Construction of a Regular Heptadecagon

Moti Ben-Ari

http://www.weizmann.ac.il/sci-tea/benari/

Version 1.0

This document presents Gauss's insight that it is possible to construct a *heptadecagon*—a regular polygon with 17 sides—using a straightedge and compass. The presentation is based upon [1]; I have added the detailed calculations leading to Gauss's formula. An actual construction from [3] is also presented; again I have added the detailed calculations.

1 Construction of regular polygons

History The ancient Greeks knew how to construct some regular polygons using a straightedge and compass: a triangle, a square, a pentagon and a regular polygon with 15 sides. Of course, given a regular polygon with n sides, it is easy to construct a polygon with $2n$ sides by bisecting the sides.

No progress was made for two thousand years until in 1796, just before his 19th birthday, Carl Friedrich Gauss awoke one morning and by "concentrated thought" figured out how to construct a regular *heptadecagon*, a regular polygon with 17 sides. This achievement inspired him to become a mathematician.

The construction of the regular heptadecagon led to the Gauss-Wantzel Theorem which states that a regular polygon with n sides can be constructed using a straightedge and compass if and only if n is the product of a power of 2 and zero or more *distinct* prime Fermat numbers $2^{2^k}+1$. The known Fermat primes are $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$. A regular polygon with 257 sides was first constructed by Magnus Georg Paucker in 1822 and Friedrich Julius Richelot 1832. In 1894 Johann Gustav Hermes claimed to have constructed a regular polygon with 65537 sides and his manuscript is saved at the University of Göttigen should you wish to check it.

The cosine of the central angle To construct a regular polygon, it is sufficient to construct a line segment of length $\cos\theta$, where θ is the central angle subtended by a chord that is a side of the polygon inscribed in a unit circle (Figure 1). Given the line segment $\overline{OB} = \cos\theta$, construct a perpendicular at B and label its intersection with the unit circle by C. Then $\cos\theta = \dfrac{\overline{OB}}{\overline{OC}} = \overline{OB}$ so $\theta = \cos^{-1}(\overline{OB})$. The chord $\overline{AC}$ is a side of the polygon.

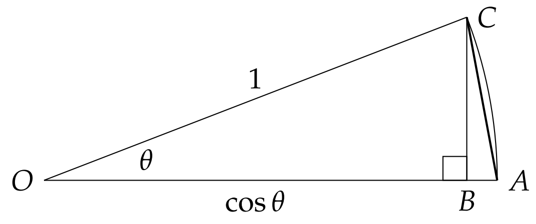


Figure 1: Constructing a side from the cosine of the central angle it subtends

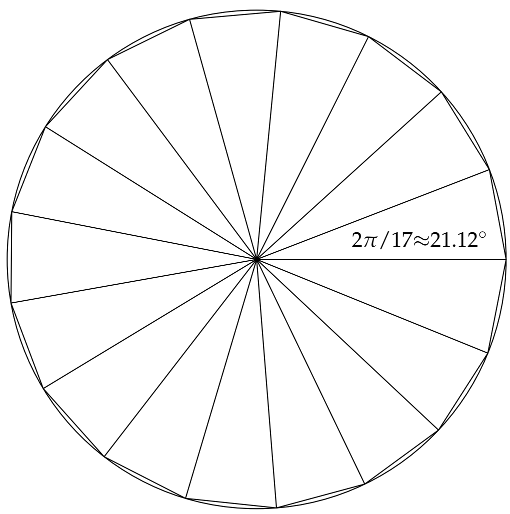


Figure 2: A heptadecagon inscribed in a unit circle

Constructible lengths Given a line segment defined to have length 1, the lengths that are constructible are those which can be obtained from line segments of known length using the operations $\{+, -, \times, \div, \sqrt{}\}$. In Appendices A, B we show that a equilateral triangle and a regular pentagon are constructible by giving expressions for $\cos\left(\frac{2\pi}{3}\right)$ and $\cos\left(\frac{2\pi}{5}\right)$ that use only the operations $\{+, -, \times, \div, \sqrt{}\}$.

Construction of a heptadecagon The central angle of a heptadecagon (Figure 2) is $\frac{2\pi}{17}$ radians or $\frac{360^\circ}{17} \approx 21.12^\circ$. Gauss showed that [1, 2]:

$$
\begin{aligned}
\cos\left(\frac{2\pi}{17}\right) &= -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} + \\
&\quad \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}\,.
\end{aligned}
$$

This value can be computed using the operations $\{+, -, \times, \div, \sqrt{}\}$ so it is constructible!

Sections 2, 3, 4 present Gauss's mathematical ideas, together with the detailed calculations. The proof does not use complex numbers explicitly, though I have added some notes on complex numbers. Section 5 shows an efficient construction of $\cos\left(\frac{2\pi}{17}\right)$.

2 Roots of unity

We use the following theorem without proof:

Fundamental Theorem of Algebra Every polynomial of degree n (with complex coefficients) has exactly n (complex) roots.

Roots of unity and regular polygons Consider the equation $x^n - 1 = 0$ for any integer $n \geq 1$. One root is $x = 1$. By the Fundamental Theorem of Algebra there are $n-1$ other roots. Denote one root by r so that $r^n = 1$. r is called a *root of unity*.

Complex numbers The root r is $\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$.
By de Moivre's formula:

$$\left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^n = \cos\left(\frac{2\cdot n\pi}{n}\right) + i\sin\left(\frac{2\cdot n\pi}{n}\right) = 1\,.$$

Consider now r^2. Then:

$$(r^2)^n = (r^n)^2 = 1^2 = 1\,.$$

It follows that:

$$1, r, r^2, \ldots, r^{n-2}, r^{n-1}$$

are n-th roots of unity.

Theorem Let n be a prime and r an n-th root of unity; then $\{1, r, r^2, \ldots, r^{n-2}, r^{n-1}\}$ are distinct, so they are *all* the n-th roots of unity.

Proof Suppose that $r^i = r^j$ for some $1 \leq i < j \leq n$, so $r^j/r^i = r^{j-i} = 1$. Let m be the smallest number $0 < m < n$ such that $r^m = 1$. Now $n = ml + k$ for some $0 < l < n$ and $0 \leq k < m$. From $1 = r^n = r^{ml+k} = (r^m)^l \cdot r^k = 1^l \cdot r^k = r^k$, we have $0 \leq k < m$ and $r^k = 1$. Since m is the smallest such positive integer, $k = 0$ and $n = ml$, so n is not prime.

We use the following theorem without proof.

Theorem Let $\{a_1, a_2, \ldots, a_{n-1}, a_n\}$ be the roots of an n-th degree polynomial $f(x)$. Then

$$f(x) = (x - a_1)(x - a_2)\cdots(x - a_{n-1})(x - a_n)\,.$$

Vieté's formula [1, p. 28] gives the coefficients of the polynomial in terms of its roots; the formula can be obtained by multiplication. You can see that the coefficient of x^{n-1} is:

$$-(a_1 + a_2 + \cdots + a_{n-1} + a_n)\,.$$

Since the coefficient of x^{n-1} in $x^n - 1$ for $n \geq 2$ is obviously zero, we have:

$$-(1 + r + r^2 + \cdots + r^{n-2} + r^{n-1}) = 0\,.$$

We will use this in the form:

$$r + r^2 + \cdots + r^{n-2} + r^{n-1} = -1\,.$$

For the heptadecagon this is:

$$r + r^2 + r^3 + r^4 + r^5 + r^6 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12} + r^{13} + r^{14} + r^{15} + r^{16} = -1\,.$$

3 Gauss's proof that a heptadecagon is constructable

What Gauss saw is the one need not work with the roots in the natural order $r, r^2, \ldots, r^{16}$. Instead, one can notice that the powers of r^3 give all the roots, but in a different order. For $k < 17$, $r^{17m+k} = (r^{17})^m \cdot r^k = 1^m \cdot r^k = r^k$, so the exponents are reduced modulo 17:

$$r^1,\ r^{1\cdot 3=3},\ r^{3\cdot 3=9},\ r^{9\cdot 3=27=10},\ r^{10\cdot 3=30=13},\ r^{13\cdot 3=39=5},\ r^{5\cdot 3=15},\ r^{15\cdot 3=45=11},$$
$$r^{11\cdot 3=33=16},\ r^{16\cdot 3=48=14},\ r^{14\cdot 3=42=8},\ r^{8\cdot 3=24=7},\ r^{7\cdot 3=21=4},\ r^{4\cdot 3=12},\ r^{12\cdot 3=36=2},\ r^{2\cdot 3=6}\,.$$

It is important that you check that this list contains each of the 16 roots exactly once.

The roots of a quadratic equation Consider the monic quadratic equation:

$$y^2 + py + q = 0\,,$$

and suppose that its roots are a, b. Then:

$$(y - a)(y - b) = y^2 - (a + b)y + ab\,.$$

Therefore, $p = -(a + b)$ and $q = ab$, so that if we are *given* $a + b$ and ab, we can write down the quadratic equation of which a, b are the roots.[1]

Let a_0 be the sum of the roots in the odd positions in the above list:

$$a_0 = r + r^9 + r^{13} + r^{15} + r^{16} + r^8 + r^4 + r^2\,,$$

and let a_1 be the sum of the roots in the even positions in the above list:

$$a_1 = r^3 + r^{10} + r^5 + r^{11} + r^{14} + r^7 + r^{12} + r^6\,.$$

We want to obtain a_0, a_1 as roots of a quadratic equation. We first compute their sum:

$$a_0 + a_1 = r + r^2 + \cdots + r^{16} = -1\,.$$

[1] Po-Shen Lo used this observation to develop a quick method for solving quadratic equations. See [4] and a document on my website.

$$
\begin{aligned}
a_0a_1 \;&=\; (r + r^9 + r^{13} + r^{15} + r^{16} + r^8 + r^4 + r^2) \times \\
&\qquad (r^3 + r^{10} + r^5 + r^{11} + r^{14} + r^7 + r^{12} + r^6) \\
&=\; \underset{1}{r^{4}} + \underset{1}{r^{11}} + \underset{1}{r^{6}} + \underset{1}{r^{12}} + \underset{1}{r^{15}} + \underset{1}{r^{8}} + \underset{1}{r^{13}} + \underset{1}{r^{7}} + \\
&\quad\; \underset{2}{r^{12}} + \underset{1}{r^{2}} + \underset{1}{r^{14}} + \underset{1}{r^{3}} + \underset{2}{r^{6}} + \underset{1}{r^{16}} + \underset{2}{r^{4}} + \underset{2}{r^{15}} + \\
&\quad\; \underset{2}{r^{16}} + \underset{3}{r^{6}} + \underset{1}{r^{1}} + \underset{2}{r^{7}} + \underset{1}{r^{10}} + \underset{2}{r^{3}} + \underset{2}{r^{8}} + \underset{2}{r^{2}} + \\
&\quad\; \underset{2}{r^{1}} + \underset{3}{r^{8}} + \underset{3}{r^{3}} + \underset{1}{r^{9}} + \underset{3}{r^{12}} + \underset{1}{r^{5}} + \underset{2}{r^{10}} + \underset{3}{r^{4}} + \\
&\quad\; \underset{3}{r^{2}} + \underset{2}{r^{9}} + \underset{4}{r^{4}} + \underset{3}{r^{10}} + \underset{2}{r^{13}} + \underset{4}{r^{6}} + \underset{2}{r^{11}} + \underset{2}{r^{5}} + \\
&\quad\; \underset{3}{r^{11}} + \underset{3}{r^{1}} + \underset{3}{r^{13}} + \underset{4}{r^{2}} + \underset{2}{r^{5}} + \underset{3}{r^{15}} + \underset{4}{r^{3}} + \underset{2}{r^{14}} + \\
&\quad\; \underset{3}{r^{7}} + \underset{3}{r^{14}} + \underset{3}{r^{9}} + \underset{4}{r^{15}} + \underset{4}{r^{1}} + \underset{4}{r^{11}} + \underset{3}{r^{16}} + \underset{4}{r^{10}} + \\
&\quad\; \underset{4}{r^{5}} + \underset{4}{r^{12}} + \underset{4}{r^{7}} + \underset{4}{r^{13}} + \underset{4}{r^{16}} + \underset{4}{r^{9}} + \underset{4}{r^{14}} + \underset{4}{r^{8}} \\
&=\; -4\,.
\end{aligned}
$$

Figure 3: The computation of a_0a_1

Now we have to work very hard to compute their product! Figure 3 contains the computation where the values of r^ir^j are written after computing $r^{(i+j) \bmod 17}$. Below each root is its number of occurrences so far; check that each root occurs exactly four times so that the value of the product is -4. Since $a_0 + a_1 = -1$ and $a_0, a_1 = -4$, a_0, a_1 are the roots of the quadratic equation:

$$y^2 + y - 4 = 0\,,$$

which are:

$$a_{0,1} = \frac{-1 \pm \sqrt{17}}{2}\,.$$

Let b_0, b_1, b_2, b_3 be the sum of every fourth root starting with r^1, r^3, r^9, r^{10}, respectively:

$$
\begin{aligned}
b_0 &= r^1 + r^{13} + r^{16} + r^4 \\
b_1 &= r^3 + r^5 + r^{14} + r^{12} \\
b_2 &= r^9 + r^{15} + r^8 + r^2 \\
b_3 &= r^{10} + r^{11} + r^7 + r^6\,.
\end{aligned}
$$

Check that $b_0 + b_2 = a_0, b_1 + b_3 = a_1$. Let us compute the corresponding products:

$$\begin{aligned}
b_0 b_2 &= (r + r^{13} + r^{16} + r^4) \times \\
&\qquad (r^9 + r^{15} + r^8 + r^2) \\
&= r^{10} + r^{16} + r^9 + r^3 + \\
&\quad r^5 + r^{11} + r^4 + r^{15} + \\
&\quad r^8 + r^{14} + r^7 + r^1 + \\
&\quad r^{13} + r^2 + r^{12} + r^6 \\
&= -1\,, \\
b_1 b_3 &= (r^3 + r^5 + r^{14} + r^{12}) \times \\
&\qquad (r^{10} + r^{11} + r^7 + r^6) \\
&= r^{13} + r^{14} + r^{10} + r^9 + \\
&\quad r^{15} + r^{16} + r^{12} + r^{11} + \\
&\quad r^7 + r^8 + r^4 + r^3 + \\
&\quad r^5 + r^6 + r^2 + r^1 \\
&= -1\,.
\end{aligned}$$

To summarize these computations:

$$\begin{aligned}
b_0 + b_2 &= a_0 \\
b_0 b_2 &= -1 \\
b_1 + b_3 &= a_1 \\
b_1 b_3 &= -1\,,
\end{aligned}$$

so b_0, b_2 are the solutions of:

$$y^2 - a_0 y - 1 = 0\,.$$

and b_1, b_3 are the solutions of:

$$y^2 - a_1 y - 1 = 0\,.$$

Using the formula for solving quadratic equations and the values previously computed for a_0, a_1, we obtain the roots b_0, b_1 (Figure 4).

Finally, let c_0, c_4 be the sum of every eighth root starting with r^1, r^{13}, respectively:[2]

$$\begin{aligned}
c_0 &= r^1 + r^{16} \\
c_4 &= r^{13} + r^4 \\
c_0 + c_4 &= r^1 + r^{16} + r^{13} + r^4 = b_0 \\
c_0 c_4 &= (r^1 + r^{16}) \cdot (r^{13} + r^4) \\
&= r^{14} + r^5 + r^{12} + r^3 = b_1\,,
\end{aligned}$$

[2]There are other sums but these two suffice.

$$
\begin{aligned}
b_0 &= \frac{a_0 + \sqrt{a_0^2 + 4}}{2} \\
&= \frac{\frac{(-1+\sqrt{17})}{2} + \sqrt{\left(\frac{(-1+\sqrt{17})}{2}\right)^2 + 4}}{2} \\
&= \frac{(-1+\sqrt{17}) + \sqrt{\left(-1+\sqrt{17}\right)^2 + 16}}{4} \\
&= \frac{(-1+\sqrt{17}) + \sqrt{34 - 2\sqrt{17}}}{4}, \\
b_1 &= \frac{a_1 + \sqrt{a_1^2 + 4}}{2} \\
&= \frac{\frac{(-1-\sqrt{17})}{2} + \sqrt{\left(\frac{(-1-\sqrt{17})}{2}\right)^2 + 4}}{2} \\
&= \frac{(-1-\sqrt{17}) + \sqrt{\left(-1-\sqrt{17}\right)^2 + 16}}{4} \\
&= \frac{(-1-\sqrt{17}) + \sqrt{34 + 2\sqrt{17}}}{4}.
\end{aligned}
$$

Figure 4: Computation of b_0, b_1

so c_0, c_4 are the roots of:

$$y^2 - b_0 y + b_1 = 0$$

It suffices to compute the root $c_0 = r^1 + r^{16}$ (Figure 6) since:

$$c_0 = r_1 + r_{16} = 2\cos\left(\frac{2\pi}{17}\right),$$

as shown in Figure 5. The y-coordinates are equal but with opposite signs and cancel, while the x-coordinate is counted twice. Therefore, the cosine of the central angle of a heptadecagon is constructible with straightedge and compass, since it is composed only of rational numbers and the operations $\{+, -, \times, \div, \sqrt{}\}$:

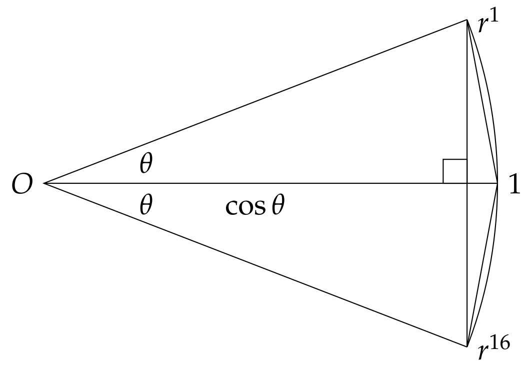


Figure 5: Constructing a side from the cosine of the central angle it subtends

$$\begin{aligned}
\cos\left(\frac{2\pi}{17}\right) &= \frac{c_0}{2} \\
&= -\frac{1}{16}+\frac{1}{16}\sqrt{17}+\frac{1}{16}\sqrt{34-2\sqrt{17}}+ \\
&\quad \frac{1}{16}\sqrt{68+12\sqrt{17}+2(-1+\sqrt{17})\sqrt{34-2\sqrt{17}}-16\sqrt{34+2\sqrt{17}}}
\end{aligned}$$

Complex numbers

$$\begin{aligned}
c_0 &= r_1 + r_{16} \\
&= \cos\left(\frac{2\pi}{17}\right)+i\sin\left(\frac{2\pi}{17}\right)+\cos\left(\frac{2\cdot 16\pi}{17}\right)+i\sin\left(\frac{2\cdot 16\pi}{17}\right) \\
&= \cos\left(\frac{2\pi}{17}\right)+i\sin\left(\frac{2\pi}{17}\right)+\cos\left(\frac{-2\pi}{17}\right)+i\sin\left(\frac{-2\pi}{17}\right) \\
&= 2\cos\left(\frac{2\pi}{17}\right).
\end{aligned}$$

$$
\begin{aligned}
c_0 &= \frac{b_0 + \sqrt{b_0^2 - 4b_1}}{2} \\
&= \frac{\frac{(-1+\sqrt{17}) + \sqrt{34-2\sqrt{17}}}{4}}{2} + \\
&\quad \frac{\sqrt{\left[\frac{(-1+\sqrt{17}) + \sqrt{34-2\sqrt{17}}}{4}\right]^2 - 4\left[\frac{(-1-\sqrt{17}) + \sqrt{34+2\sqrt{17}}}{4}\right]}}{2} \\
&= -\frac{1}{8} + \frac{1}{8}\sqrt{17} + \frac{1}{8}\sqrt{34-2\sqrt{17}} + \\
&\quad \frac{1}{8}\sqrt{\left[(-1+\sqrt{17}) + \sqrt{34-2\sqrt{17}}\right]^2 - 16\left[(-1-\sqrt{17}) + \sqrt{34+2\sqrt{17}}\right]} \\
&= -\frac{1}{8} + \frac{1}{8}\sqrt{17} + \frac{1}{8}\sqrt{34-2\sqrt{17}} + \\
&\quad \frac{1}{8}\sqrt{(-1+\sqrt{17})^2 + 2(-1+\sqrt{17})\sqrt{34-2\sqrt{17}} + (34-2\sqrt{17}) -} \\
&\quad \overline{\left[(-16-16\sqrt{17}) + 16\sqrt{34+2\sqrt{17}}\right]} \\
&= -\frac{1}{8} + \frac{1}{8}\sqrt{17} + \frac{1}{8}\sqrt{34-2\sqrt{17}} + \\
&\quad \frac{1}{8}\sqrt{68 + 12\sqrt{17} + 2(-1+\sqrt{17})\sqrt{34-2\sqrt{17}} - 16\sqrt{34+2\sqrt{17}}}
\end{aligned}
$$

Figure 6: Computation of c_0

4 Derivation of Gauss's formula

The formula we gave for $\cos\left(\frac{2\pi}{17}\right)$ is not the one given by Gauss [2, p. 458], which also appears in [1, p. 68]. I found it only in [5], where Rike gives it as an exercise to transform the formula to the one that Gauss gave. This Section gives that transformation.
Let us simplify $2(-1+\sqrt{17})\sqrt{34-2\sqrt{17}}$:

$$\begin{aligned}
2(-1+\sqrt{17})\sqrt{34-2\sqrt{17}} &= -2\sqrt{34-2\sqrt{17}}+2\sqrt{17}\sqrt{34-2\sqrt{17}}+ \\
&\quad 4\sqrt{34-2\sqrt{17}}-4\sqrt{34-2\sqrt{17}} \\
&= 2\sqrt{34-2\sqrt{17}}+2\sqrt{17}\sqrt{34-2\sqrt{17}}+ \\
&\quad -4\sqrt{34-2\sqrt{17}} \\
&= 2(1+\sqrt{17})\sqrt{34-2\sqrt{17}}-4\sqrt{34-2\sqrt{17}}\,.
\end{aligned}$$

We will remember the term $-4\sqrt{34-2\sqrt{17}}$ for now and simplify the first term by squaring it and then taking the square root:

$$\begin{aligned}
2(1+\sqrt{17})\sqrt{34-2\sqrt{17}} &= 2\sqrt{\left[(1+\sqrt{17})\sqrt{34-2\sqrt{17}}\right]^2} \\
&= 2\sqrt{(18+2\sqrt{17})(34-2\sqrt{17})} \\
&= 2\sqrt{(18\cdot 34-4\cdot 17)+\sqrt{17}(2\cdot 34-2\cdot 18)} \\
&= 2\cdot 4\sqrt{34+2\sqrt{17}}\,.
\end{aligned}$$

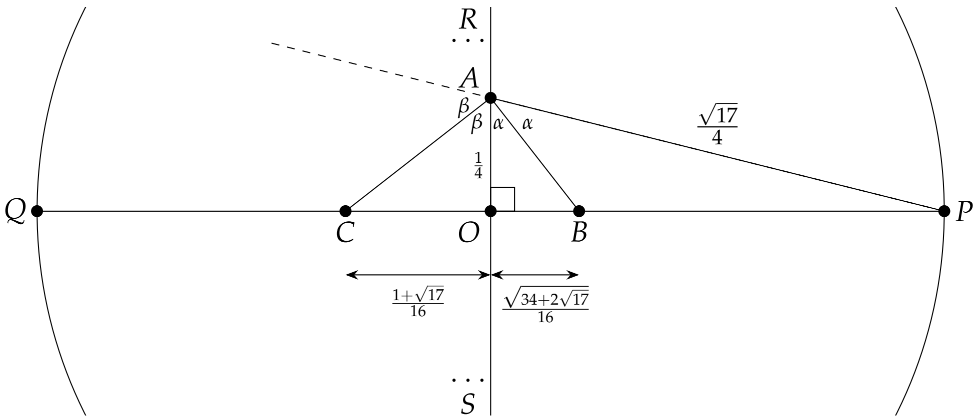
Substituting terms results in Gauss's formula:

$$\begin{aligned}
\cos\left(\frac{2\pi}{17}\right) &= -\frac{1}{16}+\frac{1}{16}\sqrt{17}+\frac{1}{16}\sqrt{34-2\sqrt{17}}+ \\
&\quad \frac{1}{16}\sqrt{68+12\sqrt{17}+2\cdot 4\sqrt{34+2\sqrt{17}}-4\sqrt{34-2\sqrt{17}}-16\sqrt{34+2\sqrt{17}}} \\
&= -\frac{1}{16}+\frac{1}{16}\sqrt{17}+\frac{1}{16}\sqrt{34-2\sqrt{17}}+ \\
&\quad \frac{1}{8}\sqrt{17+3\sqrt{17}-\sqrt{34-2\sqrt{17}}-2\sqrt{34+2\sqrt{17}}}\,.
\end{aligned}$$

5 Construction with a straightedge and compass

Several constructions are given in [8]. Here I give the construction from [3] because it directly constructs $\cos\left(\frac{2\pi}{17}\right)$. The construction uses only Pythagoras's Theorem and the Angle Bisector Theorem [7].

Construct a unit circle centered at O, and construct perpendicular diameters, $\overline{QP}, \overline{SR}$.[3]



Construct A so that $\overline{OA} = \frac{1}{4}\overline{OR}$. By Pythagoras's Theorem:

$$\overline{AP} = \sqrt{(1/4)^2 + 1^2} = \sqrt{17}/4\,.$$

Let B be the intersection of the internal bisector of $\angle OAP$ and $\overline{OP}$, and let C be the intersection of the external bisector of $\angle OAP$ and $\overline{QO}$. By the angle bisector theorem:

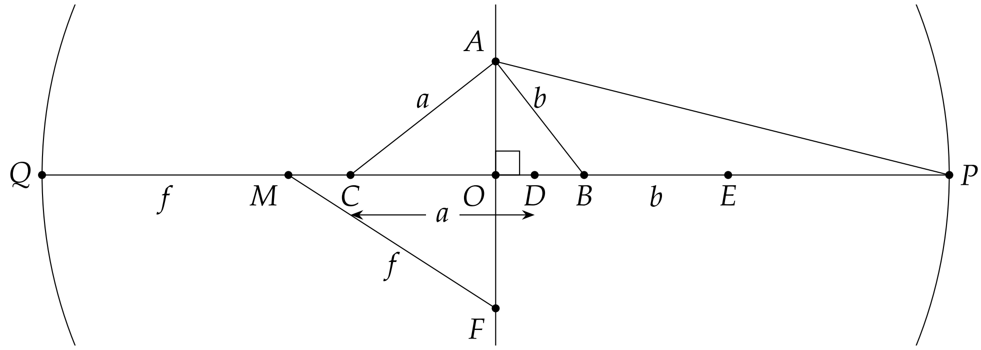
$$\begin{aligned}
\frac{\overline{OB}}{\overline{BP}} &= \frac{\overline{AO}}{\overline{AP}} \\
\frac{\overline{OB}}{1-\overline{OB}} &= \frac{1/4}{\sqrt{17}/4} \\
\overline{OB} &= \frac{1}{1+\sqrt{17}} = \frac{1}{1+\sqrt{17}} \cdot \frac{1-\sqrt{17}}{1-\sqrt{17}} \\
&= \frac{-1+\sqrt{17}}{16},
\end{aligned}$$

and:

$$\begin{aligned}
\frac{\overline{OC}}{\overline{CP}} &= \frac{\overline{AO}}{\overline{AP}} \\
\frac{\overline{OC}}{1-\overline{OC}} &= \frac{1/4}{\sqrt{17}/4} \\
\overline{OC} &= \frac{1}{-1+\sqrt{17}} = \frac{1}{-1+\sqrt{17}} \cdot \frac{1+\sqrt{17}}{1+\sqrt{17}} \\
&= \frac{1+\sqrt{17}}{16}.
\end{aligned}$$

[3] To save space I have clipped the circle so that $R = (0,1), S = (0,-1)$ do not appear.

Construct D on $\overline{OP}$ such that $\overline{CD} = \overline{CA}$:



$$\begin{aligned}
\overline{CD} = \overline{CA} &= \sqrt{\overline{OA}^2 + \overline{OC}^2} \\
&= \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1+\sqrt{17}}{16}\right)^2} \\
&= \frac{1}{16}\sqrt{34 + 2\sqrt{17}}\,.
\end{aligned}$$

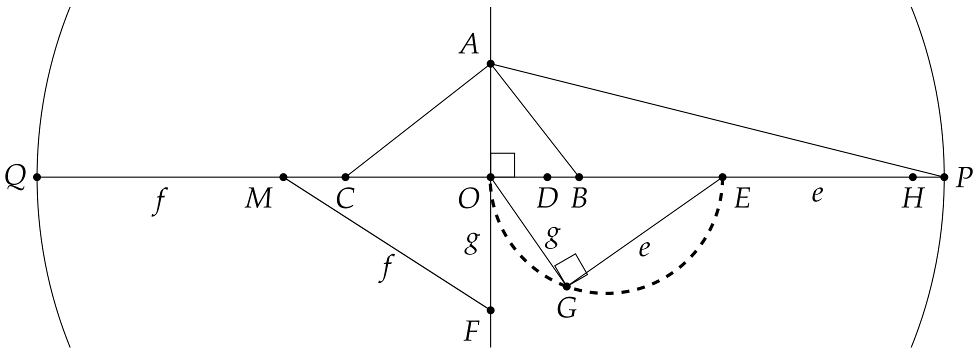
Construct E on $\overline{OP}$ such that $\overline{BE} = \overline{BA}$:

$$\begin{aligned}
\overline{BE} = \overline{BA} &= \sqrt{\overline{OA}^2 + \overline{OB}^2} \\
&= \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1-\sqrt{17}}{16}\right)^2} \\
&= \frac{1}{16}\sqrt{34 - 2\sqrt{17}}\,.
\end{aligned}$$

Construct M as the midpoint of $\overline{QD}$ and construct F on $\overline{OS}$ such that $\overline{MF} = \overline{MQ}$:

$$\begin{aligned}
\overline{MF} = \overline{MQ} &= \frac{1}{2}\overline{QD} = \frac{1}{2}(\overline{QC} + \overline{CD}) = \frac{1}{2}((1 - \overline{OC}) + \overline{CD}) \\
&= \frac{1}{2}\left[1 - \left(\frac{1+\sqrt{17}}{16}\right) + \frac{\sqrt{34+2\sqrt{17}}}{16}\right] \\
&= \frac{1}{32}\left(15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}\right)\,.
\end{aligned}$$

Construct a circle whose diameter is $\overline{OE}$. Construct a chord $\overline{OG} = \overline{OF}$. Note that $\overline{MO} = 1 - \overline{MQ} = 1 - \overline{MF}$:



$$
\begin{aligned}
\overline{OG} = \overline{OF} &= \sqrt{\overline{MF}^2 - \overline{MO}^2} = \sqrt{\overline{MF}^2 - (1-\overline{MF})^2} \\
&= \sqrt{2\overline{MF} - 1} \\
&= \sqrt{\frac{1}{16}\left(15 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}\right) - 1} \\
&= \frac{1}{4}\sqrt{-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}}.
\end{aligned}
$$

$\angle OGE$ is a right angle since it is subtended by a diameter of the circle. Construct H on $\overline{OP}$ such that $\overline{EH} = \overline{EG}$:

$$
\begin{aligned}
\overline{EH} = \overline{EG} &= \sqrt{\overline{OE}^2 - \overline{OG}^2} = \sqrt{(\overline{OB} + \overline{BE})^2 - \overline{OG}^2} \\
&= \sqrt{\left(\frac{-1+\sqrt{17}}{16} + \frac{\sqrt{34 - 2\sqrt{17}}}{16}\right)^2 - \frac{1}{16}\left(-1 - \sqrt{17} + \sqrt{34 + 2\sqrt{17}}\right)} \\
&= \frac{1}{16}\sqrt{\left((18 - 2\sqrt{17}) + 2(-1+\sqrt{17})\sqrt{34 - 2\sqrt{17}} + (34 - 2\sqrt{17})\right) +} \\
&\qquad \overline{\left(16 + 16\sqrt{17} - 16\sqrt{34 + 2\sqrt{17}}\right)} \\
&= \frac{1}{16}\sqrt{68 + 12\sqrt{17} - 16\sqrt{34 + 2\sqrt{17}} - 2(1 - \sqrt{17})\sqrt{34 - 2\sqrt{17}}}.
\end{aligned}
$$

Let us compute $\overline{OE}$:

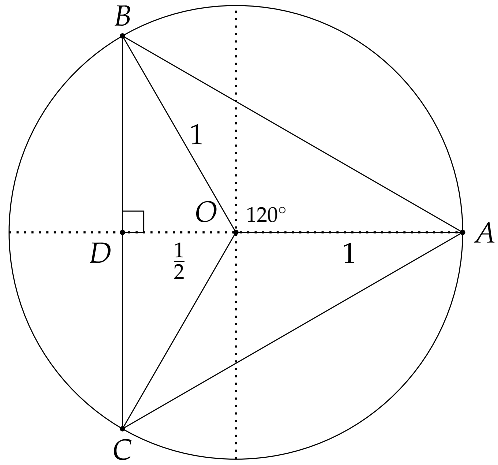
$$
\begin{aligned}
\overline{OE} = \overline{OB} + \overline{BE} &= \frac{-1 + \sqrt{17}}{16} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} \\
&= \frac{1}{16}\left(-1 + \sqrt{17} + \sqrt{34 - 2\sqrt{17}}\right).
\end{aligned}
$$

Finally, $\overline{OH} = \overline{OE} + \overline{EH}$ which is $\cos\left(\frac{2\pi}{17}\right)$.

A Constructing an equilateral triangle

The central angle of an equilateral triangle is $360°/3 = 120°$ and we can compute its cosine from the formula for the cosine of the sum of two angles:

$$\cos 120° = \cos(90° + 30°) = \cos 90° \cos 30° - \sin 90° \sin 30° = 0 \cdot \frac{\sqrt{3}}{2} - 1 \cdot \frac{1}{2} = -\frac{1}{2}\,.$$



To construct the triangle, extend the radius $\overline{OA}$ of a unit circle to D by the constructible length $\frac{1}{2}$. Construct the perpendicular throught D that intersects the circle at B, C; then $\overline{AB}, \overline{AC}, \overline{BC}$ are the sides of an equilateral triangle inscribed in the unit circle. The side is $\overline{BD} + \overline{DC}$, which is:

$$2 \cdot \sqrt{1^2 - \left(\frac{1}{2}\right)^2} = \sqrt{3}\,.$$

B Constructing a regular pentagon

The central angle of a regular pentagon The central angle is $360°/5 = 72°$. Let us compute $\cos 36°$ using the trigonometric identities for 2θ and $\theta/2$ [9]:

$$\begin{aligned} 0 = \cos 90° &= \cos(72° + 18°) \\ &= (2\cos^2 36° - 1)\sqrt{\frac{1 + \cos 36°}{2}} - 2\sin 36° \cos 36° \sqrt{\frac{1 - \cos 36°}{2}}\,. \end{aligned}$$

There is now only one angle in the formula; let $x = \cos 36°$. Then:

$$\begin{aligned} (2x^2 - 1)\sqrt{\frac{1+x}{2}} &= 2\sqrt{1 - x^2} \cdot x \cdot \sqrt{\frac{1-x}{2}} \\ (2x^2 - 1)\sqrt{1+x} &= 2\sqrt{1-x} \cdot \sqrt{1+x} \cdot x \cdot \sqrt{1-x} \\ 2x^2 - 1 &= 2x(1-x) \\ 4x^2 - 2x - 1 &= 0\,. \end{aligned}$$

Solving the quadratic equation gives:

$$\cos 36^\circ = \frac{1+\sqrt{5}}{4},$$

which can be computed using $\{+, -, \times, \div, \sqrt{}\}$ so it is constructible.

Figure 7 shows how to construct a regular pentagon from $\cos 36^\circ$. From D at distance $\cos 36^\circ$ from O construct a perpendicular to $\overline{OA}$ that intersects a unit circle at C. Construct $\overline{OC}$. Construct a line from A perpendicular to $\overline{OC}$. Its intersection with the unit circle at B defines $\overline{AB}$, the side of the inscribed pentagon.

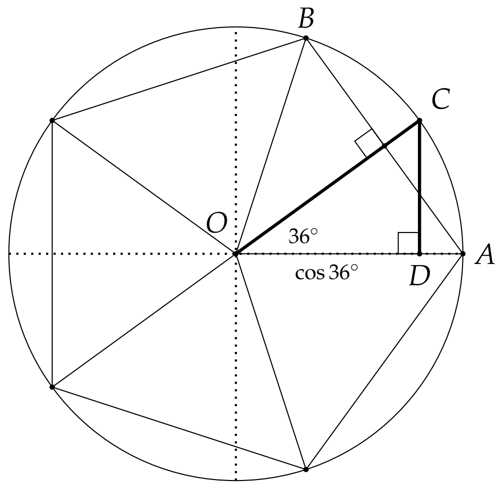
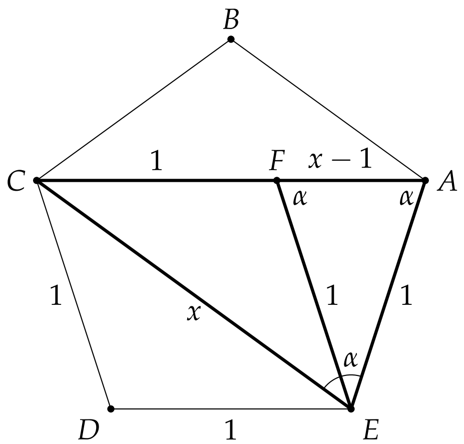


Figure 7: The central angle of a regular pentagon

Direct construction of a regular pentagon Here is a solution to Exercises 2.3.3–2.3.4 of [6, page 28], showing that a regular pentagon is constructible.

Let $ABCDE$ be a regular pentagon whose sides are of length 1.



Construct the diagonals $\overline{AC}$ and $\overline{CE}$. $\angle AED = \angle EDC$ because they are the interior angles of a regular polygon. $\overline{AE} = \overline{CD}$ because they are the sides of a regular polygon. Therefore, $ACDE$ is an isoceles trapezoid and $\overline{AC} \parallel \overline{ED}$.

Construct a line through E parallel to $\overline{DC}$ and let F be its intersection with $\overline{AC}$. It follows that $\overline{EF} = 1$. Let x be the length of the diagonals of the regular pentagon; it is easy to show that they are all equal. Then $\overline{AC} = \overline{CE} = x$, so $\triangle ACE$ is an isoceles triangle with base angles α. $\triangle AEF$ is also isoceles so $\angle AFE = \angle FAE = \alpha$. Therefore, $\triangle ACE \sim \triangle AEF$:

$$\frac{x}{1} = \frac{1}{x-1}\,.$$

Multiplying out gives the quadratic equation:

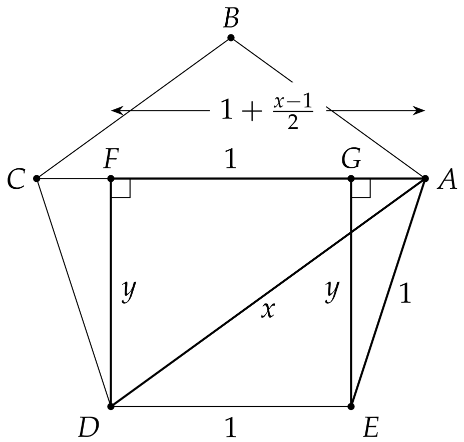
$$x^2 - x - 1 = 0\,,$$

whose positive root is:

$$\frac{1+\sqrt{5}}{2}\,.$$

This length is constructible using rational numbers and square roots. The regular pentagon is constructible since from a line segment $\overline{AC}$ of length $\frac{1+\sqrt{5}}{2}$, construct the isoceles triangle $\triangle ABC$ which gives two sides and the interior angle of a regular pentagon.

Another demonstration that a regular pentagon is constructible Let $ABCDE$ be a regular pentagon whose sides are of length 1.



Construct the diagonals $\overline{AD} = \overline{AC}$. Construct $\overline{DF} \perp \overline{AC}, \overline{EG} \perp \overline{AC}$. Let $y = \overline{DF} = \overline{EG}$. By Pythagoras's theorem on $\triangle EGA, \triangle DFA$:

$$\begin{aligned} y^2 + \left(\frac{x-1}{2}\right)^2 &= 1^2 \\ y^2 + \left(1 + \frac{x-1}{2}\right)^2 &= x^2\,. \end{aligned}$$

Eliminate y^2 and simplify to obtain $x^2 - x - 1$. Continue as in the previous solution.

References

[1] Jörg Bewersdorff. *Galois Theory for Beginners: A Historical Perspective*. American Mathematical Society, 2006.

[2] Todd W. Bressi and Paul Groth, editors. *Disquisitiones Arithmeticae*. Yale University Press, 2006.

[3] James J. Callagy. The central angle of the regular 17-gon. *The Mathematical Gazette*, 67(442):290–292, 1983. `https://wwww.jstor.org/stable/3617271`.

[4] Po-Shen Lo. A different way to solve quadratic equations, 2019. `https://www.poshenloh.com/quadratic/`.

[5] Tom Rike. Fermat numbers and the heptadecagon, 2005. `https://mathcircle.berkeley.edu/sites/default/files/BMC6/ps0506/Heptadecagon.pdf`.

[6] John Stillwell. *Mathematics and Its History (Third Edition)*. Springer, 2010.

[7] Wikipedia contributors. Angle bisector theorem — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Angle_bisector_theorem&oldid=984147660`, 2020. [Online; accessed 23-October-2020].

[8] Wikipedia contributors. Heptadecagon — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Heptadecagon&oldid=975964212`, 2020. [Online; accessed 23-October-2020].

[9] Wikipedia contributors. Pentagon — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Pentagon&oldid=983136827`, 2020. [Online; accessed 23-October-2020].