An Alternate Method for Solving Quadratic Equations

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1 Introduction

This document presents Po-Shen Loh's alternate method of solving quadratic equations [1, 2]. I have added more examples and details of the computations.

Section **??** reviews the traditional methods for solving quadratic equations. Loh's method is based on a very simple observation about the roots that is hard to see intuitively; Section **??** tries to convince the reader that the observation makes sense and then explains the method for computing roots. In Section **??** the computation is carried out for two examples. Section **??** derives the traditional formula for the roots from Loh's formulas. Section **??** shows that the method works for polynomials that have complex roots because they are irreducible over the rational numbers. Except for the last Section, the material should be accessible to all secondary-school teachers and students.

2 Traditional methods for solving quadratic equations

Every secondary-school student memorizes the formula for obtaining the roots of a quadratic equation $ax^2 + bx + c = 0$:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

For now we will work only with monic polynomials; the roots of $x^2 + bx + c = 0$ are:

$$x_1, x_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2} .$$

Another method of solving quadratic equations is by factoring the polynomials more-or-less by trial and error. Sometimes factoring is very easy:

$$\begin{aligned} x^2 - 4x + 3 &= (x - r_1)(x - r_2) = 0 \\ &= (x-1)(x-3) = 0 \\ x_1, x_2 &= 1, 3\,. \end{aligned}$$

It is much harder to factor:

$$x^2 - 2x - 24 = (x - r_1)(x - r_2) = 0\,.$$

Possible roots (r_1, r_2) are:

$$(\pm 1, \mp 24)\,, (\pm 2, \mp 12)\,, (\pm 3, \mp 8)\,, (\pm 4, \mp 6)\,.$$

It is clear that the signs of r_1, r_2 must be different so that their product is negative -24, but still we have eight possibilities to check.

3 Computing the roots

If r_1, r_2 are the roots of $x^2 + bx + c$, then:[1]

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 = x^2 + bx + c\,.$$

Even if we do not know the values of the roots, we do know that:

$$r_1 + r_2 = -b\,, \qquad r_1 r_2 = c\,.$$

Consider some values of $-b, r_1, r_2$ and let m_{12} be the average of r_1, r_2:

$-b$	r_1	r_2	m_{12}
33	12	21	$16\frac{1}{2}$
33	8	25	$16\frac{1}{2}$
33	1	32	$16\frac{1}{2}$
-4	-16	12	-2
-4	-4	0	-2
-4	-3	-1	-2

[1] Loh points out the difference between his method and factoring. In the latter, we *assume* that two roots exist. This is true because of the Fundamental Theorem of Algebra, but that is a heavy theorem for the simple task of finding the roots of a quadratic equation. In his method, we simply say: *if the roots exist*.

For any quadratic equation, the average of the two roots is constant:

$$\frac{r_1 + r_2}{2} = \frac{(-b - r_2) + r_2}{2} = \frac{-b}{2} + \frac{-r_2 + r_2}{2} = -\frac{b}{2}\,.$$

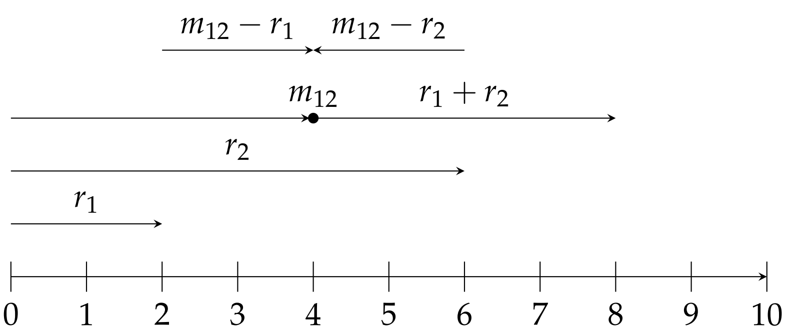
Let s be any number; then:

$$-b = -b + s + (-s) = \left(\frac{-b}{2} + s\right) + \left(\frac{-b}{2} - s\right) = r_1 + r_2\,.$$

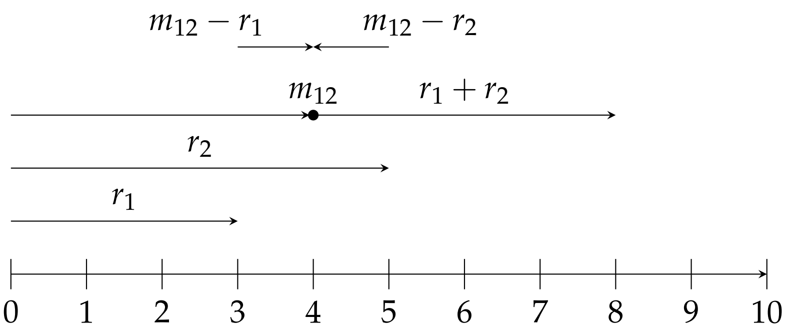
If one root is at distance s from the average m_{12}, the other root is at distance $-s$ from the average:

$-b$	r_1	r_2	m_{12}	$m_{12} - r_1$	$m_{12} - r_2$
33	12	21	$16\frac{1}{2}$	$4\frac{1}{2}$	$-4\frac{1}{2}$
33	8	25	$16\frac{1}{2}$	$8\frac{1}{2}$	$-8\frac{1}{2}$
33	1	32	$16\frac{1}{2}$	$15\frac{1}{2}$	$-15\frac{1}{2}$
-4	-16	12	-2	14	-14
-4	-4	0	-2	2	-2
-4	-3	-1	-2	1	-1

The following diagram visualizes this relationship for $r_1, r_2 = 2, 6$, where $m_{12} = 4, s = 2$:



If we use other values $r_1, r_2 = 3, 5$ for which $r_1 + r_2 = 8$, $m_{12} = 4$ remains the same, while $s = 2$ becomes $s = 1$:



The offset s seems to be arbitrary in:

$$r_1 = \left(\frac{-b}{2} + s\right), \quad r_2 = \left(\frac{-b}{2} - s\right),$$

but there is an additional constraint $r_1 r_2 = c$ where c is the constant term in the polynomial. By multiplying the two expressions we have derived for r_1, r_2, we can determine s and then r_1, r_2.

$$c = \left(-\frac{b}{2} + s\right)\left(-\frac{b}{2} - s\right).$$

4 Examples

Let us use this method on the polynomial $x^2 - 2x - 24$, where $b = -2, c = -24$:

$$\begin{aligned}
c &= \left(-\frac{b}{2} + s\right)\left(-\frac{b}{2} - s\right) \\
-24 &= (1+s)(1-s) \\
s^2 &= 25 \\
s &= 5 \\
r_1 &= 1 + 5 = 6 \\
r_2 &= 1 - 5 = -4.
\end{aligned}$$

Check:

$$(x-6)(x-(-4)) = x^2 - 6x - (-4)x + (6 \cdot -4) = x^2 - 2x - 24.$$

As another example, let us find the roots of $x^2 - 83x - 2310$:

$$\begin{aligned}
c &= \left(-\frac{b}{2} + s\right)\left(-\frac{b}{2} - s\right) \\
-2310 &= \left(\frac{83}{2} + s\right)\left(\frac{83}{2} - s\right) \\
s^2 &= \frac{6889}{4} + 2310 = \frac{16129}{4} \\
s &= \frac{127}{2} \\
r_1 &= \frac{83}{2} - \frac{127}{2} = -22 \\
r_2 &= \frac{83}{2} + \frac{127}{2} = 105.
\end{aligned}$$

Check:

$$(x+22)(x-105) = x^2 + 22x - 105x + (22 \cdot -105) = x^2 - 83x - 2310.$$

Let us compare this computation with the computation using the traditional formula that we have all memorized:

$$\begin{aligned}
\frac{-b \pm \sqrt{b^2-4c}}{2} &= \frac{-(-83) \pm \sqrt{(-83)^2 - 4 \cdot (-2310)}}{2} \\
&= \frac{83 \pm \sqrt{6889+9240}}{2} = \frac{83 \pm \sqrt{16129}}{2} \\
&= \frac{83 \pm 127}{2} \\
r_1 &= \frac{83-127}{2} = -22 \\
r_2 &= \frac{83+127}{2} = 105\,.
\end{aligned}$$

While the computation in Loh's method is similar to that of the traditional formula, it has the advantage that we can derive it immediately from the average and product of the roots. In the next section, we show that it is easy to derive the traditional formulas from this method.[2]

5 The traditional formula

With arbitrary coefficients b, c, the equations are:

$$\begin{aligned}
c = r_1, r_2 &= \left(\frac{-b}{2} + s\right)\left(\frac{-b}{2} - s\right) \\
&= \left(\frac{b^2}{4} - s^2\right) \\
s &= \sqrt{\left(\frac{b^2}{4}\right) - c} \\
r_1, r_2 &= \frac{-b}{2} \pm \sqrt{\left(\frac{b^2}{4}\right) - c}\,,
\end{aligned}$$

which can also be written:

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2-4c}}{2}\,,$$

the traditional formula for obtaining the roots of a monic polynomial.

If the polynomial is not monic, $a \neq 1$, divide it by a, substitute in the equation and simplify:

$$ax^2 + bx + c = 0$$

[2] See the previous footnote for a theoretical advantage.

$$
\begin{aligned}
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\
r_1, r_2 &= \frac{-(b/a) \pm \sqrt{(b/a)^2 - 4(c/a)}}{2} \\
&= \frac{-(b/a) \pm \sqrt{(b/a)^2 - 4(ac/a^2)}}{2} \\
&= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}
$$

The traditional formula has been derived from the observation that $r_1 + r_2 = -b$ and $r_1 r_2 = c$.

6 Irreducible polynomials

There are quadratic polynomials like $x^2 + 1$ that are irreducible over the rational numbers. According to the Fundamental Theorem of Algebra, all polynomials with complex coefficients have complex roots. Let us see how the method works for the polynomial $x^2 - 2x + 76$:

$$
\begin{aligned}
s^2 &= \frac{b^2}{4} - c = \frac{4}{4} - 76 = -75 \\
s &= \sqrt{-75} = \sqrt{-1 \cdot 25 \cdot 3} = i\,5\sqrt{3} \\
r_1, r_2 &= 1 \pm i\,5\sqrt{3}\,.
\end{aligned}
$$

Check:

$$
\begin{aligned}
&(x - (1 + i\,5\sqrt{3}))\,(x - (1 - i\,5\sqrt{3})) = \\
&x^2 \,-\, (1 + i\,5\sqrt{3})x \,-\, (1 - i\,5\sqrt{3})x \,+\, (1^2 - (i\,5\sqrt{3})^2) = \\
&x^2 - x - x + 1 - (-75) = \\
&x^2 - 2x + 76\,.
\end{aligned}
$$

References

[1] Po-Shen Lo. *A Different Way to Solve Quadratic Equations*, 2019, `https://www.poshenloh.com/quadratic/`.

[2] Po-Shen Loh. *A Simple Proof of the Quadratic Formula*, arXiv: 1910.06709, 2019, `https://arxiv.org/abs/1910.06709`.