The Mathematics of Origami for Secondary-School Students

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Contents

1 Introduction

The topic of this document is the mathematics of paper folding—origami. The goal is to expose teachers and students to the set of axioms of this theory and to use the axioms to construct the concept of *geometric locus*. The first part of the document is intended for teachers and includes the historical background, the mathematical content and the pedagogy associated with the theory of origami and its axioms. The second part offers activities for students on the concept of locus; the activities use paper folding and are based upon the axioms of origami.

2 Teachers' guide

2.1 Background

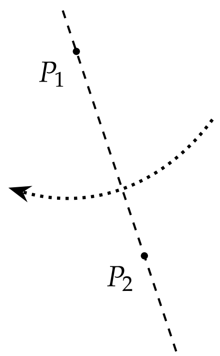
Origami is the ancient art of paper folding. Origami began to develop when paper was invented in 105 CE. In 1930 modern origami was developed by Akira Yoshizawa (1911–2005). Yoshizawa developed a method of graphically recording the process of folding. In addition, Yoshizawa developed novel methods of folding and solved geometric problems using origami. In the 1950's origami became widely known in the United States, where researchers investigated connections between origami and other fields, such as mathematics, medicine and technology.

2.2 The axioms of origami

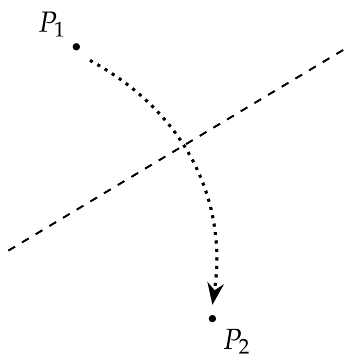
Research into the mathematics of origami began in the 1980's. There are seven axioms called the *Huzita-Hatori Axioms* that describe the possible geometric operations during the process of paper folding. The first six axioms were found by the mathematician Humiaki Huzita in 1989. The set of axioms was completed with a seventh axiom found by the mathematician Koshiro Hatori in 2001.

The axioms are expressed in terms of point and lines. The lines are straight lines created by folding a sheet of paper. In the diagrams, folds are indicated by dashed lines.

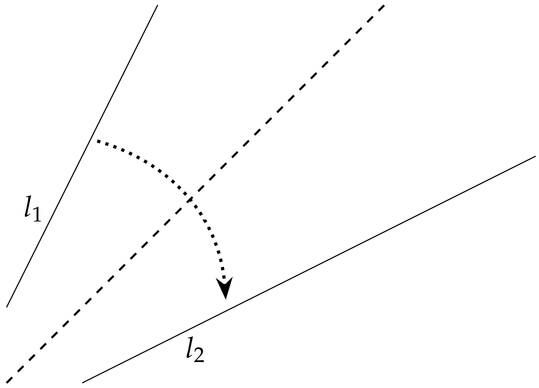
Axiom 1 Given two points P_1, P_2, there is a single fold that passes through them.



Axiom 2 Given two points P_1, P_2, there is a single fold that places P_1 onto P_2.

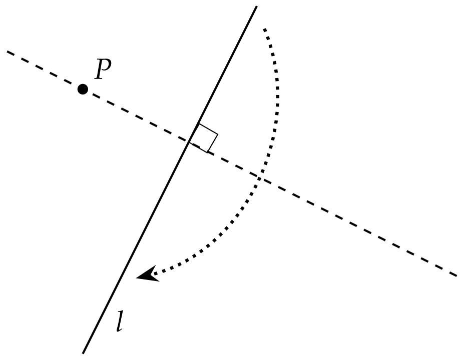


Axiom 3 Given two lines l_1, l_2, there is a fold that places l_1 onto l_2.

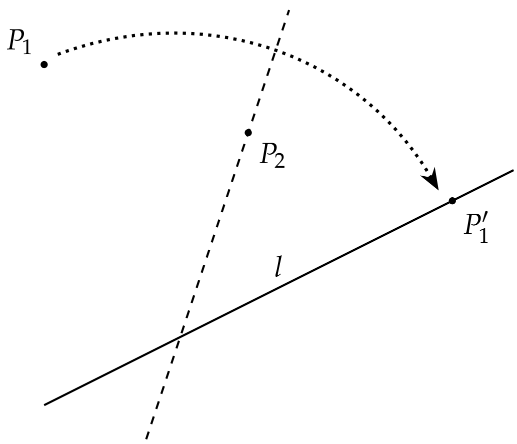


There are two cases: (a) if the lines are parallel, the line created by the fold will be parallel to the two lines and equidistant from them; (b) if the lines intersect, the line created by the fold is one of the two bisectors of the vertical angles.

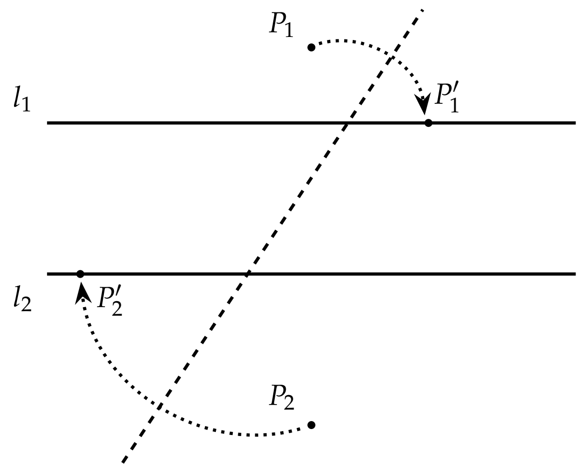
Axiom 4 Given a point P and a line l, there is a single fold perpendicular to l that passes through P.



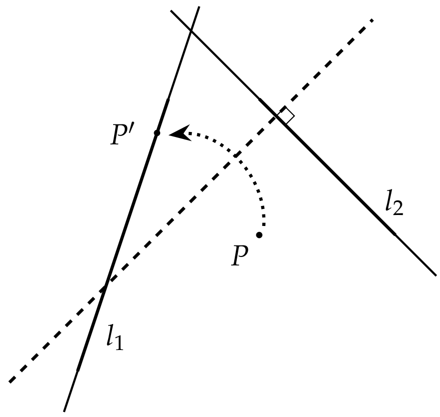
Axiom 5 Given two points P_1, P_2 and a line l, there is a fold that places P_1 onto l and passes through P_2. (There may be zero, one or two such folds as explained in Activity 4.)



Axiom 6 Given two points P_1, P_2 and two lines l_1, l_2, there is a fold that simultaneously places P_1 onto l_1 and P_2 onto l_2. (There may be zero, one, two or three such folds as explained in Activity 5.)



Axiom 7 Given a point P and two lines l_1, l_2, there is a fold that places P onto l_1 and is perpendicular to l_2.



2.3 The mathematics of origami

The first axiom systems was *Euclidean geometry*, named after the third-century mathematician who collected and extended geometrical theorems and proofs in the book *The Elements*. Euclidean geometry was based on construction by straightedge (an unmarked ruler) and compass. The straightedge can construct a line that goes through two existing points; the compass can construct a circle from a given point (its center) and a given line segment (its radius).

There were three problems for which the Greeks were unable to find constructions with straightedge and compass: (1) trisecting an arbitrary angle into three equal parts; (2) squaring a circle: given a circle, construct a square with the same area; (3) doubling a cube: given a cube, construct another cube with twice the volume. In addition, they were unable to construct a regular heptagon (a regular polygon with seven sides).

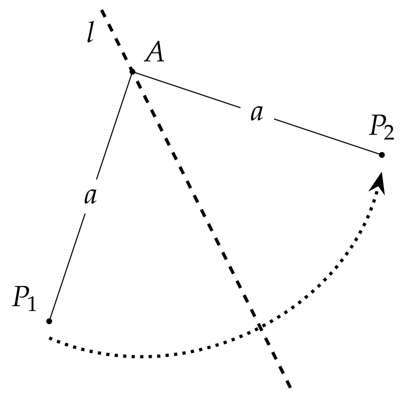
It was not until the 19-th century that the reason for their failures was understood. A straightedge and compass can only construct values that are obtainable from a line segment defined to have length 1 and the operations of $\{+, -, \times, \div, \sqrt{}\}$. Squaring a circle is certainly impossible because it requires constructing the value π which cannot be obtained from any algebraic formula. The other problems require the construction of cube roots which is impossible. For example, doubling a cube requires the construction of the value $\sqrt[3]{2}$.

The basic constructions of Euclidean geometry are: bisecting a line segment, bisecting an angle, copying a line segment, copying an angle, constructing a perpendicular to a line from a point not on the line, constructing a perpendicular to a line from a point on the line. All these constructions can be performed in origami using Axioms 1–5. The power of origami comes from Axiom 6: placing two points on two lines turns out to enable constructions that cannot be performed using straightedge and compass, in particular, constructions that require the computation of cube roots.

2.4 Proof by folding

Throughout the activities, the students will be asked to "prove by folding." The intention is that the student perform a fold that will explain why a claim is true. Before each activity, the teacher must explain to the students when a fold proves a claim. We here present an example of a simple explanation that should be shown to the students before the activities (or as necessary).

Proof by folding: Consider Axiom 2: given two points P_1, P_2, there is a single fold that places P_1 onto P_2. Look at the line constructed by this fold and an arbitrary point A on the line. The fold does not move the point, so the line segment $\overline{AP_1}$ is placed onto the line segment $\overline{AP_2}$, so their lengths are equal. We have proved by folding that if the endpoints of a line segment are copied onto the endpoints of another segment then their lengths are equal.



3 Activities for learning geometric loci with origami

3.1 Introduction

The activities are based on the origami axioms through which the students will experiment with constructing geometric loci. Paper folding can contribute to learning this concept because it provides a concrete visualization of the meaning of the concept.

How to present the activities

1. Each activity has two parts: in the first the students will get to know the axioms and the how the activities can be adapted for use in a classroom. The second part is a worksheet with the activity itself. Some students can be given the worksheets for independent study, while others may need to be guided through the steps of the activity.

2. The activities are arranged in an order that we believe is pedagogical optimal, not in the standard order of the axioms. Of course, the teacher is free to present the activities in a different order. Activity 1 is essential to what follows and should be presented first, even if the teacher chooses a different order for the subsequent activities.

3. The activities include Geogebra applications intended to deepen the students' understanding of the geometric loci constructed by the axioms.

Paper for folding

Several types of paper are appropriate for folding: (a) origami paper available in specialty shops; (b) any square or rectangular sheets of paper; (c) rectangular or square baking sheets which facilitate marking points and lines.

Each student should have at least six sheets of paper on hand (seven for the opening activity).

3.2 The structure of an activity for leaning geometric loci

Topic Geometric loci through paper folding.

Goals (a) Learning the concept and meaning of *geometric locus* through paper folding; (b) learning the origami axioms; (c) exercising methods of finding geometric loci.

Target audience Secondary-school students studying advanced mathematics.

Prerequisites Analytic geometry with emphasis on circles and parabolas.

Structure

Part 1

First phase: Acquaintance with origami

1. The teacher will start with a brief presentation of the historical background of origami, describing how simple paper folding became an advanced mathematical theory and pioneering technological tool. At this point the students can be shown part of Robert Lang's lecture [4]. Show four minutes starting from minute eleven.

2. The teacher will explain the origami axioms.

3. The teacher will acquaint the students with the basic concepts: point, line and fold.

Second phase: Familiarization and experimentation with the origami axioms For the initial encounter with the axioms we suggest doing the activities described in Appendix A *Experimenting with folding according to the origami axioms*.

Third phase: Familiarization with the concept of geometric locus The teacher will define the concept: the geometric locus is the set of points that satisfy a given condition. The locus is expressed as an expression of y in terms of x. Often the locus can be expressed in geometric terms such as a point or line with a certain property, or a geometric figure like a circle or a parabola.

Part 2

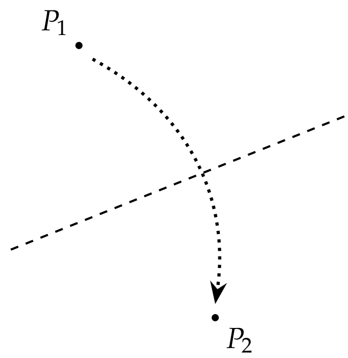
First phase: Familiarization with geometric loci through the origami axioms The teacher will distribute worksheets that ask the students to find geometric loci by paper folding (Section 4). The students can work independently or work on each step followed by a group discussion.

Second phase: Exercises Section 5 contain mathematical exercises for each axiom. The first and second phases can be combined: at the conclusion of an activity from the first phase, the students can be asked to solve the respective exercise from the second phase.

4 Worksheets for the activities

In the following activities we investigate geometric loci through the origami axioms.

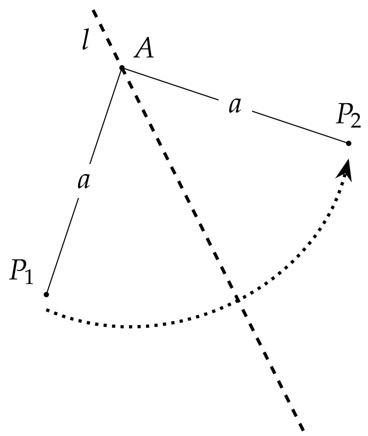
4.1 The perpendicular bisector as a geometric locus



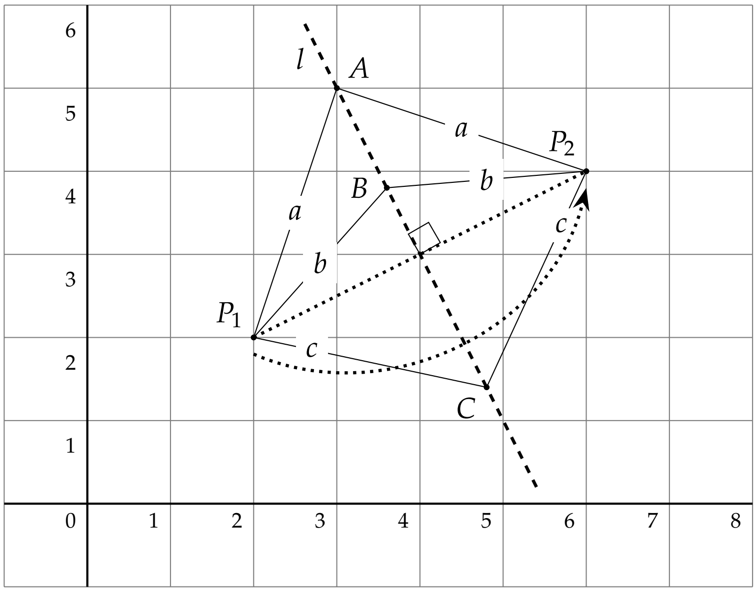
Let us look at Axiom 2: given two distinct points P_1, P_2, there is a single fold l that places P_1 onto P_2.

Use a sheet of paper and perform the following operations:

- Choose two points and label them P_1 and P_2.
- Fold the paper so that P_1 is placed onto P_2 and label the line created by the fold as l.
- Choose an arbitrary point on l and label it A.
- Show by repeating the fold that $\overline{AP_1} = \overline{AP_2}$, that is, the distance of A from P_1 equals the distance of A from P_2.



- Choose two more arbitrary points on l and label them B, C.
- Show by repeating the fold that $\overline{BP_1} = \overline{BP_2}$ and $\overline{CP_1} = \overline{CP_2}$.
- What can you conclude about *all* the points on the line l? The generalization is valid because A, B, C were *arbitrary* points on l.



Open the Geogebra application for Axiom 1 and move the slider so that the point moves on the fold—the red dashed line. Observe the distances of the points from the points P_1, P_2. The right angle and the distances are computed by Geogebra. What can you say about the distances? Do the results from the application support your conclusion from the previous paragraph?

Is it possible that there are points not on l whose distance from P_1 is equal to its distance from P_2? Prove your claim. Hint: assume that such a point exists, label it A' and derive a contradiction from the properties of the isoceles triangle $\triangle P_1 A' P_2$.

What can we conclude about the geometric locus defined by l?

Prove that l is the perpendicular bisector of $\overline{P_1 P_2}$.

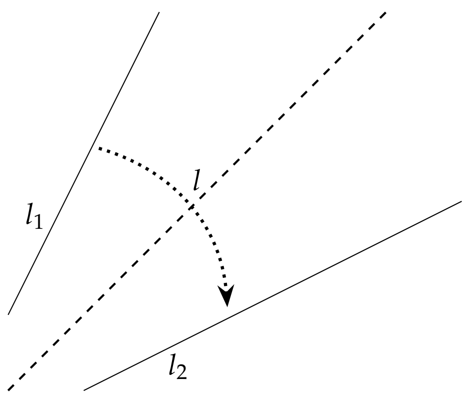
> For all points A on l, $\overline{AP_1} = \overline{AP_2}$. l is the perpendicular bisector of $\overline{P_1 P_2}$ and therefore: **the geometric locus of all points equidistant from P_1, P_2 is the perpendicular bisector of $\overline{P_1 P_2}$.**

4.2 One and two lines as loci

Consider Axiom 3: given two lines l_1, l_2 there are one or two folds that place l_1 onto l_2.

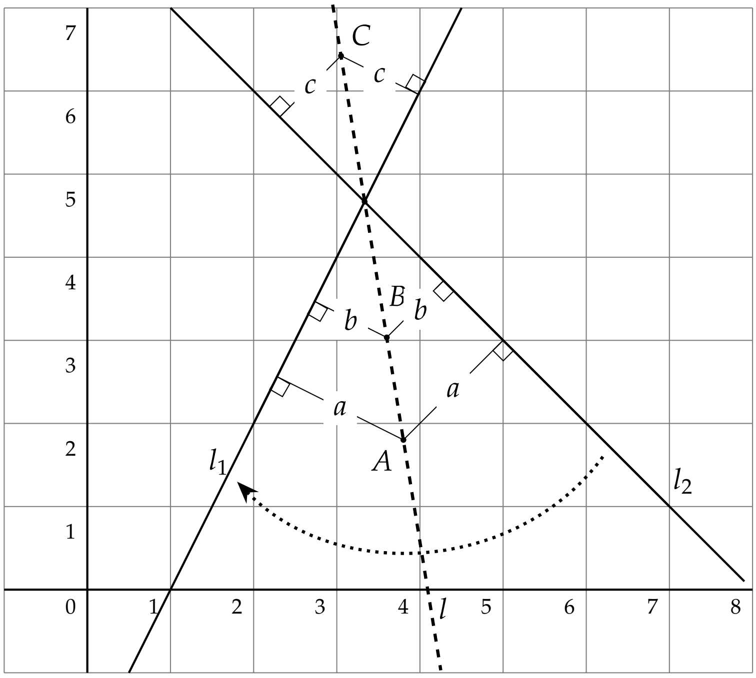
Case 1: l_1, l_2 **intersect**

- Draw two intersecting lines on a sheet of paper and label them l_1 and l_2.
- Fold the paper such that l_1 is placed onto l_2. Use a pen to mark the line created by the fold and label it l.
- Choose an arbitrary point on l and label it A.



- Construct a perpendicular line from A to l_1.
- Prove by folding that the distance from A to l_1 is equal to the distance from A to l_2. Hint: First show that there are two triangles that are congruent by side-side-side and then use the properties of congruent triangles to prove the claim.
- Can you prove the same claim for *all* points on l?

Every point on l is equidistant from l_1 and l_2.



Open the Geogebra application for Axiom 3. The size of the angles is computed by Geogebra. The perpendiculars from a point to the lines are labeled to check that the distances are equal. Experiment with the application. What property is satisfied by the fold?

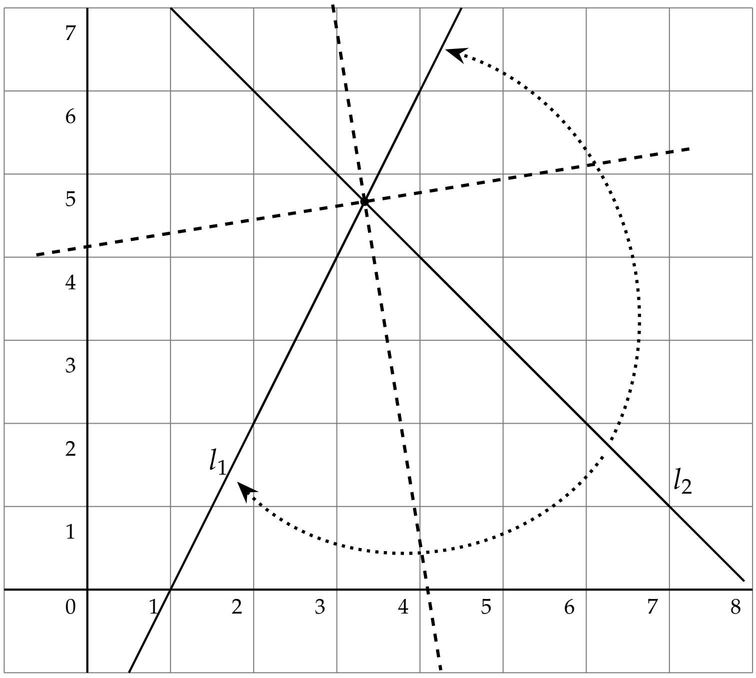
Prove that l bisects the vertical angles at the intersection of l_1 and l_2.

Can you find another fold that places l_1 onto l_2? If so, label it l'. What can you conclude about the points on l'?

Are there more folds that place l_1 onto l_2?

What is are the geometric loci represented by l and l'?

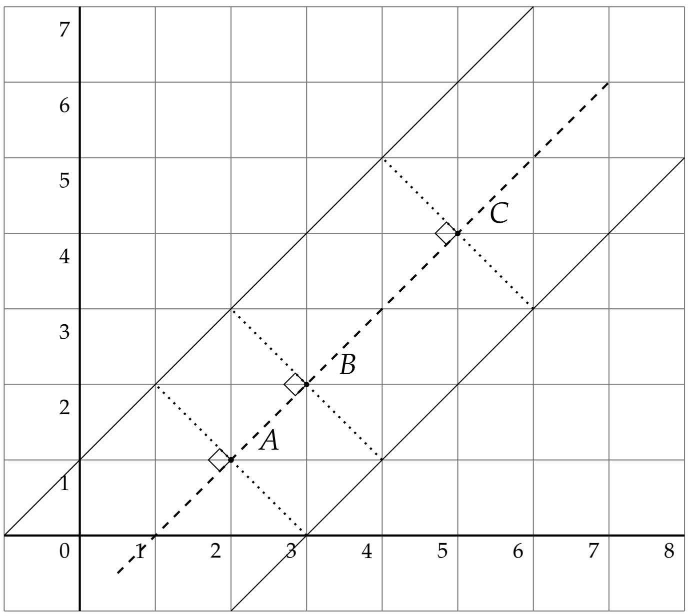
The geometric loci of all points equidistant from two intersecting lines are the bisectors of the vertical angles at the point of intersection.



Case 2: l_1 is parallel to l_2

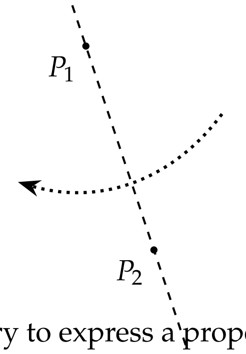
- Draw two parallel lines on a sheet of paper and label them l_1 and l_2.
- Fold the paper so that l_1 is placed onto l_2 and label the line constructed by the fold l.
- Are there additional folds that place l_1 onto l_2?
- Characterize the geometric locus of l. Hint: Choose a point A on l and show that it is equidistant from l_1 and l_2.

The geometric loci is a line parallel to l_1 and l_2 and equidistant from them.



4.3 A locus that is a line segment

Consider Axiom 1: Given two points P_1, P_2, there is a single fold l that passes through both points.



Take a sheet of paper and perform the following operations:

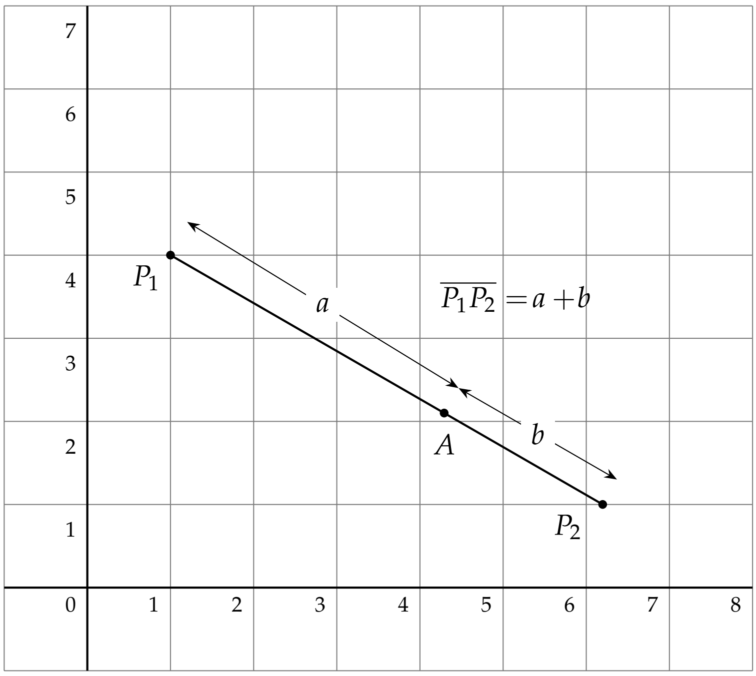
- Choose two points and label them P_1, P_2.
- Construct a fold that passes through both points.
- Mark the line constructed by the fold and label it l.

Let us consider first only points that lie on the line segment $\overline{P_1P_2}$. Try to express a properties that characterizes all the points on the segment. Hint: Choose an arbitrary point on $\overline{P_1P_2}$. What is the sum of the distances from P_1 and P_2?

Are there other points in the plane not on $\overline{P_1P_2}$ that satisfy the same property? Justify your answer. Hint: The sum of the length of two sides of a triangle is always greater than the length of the third.

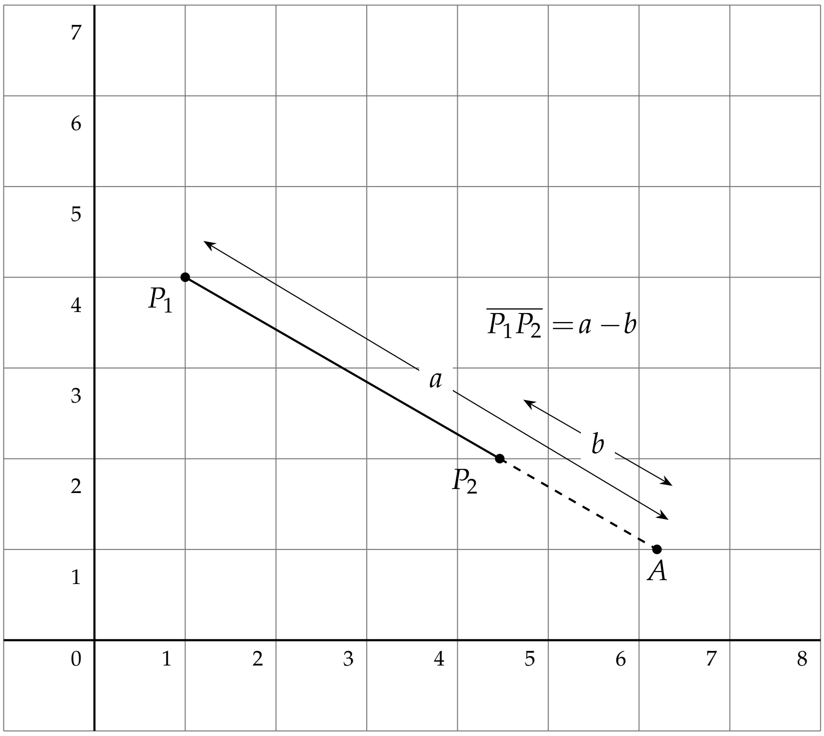
For an arbitrary point A on $\overline{P_1P_2}$, $\overline{P_1P_2} = \overline{P_1A} + \overline{AP_2}$.

The line segment $\overline{P_1P_2}$ is the geometric locus of all points the sum of whose distances from P_1 and P_2 is equal to the length of $\overline{P_1P_2}$.



Consider now the points on l that are not on $\overline{P_1P_2}$. Try to find a property that characterizes all such points. Hint: Choose an arbitrary point A on l but not on $\overline{P_1P_2}$ and look at the difference between the lengths $\overline{AP_1}$ and $\overline{AP_2}$.

Are there other points on the plane not on l that satisfy this property? Hint: The sum of the length of two sides of any triangle is always greater than the length of the third side.

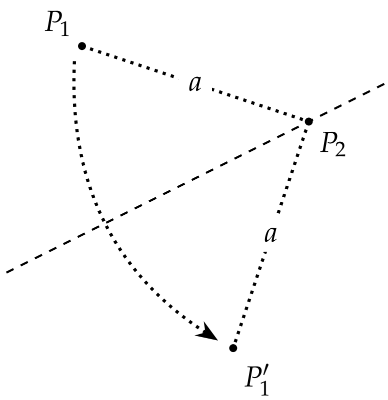


We have found two properties one of which is satisfied by every point on l. Can you characterize the geometric locus defined by l?

Conclusion
l, the line constructed by the fold, is the geometric locus of all points A such that the sum or difference of the lengths of A from P_1 and P_2 is equal to the length of $\overline{P_1P_2}$.

4.4 Loci appearing in Axiom 5

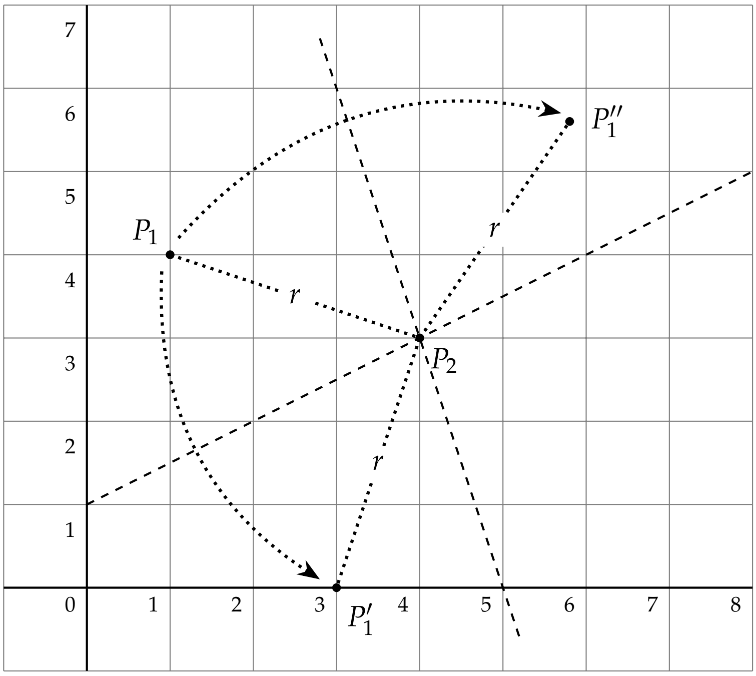
4.4.1 The circle as a locus



Take a sheet of paper and perform the following actions:

- Mark two points P_1, P_2.
- Construct a fold through P_2 but don't open the paper.
- Mark the point P_1' that P_1 lands on. Tip: make a hole through the paper at P_1 to simplify marking P_1'.
- Explain why the distance from P_1 to P_2 is equal to the distance from P_1' to P_2 (both are labeled a in the diagram).
- Open the paper and perform the actions again and again with different folds.

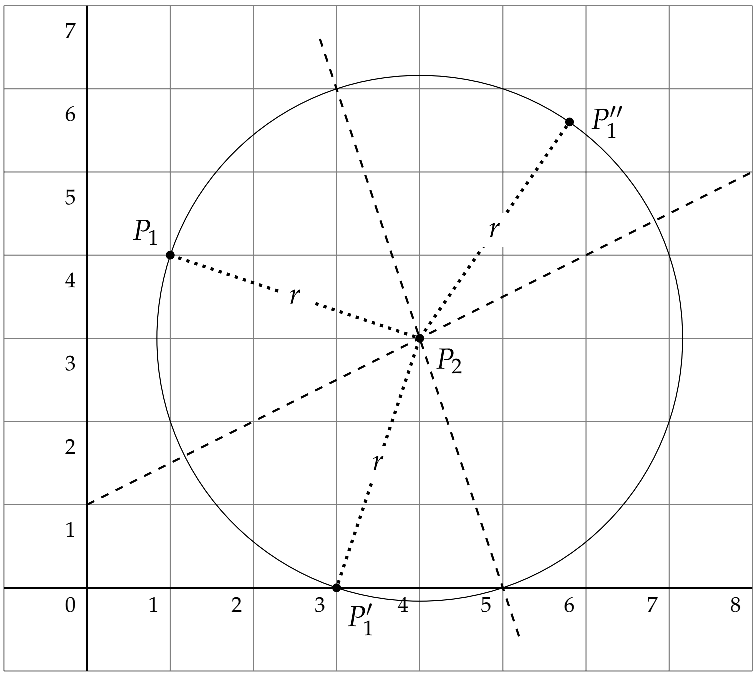
What is the common property of all the new points that are constructed?



All the points are the same distance from P_2; this distance is the length of $\overline{P_1P_2}$.

Looking at the paper, what figure is formed by the new points?

Observe the Geogebra application called `circle.ggb`. Given points P, Q, the slider can be used to change the slope of the fold that passes through point Q. P' is the reflection of point P around the fold and its place will change as the slope of the fold is changed. If you check the box labeled "Show trace of P_1'", you can see all the reflected points.[1]



[1]The track can be erased by pressing "Ctrl-F".

The reflected points form a circle. What is its center and radius?

Could there be reflected points not on the circle, that is, inside or outside the circle? Explain.

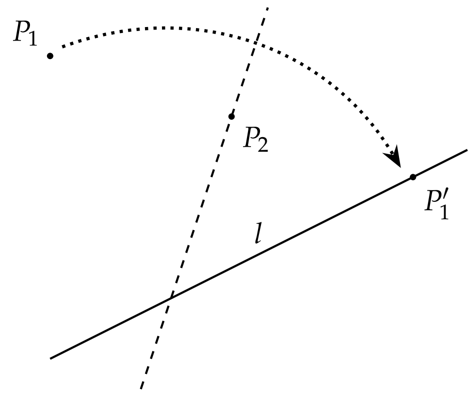
What is the geometric locus of all points at a fixed distance from a given point?

$\boxed{\text{The geometric locus of points at a fixed distance from a given point is a circle.}}$

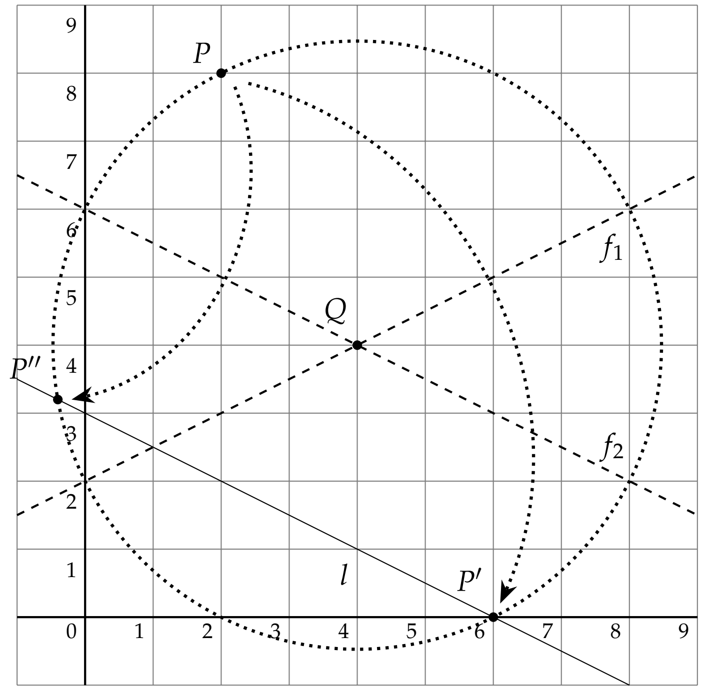
4.4.2 Individual points are loci

Consider Axiom 5: given two points P_1, P_2 and line l, there is a fold that places P_1 onto l and passes through P_2.

Previously we saw that for any fold that passes through P_2, the possible reflections for P_1 form a circle whose center is P_2 and whose radius is the length of the line segment $\overline{P_1P_2}$. Axiom 5 adds a condition.



- What is the condition?
- Of all the folds that constructed the circle, how may can possibly place P_1 onto the line l? There are three cases because the l can intersect the circle in zero, one or two points. Use the Geogebra application to help answer this question.



Explanation of the Geogebra application Given points P, Q and line l, there are two folds f_1, f_2 through Q that place P onto l at P', P''. In the previous application we saw that the geometric locus of the reflected points is a circle whose center is Q and whose radius is the length of $\overline{PQ}$. The point P', P'' define the possible folds. Note that there will be no folds if the line l does not intersect the circle and there will be one fold if the line is tangent to the circle.

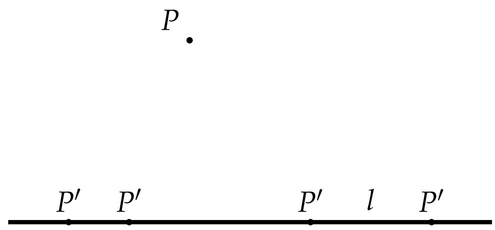
Axiom 5 describes the geometric locus of points on line l whose distance from P_2 is the length of the line segment $\overline{P_1P_2}$. What is the form of the geometric locus in each of the cases described above?

> Answers: Zero, one or two points depending on the number of intersections of l with the circle formed by the points at distance $\overline{P_1P_2}$ from P_2.

4.5 Loci appearing in Axiom 6

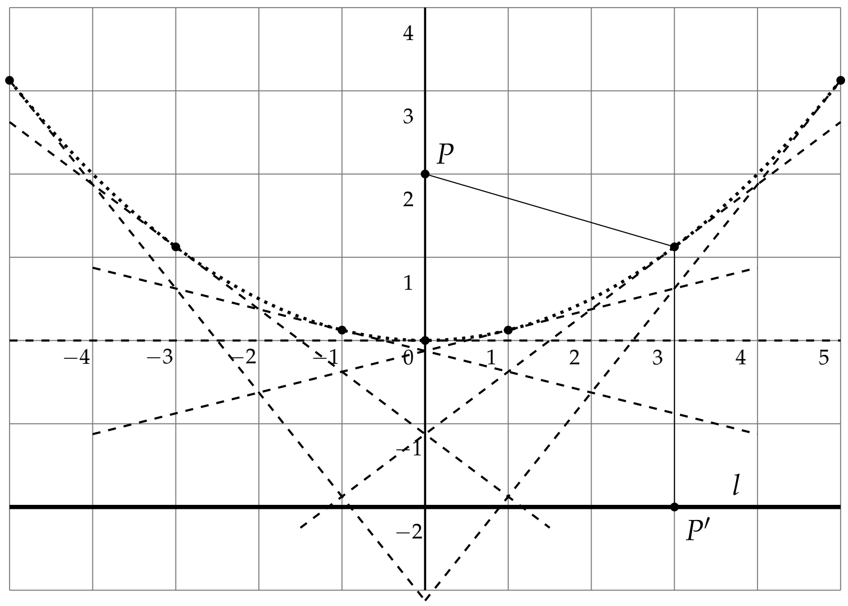
4.5.1 The parabola as a locus

Take a sheet of paper and perform the following actions:



- Mark the line l that is one edge of the paper.
- Choose a point P *near* l and a point P' *on* l.
- Fold the paper so that P' is place on P.
- Open the paper and perform this action again and again for different positions of P' on l.

Open the paper and observe the form created by the multiple folds. If necessary, perform additional folds until the form becomes clear.



Use the Geogebra application `parabola.ggb` and select “Show trace of P” and move the slider to change the position of the point P. Then select “Show locus” to see the parabola formed by the positions of point P.

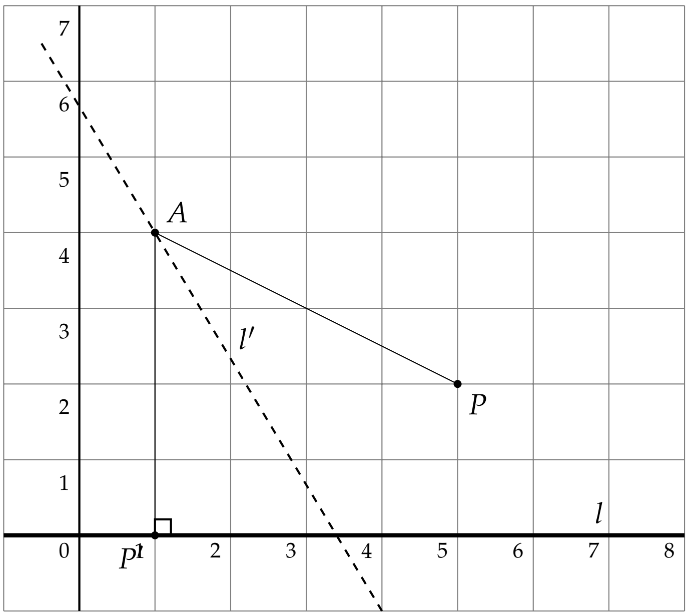
What do you think is the relationship between the folds and the parabola? Later we will prove that the folds are tangents to the parabola.

Write down the definition of a parabola? Determine the focus and directrix for the parabola defined by the folds.

How is the parabola formed?

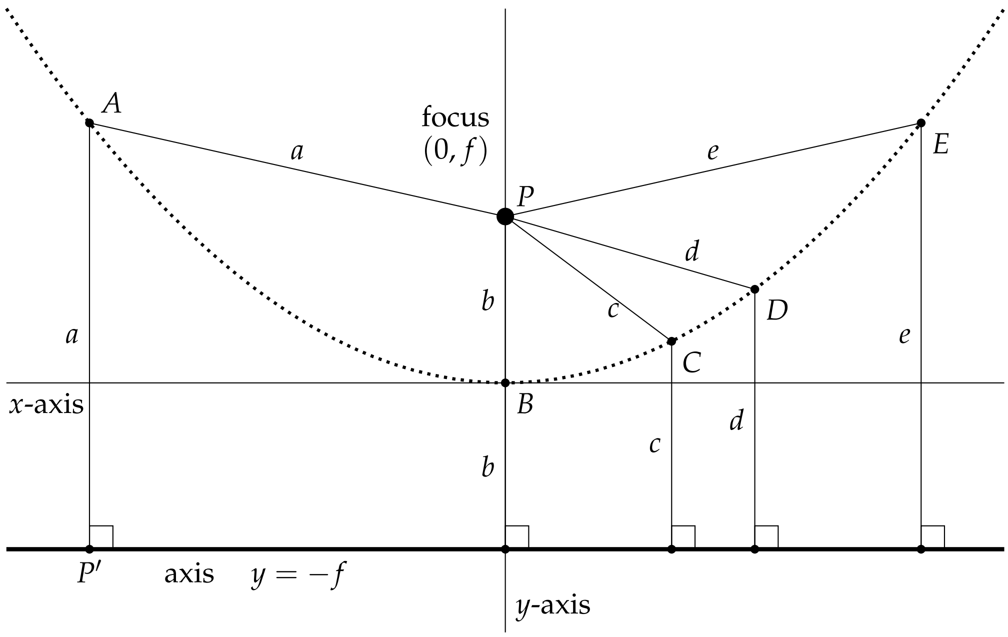
- Choose a point P' on l on the sheet of paper you have been using. Fold the paper so that P' is placed onto P. Mark the line created by the fold and label it l'.
- Construct a perpendicular to l through the point P'. Label the intersection of the perpendicular and l'.
- Show by folding that that $\overline{AP} = \overline{AP'}$. What can you conclude about the point A?

The distance of A from P is equal to the distance from A to l. Therefore, A is a point on the parabola whose focus is P and whose directrix is l.



If we take into consideration all the possible folds, we obtain a parabola.

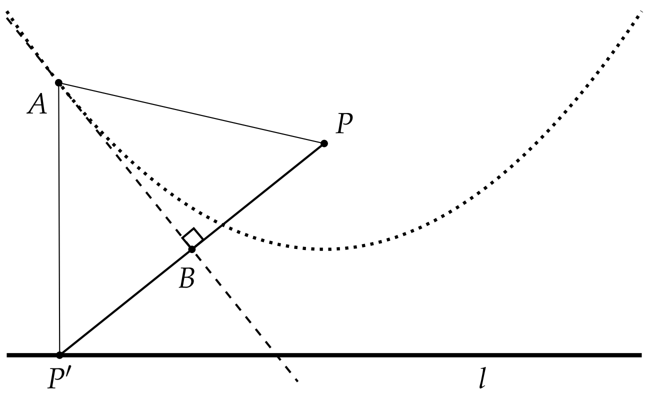
The geometric locus of all points equidistant from a given point (the focus) and a given line (the directrix) is a parabola.



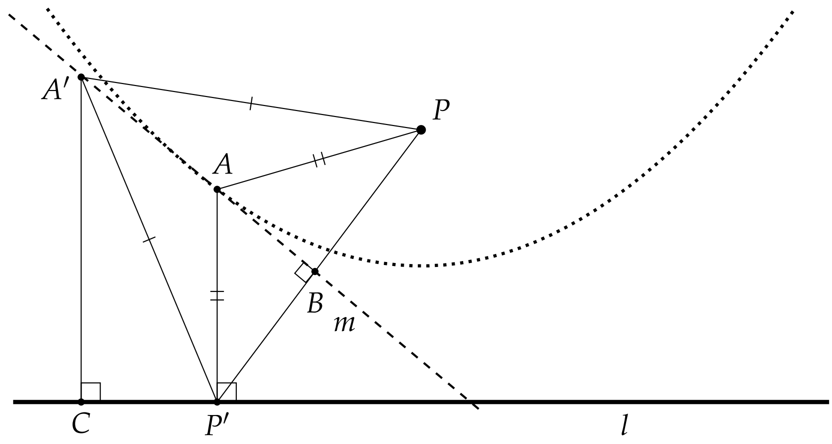
4.5.2 The tangent to a parabola

Previously we saw that the set of folds that place a given point onto a given line forms a parabola. We now prove that each of these folds is a tangent to the parabola.

- Given a parabola with focus P and directrix l, let P' be a point on l. Prove that the fold that places P onto P' is the perpendicular bisector of the line segment $\overline{PP'}$. Hint: Prove that $\triangle APB \cong \triangle AP'B$.



- Prove that the perpendicular bisector of the line segment connecting the focus with a point on the directrix is a tangent to the parabola.
- Label the perpendicular bisector by m and label its intersection with $\overline{PP'}$ by B.
- Suppose that m is not a tangent of the parabola. Then m intersects the parabola in two different points, so there is a point A' on m that is distinct from A. Drop a perpendicular from A' to l and label its intersection with l by C.
- A' is on the perpendicular bisector of $\overline{PP'}$ so $\overline{A'P} = \overline{A'P'}$ because $\triangle A'PB \cong \triangle A'P'B$.



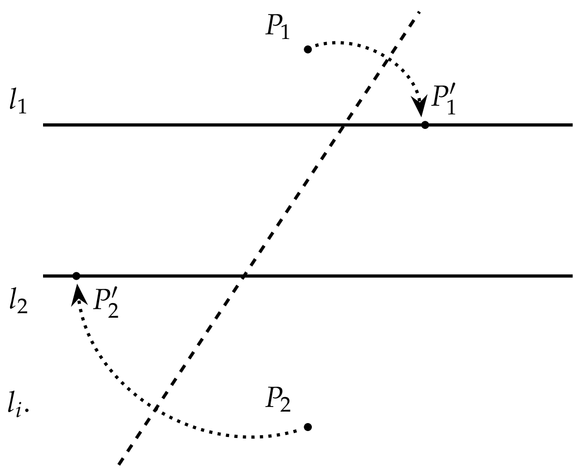
- A' is on the parabola so $\overline{A'P} = \overline{A'C}$ and $l \perp \overline{A'C}$. It follows that $\overline{A'P'} = \overline{A'C}$.
- It follows that in the right triangle $\triangle A'CP'$, the length of the side $\overline{A'C}$ is equal to the length of the hypotenuse $\overline{A'P'}$ which is impossible.
- We conclude that m cannot intersect the parabola in more than one point so it is a tangent to the parabola.

> The perpendicular bisector of the line segment connecting the focus of a parabola and a point on the directrix is a tangent to the parabola.

4.6 The locus described by Axiom 6

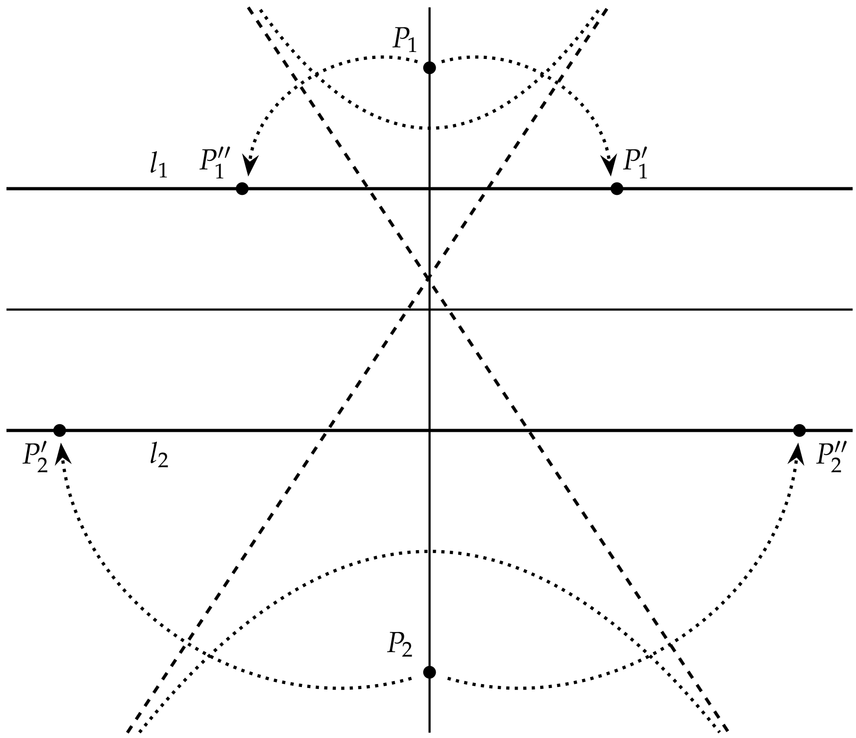
Consider Axiom 6: Given two points P_1, P_2 and two lines l_1, l_2, there is a fold that places P_1 on l_1 and P_2 on l_2.

We saw that the set of folds that place a point P on a line l create a parabola. In addition we proved that each fold is a tangent to the parabola. Therefore, the folds that place P_i on l_i, $i = 1, 2$ are the tangents to the parabolas whose foci are P_i and whose directrices are l_i.



How can a fold simultaneously satisfy these two conditions?

The fold must be a tangent of both parabolas!

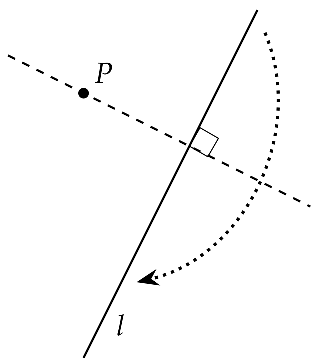


- Do all pairs of parabolas have a common tangent?
- How many common tangents can a pair of parabolas have? See Appendix B.
- What does it mean in terms of Axiom 6 for a pair of parabolas to have more than one common tangent?

> A fold resulting from Axiom 6 is a common tangent to two parabolas with foci P_1, P_1 and directrices l_1, l_2.
> Two parabolas can have zero, one, two or three common tangents.
> Therefore, there can be zero, one, two or three folds satisfying Axiom 6 for any given P_1, P_2, l_1, l_2.

4.7 The locus described by Axiom 4

Axiom 4 Given a point P and a line l, there is a single fold perpendicular to l that passes through P.



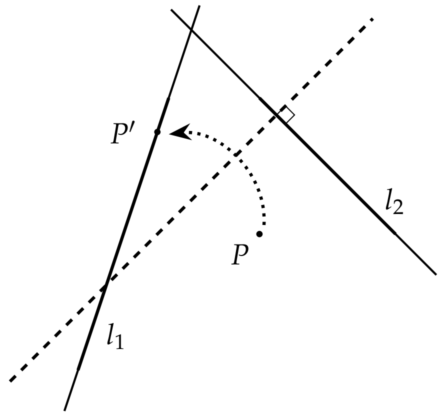
- Take a sheet of paper and mark an arbitrary point P and an arbitrary line l.
- Fold the paper so that the fold is perpendicular to l and passes through P.

- What is the geometric locus of the fold?

The fold created by Axiom 4 is the geometric locus of all points on the line perpendicular to l and passing through P.

4.8 The locus described by Axiom 7

Axiom 7 Given a point P and two lines l_1, l_2, there is a fold that places P onto l_1 and is perpendicular to l_2.



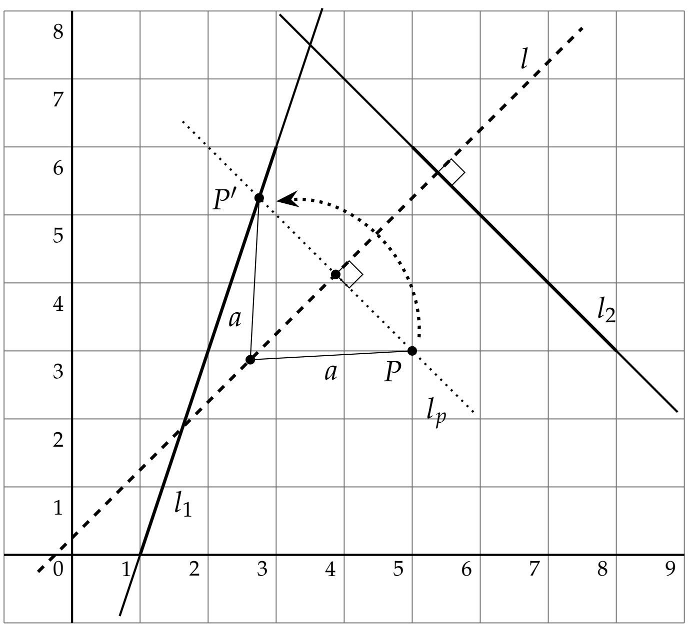
- Take a sheet of paper and mark a point P and two lines l_1, l_2.
- Fold the paper as described in Axiom 7: a fold perpendicular to l_2 that places P onto l_1.
- Leave the paper folded and mark the point upon which P is placed by P'.
- Open the paper.

Explain why every point on the fold is equidistant from p and P'. Hint: Recall from Axiom 2 and the activities that its geometric locus is the perpendicular bisector.

What characterizes the set of points on the fold created by Axiom 7?

What is the geometric locus described by those points?

The fold created by Axiom 7 is the geometric locus of all points on the line perpendicular to l_2 and equidistant from P and P', the reflection of P onto l_1.



5 Exercises for geometric loci

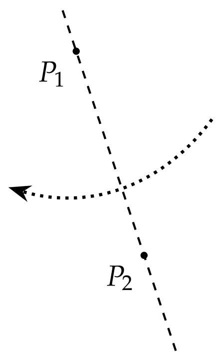
The following exercises are rather basic and can be integrated into the activities or used to summarize the activities. Some of them can be solved using algebra alone, but insights from origami can deepened the students' understanding of the axioms.

1. Find the geometric locus of all the points whose distance from the point $(-8, -6)$ is equal to their distances from $(12, 4)$.
2. Find the geometric locus of all the points whose sum or difference of their distances from $P_1 = (2, 3)$ and $P_2 = (6, 4)$ is equal to the length of the line segment $\overline{P_1 P_2}$.
3. (a) Find the geometric locus of all the points whose distance from the line $5x + 3y - 14 = 0$ is equal to their distance to the line $3x + 5y - 34 = 0$.
   (b) What is the form of the geometric locus that you found?
   (c) Explain why all the lines you found in the previous items bisect the angle between the two given lines.
4. Find the geometric locus of all points whose distance from the line $y = 2x + 6$ is equal to the distance from the line $y = 2x - 4.5$.
5. (a) Find the geometric locus of all the points whose distance from the point $(3, -6)$ is 9.
   (b) What is the form of the geometric locus that you found?
6. Find the geometric locus of all the points on the line $y = x + 1$ whose distance from the point $(2, 10)$ is 5.
7. (a) What is the form of the geometric locus of all points whose distance from the point $(0, 3)$ is equal to their distance from the line $y = -3$.
   (b) What is the geometric locus?

A Experiments with the origami axioms

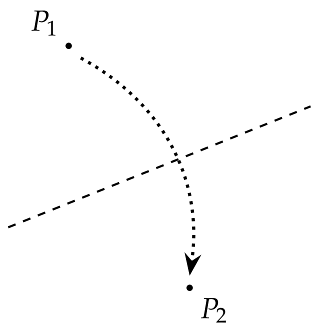
Axiom 1 Given two points P_1, P_2, there is a single fold that passes through them.

Task: Label two arbitrary points on a sheet of paper. Fold the paper so that the fold pass through the two labeled points.



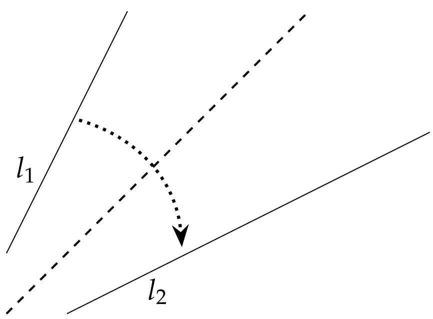
Axiom 2 Given two points P_1, P_2, there is a single fold that places P_1 onto P_2.

Task: Label two arbitrary points on a sheet of paper. Fold the paper so that one point is placed on top of the other.



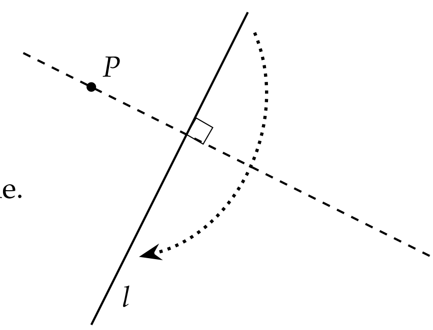
Axiom 3 Given two lines l_1, l_2, there is a fold that places l_1 onto l_2.

Task: Draw two arbitrary intersecting lines on a sheet of paper. Fold the paper so that one line is placed onto the other. Is there more than one fold that can do this?



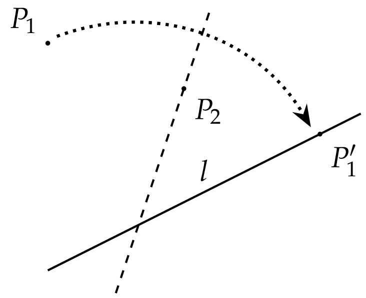
Axiom 4 Given a point P and a line l, there is a single fold perpendicular to l that passes through P.

Task: Draw an arbitrary line and an arbitrary point on a sheet of paper. Fold the paper so that the fold passes through the point and is perpendicular to the line.



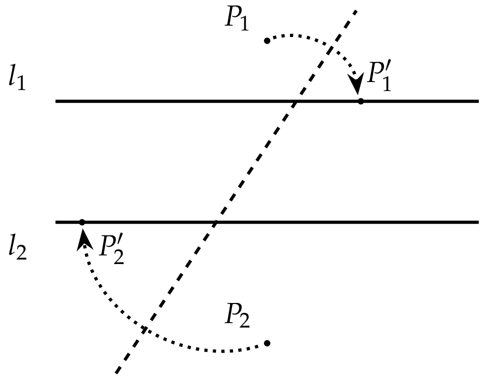
Axiom 5 Given two points P_1, P_2 and a line l, there is a fold that places P_1 onto l and passes through P_2.

Task: Draw two arbitrary points and an arbitrary line on a sheet of paper. Fold the paper so that the fold passes through one point and places the other point on the line.



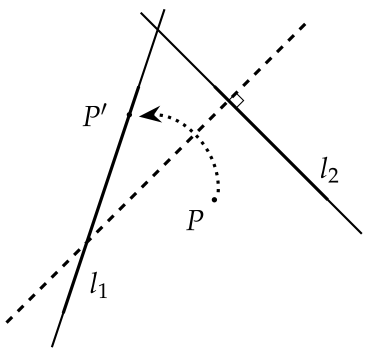
Axiom 6 Given two points P_1, P_2 and two lines l_1, l_2, there is a fold that simultaneously places P_1 onto l_1 and P_2 onto l_2.

Task: Draw two arbitrary points and two arbitrary lines on a sheet of paper. Fold the paper so that each point is one of the lines.



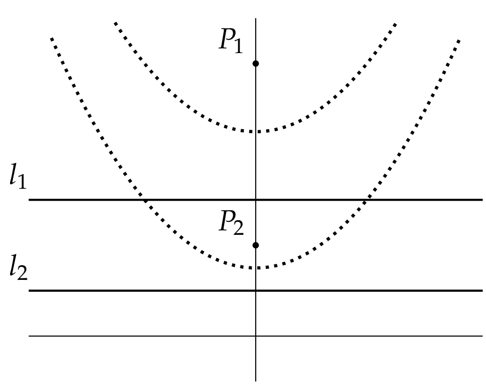
Axiom 7 Given a point P and two lines l_1, l_2, there is a fold that places P onto l_1 and is perpendicular to l_2.

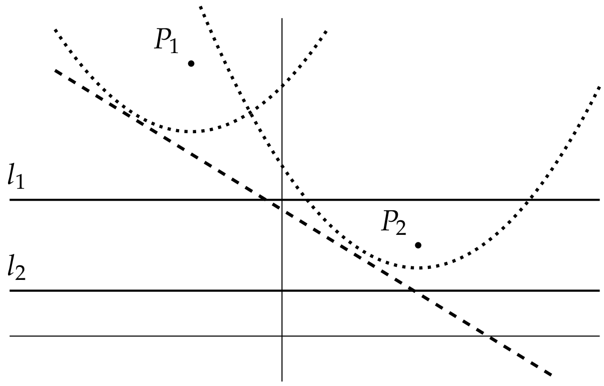
Task: Draw an arbitrary point and two arbitrary lines on a sheet of paper. Fold the paper so the point is placed onto one of the lines and the fold is perpendicular to the other line.

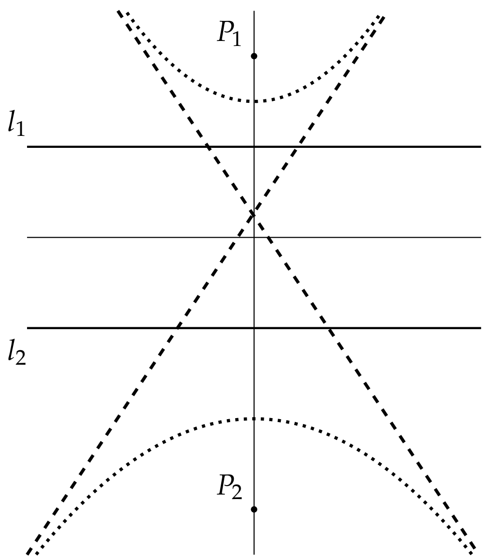


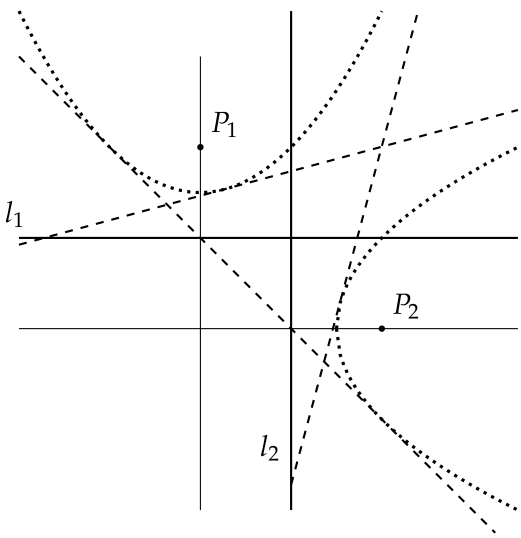
B The number of common tangents to two parabolas

The following diagrams that two parabolas can have zero, one, two or three common tangents.









C Solutions to the exercises in Section 5

Question 1 Find the geometric locus of all the points whose distance from the point $(-8, -6)$ is equal to their distances from $(12, 4)$.

Solution

The geometric locus of all points equidistant from P_1 and P_2 is the perpendicular bisector of the segment $\overline{P_1P_2}$. Let us find the equation of the perpendicular bisector of the segment connecting $(-8, -6)$ and $(12, 4)$. The slope is the negative inverse of the segment, which is -2 since the slope of the segment is:

$$\frac{4 - (-6)}{12 - (-8)} = \frac{1}{2}.$$

The midpoint of $\overline{P_1P_2}$ is:

$$\left(\frac{-8 + 12}{2}, \frac{-6 + 4}{2}\right) = (2, -1).$$

The equation of the perpendicular bisector is:

$$\begin{aligned} y - (-1) &= -2(x - 2) \\ y &= -2x + 3. \end{aligned}$$

Question 2 Find the geometric locus of all points whose sum or difference of their distances from $P_1 = (2, 3)$ and $P_2 = (6, 4)$ is equal to the length of the line segment $\overline{P_1P_2}$.

Solution In Activity 3 we saw that the geometric locus of all points whose sum or difference of their distance from P_1 and P_2 is equal to the length of $\overline{P_1P_2}$ is the line that passes through P_1 and P_2.

The slope is:

$$\frac{4 - 3}{6 - 2} = \frac{1}{4},$$

so the equation of the line is:

$$\begin{aligned} y - 3 &= \frac{1}{4}(x - 2) \\ y &= \frac{2}{3}(x + 1). \end{aligned}$$

Question 3

1. Find the geometric locus of all points whose distance from the line $5x + 3y - 14 = 0$ is equal to their distance to the line $3x + 5y - 34 = 0$.

2. What is the form of the geometric locus that you found?

3. Explain why all the lines you found in the previous items bisect the angle between the two given lines.

Solution

(1) Let (x,y) be any point that satisfies the condition. We will use the formula for finding the distance of a point from a line in the plane. We do this for each line and then equate them. The distance of (x,y) from $5x+3y-14$ is:

$$\left|\frac{5x+3y-14}{\sqrt{5^2+3^2}}\right| = \left|\frac{5x+3y-14}{\sqrt{34}}\right| .$$

The distance of (x,y) from $3x+5y-34$ is:

$$\left|\frac{3x+5y-34}{\sqrt{3^2+5^2}}\right| = \left|\frac{3x+5y-34}{\sqrt{34}}\right| .$$

Equating the formulas and multiplying by $\sqrt{34}$:

$$|5x+3y-14| = |3x+5y-34| .$$

There are two cases depending on the signs:

$$\begin{aligned} 5x+3y-14 &= 3x+5y-34 \\ x-y+10 &= 0 , \end{aligned}$$

and

$$\begin{aligned} 5x+3y-14 &= -(3x+5y-34) \\ x+y-6 &= 0 . \end{aligned}$$

(2) The geometric locus is two lines.

(3) This was proved in the Activity.

Question 4 Find the geometric locus of all points whose distance from the line $y=2x+6$ is equal to the distance from the line $y=2x-4.5$.

Solution

Both of the given lines have the same slope so they are parallel. We saw that when two lines are parallel, the geometric locus of the points equidistant from them is the line parallel to them and equidistant from them.

The line with have an equation of the form $-2x-y+c=0$. Since it is equidistant from them, we use the equation for the distance between two parallel lines and equate them:

$$\frac{|6-c|}{\sqrt{(-2)^2+(-1)^2}} = \frac{|-4.5-c|}{\sqrt{(-2)^2+(-1)^2}} .$$

It follows that:

$$|6-c| = |-4.5-c| ,$$

so the equation has the single solution $c=\dfrac{3}{4}$ and the line of the equation is $-2x-y-\dfrac{3}{4}$.

Question 5

1. Find the geometric locus of all the points whose distance from the point $(3, -6)$ is 9.

2. What is the form of the geometric locus that you found?

Solution

(1) Let (x, y) be any point that satisfies the condition. We will use the formula for finding the distance between two points:

$$\begin{aligned} 9 &= \sqrt{(x-3)^2 + (y+6)^2} \\ 81 &= (x-3)^2 + (y+6)^2 \,. \end{aligned}$$

(2) The geometric locus is a circle.

Question 6 Find the geometric locus of all points on the line $y = x + 1$ whose distance from the point $(2, 10)$ is 5.

Solution The geometric locus of all points whose distance from $(2, 10)$ is 5 is a circle whose equation is $(x-2)^2 + (y-10)^2 = 25$. We saw that the geometric locus that we seek is the point or points of intersection of the line with the circle. Substitute the equation for the line into the equation of the circle:

$$\begin{aligned} (x-2)^2 + ((x+1) - 10)^2 &= 25 \\ (x-2)^2 + (x-9)^2 &= 25 \\ x^2 - 4x + 4 + x^2 - 18x + 81 &= 25 \\ 2x^2 - 22x + 60 &= 0 \\ x = &= 5,\ 6\,. \end{aligned}$$

By substituting these values into the equation for the line we obtain that the two points of intersection are $(6, 7)$, $(5, 6)$.

Question 7

1. What is the form of the geometric locus of all point whose distance from the point $(0, 3)$ is equal to their distance from the line $y = -3$.

2. What is the geometric locus?

Solution

(1) The geometric locus is a parabola whose focus is $(0, 3)$ and whose directrix is $y = -3$.

(2) We use the formula for the equation of a parabola:

$$\begin{aligned} &x^2 = 4 \cdot 3y \\ &x^2 = 12y\,. \end{aligned}$$

D Geogebra constructions for the origami axioms

The italicized words refer to menu selections in Geogebra.

Axiom 1 Construct two *points*. Construct a *line* throught the points.

Axiom 2 Construct two *points*. Construct the *perpendicular bisector* of the line *segment* connecting the two points. Construct the *reflection* of one point relative to the line to check that it is placed on the other point.

Axiom 3 For the case where l_1, l_2 intersect: construct two *segments* that intersect or that would intersect if extended. Construct the *angle bisector* between the segments. (You can extend the segments so that they intersect but that is not necessary.) Construct the *reflection* of one line relative to the bisector and check if it is placed on the other line. (You can change the color of one line to make it clear that they are placed on each other.)

For the case where l_1, l_2 are parallel: construct two parallel *segments*. Construct a *perpendicular* to one segment and construct its *intersection* with the second segment (extended if necessary). Construct the *perpendicular bisector* of the segment between between the two lines. Construct the *reflection* of one line relative to the bisector and check that it is placed on the other line. (You can change the color of one line to make it clear that they are placed on each other.)

Axiom 4 Construct a *line* and a *point* not on the line. Construct a *perpendicular* to the line through the point. Construct a *reflection* of some point on the line relative to the perpendicular to check that it is placed on the line.

Axiom 5 Construct a *line* l and *points* P_1, P_2. Construct a *circle through point* P_1 with center P_2. Construct the *intersections* A, B of l and the circle. Construct line *segments* $\overline{P_1A}, \overline{P_1B}$. Construct a *perpendiculars* through P_2 those segments. Construct a *reflection* of P_1 relative to one of the perpendiculars to check that it is placed on the line. Repeat with the other perpendicular.

Axiom 6 Construct two *lines* l_1, l_2 and two *points* P_1, P_2. Construct a *parabola* with focus P_1 and directrix l_1 and *parabola* with focus P_2 and directrix l_2. Construct *tangents* to both parabolas. Construct the *reflections* of P_1, P_2 relative to one of the tangents and check that they are placed on the lines l_1, l_2.

Axiom 7 Construct two line *segments* l_1, l_2 and a *point* P. Construct a *parallel* to l_2 through p, and construct P'_1, the *intersection* of the parallel and l_1. (Extend l_1 if necessary.) Construct the *segment* $\overline{PP'}$ and construct its *perpendicular bisector*. Construct the *reflection* of P relative to the bisector and check that it is placed on P'.

Further reading

Applications of origami [4]. Mathematical theory of origami [1, 2]. History of origami [3]. Paper folding [5, 6].

References

[1] Roger C. Alperin. A mathematical theory of origami constructions and numbers. *New York Journal of Mathematics*, 6:119–133, 2000.

[2] Moti Ben-Ari. The mathematics of origami. `https://www.weizmann.ac.il/sci-tea/benari/mathematics#origami`, 2020.

[3] Koshiro Hatori. History of origami. `https://origami.ousaan.com/library/historye.html`, n.d.

[4] Robert Lang. The math and magic of origami. `https://www.ted.com/talks/robert_lang_the_math_and_magic_of_origami`, 2008.

[5] Robert J. Lang. Origami and geometric constructions. `http://langorigami.com/wp-content/uploads/2015/09/origami_constructions.pdf`, 1996-–2015. Accessed 26/02/2020.

[6] Liz Newton. The power of origami. `https://plus.maths.org/content/power-origami`. Accessed 26/02/2020.