33. Mascheroni's Compass Problem

> This document is Problem 33 from the book by Heinrich Dörrie: *100 Problems of Elementary Mathematics: Their History and Solution* (Dover, 1965), as reworked by Michael Woltermann.[1] I have added explanations so that students and teachers can better understand the construction. The document has been written and formatted in LaTeX, and I have redrawn the diagrams using TikZ, adding auxiliary lines and drawing diagrams incrementally for clarity.
>
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Prove that any construction that can be carried out with a compass and straight-edge can be carried out with the compass alone. The Italian L. Mascheroni (1750-1800) posed this problem to himself and solved it in a masterly fashion in his book *La geometria del compasso*, published in Pavia in 1797.

> The theorem is currently known as the Mohr-Mascheroni Theorem since it had been proved in 1672 by the Danish mathematician Georg Mohr, but his work was not widely known until the twentieth century.[3]

When we examine the separate steps by which circle and straight-edge constructions are carried out, we see that every step consists of one of the following three basic constructions:

I. Finding the point of intersection of two straight lines;

II. finding the point of intersection of a straight line and a circle;

III. finding the point(s) of intersection of two circles.

Thus we need only show that the two basic constructions I. and II. can be done with a compass alone. (Mascheroni regarded a straight line as given if two of its points are known.)

First we will solve four preliminary problems. (Dörrie talks about two, but the others are embedded in these.) In the following:

- $C(O, A)$ stands for the circle with center O through point A,
- $C(O, AB)$ stands for the circle with center O and radius AB.

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[1] `http://www2.washjeff.edu/users/mwoltermann/Dorrie/DorrieContents.htm`. I would like to thank him for giving me permission to use his work.

[2] `http://www.weizmann.ac.il/sci-tea/benari/`.

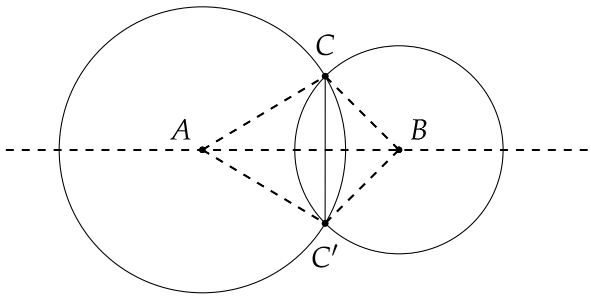
[3] A different proof can be found in: Norbert Hungerbühler. A short elementary proof of the Mohr-Mascheroni theorem. *American Mathematical Monthly* 101(8), 1994, 784–787.

Prelim 1. Reflect a point C about the line through A and B.

> Given a point C and a line AB, a reflection of C about AB is a point C' such that AB is the perpendicular bisector of the line CC'.

Solution. The reflection $C' = c(A,C) \cap c(B,C)$ (not C in general):

> The phrase "not C in general" rules out the possibility that C is on the line containing AB, in which case there is nothing to do.



Line AB is the perpendicular bisector of chord CC' of both circles

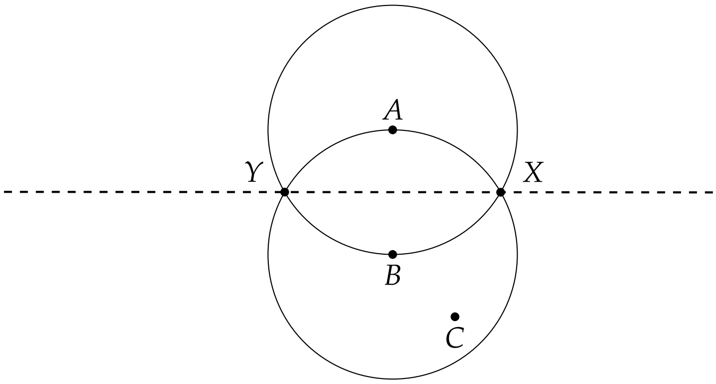
Note: Dashed lines in figures are drawn to explain the arguments, but are not used in constructions. $\square$

> The lines drawn in the diagrams serve *only* to illustrate the proofs. You must convince yourself that a compass alone is used in all the constructions.
>
> Proof that C' is the reflection of C: $\triangle ABC$ and $\triangle ABC'$ are congruent by side-side-side since AC, AC' are radii of the same circle as are BC, BC', and AB is a common side. Therefore, $\angle CAB = \angle C'AB$ so AB is the angle bisector of $\angle CAC'$. Since $\triangle CAC'$ is an isosceles triangle, the angle bisector AB is also the perpendicular bisector of the base of the triangle CC', and by definition C' is the reflection of C around AB.

Prelim 2. Construct $c(A, BC)$, given points A, B, C.

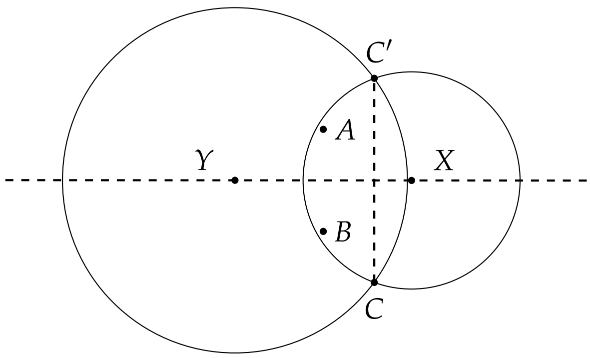
> You might ask yourself why this is a problem. Simply place the two legs of the compass on B and C, and then move the compass so that one leg is on A and draw the circle which will have radius BC. The reason is that the compass used by Euclid is a "collapsing" compass whose legs cannot maintain a distance when the compass is raised off the paper. Euclid proved what is now called the "Compass Equivalence Theorem": any construction with a fixed compass can be done with a collapsing compass. If effect, Prelim. 2 is a proof of this theorem that uses only a compass.

Solution. Let X and Y be the points of intersection of $c(A, B)$ and $c(B, A)$:

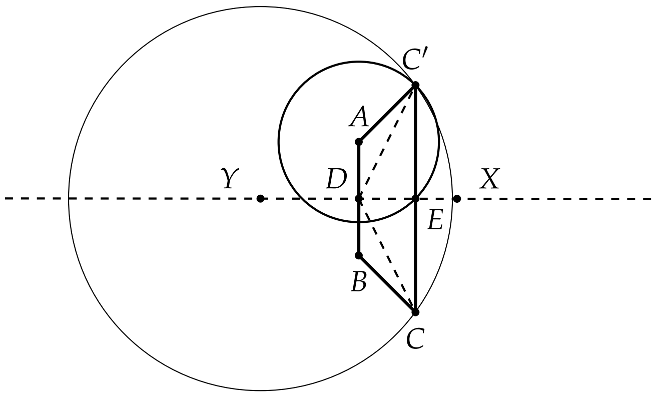


Let C' be the reflection of C about line XY:

> By Prelim. 1, this construction can be done using only a compass.



$c(A, C')$ is the desired circle:

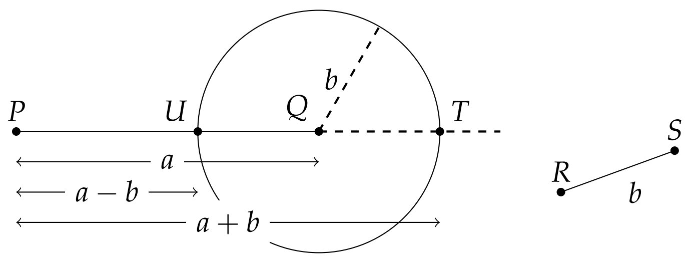


Since A is the reflection of B about XY, and reflection preserves distance, so $AC' = BC$.

> A is the reflection of B around XY, and C' is the reflection of C around the XY. By definition XY is the perpendicular bisector of CC' and AB, so $C'E = EC$, $AD = DB$ and $\angle DEC = \angle DEC' (= 90°)$. $\triangle DEC$ is congruent to $\triangle DEC'$ by side-angle-side, so $DC = DC'$ and $\angle ADC' = \angle BDC$ (they are complementary to $\angle C'DE$, $\angle CDE$). Therefore, $\triangle ADC'$ is congruent to $\triangle BDC$, so $AC' = BC$.

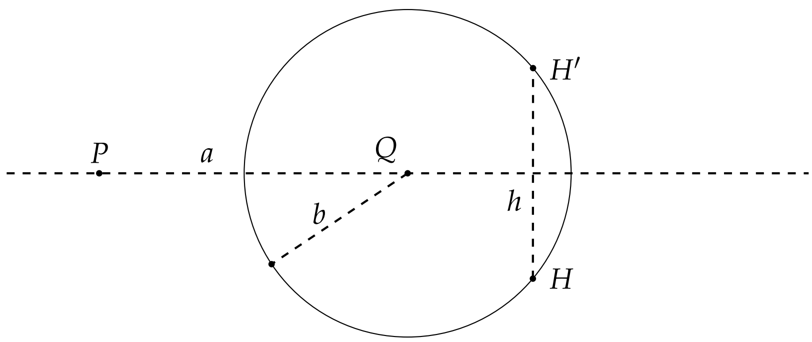
Prelim 3. Construct the sum or difference of two given segments a and b, i.e., lengthen or shorten a given segment $PQ = a$ by a segment $QX = b$. (See Prelim 2 if necessary to construct a segment of length b at Q.)

This would be trivial if we had a straight-edge. Simply extend the line segment PQ of length a with the straight-edge, copy line segment RS of length b so that Q is one end point and construct a circle of radius b. The intersections of the circle with the ray give the required line segments PU of length $a - b$ and PT of length $a + b$:

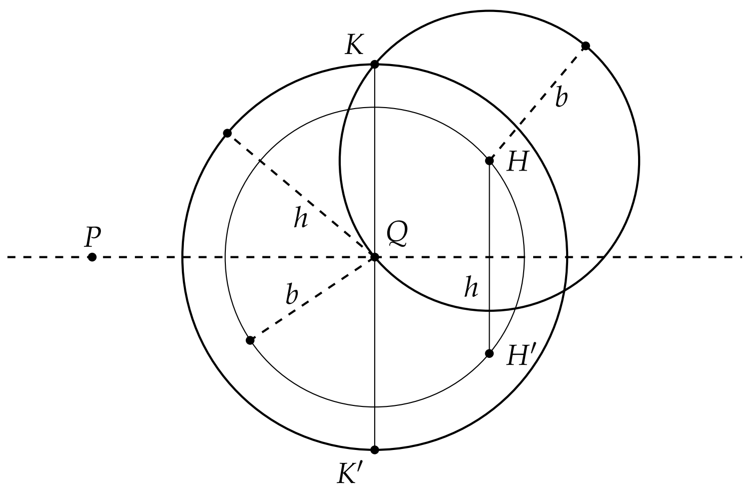


Solution.

1. Let H be any point on $c(Q, b)$, and H' its reflection about line PQ. Let h be the (length of) segment HH':

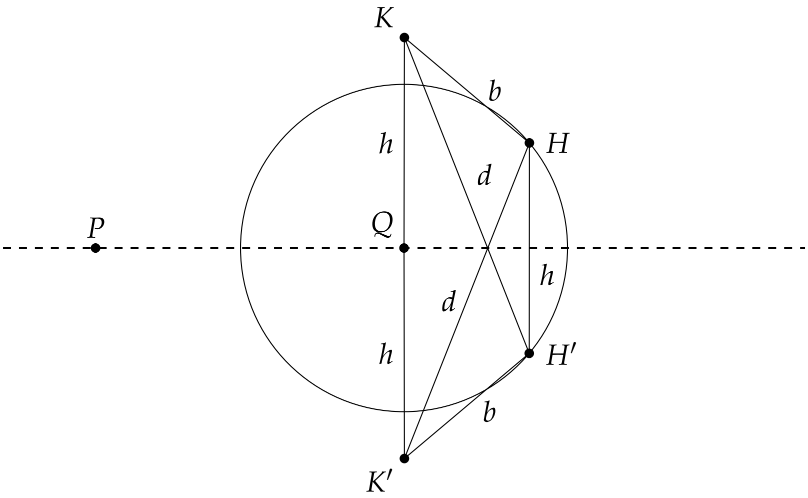


2. Let $K = c(Q, h) \cap c(H, b)$ and K' be the reflection of K about line PQ:



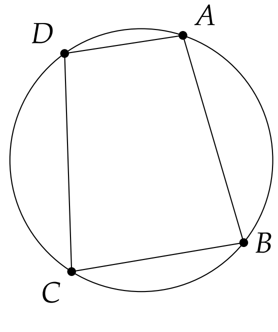
Then $KHH'K'$ is an isosceles trapezoid with legs $KH = K'H' = b$ and base $KK' = 2h$. Let $d = KH' = K'H$:

> We also have $h = HH'$ since H' is a reflection of H. Since K' is a reflection of K and reflections preserve distance, $KH = K'H'$, defined to be d.

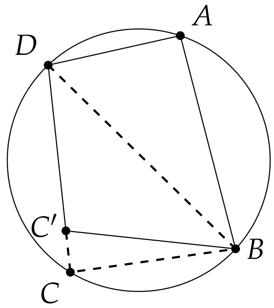


Since opposite angles of $KHH'K'$ are supplemental, $KHH'K'$ is a cyclic quadrilateral, i.e., it can be inscribed in a circle.

> Geometry textbooks give the simple proof that the opposite angles of a cyclic quadrilateral are supplementary (add up to 180°), but it is hard to find a proof of the converse, so I present both proofs here.
>
> **Opposite angles of a cyclic quadrilateral are supplementary:** An inscribed angle equals half the subtended arc, so $\angle DAB$ is half of the arc DCB and $\angle DCB$ is half of the arc DAB. But the two arcs form the entire circumference of the circle so their sum is $360°$ and $\angle DAB + \angle DCB = \frac{1}{2} \cdot 360° = 180°$, and similarly $\angle ADC + \angle AABC = 180°$



> **Quadrilateral whose opposite angles are supplementary is cyclic:** Inscribe the triangle $\triangle DAB$ in a circle (the is possible for any triangle) and suppose that C' is a point such that $\angle DAB + \angle DC'B = 180°$ but C' is *not* on the circumference of the circle. Without loss of generality, let C' be within the circle:

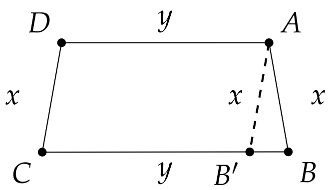


Construct a ray that extends DC' and let C be its intersection with the circle. By the forward direction of the theorem:

$$\begin{aligned}
\angle DAB + \angle DCB &= 180^\circ \\
\angle DAB + \angle DCB &= \angle DAB + \angle DC'B \\
\angle DCB &= \angle DC'B\,,
\end{aligned}$$

which is impossible if C is on the circle and C' is inside the circle.

Finally, we show that the opposite angles of an isosceles trapezoid are supplementary and therefore it is cyclic:



Construct the line AB' parallel to CD. $AB'CD$ is a parallelogram and $\triangle ABB'$ is an isosceles triangle, so $\angle C = \angle AB'B = \angle B$. Similarly, $\angle A = \angle D$. Since the sum of the internal angles of any quadrilateral is equal to 360°:

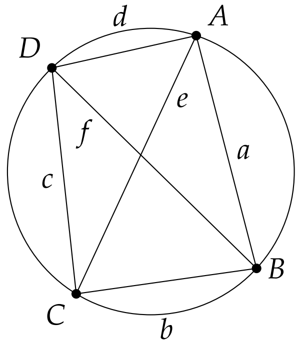
$$\begin{aligned}
\angle A + \angle B + \angle C + \angle D &= 360^\circ \\
2\angle A + 2\angle C &= 360^\circ \\
\angle A + \angle C &= 180^\circ\,,
\end{aligned}$$

and similarly $\angle B + \angle D = 180^\circ$.

Then by Ptolemy's theorem $d^2 = b^2 + 2h^2$.

Ptolemy's theorem states that for a quadrilateral inscribed in a circle, the following equality relates the lengths of the sides a, b, c, d and the lengths of the diagonals e, f:

$$ef = ac + bd\,.$$



There is a geometric proof of the theorem (see Wikipedia), but I will present a simple trigonometric proof. The law of cosines for the four triangles $\triangle ABC$, $\triangle ADC$, $\triangle DAB$, $\triangle DCB$ gives the following equations:

$$\begin{aligned} e^2 &= a^2 + b^2 - 2ab\cos\angle B \\ e^2 &= c^2 + d^2 - 2cd\cos\angle D \\ f^2 &= a^2 + d^2 - 2ad\cos\angle A \\ f^2 &= b^2 + c^2 - 2bc\cos\angle C\,. \end{aligned}$$

$\angle C = 180° - \angle A$ and $\angle D = 180° - \angle B$ because they are opposite angles of an inscribed quadrilateral, and

$$\begin{aligned} \cos\angle D &= -\cos\angle B \\ \cos\angle C &= -\cos\angle A\,. \end{aligned}$$

We can eliminate the cosine term from the first two equations and from the last two equations. After some messy arithmetic, we get:

$$\begin{aligned} e^2 &= \frac{(ac+bd)(ad+bc)}{(ab+cd)} \\ f^2 &= \frac{(ab+cd)(ac+bd)}{(ad+bc)}\,. \end{aligned}$$

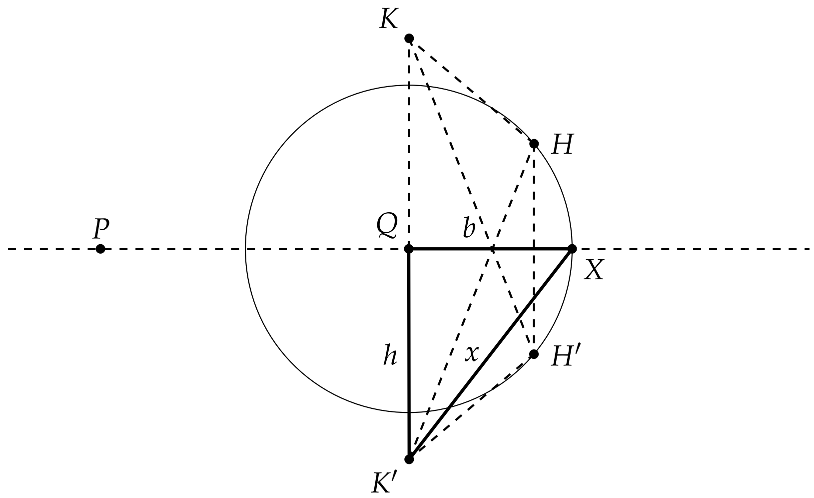
Multiply the two equations and simplify to get Ptolemy's theorem:

$$\begin{aligned} e^2 \cdot f^2 &= (ac+bd)^2 \\ ef &= (ac+bd)\,. \end{aligned}$$

For the construction on page 5, the diagonals are of length d, the legs are of length b, and the bases are of lengths h and $2h$, so Ptolemy's theorem gives $d \cdot d = b \cdot b + h \cdot 2h$ or $d^2 = b^2 + 2h^2$.

Let X be the point on line PQ that extends PQ by b. (We will eventually construct X; now we're just imagining it.)

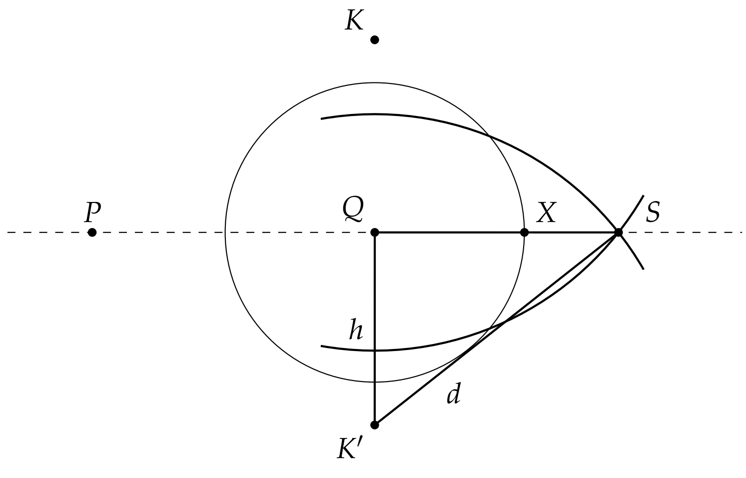
Let $x = K'X$. Since $\triangle QK'X$ is a right triangle, $x^2 = b^2 + h^2$:



It follows then that $d^2 = x^2 + h^2$ so that x is a leg of a right triangle with hypotenuse d, the other leg being h.

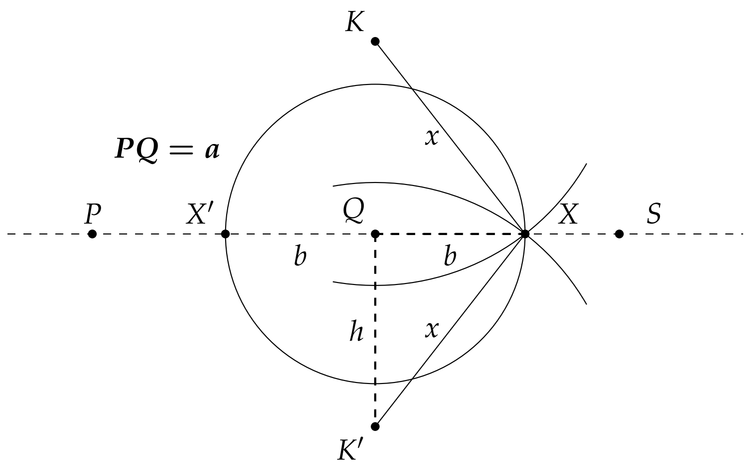
> By Ptolemy's theorem, $d^2 = b^2 + 2h^2$, so $d^2 = (x^2 - h^2) + 2h^2 = x^2 + h^2$. All the above sentence is saying is that it is *possible* to build a right triangle with sides x, h, d; such a triangle does not appear in the above diagram.

Now let $S = c(K, d) \cap c(K', d)$:



$QS^2 + h^2 = d^2$, so $QS = x$:

3. Then $X = c(K, x) \cap c(K', x)$:



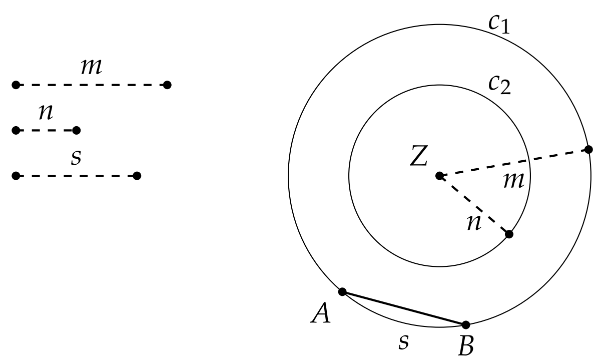
There are two Xs, one for $a+b$ and one for $a-b$. $\square$

> Recall that we want to extend PQ of length a by a length b, or decrease its length by b. Since the length of QX is $\sqrt{x^2-h^2}=b$, the length of PX is $a+b$ and the length of PX' is $a-b$.

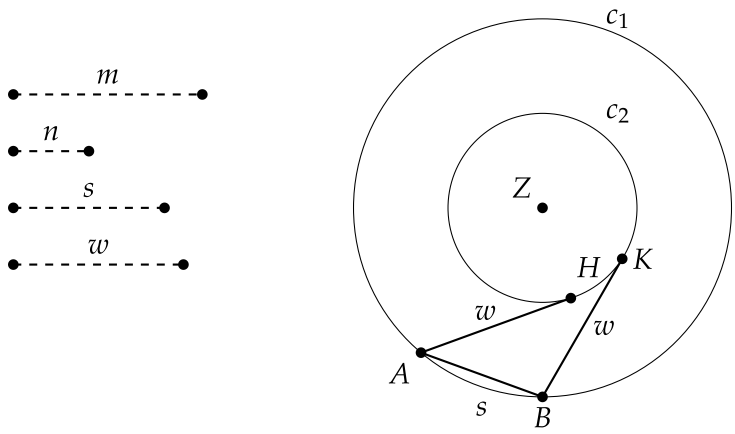
Prelim 4. Given segments of length n, m, s, construct a segment of length $x = \frac{n}{m}s$.

Solution. This solution by Mascheroni is remarkable for its brevity and simplicity. Draw two concentric circles $c_1 = c(Z, m)$ and $c_2 = c(Z, n)$ and chord $AB = s$ on c_1. (It is assumed that s falls within c_1. If not, use Prelim 3 to replace n and m by sufficiently large integer multiples $kn = N$ and $km = M$.)

> There is an implicit assumption that $m > n$. If not, just exchange the notation.
>
> The expression "it is assumed that s falls within c_1" refers to the possibility that s is within c_1 but also cuts through c_2. By using multiples this can be avoided.

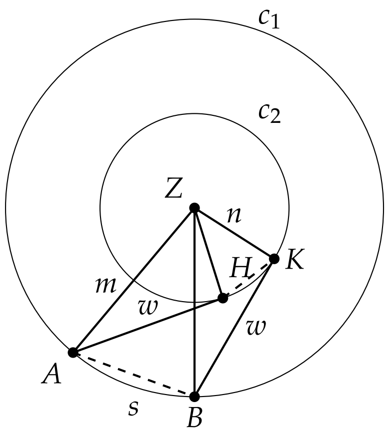


Next lay off any length w from A and B on c_2 with H and K on c_2 so that $AH = BK = w$:



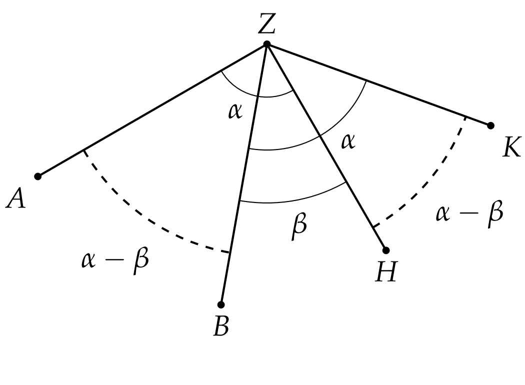
$\triangle AHZ$ and $\triangle BZK$ are congruent by SSS,

> The sides are $ZA = ZB = m$ (radius of circle c_1), $ZH = ZK = n$ (radius of circle c_2), $AH = BK = w$ (by construction):

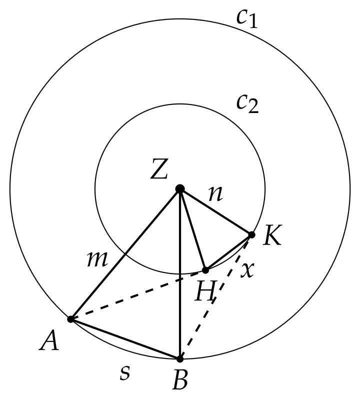


so $\angle AZH = \angle BZK$ and $\angle AZB = \angle HZK$.

> This follows by subtraction of angles, but it is somewhat hard to see in the above diagram. The following diagram clarifies the relation among the angles. Let $\alpha = \angle AZH = \angle BZK$ and $\beta = \angle BZH$; then $\angle AZB = \angle HZK = \alpha - \beta$.



and $\triangle ZAB$ and $\triangle ZHK$ are similar.



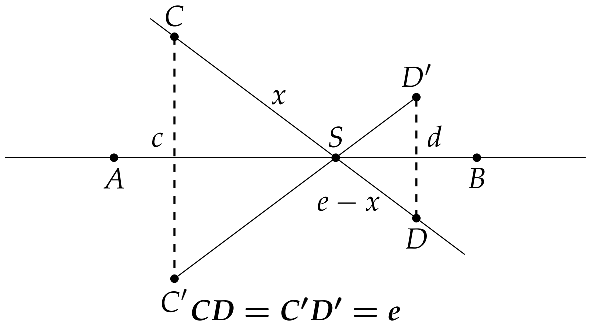
Then $\frac{m}{s} = \frac{n}{x}$ and $x = \frac{n}{m}s$. $\square$

Now for the solutions to I. and II. above.

I′. Find the point of intersection S of two straight lines AB and CD, each of which is given by two points, with compass alone.

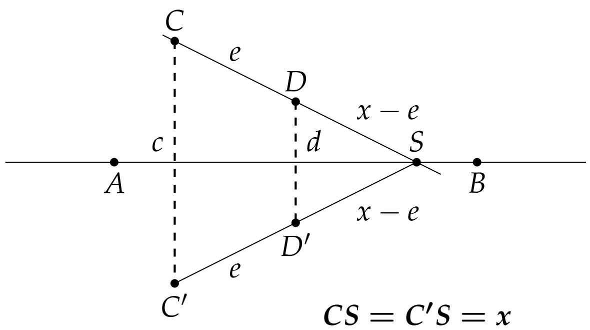
Solution. Let C' and D' be reflections of C and D about line AB respectively. The sought-for point of intersection S then lies on line $C'D'$.

> CD and $C'D'$ intersect AB at the same point S because the reflection around AB preserves distance: $C'S = CS$ and $D'S = DS$.



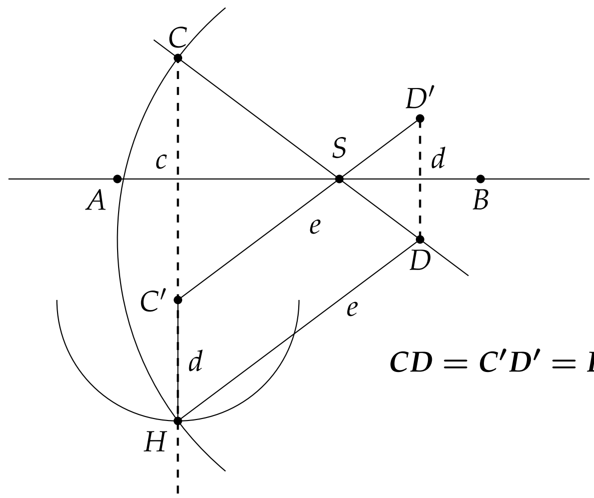
$\triangle CSC'$ and $\triangle DSD'$ are similar so $\frac{CS}{DS} = \frac{CC'}{DD'}$. With $x = CS, c = CC', d = DD'$ and $e = CD$, we get $\frac{x}{e-x} = \frac{c}{d}$ or $x = \frac{c}{c+d}e$. (If D is on the same side of AB as C, then $x = \frac{c}{c-d}e$.)

> Here is the diagram for D on the same side of AB as C:



The similarity of the triangles $\triangle CSC'$ and $\triangle DSD'$ gives $\frac{x}{x-e} = \frac{c}{d}$ and we can solve for $x = \frac{c}{c-d}e$.

$c + d = CH$ where H is the intersection point of $c(C', d)$ and $c(D, e)$ on line CC'.

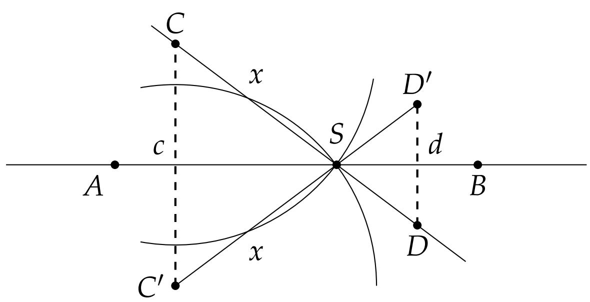


$\boldsymbol{CD = C'D' = DH = e}$

The circle $c(C', d)$ consists of the points at distance d from C'. We claim that H, the intersection of $c(C', d)$ and $c(D, e)$, is on the extension of the line segment CC', so that CH is a line segment of length $CH = CC' + C'H = c + d$.

We previously defined $C'D' = e$ and $D'D = d$. The definition of H as the intersection of the circles $c(C', d)$ and $c(D, e)$ gives $HC' = d$, $DH = e$. Therefore, the quadrilateral $C'D'DH$ is a parallelogram, since the lengths of both pairs of opposite sides are equal. DD' was constructed so that it is parallel to CC', so $C'H$, which is parallel to DD' is also parallel to C'. Since one of its end points is C', it must be on the line containing CC', and the length of CH is $c + d$).

($CH = c - d$ in case D is on the same side of AB as C.) Preliminary problem 4 then allows us to construct x, and from that S as the intersection of arcs of the circles $c(C, x)$ and $c(C', x)$.



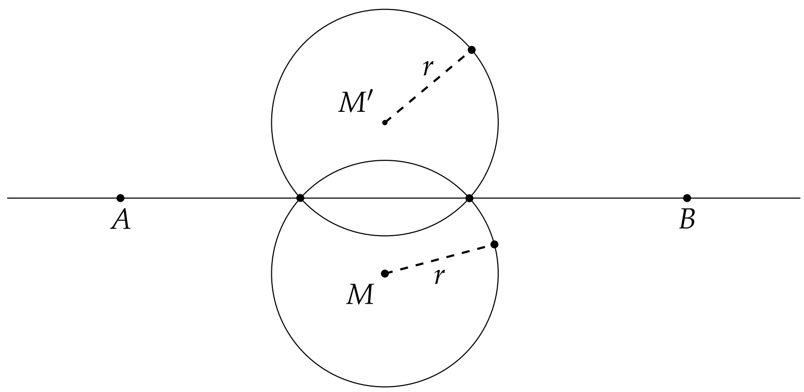
$\boldsymbol{CD = C'D' = DH = e}$

x is the length of CS which equals the length of $C'S$ because reflection preserves distances, so all we have to do is compute x, and then S will be the intersection of the circles $c(C, x), c(C', x)$. By Preliminary problem 4, we can compute $x = \frac{c}{c+d}e$ given c, e, d, where the line segment of length $c + d$ is constructed above as CH.

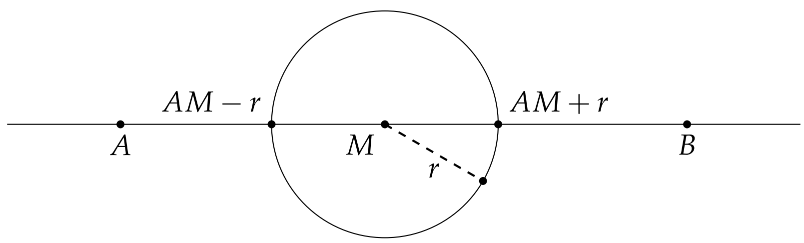
II′. Determine the point of intersection S of a given circle k and a given straight line AB with compass alone.

Solution. Let $k = c(M, r)$, and M' be the reflection of M about line AB.

Recall from Prelim. problem 1 that a reflection can be constructed with compass alone given the points A, B defining a line.



The points of intersection are the points where $c(M, r)$ and $c(M', r)$ intersect. This construction cannot be done if M is on line AB.



In this exceptional case, extend and shorten AM by r by Prelim 3; the end points of the extended and shortened segments are the desired points. □