The Mathematics of Origami

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Version 4

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Chapter 1

Introduction

This document develops the mathematics of origami using secondary-school mathematics. Equations of lines are given in the slope-intercept form $y = mx + b$.

Chapter 2 develops the mathematical formulas for the seven axioms and together with numerical examples. In the diagrams, given lines are solid, folds are dashed, auxiliary lines are dotted, and dotted arrows indicate the direction of folding the paper.

The fold operations can construct every length that can be constructed by straightedge and compass. Given a, b: $a + b, a - b, a \times b, a/b, \sqrt{a}$ can be constructed [6, Chapter 4].

Folding is more powerful because it can construct cube roots. Chapter 3 presents two methods for trisecting an arbitrary angle and Chapter 4 presents two methods for doubling a cube.

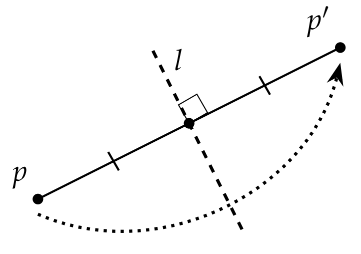
Chapter 5 explains Eduard Lill's geometric method for finding real roots of any polynomical; we will demonstrate the method for cubic polynomials. Chapter 6 presents Margharita P. Beloch's implementation of Lill' method using a fold.

Appendix A contains links to GeoGebra projects that visualize the axioms. Appendix B derives trigonometric identities for tangents that may not be familiar. Appendix C explains the geometric definition of parabolas.

Definitions

Each axiom states that a *fold* exists that will place given points and lines onto points and lines, such that certain properties hold. The term fold comes from the origami operation of folding a piece of paper, but here it is used to refer the geometric line that would be created by folding the paper.

Formal definitions are given in [7, Chapter 10]. The reader should be aware that, *by definition*, folds result in *reflections*. Given a point p, its reflection around a fold l results in a point p', such that l is the perpendicular bisector of the line segment $\overline{pp'}$:

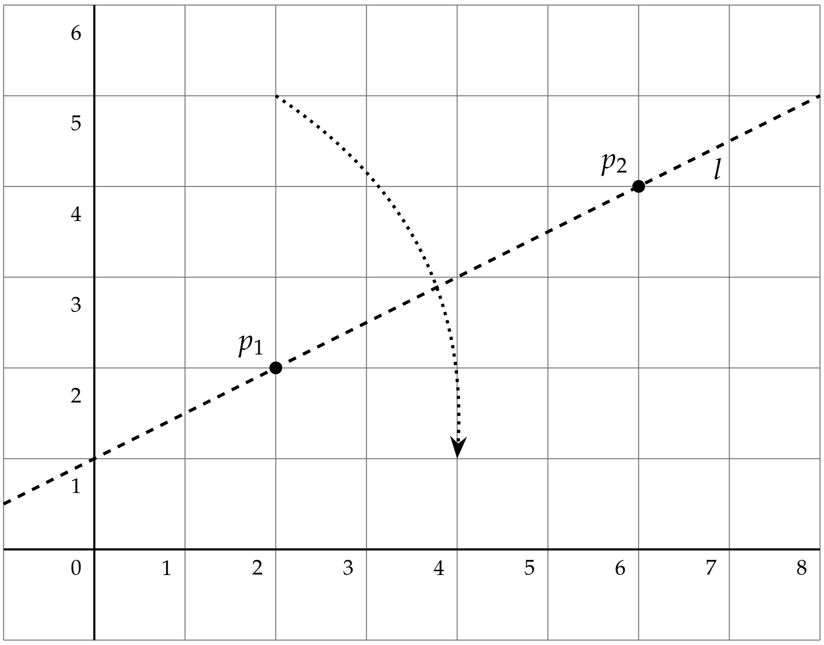


Chapter 2

Axioms

2.1 Axiom 1

Axiom Given two distinct points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, there is a unique fold l that passes through both of them.



Derivation of the equation of the fold

The equation of fold l is derived from the coordinates of p_1 and p_2: the slope is the quotient of the differences of the coordinates and the intercept is derived from p_1:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)\,. \tag{2.1}$$

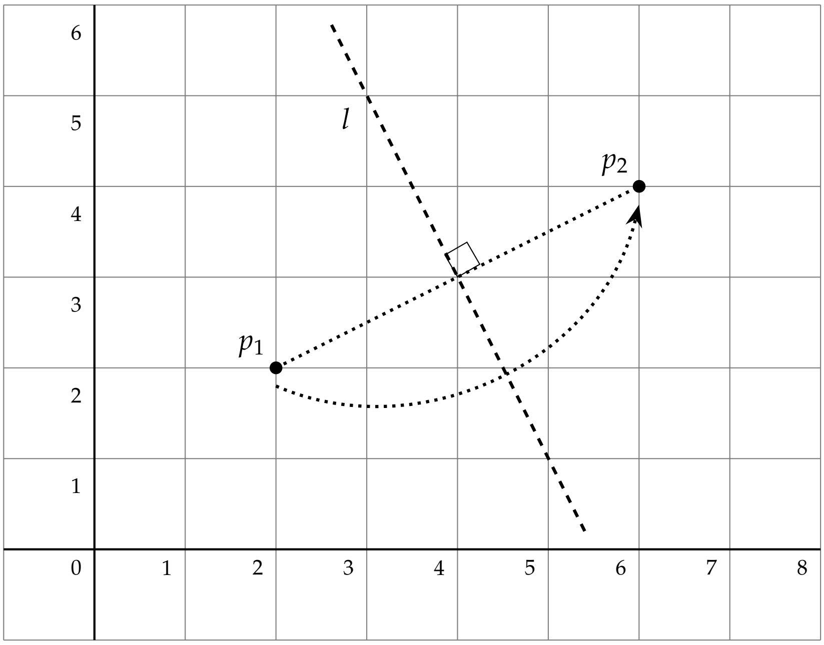
Example

Let $p_1 = (2, 2), p_2 = (6, 4)$. The equation of l is:

$$\begin{aligned} y - 2 &= \frac{4-2}{6-2}(x-2) \\ y &= \frac{1}{2}x + 1\,. \end{aligned}$$

2.2 Axiom 2

Axiom Given two distinct points $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$, there is a unique fold l that places p_1 onto p_2.



Derivation of the equation of the fold

The fold l is the perpendicular bisector of $\overline{p_1 p_2}$. Its slope is the negative reciprocal of the slope of the line connecting p_1 and p_2. l passes through the midpoint between the points:

$$y - \frac{y_1 + y_2}{2} = -\frac{x_2 - x_1}{y_2 - y_1}\left(x - \frac{x_1 + x_2}{2}\right). \tag{2.2}$$

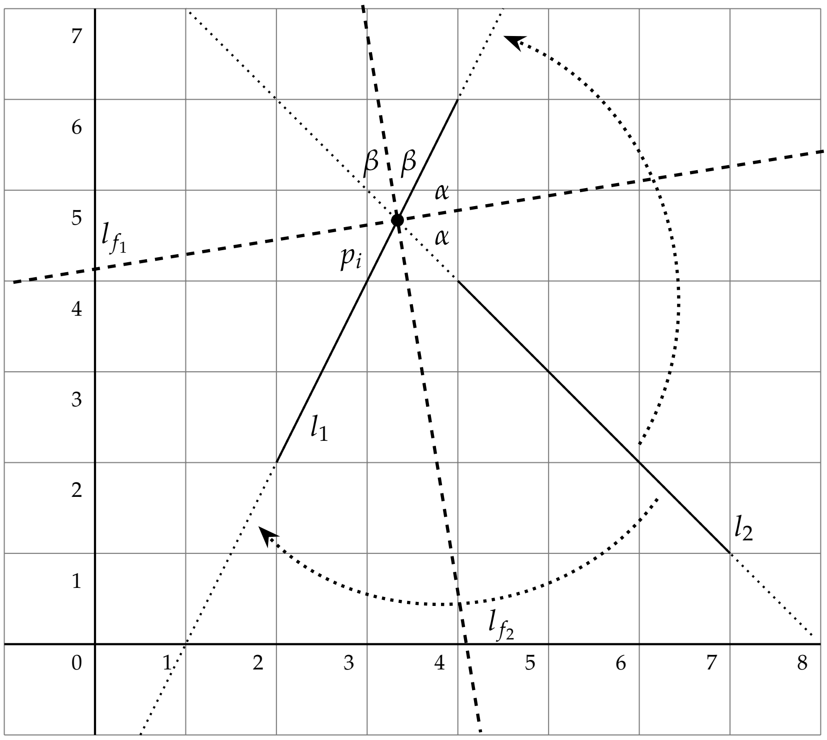
Example

Let $p_1 = (2, 2), p_2 = (6, 4)$. The equation of l is:

$$\begin{aligned} y - \left(\frac{2+4}{2}\right) &= -\frac{6-2}{4-2}\left(x - \left(\frac{2+6}{2}\right)\right) \\ y &= -2x + 11\,. \end{aligned}$$

2.3 Axiom 3

Axiom Given two lines l_1 and l_2, there is a fold l that places l_1 onto l_2.



Derivation of the equation of the fold

If the lines are parallel, let l_1 be $y = mx + b_1$ and let l_2 be $y = mx + b_2$. The fold is the line parallel to l_1, l_2 and halfway between them $y = mx + \dfrac{b_1 + b_2}{2}$.

If the lines intersect, let l_1 be $y = m_1x + b_1$ and let l_2 be $y = m_2x + b_2$.

Derivation of the point of intersection

$p_i = (x_i, y_i)$, the point of intersection of the two lines, is:

$$\begin{aligned} m_1x_i + b_2 &= m_2x_i + b_2 \\ x_i &= \frac{b_2 - b_1}{m_1 - m_2} \\ y_i &= m_1x_i + b_1\,. \end{aligned}$$

Example

Let l_1 be $y = 2x - 2$ and let l_2 be $y = -x + 8$. The point of intersection is:

$$\begin{aligned} x_i &= \frac{8 - (-2)}{2 - (-1)} = \frac{10}{3} \approx 3.33 \\ y_i &= 2 \cdot \frac{10}{3} - 2 = \frac{14}{3} \approx 4.67\,. \end{aligned}$$

Derivation of the equation of the slope of the angle bisector

The two lines form an angle at their point of intersection, actually, two pairs of vertical angles. The folds are the bisectors of these angles.

If the angle of line l_1 relative to the x-axis is θ_1 and the angle of line l_2 relative to the x-axis is θ_2, then the fold is the line which makes an angle of $\theta_b = \dfrac{\theta_1+\theta_2}{2}$ with the x-axis.

$\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$ are given and m_b, the slope of the angle bisector, is:

$$m_b = \tan\theta_b = \tan\frac{\theta_1+\theta_2}{2}\,.$$

The derivation requires the use of the following trigonometric identities:[1]

$$\begin{aligned}
\tan(\alpha_1+\alpha_2) &= \frac{\tan\alpha_1+\tan\alpha_2}{1-\tan\alpha_1\tan\alpha_2}\\
\tan\frac{\alpha}{2} &= \frac{-1\pm\sqrt{1+\tan^2\alpha}}{\tan\alpha}\,.
\end{aligned}$$

First derive m_s, the slope of $\theta_1+\theta_2$:

$$m_s = \tan(\theta_1+\theta_2) = \frac{m_1+m_2}{1-m_1m_2}\,.$$

Then derive m_b, the slope of the angle bisector:

$$\begin{aligned}
m_b &= \tan\frac{\theta_1+\theta_2}{2}\\
&= \frac{-1\pm\sqrt{1+\tan^2(\theta_1+\theta_2)}}{\tan(\theta_1+\theta_2)}\\
&= \frac{-1\pm\sqrt{1+m_s^2}}{m_s}\,.
\end{aligned}$$

Example For the lines $y = 2x-2$ and $y = -x+8$, the slope of the angle bisector is:

$$\begin{aligned}
m_s &= \frac{2+(-1)}{1-(2\cdot -1)} = \frac{1}{3}\\
m_b &= \frac{-1\pm\sqrt{1+(1/3)^2}}{1/3} = -3\pm\sqrt{10} \approx -6.16,\ 0.162\,.
\end{aligned}$$

[1]The derivation of these identities is given in Appendix B.

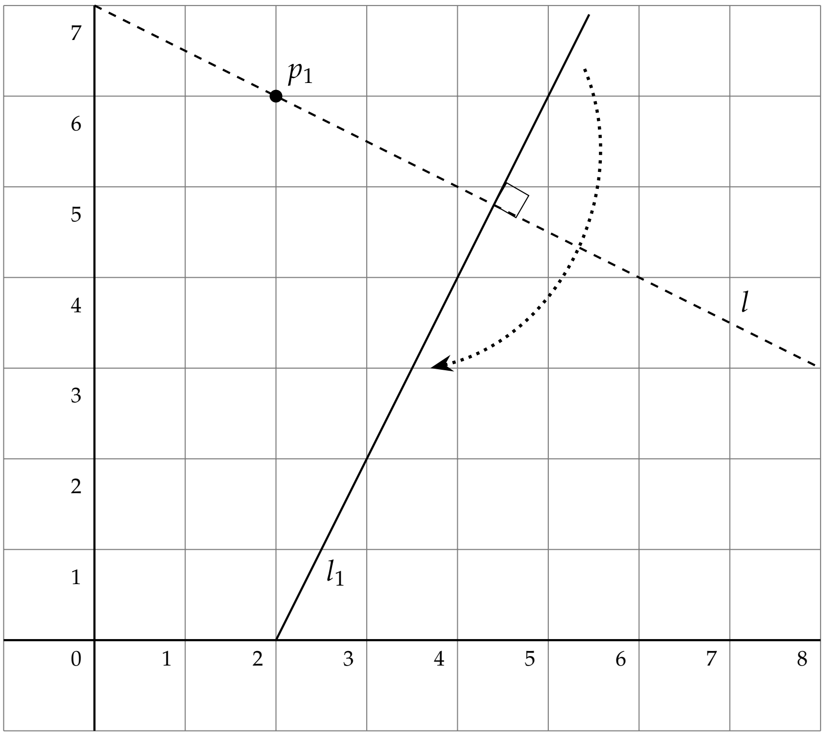
Derivation of the equation of the fold

Let us derive equation of the fold l_{f_1} with the positive slope; we know the coordinates of the intersection of the two lines $m_i = \left(\frac{10}{3}, \frac{14}{3}\right)$:

$$\begin{aligned}
\frac{14}{3} &= (-3+\sqrt{10}) \cdot \frac{10}{3} + b \\
b &= \frac{44 - 10\sqrt{10}}{3} \\
y &= (-3+\sqrt{10})x + \frac{44-10\sqrt{10}}{3} \approx 0.162x + 4.13\,.
\end{aligned}$$

2.4 Axiom 4

Axiom Given a point p_1 and a line l_1, there is a unique fold l perpendicular to l_1 that passes through point p_1.



Derivation of the equation of the fold

Let l_1 be $y = m_1 x + b_1$ and let $p_1 = (x_1, y_1)$. l is perpendicular to l_1 so its slope is $-\frac{1}{m_1}$. Since it passes through p_1, we can compute the intercept b and write down its equation:

$$\begin{aligned} y_1 &= -\frac{1}{m} x_1 + b \\ b &= \frac{(my_1 + x_1)}{m} \\ y &= -\frac{1}{m} x + \frac{(my_1 + x_1)}{m} \,. \end{aligned}$$

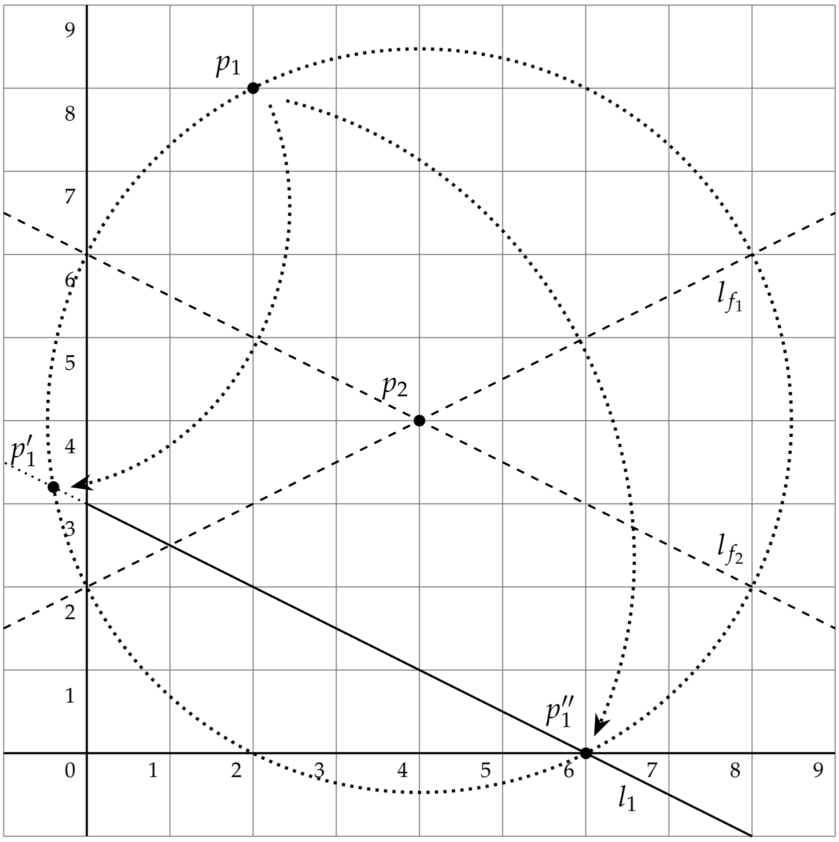
Example

Let $p_1 = (2, 6)$ and let l_1 be $y = 2x - 4$. The equation of the fold l is:

$$y = -\frac{1}{2} x + \frac{2 \cdot 6 + 2}{2} = -\frac{1}{2} x + 7 \,.$$

2.5 Axiom 5

Axiom Given two points p_1, p_2 and a line l_1, there is a fold l that places p_1 onto l_1 and passes through p_2.



For a given pair of points and a line, there may be zero, one or two folds.

Derivation of the equations of the reflections

Let l be a fold through p_2 and p'_1 be the reflection of p_1 around l. The length of $\overline{p_1p_2}$ equals the length of $\overline{p_2p_1}'$. The locus of points at distance $\overline{p_1p_2}$ *from* p_2 is the circle centered at p_2 whose radius is the length of $\overline{p_1p_2}$. The intersections of this circle with the line l_1 give the possible points p'_1.

Let l_1 be $y = m_1x + b_1$ and let $p_1 = (x_1, y_1)$, $p_2 = (x_2, y_2)$. The equation of the circle centered at p_2 with radius the length of $\overline{p_1p_2}$ is:

$$(x - x_2)^2 + (y - y_2)^2 = r^2\,, \quad \text{where}$$
$$r^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2\,.$$

Substituting the equation of the line into the equation for the circle:

$$(x - x_2)^2 + ((m_1x + b_1) - y_2)^2 = (x - x_2)^2 + (m_1x + (b_1 - y_2))^2 = r^2\,,$$

we obtain a quadratic equation for the x-coordinates of the possible intersections:

$$x^2(1+m_1^2) \;+\; 2(-x_2+m_1b-m_1y_2)x \;+\; (x_2^2+(b_1^2-2b_1y_2+y_2^2)-r^2)=0\,. \tag{2.3}$$

The quadratic equation has at most two solutions x_1', x_1'' and we can compute $y_1', y_1'')$ from $y = m_1x + b_1$. The reflected points are $p_1' = (x_1', y_1')$, $p_1'' = (x_1'', y_1'')$.

Example

Let $p_1 = (2,8)$, $p_2 = (4,4)$ and let l_1 be $y = -\dfrac{1}{2}x + 3$. The equation of the circle is:

$$(x-4)^2 + (y-4)^2 = r^2 = (4-2)^2 + (4-8)^2 = 20\,.$$

Substitute the equation of the line into the equation of the circle and simplify to obtain a quadratic equation for the x-coordinates of the intersections (or use Equation 2.3):

$$\begin{aligned}
(x-4)^2 + \left(\left(-\frac{1}{2}x+3\right)-4\right)^2 &= 20\\
\frac{5}{4}x^2 - 7x - 3 &= 0\\
5x^2 - 28x - 12 &= 0\\
(5x+2)(x-6) &= 0\,.
\end{aligned}$$

The two points of intersection are:

$$p_1' = \left(-\frac{2}{5}, \frac{16}{5}\right) = (-0.4, 3.2)\,,\quad p_1'' = (6,0)\,.$$

Derivation of the equations of the folds

The folds will be the perpendicular bisectors of $\overline{p_1p_1'}$ and $\overline{p_1p_1''}$. The equation of a perpendicular bisector is given by Equation 2.2, repeated here with for p_1':

$$y - \frac{y_1+y_1'}{2} = -\frac{x_1'-x_1}{y_1'-y_1}\left(x - \frac{x_1+x_1'}{2}\right)\,. \tag{2.4}$$

Example

For $p_1 = (2,8)$ and $p_1' = \left(-\dfrac{2}{5}, \dfrac{16}{5}\right)$, the equation of the fold l_{f_1} is:

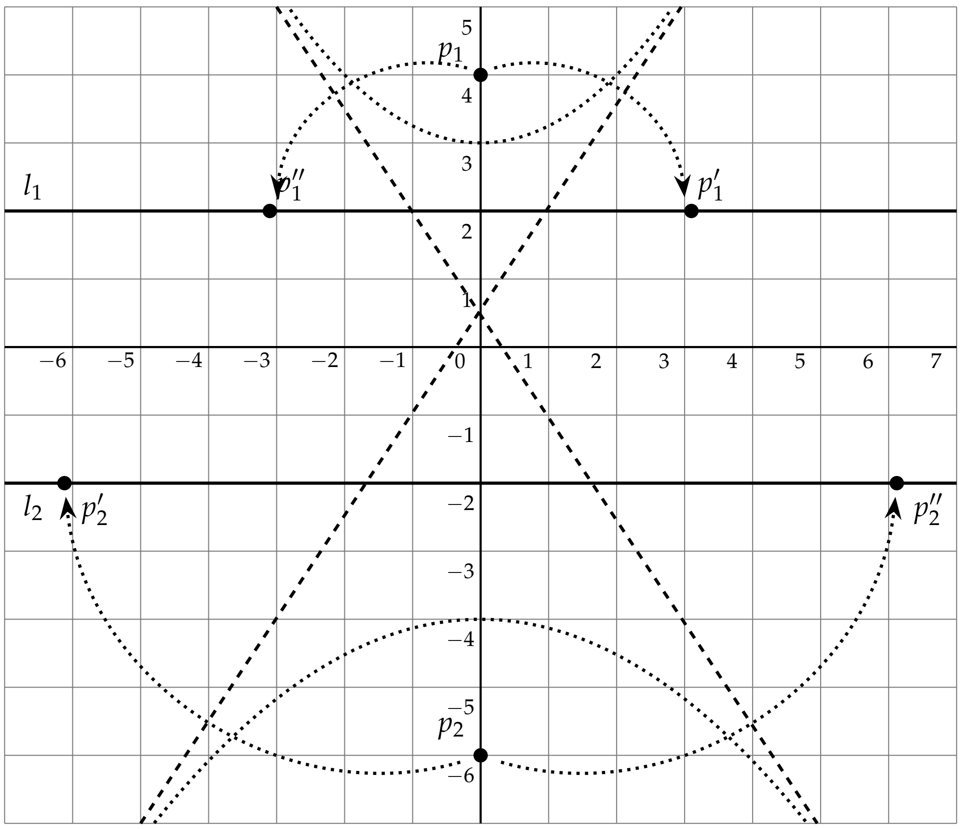
$$\begin{aligned}
y - \frac{8+(16/5)}{2} &= -\frac{(-2/5)-2}{(16/5)-8}\left(x - \frac{2+(-2/5)}{2}\right)\\
y &= -\frac{1}{2}x + 6\,.
\end{aligned}$$

For $p_1 = (2,8)$ and $p_1'' = (6,0)$, the equation of the fold l_{f_2} is:

$$\begin{aligned}
y - \frac{8+0}{2} &= -\frac{6-2}{0-8}\left(x - \frac{2+6}{2}\right)\\
y &= \frac{1}{2}x + 2\,.
\end{aligned}$$

2.6 Axiom 6

Axiom Given two points p_1 and p_2 and two lines l_1 and l_2, there is a fold l that places p_1 onto l_1 and p_2 onto l_2.



For a given pair of points and pair of lines, there may be zero, one, two or three folds. This is proved in [7, Chapter 10]; in Appendix D we give graphic examples of each of the four cases.

A fold that places p_i onto l_i is a line such that the distance from p_i to the line is equal to the distance from l_i to the line. The locus of points that are equidistant from a point p_i and a line l_i is a parabola with focus p_i and directrix l_i. A fold is any line tangent to that parabola. A detailed justification of this claim is given in Appendix C.

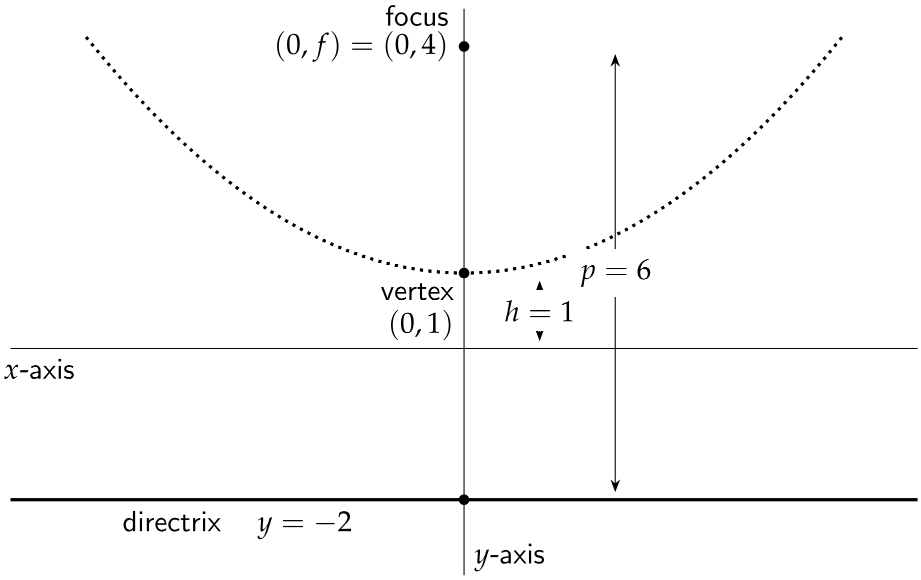
For a fold to simultaneously place p_1 onto l_1 and p_2 onto l_2, it must be a tangent common to the two parabolas.

The formula for an arbitrary parabola is quite complex, so we limit the presentation to parabolas with the the y-axis as the axis of symmetry. This is not a significant limitation because for any parabola there is a rigid motion that moves the parabola so that its axis of symmetry is the y-axis.

An example will also be given where one of the parabolas has the x-axis as its axis of symmetry.

Derivation of the equation a fold

Let $(0, f)$ be the focus of a parabola with directrix $y = d$. Define $p = f - d$, the signed length of the line segment between the focus and the directrix.[2] If the vertex of the parabola is on the x-axis, the equation of the parabola is $y = \dfrac{x^2}{2p}$. To move the parabola up or down the y-axis so that its vertex is at $(0, h)$, add h to the equation of the parabola: $y = \dfrac{x^2}{2p} + h$.



Define $a = 2ph$ so that the equation of the parabola is:

$$y = \frac{x^2}{2p} + \frac{a}{2p}$$
$$x^2 - 2py + a = 0\,.$$

The equation of the parabola in the diagram above is:

$$x^2 - 2 \cdot 6 y + 2 \cdot 6 \cdot 1 = 0$$
$$x^2 - 12y + 12 = 0\,.$$

Substitute the equation of an *arbitrary* line $y = mx + b$ into the equation for the parabola to obtain an equation for the points of intersection of the line and the parabola:

$$x^2 - 2p(mx + b) + a = 0$$
$$x^2 + (-2mp)x + (-2pb + a) = 0\,.$$

The line will be tangent to the parabola iff this quadratic equation has *exactly one* solution iff its discriminant is zero:

$$(-2mp)^2 \; - \; 4 \cdot 1 \cdot (-2pb + a) = 0\,,$$

[2]We have been using the notation p_i for points; the use of p here might be confusing but it is the standard notation. The formal name for p is one-half the *latus rectum*.

which simplifies to:

$$m^2p^2 + 2pb - a = 0\,. \tag{2.5}$$

This is the equation with variable m for the slopes of tangents to the parabola. There are an infinite number of tangents because for each m, there is some b that makes the line a tangent by moving it up or down.[3]

To obtain the common tangents to both parabolas, the equations for the two parabolas have two unknowns and can be solved for m and b.

Example

Parabola 1: focus $(0,4)$, directrix $y = 2$, vertex $(0,3)$, $p = 2$, $a = 2 \cdot 2 \cdot 3 = 12$. The equation of the parabola is:

$$x^2 - 2 \cdot 2y + 12 = 0\,.$$

Substituting into Equation 2.5 and simplifying:

$$m^2 + b - 3 = 0\,.$$

Parabola 2: focus $(0,-4)$, directrix $y = -2$, vertex $(0,-3)$, $p = -2$, $a = 2 \cdot -2 \cdot -3 = 12$. The equation of the parabola is:

$$x^2 - 2 \cdot (-2)y + 12 = 0\,.$$

Substituting into Equation 2.5 and simplifying:

$$m^2 - b - 3 = 0\,.$$

The solutions of the two equations:

$$\begin{aligned} m^2 + b - 3 &= 0 \\ m^2 - b - 3 &= 0\,, \end{aligned}$$

are $m = \pm\sqrt{3} \approx \pm 1.73$ and $b = 0$. There are two common tangents that are the folds:

$$y = \sqrt{3}x\,, \quad y = -\sqrt{3}x\,.$$

Example

Parabola 1 is unchanged.

Parabola 2: focus $(0,-6)$, directrix $y = -2$, vertex $(0,-4)$, $p = -4$, $a = 2 \cdot -4 \cdot -4 = 32$. The equation of the parabola is:

$$x^2 - 2 \cdot (-4)y + 32 = 0\,.$$

Substituting into Equation 2.5 and simplifying:

$$2m^2 - b - 4 = 0\,.$$

[3] Except of course for a line parallel to the axis of symmetry.

The solutions of the two equations:

$$\begin{aligned}m^2 + b - 3 &= 0\\ 2m^2 - b - 4 &= 0\,,\end{aligned}$$

are $m = \pm\sqrt{\dfrac{7}{3}} \approx \pm 1.53$ and $b = \dfrac{2}{3}$. There are two common tangents that are folds:

$$y = \sqrt{\frac{7}{3}}x + \frac{2}{3}\,, \quad y = -\sqrt{\frac{7}{3}}x + \frac{2}{3}\,.$$

Example

Let us now define a parabola whose axis of symmetry is the x-axis.

Parabola 1 is unchanged.

Parabola 2: focus $(4, 0)$, directrix $x = 2$, vertex $(3, 0)$, $p = 2$, $a = 2 \cdot 2 \cdot 3 = 12$. The equation of the parabola is:

$$y^2 - 4x + 12 = 0\,.$$

Note that this is an equation with x and y^2 instead of x^2 and y, so we can't use Equation 2.5 and we must perform the derivation again.

Substitute the equation for a line:

$$\begin{aligned}&(mx + b)^2 - 4x + 12 = 0\\ &m^2x^2 + (2mb - 4)x + (b^2 + 12) = 0\,,\end{aligned}$$

set the discriminant equal to zero and simplify:

$$\begin{aligned}&(2mb - 4)^2 \;-\; 4m^2(b^2 + 12) = 0\\ &-3m^2 - mb + 1 = 0\,.\end{aligned}$$

If we try to solve the two equations:

$$\begin{aligned}m^2 + b - 3 &= 0\\ -3m^2 - mb + 1 &= 0\,,\end{aligned}$$

we obtain a cubic equation with variable m:

$$m^3 - 3m^2 - 3m + 1 = 0\,. \tag{2.6}$$

Since a cubic equation has at least one and at most three (real) solutions, there can be one, two or three common tangents. There can also be no common tangents if the two equations have no solution, for example, if one parabola is "contained" with another: $y = x^2$, $y = x^2 + 1$.

The formula for solving cubic equations is quite complicated, so I used a calculator on the internet and obtained three solutions:

$$m = 3.73\,,\ m = -1\,,\ m = 0.27\,.$$

Choosing $m = 0.27$, $b = 3 - m^2 = 2.93$, and the equation of the fold is:

$$y = 0.27x + 2.93\,.$$

From the form of Equation 2.6, we might guess that 1 or -1 is a solution:

$$\begin{aligned} 1^3 - 3 \cdot 1^2 - 3 \cdot 1 + 1 &= -4 \\ (-1)^3 - 3 \cdot (-1)^2 - 3 \cdot (-1) + 1 &= 0\,. \end{aligned}$$

Divide Equation 2.6 by $m - (-1) = m + 1$ to obtain the quadratic equation $m^2 - 4m + 1$ whose roots are $2 \pm \sqrt{3} \approx 3.73, 0.27$.

Derivation of the equations of the reflections

We derive the position of the reflection $p_1' = (x_1', y_1')$ of $p_1 = (x_1, y_1)$ around some tangent line l_t whose equation is $y = m_t x + b_t$. The derivation is identical for any tangent and for p_2.

To reflect p_1 around l_t, we find the line l_p with equation $y = m_p x + b_p$ that is perpendicular to l_t and passes through p_1;

$$\begin{aligned} y &= -\frac{1}{m_t}x + b_p \\ y_1 &= -\frac{1}{m_t}x_1 + b_p \\ y &= \frac{-x}{m_t} + \left(y_1 + \frac{x_1}{m_t}\right)\,. \end{aligned}$$

Next we find the intersection $p_t = (x_t, y_t)$ of l_t and l_p:

$$\begin{aligned} m_t x_t + b_t &= \frac{-x_t}{m_t} + \left(y_1 + \frac{x_1}{m_t}\right) \\ x_t &= \frac{\left(y_1 + \dfrac{x_1}{m_t} - b_t\right)}{\left(m_t + \dfrac{1}{m_t}\right)} \\ y_t &= m_t x_t + b_t\,. \end{aligned}$$

The reflection p_1' is easy to derive because the intersection p_t is the midpoint between p_1 and its reflection p_1':

$$x_t = \frac{x_1 + x_1'}{2}\,,\quad y_t = \frac{y_1 + y_1'}{2}$$

$$x_1' = 2x_t - x_1\,,\quad y_1' = 2y_t - y_1\,.$$

Example

$p_1 = (0, 4)$, l_1 is $y = \sqrt{3}x$:

$$x_t = \frac{\left(4 + \frac{0}{\sqrt{3}} - 0\right)}{\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)} = \sqrt{3}$$

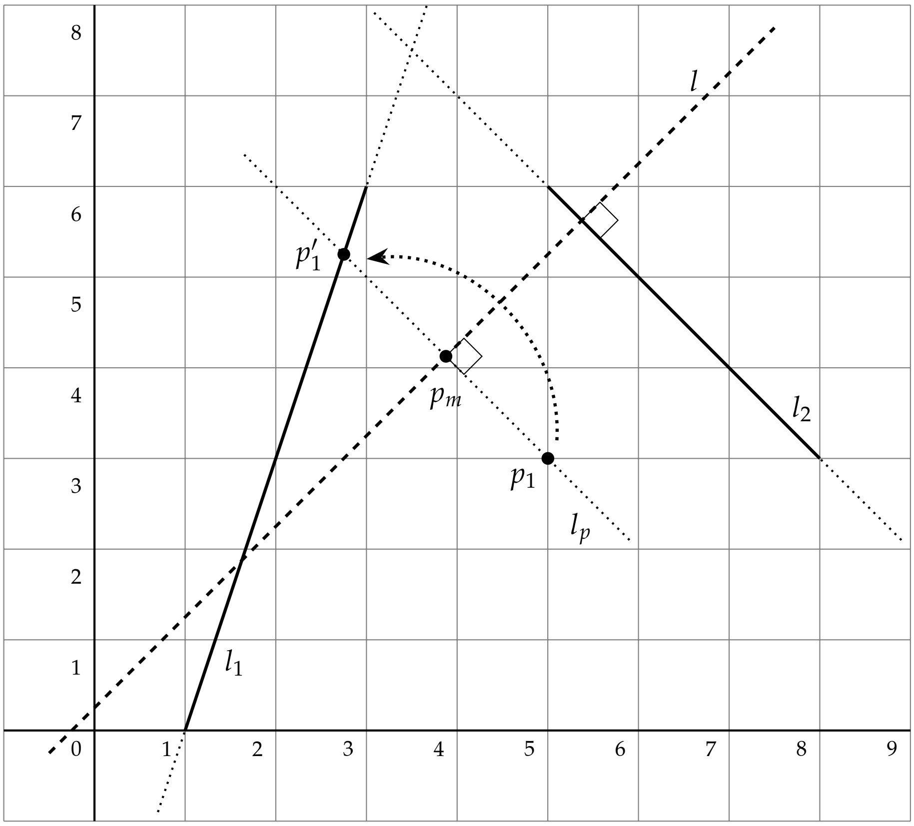
$$y_t = \sqrt{3}\sqrt{3} + 0 = 3$$

$$x_1' = 2x_t - x_1 = 2\sqrt{3} - 0 = 2\sqrt{3} \approx 3.46$$

$$y_1' = 2y_t - y_1 = 2 \cdot 3 - 4 = 2\,.$$

2.7 Axiom 7

Axiom Given one point p_1 and two lines l_1 and l_2, there is a fold l that places p_1 onto l_1 and is perpendicular to l_2.



Derivation of the equation of the fold

Let $p_1 = (x_1, y_1)$, let l_1 be $y = m_1 x + b_1$ and let l_2 be $y = m_2 x + b_2$.

Since the fold l is perpendicular to l_2, and the line l_p containing $\overline{p_1 p_1'}$ is perpendicular to l, it follows that l_p parallel to l_2:

$$y = m_2 x + b_p\,.$$

l_p passes through p_1 so $y_1 = m_2 x_1 + b_p$ and its equation is:

$$y = m_2 x + (y_1 - m_2 x_1)\,.$$

$p_1' = (x_1', y_1')$, the reflection of p_1 around the fold l, is the intersection of l_1 and l_p:

$$\begin{aligned}
m_1 x_1' + b_1 &= m_2 x_1' + (y_1 - m_2 x_1) \\
x_1' &= \frac{y_1 - m_2 x_1 - b_1}{m_1 - m_2} \\
y_1' &= m_1 x_i' + b_1\,.
\end{aligned}$$

The midpoint $p_m = (x_m, y_m)$ of l_p is on the fold l:

$$(x_m, y_m) = \left(\frac{x_1 + x_1'}{2}, \frac{y_1 + y_1'}{2} \right) .$$

The equation of the fold l is the perpendicular bisector of $\overline{p_1 p_1'}$. First compute the intercept of l which passes through p_m:

$$y_m = -\frac{1}{m_2} x_m + b_m$$

$$b_m = y_m + \frac{x_m}{m_2} .$$

The equation of the fold l is:

$$y = -\frac{1}{m_2} x + \left(y_m + \frac{x_m}{m_2} \right) .$$

Example

Let $p_1 = (5, 3)$, let l_1 be $y = 3x - 3$ and let l_2 be $y = -x + 11$.

$$x_1' = \frac{3 - (-1) \cdot 5 - (-3)}{3 - (-1)} = \frac{11}{4}$$

$$y_1' = 3 \cdot \frac{11}{4} + (-3) = \frac{21}{4}$$

$$p_m = \left(\frac{5 + \frac{11}{4}}{2}, \frac{3 + \frac{21}{4}}{2} \right) = \left(\frac{31}{8}, \frac{33}{8} \right) .$$

The equation of the fold l is:

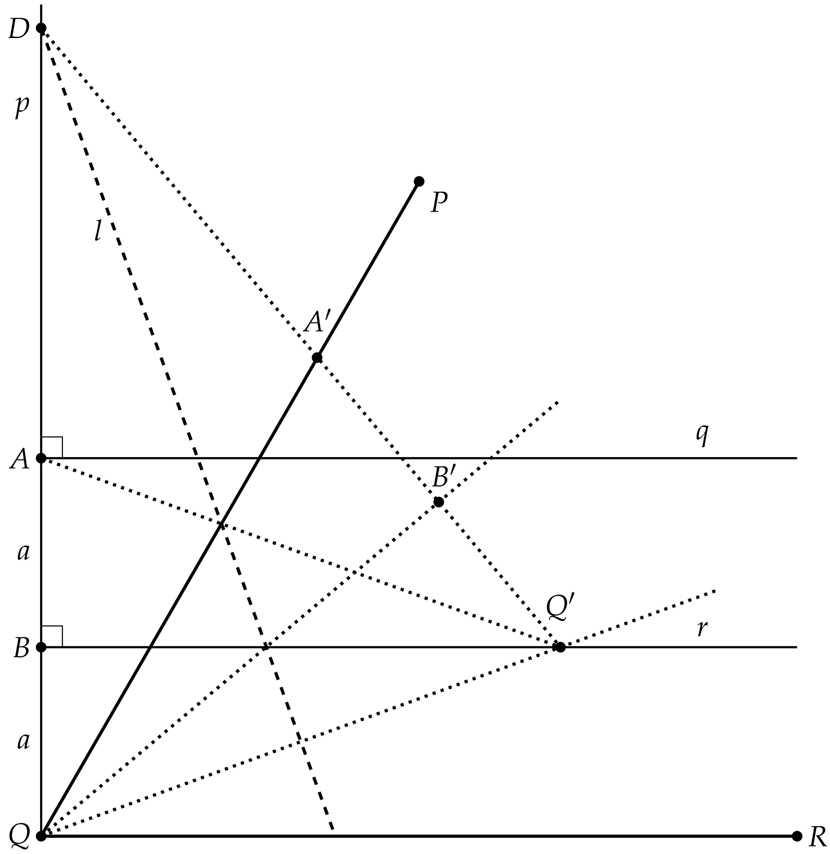
$$y = -\frac{1}{-1} \cdot x + \left(\frac{33}{8} + \frac{\frac{31}{8}}{-1} \right) = x + \frac{1}{4} .$$

Chapter 3

Trisecting an Angle

3.1 Abe's trisection of an angle

3.1.1 The construction

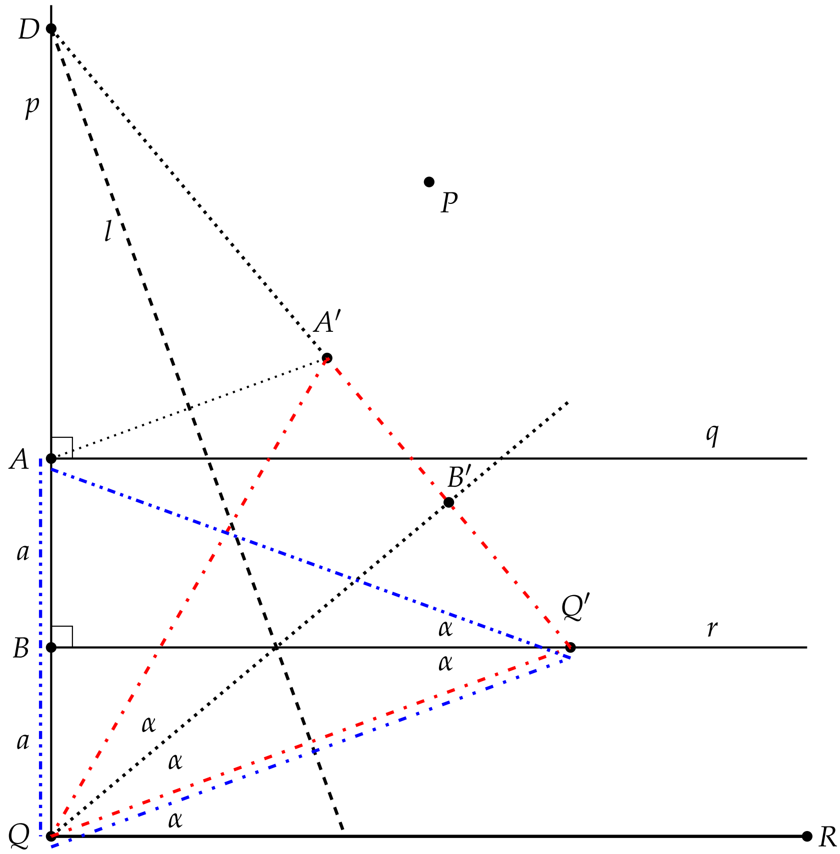


Given an acute angle $\angle PQR$, let p be the perpendicular to $\overline{QR}$ at Q. Let q be a perpendicular to p that intersects $\overline{PQ}$ at point A, and let r be the perpendicular to p at B that is halfway between Q and A.

Using Axiom 6, construct a fold l that places A at A' on $\overline{PQ}$ and Q at Q' on r. Let B' be the reflection of B around l.

Draw the lines $\overline{QB'}$ and QQ'. We claim that $\angle PQB'$, $\angle B'QQ'$ and $\angle Q'QR$ are a trisection of $\angle PQR$.

3.1.2 First proof



Since A', B', Q' are all reflections around the same line l of the points A, B, Q on one line DQ, they are all on one line $\overline{DQ'}$. By construction, $\overline{AB} = \overline{BQ}$, $\overline{BQ'}$ is perpendicular to AQ; $\overline{BQ'}$ is a common side, so $\triangle ABQ' \cong \triangle QBQ'$ by side-angle-side. Therefore, $\angle AQ'B = \angle QQ'B = \alpha$, since $\overline{Q'B}$ is the perpendicular bisector of the isoceles triangle $\triangle AQ'Q$.

By alternating interior angles, $\angle Q'QR = \angle QQ'B = \alpha$.

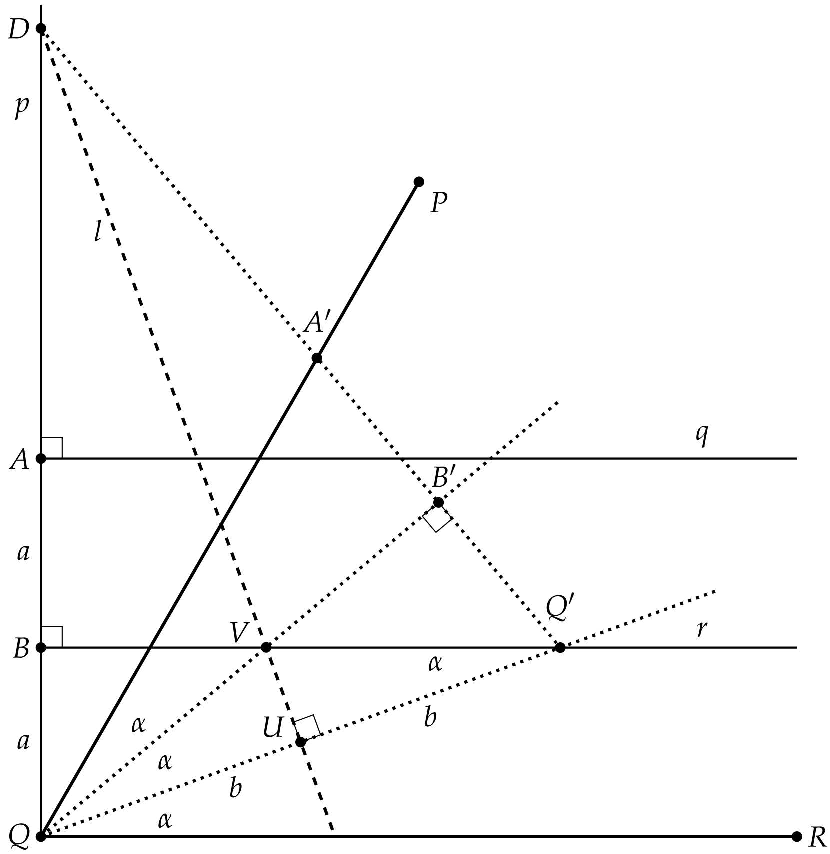
By reflection, $\triangle AQ'Q \cong \triangle A'QQ'$.[1]

> The fold l is the perpendicular bisector of both $\overline{AA'}$ and $\overline{QQ'}$; drop perpendiculars from A and A' to $\overline{QQ'}$; then $\overline{AQ} = \overline{A'Q'}$ follows by congruent right triangles. $\overline{AA'Q'Q}$ is an isoceles trapezoid so its diagonals are equal $\overline{AQ'} = \overline{A'Q}$.

Therefore, $\overline{QB'}$, the reflection of $\overline{Q'B}$, is the perpendicular bisector of an isoceles triangle and $\angle A'QB' = \angle Q'QB' = \angle QQ'B = \alpha$.

[1] The two triangles have been emphasized using different patterns of dashes and dots, as well as using color.

3.1.3 Second proof

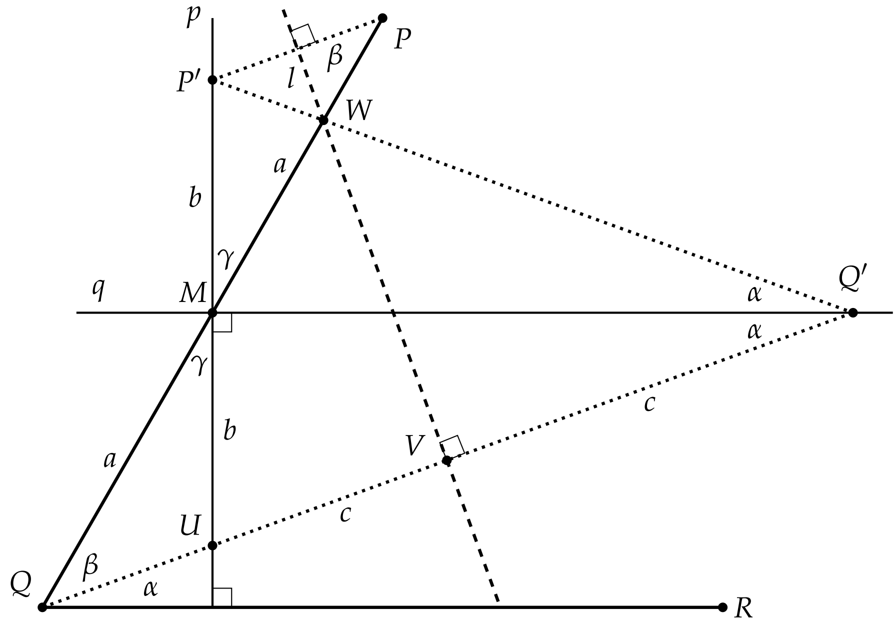


Since l is a fold, it is the perpendicular bisector of $\overline{QQ'}$. Denote the intersection of l with $\overline{QQ'}$ by U, and its intersection with $\overline{QB'}$ by V. $\triangle VUQ \cong \triangle VUQ'$ by side-angle-side since $\overline{VU}$ is a common side, the angles at U are right angles and $\overline{QU} = \overline{Q'U} = b$. Therefore, $\angle VQU = \angle VQ'U = \alpha$ and then $\angle Q'QR = \angle VQ'U = \alpha$ by alternating interior angles.

As in Proof 1, A', B', Q' are all reflections around l, so they are all on one line $\overline{DQ'}$, and $\overline{A'B'} = \overline{AB} = \overline{BQ} = \overline{B'Q'} = a$. Then $\triangle A'B'Q \cong \triangle Q'B'Q$ and $\angle A'QB' = \angle Q'QB' = \alpha$.

3.2 Martin's trisection of an angle

3.2.1 The construction



Given the acute angle $\angle PQR$, let M be the midpoint of $\overline{PQ}$. Construct p the perpendicular to $\overline{QR}$ through M and construct q perpendicular to p through M. q is parallel to $\overline{QR}$.

Using Axiom 6, construct a fold l that places P at P' on p and Q at Q' on q. More than one fold may be possible; choose the one that intersects $\overline{PM}$.

Draw the lines $\overline{PP'}$ and $\overline{QQ'}$. Denote the intersection of $\overline{QQ'}$ with p by U and its intersection with l by V. Denote the intersection of $\overline{PQ}$ and $P'Q'$ with l by W.[2]

3.2.2 Proof

$\triangle QMU \cong \triangle PMP'$ by angle-side-angle: $\angle P'PM = \angle UQM = \beta$ by alternate interior angles; $\overline{QM} = \overline{MP} = a$ since M is the midpoint of $\overline{PQ}$; $\angle QMU = \angle PMP'$ are vertical angles. Therefore, $\overline{P'M} = \overline{MU} = b$.

$\triangle P'MQ' \cong \triangle UMQ'$ by side-angle-side: we have shown that $\overline{P'M} = \overline{MU} = b$; the angles at M are right angles; $\overline{MQ'}$ is a common side. Since the altitude of the isoceles triangle $\triangle P'Q'U$ is the bisector of $\angle P'Q'U$, so $\angle P'Q'M = \angle UQ'M = \alpha$.

$\triangle QWV \cong \triangle Q'WV$ by side-angle-side: $\overline{QV} = \overline{VQ'} = c$ and the angles at V are right angles since the fold is the perpendicular bisector of $\overline{QQ'}$; $\overline{VW}$ is a common side. Therefore, $\angle WQV = \beta = \angle WQ'V = 2\alpha$. By alternate interior angles $\angle Q'QR = \angle MQ'Q = \alpha$. We have $\angle PQR = \beta + \alpha = 2\alpha + \alpha = 3\alpha$ so $\angle Q'QR$ is one-third of $\angle PQR$.

[2]It is not immediate that both $\overline{PQ}$ and $P'Q'$ intersect l at the same point. $\triangle PP'W \sim \triangle QQ'W$ so the altitudes divide the vertical angles $\angle PWP'$, $\angle QWQ'$ similarly and thus must be on the same line.

Chapter 4

Doubling a Cube

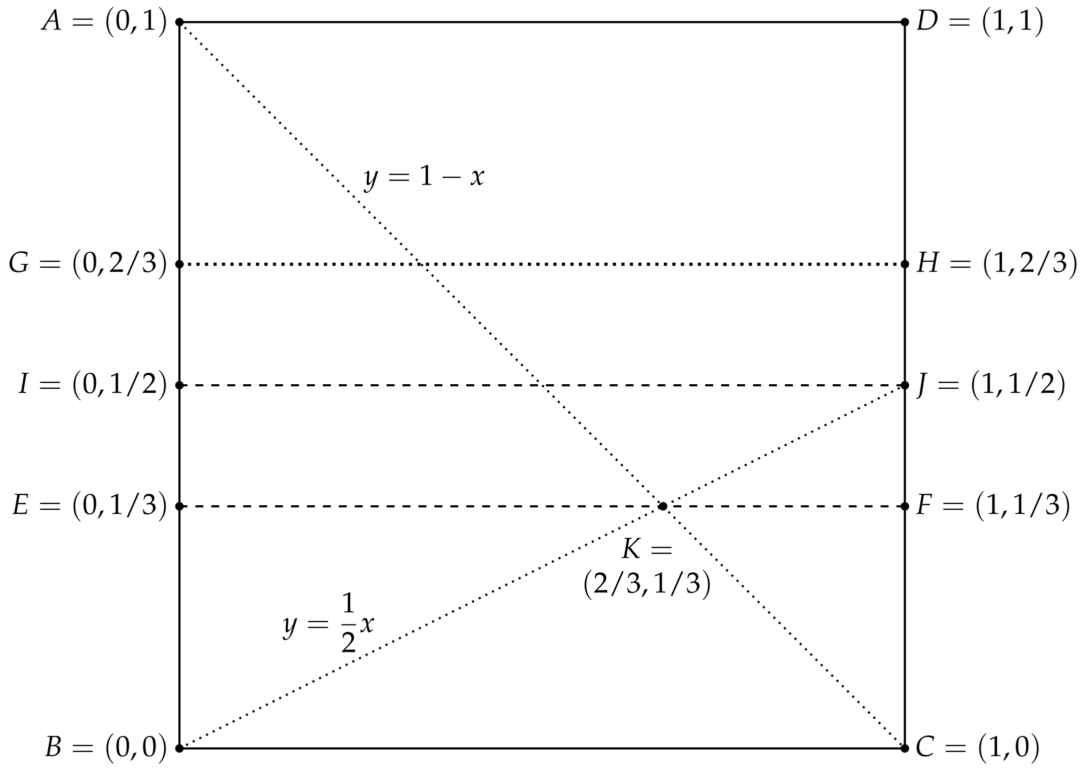
4.1 Messer's doubling of a cube

A cube of volume V has sides of length $\sqrt[3]{V}$. The volume of a cube with twice the volume is $2 \cdot V$, so we need to construct the length $\sqrt[3]{2 \cdot V} = \sqrt[3]{2} \cdot \sqrt[3]{V}$. If we can construct $\sqrt[3]{2}$, we can multiply by the given length $\sqrt[3]{V}$ to double the cube.

4.1.1 Dividing a length into thirds

Lang [5] shows efficient constructs for rational fractions of the length of the side of a square (piece of paper). Here, we need to divide the side of the square into thirds.

First, fold the square in half to locate the point $J = (1, 1/2)$. Next, draw the lines $\overline{AC}$ and $\overline{BJ}$.



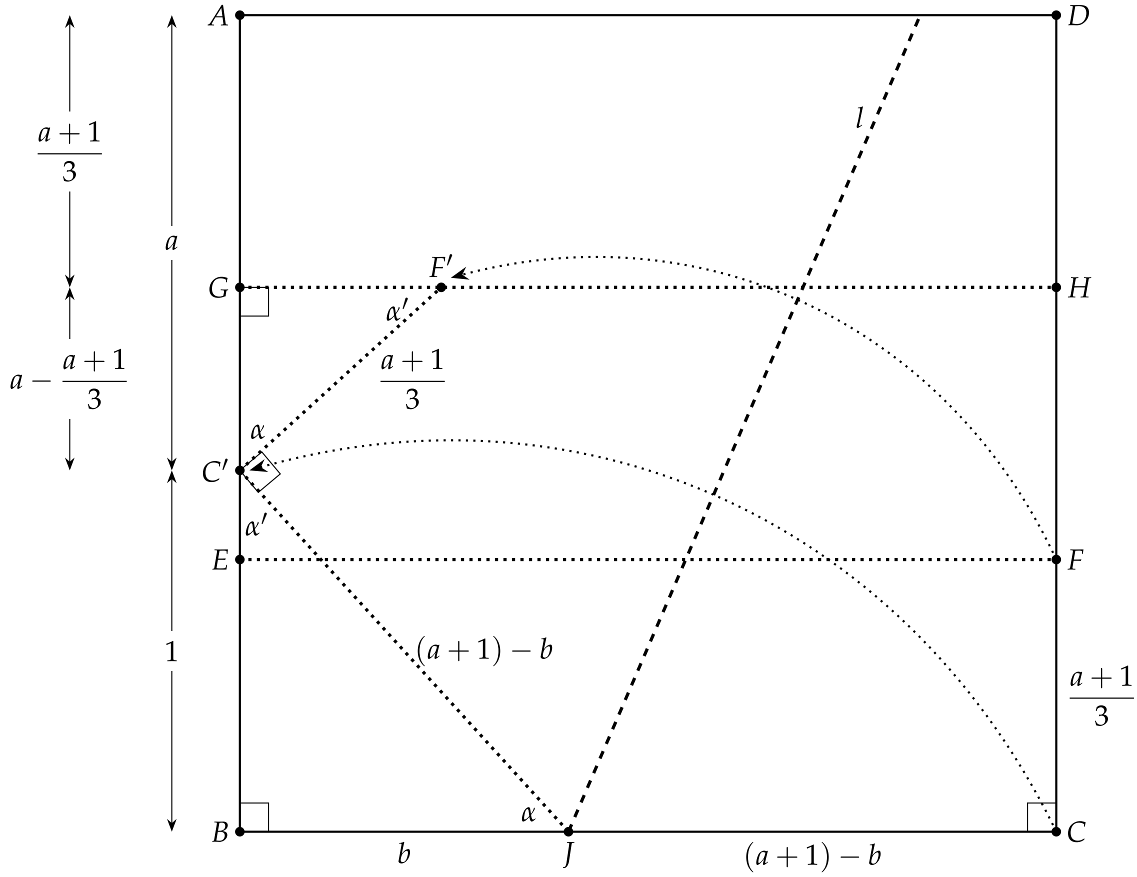
The coordinates of their point of intersection K is obtained by solving the two equations:

$$\begin{aligned} y &= 1 - x \\ y &= \frac{1}{2}x\,. \end{aligned}$$

The result is $x = 2/3, y = 1/3$.

Construct the line $\overline{EF}$ perpendicular to $\overline{AB}$ that goes K, and construct the reflection $\overline{GH}$ of $\overline{BC}$ around $\overline{EF}$. The side of the square has now been divided into thirds.

4.1.2 Building $\sqrt[3]{2}$



Label the side of the square by $a+1$. The construction will show that $a = \sqrt[3]{2}$.

Using Axiom 6 place C at C' on $\overline{AB}$ and F at F' on $\overline{GH}$. Denote by J the point intersection of the fold with $\overline{BC}$ and denote by b the length of $\overline{BJ}$. The length of $\overline{JC}$ is $(a+1)-b$.

When the fold is performed, the line segment $\overline{JC}$ is reflected onto the line segment $\overline{JC'}$ of the same length, and $\overline{CF}$ is folded onto the line segment $\overline{C'F'}$ of the same length. A simple computation shows that the length of $\overline{GC'}$ is:

$$a - \frac{a+1}{3} = \frac{2a-1}{3}\,. \tag{4.1}$$

Finally, since $\angle FCJ$ is a right angle, so is $\angle F'C'J$.

$\triangle C'BJ$ is a right triangle so by Pythagoras's theorem:

$$\begin{aligned}
1^2 + b^2 &= ((a+1) - b)^2 \\
a^2 + 2a - 2(a+1)b &= 0 \\
b &= \frac{a^2+2a}{2(a+1)}\,.
\end{aligned}$$

$\angle GC'F' + \angle F'C'J + \angle JC'B = 180^\circ$ since they form the straight line $\overline{GB}$. Denote $\angle GC'F'$ by α.

$$\angle JC'B = 180^\circ - \angle F'C'J - \angle GC'F' = 180^\circ - 90^\circ - \angle GC'F' = 90^\circ - \angle GC'F = 90^\circ - \alpha\,,$$

which we denote by α'. The triangles $\triangle C'BJ$, $\triangle F'GC'$ are right triangles, so $\angle C'JB = \alpha$ and $\angle C'F'G = \alpha'$. Therefore, the triangles are similar and using Equation 4.1 we have:

$$\frac{b}{(a+1)-b} = \frac{\dfrac{2a-1}{3}}{\dfrac{a+1}{3}}\,.$$

Substituting for b:

$$\begin{aligned}
\frac{\dfrac{a^2+2a}{2(a+1)}}{(a+1) - \dfrac{a^2+2a}{2(a+1)}} &= \frac{2a-1}{a+1} \\
\frac{a^2+2a}{a^2+2a+2} &= \frac{2a-1}{a+1}\,.
\end{aligned}$$

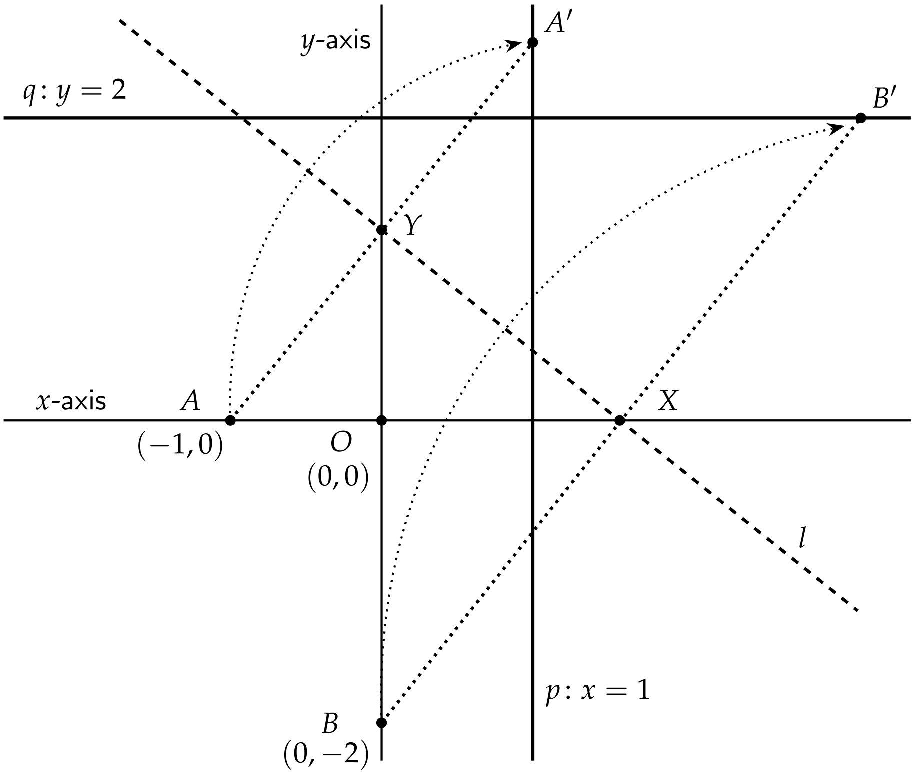
Simplifying results in $a^3 = 2$ and $a = \sqrt[3]{2}$.

4.2 Beloch's doubling of a cube

In 1936 Margharita P. Beloch formalized Axiom 6 (often called the *Beloch fold*) and showed that it could be used to solve cubic equations. Here we give her construction for doubling the cube. The solution of cubic equations is discussed in Chapters 5, 6.

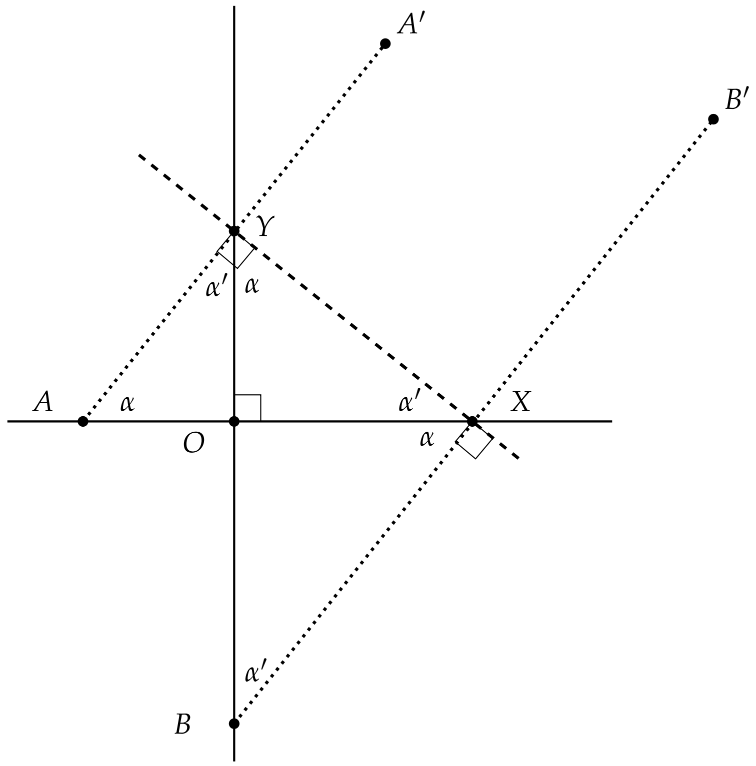
4.2.1 The construction

Place point A at $(-1, 0)$ and point B at $(0, -2)$. Let p be the line with equation $x = 1$ and let q be the line with equation $y = 2$. Using Axiom 6 construct a fold l that places A at A' on p and B at B' on q. Denote the intersection of the fold and the y-axis by Y and the intersection of the fold and x-axis by X.



4.2.2 Proof

Let us extract a simplified diagram:



The fold is the perpendicular bisector of $\overline{AA'}$ and $\overline{BB'}$. Therefore, $\angle AYX$ and $\angle YXB$ are right angles and $\overline{AA'}$ is parallel to $\overline{BB'}$. By alternate interior angles $\angle YAO = \angle BXO = \alpha$. If an acute angle in a right triangle is α, the other acute angle must be $90° - \alpha$, which we denote α'. The labeling of the angles in all the triangles in the diagram follows immediately.

We have three similar triangles $\triangle AOY \sim \triangle YOX \sim \triangle XOB$. $\overline{OA} = 1$, $\overline{OB} = 2$ are given, so:

$$\frac{\overline{OY}}{\overline{OA}} = \frac{\overline{OX}}{\overline{OY}} = \frac{\overline{OB}}{\overline{OX}}$$
$$\frac{\overline{OY}}{1} = \frac{\overline{OX}}{\overline{OY}} = \frac{2}{\overline{OX}}$$
$$\overline{OY}^2 = \overline{OX} = \frac{2}{\overline{OY}},$$

resulting in $\overline{OY}^3 = 2$ and $\overline{OY} = \sqrt[3]{2}$.

Chapter 5

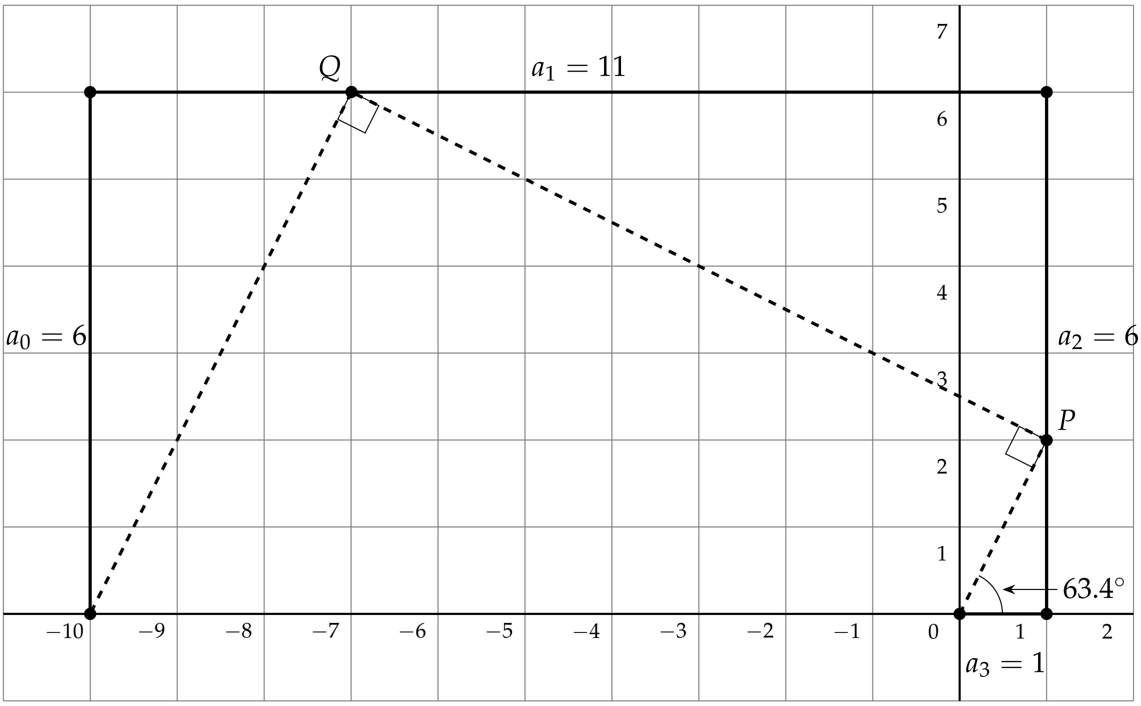
Lill's Method for Finding Roots

5.1 Magic

Construct a path consisting of four line segments $\{a_3, a_2, a_1, a_0\}$ of lengths:

$$\{a_3 = 1, a_2 = 6, a_1 = 11, a_0 = 6\},$$

starting from the origin, first along the positive direciton of the x-axis and turning 90° counterclockwise between segments. Construct a second path as follows: draw a line from the origin at an angle of 63.4° and mark its intersection with a_2 by P. Turn left 90°, draw a line and and mark its intersection with a_1 by Q. Turn left 90° once again, draw a line and note that it intersects the end of the first path at $(-10, 0)$.



Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 = x^3 + 6x^2 + 11x + 6$. Compute $\tan 63.4° = 2$, the tangent of the angle at the start of the second path. Then:

$$p(-\tan 63.4°) = (-2)^3 + 6(-2)^2 + 11(-2) + 6 = 0\,.$$

Congratulations! You have found a root of the cubic polynomial $x^3 + 6x^2 + 11x + 6$.

5.2 Introduction

This example demonstrates a method discovered by Eduard Lill in 1867 for graphically finding the real roots of any polynomial [3, 4, 9]. We limit the presentation to cubic polynomials.

Clearly, this is not an algebraic method of computing roots of cubic equations; in the example, we are essentially verifying that -2 is a root. Lill's method has seen renewed interest because it can be implemented using origami [4].

In Sections 5.3–5.4 we continue the initial example to find additional roots and to show that if an angle α is such that $-\tan\alpha$ is *not* a root, then the construction doesn't work.

Section 5.5 presents the full specification of Lill's method. Some of the description may be difficult to understand, but will be clarified when demonstrated by additional examples in Sections 5.6–5.8. Since Lill's method can find a real root of any cubic polynomial, it can be used to trisect an angle. Since it can find $\sqrt[3]{2}$ as a root of $x^3 - 2$, it can double a cube as shown in Section 5.9.

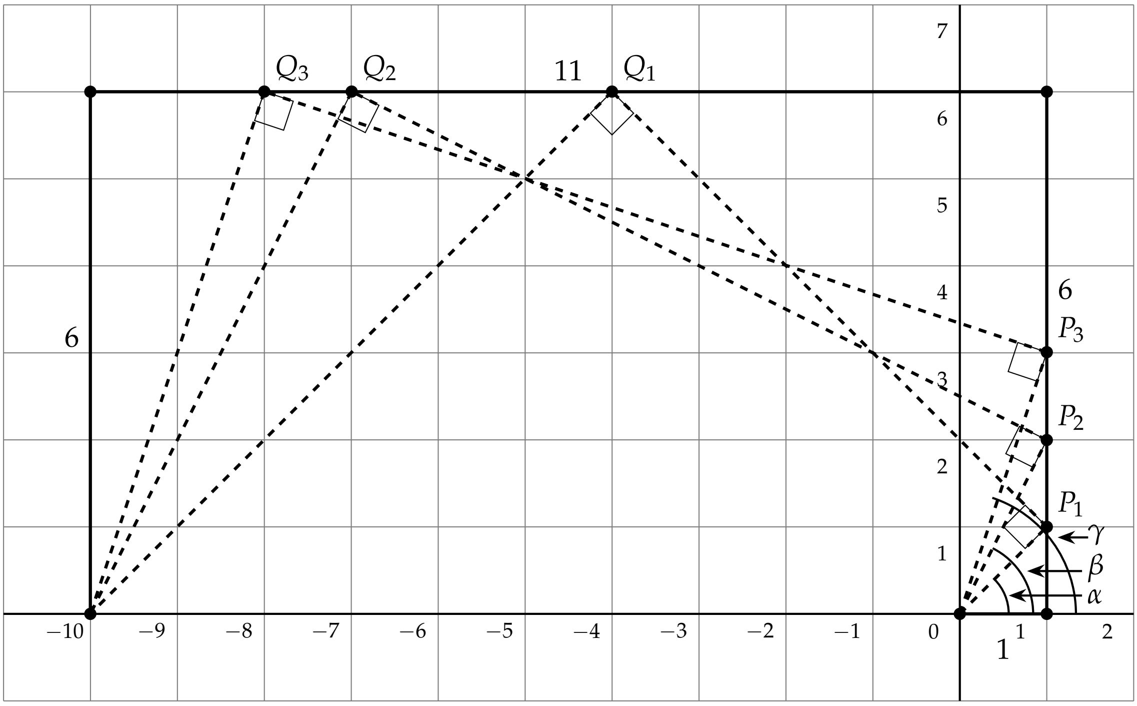
Section 5.10 gives a proof that Lill's method can find the real roots of any cubic polynomial. The proof for arbitrary polynomials follows the same structure.

5.3 Multiple roots

Let us continue the example above. The polynomial $p(x) = x^3 + 6x^2 + 11x + 6$ has three roots $-1, -2, -3$. Computing the arc tangent of the negation of the roots gives:

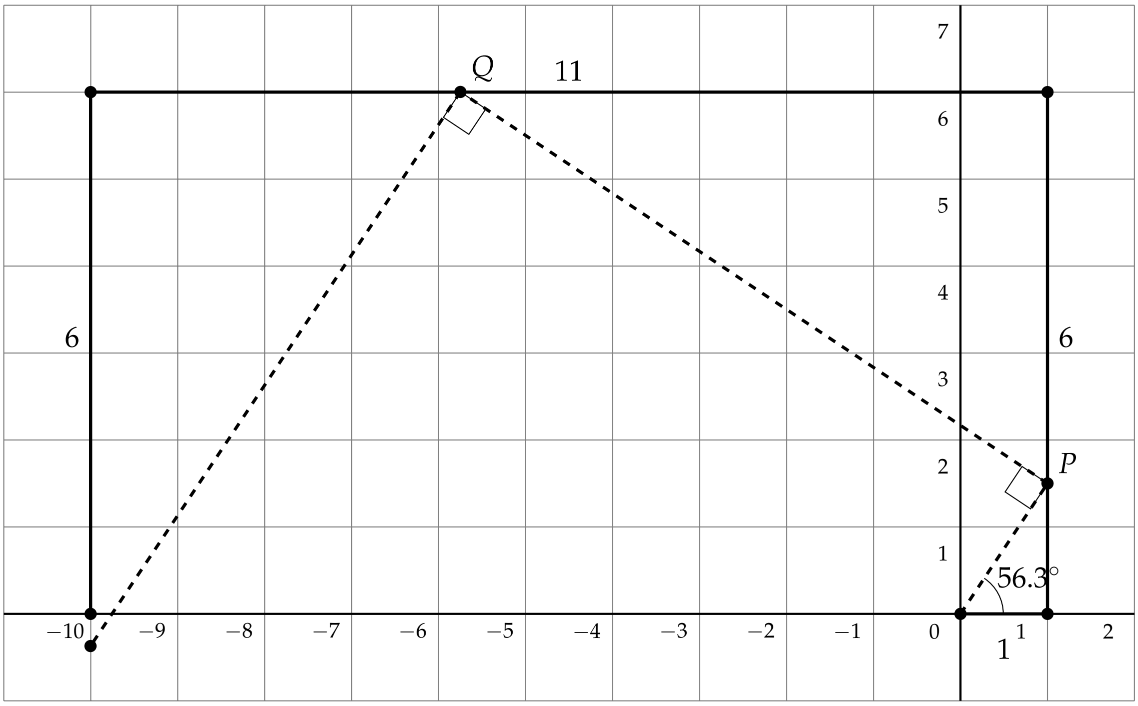
$$\alpha = -\tan^{-1} -1 = 45^\circ, \quad \beta = -\tan^{-1} -2 = 63.4^\circ, \quad \gamma = -\tan^{-1} -3 = 71.6^\circ .$$

In the diagram below we see that for each of the three angles, the second path intersects the end of the first path.



5.4 Paths that do not lead to roots

Perhaps the second path intersects the end of the first path for *any* initial angle, for example, 56.3°. In the following diagram, the second path intersects the extension of the line segment for the coefficient a_0, but not at $(-10, 0)$, the end of the first path. We conclude that $-\tan 56.3 = -1.5$ is *not* a root of the equation.



5.5 Specification of Lill's method

Examining the examples below will help understand the details.

- Start with an arbitrary cubic polynomial: $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.
- Construct the first path as follows:
  - For each coefficient a_3, a_2, a_1, a_0 (in that order) draw a line segment starting at the origin $O = (0, 0)$ in the positive direction of the x-axis. Turn 90° counterclockwise between each segment.
- Construct the second path as follows:
  - We use a_i to denote the side of length a_i.
  - Construct a line from O at an angle of θ with the positive x-axis that intersects a_2 at point P.
  - Turn $\pm 90°$ and construct a line from P that intersects a_1 at Q.
  - Turn $\pm 90°$ and construct a line from Q that intersects a_0 at R.
  - If R is the end point of the first path, then $-\tan\theta$ is a root of $p(x)$.
- Special cases:
  - When drawing the line segments of the first path, if a coefficient is negative, draw the line segment *backwards*.
  - When drawing the line segments of the first path, if a coefficient is zero, do not draw a line segment but continue with the next $\pm 90°$ turn.
- Notes:
  - "Intersects a_i" includes "intersects the line segment a_i or any extension of it" or "intersects the line that contains the line segment a_i".
  - When building the second path, choose to turn left or right by 90° so that there is an intersection with the next segment of the first path.

5.6 Negative coefficients

Section 2.6 gave an example of the use of Axiom 6 which resulted in the polynomial $p(x) = x^3 - 3x^2 - 3x + 1$ that has negative coefficients.

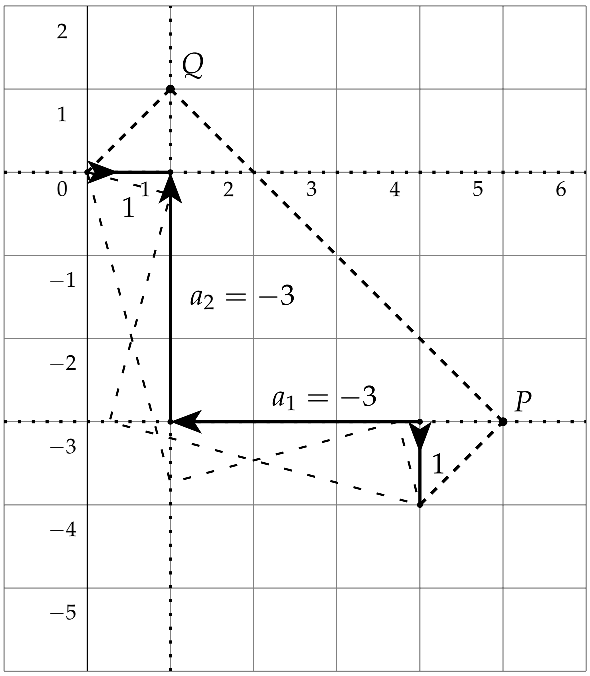
We start by drawing a segment of length 1 to the right. Next we turn 90° to face up, but the coefficient is negative, so we draw a segment of length 3 *down*. After turning 90° to the left, the coefficient is again negative, so we draw a segment of length 3 to the right. Finally, we turn down and draw a segment of length 1.

We start the second path with a line angled 45° with the x-axis. It intersects the *extension* of the line segment for a_2 at $(1, 1)$. Turning $-90°$ (to the right), the line intersects the *extension* of the line segment for a_1 at $(5, -3)$. Turning $-90°$ again, the line intersects the end of the first path at $(4, -4)$.

Since $-\tan 45° = -1$, a real root of the polynomial is -1:

$$p(-1) = (-1)^3 - 3(-1)^2 - 3(-1) + 6 = 0\,.$$

The loosely dashed lines in the Figure will be discussed in Section 5.8.



5.7 Zero coefficients

a_2, the coefficient of the x^2 term in the polynomial $x^3 - 7x - 6 = 0$, is zero. For a zero coefficient, we "draw" a line segment of length 0, that is, we do not draw a line, but we still make the $\pm 90°$ turn before "drawing" it, as indicated by the arrow pointed up at point $(1,0)$ in the diagram. Next make an additional turn and draw a line of length -7, that is, of length 7 backwards, to point $(8,0)$. Finally, turn again and draw a line of length -6 to point $(8,6)$.

There are three second paths that intersect the end of the first path. They start with angles of:

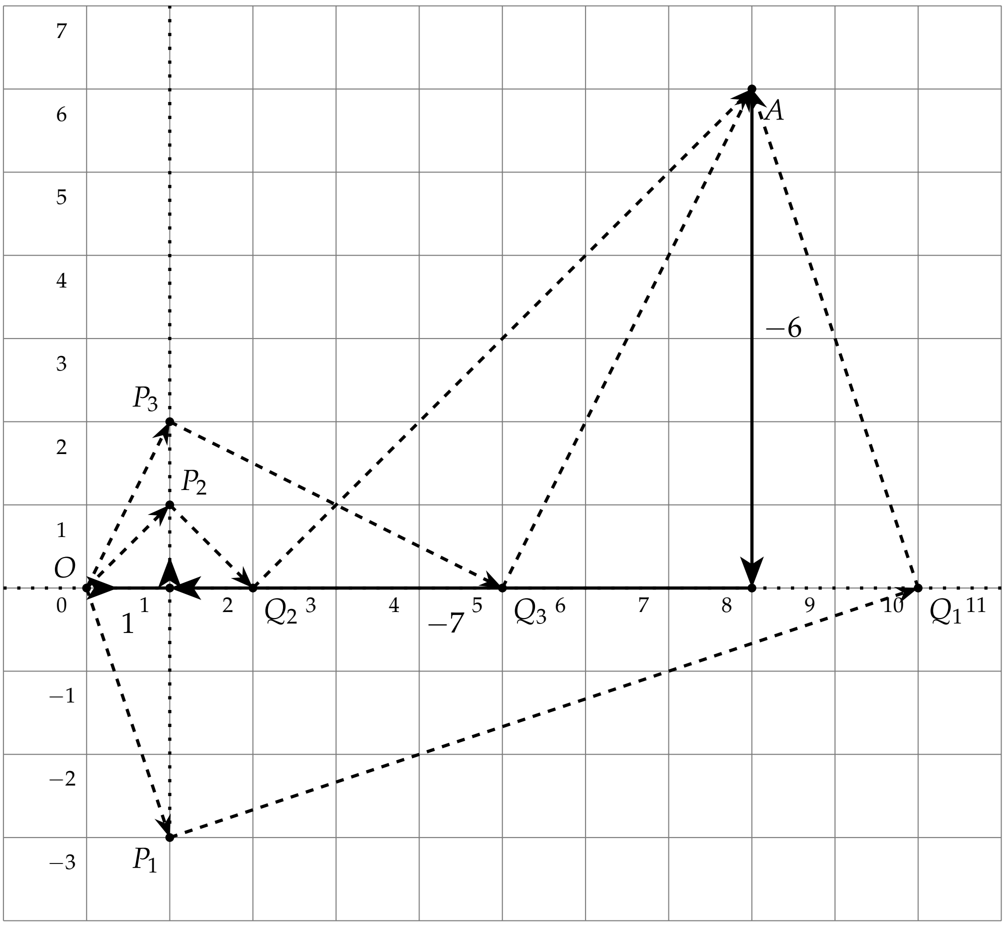
$$\alpha = 45°, \quad \beta = 63.4°, \quad \gamma = -71.6°\,.$$

We conclude that there are three real roots:

$$-\tan 45° = -1, \quad -\tan 63.4° = -2, \quad -\tan(-71.6°) = 3\,.$$

Check:

$$(x+1)(x+2)(x-3) = (x^2+3x+2)(x-3) = x^3 - 7x - 6\,.$$



5.8 Non-integer roots

Consider the polynomial $p(x) = x^3 - 2x + 1$. The first path goes from $(0,0)$ to $(1,0)$ and then turns up. The coefficient of x^2 is zero so no segment is drawn and the path turns left. The next segment is of length -2 so it goes backwards from $(1,0)$ to $(3,0)$. Finally, the path turns down and a line of length 1 is drawn from $(3,0)$ to $(3,-1)$.

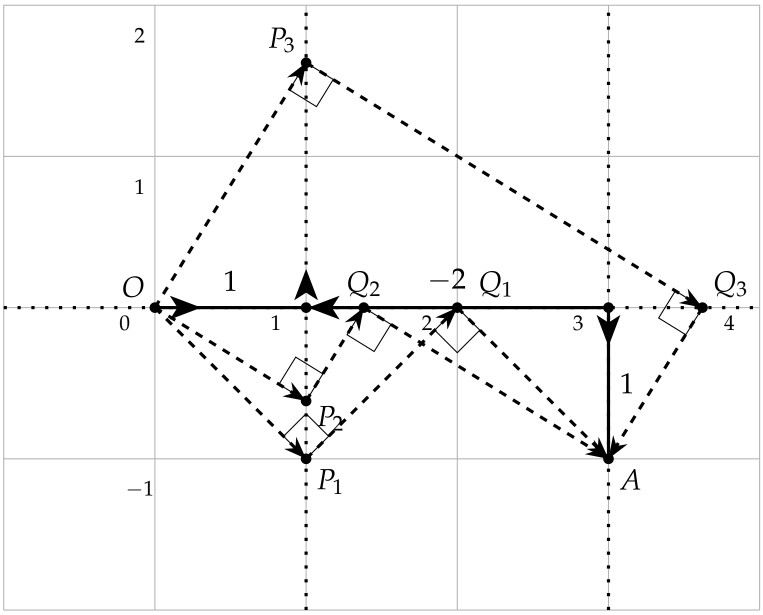
It is easy to see that 1 is a root of $p(x)$. Since $-\tan^{-1} -45^\circ = -1$, there is a path $\overline{OP_1Q_1A}$.

If we divide $p(x)$ by $x-1$, we obtain the quadratic polynomial $x^2 + x - 1$ whose roots are:

$$\frac{-1 \pm \sqrt{5}}{2} \approx 0.62,\ -1.62\,.$$

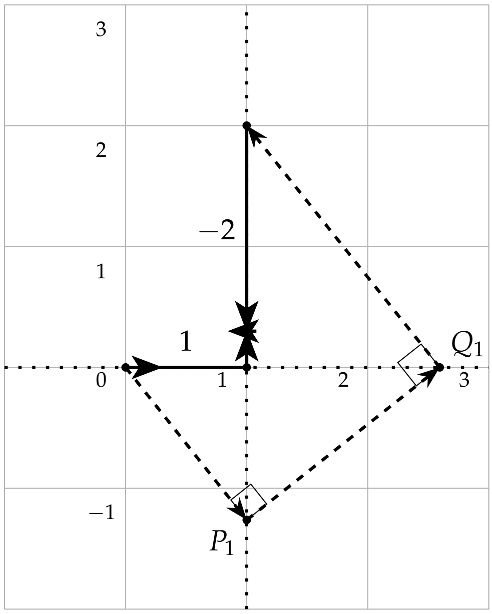
Therefore, there are two additional second paths: one starting with an angle of -31.8° since $-\tan^{-1} 0.62 = -31.8^\circ$, and one starting with 58.3° since $-\tan^{-1} 1.62 = 58.3^\circ$.

Similarly, the polynomial in Section 5.6 has roots $2 \pm \sqrt{3} \approx 3.73, 0.27$. The corresponding angles are -75° and -15°, because $-\tan(-75^\circ) \approx 3.73$ and $-\tan(-15^\circ) \approx 0.27$.



5.9 The cube root of two

To double a cube we need to compute $\sqrt[3]{2} \approx 1.26$ which is a root of the cubic polynomial $x^3 - 2$. In the construction of the first path, we turn left twice without drawing any line segments, because a_2 and a_1 are both zero. Then we turn left again (to face down) and draw backwards because $a_0 = -2$ is negative. The first segment of the second path is drawn at an angle of $-51.6°$ and $-\tan(-51.6°) \approx 1.26 \approx \sqrt[3]{2}$.



5.10 Proof of Lill's method

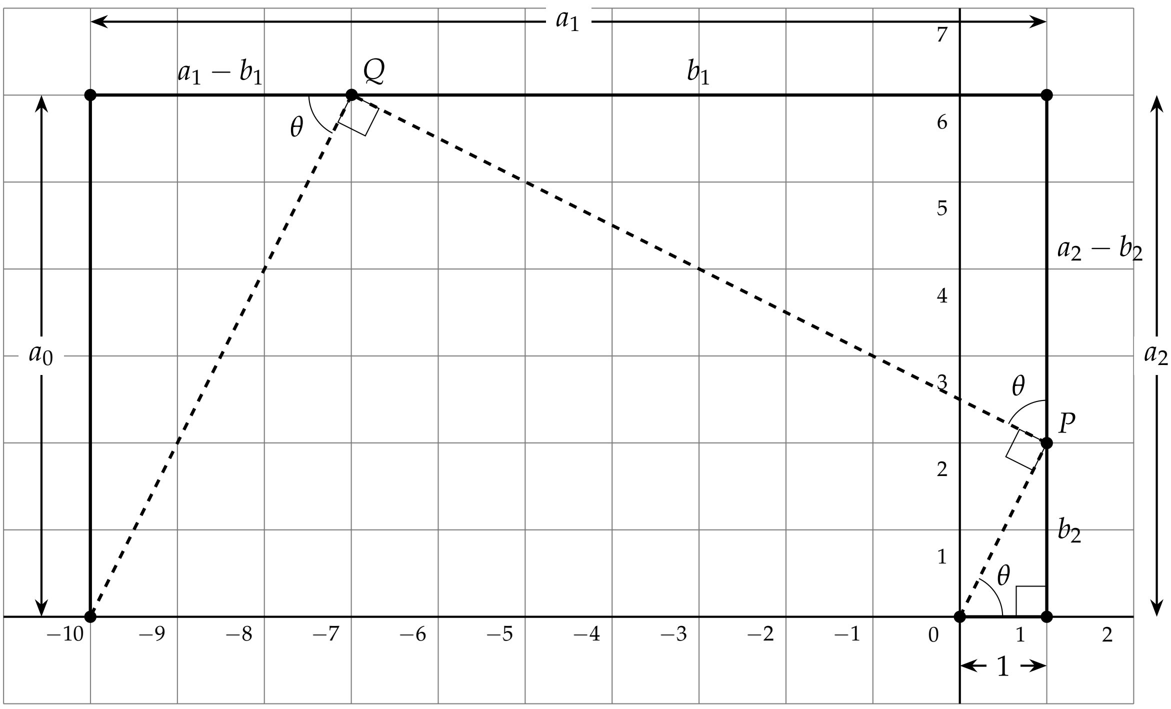
We limit ourselves to monic cubic polynomials $p(x) = x^3 + a_2x^2 + a_1x + a_0$.[1] In the diagram below, segments of the first path are labeled with coefficients and with $b_2, b_1, a_2 - b_2, a_1 - b_1$. Since the sum of the angles of a triangle is 180°, in a right triangle if one acute angle is θ, the other is $90° - \theta$. Therefore, the angle above P and the angle to the left of Q are equal to θ. We now derive a sequence of formulas for $\tan\theta$:

$$\begin{aligned}
\tan\theta &= \frac{b_2}{1} = b_2 \\
\tan\theta &= \frac{b_1}{a_2 - b_2} = \frac{b_1}{a_2 - \tan\theta} \\
b_1 &= \tan\theta(a_2 - \tan\theta) \\
\tan\theta &= \frac{a_0}{a_1 - b_1} = \frac{a_0}{a_1 - \tan\theta(a_2 - \tan\theta)}\,.
\end{aligned}$$

Simplifying the last equation gives:

$$\begin{aligned}
(\tan\theta)^3 - a_2(\tan\theta)^2 + a_1(\tan\theta) - a_0 &= 0 \\
-(\tan\theta)^3 + a_2(\tan\theta)^2 - a_1(\tan\theta) + a_0 &= 0 \\
(-\tan\theta)^3 + a_2(-\tan\theta)^2 + a_1(-\tan\theta) + a_0 &= 0\,.
\end{aligned}$$

We conclude that $-\tan\theta$ is a real root of $p(x) = x^3 + a_2x^2 + a_1x + a_0$.



[1] If the polynomial is not monic, divide it by a_3 and the resulting monic polynomial has the same roots.

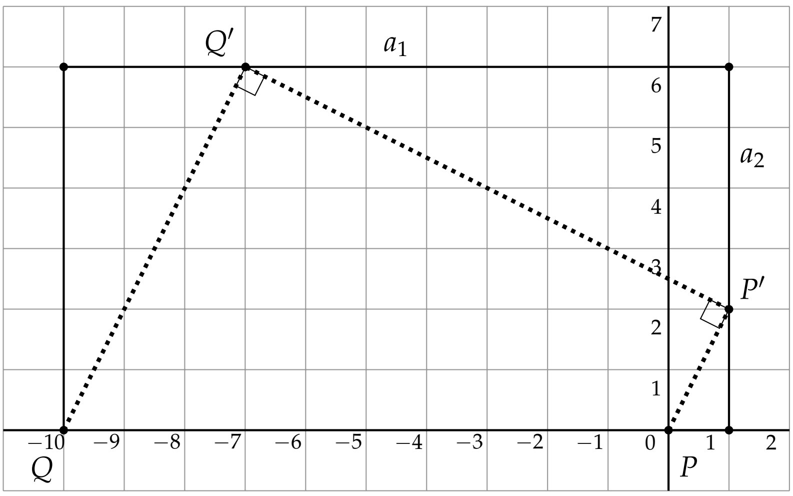
Chapter 6

Beloch's Fold and Beloch's Square

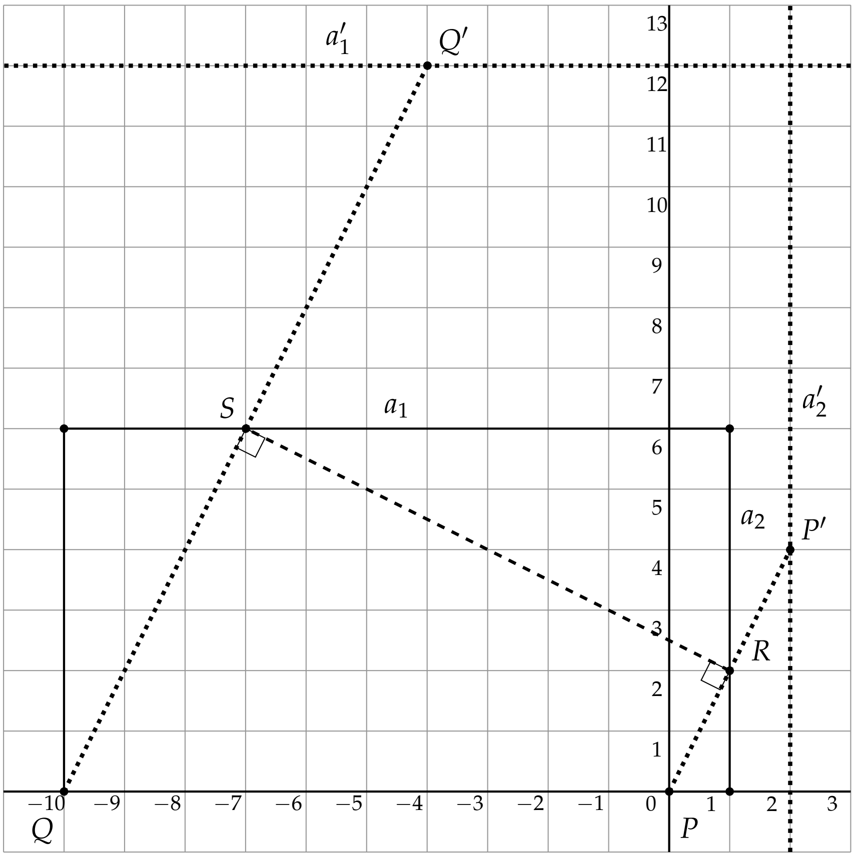
6.1 The Beloch fold

Margharita P. Beloch discovered a remarkable connection between origami and Lill's method for finding roots of polynomials. She found that one application of the operation of origami Axiom 6 (Section 2.6) applied to the first path of Lill's method can obtain a real root of any cubic polynomial. The operation is often called the *Beloch fold*.

Consider the polynomial $p(x) = x^3 + 6x^2 + 11x + 6$ from Section 5.1. In the following diagram we have emphasized the second path and renamed some vertices. All we have to do to solve the equation is to perform a Beloch fold to simultaneously place the points P', Q' on the line segments of lengths a_2, a_1, respectively. Unfortunately, if you perform the fold, the path does not solve the equation: Q' is way off to the right, so the angles at P' and Q' are not right angles.

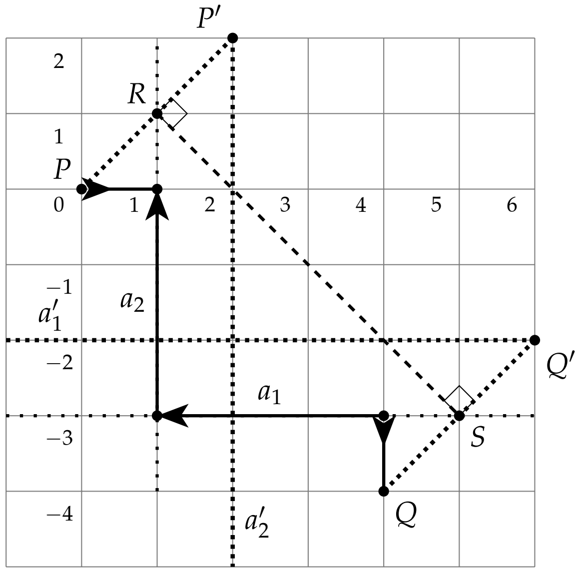


Recall that a fold is the perpendicular bisector of the line segment between any point and its reflection around the fold. We want $\overline{P'Q'}$ to be a fold so that it will be perpendicular to both $\overline{QQ'}$ and $\overline{PP'}$. If $\overline{P'Q'}$ is the perpendicular bisector of $\overline{QQ'}$ and $\overline{PP'}$, then P', Q', the reflections of P, Q, must be the same distance away from the fold as P and Q, respectively. With some change of notation we have the following diagram.



A line a_2' is drawn so that it is parallel to a_2 and the same distance from a_2 as a_2 is from P. Similarly, line a_1' is drawn so that it is parallel to a_1 and the same distance from a_1 as a_1 is from Q. Apply Axiom 6 to simultaneously place P at P' on a_2' and to place Q at Q' on a_1'. The fold $\overline{RS}$ is the perpendicular bisector of the lines $\overline{PP'}$ and $\overline{QQ'}$; therefore, the angles at R and S are right angles as required.

Let us try the Beloch fold on the polynomial $x^3 - 3x^2 - 3x + 1$ from Section 5.6. a_2 is the vertical line segment of length 3 whose equation is $x = 1$, and its parallel line is a_2' whose equation is $x = 2$, because P is at a distance of 1 from a_2. a_1 is the horizontal line segment of length 3 whose equation is $y = -3$, and its parallel line is a_1' whose equation is $y = -2$ because Q is at a distance of 1 from a_1. The fold RS is the perpendicular bisector of both $\overline{PP'}$ and $\overline{QQ'}$. The line $\overline{PRSQ}$ is the same as the second path in Section 5.6.

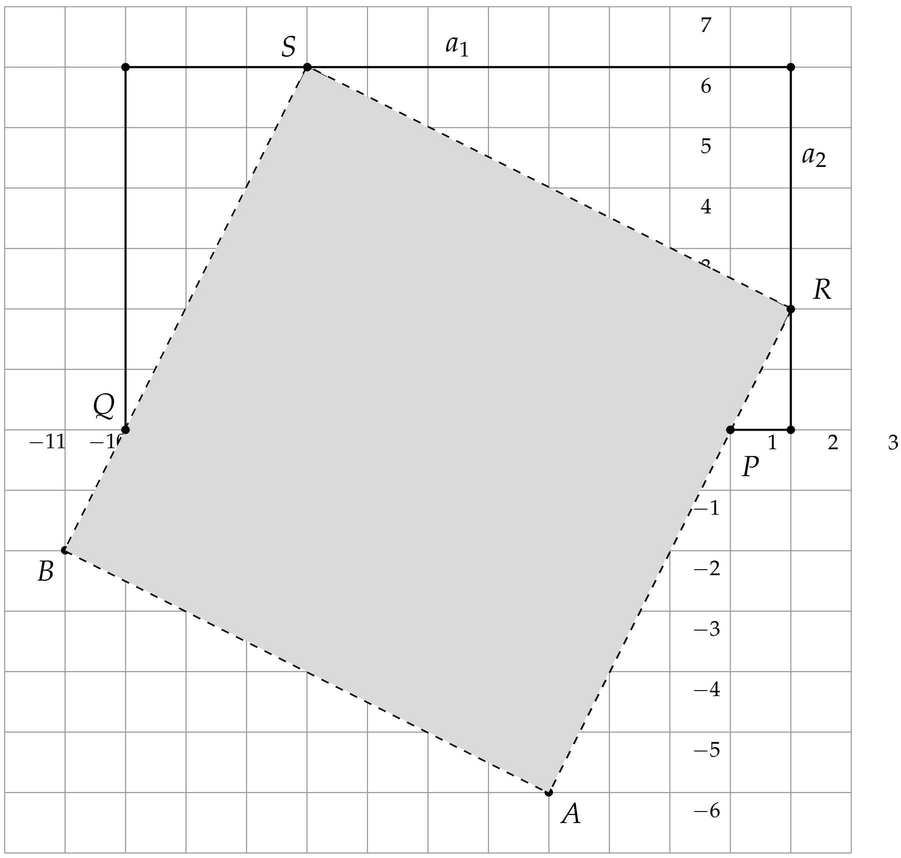


6.2 The Beloch square

This construction in the previous section can be expressed in terms of a *Beloch square*: Given two points P, Q and two lines a_2, a_1, construct a square $\overline{ARSB}$, such that:

- One side is $\overline{RS}$ where R lies on a_2 and S lies on a_1;
- P lies on $\overline{RA}$ and Q lies on $\overline{SB}$.

The following diagram extends the construction for $x^3 + 6x^2 + 11x + 6$ to show the Beloch square. The length of RS is $\sqrt{80} = 4\sqrt{5} \approx 8.94$. We can construct the square by adding three sides of this length.



Chapter 7

Constructing a Nonagon

7.1 Construction of regular polygons

The Gauss-Wantzel Theorem states that a regular polygon is constructible by straightedge and compass if the number of sides is:

$$n = 2^k \cdot p_1 \cdot \cdots \cdot p_m \,,$$

where the p_i's (if any) are distinct Fermat primes of the form $F_m = 2^{2^m} + 1$. There are five known Fermat primes: $F_0 = 3, F_1 = 5, F_2 = 17, F_3 = 257, F_4 = 65537$.[1] Therefore, a *nonagon*, a regular polygon with nine sides, is not constructible.

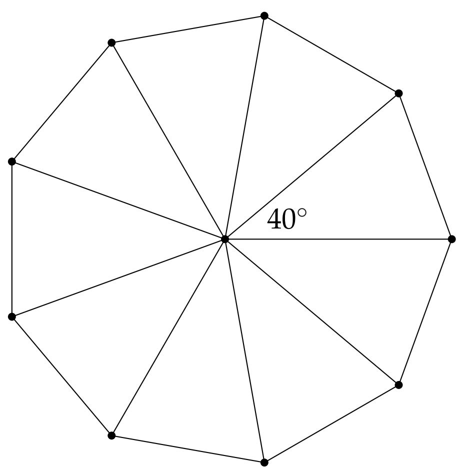
Regular polygons with:

$$n = 2^i \cdot 3^j \cdot p_1 \cdot \cdots \cdot p_m$$

sides can be constructed using origami, where the p_i's (if any) are distinct primes of the form $2^k \cdot 3^l + 1$ [1]. Here we construct a nonagon using Lill's method and Beloch's fold.

7.2 The cubic equation for a nonagon

A regular n-gon can be constructed by constructing its central angle $360°/n$. For a nonagon the central angle is $\theta = 360°/9 = 40°$:



[1]At age 19 Gauss constructed the regular 17-gon and decided (fortunately for us) that he would become a mathematician. The regular 257-gon was constructed by Magnus Georg Paucker in 1822 and by Friedrich Julius Richelot in 1832. Johann Gustav Hermes claimed to have constructed the regular 65537-gon in 1894. Should you wish to check its correctness, his manuscript is preserved at the University of Göttigen.

40° is one-third of 120°, which is constructible by appending a 30° angle (obtained by bisecting an angle of an equilateral triangle) to a 90° angle (obtained by constructing a perpendicular). Since angles can be trisected using origami, 40° can be constructed using origami. Here we present a different construction that solves a cubic equation.

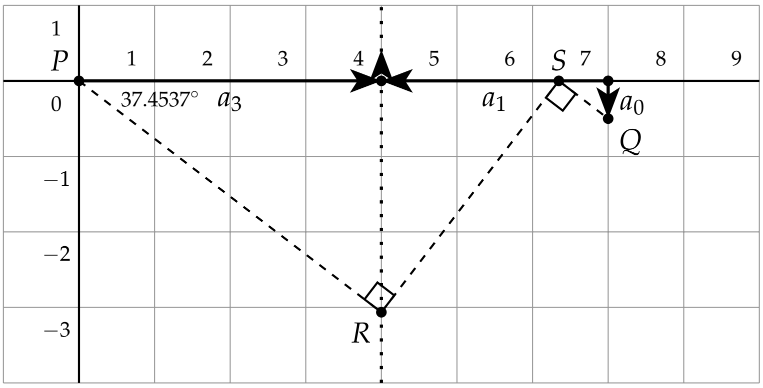
It is a simple calculation using the formula for $\cos(\alpha + \beta)$ to show that:

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta\,.$$

Let $x = \cos\theta$ and $a = \cos 3\theta$. If we can solve the cubic equation $4x^3 - 3x - a = 0$ for x, the angle itself can be constructed by constructing a right triangle with a side the length of x and a hypotenuse of length 1 (see the construction below).

For the nonagon the equation is $4x^3 - 3x + \frac{1}{2} = 0$ since $\cos 120^\circ = -\frac{1}{2}$.

Let us construct a path for the equation as required for Lill's method:



The coefficient a_2 is zero so the 90° turn is performed but no line segment is constructed in that direction.

The second path starts from P at an angle of -37.4537° and is then reflected by $\pm 90^\circ$ as specified in Lill's method. The second path intersects Q so we know that $x = -\tan -37.4537^\circ = 0.766044$ is a root of $4x^3 - 3x + \frac{1}{2}$.

7.3 Solving the equation with Beloch's fold

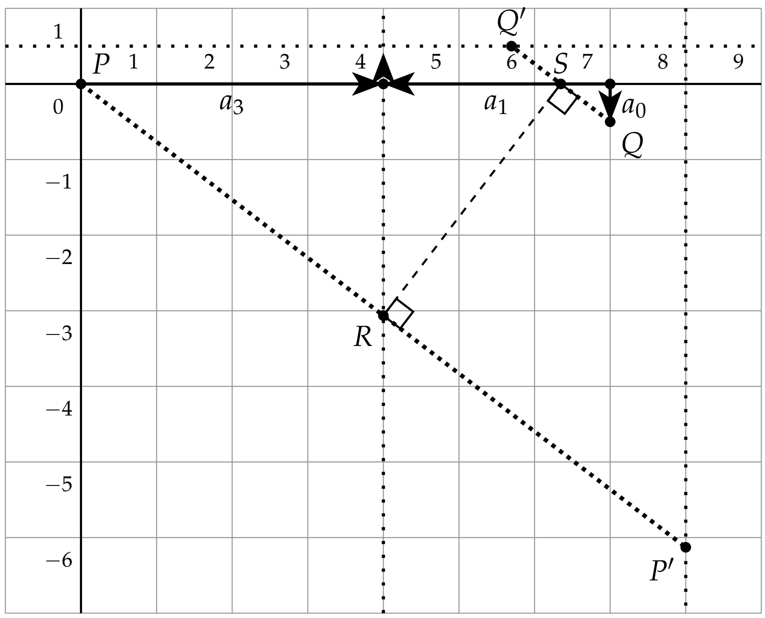
I cheated above because I know that a root of the equation is $x = -\tan -37.4537^\circ = 0.766044$ and drew the second path using this angle. You can check that:

$$4(0.766044)^3 - 3(0.766044) + 0.5 \approx 0\,.$$

The answer can be obtained using Beloch's fold as described in Chapter 6. We start by drawing a line parallel to a_2[2] at the same distance from a_2 as a_2 is from P. Similarly, we draw

[2]Since $a_2 = 0$ the line is drawn parallel to a line that would be drawn if a_2 were not equal to 0.

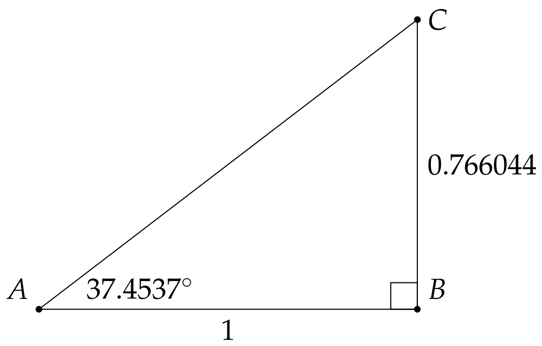
a line parallel to a_1 at the same distance from a_1 as a_1 is from Q. Beloch's fold (the line $\overline{RS}$) simultaneously places P at P' on the line parallel to a_2 and Q at Q' on the line parallel to a_1. This constructs the angle $\angle SPQ = 37.4537°$.



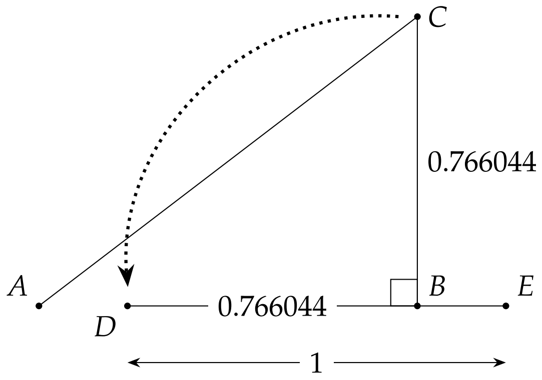
7.4 Constructing the central angle of the nonagon

We have that $x = \cos\theta = 0.766044$ is a root of the equation, so we need to construct $\cos^{-1} 0.766044 = 40°$ in order to construct a nonagon. We use the previous construction of the angle 37.4537° whose tangent is 0.766044.

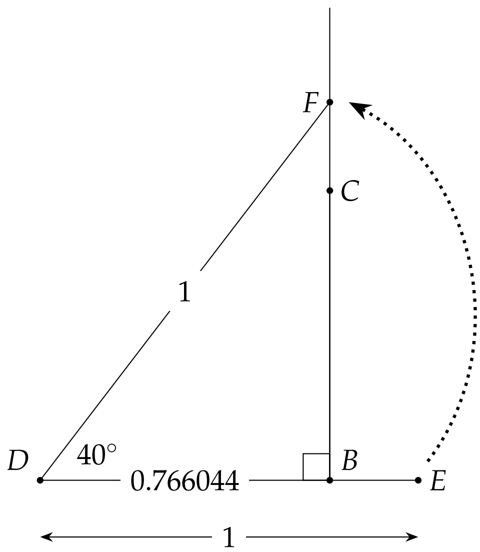
Given a line segment of length 1 and the angle 37.4537°, the right triangle with this angle whose adjacent side is 1 will have an opposite side of 0.766044 by definition of tangent:



Let us fold the side $\overline{CB}$ onto the side $\overline{AB}$. Then translate $\overline{AB}$ to the right so that its endpoints are D, E ([1] explains how to translate a line segment).



Now fold $\overline{DE}$ onto the extension of $\overline{CB}$ so that the point E is placed onto this line at F. Then $\angle EDF = \cos^{-1} \dfrac{0.766044}{1} = 40°$.



References

The following references were used in the preparation of this document.

The axioms are given in the Wikipedia article [10], together with parametric equations for the first five axioms. Lee [6, Chapter 4] is a good overview of the mathematics of origami, while Martin [7, Chapter 10] is a formal development. Lang [5] shows how rational numbers, some irrational numbers and approximations to others can be constructing in origami. Trisecting an angle and doubling a cube are described by Newton [8]. Ben-Lulu [2] provides a different proof of the trisection. The constructions for doubling the cube is from Newton [8] and Lee [6]. Hull [4] presents Lill's method for solving polynomial equations and Beloch's implementation of the method for cubic equations. Bradford [3] has numereous visualizations of Lill's method.

[1] Roger C. Alperin. A mathematical theory of origami constructions and numbers. *New York Journal of Mathematics*, 6:119–133, 2000.

[2] Oriah Ben-Lulu. Angle trisections in various axiom systems. Weizmann Institute of Science, 2020. (in Hebrew).

[3] Phillips Verner Bradford. Visualizing solutions to n-th degree algebraic equations using right-angle geometric paths. Archived May 2, 2010, at the Wayback Machine, `https://web.archive.org/web/20100502013959/http://www.concentric.net/~pvb/ALG/rightpaths.html`, 2010.

[4] Thomas C. Hull. Solving cubics with creases: The work of Beloch and Lill. *American Mathematical Monthly*, 118(4):307–315, 2011.

[5] Robert J. Lang. Origami and geometric constructions. `http://langorigami.com/wp-content/uploads/2015/09/origami_constructions.pdf`, 1996-–2015. Accessed 26/02/2020.

[6] Hwa Young Lee. Origami-constructible numbers. Master's thesis, University of Georgia, 2017.

[7] George E. Martin. *Geometric Constructions*. Springer, 1998.

[8] Liz Newton. The power of origami. `https://plus.maths.org/content/power-origami`. Accessed 26/02/2020.

[9] M. Riaz. Geometric solutions of algebraic equations. *American Mathematical Monthly*, 69(7):654–658, 1962.

[10] Wikipedia contributors. Huzita–Hatori axioms — Wikipedia, the free encyclopedia. `https://en.wikipedia.org/w/index.php?title=Huzita%E2%80%93Hatori_axioms&oldid=934987320`, 2020. Accessed 26/02/2020.

Appendix A

GeoGebra links

Axiom 1	`https://www.geogebra.org/m/fq9d5hms`
Axiom 2	`https://www.geogebra.org/m/fgmfss27`
Axiom 3	`https://www.geogebra.org/m/ek3mqupw`
Axiom 4	`https://www.geogebra.org/m/renzzbdg`
Axiom 5	`https://www.geogebra.org/m/aszn9ywu`
Axiom 6	`https://www.geogebra.org/m/bxe5e5ku`
Axiom 7	`https://www.geogebra.org/m/yeq5gmeg`
Abe's trisection	`https://www.geogebra.org/m/dxrcvjam`
Martin's trisection	`https://www.geogebra.org/m/caky7edd`
Messer's doubling of the cube	`https://www.geogebra.org/m/mrcwjqh8`
Beloch's doubling of the cube	`https://www.geogebra.org/m/enzmmwua`

Due to a bug in Geogebra, in projects that use Axiom 6, points defined by reflection around the common tangent are not saved or are saved incorrectly.

Appendix B

Derivation of trigonometric identities

The trigonometric identifies for tangent used in the proof of Axiom 3 can be derived from identities for the sine and cosine:

$$\begin{aligned}
\tan(\theta_1 + \theta_2) &= \frac{\sin(\theta_1 + \theta_2)}{\cos(\theta_1 + \theta_2)} \\
&= \frac{\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2}{\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2} \\
&= \frac{\sin\theta_1 + \cos\theta_1 \tan\theta_2}{\cos\theta_1 - \sin\theta_1 \tan\theta_2} \\
&= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}\,.
\end{aligned}$$

We use this formula with $\theta = (\theta/2) + (\theta/2)$ to obtain a quadratic equation in $\tan(\theta/2)$:

$$\tan\theta = \frac{\tan(\theta/2) + \tan(\theta/2)}{1 - \tan^2(\theta/2)}$$

$$\tan\theta\,(\tan(\theta/2))^2 \;+\; 2\,(\tan(\theta/2)) \;-\; \tan\theta = 0\,.$$

Its solutions are:

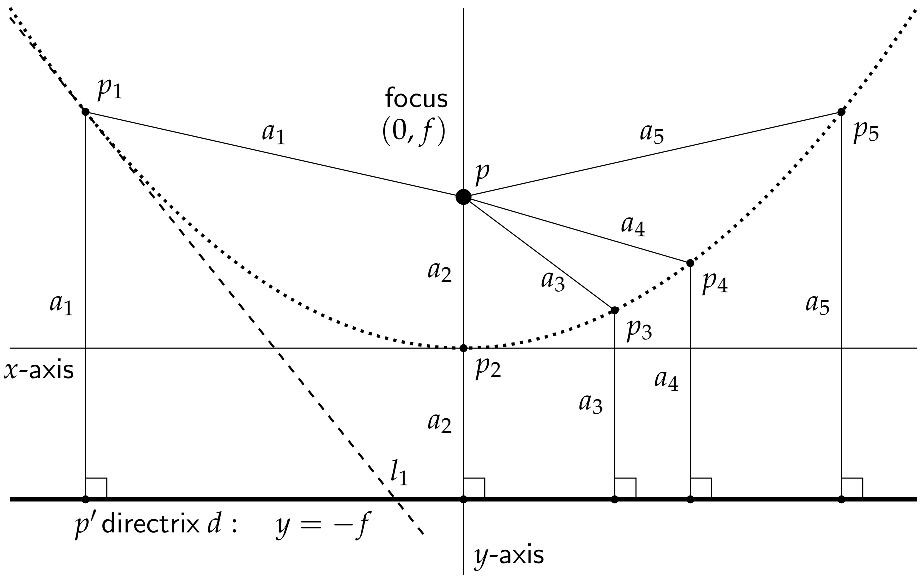
$$\tan(\theta/2) = \frac{-1 \pm \sqrt{1 + \tan^2\theta}}{\tan\theta}\,.$$

Appendix C

Parabolas

Students are usually introduced to parabolas as the graphs of second degree equations $y = ax^2 + bx + c$. However, parabolas can be defined geometrically: given a point, the *focus*, and a line, the *directrix*, the locus of points equidistant from the focus and the directrix defines a parabola.

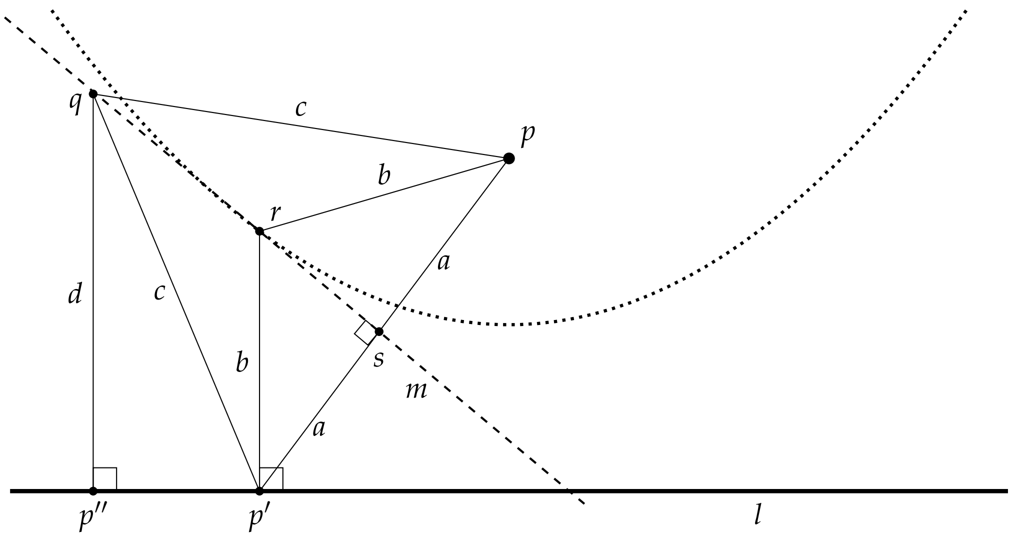
The following diagram shows the focus—the large dot at $p = (0, f)$, and the directrix—the thick line d whose equation is $y = -f$. The resulting parabola is shown as a dotted curve. Its *vertex* p_2 is at the origin of the axes.



We have selected five points p_i, $i = 1, \ldots, 5$ on the parabola. Each point p_i is at a distance of a_i both from the focus and from the directrix. Drop a perpendicular from p_i to the directrix and let p'_i be the intersection of the perpendicular and the directrix. Using Axiom 2, construct the line l_i by folding p onto p'_i. Since p_i is on the parabola, $\overline{p'p_i} = \overline{p_i p} = a_i$. The diagram shows the fold l_1 through p_1.

Theorem The folds are tangents to the parabola.

Proof (Oriah Ben Lulu) In the following diagram, the focus is p, the directrix is l, p' is a point on the directrix and m is the fold that places p on p'. By definition, m is the perpendicular bisector of $\overline{pp'}$. Let s be the intersection of $\overline{pp'}$ and m; then $\overline{ps} = \overline{p's} = a$ and $m \perp \overline{pp'}$.



Let r be the intersection of a perpendicular to l through p' and the fold m. Then $\triangle psr \cong \triangle p'sr$ by side-angle-side, since $\overline{ps} = \overline{p's}$, $\angle psr = \angle p'sr = 90°$ and $\overline{rs}$ is a common edge. It follows that $\overline{pr} = \overline{p'r} = b$ and therefore r must be on the parabola.

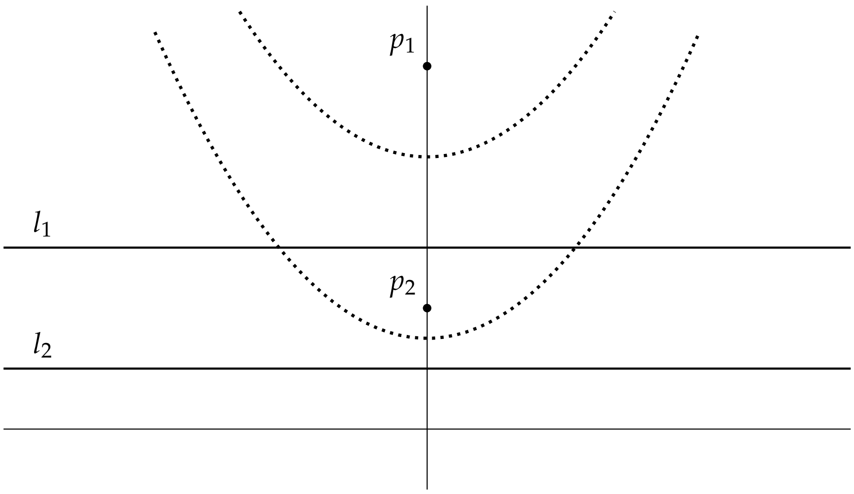
Choose a point p'' on the directrix that is *distinct* from p' and suppose that m is also the fold that places p on p''. Let q be the intersection of the perpendicular to l through p'' and the fold m. As before, we can prove that $\overline{pq} = \overline{p'q} = c$. Let $\overline{qp''} = d$. If q is on the parabola then $d = \overline{qp''} = \overline{qp} = c$. But c is the hypotenuse of the right triangle $\triangle qp''p'$ and cannot be equal to one of its sides d.

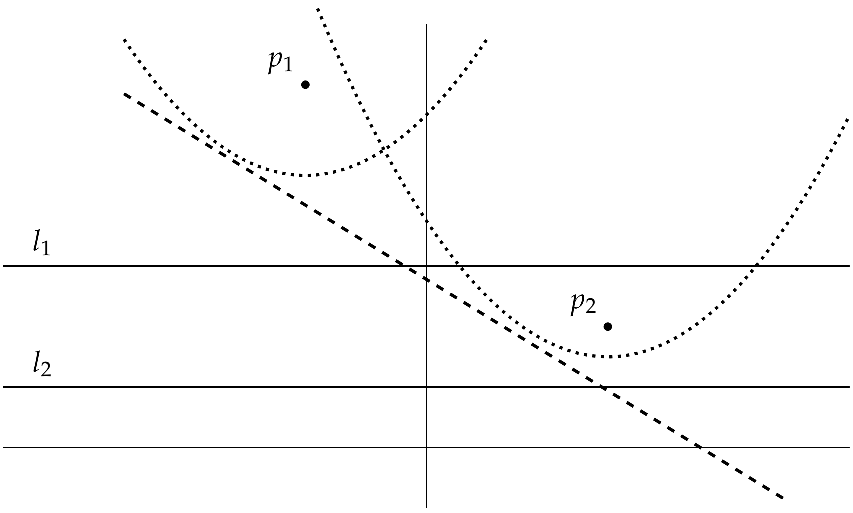
We have proved that m intersects the parabola in only one point so it is a tangent to the parabola.

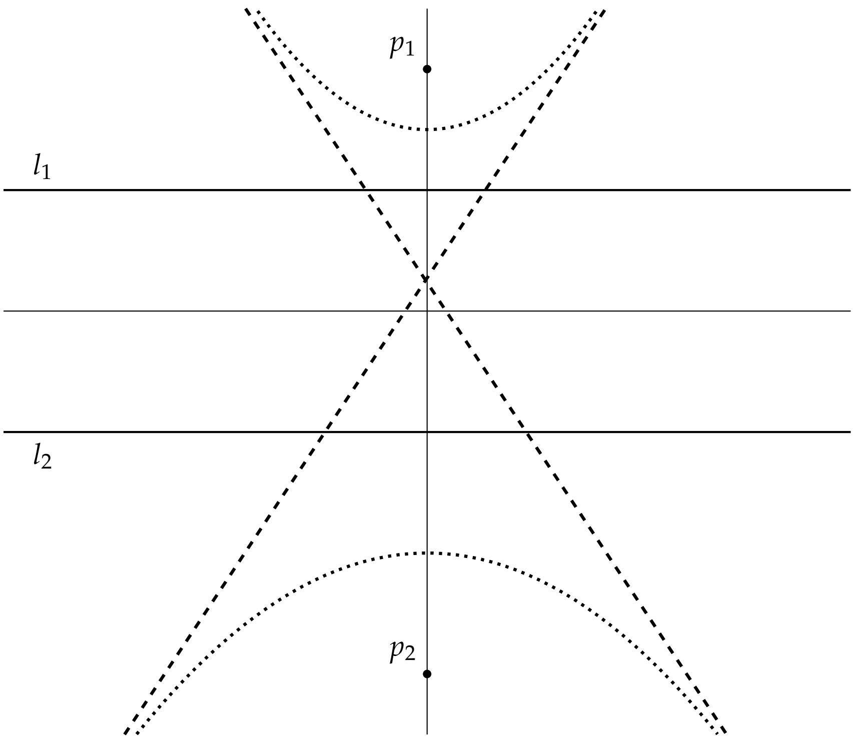
Appendix D

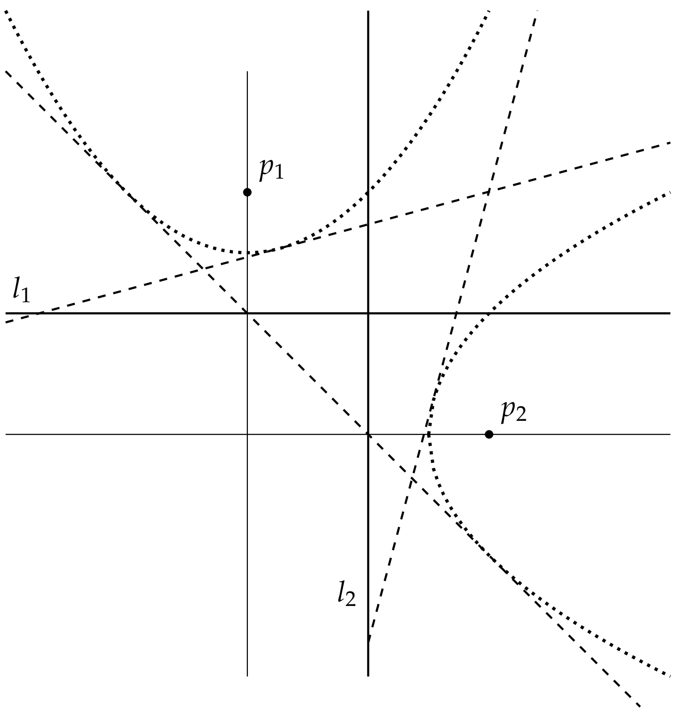
Tangents Common to Two Parabolas

Here are diagrams showing the four cases:






p_1
l_1
l_2
p_2


p_1
l_1
p_2
l_2