$\sqrt{\mathbf{x}+\mathbf{5}} = \mathbf{5} - \mathbf{x}^2$

Moti Ben-Ari

http://www.weizmann.ac.il/sci-tea/benari/

Solve for x. Square both sides of the equation and collect terms:

$$f(x) = x^4 - 10x^2 - x + 20 = 0\,.$$

Does the polynomial of degree four factor as two polynomials of degree two with integer coefficients? If so, *the coefficients of the x terms must be equal and of opposite signs*, since there is no x^3 term! Let n be a positive integer and k_1, k_2 be any integers:

$$f(x) = (x^2 - nx + k_1)\,(x^2 + nx + k_2)\,.$$

Carry out the multiplication:

$$\begin{array}{lllllll} f(x) = & x^4 & +nx^3 & +k_2x^2 & & \\ & & -nx^3 & -n^2x^2 & -nk_2x & \\ & & & +k_1x^2 & +nk_1x & +k_1k_2\,. \end{array}$$

Equating the coefficients results in three equations in three unknowns:

$$\begin{array}{rcl} (k_1 + k_2) - n^2 & = & -10 \\ n(k_1 - k_2) & = & -1 \\ k_1k_2 & = & 20\,. \end{array}$$

From the last two equations and the choice of n as a positive integer, it is clear that:

$$k_1 = 4,\, k_2 = 5 \;\text{ or }\; k_1 = -5,\, k_2 = -4\,.$$

Only $k_1 = -5$, $k_2 = -4$ satisfy the first equation for the coefficient of the x^2 term:

$$f(x) = (x^2 - x - 5)\,(x^2 + x - 4)\,.$$

$f(x)$ can be zero if either factor is zero. Solving the quadratic equations gives four possible solutions:

$$\frac{1 \pm \sqrt{21}}{2}\;,\quad \frac{-1 \pm \sqrt{17}}{2}\,.$$

Because of the square root in $\sqrt{x+5}$, we have $5 - x^2 \geq 0$ and:

$$-2.24 \approx -\sqrt{5} \leq x \leq \sqrt{5} \approx 2.24$$

By numerically computing the roots, we see that there are only two solutions:

$$\frac{1 - \sqrt{21}}{2} \approx -1.79\;,\quad \frac{-1 + \sqrt{17}}{2} \approx 1.56\,.$$