a > b - c : \supset . a - (b - c) = a + c - b.$

12. $a, b, a', b' \in \mathrm{N} . a = a' . b = b' : \supset : a < b . = . a' < b'.$

Dem. Hyp $. \supset . b - a = b' - a' . \supset . b - a \in \mathrm{N} = b' - a' \in \mathrm{N} . \supset .$ Thes.

13. $a, b \in \mathrm{N} . \supset . a < a + b.$

Dem. Hyp . $P8 : \supset : a + b - a = b : \supset . a + b - a \in \mathrm{N}$. P5 $: \supset :$ Thesis.

14. $a, b, c \in \mathrm{N} . a < b . b < c : \supset . a < c.$

Dem. Hyp $. \supset : b - a \in \mathrm{N} . c - b \in : \supset : (b - a) + (c - b) \in \mathrm{N} : \supset : c - a \in \mathrm{N} : \supset .$

Thesis.

15. $a, b, c \in \mathrm{N} . \supset : a < b . = . a + c < b + c.$

Dem. Hyp $. \supset : a < b . = . b - a \in \mathrm{N} . = . (b + c) - (a + c) \in \mathrm{N} . = . a + c < b + c.$

16. $a, b, a', b' \in \mathrm{N} . a < b . a' < b' : \supset . a + a' < b + b'.$

Dem. Hyp $. \supset : a + a' < b + a' . b + a' < b + b' : \supset .$ Thesis.

17. $a, b, c \in \mathrm{N} . a < b < c : \supset . c - a > c - b.$

Dem. Hyp $. \supset b - a \in \mathrm{N} . c - b \in \mathrm{N} . (c - b) + (b - a) = c - a : \supset .$ Thesis.

18. $a \in \mathrm{N} . \supset : a = 1 . \cup . a > 1.$

Dem. $1 \in [a \in]$ Thesis.

$a \in \mathrm{N}$. P13 $: \supset : a + 1 > 1 : \supset : a + 1 \in [a \in]$ Thesis.

$(1)(2) . \supset .$ Theor.

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19. $a, b \in \mathrm{N} . \supset . a + b -= b.$

Dem. $a \in \mathrm{N}$. §1 P8 $: \supset : a + 1 -= 1 : \supset : 1 \in [b \in]$ Thesis. (1)

$a \in \mathrm{N} . b \in \mathrm{N} . b \in [b \in]$ Ts $: \supset : a + b -= b$. §1 P17 (2)

$: \supset : a + (b + 1) -= b + 1 : \supset : b + 1 \in [b \in]$ Ts.

$(1)(2) . \supset .$ Theor.

20. $a, b \in \mathrm{N} . a < b . a = b := \Lambda.$

Dem. Hyp $: \supset : b - a \in \mathrm{N} . (b - a) + a = a$. P19 $: \supset : \Lambda.$

21. $a, b \in \mathrm{N} . a > b . a = b := \Lambda.$

22. $a, b \in \mathrm{N} . a > b . a < b := \Lambda.$

23. $a, b \in \mathrm{N} : \supset : a < b . \cup . a = b . \cup . a > b.$

Dem $a \in \mathrm{N}$. P18 $: \supset . 1 \in [b \in]$ Ts. (1)

$a, b \in \mathrm{N} . a < b : \supset . a < b + 1.$ (2)

$a, b \in \mathrm{N} . a = b : \supset . a < b + 1.$ (3)

$a, b \in \mathrm{N} . a > b : \supset : a - b \in \mathrm{N}$. P18 $: \supset : a - b = 1 . \cup . a - b > 1.$ (4)

$a, b \in \mathrm{N} . a - b = 1 : \supset . a = b + 1.$ (5)

$a, b \in \mathrm{N} . a - b > 1 : \supset . a > b + 1.$ (6)

$a, b \in \mathrm{N} . a > b . (4)(5)(6) : \supset : a = b + 1 . \cup . a > b + 1.$ (7)

$a, b \in \mathrm{N} : a > b . \cup . a = b . \cup . a > b : (2)(3)(7) \therefore \supset \therefore a < b + 1 . \cup . a =$ (8)

$b + 1 . \cup . a > b + 1.$

$a, b \in \mathrm{N} . b \in [b \in]$ Ts $. (8) : \supset : b + 1 \in [b \in]$ Ts. (9)

$(1)(9) . \supset .$ Theor.

§3. Maxima and minima.

E

Let $a \in \mathrm{K\,N}$, that is, let a be a cla

§3. De maximis et minimis.

Explicationes.

Sit $a \in KN$, hoc est sit a quaedam numerorum classis; tunc Ma legatur *maximus inter* a, et $\text{Ɯ}a$ legatur *minimus inter* a.

Definitiones.

1. $a \in K\,\mathrm{N} . \supset : Ma = [x \in](x \in a \therefore a . \in > x := \Lambda.$

2. $a \in K\,\mathrm{N} . \supset : \text{Ɯ}a = [x \in](x \in a \therefore a . \in < x := \Lambda.$

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Theoremata.

3. $n \in \mathrm{N} . a \in K\,\mathrm{N} . a -= \Lambda . a \ni > n = \Lambda : \supset . Ma \in N.$

Dem. $a \in K\,\mathrm{N} . a -= \Lambda . a \ni > 1 = \Lambda : \supset : a = 1 : \supset . Ma = 1 : \supset . Ma \in N.$ (1)

$(1) \supset : 1 \in [n \in]$ (Hp $\supset$ Ts). (2)

$n \in \mathrm{N} . a \in K\,\mathrm{N} . a \ni > n + 1 = \Lambda . n + 1 \in a : \supset : n + 1 = Ma : \supset : Ma \in N.$ (3)

$n \in \mathrm{N} . a \in K\,\mathrm{N} . a \ni > n + 1 = \Lambda . n + 1 -\in a : \supset : a \ni > n = \Lambda.$ (4)

$n \in [n \in]$ (Hp $\supset$ Ts) $. a \in K\,\mathrm{N} . a \ni > n + 1 = \Lambda . n + 1 -\in a : \supset : Ma \in N.$ (5)

$n \in [n \in]$ (Hp $\supset$ Ts) $. (6) : \supset . (n + 1) \in [n \in]$ (Hp $\supset$ Ts). (7)

$(2)(7)$. §1 P9 $: \supset : n \in \mathrm{N} . \supset .$ Hp $\supset$ Ts. (Th.)

4. $a \in K\,\mathrm{N} . a -= \Lambda : \supset . \text{Ɯ}a \in N.$

5. $a \in K\,\mathrm{N} . \supset : \text{Ɯ}a = M[x \in](a \ni < x = \Lambda).$

§4. De multiplicatione.

Definitiones.

1. $a \in \mathrm{N} . \supset . a \times 1 = a.$

2. $a, b \in \mathrm{N} . \supset . a \times (b + 1) = a \times b + a.$

$ab = a \times b ; ab + c = (ab) + c ; abc = (ab)c$

Theoremata.

3. $a, b \in \mathrm{N} . \supset . ab \in N.$

Dem. $a, b \in \mathrm{N}$. P1 $: \supset : a \times 1 \in$

$a, b \in \mathrm{N} . b \in [b \in]$ Ts $: \supset$

$: \supset : a(b + 1) \in \mathrm{N} : \supset :$

$(1)(2) . \supset .$ Theor.

4. $a, b, c \in \mathrm{N} . \supset . (a + b)$

Nota. Haec est prop. 5^a E

Dem. $a, b \in \mathrm{N}$. P1 $: \supset : 1 \in$

$a, b, c \in \mathrm{N} . c \in [c \in]$

$: \supset : (a + b)c + a +$

$: \supset : (a + b)(c + 1) =$

Ts.

$(1)(2) . \supset .$ Theor.

5. $a \in \mathrm{N} . \supset . 1 \times a = a.$

Dem. $1 \in [a \in]$ Ts.

$a \in [a \in]$ Ts $. \supset . 1 \times a$

$a + 1 . \times . 1 \times (a +$

$(1)(2) . \supset .$ Theor.

6. $a, b \in \mathrm{N} . \supset . ba + a =$

7. $a, b \in \mathrm{N} . \supset . ab = ba.$

Dem. $a \in \mathrm{N}$. P5 . P1 $: \supset . a$

$a, b \in \mathrm{N} . b \in [b \in]$ Ts

$: \supset : ab = ba : \supset :$

.P6$: \supset : a(b + 1) =$

$(1)(2) . \supset .$ Theor.

8. $a, b, c \in \mathrm{N} . \supset . a(b +$

Dem. P4 . P7 $: \supset .$ Theor.

9. $a, b, c \in \mathrm{N} . a = b : \supset :$

Dem. $a, b \in \mathrm{N} . a = b :: \supset ::$

$. \supset : ac = bc . a =$

$a(c + 1) = b(c + 1$

$:: \supset : c \in \mathrm{N} . \supset .$ Ts

10. $a, b, c \in \mathrm{N} : a < b : \supset$

Dem. Hyp $. \supset : b - a \in \mathrm{N} . ($

$bc : \supset : (b - a)c =$

11. $a, b, c \in \mathrm{N} . a < b : \supset :$

Dem. Hyp $. \supset : b - a \in \mathrm{N}$.

$: \supset : bc - ac \in \mathrm{N} :$

12. $a, b, c \in \mathrm{N} . \supset \therefore a < b$

$a > b . = . ac > bc$

13. $a, b, a', b' \in \mathrm{N} . a < a$

14. $a, b \in \mathrm{N} : \supset . ab . > \cup$

15. $a, b, c \in \mathrm{N} . \supset . a(bc) =$

Dem. $a, b \in \mathrm{N}$. P1 $: \supset : 1 \in$

$a, b, c \in \mathrm{N} . c \in [c \in]$

$a(bc) + ab = abc$

$\supset : a(b(c + 1)) = a$

$(1)(2) . \supset .$ Theor.

§5. I

1. $m,p,q \in \mathrm{N} \,.\supset.\, m\frac{p}{q} = mp/q.$

2. $p,q,p',q' \in \mathrm{N} \,.\supset::\, \frac{p}{q} = \frac{p'}{q'} \,.=\therefore\, x \in \mathrm{N} \,.\, x\frac{p}{q}, x\frac{p'}{q'} \in \mathrm{N} :$
$\supset_x .\, x\frac{p}{q} = x\frac{p'}{q'}.$

3. $R =:: [x\,\epsilon] \therefore p,q \in \mathrm{N} \,.\, \frac{p}{q} = x : -\!= \Lambda.$

4. $p \in \mathrm{N} \,.\supset.\, \frac{p}{1} = p.$

Theorems.

Theoremata.

5. $p,q,p',q' \in \mathrm{N} \,.\supset::\, \frac{p}{q} = \frac{p'}{q'} \,.=.\, pq' = p'q.$ (Eucl. VII, 19)

Dem. Hp. $\frac{p}{q} = \frac{p'}{q'} : \supset \therefore qq', qq'\frac{p}{q}, qq'\frac{p'}{q'} \in \mathrm{N}$. P2 (1)
$\therefore \supset \therefore qq'\frac{p}{q} = qq'\frac{p'}{q'} \,.\, qq'\frac{p}{q} = pq' \,.\, qq'\frac{p'}{q'} =$
$p'q \therefore \supset \therefore pq' = p'q.$
Hp. $pq' = p'q \therefore \supset \therefore x \in \mathrm{N} \,.\, x\frac{p}{q}, x\frac{p'}{q'} \in \mathrm{N} : \supset_x : xp'q' =$ (2)
$xp'q : \supset : (x\frac{p}{q})qq' = (x\frac{p'}{q'})qq' : \supset : x\frac{p}{q} = x\frac{p'}{q'}.$
(1)(2) $.\supset.$ Th.

6. $m,p,q \in \mathrm{N} \,.\supset.\, \frac{p}{q} = \frac{mp}{mq}.$ (Eucl. VII, 17)

7. $p,q \in \mathrm{N} \,.\, m \in \mathrm{N} \,.\, mDp \,.\, mDq : \supset.\, \frac{p}{q} = \frac{p/m}{q/m}.$

8. $p,q,p',q' \in \mathrm{N} \,.\, p\pi q \,.\, p'\pi q' \,.\, \frac{p}{q} = \frac{p'}{q'} : \supset : p = p' \,.\, q =$
$q'.$

9. $p,q,p',q' \in \mathrm{N} \,.\, p'\pi q' \,.\, \frac{p}{q} = \frac{p'}{q'} : \supset : p'/p = q'/q =$
$M \ni D(p,q).$

10. $p,q,p',q' \in \mathrm{N} \,.\, \frac{p}{q} = \frac{p'}{q'} \,.\, p\pi q \,.\, q' < q := \Lambda.$ (Eucl. VII, 21)

11. $p,q,p',q' \in \mathrm{N} : \supset : \frac{p}{q} = \frac{p'}{q'} \,.=.\, \frac{p}{p'} = \frac{q}{q'} \,.=.\, \frac{q}{p} = \frac{q'}{p'}.$ (Eucl. VII, 13)

12. $p,q \in \mathrm{N} \,.\supset:: [m\,\epsilon] : m \in \mathrm{N} \,.\, m\frac{p}{q} \in \mathrm{N} \therefore -\!= \Lambda.$

12'. $a\breve{a} \in \mathrm{R} \,.\supset:: [m\,\epsilon] : m \in \mathrm{N} \,.\, ma \in \mathrm{N} \therefore -\!= \Lambda.$

14

13. $p,q,p',q' \in \mathrm{N} \,.\supset:: [(r,s,l)\,\epsilon] : r,s,t \in \mathrm{N} \,.\, \frac{p}{q} = \frac{r}{t} \,.\, \frac{p'}{q'} =$
$\frac{s}{t} \therefore -\!= \Lambda.$

13'. $a,b \in \mathrm{R} \,.\supset:: [(r,s,t)\,\epsilon] : r,s,t \in \mathrm{N} \,.\, a = \frac{r}{t} \,.\, b = \frac{s}{t} \therefore -\!=$
$\Lambda.$

14. $a,b,c \in \mathrm{R} \,.\supset:: [(m,n,p,q)\,\epsilon] : m,n,p,q \in \mathrm{N} \,.\, a =$
$\frac{m}{q} \,.\, b = \frac{n}{q} \,.\, c = \frac{p}{q} \therefore -\!= \Lambda.$

15. $p,q,r \in \mathrm{N} \,.\, a = \frac{p}{r} \,.\, b = \frac{q}{r} : \supset : a = b \,.=.\, p = q.$

16. $m \in \mathrm{N} \,.\, a,b \in \mathrm{R} \,.\, a = b \,.\, ma \in \mathrm{N} : \supset.\, mb \in N.$

17. $a,b,c \in \mathrm{R} \,.\supset \therefore a = a.$
$\supset \therefore a = b \,.=.\, b = a.$
$\supset \therefore a = b \,.\, b = c : \supset.\, a = c.$

18. $N \supset \mathrm{R}.$

Definitions.

Definitiones.

19. $a,b \in \mathrm{R} \,.\supset:: a < b \,.=\therefore x \in \mathrm{N} \,.\, xa, xb \in \mathrm{N} : \supset.\, xa <$
$xb.$

20. $a,b \in \mathrm{R} \,.\supset : b > a \,.=.\, a < b.$

Theorems.

Theoremata.

21. $p,q,r \in \mathrm{N} \,.\, a = \frac{p}{r} \,.\, b = \frac{q}{r} : \supset : a < b \,.=.\, p < q.$

22. $p,q,p',q' \in \mathrm{N} \,.\supset : \frac{p}{q} < \frac{p'}{q'} \,.=.\, pq' < p'q.$

23. $p,q,r \in \mathrm{N} \,.\, a = \frac{r}{p} \,.\, b = \frac{r}{q} : \supset : a < b \,.=.\, p > q.$

24. $p,q,p',q' \in \mathrm{N} \,.\, \frac{p}{q} < \frac{p'}{q'} : \supset.\, \frac{p}{q} < \frac{p+p'}{q+q'} < \frac{p'}{q'}.$

25. $a \in \mathrm{R} \,.\supset \therefore ?. \ni > a : -\!= \Lambda.$

26. $a \in \mathrm{R} \,.\supset \therefore \mathrm{R} \,.\, \ni < a : -\!= \Lambda.$

27. $a,b \in \mathrm{R} \,.\, a < b : \supset \therefore \mathrm{R} \,.\, \ni > a \,.\, \ni < b : -\!= \Lambda.$

28. $a,b \in \mathrm{R} : \supset \therefore a < b \,.\, a = b := \Lambda.$
$\supset \therefore a > b \,.\, a = b := \Lambda.$
$\supset \therefore a < b \,.\, a > b := \Lambda.$
$\supset \therefore a -\!< b \,.\, a -\!= b \,.\, a -\!> b := \Lambda.$

29. $a,b,c \in \mathrm{R} : \supset \therefore a < \cup = b \,.\, b < c : \supset : a < c.$
$\supset \therefore a < b \,.\, b < \cup = c : \supset : a < c.$

15

Defi

Definitiones.

30. $a,b \in \mathrm{R} \,.\supset.\, a + b = [c\,\epsilon](c \in \mathrm{R} \therefore x \in \mathrm{N} \,.\, xa, xb, xc \in$
$\mathrm{N} : \supset_x .\, xa + xb = xc).$

31. $a,b \in \mathrm{R} \,.\supset:: b - a = \therefore [x\,\epsilon](x \in \mathrm{R} \,.\, a + x = b).$

32. $a,b \in \mathrm{R} \,.\supset.\, ab = [c\,\epsilon](c \in \mathrm{R} \therefore x \in \mathrm{N} \,.\, xa, (xa)b, xc \in$
$\mathrm{N} : \supset_x .(xa)b = xc).$

33. $a,b \in \mathrm{R} \,.\supset.\, b/a = [x\,\epsilon](x \in \mathrm{R} \,.\, ax = b).$

Theoremata.

34. $p,q,r \in \mathrm{N} \,.\supset \frac{p}{r} + \frac{q}{r} = \frac{p+q}{r}.$

35. $a,b \in \mathrm{R} \,.\supset.\, a + b \in \mathrm{R}.$

36. $p,q,r \in \mathrm{N} \,.\, p < q : \supset.\, \frac{q}{r} - \frac{p}{r} = \frac{q-p}{r}.$

37. $a,b \in \mathrm{R} \,.\, a < b : \supset.\, b - a \in \mathrm{R}.$

38. $p,q,p',q' \in \mathrm{N} \,.\supset.\, \frac{p}{q}\frac{p'}{q'} = \frac{pp'}{qq'}.$

39. $a,b \in \mathrm{R} \,.\supset.\, ab \in \mathrm{R}.$

40. $p,q,p',q' \in \mathrm{N} \,.\supset.\, \frac{p}{q}/\frac{p'}{q'} = \frac{pq'}{p'q}.$

41. $a,b \in \mathrm{R} \,.\supset.\, b/a \in \mathrm{R}.$

42. $p,q \in \mathrm{N} \,.\supset.\, \frac{p}{q} = \frac{p}{q}.$

§9. Rationalum systemata. Irrationales.

Explicatio.

Si $a \in \mathrm{K\,R}$, signum $\mathrm{T}\,a$ legitur *terminus summus*, vel *limes summus*

ens relationes ac operationes tantum definimus.

Definitiones.

1. $a \in \mathrm{K\,R} \,.\, x \in \mathrm{R} : \supset:: x < \mathrm{T}\,a \,.= \therefore$

2. $a \in \mathrm{K\,R} \,.\, x \in \mathrm{R} : \supset::: x = \mathrm{T}\,a \,.=:.$
$\mathrm{R} \,.\, u < x : \supset_x \therefore a \,.\, \ni > u : -\!=$

3. $a \in \mathrm{K\,R} \,.\, x \in \mathrm{R} : \supset \therefore x > \mathrm{T}\,a \,.=: x$
$\mathrm{T}\,a.$

Theorema.

4. $x \in \mathrm{R} \,.\supset:: x = \therefore$

Signum Q legitur *quan*

et ∞ exceptis.

5.

6.

7.

8.

45’. $a \in \mathrm{K}\,\mathrm{Q} \,.\supset.\, \mathbf{E}\,\mathbf{L}\,a = \mathbf{I}\,a \cup \mathbf{E}\,a$

46. $a \in \mathrm{K}\,\mathrm{Q} \,.\supset.\, \mathbf{E}\,\mathbf{I}\,a = \mathbf{-}(\mathbf{I}\,a \cup \mathbf{L}\,\mathbf{I}\,a)$

46’. $a \in \mathrm{K}\,\mathrm{Q} \,.\supset.\, \mathbf{EE}\,a = \mathbf{-}(\mathbf{E}\,a \cup \mathbf{L}\,\mathbf{E}\,a)$ 46’.

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1 H. Kennedy, *Peano. Life and Works of Giuseppe Peano*, San Francisco: Peremptory Publications, 2002, p. 41.

2 G. Peano, *Arithmetices principia, nova methodo exposita*, Bocca, Torino, 1889.

3 These English translations listed are the only ones (to my knowledge) and all the English translations listed in:

I. Grattan-Guiness (ed.), *Landmark Writings in Western Mathematics 1640-1940*, Amsterdam: Elsevier, 2005, p. 614.

4 G. Peano, (1889), “The principles of arithmetic presented by a new method" in: J. van Heijenoort (ed.), *From Frege to Gödel. A source book in mathematical logic. 1879-1931*, Cambridge: Harvard University Press, 1967, p. 83-97.

5 G. Peano, *Selected works of Giuseppe Peano*, H. Kennedy (ed.), London: George Allen & Unwin, 1973, p. 101-134.

6 Written by Vincent Verheyen. Last updated on 17/8/2015. I encourage you to use your reason for good. If you want my support, please contact me via http://vincentverheyen.com/contact. It is possible to contribute to the flourishing of knowledge

even when you have an intelligence like mine. Thank you and good luck studying.

I would like to thank Mauro Allegranzo and acknowledge his support of this work and his various comments during its creation.

7 **Giuseppe Peano’s footnote (original):**

Boole:

The mathematical analysis of logic ..., Cambridge, 1847.

The calculus of logic, Camb. and Dublin Math. Journal, 1848.

An investigation of the laws of thought ..., London, 1854.

E. Schröder:

Der Operationskreis des Logikkalculus, Leipzig, 1877.

Ipse iam nonnulla quae ad logicam pertinent tractavit in praecedenti opera.

Lehrbuch der Arithmetik und Algebra ..., Leipzig, 1873.

Boole e Schröder theorias brevissime exposui in meo libro *Calcolo geometrico* ..., Torino, 1888.

Vide:

C. S. Pierce, *On the Algebra of logic*; American Journal, III, 15; VII, 180.

Jevons, *The principles of science*, London, 1883.

Mc.Coll., *The calculus of equivalent statements*, Proceedings of the London Math. Society, 1878, Vol. IX, 9. Vol X, 16.

7 **Giuseppe Peano’s footnote (translated):**

Boole:

The mathematical analysis of logic ... (Cambridge, 1847.)

‘The calculus of logic,’ *Camb. and Dublin Math. J.*, 3 (1848), 193-98.

An investigation of the laws of thought ... (London, 1854).

E. Schröder:

Der Operationskreis des Logikkalculus (Leipzig, 1877).

He had already treated several matters pertaining to logic in a preceding work.

Lehrbuch der Arithmetik und Algebra ... (Leipzig, 1873).

I gave a very brief presentation of the theories of Boole and Schröder in my book *Calcolo geometric* etc. (Torino, 1888).

Cf:

C. S. Pierce, ‘On the Algebra of logic,’ *American J. Math.*, 3 (1880), 15-57; 7 (1885), 180-202.

Jevons, *The principles of science* (London, 1883).

Mc.Coll., ‘The calculus of equivalent statements,’ *Proc. London Math. Soc.*, 9 (1878), 9-20; 10 (1878), 16-28.

8 The 2 other translation, mentioned at the beginning of this current document, translated "ähnlich" literally to "similar", instead of "equivalent". However, additional information can be found in a footnote of the first translation:

"Today "similar" has another meaning and instead we would say "equivalent"."

G. Peano, (1889), “The principles of arithmetic presented by a new method" in: J. van Heijenoort (ed.), *From Frege to Gödel. A source book in mathematical logic. 1879-1931*, Cambridge: Harvard University Press, 1967, p. 93.