How to Do Trigonometry Without Memorizing (Almost) Anything

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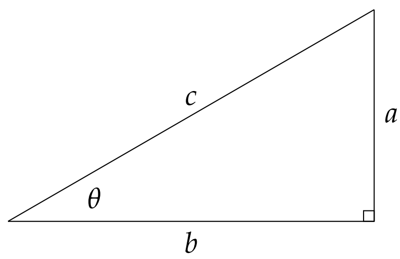
Trigonometry facilitates geometric reasoning using algebraic computation. To the student trigonometry can appear as a large set of obscure formulas to be memorized. The purpose of this document is to show that trigonometric identities can be obtained by geometric reasoning with little memorization.

The appendices contains proofs of the law of sines and the law of cosines. Although these formulas are easy to memorize, it is useful to see how they can be proved using only geometric facts.

1 Basic definitions

We have to start with definitions that must be memorized. In a right triangle:



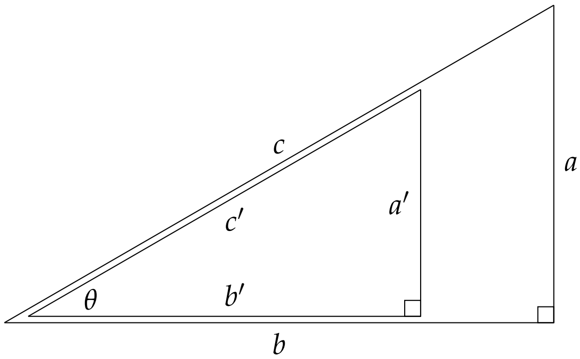
the sine and cosine of an angle are defined as the ratios of the sides of the triangle with the hypotenuse:[1]

$$\sin\theta = \frac{a}{c}, \quad \cos\theta = \frac{b}{c}\,.$$

[1]This document does not discuss the tangent. Since $\tan\theta = \sin\theta/\cos\theta$, identities for tangent can be obtained from those for sine and cosine.

The ratios can be verbally expressed as “sine is opposite over hypotenuse” and “cosine is adjacent over hypotenuse.”

The trigonometric functions are functions of the angle alone and do not depend on the size of the triangle. Given two right triangles with the same angles:



the similarity of the triangles gives:

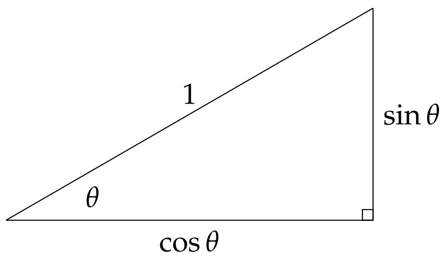
$$\frac{c'}{c} = \frac{a'}{a} = \frac{b'}{b},$$

and then:

$$\frac{a}{c} = \frac{a'}{c'} = \sin\theta$$

$$\frac{b}{c} = \frac{b'}{c'} = \cos\theta\,.$$

Since we are free to choose the length of one of the sides, it will make our lives easier if we choose the length of the hypotenuse to be 1:

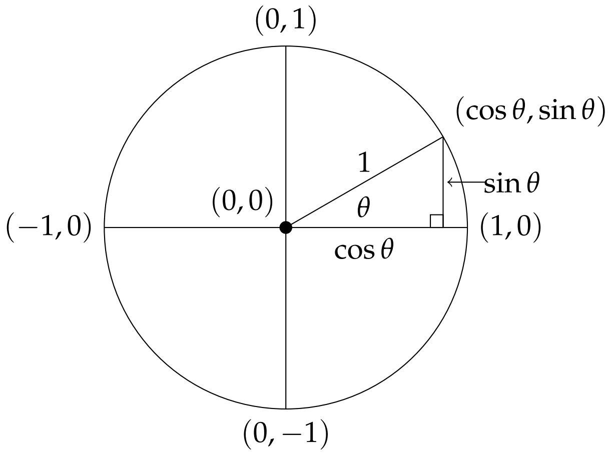


Therefore, the denominator of the sine and cosine ratios is 1 and can be ignored. The functions $\sin\theta$ and $\cos\theta$ give the lengths of the opposite and adjacent sides of the triangle. From Pythagoras’s theorem we have:

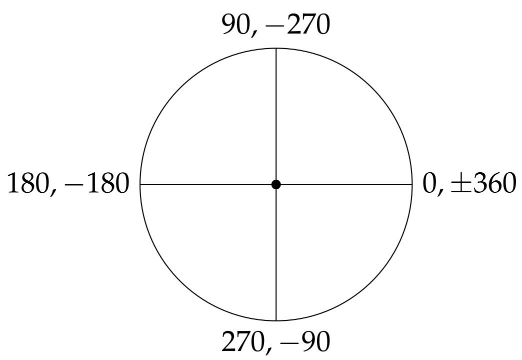
$$\sin^2\theta + \cos^2\theta = 1^2 = 1$$

2 The unit circle

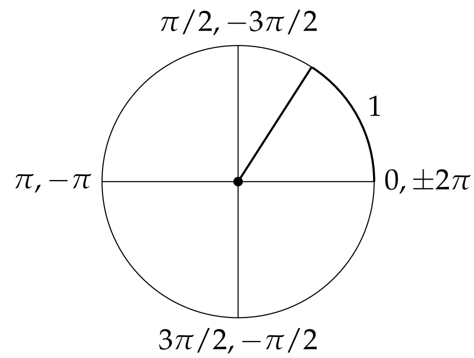
By taking the hypotenuse of the right angle to be 1, we can transform the entire presentation of trigonometry into the *unit circle* in the coordinate system of the plane. The values $\cos\theta$ and $\sin\theta$ are not only the lengths of the sides of the triangle, but also the coordinates $(\cos\theta, \sin\theta)$ of the intersection of the ray from the origin with the unit circle:



The unit of angles is the *degree*. In a circle, angles are measured starting from the positive x-axis, counterclockwise for positive angles and clockwise for negative angles. There are 360 degrees in a circle (notation 360°). The axes of the Cartesian coordinate system naturally divide the unit circle into four *quadrants*:



An alternate unit of angles is the *radian*. One radian is the angle that subtends an arc on a circle whose length is equal to the radius. Since the radius of the unit circle is 1, its circumference is of length 2π, and as a ray traces out the entire circumference (counterclockwise) it moves from an angle of 0 radians to an angle of 2π radians:



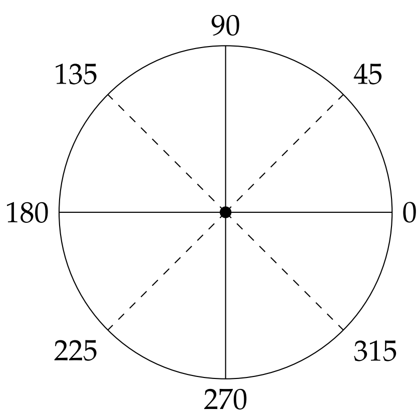
One radian equals approximately 57.3 degrees.

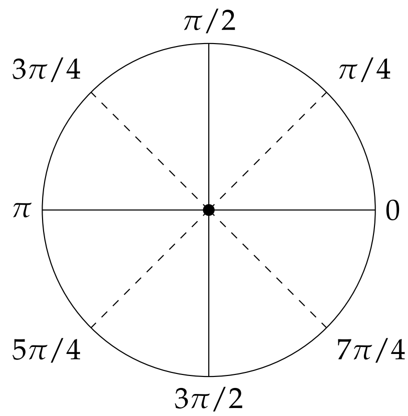
From the coordinates of the intersection of the x- and y-axes with the unit circle, we can read off the values of sine and cosine for these angles:

Angle (degrees)	Angle (radians)	sin	cos
0	0	0	1
90	$\pi/2$	1	0
180	π	0	-1
270	$3\pi/2$	-1	0

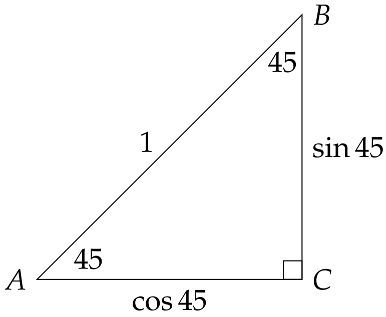
3 Dividing the unit circle into 8 segments

We have seen that the axes divide the unit circle into 4 quadrants. It is useful to divide the unit circle into 6, 8 and 12 segments, and to learn the sine and cosine of the corresponding angles. First, divide each quadrant in half to produce 8 segments, where the angle of each segment is 45° or $\pi/4$ radians:





What are the sine and cosine of 45°? In the triangle:



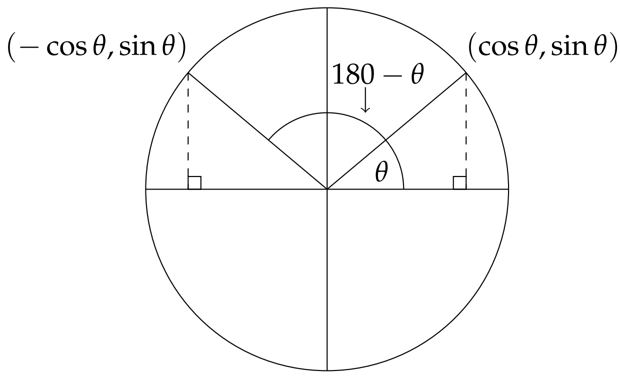
If the angle $\angle BAC$ is $45°$, the opposite angle $\angle ABC$ must also be $45°$ so that the sum of the angles in the triangle is $180°$. The triangle is isoceles so the sine and the cosine are equal. By Pythagoras's theorem:

$$\begin{aligned}
\sin^2 45 + \cos^2 45 &= 1 \\
2\sin^2 45 &= 1 \\
\sin 45 &= \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{2}{2} = \frac{\sqrt{2}}{2} \\
\cos 45 &= \sin 45 = \frac{\sqrt{2}}{2}.
\end{aligned}$$

4 The sine and cosine of angles greater than $90°$

Now that we know the sine and cosine of $45°$, we can ask about the other three symmetrical angles $135°, 225°, 315°$. With the help of our friend the unit circle, we can immediately find their sine and cosine.

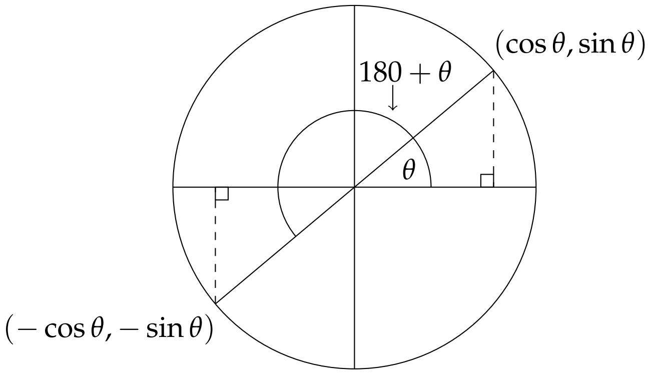
We first do so for an arbitrary angle θ in the first quadrant. Since the projections of the rays on the x- and y-axes are the same, we only have to change the sign of the result. For the second quadrant:



so:

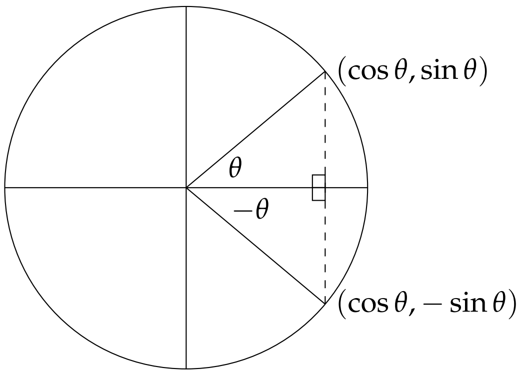
$$\begin{aligned}
\cos 135 &= \cos(180 - 45) = -\cos 45 = -\frac{\sqrt{2}}{2} \\
\sin 135 &= \sin(180 - 45) = \sin 45 = \frac{\sqrt{2}}{2}.
\end{aligned}$$

For the third quadrant:



$$\cos 225 = \cos(180 + 45) = -\cos 45 = -\frac{\sqrt{2}}{2}$$
$$\sin 225 = \sin(180 + 45) = -\sin 45 = -\frac{\sqrt{2}}{2}.$$

For the fourth quadrant, it convenient to use the negative angle $-\theta$ instead of the positive angle $360 - \theta$:



$$\cos 315 = \cos(-45) = \cos 45 = \frac{\sqrt{2}}{2}$$
$$\sin 315 = \sin(-45) = -\sin 45 = -\frac{\sqrt{2}}{2}.$$

Summarizing the results in a table:

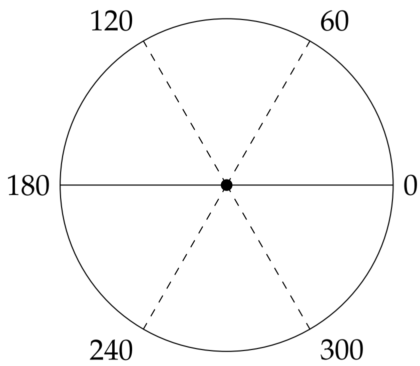
Angle (degrees)	Angle (radians)	sin	cos
θ	θ	$\sin\theta$	$\cos\theta$
$180 - \theta$	$\pi - \theta$	$\sin\theta$	$-\cos\theta$
$180 + \theta$	$\pi + \theta$	$-\sin\theta$	$-\cos\theta$
$-\theta$	θ	$-\sin\theta$	$\cos\theta$

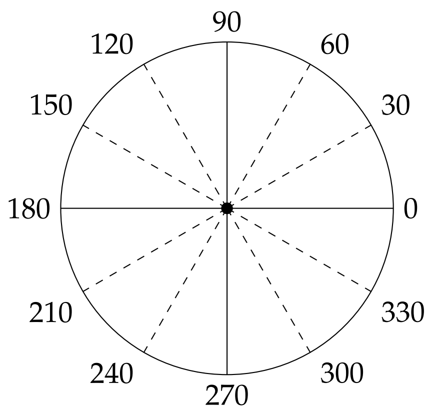
For 45°:

Angle (degrees)	Angle (radians)	sin	cos
45	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
135	$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$
225	$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$
315	$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$

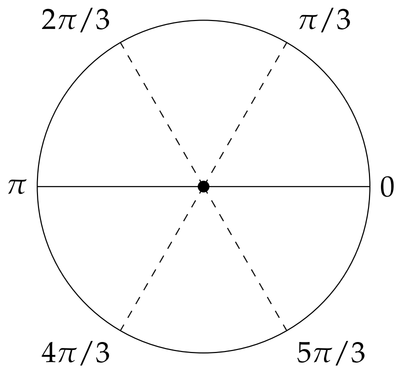
5 The sine and cosine of 30° and 60°

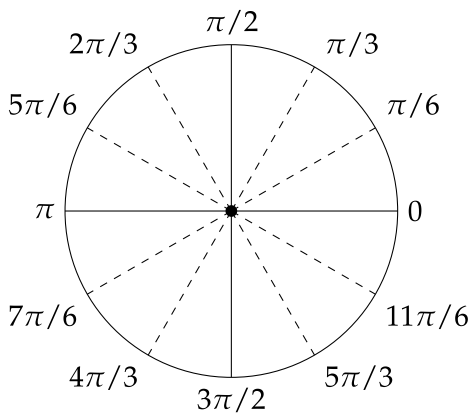
The unit circle can also be divided into 6 segments of 60° and 12 segments of 30°:



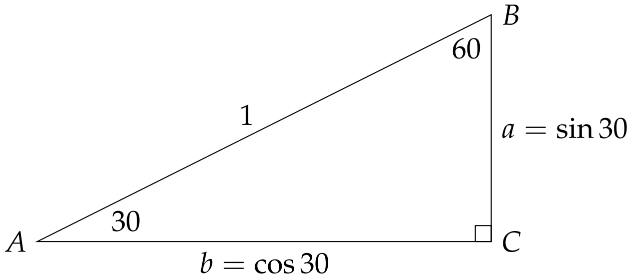


In radians:

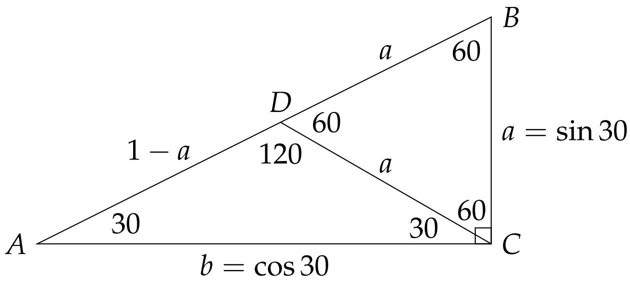




Let us first compute the sine of 30°. In the right triangle:



draw a line from C to the hypotenuse that makes an angle of 30° with the line AC:



Using various geometric facts about the angles in a triangle, we have inferred the other angles in the figure. Since $\triangle BCD$ is equilateral, all its sides are equal to $a = \sin 30$. Recall that $AB = 1$ since we are in the unit circle, so $DA = 1 - a$. Since $\triangle ACD$ is isoceles, $a = 1 - a$. Therefore:

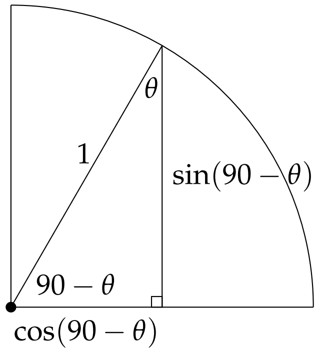
$$\begin{aligned}\sin 30 &= a = 1 - a \\ &= \frac{1}{2}.\end{aligned}$$

From $\sin^2\theta + \cos^2\theta = 1$, we obtain:

$$\cos 30 = \sqrt{1 - \sin^2 30} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

6 The sine and cosine of $(90 - \theta)$

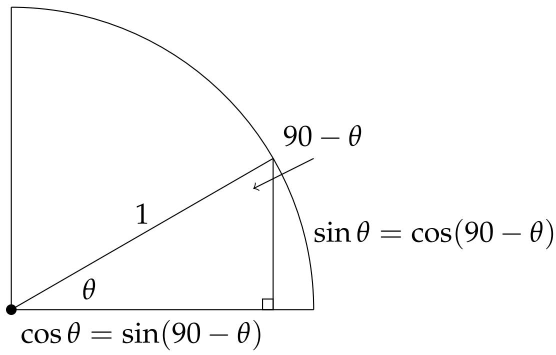
Let us now turn to the computation of the sine and cosine of 60°. Since $60 = 90 - 30$, we suspect that there might be a simple relation between the trigonometric functions of 60° and 30°. This becomes clear when we draw an angle of $90 - \theta$ in the unit circle:



Since the angle where the triangle intersects the unit circle is θ, the trigonometric functions of $90 - \theta$ can be obtained from those of θ by switching "opposite" and "adjacent":

$$\begin{aligned} \cos(90 - \theta) &= \sin\theta \\ \sin(90 - \theta) &= \cos\theta \,. \end{aligned}$$

Another way of seeing this is to note that the triangle is congruent to the triangle we used to compute $\sin\theta$ and $\cos\theta$:



It follows that:

$$\begin{aligned} \cos 60 = \cos(90 - 30) = \sin 30 = \frac{1}{2} \\ \sin 60 = \sin(90 - 30) = \cos 30 = \frac{\sqrt{3}}{2} \,. \end{aligned}$$

The trigonometric functions for multiples of 30°, 60° can be easily computed by examining the unit circle:

Angle (degrees)	Angle (radians)	sin	cos
0	0	0	1
30	$\pi/6$	$1/2$	$\sqrt{3}/2$
60	$\pi/3$	$\sqrt{3}/2$	$1/2$
90	$\pi/2$	1	0
120	$2\pi/3$	$\sqrt{3}/2$	$-1/2$
150	$5\pi/6$	$1/2$	$-\sqrt{3}/2$
180	π	0	-1
210	$7\pi/6$	$-1/2$	$-\sqrt{3}/2$
240	$4\pi/3$	$-\sqrt{3}/2$	$-1/2$
270	$3\pi/2$	-1	0
300	$5\pi/3$	$-\sqrt{3}/2$	$1/2$
330	$11\pi/6$	$-1/2$	$\sqrt{3}/2$

7 Multiple angle formulas

Unfortunately, you have to memorize the formula for the sine of the sum of two angles:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta\,.$$

The proof is not hard but it is difficult to reproduce on a moment's notice. The other formulas follow easily. The sine of the difference of two angles is:

$$\sin(\alpha - \beta) = \sin(\alpha + (-\beta)) = \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta\,.$$

For the cosine:

$$\begin{aligned}
\cos(\alpha + \beta) &= \sin(90 - (\alpha + \beta)) \\
&= \sin((90 - \alpha) - \beta) \\
&= \sin(90 - \alpha)\cos\beta - \cos(90 - \alpha)\sin\beta \\
&= \cos\alpha\cos\beta - \sin\alpha\sin\beta\,,
\end{aligned}$$

and:

$$\cos(\alpha - \beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\,.$$

The double angle formulas follow from the formulas for the sum of two angles:

$$\begin{aligned}
\sin 2\alpha &= \sin(\alpha + \alpha) \\
&= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha \\
&= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos(\alpha + \alpha) \\
&= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha \\
&= \cos^2\alpha - \sin^2\alpha \\
&= \cos^2\alpha - (1 - \cos^2\alpha) \\
&= 2\cos^2\alpha - 1 \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\
&= (1 - \sin^2\alpha) - \sin^2\alpha \\
&= 1 - 2\sin^2\alpha\,.
\end{aligned}$$

8 Summary

We started from the definition of the trigonometric functions sine and cosine in terms of the ratios of the sides of a right triangle. Since the absolute values of the sides do not matter, we took the hypotenuse to be 1. From Pythagoras's theorem:

$$\sin^2\theta + \cos^2\theta = 1$$

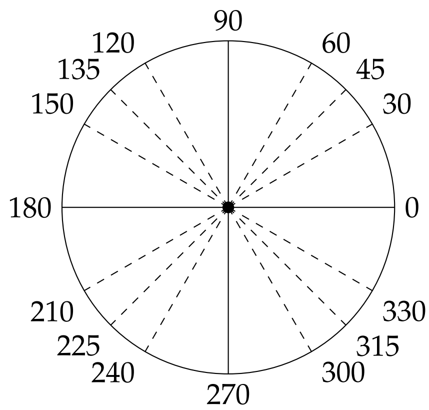
and:

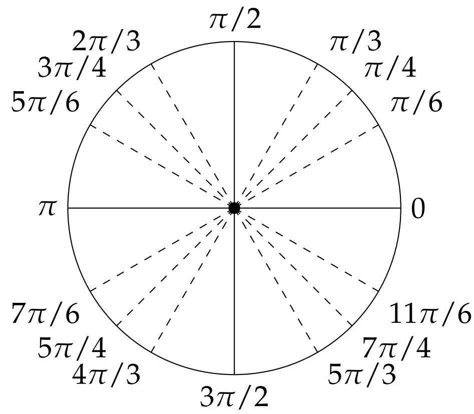
$$\sin 45 = \cos 45 = \frac{\sqrt{2}}{2}\,.$$

Using an elementary construction and geometric facts about triangles, we showed that:

$$\sin 30 = \cos 60 = \tfrac{1}{2}$$
$$\cos 30 = \sin 60 = \tfrac{\sqrt{3}}{2}\,.$$

Given the sine and cosine of any angle θ in the first quadrant, our friend the unit circle enabled us to immediately compute the trigonometric functions of any angle obtained from θ by adding or subtracting multiples of $90°$. In particular, we know the sine and cosine of the following angles:



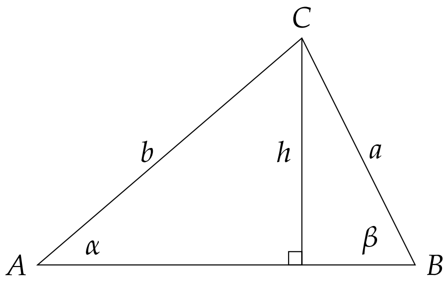


All this without memorizing anything except the definitions of sine and cosine!

The formulas for multiple angles are easily derived from the formulas for $\sin(\alpha + \beta)$, but you will have to memorize that one formula because it is not easy to derive.

A The law of sines

The law of sines is not difficult to memorize, but it is worthwhile to see how easily it can be derived. Given a triangle $\triangle ABC$, drop an altitude from one vertex:



Since:

$$\begin{aligned}\sin\alpha &= \frac{h}{b}\\ \sin\beta &= \frac{h}{a},\end{aligned}$$

we have:

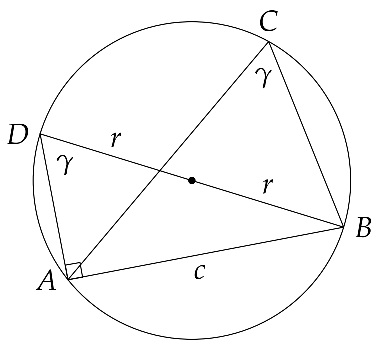
$$\begin{aligned}b\sin\alpha &= a\sin\beta\\ \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b}.\end{aligned}$$

A similar construction gives the equality of these ratios with $\dfrac{\sin\gamma}{c}$.

B The law of sines in a circle

There is an alternate proof of the law of sines that also relates the ratios of the sines to the sides of the triangle to the radius of a circumscribed circle.[2]

Consider the triangle $\triangle ABC$ and its circumscribed circle.[3] (Every triangle can be circumscribed by a circle whose center is the intersection of the perpendicular bisectors of its sides.) Find the point D on the circle such that DB passes through the center of the circle. Draw the line AD:



[2] I would like to thank Avital Elbaum Cohen for bringing this proof to my attention.

[3] This proof is for an acute triangle. The proof for an obtuse triangle is similar.

Since $\angle ADB$ and $\angle ACB$ subtend the same chord AB, they are equal. Since $\angle DAB$ subtends a diameter it is equal to $90°$. Therefore:

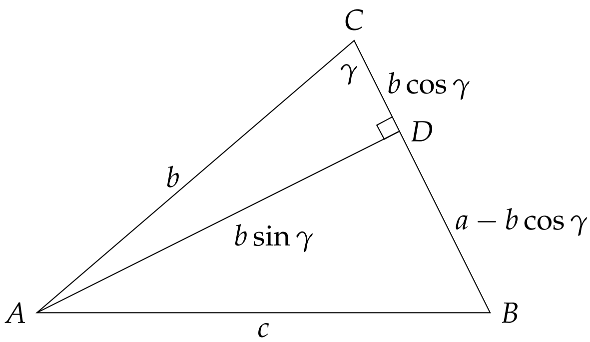
$$\sin\gamma = \frac{c}{2r}\,.$$

A similar construction can be done for the other vertices of the triangle giving:

$$\frac{1}{2r} = \frac{\sin\gamma}{c} = \frac{\sin\beta}{b} = \frac{\sin\alpha}{a}\,.$$

C The law of cosines

The law of cosines is also not hard to memorize and its proof is not obvious. Nevertheless, we give a proof to show that only geometric facts and the definitions of sine and cosine are needed.[4] Drop an altitude from one vertex:



In the right triangle $\triangle ADC$, the hypotenuse is b and we can compute the lengths of the other two sides by trigonometry. The length of DB is the length of CB (a, not denoted in the diagram) minus the length of CD which we just computed. Using Pythagoras's theorem on the triangle $\triangle ABD$:

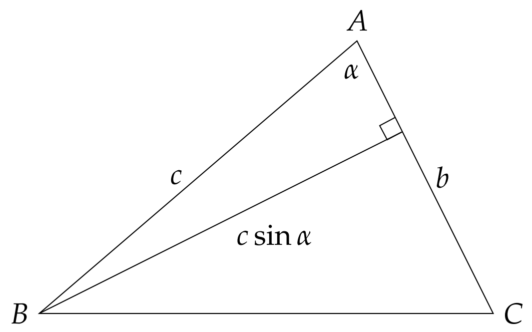
$$\begin{aligned} c^2 &= (a - b\cos\gamma)^2 + b^2\sin^2\gamma \\ &= a^2 - 2ab\cos\gamma + b^2\cos^2\gamma + b^2\sin^2\gamma \\ &= a^2 - 2ab\cos\gamma + b^2(\cos^2\gamma + \sin^2\gamma) \\ &= a^2 + b^2 - 2ab\cos\gamma\,. \end{aligned}$$

Pythagoras's theorem is obtained by setting $\gamma = 90°$, so we see that the law of cosines is a generalization of Pythagoras's theorem.

[4]This proof is for an acute triangle. The proof for an obtuse triangle is similar.

D The area of a triangle

The same construction:



gives the generalized formula for the area of a triangle:

$$S(\triangle ABC) = \frac{1}{2} \cdot \mathit{base} \cdot \mathit{height} = \frac{1}{2} bc \sin \alpha \, .$$

If α is a right angle, $S(\triangle ABC) = \frac{1}{2} bc$, as expected.