Are Triangles with the Equal Area and Perimeter Congruent?

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Are triangles with the equal area and equal perimeter congruent? Not necessarily: the triangles with sides $(17, 25, 28)$ and $(20, 21, 27)$ both have perimeter 70 and area 210. Barabash [1] shows that given a equilateral triangle, there are non-congruent triangles with the same area and perimeter; however, her proof is not constructive. This document (based on [2]) shows that given a triangle with rational sides, it is possible to construct a non-congruent triangle with *rational* sides and the same area and perimeter.

As a bonus, an elegant proof of Heron's formula is obtained.

[1] Barabash, Marita. A Non-Visual Counterexample in Elementary Geometry. *The College Mathematics Journal* 36(5), 2005.

[2] McCallum, William. *A tale of two triangles: Heron triangles and elliptic curves*, 2012, `http://blog.kleinproject.org/?p=4`.

1 From triangles to elliptic curves

Figure 1 shows O, the *incenter* of an arbitrary triangle $\triangle ABC$, which is the intersection of the bisectors of the three angles. To prove that the bisectors intersect in a single point, note that an angle bisector is the locus of points equidistant from the sides and conversely the locus of points equidistant from the sides is an angle bisector. O is equidistant from AB and AC and also equidistant from AB and BC. Therefore, O is equidistant from AC and BC, so it is on the angle bisector of $\angle C$.

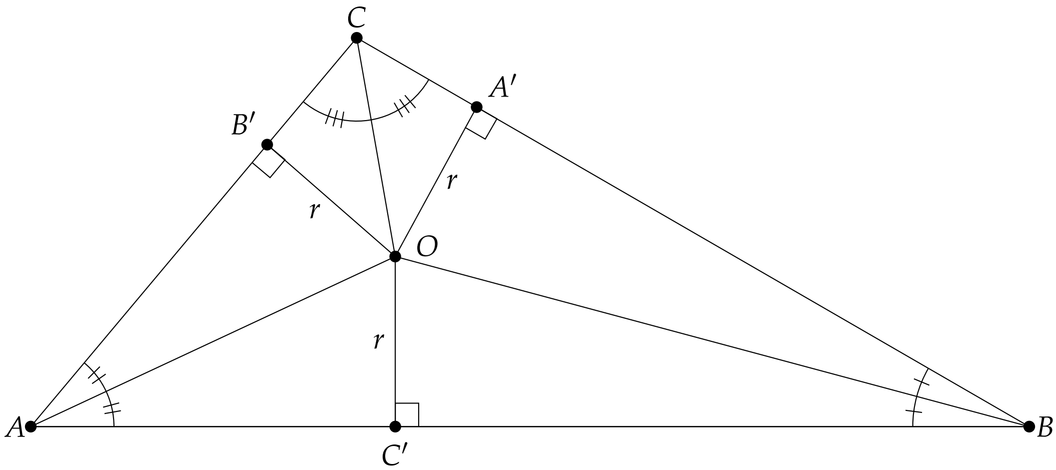

Figure 1: The incenter of a triangle

Drop altitudes from O to the sides. This results in pairs of congruent *right* triangles $\{\triangle AOB', \triangle AOC'\}, \{\triangle BOA', \triangle BOC'\}, \{\triangle COA', \triangle COB'\}$, since each pair shares a joint hypotenuse and the bisected angles are equal. It follows that the lengths of the altitudes are equal which we denote by r. O is the center of the inscribed circle with radius r.

Figure 2 shows the sides a, b, c divided into segments u, v, w, and the angles $\alpha/2, \beta/2, \gamma/2$ of the three pairs of triangles around the incenter O. The area of $\triangle ABC$ is the sum of the areas of $\triangle AOC, \triangle BOC, \triangle AOB$. r is the height of all the triangles so the area is:

$$A = \frac{1}{2}(w+v)r + \frac{1}{2}(v+u)r + \frac{1}{2}(u+w)r = \frac{1}{2}\cdot 2(u+v+w)r = rs\,, \tag{1}$$

since the semi-perimeter is:

$$s = \frac{1}{2}\cdot 2(u+v+w) = u+v+w\,. \tag{2}$$

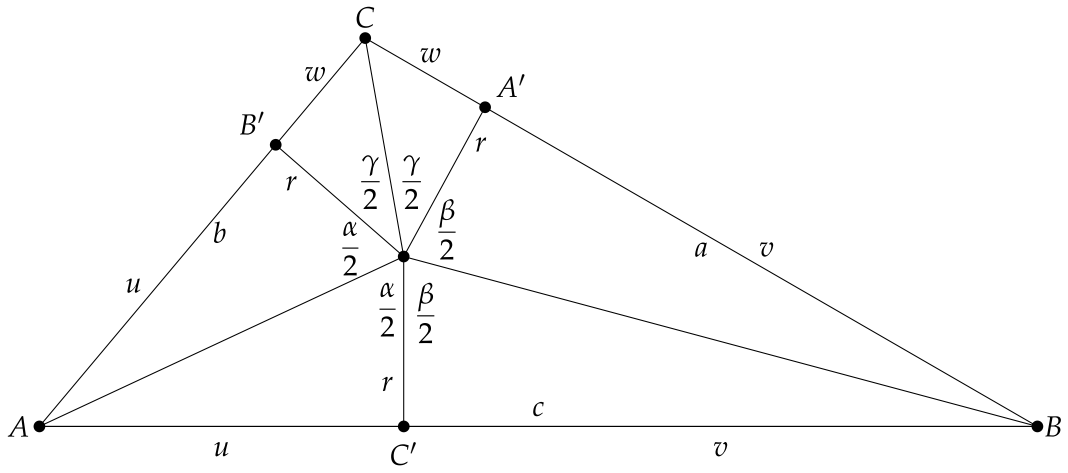

Figure 2: Angles and line segments formed by the altitudes

The lengths of u, v, w can be computed from the angles and r:

$$\begin{aligned} \tan\frac{\alpha}{2} &= \frac{u}{r} & (3)\\ \tan\frac{\beta}{2} &= \frac{v}{r} & (4)\\ \tan\frac{\gamma}{2} &= \frac{w}{r}\,. & (5) \end{aligned}$$

s can now be expressed in terms of the tangents:

$$s = u+v+w = r\tan\frac{\alpha}{2} + r\tan\frac{\beta}{2} + r\tan\frac{\gamma}{2} = r\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right)\,,$$

and by Equation 1 the area is:

$$A = rs = r^2\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right)\,. \tag{6}$$

From $A = rs$ we have $r = A/s$, so Equation 6 can be written as:

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \frac{A}{r^2} = \frac{A}{(A/s)^2} = \frac{s^2}{A}\,. \tag{7}$$

Since the sum of the angles α, β, γ is 2π:

$$\begin{aligned}
\gamma &= 2\pi - (\alpha + \beta) && (8)\\
\gamma/2 &= \pi - (\alpha/2 + \beta/2) && (9)\\
\tan\gamma/2 &= \tan(\pi - (\alpha/2 + \beta/2)) && (10)\\
\tan\gamma/2 &= -\tan(\alpha/2 + \beta/2) && (11)\\
\tan\gamma/2 &= \frac{\tan\alpha/2 + \tan\beta/2}{\tan\alpha/2\ \tan\beta/2 - 1}\,. && (12)
\end{aligned}$$

Here is proof of the formula for the tangent of the sum of two angles:

$$\begin{aligned}
\tan(\theta+\phi) &= \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} && (13)\\
\tan(\theta+\phi) &= \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi} && (14)\\
\tan(\theta+\phi) &= \frac{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\phi}{\cos\phi}}{1 - \dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi}} && (15)\\
\tan(\theta+\phi) &= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\,, && (16)
\end{aligned}$$

where Equation 15 is obtained by dividing by $\cos\theta\cos\phi$.

Let us simplify the notation by defining variables for the tangents:

$$\begin{aligned}
x &= \tan\frac{\alpha}{2}\\
y &= \tan\frac{\beta}{2}\\
z &= \tan\frac{\gamma}{2}\,.
\end{aligned}$$

By Equation 12 we can replace $z = \tan\gamma/2$ by an expression in x, y:

$$z = \frac{x+y}{xy-1}\,. \qquad (17)$$

With this notation, Equation 7 becomes:

$$x + y + \frac{x+y}{xy-1} = \frac{s^2}{A}\,. \qquad (18)$$

Given fixed values of A and s, are there multiple solutions to Equation 18?

For the right triangle $(3, 4, 5)$:

$$\frac{s^2}{A} = \frac{\left(\frac{1}{2}(3+4+5)\right)^2}{\frac{1}{2}\cdot 3\cdot 4} = \frac{6^2}{6} = 6\,. \qquad (19)$$

If there is another solution to the equation:

$$x + y + \frac{x + y}{xy - 1} = 6\,, \tag{20}$$

there is another triangle with area 6 and semi-perimeter 6. Equation 20 can be written as:

$$x^2y + xy^2 - 6xy + 6 = 0\,. \tag{21}$$

This is an equation for an *elliptic curve*. Elliptic curves were used by Andrew Wiles' in his proof of Fermat's last theorem. They have also been used in public-key cryptography.

2 Solving the equation for the elliptic curve

A portion of the graph of Equation 21 is shown in the Figure 3. Any point on the closed curve in the first quadrant is a solution to the equation. Only points in the first quadrant are of interest because we want positive values for the lengths of the sides of the triangle. The points A, B, D correspond to the triangle $(3, 4, 5)$ as we shall show below. To find additional (*rational*) solutions, the *method of two secants* is used.[1]

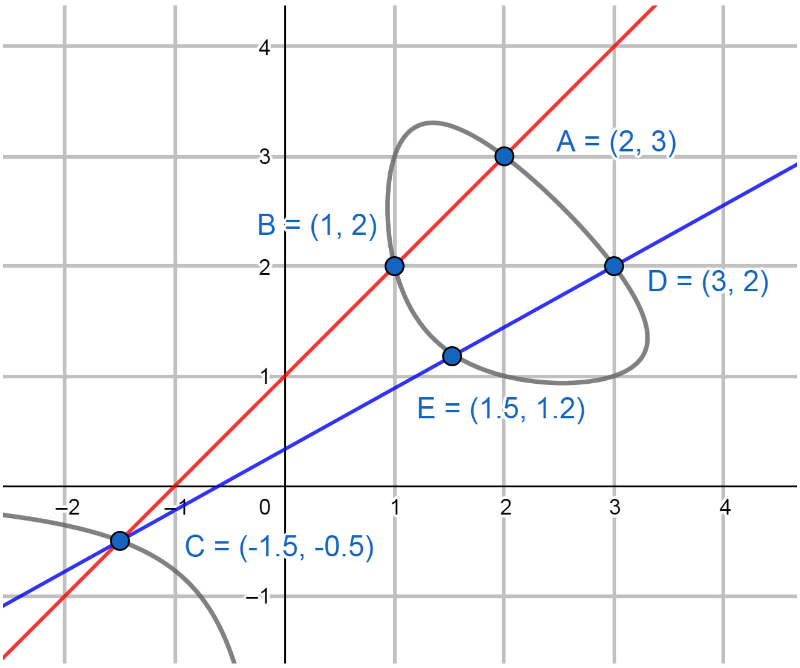

Figure 3: The graph of $x^2y + xy^2 - 6xy + 6 = 0$ with two secants

Draw a secant through the points $A = (2, 3)$ and $B = (1, 2)$. We will show that it intersects the curve at $C = (-1.5, -0.5)$, but this does not give a solution because the values are

[1]McCallum [2] notes that there are an infinite number of rational solutions.

negative. If we draw a second secant from C to $D = (3, 2)$, the intersection with the curve at E does give a new solution.[2]

The equation of the (red) line through A, B is $y = x + 1$. Substitute for y in Equation 21:

$$x^2(x+1) + x(x+1)^2 - 6x(x+1) + 6 = 0$$

and simplify:

$$2x^3 - 3x^2 - 5x + 6 = 0\,.$$

From the points A, B, we know two roots $x = 2, x = 1$, so we can factor the cubic polynomial as:

$$(x-2)(x-1)(ax+b) = 0\,,$$

where only the third root is unknown. Multiply the factors and we immediately see that a, the coefficient of the cubic term x^3, must be 2, and $2b$, the constant term, must be 6. Therefore, the third factor is $2x + 3$ which gives the third root $x = -\frac{3}{2}$ and $y = x + 1 = -\frac{1}{2}$. This is the point C in the graph.

Let us now draw a second secant (the blue line) through $C = (-\frac{3}{2}, -\frac{1}{2})$ and $D = (3, 2)$. The equation of the line is:

$$y = \frac{5}{9}x + \frac{1}{3}\,. \tag{22}$$

Substitute for y in Equation 21:

$$x^2\left(\frac{5}{9}x + \frac{1}{3}\right) + x\left(\frac{5}{9}x + \frac{1}{3}\right)^2 - 6x\left(\frac{5}{9}x + \frac{1}{3}\right) + 6 = 0\,,$$

and simplify:

$$\frac{70}{81}x^3 - \frac{71}{27}x^2 - \frac{17}{9}x + 6 = 0\,.$$

Again, we have two roots $x = 3, x = -\frac{3}{2}$, so we can factor the cubic polynomial as:

$$(x-3)(x+\frac{3}{2})(ax+b) = 0\,.$$

Equating the coefficient of the cubic term and equating the constant term give:

$$\frac{70}{81}x - \frac{4}{3} = 0\,,$$

so:

$$x = \frac{81}{70} \cdot \frac{4}{3} = \frac{27 \cdot 4}{70} = \frac{54}{35}\,.$$

y can be computed from Equation 22 and the coordinates of E are:

$$\left(\frac{54}{35}, \frac{25}{21}\right)\,.$$

Finally, compute z from Equation 17:

$$z = \frac{x+y}{xy-1} = \left(\frac{54}{35} + \frac{25}{21}\right) \Big/ \left(\frac{54}{35}\frac{25}{21} - 1\right) = \frac{2009}{615} = \frac{49}{15}\,.$$

[2] $(1.5, 1.2)$ is an approximation displayed by GeoGebra. We will compute the exact coordinates of E below.

3 From solutions to the elliptic curve to triangles

From x, y, z, a, b, c, the sides of the triangle $\triangle ABC$ can be computed:

$$\begin{aligned} a &= w + v = r(z + y) = (z + y) \\ b &= u + w = r(x + z) = (x + z) \\ c &= u + v = r(x + y) = (x + y)\,, \end{aligned}$$

since $r = \dfrac{A}{s} = \dfrac{6}{6} = 1$.

For the solution $A = (2, 3)$ of the elliptic curve, the value of z is:

$$z = \frac{x + y}{xy - 1} = \frac{2 + 3}{2 \cdot 3 - 1} = 1\,,$$

and the sides of the triangle are:

$$\begin{aligned} a &= z + y = 1 + 3 = 4 \\ b &= x + z = 2 + 1 = 3 \\ c &= x + y = 2 + 3 = 5\,, \end{aligned}$$

the right triangle with $s = A = 6$. Computing the sides corresponding to B and D gives the same triangle.

For E:

$$\begin{aligned} a &= z + y = \frac{49}{15} + \frac{25}{21} = \frac{156}{35} \\ b &= x + z = \frac{54}{35} + \frac{49}{15} = \frac{101}{21} \\ c &= x + y = \frac{54}{35} + \frac{25}{21} = \frac{41}{15}\,, \end{aligned}$$

which I am sure that you found by trial and error!

Let us check the result. The semi-perimeter is:

$$s = \frac{1}{2}\left(\frac{156}{35} + \frac{101}{21} + \frac{41}{15}\right) = \frac{1}{2}\left(\frac{468 + 505 + 287}{105}\right) = \frac{1}{2}\left(\frac{1260}{105}\right) = 6\,,$$

and the area can be computed using Heron's formula:

$$\begin{aligned} A &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{6\left(6 - \frac{156}{35}\right)\left(6 - \frac{101}{21}\right)\left(6 - \frac{41}{15}\right)} \\ &= \sqrt{6 \cdot \frac{54}{35} \cdot \frac{25}{21} \cdot \frac{49}{15}} \\ &= \sqrt{\frac{396900}{11025}} \\ &= \sqrt{36} = 6\,. \end{aligned}$$

4 A proof of Heron's formula

The triple tangent formula states that if $\phi + \theta + \psi = \pi$ then:

$$\tan\phi + \tan\theta + \tan\psi = \tan\phi\tan\theta\tan\psi\,. \tag{23}$$

The proof follows immediately from Equation 16:

$$\begin{aligned}
\tan\psi &= \tan(\pi - (\phi + \theta)) \\
&= -\tan(\phi + \theta) \\
&= \frac{\tan\phi + \tan\theta}{\tan\phi\tan\theta - 1} \\
\tan\phi\tan\theta\tan\psi - \tan\psi &= \tan\phi + \tan\theta \\
\tan\phi\tan\theta\tan\psi &= \tan\phi + \tan\theta + \tan\psi\,.
\end{aligned}$$

From Equations 3–6, and $r = A/s$:

$$\begin{aligned}
A &= r^2\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right) \\
&= r^2\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\right) \\
&= r^2\left(\frac{u}{r}\frac{v}{r}\frac{w}{r}\right) \\
&= \frac{u\,v\,w}{r} \\
&= \frac{s}{A}\,u\,v\,w \\
A^2 &= s\,u\,v\,w\,.
\end{aligned}$$

From Equation 2 and the definitions of a, b, c, u, v, w in Figure 2:

$$\begin{aligned}
s - a &= (u + v + w) - (w + v) = u \\
s - b &= (u + v + w) - (u + w) = v \\
s - c &= (u + v + w) - (u + v) = w\,,
\end{aligned}$$

and Heron's formula follows:

$$\begin{aligned}
A^2 &= s\,u\,v\,w \\
&= s(s-a)(s-b)(s-c) \\
A &= \sqrt{s(s-a)(s-b)(s-c)}\,.
\end{aligned}$$