A Regular Heptadecagon is Constructible

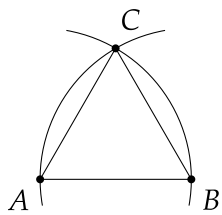
Moti Ben-Ari

Introduction

For centuries people learned mathematics by studying Euclid's *Elements*. A focus of the *Elements* is on the construction of geometric figures using a straightedge (a ruler with no markings) and a compass. The *Elements* gives constructions of regular polygons with $n = 3, 4, 5, 15$ sides (and polygons with $2^k n$ sides), but not until two thousand years later was the construction of another regular polygon discovered. In 1796 Carl Friedrich Gauss awoke one morning just before his 19th birthday, and by "concentrated thought" discovered that a regular heptadecagon (a regular polygon with 17 sides) is constructible. It was a revolutionary development because Gauss proved the constructibility *algebraically* not geometrically.

Constructibility

The first proposition in the *Elements* is that an equilateral triangle can be constructed. (The word "construct" will be used as an abbreviation for "construct by straightedge and compass.") Given a line segment $\overline{AB}$ draw two circles whose centers are A, B and whose radii are the length of $\overline{AB}$. C, the intersection of the circles, defines the third vertex of the triangle:

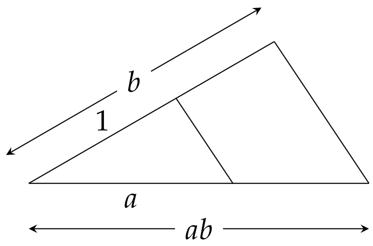


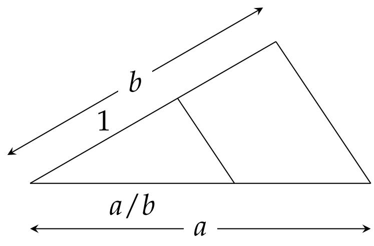
Define a real number x to be *constructible* if and only if starting with a line segment defined to be of length 1 it is possible to construct a line segment of length x.

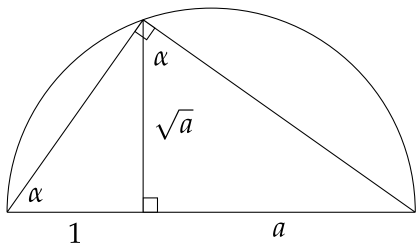
Theorem: x is constructible if and only if it is the result of evaluating an expression composed from the integer 1 using the operations $\{+, -, \times, /, \sqrt{}\}$.

Here is an outline of the proof:

- Lines are defined by linear equations and circles by quadratic equations, so the intersections of lines and circles are the solutions of two such equations. Expressions for the coordinates of the intersections (and hence the length of line segments between two points) must also be composed from integers and the operations $\{+, -, \times, /, \sqrt{}\}$.
- Given two line segments, their sum and difference can be obtained by copying one onto the other or onto an extension of the other. The following diagrams show how to construct products, differences and square roots using similar triangles.





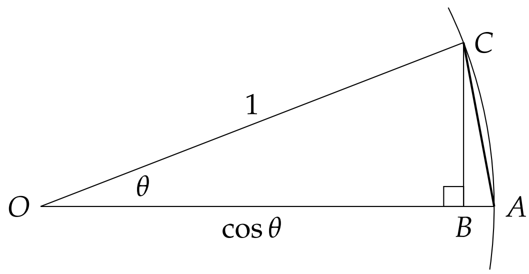


Impossible constructions

The Greeks were unable to trisect an angle (divide a given angle into three equal parts), square a circle (construct a square with the same area as a given circle) and double a cube (construct a cube with twice the volume of a given cube). During the nineteenth century it was proved that these constructions are impossible. Of equal interest was the construction of regular polygons. The Greeks were unable to construct regular polygons with $n = 7, 9, 11, 13, 14, 17, 18, \ldots$ sides.

The mathematics of constructibility

A regular polygon can be inscribed in a unit circle. The polygon can be constructed if and only the central angle subtended by one of its sides (or the cosine of the angle) can be constructed. In the following diagram $\overline{AC}$ is a side of the polygon.



The central angle of an equilateral triangle is $120°$ and $\cos 120° = -1/2$ is constructible. The central angle of a regular pentagon is $72°$ and it is not too hard to show that $\cos 72° = (\sqrt{5} - 1)/4$, so a regular pentagon is constructible. A regular pentadecagon (a regular polygon with 15 sides) is constructible since its central angle is $360°/15 = 24° = (120° - 72°)/2$. We want to show that the angle $360°/17$ or its cosine is constructible.

Mathematical prerequisites

With one exception we will only use secondary-school algebra: multiplication of variables with exponents, integer division with remainder, and finding the roots of quadratic equations. The exception is the Fundamental Theorem of Algebra, which states that an n-th degree polynomial with complex coefficients has n complex roots. Actually, we only need one simple case of the theorem: the polynomial $x^n - 1$ has (at least) one root $r \neq 1$ or, even more narrowly, that $x^{17} - 1$ has (at least) one root $r \neq 1$

The roots of unity

The roots of $x^n - 1$ are called the *n-th roots of unity*. If r is an n-th root of unity then $(r^2)^n = (r^n)^2 = 1^2 = 1$ so r^2 is also an n-th root of unity. It follows that $1, r, r^2, \ldots, r^{n-2}, r^{n-1}$ are all n-th roots of unity. What we don't know is if they are distinct and thus all the n-th roots of unity. For example, the roots of the polynomial $x^4 - 1$ are $1, -1, i, -i$, but $(-1)^2 = 1$, $(-1)^3 = -1$, $(-1)^4 = 1$ so the powers of $r = -1$ are not distinct and do not give all the fourth roots of unity.

Theorem: If n is a prime number and r is an n-the root of unity then $1, r, r^2, \ldots, r^{n-2}, r^{n-1}$ are distinct.

Proof: If $r^i = r^j$ for some $0 \leq i < j \leq n-1$ then $r^j / r^i = r^{j-i} = 1$. Let m be the smallest positive integer such that $r^m = 1$. By the division formula:

$$1 = r^n = r^{ml+k} = (r^m)^l \cdot r^k = 1^l \cdot r^k = r^k\,,$$

where $0 \leq k < m$, but $0 < k$ and $r^k = 1$ contradict the assumption that m was the smallest such positive number. Therefore, $k = 0$ and $n = ml$ is not prime.

From roots back to coefficients of polynomials

Suppose that we know the values of r_1, r_2, two roots of a quadratic polynomial $x^2 + bx + c$. What are the coefficients? By computation:

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 = x^2 + bx + c\,,$$

so

$$b = -(r_1 + r_2), \qquad c = r_1 r_2\,, \tag{1}$$

and similarly the coefficients of any polynomial can be computed if the roots are known. Given that the roots of $x^{17} - 1$ are $\{1, r, r^2, \cdots, r^{15}, r^{16}\}$, we can find and coefficient of the polynomial by multiplying $(x-1)(x-r^1)(x-r^2)\cdots(x-r^{15})(x-r^{16})$. The coefficient of x^{16} is:

$$-(1 + r^1 + r^2 + \cdots + r^{15} + r^{16})\,,$$

which is 0 in $x^{17} - 1$ so:

$$r^1 + r^2 + \cdots + r^{15} + r^{16} = -1\,. \tag{2}$$

Gauss's proof that the heptadecagon is constructible

Gauss's insight was that we need not work with the roots in their natural order $r, \cdots, r^{16}$, but that other powers of roots can generate all the roots. Starting from r, repeatedly raising it to the third power (modulo 17) gives:

$$r^1,\ (r^1)^3 = r^3,\ (r^3)^3 = r^9,\ (r^9)^3 = r^{27} = r^{10},\ (r^{10})^3 = r^{30} = r^{13},\ \ldots\,.$$

The reader is invited to compute the roots and check that the following list contains all the roots (except 1) exactly once:

$$r^1\ r^3\ r^9\ r^{10}\ r^{13}\ r^5\ r^{15}\ r^{11}\ r^{16}\ r^{14}\ r^8\ r^7\ r^4\ r^{12}\ r^2\ r^6\,.$$

Write the roots as follows in order to emphasize the roots in the odd and even positions:

$$\begin{array}{cccccccccccccccc} r^1 & & r^9 & & r^{13} & & r^{15} & & r^{16} & & r^8 & & r^4 & & r^2 & \\ & r^3 & & r^{10} & & r^5 & & r^{11} & & r^{14} & & r^7 & & r^{12} & & r^6\,. \end{array}$$

Let a_0, a_1 be the sums of the roots in the odd and even positions, respectively:

$$\begin{aligned} a_0 &= r + r^9 + r^{13} + r^{15} + r^{16} + r^8 + r^4 + r^2 \\ a_1 &= r^3 + r^{10} + r^5 + r^{11} + r^{14} + r^7 + r^{12} + r^6\,. \end{aligned}$$

Compute $a_0 + a_1$ using Equation 2:

$$a_0 + a_1 = r + r^2 + \cdots + r^{16} = -1\,.$$

Now compute $a_0 a_1$ and simplify. It takes quite a lot of work, but the result is the sum of four copies of Equation 2 so:

$$a_0 a_1 = -4\,.$$

Given that a_0, a_1 are roots, by Equation 1 they are the roots of the polynomial:

$$y^2 + y - 4 = 0\,,$$

and their values are:

$$a_0, a_1 = \frac{-1 \pm \sqrt{17}}{2}\,.$$

Let b_0, b_1, b_2, b_3 be the sums of every fourth root starting from r^1, r^3, r^9, r^{10}, respectively:

$$\begin{aligned} b_0 &= r^1 + r^{13} + r^{16} + r^4 \\ b_1 &= r^3 + r^5 + r^{14} + r^{12} \\ b_2 &= r^9 + r^{15} + r^8 + r^2 \\ b_3 &= r^{10} + r^{11} + r^7 + r^6\,. \end{aligned}$$

Check that $b_0 + b_2 = a_0, b_1 + b_3 = a_1$ and compute the corresponding products which are $b_0 b_2 = b_1 b_3 = -1$. Therefore, b_0, b_2 are the solutions of $y^2 - a_0 y - 1 = 0$, and b_1, b_3 are the solutions of $y^2 - a_1 y - 1 = 0$. Using the values previously computed for a_0, a_1 we can compute the roots b_0, b_1. We will spare the readers the messy algebra and just give the results:

$$b_0, b_1 = \frac{(-1 \pm \sqrt{17}) + \sqrt{34 \mp 2\sqrt{17}}}{4}\,.$$

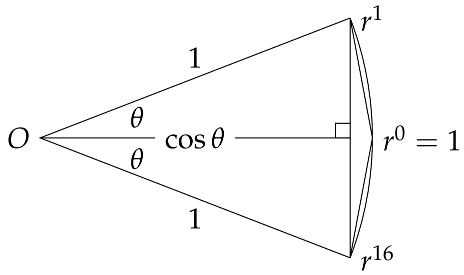
(For b_0 take plus and then minus and for b_1 take minus and then plus.)

Finally, let c_0, c_4 be the sums of every eighth root starting with r^1, r^{13}:

$$\begin{aligned} c_0 &= r^1 + r^{16} \\ c_4 &= r^{13} + r^4 \\ c_0 + c_4 &= = b_0 \\ c_0 c_4 &= b_1 , \end{aligned}$$

so c_0, c_4 are the roots of $y^2 - b_0 y + b_1 = 0$.

For $\theta = (360^\circ/17)$, $\cos\theta = c_0/2 = (r^1 + r^{16})/2$:



so it suffices to compute the root c_0. The result is:

$$\begin{aligned} \cos\left(\frac{360^\circ}{17}\right) &= \frac{c_0}{2} \\ &= -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} \; + \\ &\quad \frac{1}{16}\sqrt{68 + 12\sqrt{17} + 2(-1 + \sqrt{17})\sqrt{34 - 2\sqrt{17}} - 16\sqrt{34 + 2\sqrt{17}}} \, . \end{aligned}$$

The cosine of the central angle of a heptadecagon is constructible since it is the value of an expression composed of integers and the operations $\{+, -, \times, /, \sqrt{}\}$!

The formula that usually appears in the literature is:

$$\begin{aligned} \cos\left(\frac{360^\circ}{17}\right) &= -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} \\ &\quad + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}} \, . \end{aligned}$$

We leave it to the reader to derive this formula from the one we derived above.

Gauss-Wantzel Theorem

The construction of the regular heptadecagon led to the Gauss-Wantzel theorem, which states that a regular polygon with n sides is constructible if and only if n is the product of a power of 2 and zero or more *distinct* Fermat numbers $2^{2^k} + 1$ which are prime. The known Fermat primes are:

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537 \, .$$

A regular polygon with 257 sides was constructed by Magnus Georg Paucker in 1822 and by Friedrich Julius Richelot 1832. In 1894 Johann Gustav Hermes claimed to have constructed a regular polygon with 65537 sides.

Conclusion

Until the eighteenth or nineteenth century, mathematical theorems were proved geometrically. Even Newton who invented the calculus used it as a means of discovery and he insisted that a proof must be geometric. This is not surprising since calculus was not placed on a firm theoretical foundation until the late nineteenth century.

Gauss proved the constructibility of the heptadecagon without giving a geometric construction! In fact, the first constructions were not published until almost a century later (a modern construction is given in [3]). His algebraic solution led to the growing ascendancy of algebra during the nineteenth century.

References

[1] Mordechai Ben-Ari. *Mathematical Surprises*. Springer, 2020. Open access from: `https://link.springer.com/book/10.1007/978-3-031-13566-8`.

[2] Jörg Bewersdorff. *Galois Theory for Beginners: A Historical Perspective (Second Edition)*. American Mathematical Society, 2019.

[3] James J. Callagy. The central angle of the regular 17-gon. *The Mathematical Gazette*, 67(442):290–292, 1983. `https://www.jstor.org/stable/3617271`.