

# The Five-Color Theorem

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## 1 Planar maps and graphs

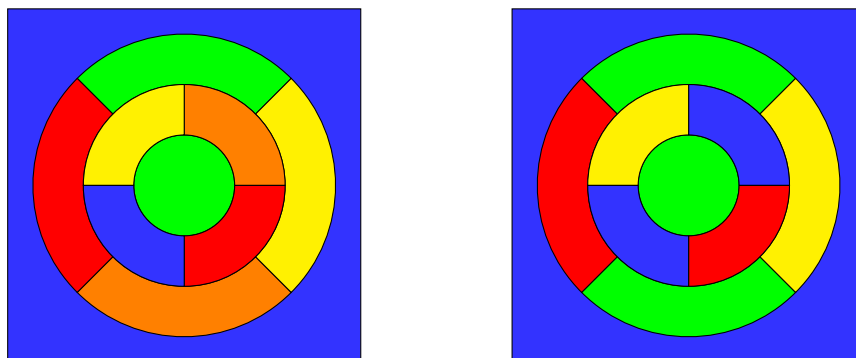
**Theorem 1** *A planar map can be colored with four colors so that no adjacent regions share the same color.*

The proof of this theorem is extremely difficult; here we prove the much simpler five-color theorem, originally proved in the nineteenth century.

**Theorem 2** *A planar map can be colored with five colors so that no adjacent regions share the same color.*

**Definition 1** *A planar map is a set of regions in the plane sharing boundaries. A coloring of a map is an assignment of a color to each region such that regions sharing a boundary are assigned different colors.<sup>1</sup>*

The following diagrams show a planar map with ten regions. The left diagram shows a five-coloring and the right diagram shows a four-coloring.



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<sup>1</sup>Regions not sharing a boundary may be considered as “the same,” for example, state of Alaska and the state of Washington both belong to the United States, although they do not share a boundary and it is impossible to drive from one state to the other without passing through a different country (Canada). In the mathematical problem, such regions can be colored with the same color or different colors.

**Definition 2** A graph is a set of vertices  $V$  and a set of edges  $E$ , such that each edge is incident with exactly two vertices.

A planar graph is a graph such that no edges cross each other. In a planar graph, areas enclosed by a set of edges are called faces.

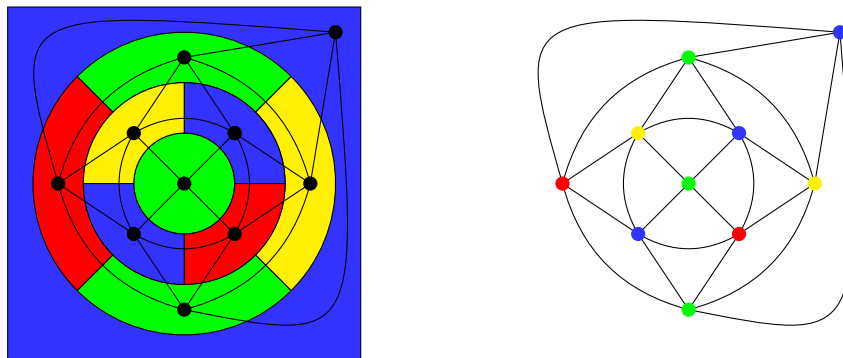
A coloring of a planar graph is an assignment of colors to vertices such that no two vertices of the same color are connected by an edge.

Planar maps and planar graphs are dual and it is convenient to treat coloring problems in graphs rather than maps.

**Theorem 3** Given a planar map, a planar graph can be constructed such for each coloring of the regions of the map, there is a coloring of the vertices of the graph, and conversely.

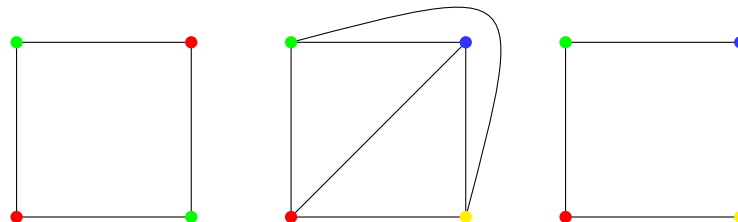
**Proof** Construct one vertex for each region and construct an edge between two vertices iff the regions share a boundary. ■

The following diagram shows how a planar graph can be constructed and colored based on the planar map shown above.



We can further limit our graphs to those whose faces are *triangular*.<sup>2</sup>

The left diagram below shows that a square can be two-colored, but if it is triangulated (center), three colors are necessary. However, the goal is to prove that *all* graphs can be five-colored, so if the triangulated graph is five-colored, so is the original graph, because deleting the extra edge does not invalidate the coloring (right).



<sup>2</sup>The faces are not necessarily *triangles* because the edges may be curved. By Fáry's theorem, any triangular planar graph can be transformed to an equivalent graph with straight edges.

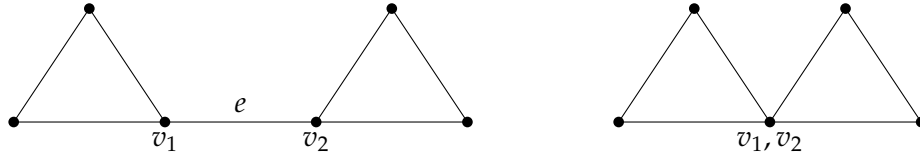
## 2 Euler's formula

**Theorem 4 (Euler)** Let  $G$  be a connected planar graph with  $V$  vertices,  $E$  edges and  $F$  faces. Then  $V - E + F = 2$ .

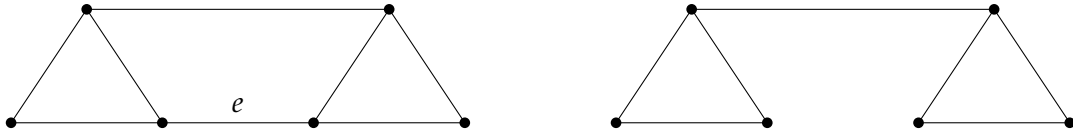
**Proof** By induction on the number of edges. If the number of edges in the connected planar graph is zero, there is only a single vertex and a single face, so  $1 - 0 + 1 = 2$ .

Let  $G$  be a connected planar graph with  $V$  vertices,  $E$  edges and  $F$  faces, and remove an edge  $e$  connecting vertices  $v_1, v_2$ . There are two cases:

**Case 1** The graph becomes disconnected. Identify  $v_1$  with  $v_2$ . The resulting graph  $G'$  is a planar connected graph and has fewer edges than  $G$ , so by the induction hypothesis,  $(V - 1) - (E - 1) + F = 2$  since the number of vertices is also reduced by one. Simplifying, we get  $V - E + F = 2$  for  $G$ .



**Case 2** The graph remains connected. The resulting graph  $G'$  has fewer edges than  $G$  so by the induction hypothesis,  $V - (E - 1) + (F - 1) = 2$  since removing the edge joins two faces into one. Simplifying, we get  $V - E + F = 2$  for  $G$ .



■

**Theorem 5** Let  $G$  be a connected, triangulated planar graph. Then  $E = 3V - 6$ .

For example, the planar graph in Section 1 has 10 vertices and  $24 = 3 \cdot 10 - 6 = 24$  edges.

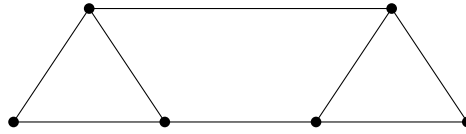
**Proof** Each face is bounded by three edges, so  $E = 3F/2$  because each edge has been counted twice, once for each face it bounds. By Euler's formula:

$$\begin{aligned} E &= V + F - 2 \\ E &= V + 2E/3 - 2 \\ E &= 3V - 6. \end{aligned}$$

■

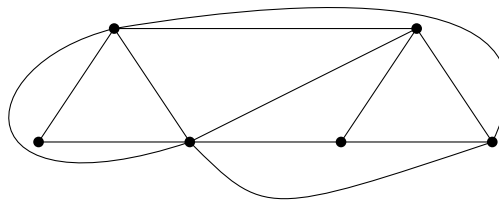
**Theorem 6** Let  $G$  be a connected planar graph. Then  $E \leq 3V - 6$ .

For the following graph,  $E = 8 < 3 \cdot 6 - 6 = 12$ .



**Proof** Triangulate  $G$  to obtain  $G'$ . In  $G'$ ,  $E = 3V - 6$  by Theorem 6. Now remove edges from  $G'$  to obtain  $G$ . The number of vertices does not change so  $E \leq 3V - 6$ . ■

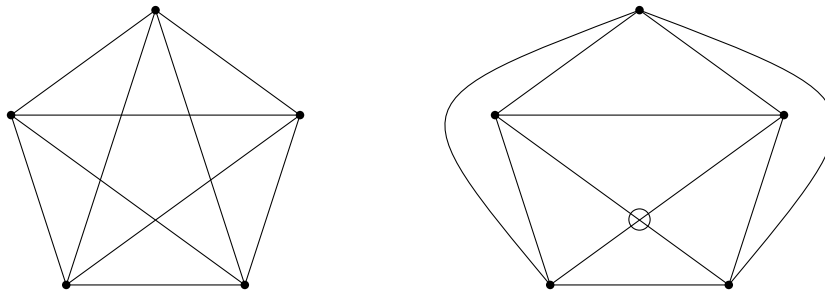
Here is the triangulated graph for which  $E = 12 = 3 \cdot 6 - 6 = 12$ .



### 3 Non-planar graphs

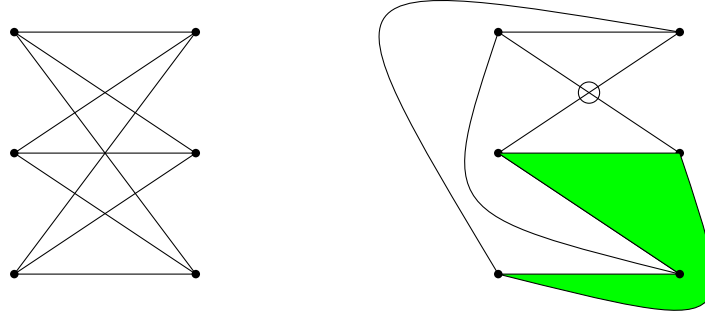
Let us take a short detour to show how Theorems 4 and 6 can be used to prove that certain graphs are not planar.

**Theorem 7**  $K_5$ , the complete graph on five vertices, is not planar.



**Proof** For  $K_5$ ,  $V = 5$  and  $E = 10$ . But  $10 \not\leq 3 \cdot 5 - 6 = 9$ . ■

**Theorem 8**  $K_{3,3}$ , the bipartite graph with three vertices on each side, is not planar.



**Proof**  $V = 6$  and  $E = 9$ . By Theorem 4,  $F = E - V + 2 = 9 - 6 + 2 = 5$ . But each face is bounded by four edges, so  $E = 4F/2 = 10 \neq 9$ . ■

## 4 The degrees of the vertices

**Definition 3**  $d(v)$ , the degree of vertex  $v$ , is the number of edges incident with  $v$ .

For the graph in Section 1, there are 8 vertices in the two annuli, each of degree 5; the vertices of the outer face and the inner circular face are each of degree 4. Therefore:

$$\sum_{v \in V} d(v) = 5 \cdot 8 + 4 \cdot 2 = 48.$$

To get the total number of edges, divide by 2 because each edge was counted twice, once for each of the vertices it is incident to.

By generalizing the argument we get:

**Theorem 9** Let  $d_i, i = 1, 2, 3, \dots, k$  be the number of vertices of degree  $i$  in a connected planar graph  $G$  with  $V$  vertices and  $E$  edges, where  $k$  is the highest degree of a vertex in  $V$ . Then:

$$\sum_{v \in V} d(v) = \sum_{i=1}^k i \cdot d_i = 2E.$$

**Theorem 10** Let  $G$  be a connected planar graph with  $E$  edges and  $V$  vertices, and let  $d_i, i = 1, \dots, k$  be the number of vertices of degree  $i$ , where  $k$  is the highest degree of a vertex in  $V$ . Then there must be a vertex  $v$  in  $V$  such that  $d(v) \leq 5$ .

**Proof 1** Clearly, if there are  $d_1$  vertices of degree 1,  $d_2$  vertices of degree 2,  $\dots$ ,  $d_k$  vertices of degree  $k$ , then  $V = \sum_{i=1}^k d_i$ . From Theorems 6 and 9:

$$\sum_{i=1}^k i \cdot d_i = 2E \leq 2(3V - 6) = 6V - 12 = 6 \sum_{i=1}^k d_i - 12.$$

Therefore:

$$\sum_{i=1}^k i \cdot d_i \leq 6 \sum_{i=1}^k d_i - 12,$$

and:

$$\sum_{i=1}^k (6 - i) d_i > 12.$$

Since  $12 > 0$ , for least one  $i$ ,  $6 - i > 0$  and for that  $i$ ,  $i < 6$ . ■

**Proof 2** Let us compute the *average* degree of the vertices, which is the sum of the degrees divided by the number of vertices:

$$d_{avg} = \frac{\sum_{i=1}^k i \cdot d_i}{V}.$$

But the sum of the degrees is twice the number of edges which by Theorem 6 gives:

$$d_{avg} = \frac{2E}{V} \leq \frac{6V - 12}{V} = 6 - \frac{6}{V} < 6.$$

If the average degree is less than six, there must be a vertex of degree less than six. ■

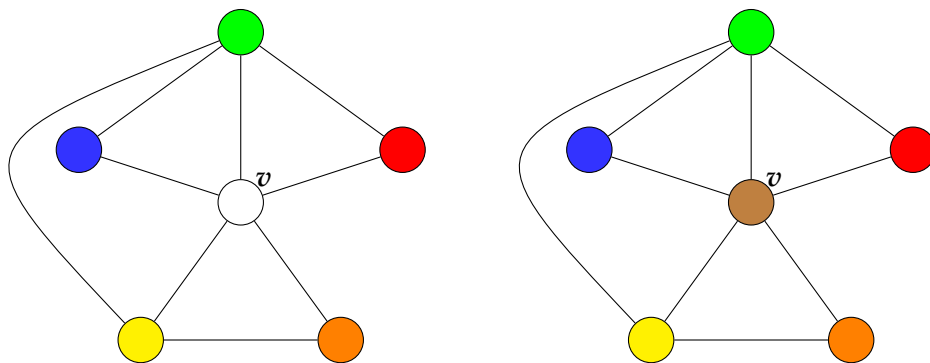
For the graph in Section 1, the sum of the degrees is  $8 \cdot 5 + 2 \cdot 4 = 48$ . There are 10 vertices, so the average degree is  $\frac{48}{10} = 4.8$  and there must be a vertex of degree 4 or less.

## 5 The six-color theorem

**Theorem 11** Any planar graph  $G$  can be six-colored.

**Proof** By induction on the number of vertices in  $G$ . Clearly, if the graph has six vertices or fewer, six colors suffice.

For the inductive step, let  $G$  be a planar graph. By Theorem 10 it has a vertex  $v$  with degree 5 or fewer. Delete vertex  $v$  to obtain the graph  $G'$ . By the induction hypothesis,  $G'$  can be six-colored, but  $v$  has at most 5 neighbors and at most 5 colors are used to color them, so  $v$  can be colored using the sixth color. ■



## 6 The five-color theorem

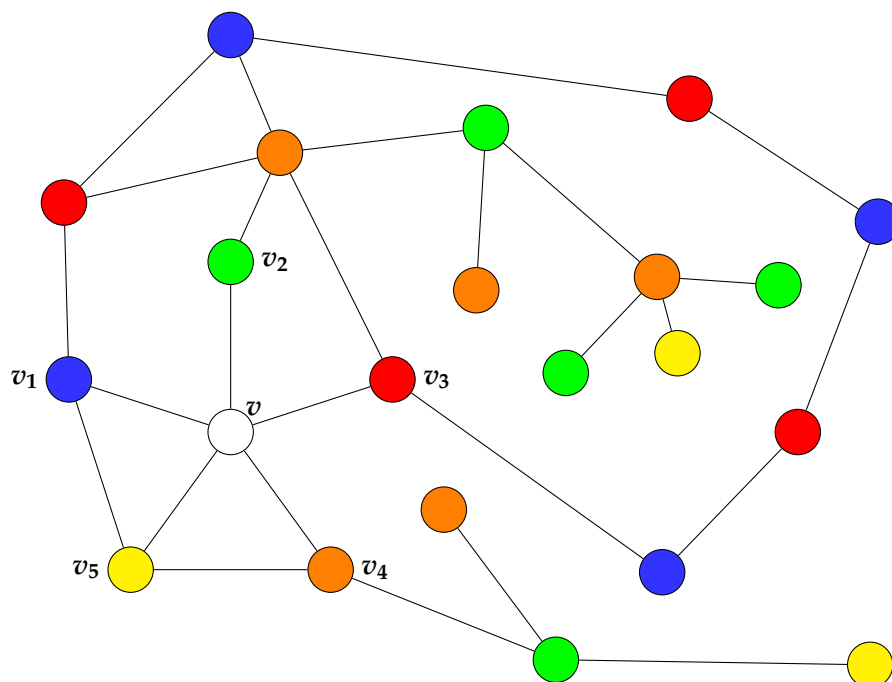
**Definition 4** Let  $G$  be a colored planar graph.  $G'$  is a chain iff  $G'$  is a maximal, two-colored subgraph of  $G$ .<sup>3</sup>

**Theorem 12** Any planar graph  $G$  can be five colored.

**Proof** By induction on the number of vertices. The base case is any planar graph with five vertices or less.

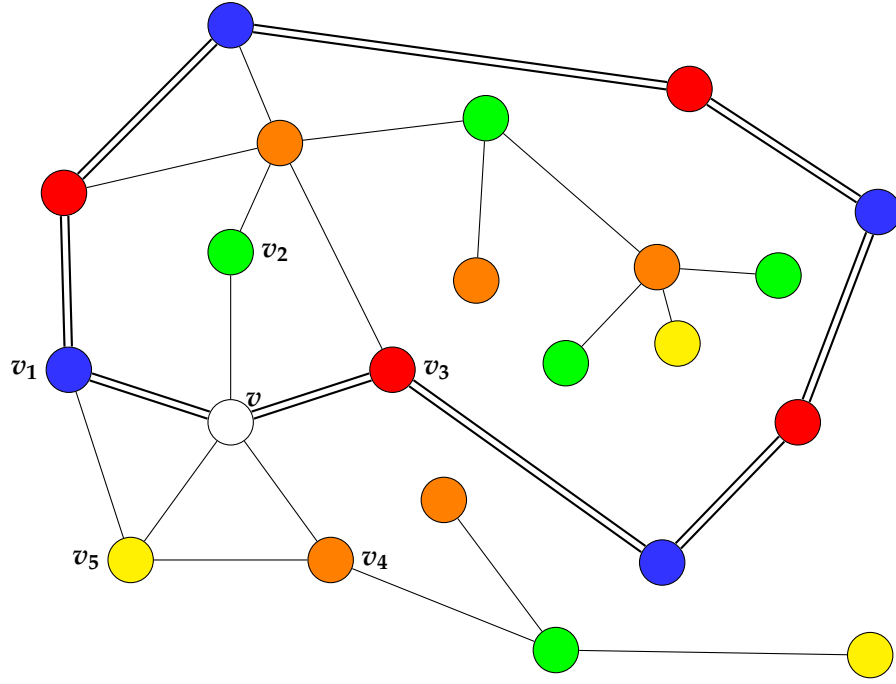
For the inductive step, let  $G$  be a planar graph. By Theorem 10 it has a vertex  $v$  with degree 5 or less. Delete the vertex  $v$  to obtain  $G'$ . By the induction hypothesis,  $G'$  can be five-colored. In  $G$ , if the degree of  $v$  is less than 5, or if  $v_1, \dots, v_5$ , the neighbors of  $v$ , are colored with four colors or fewer,  $v$  can be colored with the fifth color.

Otherwise,  $v_1, \dots, v_5$  are colored with different colors in  $G'$ .

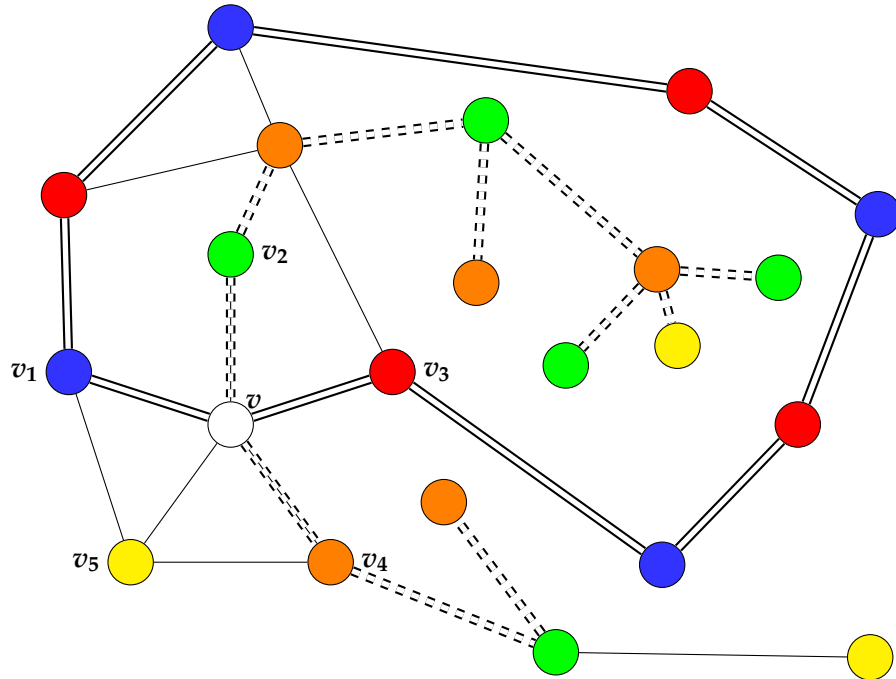


Consider the vertex  $v_1$  colored blue and the vertex  $v_3$  colored red, and consider the blue-red chain in which they are contained. By adding the vertex  $v$  and the edges  $\overline{vv_1}, \overline{vv_3}$  to the chain we obtain a closed path  $P$  (denoted in the diagram below by a double line) that divides the plane into an “inside” region and an “outside” region.

<sup>3</sup>This is also called a *Kempe chain* because it was introduced by Alfred Kempe in his (incorrect) proof of the four-color theorem (see Section 7).



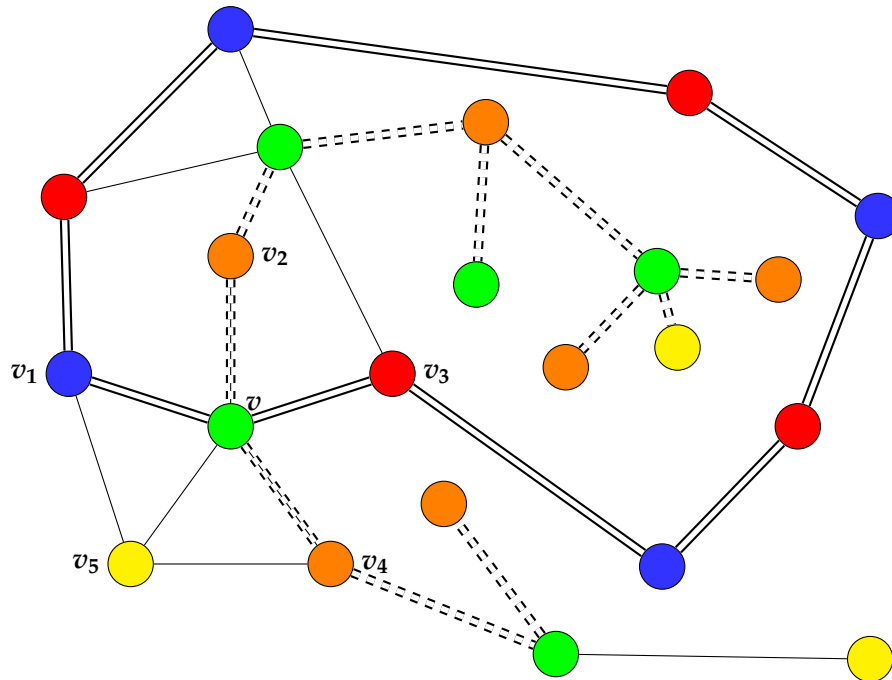
Consider the vertex  $v_2$  colored green and the vertex  $v_4$  colored orange. These vertices *cannot* be contained in a single green-orange chain, because  $v_2$  is *inside*  $P$  and  $v_4$  is *outside*  $P$ , so any path connecting them must cross  $P$ , contradicting the assumption that the graph is planar.<sup>4</sup> In the following diagram the two *unconnected* green-orange chains which contain  $v_2$  and  $v_4$  are denoted with a double dashed line.



<sup>4</sup>This follows from the *Jordan curve theorem* which is intuitively obvious but very difficult to prove.



We can exchange the colors on the chain containing  $v_2$ , and this will not change the fact that  $G'$  is five-colored. Since  $v_2$  and  $v_4$  are now both colored orange,  $v$  can be colored green to obtain a five-coloring of  $G$ . ■



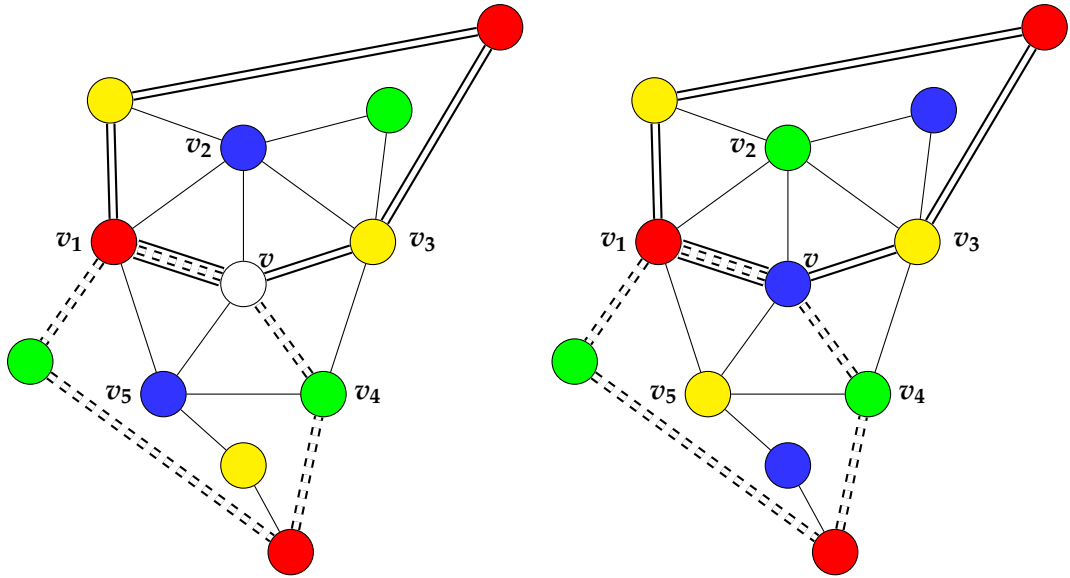
## 7 Kempe's incorrect proof of the four-color theorem

The four-color theorem was posed and conjectured to hold in 1852. In 1879, Alfred B. Kempe published a proof of the theorem, but eleven years later, in 1890, Percy J. Heawood found an error in the proof. Nevertheless, Kempe's work is important because: (1) it provided a correct proof of the five-color theorem, and (2) his proof contained the basic ideas that were used by Kenneth Appel and Wolfgang Haken in their correct proof published in 1976.

**Proof** The base case of the induction and most of the proof is the same as that of the five-color theorem. The new case that must be considered is a vertex  $v$  with five neighbors which, by the inductive hypothesis, are colored with four colors after removing  $v$ .

In the left diagram below, there are two vertices  $v_2, v_5$  colored blue. Consider the blue-green chain containing  $v_2$  and the blue-yellow chain containing  $v_5$ . The blue-green chain is contained within the closed path defined by the red-yellow chain containing  $v_1, v_3$  and the blue-yellow chain is contained within the closed path defined by the red-green chain containing  $v_1, v_4$ .

Now exchange the colors on the blue-green chain and on the blue-yellow chain (right diagram below). The result is that the neighbors of  $v$  are colored with the three colors red, green and yellow, leaving blue free to color  $v$ .



Heawood noted that the closed paths defined by the red-yellow and red-green chains can share red vertices ( $v_1$  and the red vertex below  $v_4$  in the left diagram below). When the colors are exchanged in the blue-green and blue-yellow chains, it is possible for blue vertices to be connected (right diagram below), so the coloring is no longer correct.

