How to (Almost) Square a Circle

Moti Ben-Ari

Department of Science Teaching

Weizmann Institute of Science

http://www.weizmann.ac.il/sci-tea/benari/

Approximations to π

The Greeks first studied geometrical constructions that use a straightedge and compass. They were not able to solve three problems: trisecting an angle, duplicating a cube (given a cube, construct another cube with twice the volume) and squaring a circle (given a circle, construct a square with the same area). In the nineteenth century it was proved that these constructions are impossible.

It can be shown algebraically that given a line segment defined to have length 1, the constructible numbers (lengths) are those obtainable from that line segment using the operators $+, -, \times, \div, \sqrt{}$. Since cube roots cannot be constructed, it follows that it is impossible to duplicate an arbitrary cube, because to duplicate a cube of volume 1 requires solving the equation $x^3 - 2 = 0$, that is, constructing the length $\sqrt[3]{2}$. For the same reason, it is impossible to trisect an arbitrary angle, because to trisect 60° requires solving the cubic equation $8x^3 - 6x - 1 = 0$.

The case of squaring the circle is more difficult and was not solved until 1882. The area of the unit circle is $\pi r^2 = \pi$, so it is required to construct a square whose side is $\sqrt{\pi}$. However, π is *transcendental*, meaning that it is not the solution of *any* algebraic equation. The proof is extremely complex involving concepts from analysis and complex numbers.

There are simple approximations to π, in particular:

$$\frac{355}{113} = 3.14159292\,,$$

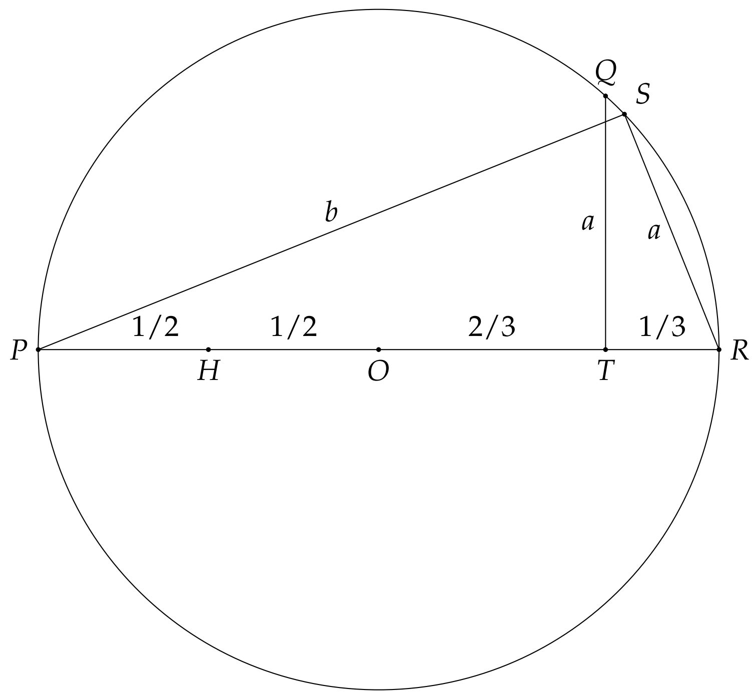
which differs from $\pi \approx 3.14159265$ by only 2.67×10^{-7}. Consider the radius of the earth at the equator which is $6,378.1$ km. Computing the circumference using $355/113$ gives $40,074.78761$ km, while using π the result is $40,074.78421$ km, a difference of less than 4 meters!

The rational number $355/113$ could be constructed by extending the line segment of length one 113 times, extending it again to obtain a line segment of length 355, and constructing a segment whose length is the quotient of the two. This note presents a short construction of $355/113$ published by Ramanujan in the *Journal of the Indian Mathematical Society*, 1913, p. 138. We present the construction in incremental stages, asking you to perform the computations in each stage as exercises. Answers are given at the end of the note, followed by Ramanujan's article.

Ramanujan (1887–1920) grew up in what is now the state of Tamil Nadu in India, but quickly advanced beyond the mathematical level of the local schools and colleges. He sent his work to the English mathematician G.H. Hardy who invited him to England. Ramanujan arrived in 1914 but could not return to India until after World War I. He suffered greatly from the cold climate and unfamiliar food and died at age 32 shortly after returning to India. Ramanujan's mathematics is still the subject of research one hundred years later. A biography of Ramanujan is: Robert Kanigel. *The Man Who Knew Infinity: A Life of the Genius Ramanujan*, 1991.

Stage 1

- Construct a unit circle centered at O and let PR be a diameter.
- Mark point H that bisects PO and mark point T that trisects RO.
- Construct a perpendicular at T. Its intersection with the circle is Q.
- Construct a chord $RS = QT$.



Exercise 1 *Construct the length* $TR = \dfrac{1}{3}$.

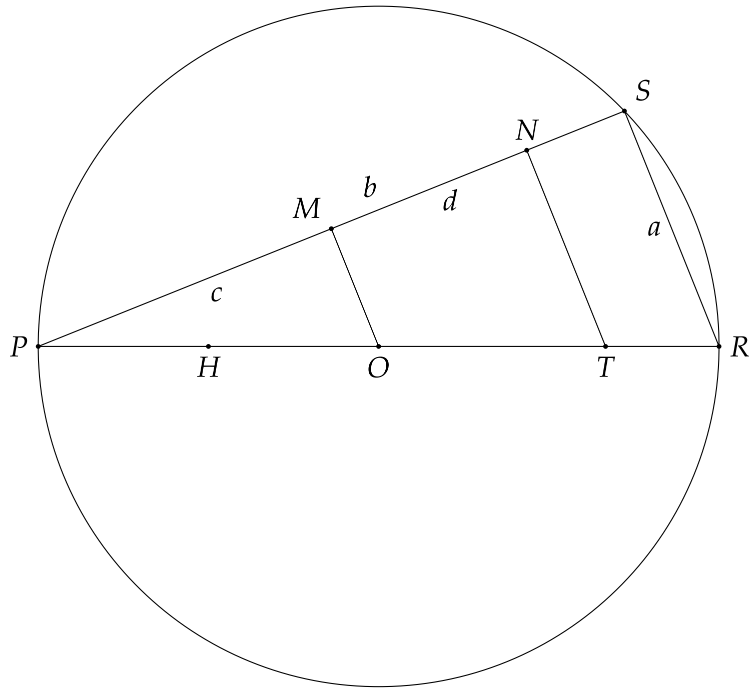
Exercise 2 *Compute the length of* QT.

Exercise 3 *Compute the length of* PS.

Exercise 4 *Construct chord* $RS = QT$.

Stage 2

- Construct NT and OM parallel to RS.

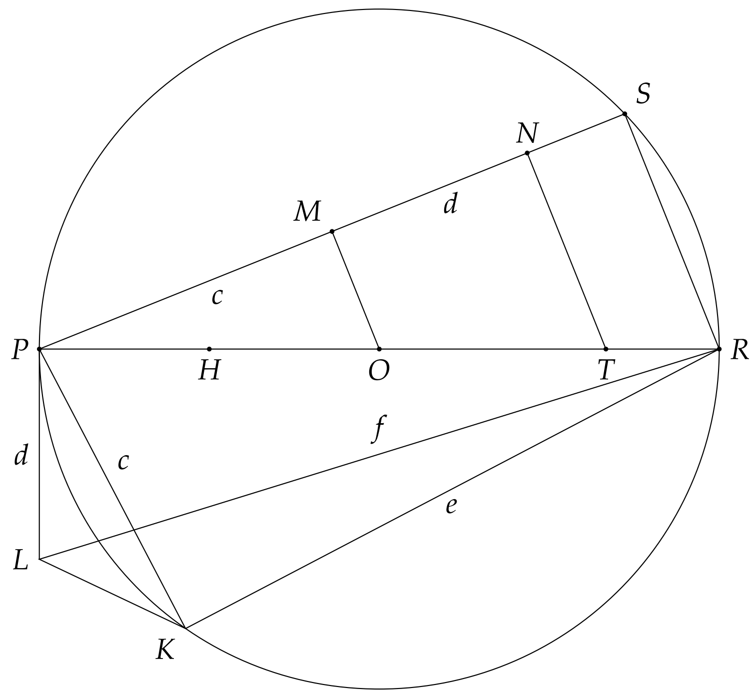


Exercise 5 *Compute the length of PM.*

Exercise 6 *Compute the length of MN.*

Stage 3

- Construct the chord $PK = PM$.
- Construct the tangent $PL = MN$.
- Connect the points K, L, R.

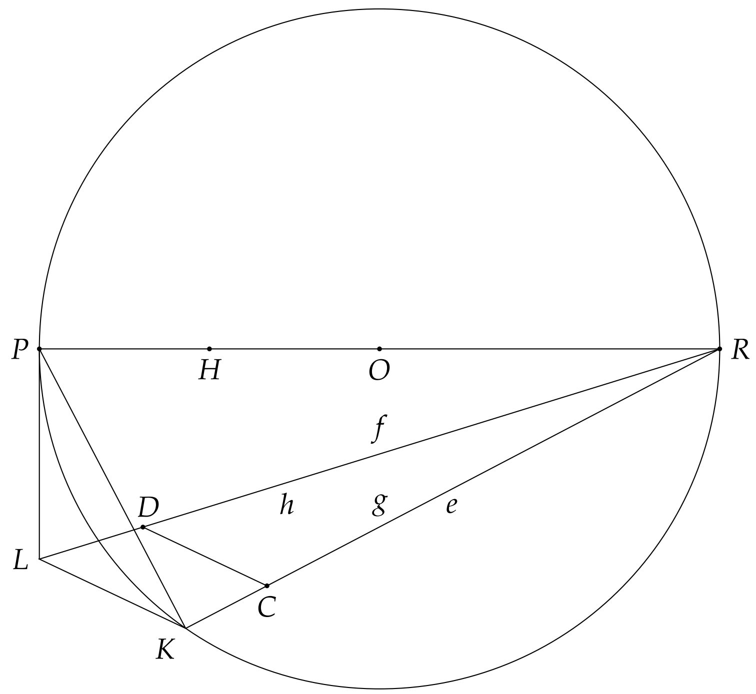


Exercise 7 *What do you know about $\triangle PKR$? Compute the length of RK.*

Exercise 8 *What do you know about $\triangle LPR$? Compute the length of RL.*

Stage 4

- Mark the point C such that $RC = RH$.
- Construct CD parallel to KL.



Exercise 9 *Compute the length of RC.*

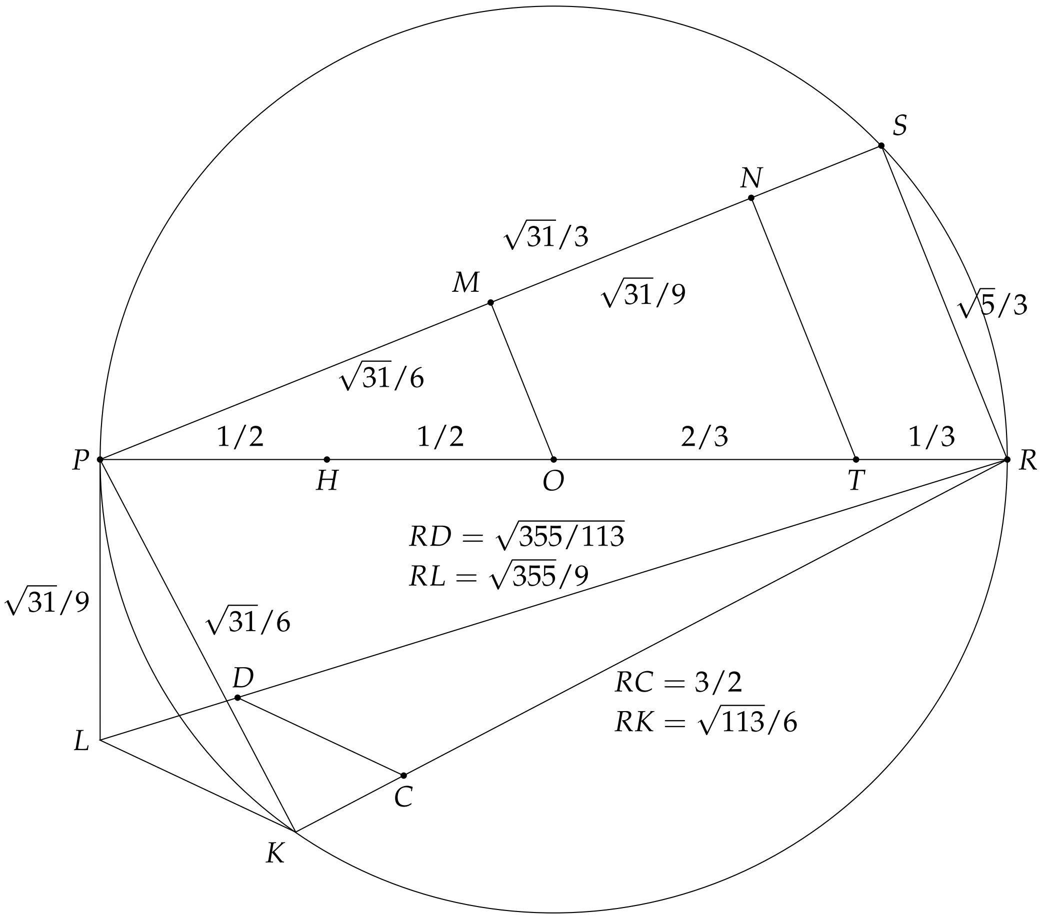
Exercise 10 *Compute the length of RD.*

Exercise 11 *Construct a square whose side is of length RD.*

Exercise 12 *Compute RD^2, the area of the square, both as a fraction and as a decimal number.*

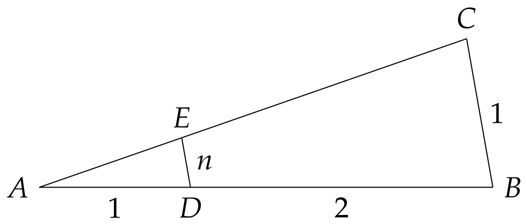
Summary

Here is the complete construction with all lengths labeled.



Answers to the Exercises

1. Construct $\triangle ABC$ with the lengths shown and then construct DE parallel to BC.



By similar triangles:

$$\frac{n}{1} = \frac{1}{3}.$$

2. By Pythagoras's theorem on $\triangle QOT$:

$$QT = \sqrt{1^2 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}.$$

3. $\triangle PSR$ is a right triangle because it subtends a diameter. By Pythagoras's theorem:

$$PS = \sqrt{2^2 - \left(\frac{\sqrt{5}}{3}\right)^2} = \sqrt{4 - \frac{5}{9}} = \frac{\sqrt{31}}{3}.$$

4. Set the compass legs on QT and draw a circle with this radius and with center R.

5. By similar triangles:

$$\frac{PM}{PO} = \frac{PS}{PR}, \quad \frac{PM}{1} = \frac{\sqrt{31}/3}{2}, \quad PM = \frac{\sqrt{31}}{6}.$$

6. By similar triangles:

$$\frac{PN}{PT} = \frac{PS}{PR}, \quad \frac{PN}{5/3} = \frac{\sqrt{31}/3}{2}, \quad PN = \frac{5\sqrt{31}}{18}.$$

$$MN = PN - PM = \sqrt{31}\left(\frac{5}{18} - \frac{1}{6}\right) = \frac{\sqrt{31}}{9}.$$

7. $\triangle PKR$ is a right triangle because it subtends a diameter. By Pythagoras's theorem:

$$RK = \sqrt{2^2 - \left(\frac{\sqrt{31}}{6}\right)^2} = \frac{\sqrt{113}}{6}.$$

8. $\triangle LPR$ is a right triangle because it PL is a tangent. By Pythagoras's theorem:

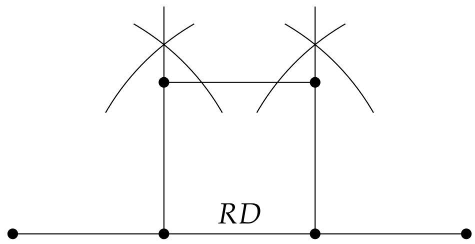
$$RL = \sqrt{2^2 + \left(\frac{\sqrt{31}}{9}\right)^2} = \frac{\sqrt{355}}{9}.$$

9. $RC = RH = \dfrac{1}{3} + \dfrac{2}{3} + \dfrac{1}{2} = \dfrac{3}{2}$.

10. Since CD is parallel to LK, by similar triangles:

$$\frac{RD}{RC} = \frac{RL}{RK}, \quad \frac{RD}{3/2} = \frac{\sqrt{355}/9}{\sqrt{113}/6}, \quad RD = \sqrt{\frac{355}{113}}\,.$$

11. Given a line segment of length RD, extend it to a line segment of length $3RD$. Construct perpendiculars of length RD on the points at the ends of the middle segment. Draw a line beteen the end points of these perpendiculars.



12.
$$\frac{355}{113} = 3.14159292\,.$$

Squaring the circle

(*Journal of the Indian Mathematical Society*, *v*, 1913, 138)

Let PQR be a circle with centre O, of which a diameter is PR. Bisect PO at H and let T be the point of trisection of OR nearer R. Draw TQ perpendicular to PR and place the chord $RS = TQ$.

Join PS, and draw OM and TN parallel to RS. Place a chord $PK = PM$, and draw the tangent $PL = MN$. Join RL, RK and KL. Cut off $RC = RH$. Draw CD parallel to KL, meeting RL at D.

Then the square on RD will be equal to the circle PQR approximately.

For $$RS^2 = \frac{5}{36}d^2,$$

where d is the diameter of the circle.

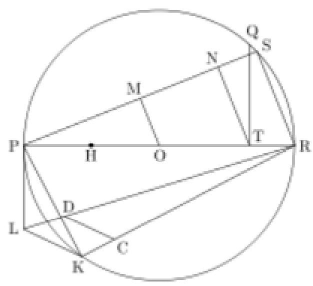
Therefore $$PS^2 = \frac{31}{36}d^2.$$

But PL and PK are equal to MN and PM respectively.

Therefore $$PK^2 = \frac{31}{144}d^2, \text{ and } PL^2 = \frac{31}{324}d^2.$$

Hence $$RK^2 = PR^2 - PK^2 = \frac{113}{144}d^2,$$

and $$RL^2 = PR^2 + PL^2 = \frac{355}{324}d^2.$$



But $$\frac{RK}{RL} = \frac{RC}{RD} = \frac{3}{2}\sqrt{\frac{113}{355}},$$

and $$RC = \frac{3}{4}d.$$

Therefore $$RD = \frac{d}{2}\sqrt{\frac{355}{113}} = r\sqrt{\pi}, \text{ very nearly.}$$

Note.—If the area of the circle be $140,000$ square miles, then RD is greater than the true length by about an inch.