The Mathematics of Origami

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Contents

1 Introduction

This document develops the mathematics of origami using secondary-school mathematics [2]. Equations of lines are given in the slope-intercept form $y = mx + b$.

Sections 3–9 develop the mathematical formulas for the seven axioms and together with numerical examples. In the diagrams, given lines are solid, folds are dashed, auxiliary lines are dotted, and dotted arrows indicate the direction of folding the paper.

The folding operations can construct every length that can be constructed by straightedge and compass: given lengths a, b, the lengths $a + b, a - b, a \times b, a/b, \sqrt{a}$ can be constructed [5, Chapter 4].

Folding is more powerful because it can construct cube roots. The proof is somewhat complex, so we limit ourselves to showing that folding can trisect an angle and double a cube. Sections 10–11 present two methods for trisecting an arbitrary angle and Sections 12–13 present two methods for doubling a cube by computing $\sqrt[3]{2}$.

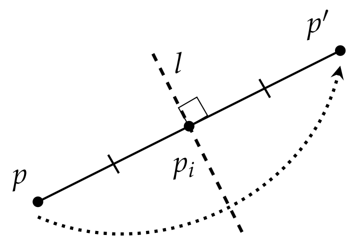
A guide to the literature used in preparing this document is given in Section 14.

Appendix A contains links to GeoGebra projects demonstrating the axioms. Appendix B derives trigonometric identities for tangents that may not be familiar. Appendix C explains the geometric definition of parabolas.

2 Definitions

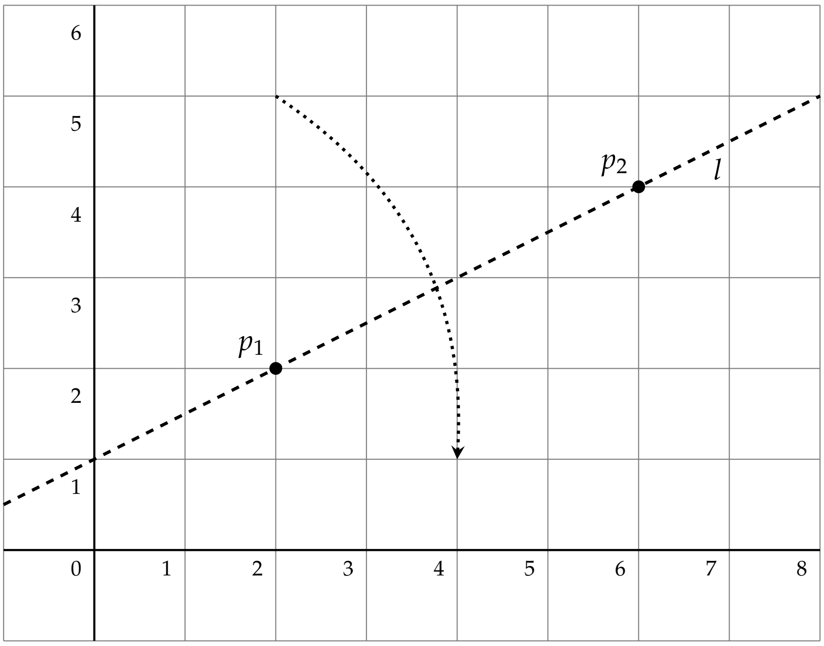
Each axiom states that a *fold* exists that will place given points and lines onto points and lines, such that certain properties hold. The term fold comes from the origami operation of folding a piece of paper, but here it is used to refer the the line created by folding the paper.

Formal definitions are given in [6, Chapter 10]. The reader should be aware that, *by definition*, folds result in *reflections*. Given a point p, its reflection around a fold l results in a point p', such that l is the perpendicular bisector of the line segment $\overline{pp'}$:



3 Axiom 1

Axiom Given two distinct points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, there is a unique fold l that passes through both of them.



Derivation of the equation of the fold

The equation of fold l is derived from the coordinates of p_1 and p_2: the slope is the quotient of the differences of the coordinates and the intercept is derived from p_1:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\,. \tag{1}$$

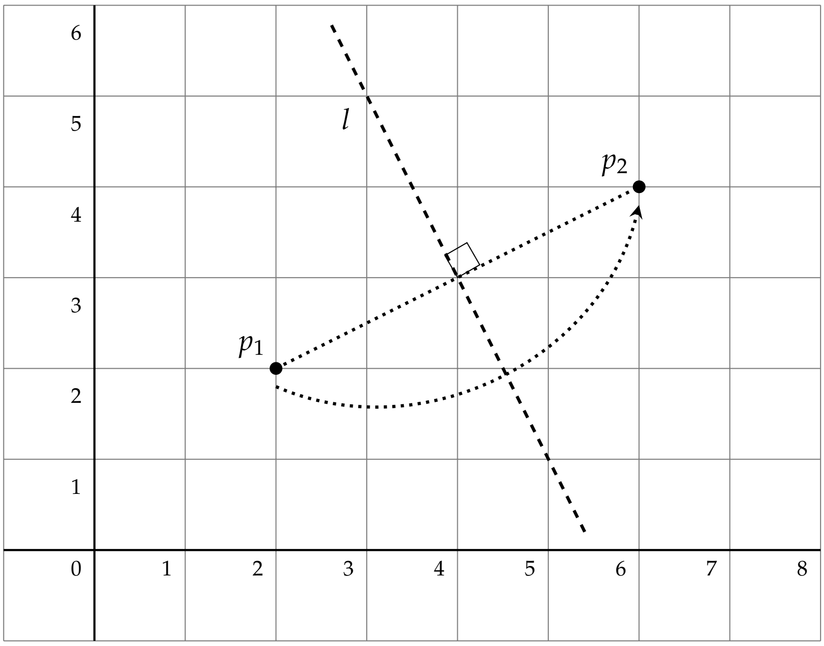
Example

Let $p_1 = (2, 2), p_2 = (6, 4)$. The equation of l is:

$$\begin{aligned} y - 2 &= \frac{4-2}{6-2}(x-2) \\ y &= \frac{1}{2}x + 1\,. \end{aligned}$$

4 Axiom 2

Axiom Given two distinct points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, there is a unique fold l that places p_1 onto p_2.



Derivation of the equation of the fold

The line l is the perpendicular bisector of $\overline{p_1 p_2}$. Its slope is the negative inverse of the slope of the line connecting p_1 and p_2. The line passes through the midpoint between the points:

$$y - \frac{y_1 + y_2}{2} = -\frac{x_2 - x_1}{y_2 - y_1}\left(x - \frac{x_1 + x_2}{2}\right). \tag{2}$$

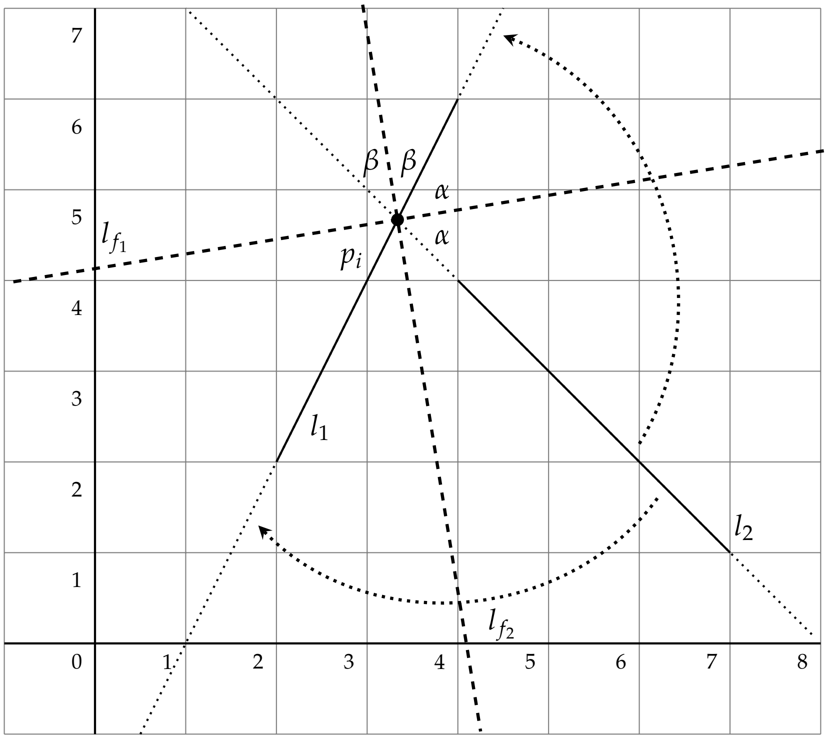
Example

Let $p_1 = (2, 2), p_2 = (6, 4)$. The equation of l is:

$$\begin{aligned} y - \left(\frac{2+4}{2}\right) &= -\frac{6-2}{4-2}\left(x - \left(\frac{2+6}{2}\right)\right) \\ y &= -2x + 11\,. \end{aligned}$$

5 Axiom 3

Axiom Given two lines l_1 and l_2, there is a fold l that places l_1 onto l_2.



Derivation of the equation of the fold

If the lines are parallel, let l_1 be $y = mx + b_1$ and let l_2 be $y = mx + b_2$. The fold is the line parallel to l_1, l_2 and halfway between them $y = mx + \frac{b_1 + b_2}{2}$.

If the lines intersect, let l_1 be $y = m_1x + b_1$ and let l_2 be $y = m_2x + b_2$.

Derivation of the point of intersection

$p_i = (x_i, y_i)$, the point of intersection of the two lines, is:

$$\begin{aligned} m_1x_i + b_2 &= m_2x_i + b_2 \\ x_i &= \frac{b_2 - b_1}{m_1 - m_2} \\ y_i &= m_1x_i + b_1 \,. \end{aligned}$$

Example

Let l_1 be $y = 2x - 2$ and let l_2 be $y = -x + 8$. The point of intersection is:

$$\begin{aligned} x_i &= \frac{8 - (-2)}{2 - (-1)} = \frac{10}{3} \approx 3.33 \\ y_i &= 2 \cdot \frac{10}{3} - 2 = \frac{14}{3} \approx 4.67 \,. \end{aligned}$$

Derivation of the equation of the slope of the angle bisector

The two lines form an angle at their point of intersection, actually, two pairs of vertical angles. The folds are the bisectors of these angles.

If the angle of line l_1 relative to the x-axis is θ_1 and the angle of line l_2 relative to the x-axis is θ_2, then the fold is the line which makes an angle of $\theta_b = \dfrac{\theta_1 + \theta_2}{2}$ with the x-axis.

$\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$ are given and m_b, the slope of the angle bisector, is:

$$m_b = \tan\theta_b = \tan\frac{\theta_1 + \theta_2}{2}\,.$$

The derivation requires the use of the following trigonometric identities:[1]

$$\begin{aligned}
\tan(\alpha_1 + \alpha_2) &= \frac{\tan\alpha_1 + \tan\alpha_2}{1 - \tan\alpha_1 \tan\alpha_2} \\
\tan\frac{\alpha}{2} &= \frac{-1 \pm \sqrt{1 + \tan^2\alpha}}{\tan\alpha}\,.
\end{aligned}$$

First derive m_s, the slope of $\theta_1 + \theta_2$:

$$m_s = \tan(\theta_1 + \theta_2) = \frac{m_1 + m_2}{1 - m_1 m_2}\,.$$

Then derive m_b, the slope of the angle bisector:

$$\begin{aligned}
m_b &= \tan\frac{\theta_1 + \theta_2}{2} \\
&= \frac{-1 \pm \sqrt{1 + \tan^2(\theta_1 + \theta_2)}}{\tan(\theta_1 + \theta_2)} \\
&= \frac{-1 \pm \sqrt{1 + m_s^2}}{m_s}\,.
\end{aligned}$$

Example For the lines $y = 2x - 2$ and $y = -x + 8$, the slope of the angle bisector is:

$$\begin{aligned}
m_s &= \frac{2 + (-1)}{1 - (2 \cdot -1)} = \frac{1}{3} \\
m_b &= \frac{-1 \pm \sqrt{1 + (1/3)^2}}{1/3} = -3 \pm \sqrt{10} \approx -6.16,\ 0.162\,.
\end{aligned}$$

[1] The derivation of these identities is given in Appendix B, using the more familiar identities for sin and cos.

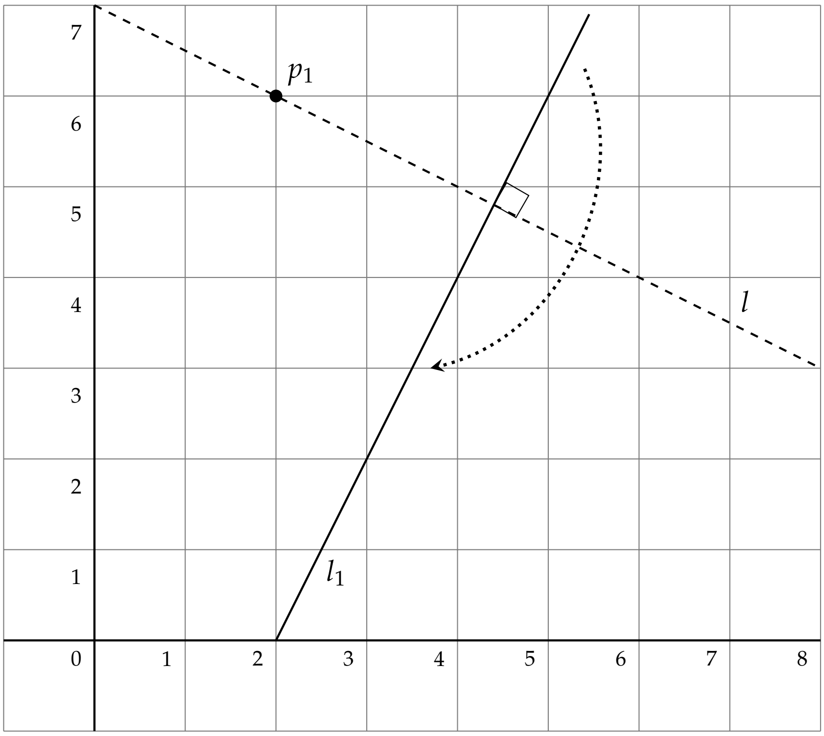
Derivation of the equation of the fold

Let us derive equation of the fold l_{f_1} with the positive slope; we know the coordinates of the intersection of the two lines $m_i = \left(\frac{10}{3}, \frac{14}{3}\right)$:

$$
\begin{aligned}
\frac{14}{3} &= (-3+\sqrt{10}) \cdot \frac{10}{3} + b \\
b &= \frac{44-10\sqrt{10}}{3} \\
y &= (-3+\sqrt{10})x + \frac{44-10\sqrt{10}}{3} \approx 0.162x + 4.13\,.
\end{aligned}
$$

6 Axiom 4

Axiom Given a point p_1 and a line l_1, there is a unique fold l perpendicular to l_1 that passes through point p_1.



Derivation of the equation of the fold

Let l_1 be $y = m_1 x + b_1$ and let $p_1 = (x_1, y_1)$. l is perpendicular to l_1 so its slope is $-\dfrac{1}{m_1}$. Since it passes through p_1, we can compute the intercept b and write down its equation:

$$\begin{aligned} y_1 &= -\frac{1}{m}x_1 + b \\ b &= \frac{(my_1 + x_1)}{m} \\ y &= -\frac{1}{m}x + \frac{(my_1 + x_1)}{m} \,. \end{aligned}$$

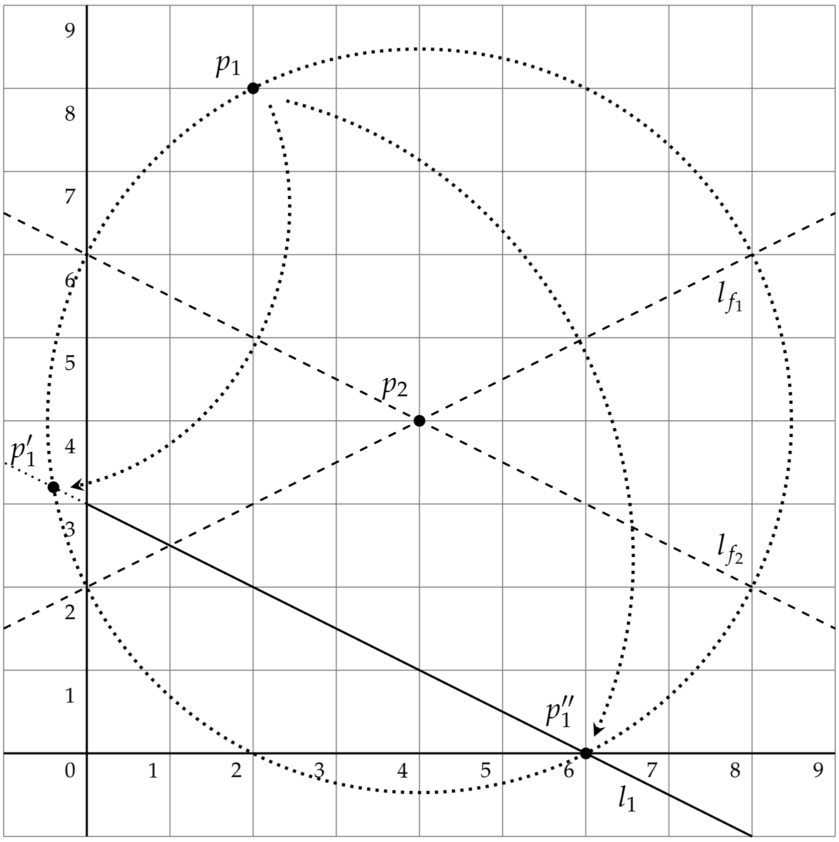
Example

Let $p_1 = (2, 6)$ and let l_1 be $y = 2x - 4$. The equation of the fold l is:

$$y = -\frac{1}{2}x + \frac{2 \cdot 6 + 2}{2} = -\frac{1}{2}x + 7 \,.$$

7 Axiom 5

Axiom Given two points p_1, p_2 and a line l_1, there is a fold l that places p_1 onto l_1 and passes through p_2.



For a given pair of points and a line, there may be zero, one or two folds.

Derivation of the equations of the reflections

Let l be a fold through p_2 and p_1' be the reflection of p_1 around l. The length of $\overline{p_1p_2}$ equals the length of $\overline{p_2p_1}'$. The locus of points at distance p_1p_2 *from* p_2 is the circle centered at p_2 whose radius is the length of $\overline{p_1p_2}$. The intersections of this circle with the line l_1 give the possible points p_1'.

Let l_1 be $y = m_1x + b_1$ and let $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$. The equation of the circle centered at p_2 with radius the length of $\overline{p_1p_2}$ is:

$$(x - x_2)^2 + (y - y_2)^2 = r^2 , \quad \text{where}$$
$$r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 .$$

Substituting the equation of the line into the equation for the circle:

$$(x - x_2)^2 + ((m_1x + b_1) - y_2)^2 = (x - x_2)^2 + (m_1x + (b_1 - y_2))^2 = r^2 ,$$

we obtain a quadratic equation for the x-coordinates of the possible intersections:

$$x^2(1+m_1^2) \;+\; 2(-x_2+m_1 b-m_1 y_2)x \;+\; (x_2^2+(b_1^2-2b_1 y_2+y_2^2)-r^2)=0\,. \qquad (3)$$

There will be at most two solutions $p_1' = (x_1', y_1')$, $p_1'' = (x_1'', y_1'')$, where y_1', y_1'' are obtained from $y = m_1 x + b_1$ for $x = x_1', x = x_1''$.

Example

Let $p_1 = (2,8)$, $p_2 = (4,4)$ and let l_1 be $y = -\dfrac{1}{2}x + 3$. The equation of the circle is:

$$(x-4)^2+(y-4)^2 = r^2 = (4-2)^2+(4-8)^2 = 20\,.$$

Substitute the equation of the line into the equation of the circle and simplify to obtain a quadratic equation for the x-coordinates of the intersections (or use Equation 3):

$$\begin{aligned}
(x-4)^2+\left(\left(-\frac{1}{2}x+3\right)-4\right)^2 &= 20\\
\frac{5}{4}x^2-7x-3 &= 0\\
5x^2-28x-12 &= 0\,.
\end{aligned}$$

This quadratic equation factors into $(5x+2)$ and $(x-6)$, giving two points of intersection:

$$p_1' = \left(-\frac{2}{5}, \frac{16}{5}\right) = (-0.4, 3.2)\,, \quad p_1'' = (6,0)\,.$$

Derivation of the equations of the folds

The folds will be the perpendicular bisectors of $\overline{p_1 p_1'}$ and $\overline{p_1 p_1''}$. The equation of a perpendicular bisector is given by Equation 2, repeated here with for p_1':

$$y - \frac{y_1+y_1'}{2} = -\frac{x_1'-x_1}{y_1'-y_1}\left(x-\frac{x_1+x_1'}{2}\right)\,. \qquad (4)$$

Example

For $p_1 = (2,8)$ and $p_1' = \left(-\dfrac{2}{5}, \dfrac{16}{5}\right)$, the equation of the fold l_{f_1} is:

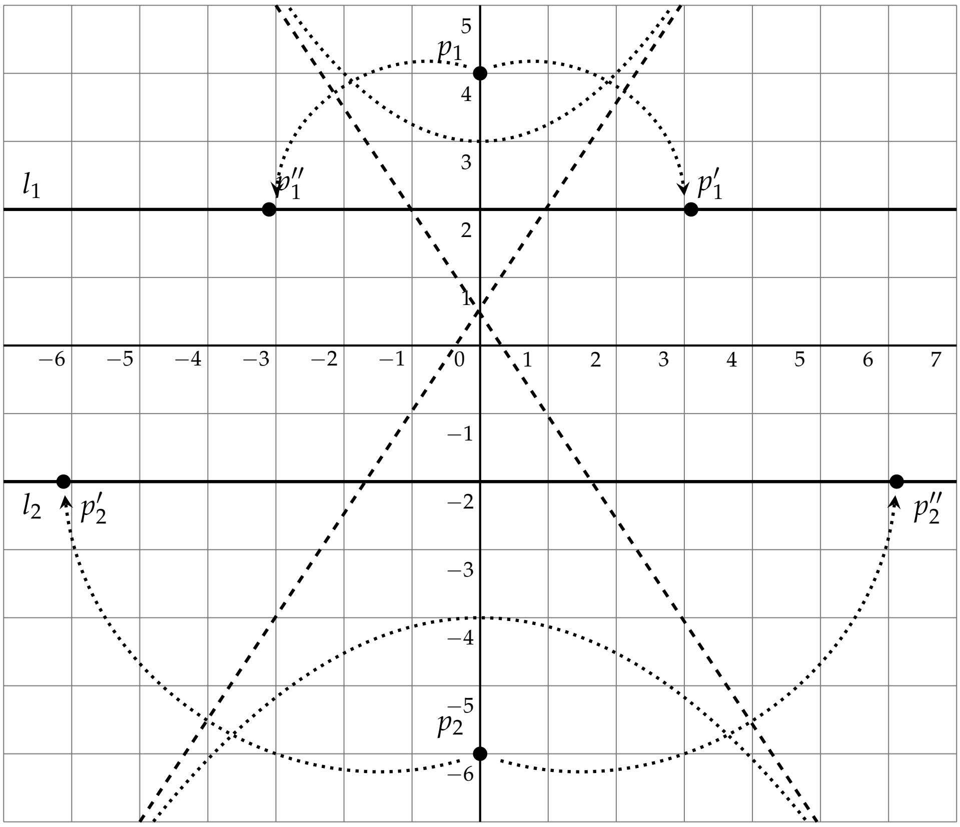
$$\begin{aligned}
y-\frac{8+(16/5)}{2} &= -\frac{(-2/5)-2}{(16/5)-8}\left(x-\frac{2+(-2/5)}{2}\right)\\
y &= -\frac{1}{2}x+6\,.
\end{aligned}$$

For $p_1 = (2,8)$ and $p_1'' = (6,0)$, the equation of the fold l_{f_2} is:

$$\begin{aligned}
y-\frac{8+0}{2} &= -\frac{6-2}{0-8}\left(x-\frac{2+6}{2}\right)\\
y &= \frac{1}{2}x+2\,.
\end{aligned}$$

8 Axiom 6

Axiom Given two points p_1 and p_2 and two lines l_1 and l_2, there is a fold l that places p_1 onto l_1 and p_2 onto l_2.



For a given pair of points and pair of lines, there may be zero, one, two or three folds.

A fold that places p_i onto l_i is a line such that the distance from p_i to the line is equal to the distance from l_i to the line. The locus of points that are equidistant from a point p_i and a line l_i is a parabola with focus p_i and directrix l_i. A fold is any line tangent to that parabola. A detailed justification of this claim is given in Appendix C.

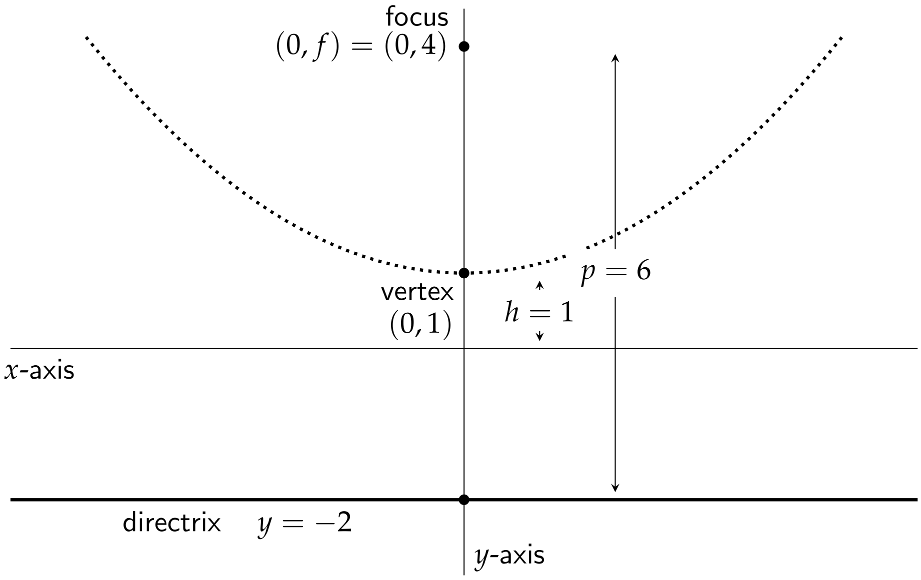
For a fold to simultaneously place p_1 onto l_1 and p_2 onto l_2, it must be a tangent common to the two parabolas.

The formula for an arbitrary parabola is quite complex, so we limit the presentation to parabolas with the the y-axis as the axis of symmetry. This is not a significant limitation because for any parabola there is a rigid motion that moves the parabola so that its axis of symmetry is the y-axis.

An example will also be given where one of the parabolas has the x-axis as its axis of symmetry.

Derivation of the equation a fold

Let $(0, f)$ be the focus of a parabola with directrix $y = d$. Define $p = f - d$, the signed length of the line segment between the focus and the directrix.[2] If the vertex of the parabola is on the x-axis, the equation of the parabola is $y = \dfrac{x^2}{2p}$. To move the parabola up or down the y-axis so that its vertex is at $(0, h)$, add h to the equation of the parabola: $y = \dfrac{x^2}{2p} + h$.



Define $a = 2ph$ so that the equation of the parabola is:

$$\begin{aligned} y &= \frac{x^2}{2p} + \frac{a}{2p} \\ x^2 - 2py + a &= 0\,. \end{aligned}$$

The equation of the parabola in the diagram above is:

$$\begin{aligned} x^2 - 2 \cdot 6y + 2 \cdot 6 \cdot 1 &= 0 \\ x^2 - 12y + 12 &= 0\,. \end{aligned}$$

Substitute the equation of an *arbitrary* line $y = mx + b$ into the equation for the parabola to obtain an equation for the points of intersection of the line and the parabola:

$$\begin{aligned} x^2 - 2p(mx + b) + a &= 0 \\ x^2 + (-2mp)x + (-2pb + a) &= 0\,. \end{aligned}$$

The line will be tangent to the parabola iff this quadratic equation has *exactly one* solution iff its discriminant is zero:

$$(-2mp)^2 \;-\; 4 \cdot 1 \cdot (-2pb + a) = 0\,,$$

[2] We have been using the notation p_i for points; the use of p here might be confusing but it is the standard notation. The formal name for p is one-half the *latus rectum*.

which simplifies to:

$$m^2p^2 + 2pb - a = 0\,. \tag{5}$$

This is the equation with variable m for the slopes of tangents to the parabola. There are an infinite number of tangents because for each m, there is some b that makes the line a tangent by moving it up or down.

To obtain the common tangents to both parabolas, the equations for the two parabolas have two unknowns and can be solved for m and b.

Example

Parabola 1: focus $(0,4)$, directrix $y = 2$, vertex $(0,3)$, $p = 2$, $a = 2 \cdot 2 \cdot 3 = 12$. The equation of the parabola is:

$$x^2 - 2 \cdot 2y + 12 = 0\,.$$

Substituting into Equation 5 and simplifying:

$$m^2 + b - 3 = 0\,.$$

Parabola 2: focus $(0,-4)$, directrix $y = -2$, vertex $(0,-3)$, $p = -2$, $a = 2 \cdot -2 \cdot -3 = 12$. The equation of the parabola is:

$$x^2 - 2 \cdot (-2)y + 12 = 0\,.$$

Substituting into Equation 5 and simplifying:

$$m^2 - b - 3 = 0\,.$$

The solutions of the two equations:

$$\begin{aligned} m^2 + b - 3 &= 0 \\ m^2 - b - 3 &= 0\,, \end{aligned}$$

are $m = \pm\sqrt{3} \approx \pm 1.73$ and $b = 0$. There are two common tangents that are the folds:

$$y = \sqrt{3}x\,, \quad y = -\sqrt{3}x\,.$$

Example

Parabola 1 is unchanged.

Parabola 2: focus $(0,-6)$, directrix $y = -2$, vertex $(0,-4)$, $p = -4$, $a = 2 \cdot -4 \cdot -4 = 32$. The equation of the parabola is:

$$x^2 - 2 \cdot (-4)y + 32 = 0\,.$$

Substituting into Equation 5 and simplifying:

$$2m^2 - b - 4 = 0\,.$$

The solutions of the two equations:

$$
\begin{aligned}
&m^2 + b - 3 = 0\\
&2m^2 - b - 4 = 0\,,
\end{aligned}
$$

are $m = \pm\sqrt{\dfrac{7}{3}} \approx \pm 1.53$ and $b = \dfrac{2}{3}$. There are two common tangents that are folds:

$$
y = \sqrt{\frac{7}{3}}x + \frac{2}{3}\,, \quad y = -\sqrt{\frac{7}{3}}x + \frac{2}{3}\,.
$$

Example

Let us now define a parabola whose axis of symmetry is the x-axis.

Parabola 1 is unchanged.

Parabola 2: focus $(4,0)$, directrix $x = 2$, vertex $(3,0)$, $p = 2$, $a = 2 \cdot 2 \cdot 3 = 12$. The equation of the parabola is:

$$
y^2 - 4x + 12 = 0\,.
$$

Note that this is an equation with x and y^2 instead of x^2 and y, so we can't use Equation 5 and we must perform the derivation again.

Substitute the equation for a line:

$$
\begin{aligned}
&(mx + b)^2 - 4x + 12 = 0\\
&m^2x^2 + (2mb - 4)x + (b^2 + 12) = 0\,,
\end{aligned}
$$

set the discriminant equal to zero and simplify:

$$
\begin{aligned}
&(2mb - 4)^2 \,-\, 4m^2(b^2 + 12) = 0\\
&-3m^2 - mb + 1 = 0\,.
\end{aligned}
$$

If we try to solve the two equations:

$$
\begin{aligned}
&m^2 + b - 3 = 0\\
&-3m^2 - mb + 1 = 0\,,
\end{aligned}
$$

we obtain a cubic equation with variable m:

$$
m^3 - 3m^2 - 3m + 1 = 0\,. \tag{6}
$$

Since a cubic equation has at most three (real) solutions, there can be zero, one, two or three common tangents.

The formula for solving cubic equations is quite complicated, so I used a calculator on the internet and obtained three solutions:

$$m = 3.73\,,\ m = -1\,,\ m = 0.27\,.$$

Choosing $m = 0.27$, $b = 3 - m^2 = 2.93$, and the equation of the fold is:

$$y = 0.27x + 2.93\,.$$

From the form of Equation 6, we might guess that 1 or -1 is a root:

$$\begin{aligned}
&1^3 - 3\cdot 1^2 - 3\cdot 1 + 1 = -4 \\
&(-1)^3 - 3\cdot(-1)^2 - 3\cdot(-1) + 1 = 0\,.
\end{aligned}$$

Divide Equation 6 by $m - (-1) = m + 1$ to obtain the quadratic equation $m^2 - 4m + 1$ whose roots are $2 \pm \sqrt{3} \approx 3.73, 0.27$.

Derivation of the equations of the reflections

For clarity, we derive the position of the reflection $p_1' = (x_1', y_1')$ of $p_1 = (x_1, y_1)$ around some tangent line l_t whose equation is $y = m_t x + b_t$. The derivation is identical for any tangent and for p_2.

To reflect p_1 around l_t, we find the line l_p with equation $y = m_p x + b_p$ that is perpendicular to l_t and passes through p_1;

$$\begin{aligned}
y &= -\frac{1}{m_t}x + b_p \\
y_1 &= -\frac{1}{m_t}x_1 + b_p \\
y &= \frac{-x}{m_t} + \left(y_1 + \frac{x_1}{m_t}\right)\,.
\end{aligned}$$

Next we find the intersection $p_t = (x_t, y_t)$ of l_t and l_p:

$$\begin{aligned}
m_t x_t + b_t &= \frac{-x_t}{m_t} + \left(y_1 + \frac{x_1}{m_t}\right) \\
x_t &= \frac{\left(y_1 + \dfrac{x_1}{m_t} - b_t\right)}{\left(m_t + \dfrac{1}{m_t}\right)} \\
y_t &= m_t x_t + b_t\,.
\end{aligned}$$

The reflection p_1' is easy to derive because the intersection p_t is the midpoint between p_1 and its reflection p_1':

$$x_t = \frac{x_1 + x_1'}{2}\,, \quad y_t = \frac{y_1 + y_1'}{2}$$

$$x_1' = 2x_t - x_1\,, \quad y_1' = 2y_t - y_1\,.$$

Example

$p_1 = (0, 4)$, l_1 is $y = \sqrt{3}x$:

$$x_t = \frac{\left(4 + \frac{0}{\sqrt{3}} - 0\right)}{\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)} = \sqrt{3}$$

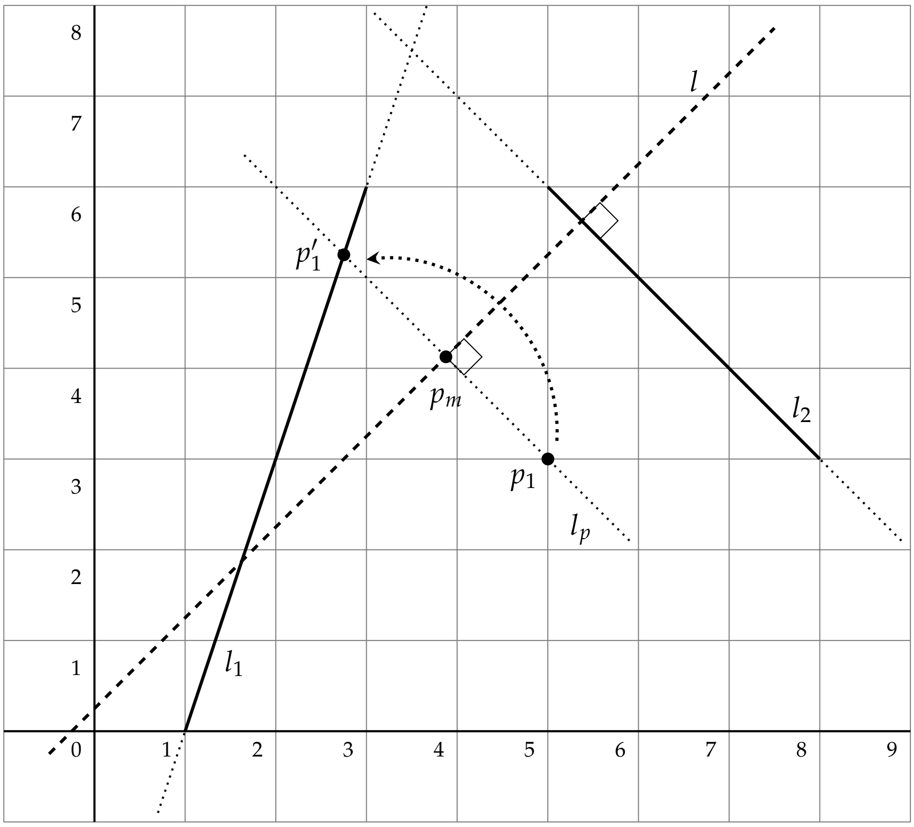
$$y_t = \sqrt{3}\sqrt{3} + 0 = 3$$

$$x_1' = 2x_t - x_1 = 2\sqrt{3} - 0 = 2\sqrt{3} \approx 3.46$$

$$y_1' = 2y_t - y_1 = 2 \cdot 3 - 4 = 2\,.$$

9 Axiom 7

Axiom Given one point p_1 and two lines l_1 and l_2, there is a fold l that places p_1 onto l_1 and is perpendicular to l_2.



Derivation of the equation of the fold

Let $p_1 = (x_1, y_1)$, let l_1 be $y = m_1 x + b_1$ and let l_2 be $y = m_2 x + b_2$.
Since the fold l is perpendicular to l_2, and the line l_p containing $\overline{p_1 p_1'}$ is perpendicular to l, it follows that l_p parallel to l_2:

$$y = m_2 x + b_p \,.$$

l_p passes through p_1 so $y_1 = m_2 x_1 + b_p$ and its equation is:

$$y = m_2 x + (y_1 - m_2 x_1) \,.$$

$p_1' = (x_1', y_1')$, the reflection of p_1 around the fold l, is the intersection of l_1 and l_p:

$$\begin{aligned} m_1 x_1' + b_1 &= m_2 x_1' + (y_1 - m_2 x_1) \\ x_1' &= \frac{y_1 - m_2 x_1 - b_1}{m_1 - m_2} \\ y_1' &= m_1 x_i' + b_1 \,. \end{aligned}$$

The midpoint $p_m = (x_m, y_m)$ of l_p is on the fold l:

$$(x_m, y_m) = \left(\frac{x_1 + x_1'}{2}, \frac{y_1 + y_1'}{2}\right).$$

The equation of the fold l is the perpendicular bisector of $\overline{p_1 p_1'}$. First compute the intercept of l which passes through p_m:

$$y_m = -\frac{1}{m_2} x_m + b_m$$
$$b_m = y_m + \frac{x_m}{m_2}.$$

The equation of the fold l is:

$$y = -\frac{1}{m_2} x + \left(y_m + \frac{x_m}{m_2}\right).$$

Example

Let $p_1 = (5, 3)$, let l_1 be $y = 3x - 3$ and let l_2 be $y = -x + 11$.

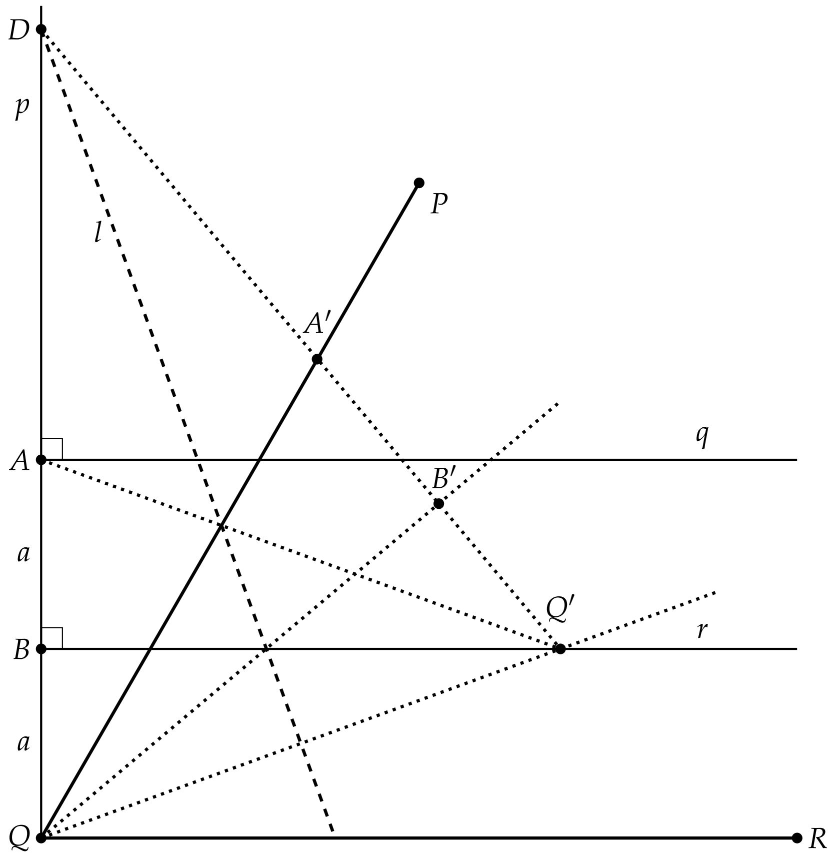
$$x_1' = \frac{3 - (-1) \cdot 5 - (-3)}{3 - (-1)} = \frac{11}{4}$$
$$y_1' = 3 \cdot \frac{11}{4} + (-3) = \frac{21}{4}$$
$$p_m = \left(\frac{5 + \frac{11}{4}}{2}, \frac{3 + \frac{21}{4}}{2}\right) = \left(\frac{31}{8}, \frac{33}{8}\right).$$

The equation of the fold l is:

$$y = -\frac{1}{-1} \cdot x + \left(\frac{33}{8} + \frac{\frac{31}{8}}{-1}\right) = x + \frac{1}{4}.$$

10 Abe's trisection of an angle

10.1 The construction

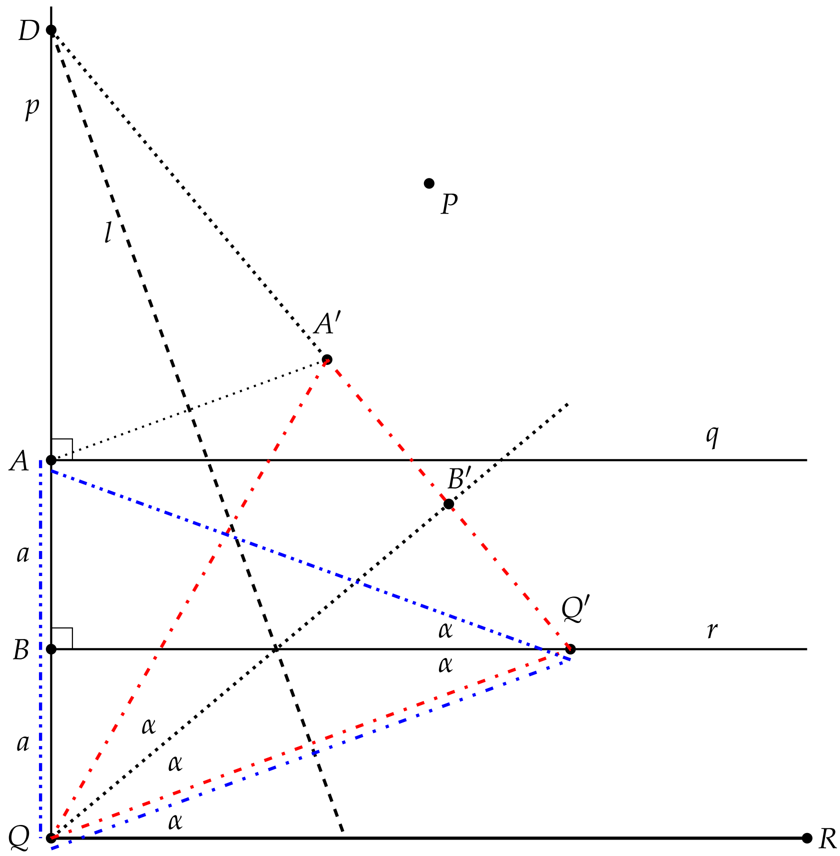


Given the acute angle $\angle PQR$, let p be the perpendicular to $\overline{QR}$ at Q. Let q be a perpendicular to p that intersects $\overline{PQ}$ and let r be the perpendicular to p that is halfway between Q and A. r intersects p at point B.

Using Axiom 6, construct a fold l that places A at A' on $\overline{PQ}$ and Q at Q' on r. Let B' be the reflection of B around l.

Draw the lines $\overline{QB'}$ and QQ'. We claim that $\angle PQB'$, $\angle B'QQ'$ and $Q'QR$ are a trisection of $\angle PQR$.

10.2 First proof



Since A', B', Q' are all reflections around the same line l, they are all on one line $\overline{DQ'}$. By construction, $\overline{AB} = \overline{BQ}$, $\overline{BQ'}$ is perpendicular to AQ and $\overline{BQ'}$ is a common side, so $\triangle ABQ' \cong QBQ'$ by side-angle-side. Therefore, $\angle AQ'B = \angle QQ'B = \alpha$, since $\overline{Q'B}$ is the perpendicular bisector of the isoceles triangle $\triangle AQ'Q$.

By alternating interior angles, $\angle Q'QR = \angle QQ'B = \alpha$.

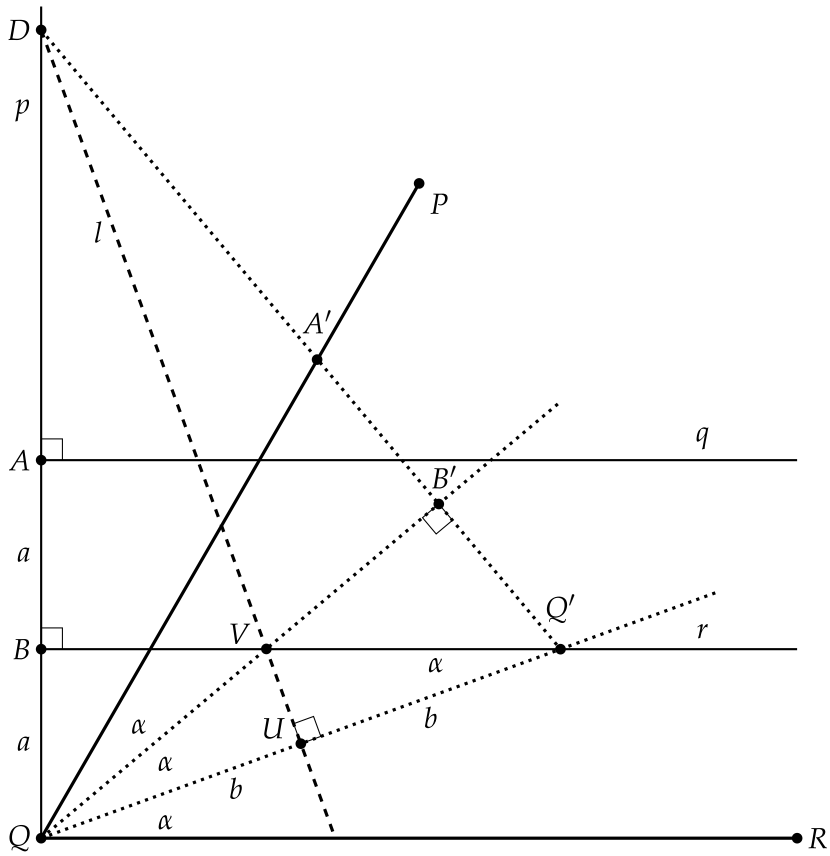
By reflection, $\triangle AQ'Q \cong \triangle A'QQ'$.[3]

> The fold l is the perpendicular bisector of both $\overline{AA'}$ and $\overline{QQ'}$; drop perpendiculars from A and A' to $\overline{QQ'}$; then $\overline{AQ} = \overline{A'Q'}$ follows by congruent right triangles. $\overline{AA'Q'Q}$ is an isoceles trapezoid so its diagonals are equal $\overline{AQ'} = \overline{A'Q}$.

Therefore, $\overline{QB'}$, the reflection of $\overline{Q'B}$, is the perpendicular bisector of an isoceles triangle and $\angle A'QB' = \angle Q'QB' = \angle QQ'B = \alpha$.

[3]The two triangles have been emphasized using different patterns of dashes and dots, as well as using color.

10.3 Second proof

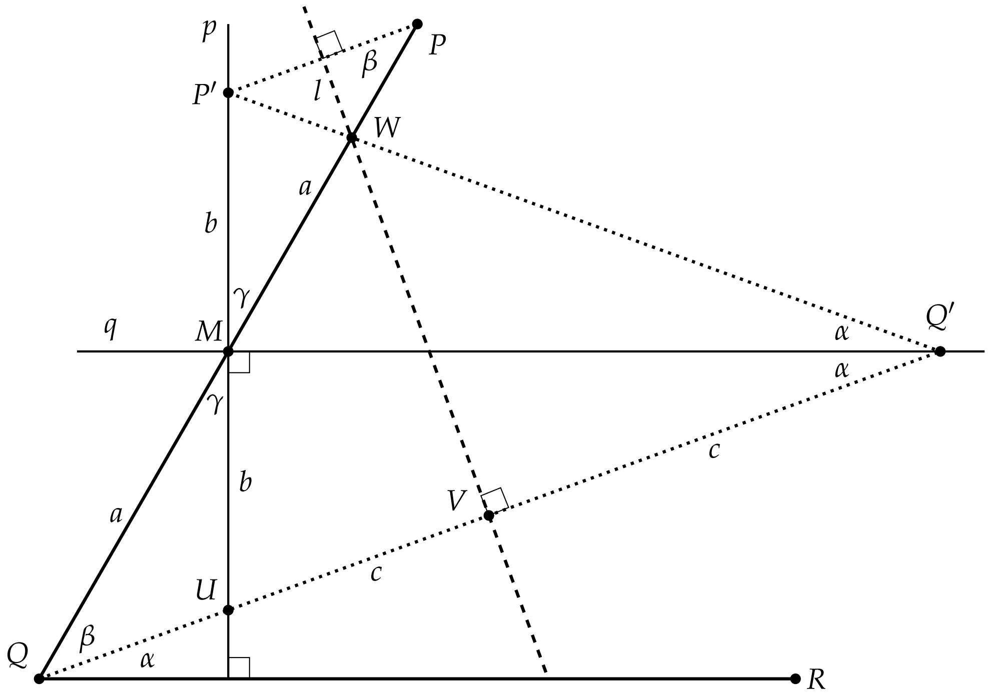


Since l is a fold, it is the perpendicular bisector of $\overline{QQ'}$. Denote the intersection of t with $\overline{QQ'}$ by U, and its intersection with $\overline{QB'}$ by V. $\triangle VUQ \cong \triangle VUQ'$ by side-angle-side since $\overline{VU}$ is a common side. $\angle VQU = \angle VQ'U = \alpha$ and then $\angle Q'QR = \angle VQ'U = \alpha$ by alternating interior angles.

As in Proof 1, A', B', Q' are all reflections around l, so they are all on one line $\overline{DQ'}$, and $\overline{A'B'} = \overline{AB} = \overline{BQ} = \overline{B'Q'} = a$. Then $\triangle A'B'Q = \triangle Q'B'Q$ and $\angle A'QB' = \angle Q'QB' = \alpha$.

11 Martin's trisection of an angle

11.1 The construction



Given the acute angle $\angle PQR$, let M be the midpoint of $\overline{PQ}$. Construct p the perpendicular to $\overline{QR}$ through M and construct q perpendicular to p through M. q is parallel to $\overline{QR}$.

Using Axiom 6, construct a fold l that places P at P' on p and Q at Q' on q. More than one fold may be possible; choose the one that intersects $\overline{PM}$.

Draw the lines $\overline{PP'}$ and $\overline{QQ'}$. They are bisected by the fold. Denote the intersection of $\overline{QQ'}$ with p by U and its intersection with l by V. Draw $P'Q'$.

11.2 Proof

By alternate interior angles $\angle Q'QR = \angle MQ'Q$. Since l is the perpendicular bisector of both PP' and QQ', $PP' \parallel QQ'$, and by alternate interior angles $\angle P'PQ = \angle PQQ' = \beta$.

$\triangle QMU \cong \triangle PMP'$ by angle-side-angle: M is the midpoint of $\overline{PQ}$ so $\overline{QM} = \overline{MP} = a$; we have shown that $\angle P'PM = \angle MQU = \beta$; $\angle QMU = \angle PMP'$ are vertical angles. Therefore, $\overline{P'M} = \overline{MU} = b$.

$\triangle P'MQ' \cong \triangle UMQ'$ by side-angle-side: we have shown that $\overline{P'M} = \overline{MU} = b$; the angles at M are right angles; $\overline{MQ'}$ is a common side. Since the altitude of the isoceles triangle $\triangle P'Q'U$ is the bisector of $\angle P'Q'U$, $\angle P'Q'M = \angle UQ'M = \alpha$.

$\triangle QWV \cong \triangle SWV$ by side-angle-side: $\overline{QV} = \overline{VQ'} = c$; the angles at V are right angles since l is the perpendicular bisector of $\overline{QQ'}$; $\overline{VW}$ is a common side. Therefore, $\angle WQV = \beta = \angle WQ'V = 2\alpha$. We have $\angle PQR = \beta + \alpha = 2\alpha + \alpha = 3\alpha$ so $\overline{QQ'}$ trisects $\angle PQR$.

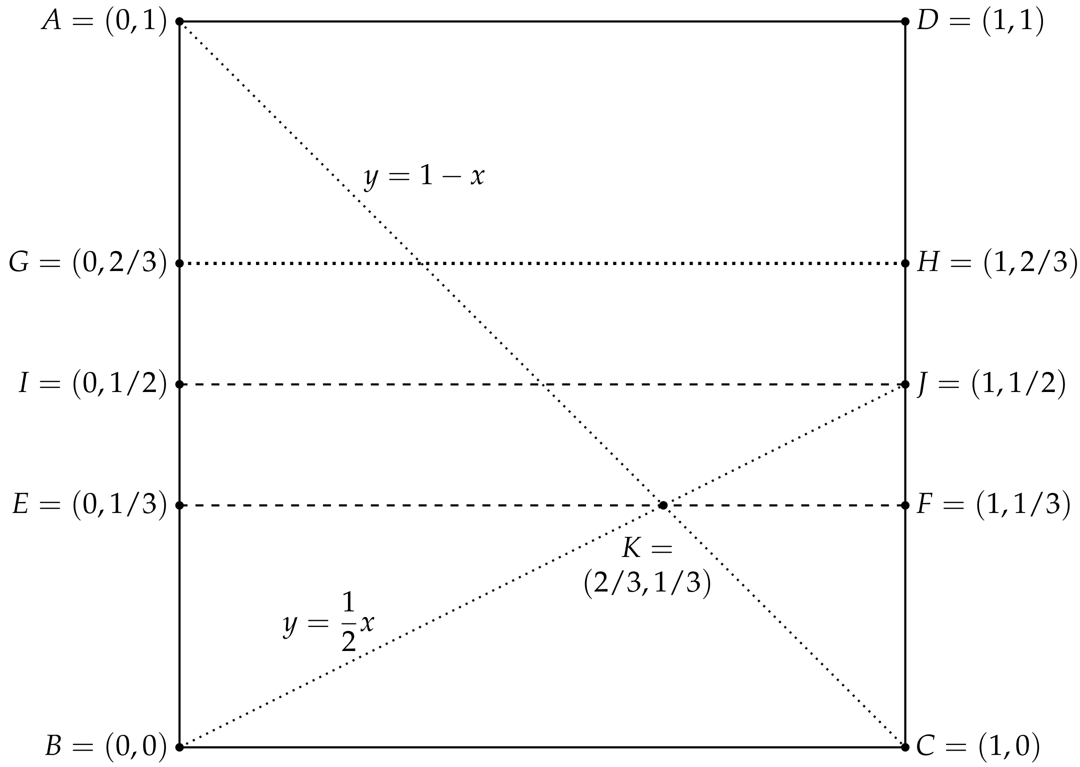
12 Messer's doubling of a cube

A cube of volume V has sides of length $\sqrt[3]{V}$. The volume of a cube with twice the volume is $2 \cdot V$, so we need to construct the length $\sqrt[3]{2 \cdot V} = \sqrt[3]{2} \cdot \sqrt[3]{V}$. If we can construct $\sqrt[3]{2}$, we can multiply by the given length $\sqrt[3]{V}$ to double the cube.

12.1 Dividing a length into thirds

Lang [4] shows efficient constructs for obtaining rational fractions of the length of the side of a square (piece of paper). Here, we need to divide the side of the square into thirds.

First, fold in half to locate the point $J = (1, 1/2)$. Next, draw the lines $\overline{AC}$ and $\overline{BJ}$.



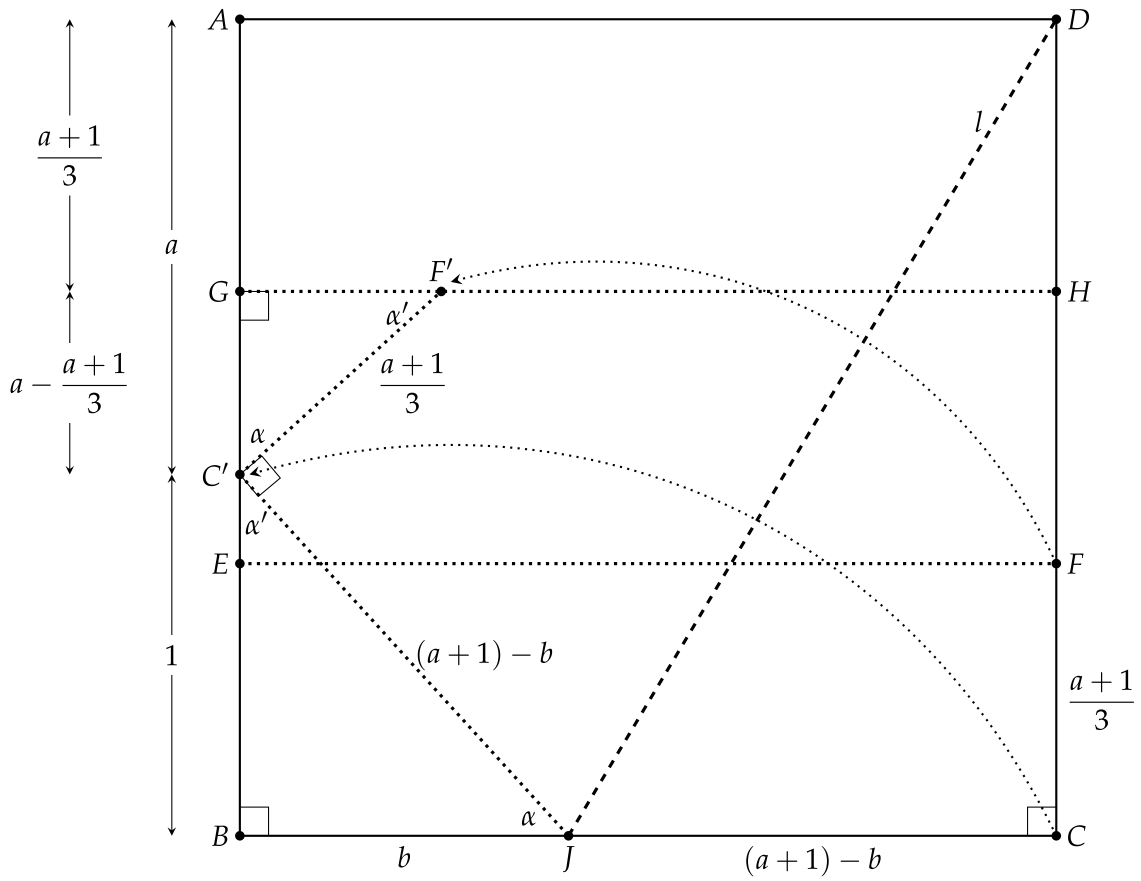
The coordinates of their point of intersection K is obtained by solving the two equations:

$$\begin{aligned} y &= 1 - x \\ y &= \frac{1}{2}x\,. \end{aligned}$$

The result is $x = 2/3, y = 1/3$.

Construct the line $\overline{EF}$ perpendicular to $\overline{AB}$ that goes K, and construct the reflection $\overline{GH}$ of $\overline{BC}$ around $\overline{EF}$. The side of the square has now been divided into thirds.

12.2 Computing $\sqrt[3]{2}$



Label the side of the square by $a+1$. We will show that $a = \sqrt[3]{2}$.

Using Axiom 6 place C at C' on $\overline{AB}$ and F at F' on $\overline{GH}$. Denote by J the point intersection of the fold with $\overline{BC}$ and denote by b the length of $\overline{BJ}$. The length of $\overline{JC}$ is $(a+1)-b$.

When the fold is performed, the line segment $\overline{JC}$ is reflected onto the line segment $\overline{C'J}$ of the same length, and $\overline{CF}$ is folded onto the line segment $\overline{C'F'}$ of the same length. A simple computation shows that the length of $\overline{GC'}$ is $a - \frac{a+1}{3} = \frac{2a-1}{3}$. Finally, since $\angle FCJ$ is a right angle, so is $\angle F'C'J$.

$\triangle C'BJ$ is a right triangle so by Pythagoras's theorem:

$$\begin{aligned}
1^2 + b^2 &= ((a+1)-b)^2 \\
&= a^2 + 2a + 1 - 2(a+1)b + b^2 \\
0 &= a^2 + 2a - 2(a+1)b \\
b &= \frac{a^2+2a}{2(a+1)}\,.
\end{aligned}$$

$\angle GC'F' + \angle F'C'J + \angle JC'B = 180°$ since they form the straight line $\overline{GB}$. Denote $\angle GC'F'$ by α.

$$\angle JC'B = 180° - \angle F'C'J - \angle GC'F' = 180° - 90° - \angle GC'F' = 90° - \angle GC'F = 90° - \alpha\,,$$

which we denote by α'. The triangles $\triangle C'BJ$, $\triangle F'GC'$ are right triangles, so $\angle C'JB = \alpha$ and $\angle C'F'G = \alpha'$. Therefore, the triangles are similar and we have:

$$\frac{b}{(a+1)-b} = \frac{\frac{2a-1}{3}}{\frac{a+1}{3}}.$$

Substituting for b:

$$\begin{aligned}
\frac{\frac{a^2+2a}{2(a+1)}}{(a+1)-\frac{a^2+2a}{2(a+1)}} &= \frac{2a-1}{a+1} \\
\frac{a^2+2a}{a^2+2a+2} &= \frac{2a-1}{a+1}
\end{aligned}$$

Simplifying results in $a^3 = 2$, $a = \sqrt[3]{2}$.

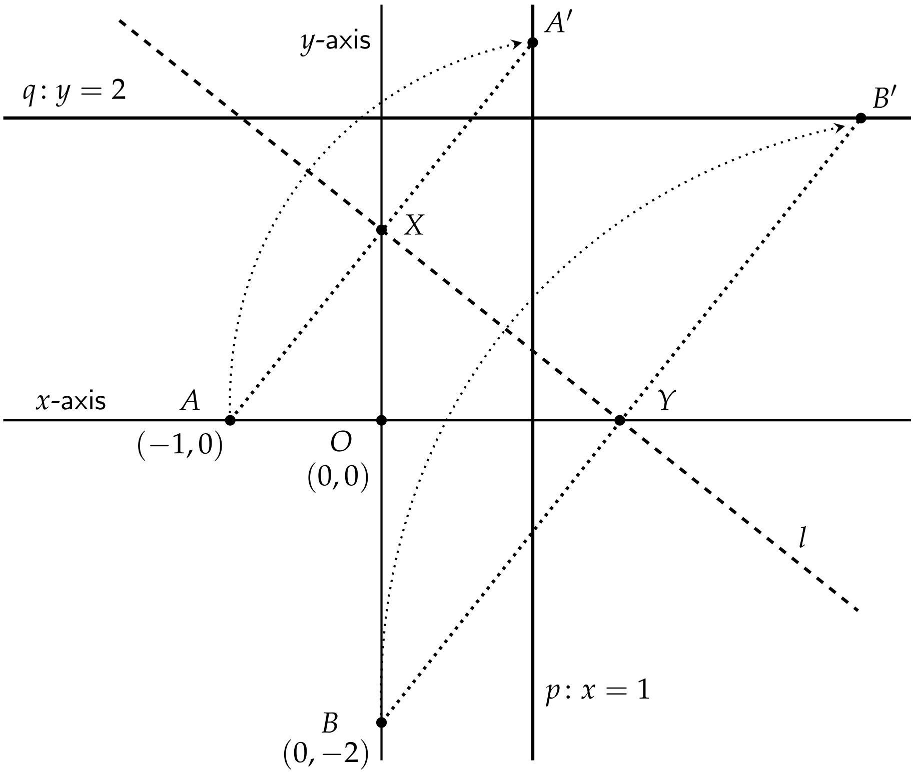
13 Beloch's doubling of a cube

In 1936 Margharita P. Beloch was the first to formalize Axiom 6 (often called the *Beloch fold* in her honor) and to show that it could be used to solve cubic equations. Here we give her construction for doubling the cube.

13.1 The construction

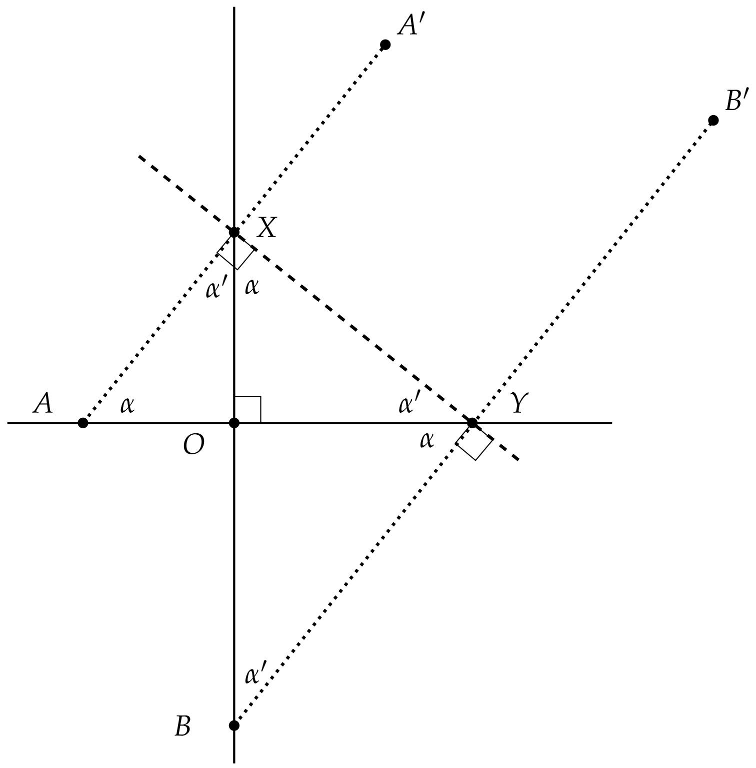
Place point A at $(-1, 0)$ and point B at $(0, -2)$. Let p be the line with equation $x = 1$ and let q be the line with equation $y = 2$.

Using Axiom 6 construct a fold l that places A at A' on p and B at B' on q. Denote the intersection of the fold and the y-axis by X and the intersection of the fold and x-axis by Y.



13.2 Proof

Let us extract a simplified diagram:



The fold is the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$. Therefore, $\angle AXY$ and $\angle XYB$ are right angles and $\overline{AA'}$ is parallel to $\overline{BB'}$. By alternate interior angles $\angle XAO = \angle BYO = \alpha$. If an acute angle in a right triangle is α, the other acute angle must be $90^\circ - \alpha$, which we denote α'. The labeling of the angles in all the triangles in the diagram follows immediately.

We have three similar triangles $\triangle AOX \sim \triangle XOY \sim \triangle YOB$. $\overline{OA} = 1, \overline{OB} = 2$ are given, so:

$$\frac{\overline{OX}}{\overline{OA}} = \frac{\overline{OY}}{\overline{OX}} = \frac{\overline{OB}}{\overline{OY}}$$
$$\frac{\overline{OX}}{1} = \frac{\overline{OY}}{\overline{OX}} = \frac{2}{\overline{OY}}$$
$$\overline{OX}^2 = \overline{OY} = \frac{2}{\overline{OX}}\,,$$

resulting in $\overline{OX}^3 = 2$ and $\overline{OX} = \sqrt[3]{2}$.

14 References

The following references were used in the preparation of this document.

The axioms are given in the Wikipedia article [8], together with parametric equations for the first five axioms. Lee [5, Chapter 4] is a good overview of the mathematics of origami, while Martin [6, Chapter 10] is a formal development. Lang [4] shows how rational numbers, some irrational numbers and approximations to others can be constructing in origami. Trisecting an angle and doubling a cube are described by [7] and Ben-Lulu [1] provides a different proof of the trisection. The construction for doubling the cube is from Newton [7] and Lee [5]. Hull [3] presents Beloch's work on solving cubic equations with origami.

[1] Oriah Ben-Lulu. Angle trisections in various axiom systems. Weizmann Institute of Science, 2020. (in Hebrew).

[2] Ann Xavier Gantert. *Geometry*. Perfection Learning, 2008.

[3] Thomas C. Hull. Solving cubics with creases: The work of Beloch and Lill. *American Mathematical Monthly*, 118:307–315, 2011.

[4] Robert J. Lang. Origami and geometric constructions. `http://langorigami.com/wp-content/uploads/2015/09/origami_constructions.pdf`, 1996-–2015. Accessed 26/02/2020.

[5] Hwa Young Lee. Origami-constructible numbers. Master's thesis, University of Georgia, 2017.

[6] George E. Martin. *Geometric Constructions*. Springer, 1998.

[7] Liz Newton. The power of origami. `https://plus.maths.org/content/power-origami`. Accessed 26/02/2020.

[8] Wikipedia contributors. Huzita–Hatori axioms — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Huzita%E2%80%93Hatori_axioms&oldid=934987320`, 2020. Accessed 26/02/2020.

A GeoGebra links

Axiom 1	`https://www.geogebra.org/m/fq9d5hms`
Axiom 2	`https://www.geogebra.org/m/fgmfss27`
Axiom 3	`https://www.geogebra.org/m/ek3mqupw`
Axiom 4	`https://www.geogebra.org/m/renzzbdg`
Axiom 5	`https://www.geogebra.org/m/aszn9ywu`
Axiom 6	`https://www.geogebra.org/m/bxe5e5ku`
Axiom 7	`https://www.geogebra.org/m/yeq5gmeg`
Abe's trisection	`https://www.geogebra.org/m/dxrcvjam`
Martin's trisection	`https://www.geogebra.org/m/caky7edd`
Messer's doubling of the cube	`https://www.geogebra.org/m/mrcwjqh8`
Beloch's doubling of the cube	`https://www.geogebra.org/m/enzmmwua`

Due to a bug in Geogebra, in projects that use Axiom 6, points defined by reflection around the common tangent are not saved or are saved incorrectly.

B Derivation of the trigonometric identities

The trigonometric identifies for tangent used in the proof of Axiom 3 can be derived from identifies for the sine and cosine:

$$\begin{aligned}
\tan(\theta_1+\theta_2) &= \frac{\sin(\theta_1+\theta_2)}{\cos(\theta_1+\theta_2)} \\
&= \frac{\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2} \\
&= \frac{\sin\theta_1+\cos\theta_1\tan\theta_2}{\cos\theta_1-\sin\theta_1\tan\theta_2} \\
&= \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2}\,.
\end{aligned}$$

We use this formula with $\theta = (\theta/2) + (\theta/2)$ to obtain a quadratic equation in $\tan(\theta/2)$:

$$\tan\theta = \frac{\tan(\theta/2)+\tan(\theta/2)}{1-\tan^2(\theta/2)}$$

$$\tan\theta\,(\tan(\theta/2))^2 \;+\; 2\,(\tan(\theta/2)) \;-\; \tan\theta = 0\,.$$

Its solutions are:

$$\tan(\theta/2) = \frac{-1\pm\sqrt{1+\tan^2\theta}}{\tan\theta}\,.$$

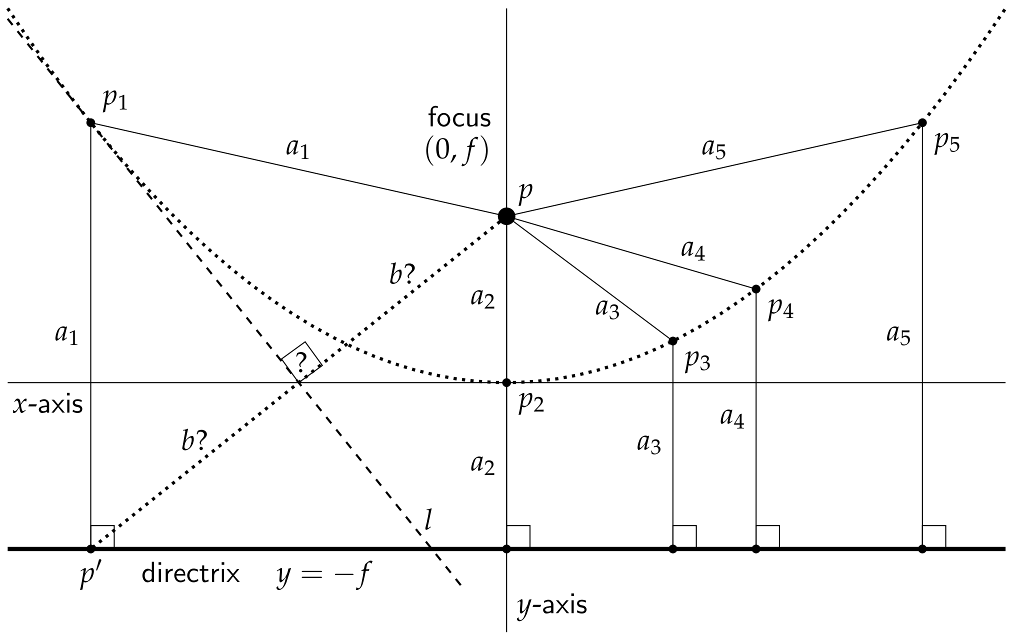
C Parabolas

Students are usually introduced to parabolas as the graphs of second degree equations:

$$y = ax^2 + bx + c\,.$$

However, parabolas can be defined geometrically: given a point, the *focus*, and a line, the *directrix*, the locus of points equidistant from the focus and the directrix defines a parabola.

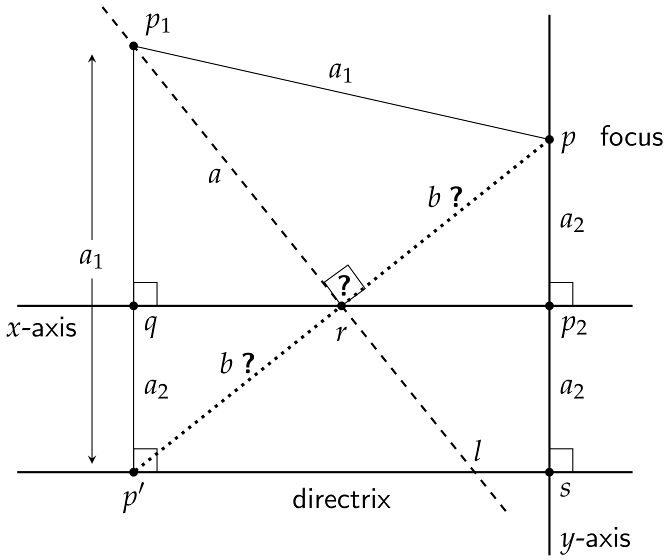
The following diagram shows the focus—the large point at $p = (0, f)$, and the directrix—the thick line whose equation is $y = -f$. The resulting parabola is shown as a dotted curve. Its vertex p_2 is at the origin of the axes.



We have selected five points p_i, $i = 1, \ldots, 5$ on the parabola. Each point p_i is at a distance of a_i both from the focus and from the directrix.

Consider the point p' that is the intersection of the perpendicular from p_1 to the directrix. Since p_1 is on the parabola $\overline{p'p_1} = \overline{p_1p} = a_1$. We claim that the tangent l to the parabola at p_1 (dashed line) is a fold that reflects p onto p'.

We have to prove the l is the perpendicular bisector of $\overline{pp'}$. Let us extract a simplified diagram:



- The directrix is parallel to the x-axis, the focus p is on the y-axis and $\overline{p_1p'}$ is perpendicular to the directrix. Therefore, $\angle p'qr$ and $\angle pp_2r$ are right angles.
- $\overline{qp'}$ and $\overline{p_2s}$ are opposite sides of a rectangle, so $\overline{qp'} = \overline{p_2s}$, which in turn is equal to $\overline{pp_2}$ since p_2 is on the parabola and thus equidistant from p and s.
- $\angle qrp'$ and $\angle p_2rp$ are equal vertical angles.
- The right triangles $\triangle qrp'$ and $\triangle p_2rp$ have one acute angle equal and one side equal so they are congruent. Therefore, $\overline{p'r} = \overline{rp}$ and $\overline{p_1r}$ is the median of $\triangle pp_1p'$.
- p_1 is on the parabola so $\overline{pp_1} = \overline{p_1p'}$. Therefore, $\triangle pp_1p'$ is an isoceles triangle.
- In the isoceles triangle $\triangle pp_1p'$, the median $\overline{p_1r}$ is also the perpendicular bisector of $\overline{pp'}$.
- Line l contains the line segment $\overline{p_1r}$ and is the perpendicular bisector of $\overline{pp'}$.