33. Mascheroni's Compass Problem

> The construction presented in this document is taken from the book by Heinrich Dörrie: *100 Problems of Elementary Mathematics: Their History and Solution* (Dover, 1965), as reworked by Michael Woltermann.[1] The document is written in LaTeX and I have redrawn the diagrams using Ti*k*Z, often drawing a diagram incrementally for clarity. I have added explanations so that students and teachers can better understand the construction.
>
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Prove that any construction that can be carried out with a compass and straight-edge can be carried out with the compass alone. The Italian L. Mascheroni (1750-1800) posed this problem to himself and solved it in a masterly fashion in his book *La geometria del compasso*, published in Pavia in 1797.

> The theorem is known as the Mohr-Mascheroni Theorem since it had been proved in 1672 by the Danish mathematician Georg Mohr, but his work was not widely known until the twentieth century.

When we examine the separate steps by which circle and straight-edge constructions are carried out, we see that every step consists of one of the following three basic constructions:

I. Finding the point of intersection of two straight lines;

II. finding the point of intersection of a straight line and a circle;

III. finding the point(s) of intersection of two circles.

Thus we need only show that the two basic constructions I. and II. can be done with a compass alone. (Mascheroni regarded a straight line as given if two of its points are known.)

First we will solve four preliminary problems. (Dörrie talks about two, but the others are embedded in these.) In the following:

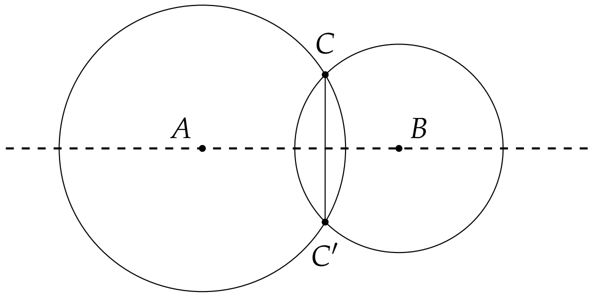
- $C(O, A)$ stands for the circle with center O through point A,
- $C(O, AB)$ stands for the circle with center O and radius AB.

Prelim 1. Reflect a point C about the line through A and B.

[1] http://www2.washjeff.edu/users/mwoltermann/Dorrie/DorrieContents.htm.
[2] http://www.weizmann.ac.il/sci-tea/benari/.

> Given a point C and a line AB, a reflection of C about AB is a point C' such that AB is the perpendicular bisector of the line CC'.

Solution. The reflection $C' = c(A, C) \cap c(B, C)$ (not C in general):



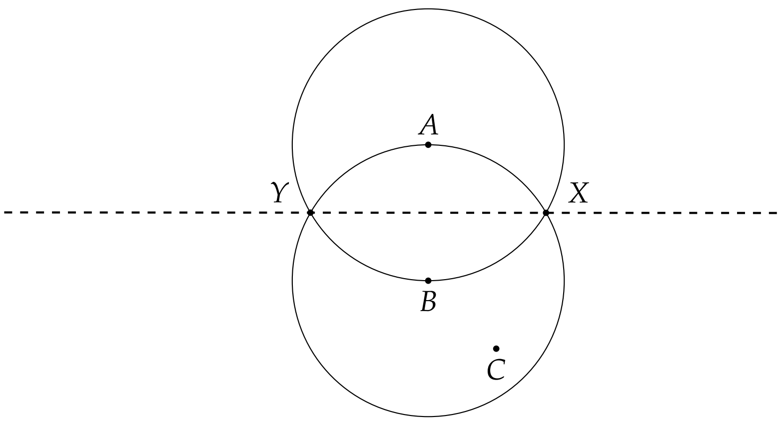
Line AB is the perpendicular bisector of chord CC' of both circles

Note: Dashed lines in figures are drawn to explain the arguments, but are not used in constructions. $\square$

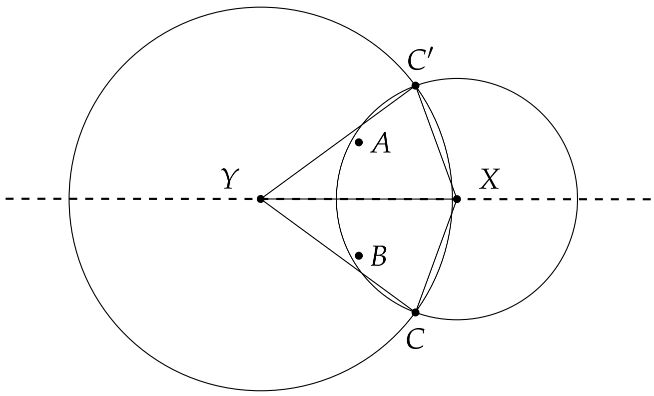
> This is important: the lines drawn in the diagrams serve only to illustrate the proofs; you must convince yourself that only a compass is used in all the constructions. I have added and modified lines, both solid and dashed, to clarify the diagrams.

Prelim 2. Construct $c(A, BC)$, given points A, B, C.

Solution. Let X and Y be the points of intersection of $c(A, B)$ and $c(B, A)$:

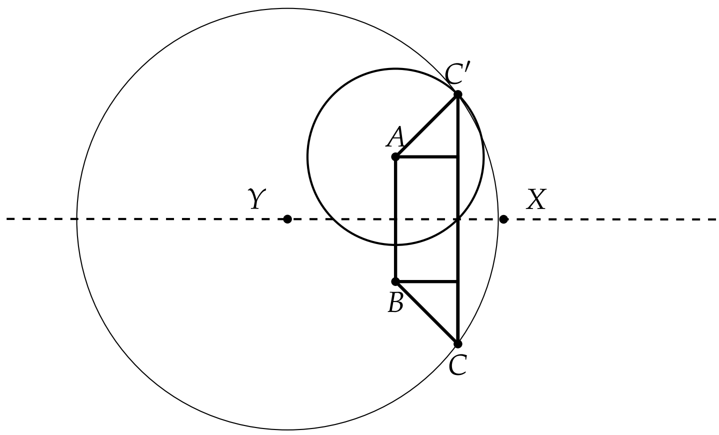


Let C' be the reflection of C about line XY:



> Since $XC = XC', YC = YC', XY = XY$, the triangles $\triangle XYC, XYC'$ are congruent; therefore, the length of the altitude from C to XY equals the length of the altitude from C' to XY. It follows that XY is the perpendicular bisector of CC', so C' is the reflection of C about XY.

$c(A, C')$ is the desired circle:



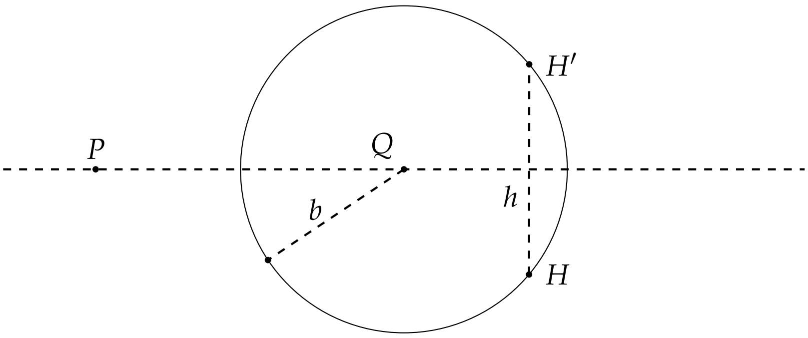
(Since A is the reflection of B about XY, and reflection preserves distance, so $AC' = BC$.)
$\square$

> Reasoning similar to that above that C' is a reflection of C shows that A is the reflection of B about XY. The thick lines show how $AC' = BC$ can be proved using congruent triangles.
>
> In general, it is a theorem that reflection preserves distance. This is proved in high-school textbooks, such as Theorem 6.1 of Ann Xavier Gantert, *Geometry* (AMSCO, 2008).

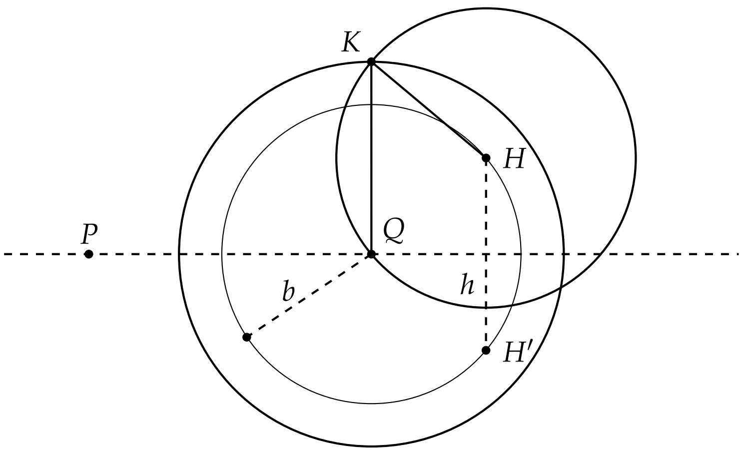
Prelim 3. Construct the sum or difference of two given segments a and b, i.e., lengthen or shorten a given segment $PQ = a$ by a segment $QX = b$. (See Prelim 2 if necessary to construct a segment of length b at Q.)

Solution.

1. Let H be any point on $c(Q, b)$, and H' its reflection about line PQ. Let h be the (length of) segment HH':



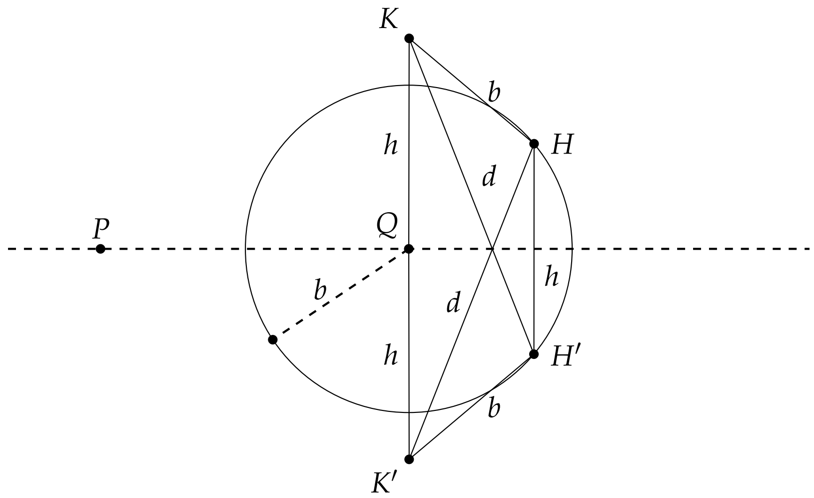
2. Let $K = c(Q, h) \cap c(H, b)$:



and K' be the reflection of K about line PQ.

Then $KHH'K'$ is an isosceles trapezoid with legs $KH = K'H' = b$ and base $KK' = 2h$. Let $d = KH' = K'H$:

> H' is a reflection of H and K' is a reflection of K. Since reflections preserve distance, $KH = K'H'$.



Since opposite angles of $KHH'K'$ are supplemental, $KHH'K'$ is a cyclic quadrilateral, i.e., it can be inscribed in a circle.

> Geometry textbooks give the simple proof that the opposite angles of a cyclic quadrilateral are supplementary (add up to 180°), but it is hard to find a proof of the converse, so I present the proofs here:
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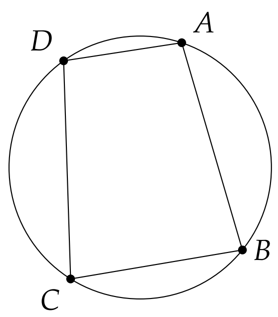
> 
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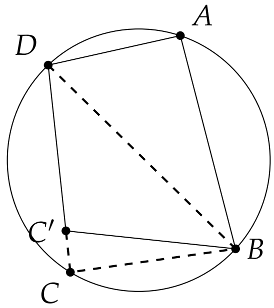
> Proof that the opposite angles of a cyclic quadilateral are supplementary: Recall that an inscribed angle equals half the subtended arc. So $\angle DAB$ is half of the arc DCB and $\angle DCB$ is half of the arc DAB. But the two arcs form the entire circumference of the circle so their sum is 360° and $\angle DAB + \angle DCB = \frac{1}{2} \cdot 360^\circ = 180^\circ$.
>
> Proof that a quadilateral whose opposite sides are supplementary is cyclic: Any triangle can be inscribed in a circle. Suppose that the triangle $\triangle DAB$ is inscribed in the circle and suppose that C' is a point such that $\angle DAB + \angle DC'B = 180^\circ$ but C' is *not* on the circumference of the circle. Without loss of generality, let C' be within the circle:

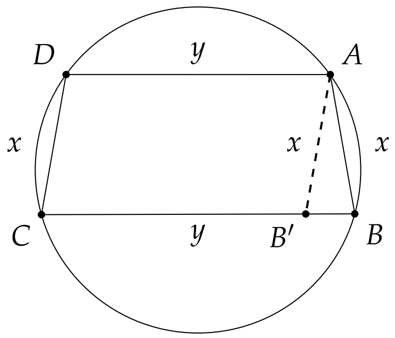


Construct a ray that extends DC' and let C be its intersection with the circle. By the forward direction of the theorem:

$$\begin{aligned} \angle DAB + \angle DCB &= 180^\circ = \angle DAB + \angle DC'B \\ \angle DCB &= \angle DC'B\,, \end{aligned}$$

which is impossible if C and C' are distinct points.

Let us complete the proof by showing that the opposite angles of an isosceles trapezoid are supplementary and therefore that an isosceles trapezoid is cycle:



Construct the line AB' parallel to CD. Then $AB'CD$ is a parallelogram and $\triangle ABB'$ is an isosceles triangle. Therefore, $\angle C = \angle AB'B = \angle B$. A similar proof shows that $\angle A = \angle D$. Since the sum of the internal angles of any quadrilateral is equal to 360°:

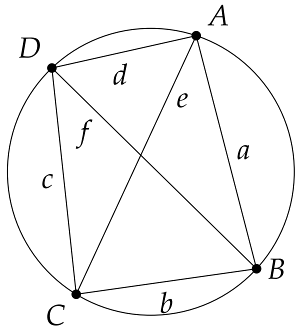
$$\begin{aligned} \angle A + \angle B + \angle C + \angle D &= 360^\circ \\ 2\angle A + 2\angle C &= 360^\circ \\ \angle A + \angle C &= 180^\circ\,, \end{aligned}$$

and similarly, $\angle B + \angle D = 180^\circ$.

Then by Ptolemy's theorem $d^2 = b^2 + 2h^2$.

Ptolemy's theorem states that for a quadrilateral inscribed in a circle, the following equality relates the lengths of the sides a, b, c, d and the lengths of the diagonals e, f:

$$ef = ac + bd\,.$$



There is a geometric proof of the theorem (see Wikipedia), but I prefer to present a simple trigonometric proof. The law of cosines for the four triangles $\triangle ABC, \triangle ADC, \triangle DAB, \triangle DCB$ gives the following four equations:

$$\begin{aligned}
e^2 &= a^2 + b^2 - 2ab\cos\angle B \\
e^2 &= c^2 + d^2 - 2cd\cos\angle D \\
f^2 &= a^2 + d^2 - 2ad\cos\angle A \\
f^2 &= b^2 + c^2 - 2bc\cos\angle C\,.
\end{aligned}$$

The opposite angles of an inscribed quadrilateral are supplementary $\angle C = 180° - \angle A$ and $\angle D = 180° - \angle B$, so $\cos\angle D = -\cos\angle B$ and $\cos\angle C = -\cos\angle A$, and we can eliminate the cosines from the first two equations and from the last two equations. After some messy arithmetic, we get:

$$\begin{aligned}
e^2 &= \frac{(ac+bd)(ad+bc)}{(ab+cd)} \\
f^2 &= \frac{(ab+cd)(ac+bd)}{(ad+bc)}\,.
\end{aligned}$$

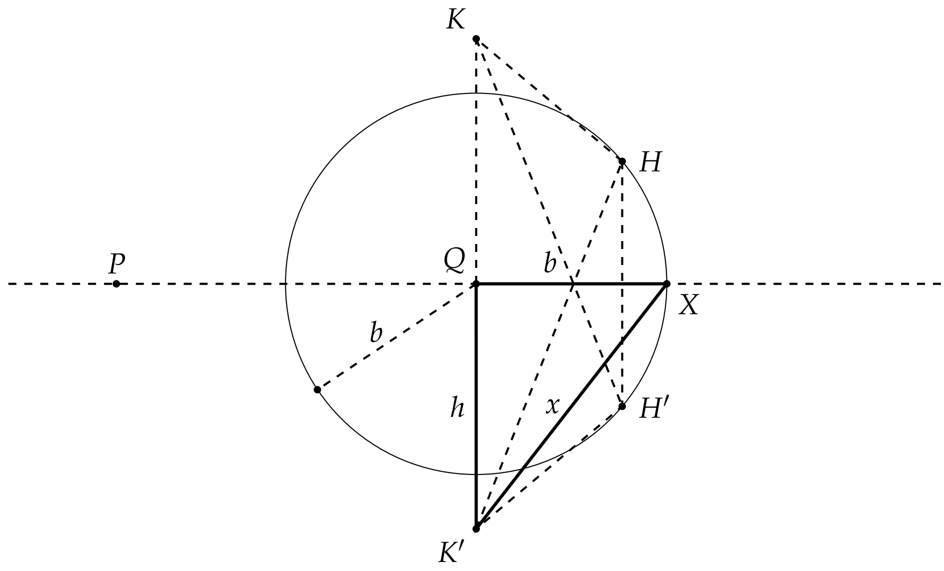
Multiply the two equations and simplify to get Ptolemy's theorem:

$$\begin{aligned}
e^2 \cdot f^2 &= (ac+bd)^2 \\
ef &= (ac+bd)\,.
\end{aligned}$$

For the construction on page 5, the diagonals are of length d, the legs are of length b and the parallel lines are of lengths h and $2h$, so Ptolemy's theorem gives $d \cdot d = b \cdot b + h \cdot 2h$ or $d^2 = b^2 + 2h^2$.

Let X be the point on line PQ that extends PQ by b. (We will eventually construct X; now we're just imagining it.)

Let $x = K'X$. Since $\triangle QK'X$ is a right triangle, $x^2 = b^2 + h^2$:

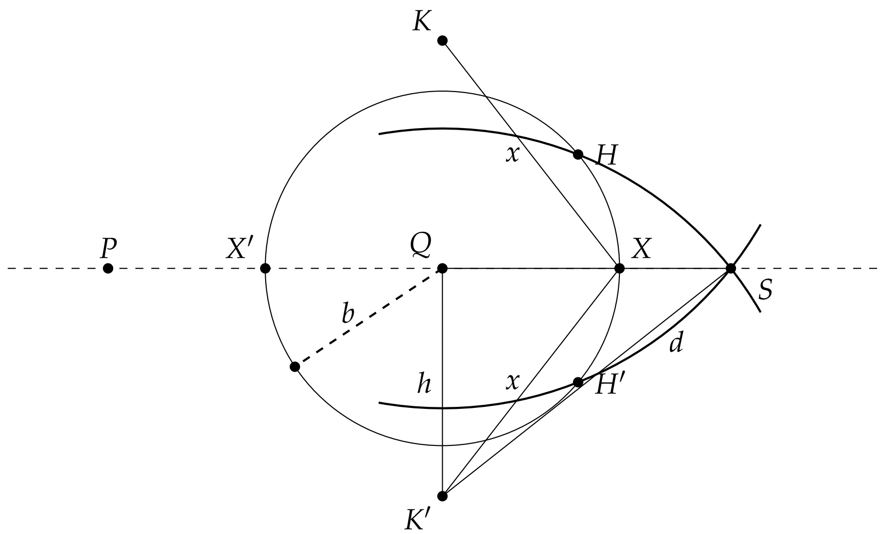


It follows then that $d^2 = x^2 + h^2$ so that x is a leg of a right triangle with hypotenuse d, the other leg being h.

> By Ptolemy's theorem, $d^2 = b^2 + 2h^2$, so $d^2 = (x^2 - h^2) + 2h^2 = x^2 + h^2$. All the previous sentence is saying is that it is possible to build a right triangle with sides x, h, d; such a triangle does not appear in the above diagram.

Now let $S = c(K, d) \cap c(K', d)$.

$QS^2 + h^2 = d^2$, so $QS = x$:



3. Then $X = c(K, x) \cap c(K', x)$.

There are two Xs, one for $a + b$ and one for $a - b$. $\square$

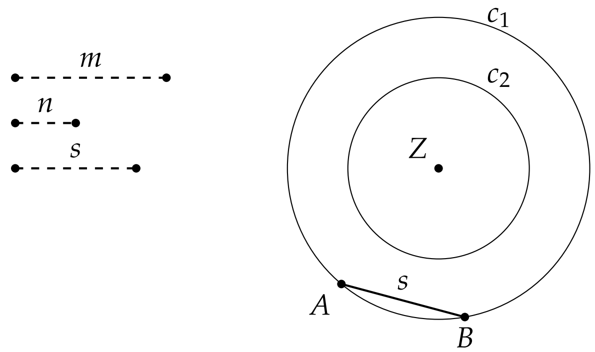
Recall what we are trying to construct: extend PQ of length a by a length b. Since the length of QX is b, the length of PX is $a+b$. Similarly, the length of PX' is $a-b$.

Prelim 4. Given segments of length n, m, s, construct a segment of length $x = \frac{n}{m}s$.

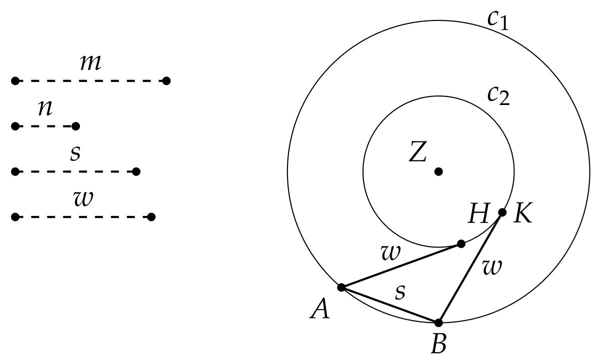
Solution. This solution by Mascheroni is remarkable for its brevity and simplicity. Draw two concentric circles $c_1 = c(Z, m)$ and $c_2 = c(Z, n)$ and chord $AB = s$ on c_1. (It is assumed that s falls within c_1. If not, use Prelim 3 to replace n and m by sufficiently large integer multiples $kn = N$ and $km = M$.)

There is an implicit assumption that $m > n$. If not, just exchange the notation.

The expression "it is assumed that s falls within c_1" refers to the possibility that s is within c_1 but also cuts through c_2. By using multiples this can be avoided.

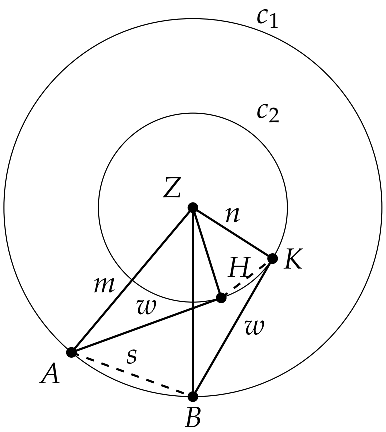


Next lay off any length w from A and B on c_2 with H and K on c_2 so that $AH = BK = w$:



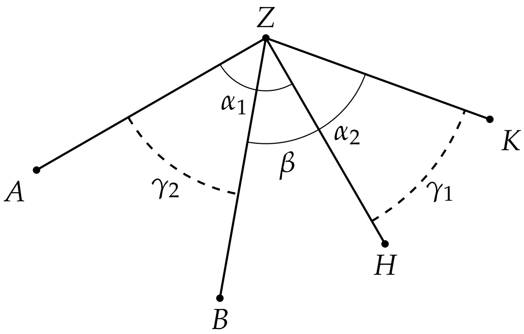
$\triangle AHZ$ and $\triangle BKZ$ are congruent by SSS,

The sides are $ZA = ZB = m$ (radius of circle c_1), $ZH = ZK = n$ (radius of circle c_2, $AH = BK = w$ (by construction).

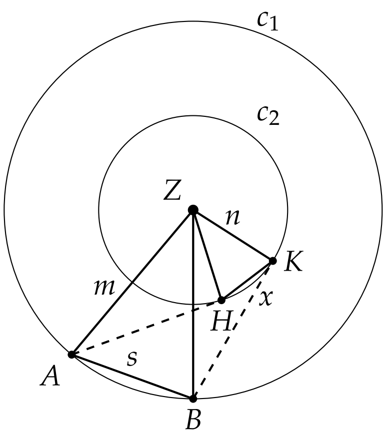


so $\angle AZH = \angle BZK$ and $\angle AZB = \angle HZK$.

> This follows by subtraction of angles, but it is somewhat hard to see in the diagram. The following diagram should clarify the construction by displaying only the angles. Since $\alpha_1 = \angle AZH = \angle BZK = \alpha_2$ by congruent triangles, we have $\gamma_1 = \alpha_1 - \beta = \alpha_2 - \beta = \gamma_2$.



and $\triangle ZAB$ and $\triangle ZHK$ are similar.



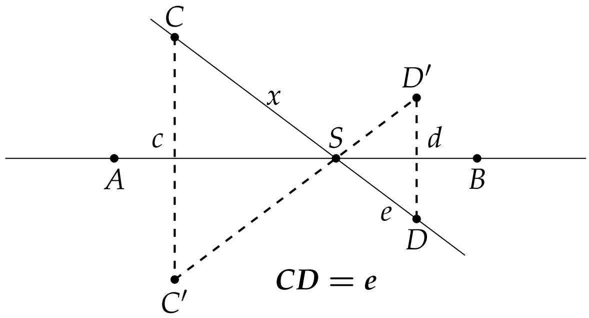
Then $\frac{m}{s} = \frac{n}{x}$ and $x = \frac{n}{m}s$. $\square$

Now for the solutions to I. and II. above.

I′. Find the point of intersection S of two straight lines AB and CD, each of which is given by two points, with compass alone.

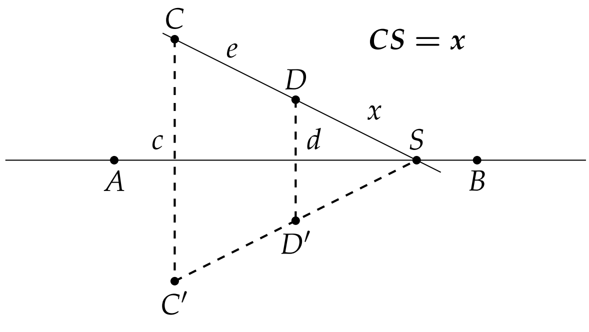
Solution. Let C' and D' be reflections of C and D about line AB respectively. The sought-for point of intersection S then lies on line $C'D'$.

> CD and $C'D'$ intersect AB at the same point S because reflection preserves distance. Given point S defined by the intersection of AB (the line of reflections) and CD, $C'S = CS$ and $D'S = DS$.



$\triangle CSC'$ and $\triangle DSD'$ are similar so $\frac{CS}{DS} = \frac{CC'}{DD'}$. With $x = CS, c = CC', d = DD'$ and $e = CD$, we get $\frac{x}{e-x} = \frac{c}{d}$ or $x = \frac{c}{c+d}e$. (If D is on the same side of AB as C, then $x = \frac{c}{c-d}e$.)

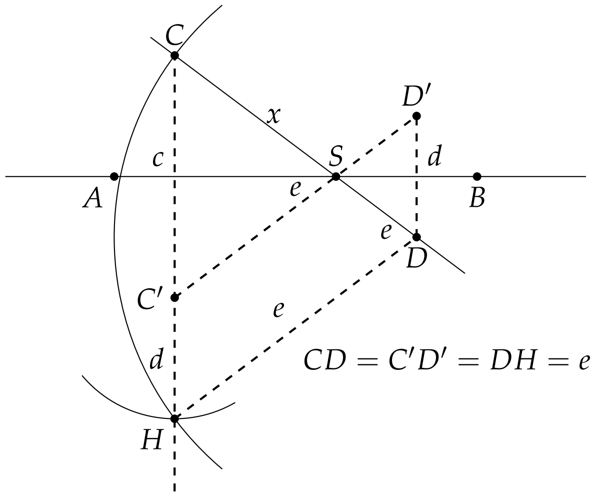
> Here is the diagram corresponding to this situation. The similarity of the triangles $\triangle CSC'$ and $\triangle DSD'$ gives $\frac{x}{x-e} = \frac{c}{d}$ and we can solve for $x = \frac{c}{c-d}e$.



$c + d = CH$ where H is the intersection point of $c(C', d)$ and $c(D, e)$ on line CC'.

> The proof could simply quote Preliminary problem 3 which claims that given two lengths a line segment whose length is their sum and difference can be computed, but this construction is simpler.

The circle $c(C', d)$ defines the points at distance d from C'. We need to construct the intersection of this cricle with the line containing CC'. The quadrilateral $C'D'DH$ is a parallelogram since the lengths of both opposite sides are equal. Since DD' is parallel to CC', $C'H$ is also parallel to DD' and therefore on the line containing CC'. Since e is the distance CD, then H, the intersection of $c(D, e)$ with the line containing CC', is also at a distance e from D.

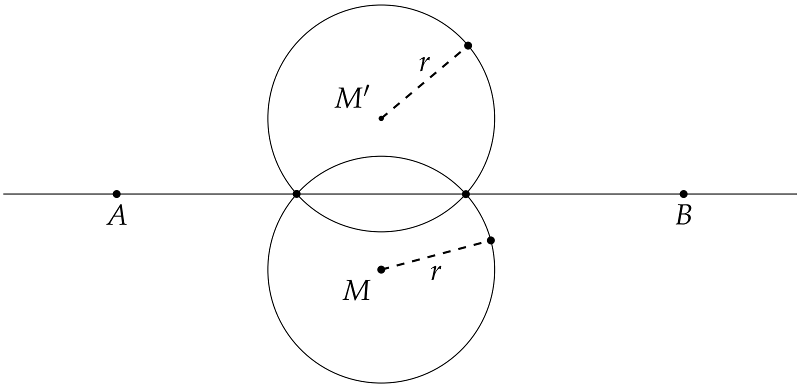


($CH = c - d$ in case D is on the same side of AB as C.) Preliminary problem 4 then allows us to construct x, and from that S as the intersection of arcs of the circles $c(C, x)$ and $c(C', x)$.

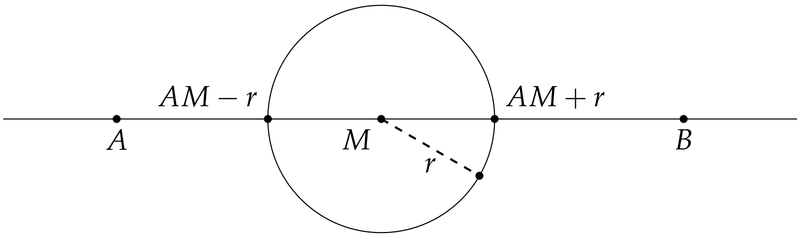
> x is the length of CS which equals the length of $C'S$ because reflection preserves distances, so all we have to do is compute x, and then S will be the intersections of the circles $c(C, x), c(C', x)$. By Preliminary problem 4, we can compute $x = \frac{c}{c+d}e$ given c, e, d, where the line segment of length $c + d$ is constructed above as CH.

II'. Determine the point of intersection S of a given circle k and a given straight line AB with compass alone.

Solution. Let $k = c(M, r)$, and M' be the reflection of M about line AB.



The points of intersection are the points where $c(M, r)$ and $c(M', r)$ intersect. This construction cannot be done if M is on line AB.



In this exceptional case, extend and shorten AM by r by Prelim 3; the end points of the extended and shortended sements are the desired points. $\square$