34. Steiner's Straight-edge Problem

This document is Problem 34 from the book by Heinrich Dörrie: *100 Problems of Elementary Mathematics: Their History and Solution* (Dover, 1965), as reworked by Michael Woltermann.[1] I have added explanations so that students and teachers can better understand the construction. The document has been written and formatted in LaTeX, and I have redrawn the diagrams using TikZ, adding auxiliary lines and drawing diagrams incrementally for clarity.

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Prove that every construction that can be done with compass and straight-edge can be done with straight-edge alone given a fixed circle in the plane.

As far back as 1759 Lambert had solved a whole series of geometric constructions with straight-edge alone in his book *Freie Perspektive*, published in Zürich that year. He is also the source of the term "straight-edge geometry". After Lambert, French mathematicians, primarily Poncelet and Brianchon, took up straight-edge geometry, particularly after the publication of Mascheroni's *Geometria del compasso* gave a new stimulus to these studies, and they attempted to find as many constructions as possible with straight-edge alone.

With a straight-edge alone, it is possible to represent only rational algebraic expressions (so not for example, $\sqrt{ab}$).

> In the nineteenth century it was shown that any length constructed with a straight-edge and compass can be obtained from a given unit length by the operations of rational arithmetic ($+, -, \times, \div$) together with the operation of computing square roots. This was used to prove that it is impossible to trisect an angle or double a cube, because they require the construction of a length that is a cube root. The sentence above explains the reason why a straight-edge alone is not sufficient to perform all constructions.

This suggested to Poncelet that an additional fixed circle (with its center) must be given in order to draw with straight-edge alone all algebraic expressions that can be constructed with compass and straight-edge. This was confirmed by Jakob Steiner (1796-1863), the greatest geometer since the days of Appolonius, in his celebrated book *Die geometrischen Konstruktionen asgeführt mittels der geraden Linie und Eines festen Kreises* (Geometrical constructions carried out with straight lines and a fixed circle), Berlin, 1833.

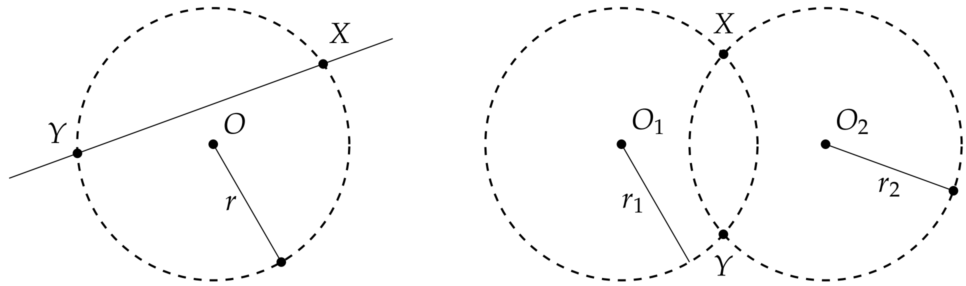
The solution presented here is based on that in Steiner's book, except that we have eliminated everything that is not strictly essential for the purpose at hand, and we have

[1] `http://www2.washjeff.edu/users/mwoltermann/Dorrie/DorrieContents.htm`. I would like to thank him for giving me permission to use his work.

[2] `http://www.weizmann.ac.il/sci-tea/benari/`.

also made it somewhat more elementary. Since in straight-edge geometry, the intersection of two straight lines is known directly, we need only demonstrate that the two fundamental problems II and III of No. 33 can be solved using a straight-edge and fixed circle.

> From Problem 33: When we examine the separate steps by which circle and straight-edge constructions are carried out, we see that every step consists of one of the following three basic constructions:
>
> I. Finding the point of intersection of two straight lines;
>
> II. finding the point(s) of intersection of a straight line and a circle;
>
> III. finding the point(s) of intersection of two circles.
>
> The following notation from Problem 33 is used here:
>
> - $c(O, A)$ stands for the circle with center O through point A,
> - $c(O, AB)$ stands for the circle with center O and radius AB.
>
> What does it mean to perform a construction with straight-edge alone? A circle is defined by a point O (its center) and a line segment (whose length is the radius r) one of whose endpoints is the center. The following diagram shows what it means to determine X, Y, the points of intersection of a line and a circle, and of two circles. Since we don't have a compass, the dashed lines in the diagram don't actually appear in the construction. Similarly, any circle in the diagrams of this document only help understand a construction and its proof.



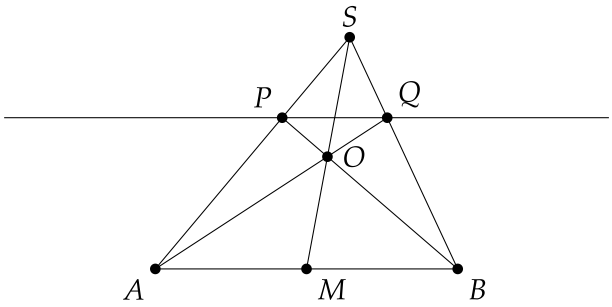
As in the solution of Mascheroni's problem, we must first solve several preliminary problems, in this case five of them.

Prelim 1. Construct a line through point P parallel to a given line. (P is not on the given line.)

Solution. Steiner considers two cases:

- **1a.** two points A and B and their midpoint M on the given line are known. We will call the straight line a "directed straight line" in this case.
- **1b.** the given straight line is arbitrary.

1a. Draw AP and let S be a point on AP extended. Connect S with M and B. Let O be the intersection point of BP and MS. Finally let line AO meet BS at Q.

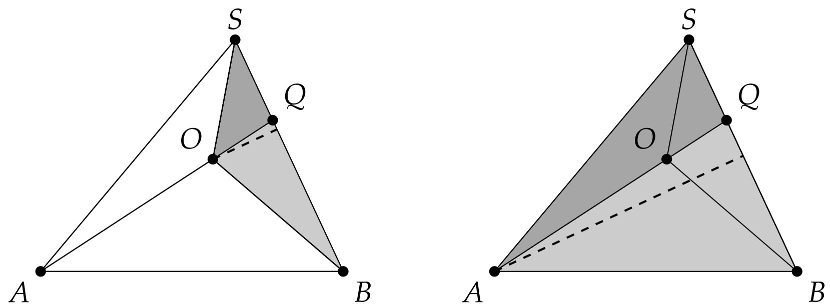


Note the order in which the lines are constructed. The line segment AB and its midpoint M are given, as is P, the point through which the parallel line is to be drawn. S is an arbitrary point on the extension of AP. Once S is chosen, the lines SB and SM can be constructed. Then BP is constructed and its intersection with SM is defined to be O. Next, a ray is from A through O intersects SB at Q and the line PQ is constructed. It appears from the diagram that PQ is parallel to AB, but that is precisely what has to be proved.

By Ceva's Theorem,

$$\frac{AM}{MB}\frac{BQ}{QS}\frac{SP}{PA} = 1$$

To prove Ceva' Theorem, examine the following diagrams.



Since the area of a triangle is proportional to its height and base, if the height of two triangles are equal, their areas are proportional to the bases:

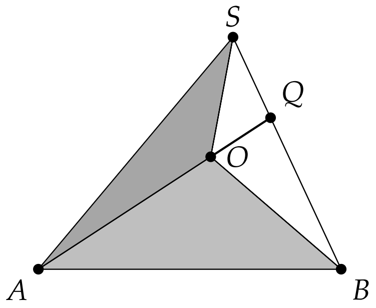
$$A_1 = \frac{1}{2}hb_1, \quad A_2 = \frac{1}{2}hb_2, \quad \frac{A_1}{A_2} = \frac{b_1}{b_2}\,.$$

Consider the gray triangles each of the above diagrams. They have the same height (the altitudes from O to SB and from A to SB), so:[3]

$$\frac{\triangle BQO}{\triangle SQO} = \frac{BQ}{QS}\,, \qquad \frac{\triangle BQA}{\triangle SQA} = \frac{BQ}{QS}\,.$$

[3]In this proof, the name of a triangle is used as an abbreviation for its area.

By subtracting the areas, the ratio of the area of the gray triangles in the following diagram is also equal to $\frac{BS}{QS}$.



$$\frac{BQ}{QS} = \frac{\triangle BQA - \triangle BQO}{\triangle SQA - \triangle SQO} = \frac{\triangle BOA}{\triangle SOA}\,.$$

This might look strange at first; the computation (using a simpler notation) is:

$$\begin{aligned}
\frac{c}{d} &= \frac{a}{b}\\
\frac{e}{f} &= \frac{a}{b}\\
c-e &= \frac{ad}{b} - \frac{af}{b}\\
c-e &= \frac{a}{b}(d-f)\\
\frac{c-e}{d-f} &= \frac{a}{b}\,.
\end{aligned}$$

Similarly, we can prove:

$$\frac{AM}{MB} = \frac{\triangle AOS}{\triangle BOS}\,, \qquad \frac{SP}{PA} = \frac{\triangle SOB}{\triangle AOB}\,,$$

so:

$$\frac{AM}{MB}\frac{BQ}{QS}\frac{SP}{PA} = \frac{\triangle AOS}{\triangle BOS}\frac{\triangle BOA}{\triangle SOA}\frac{\triangle SOB}{\triangle AOB} = 1\,,$$

since the areas in the numerator and denominator cancel (keep in mind that the order of the vertices in the name of a triangle does not matter).

from which it follows that $\frac{BQ}{QS} = \frac{AP}{PS}$

Recall that M is the midpoint of AB so $AM = MB$ and $\frac{AM}{MB} = 1$.

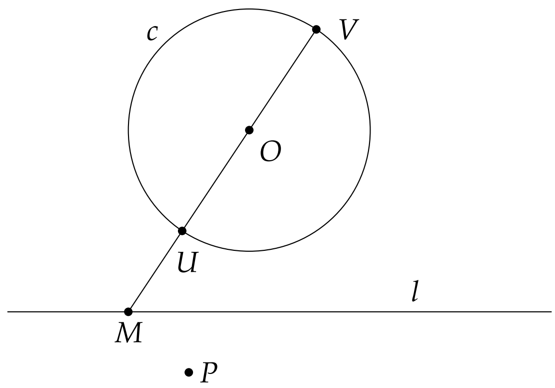
and then $\frac{BS}{QS} = \frac{AS}{PS}$.

$$\begin{aligned}
BS &= BQ + QS\\
\frac{BS}{QS} &= \frac{BQ}{QS} + \frac{QS}{QS} = \frac{BQ}{QS} + 1\\
AS &= AP + PS\\
\frac{AS}{PS} &= \frac{AP}{PS} + \frac{PS}{PS} = \frac{AP}{PS} + 1\\
\frac{BS}{QS} &= \frac{AS}{PS} \qquad\qquad \text{since } \frac{BQ}{QS} = \frac{AP}{PS}\,.
\end{aligned}$$

Thus $\triangle ABS \sim \triangle PQS$ and line PQ is parallel to line AB (since $\angle ABS = \angle PQS$).

1b. Let l be the line, $c = c(O, r)$ be the fixed circle, and P a point off l.

> The fixed circle is the single arbitrary circle that is assumed to exist. Convince yourself, both here and later, that the construction does not depend on the location of the center of the circle or its radius.

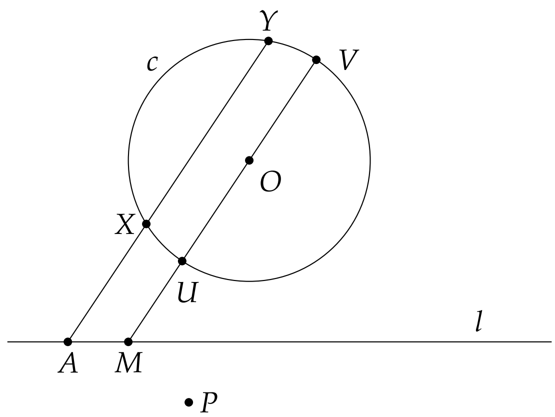


Let line MO intersect c in points U and V, making MO a directed straight line.

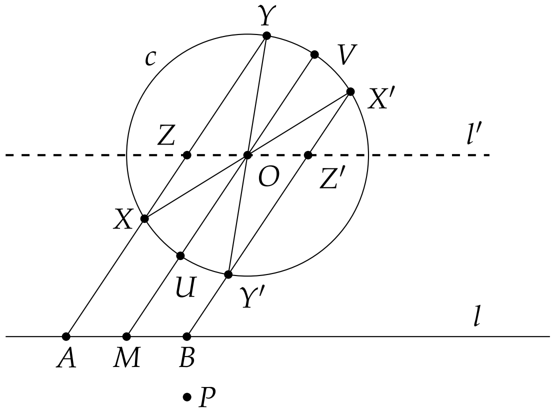
> M is an arbitrary point on line l. The line containing MO is a directed straight line because it contains the points U, V and their midpoint O (UV is a diameter of the circle and O is its center).

Use 1a to construct a line through a point A on l parallel to MO and intersecting c in points X and Y:

> A is chosen arbitrarily on line l.



Let XOX' and YOY' be diameters of c, and let line $X'Y'$ meet l at B.



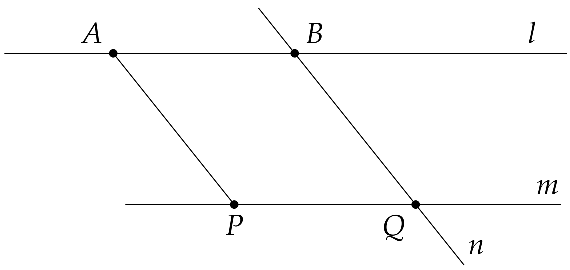
Then $AM = MB$ and l is a directed straight line. The parallel to l through P can then be constructed in accordance with 1a. $\square$

OX, OX', OY, OY' are all radii of the circle and $\angle XOY = \angle X'OY'$ since they are opposite angles. Therefore, $\triangle XOY$ and $\triangle X'OY'$ are congruent by side-angle-side. Let l' be a line through O parallel to l that intersects XY at Z and $X'Y'$ at Z'. Triangles $\triangle XOZ$ and $\triangle X'OZ'$ are congruent by angle-side-angle, so $ZO = OZ'$. Therefore, $AMOZ$ and $BMOZ'$ are parallelograms, so $AM = ZO = OZ' = MB$.

Corollary. Shift AB parallel to itself so that one of its endpoints lies on a given point P (off line AB).

Solution. Let m be the parallel through P to AB, and n be the parallel through B to AP. Let $Q = m \cap n$.

The lines m, n can be constructed by Prelim. 1.



Then PQ is the desired line segment.

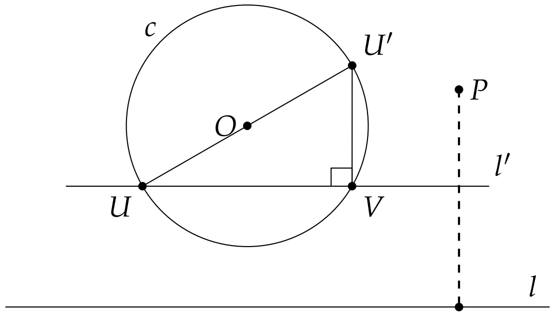
$ABQP$ is a parallelogram and opposite sides are equal $AB = PQ$.

A similar construction shifts AB so that B lies on P. $\square$

Prelim 2. Construct a perpendicular through a point P to a given line l.

Solution. Draw l' parallel to l so that it cuts c at U and V. Draw the diameter UOU' and chord VU'.

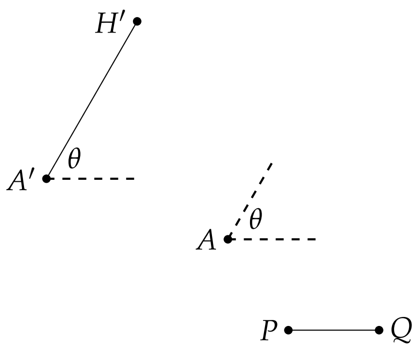
c is the fixed circle.



$\angle UVU'$ is an inscribed angle in a semicircle, hence a right angle. Thus VU' is perpendicular to UV and l. Finally construct the parallel to VU' through P in accordance with 1; this parallel is the desired perpendicular. $\square$

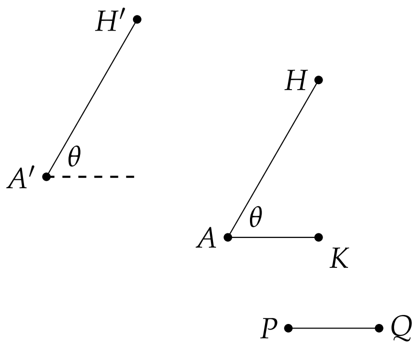
Prelim 3. Construct a segment AS at a given point A of length PQ in a given direction.

> We are given a line segment $A'H'$ representing a direction θ from some reference axis, a line segment PQ and a point A. The goal is to construct a line segment one of whose endpoints is A of length PQ and in the direction θ.



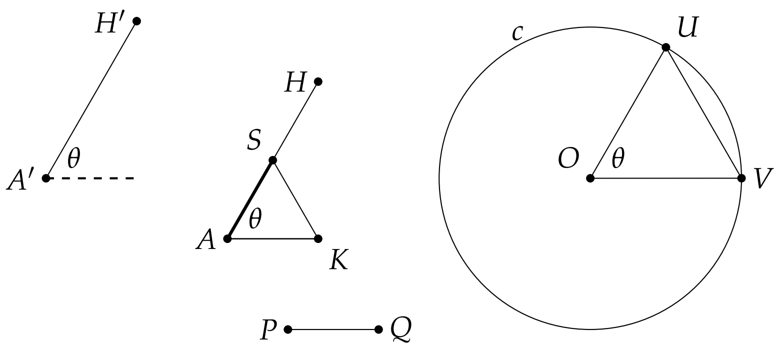
Solution. If necessary use a parallel shift by the Corollary to 1 above so that the given direction is vector $\overrightarrow{AH}$.

Use it again to displace PQ parallel to itself to AK.



> Now, the length of AK equals the length of PQ and line AH is parallel to $A'H'$. It remains to construct a line segment AS of length equal to AK on the line containing AH.

Then draw two radii OU and OV of c in directions AH and AK.

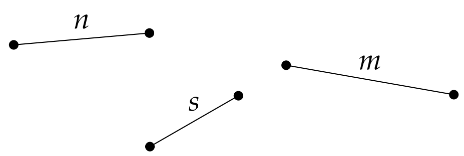


Finally draw the parallel to UV through K; its intersection S with line AH is the desired point. □

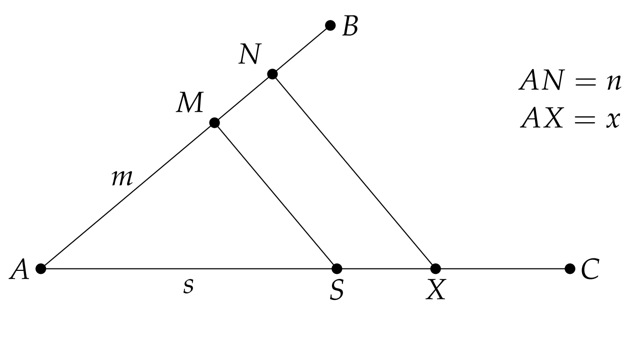
Since AH is parallel to OU and AK is parallel to OV, $\angle SAK = \angle HAK = \theta = \angle UOV$. Since SK is parallel to UV, $\triangle SAK$ is similar to $\triangle UOV$. Since $\triangle UOV$ is isosceles (OU, OV are radii of a circle), $\triangle SAK$ is isosceles and $AS = AK = PQ$.

Prelim 4. Given segments of length n, m, s, construct a segment of length $x = \frac{n}{m}s$.

Initially, we have three line segments n, m, s located at arbitrary positions and in arbitrary directions in the plane.



Solution. From any point A, draw two rays AB and AC, and (use Prelim 3) to mark off distances $AM = m$, $AN = n$ on AB and $AS = s$ on AC. Let the parallel through N to MS intersect AC at X.



Then $x = \frac{n}{m}s$. □

Since $\triangle MAS$ is similar to $\triangle NAX$, $\frac{m}{n} = \frac{s}{x}$.

Prelim 5. Given segments of length a and b, construct a segment of length $\sqrt{ab}$.

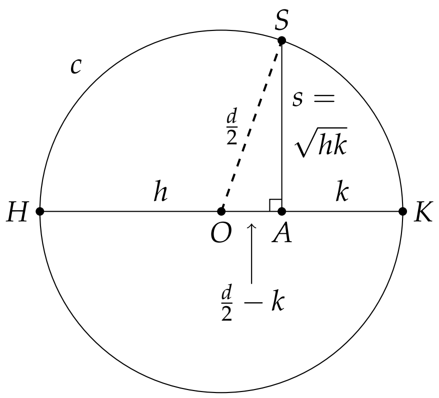
Solution. Let $x = \sqrt{ab}$, d be the diameter of fixed circle c and $t = a + b$. Note that t is constructible by Prelim 3.

> The aim of this construction is to show how to express $x = \sqrt{ab} = \frac{n}{m}s$ in order to use the result of Prelim. 4. n will be d which is given and m will be $t = a + b$ which can be constructed from the given lengths a, b as shown in Prelim. 2. The hard part is to find a value s that can be constructed. This is done by defining h, k in terms of a, b, t, d, then defining $s = \sqrt{hk}$ and showing how a line segment of this length can be constructed.

With $h = \frac{d}{t}a$, $k = \frac{d}{t}b$ and $s = \sqrt{hk}$, it follows that $x = \sqrt{ab} = \sqrt{\frac{th}{d}\frac{tk}{d}} = \frac{t}{d}s$. Note that $h + k = d$,

$$h + k = \frac{d}{t}a + \frac{d}{t}b = \frac{d(a+b)}{t} = \frac{dt}{t} = d\,.$$

By Prelim. 3, we can construct segment $HA = h$ on diameter HK of c; then $AK = k$. Use Prelim 2 to construct the perpendicular to HK through A, and call its intersection with c point S.



> We have to show that $SA = s = \sqrt{hk}$. The line segment OS is a radius, so its length is $\frac{d}{2}$. By construction, the length of OA is $\frac{d}{2} - k$. By Pythagoras's Theorem:

$$\begin{aligned} SA^2 &= \frac{d}{2}^2 - (\frac{d}{2} - k)^2 \\ &= dk - k^2 \\ &= k(d-k) \\ &= kh, \qquad \text{since } h + k = d, \\ s &= SA = \sqrt{hk}\,. \end{aligned}$$

Then $x = \frac{t}{d}s$ is constructible by Prelim 4. $\square$

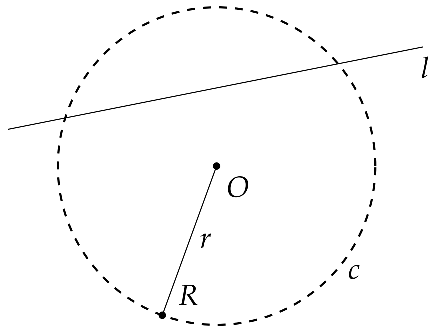
The solution to the two basic construction problems is now simple.

II. *Construct the point of intersection of a given line and a given circle.*

> The "given" circle is *not* the "fixed" circle that is assumed to exist, but a circle defined by its center and radius.

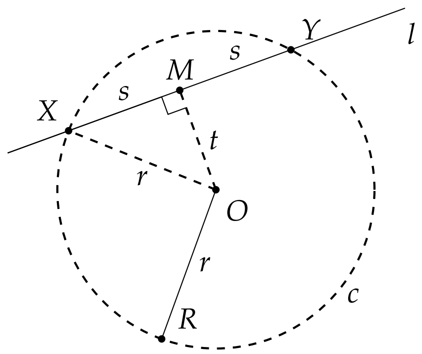
Solution. Let l be the given line and $c(O, r)$ be the given circle. We must construct X and Y, the points of intersection of l and $c(O, r)$.

> Initially, we just have the line l, and the center O and the radius r that define the circle. The second endpoint of the radius is defined to be one of the intersection points of the circle with the line l.



Let $2s$ be the length of chord XY, M be its midpoint and t be the distance OM.

> By Prelim. 2, M can be constructed as the intersection of the perpendicular from O to line l. X, Y, s are just definitions; we haven't constructed them yet.

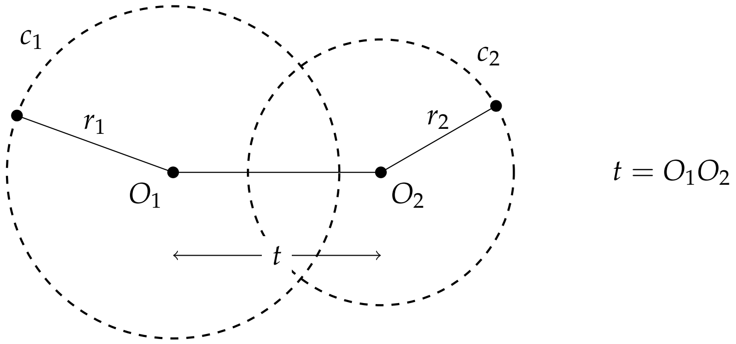


From right triangle $\triangle OMX$, $s^2 = r^2 - t^2$ or $s = \sqrt{(r+t)(r-t)}$ is constructible by Prelim 2, and $r \pm t$ by Prelim 3. s can then be constructed by Prelim 5, and by Prelim 3 again, X and Y can be found. $\square$

> Define the length of OM to be t. Prelim. 3 describes the construction of line segments of length t from a point O in the directions OR and RO, thus constructing line segments of length $r \pm t$. Prelim. 5 describes the construction of a line segment of length $s = \sqrt{(r+t)(r-t)}$ and finally Prelim. 3 constructs line segments of length s from point M in both directions along line l, thus constructing the points X and Y.

III. *Construct the points of intersection of two given circles.*

Solution. Let $c_1(O_1, r)$ and $c_2(O_2, r)$ be the two given circles, and X and Y be the points of intersection (to be constructed).

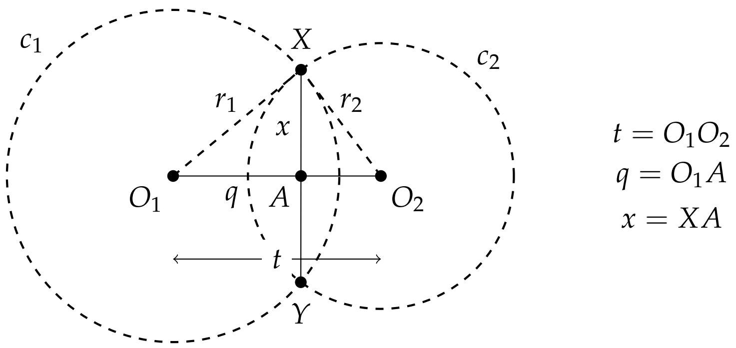


The circles are dashed to indicate that they haven't been constructed (and won't be constructed because we don't have a compass). O_1, O_2 are actual points so we can construct the line O_1O_2 with the straight-edge.

Let A be the point of intersection of XY and the line of the centers O_1O_2.

Again, just definitions. A, X, Y have not yet been constructed.

Let t, q and x be the distances O_1O_2, O_1A and XA respectively.



We will show that q and x are constructible (in straight-edge geometry); then Prelim 3 will allow us to construct A (at a distance of q from O_1), Prelim 2 the perpendicular to O_1O_2 at A, and Prelim 3 X and Y (at a distance of x from A) on this perpendicular.

X and Y are actually on the intersections of the circles since they are distances r_1, r_2 from O_1, O_2.

Finding q. Apply the law of cosines to $\triangle O_1O_2X$ to get

$$\begin{aligned}
r_2^2 &= r_1^2 + t^2 - 2r_1 t \cos\angle XO_1O_2 \\
&= r_1^2 + t^2 - 2t(r_1 \cos\angle XO_1O_2) \\
&= r_1^2 + t^2 - 2tq \\
2tq &= (r_1^2 + t^2) - r_2^2 .
\end{aligned}$$

Set $d = \sqrt{r_1^2 + t^2}$ so that $q = \frac{(d+r_2)(d-r_2)}{2t}$. d, the hypotenuse of a right triangle with legs of length r_1 and t, is constructible by Prelims 2 and 3, $n = d + r_2$, $m = 2t$, $s = d - r_2$ are constructible by Prelim 3, and $q = \frac{n}{m}s$ is constructible by Prelim 4.

> The right triangle with sides d, r_1, t doesn't appear in the above diagram. We construct it somewhere in the plane for the purpose of computing d from the given lengths r_1, t and then q from d, r_2, t.

Finding x. From (right) $\triangle AO_1X$, $x^2 = r_1^2 - q^2$, and $x = \sqrt{(r_1+q)(r_1-q)}$. Prelim 3 provides a construction of $h = r_1 + q$ and $k = r_1 - q$, and Prelim 5 a construction of $x = \sqrt{hk}$. $\square$