Surprising Constructions with Straightedge and Compass

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Introduction

I don't remember when I first saw the article by Godfried Toussaint [10] on the "collapsing compass," but it make a deep impression on me. It never occurred to me that the modern compass is not the one that Euclid wrote about. In this document, I present the collapsing compass and other surprising geometric constructions. The mathematics used is no more advanced than secondary-school mathematics, but some of the proofs are rather intricate and demand a willingness to deal with complex constructions and long proofs. The chapters are ordered in ascending levels of difficult (according to my evaluation).

The collapsing compass Euclid showed that every construction that can be done using a compass with fixed legs can be done using a collapsing compass, which is a compass that cannot maintain the distance between its legs. The presentation is not difficult and uses only the geometry of circles and triangles. Over many years numerous incorrect proofs have been given based on incorrect diagrams. In order to emphasize that one should not trust diagrams, I have included the famous "proof" that every triangle is isoceles.

Trisecting an angle The Greeks sought a construction to trisect an angle into three equal parts. Only in the nineteenth century was this proved to be impossible. In fact, this result is of no practical importance because an angle can be trisected using tools only slightly more advanced than a straightedge and compass. The chapter presents three such constructions, two of which are very easy, while the third requires some knowledge of trigonometry and limits.

Squaring a circle Another problem posed by the Greeks was that of squaring a circle: given a circle, construct a square with the same area. This is equivalent to constructing a line segment of length π, which has been proved to be impossible. This chapter presents three constructions of approximations to π, one by Ada Kochansky from 1685 and two by Ramanujan from 1913. Ramanujan's approximations are very close to the precise value of π.

Construction with only a compass Who says that both a straightedge and a compass are needed? Hundreds of years ago, Lorenzo Mascheroni and Georg Mohr showed that it is possible to limit oneself to only a compass. The proof is not very difficult, but it is very long and requires patience to follow.

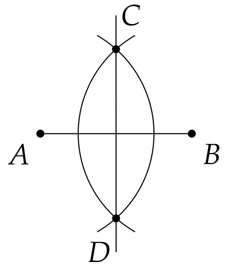
Construction with only a straightedge Is a straightedge sufficient? The answer is no because a straightedge can "compute" only linear functions, whereas a compass can "compute" quadratic functions. In 1833 Jacob Steiner proved that a straightedge if sufficient provided that somewhere in the plane a single circle exists. The proof uses only geometry but is very long.

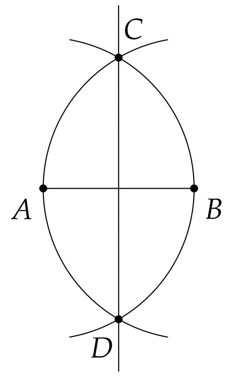
Triangles with the same area and perimeter This chapter deals with a geometric problem that is not a construction, but it is fascinating. Do there exists two non-congruent triangles that have the same perimeter and the same area? The answer is yes but finding such pairs requires a long journey into trigonometry. I have added to this chapter an elegant proof of Heron's formula for the area of a triangle.

Chapter 1
Help, My Compass Collapsed!

1.1 Fixed compasses and collapsing compasses

In a modern compass used for geometric constructions the distance between the two legs can be fixed so that it is possible to copy a line segment or a circle from one position to another. We will call such a compass a *fixed compass*. I have seen geometry textbooks that present the construction a perpendicular bisector to a line segment as follows: construct two circles centered at the ends of the line segment such that the radii are equal and *greater than half the length of the segment* (left diagram):

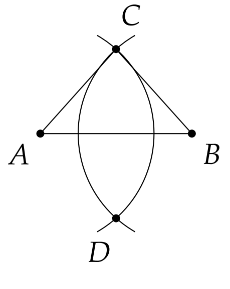


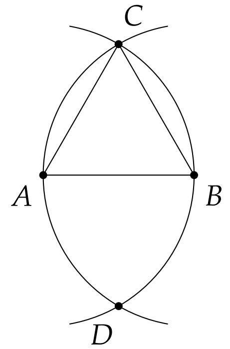


Euclid used a *collapsing compass* whose legs fold up when the compass is lifted off the paper. Teachers often use a collapsing compass consisting of a piece of chalk tied to a string. It is impossible to maintain a fixed radius when the chalk is removed from the blackboard. The right diagram above shows how to construct a perpendicular bisector with a collapsing compass: the length of the segment AB is, of course, equal to the length of the segment BA, so the radii of the two circles are equal.

The proof that the line constructed is the perpendicular bisector is not at all elementary because relatively advanced concepts like congruent triangles have to be used. However, the proof that the same construction results in an equilateral triangle is very simple (right diagram below). The length of AC equals the length of AB since they are radii of the same circle, and for the same reason the length of BC is equal to the length of BA. We have:

$$AC = AB = BA = BC\,.$$



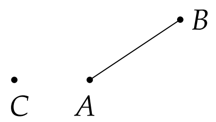


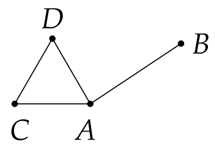
The left diagram above shows that for the construction with the fixed compass the triangle will be isosceles, but not necessarily equilateral.

This construction of an equilateral triangle is the first proposition in Euclid's *Elements*. The second proposition shows how to copy a given line segment AB to a segment of the same length, one of whose end points is a given point C. Therefore, a fixed compass adds no additional capability. Toussaint [10] showed that many incorrect constructions for this proposition have been given. In fact, it was Euclid who gave a correct construction! The following section presents Euclid's construction and the proof of its correctness. Then I show an incorrect construction that can be found even in modern textbooks.

1.2 Euclid's construction for copying a line segment

Theorem: Given a line segment AB and a point C, a line segment can be constructed (using a collapsing compass) at C whose length is equal to the length of AB:





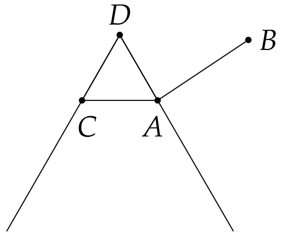
Construction:

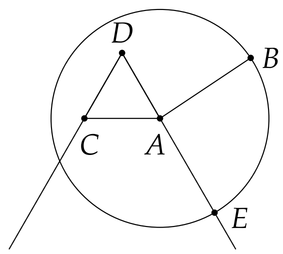
Construct the line segment from A to C.

Construct an equilateral triangle whose base is AC (right diagram above). Label the third vertex D. By Euclid's first proposition, the triangle can be constructed using a collapsing compass.

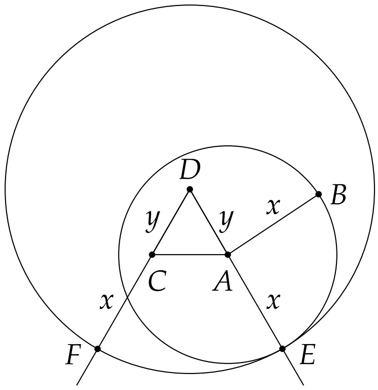
Construct a ray that is a continuation of DA and a ray that is a continuation of DC (left diagram below).

Construct a circle centered at A with radius AB. Label the intersection of the circle and the ray DA by E (right diagram below).





Construct a circle centered at D with radius DE. Label the intersection of the circle and the ray DC by F:



Claim: The length of the line segment CF is equal to the length of AB.

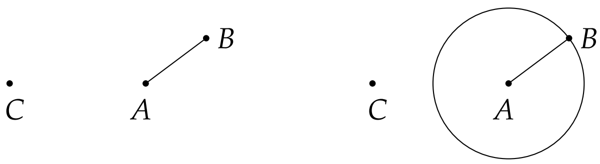
Proof: $DC = DA$ because $\triangle ACD$ is equilateral. $AE = AB$ because they are radii of the same circle centered at A. $DF = DE$ because they are radii of the same circle centered at D. Therefore, the length of the line segment CF is:

$$CF = DF - DC = DE - DC = DE - DA = AE = AB\,.$$

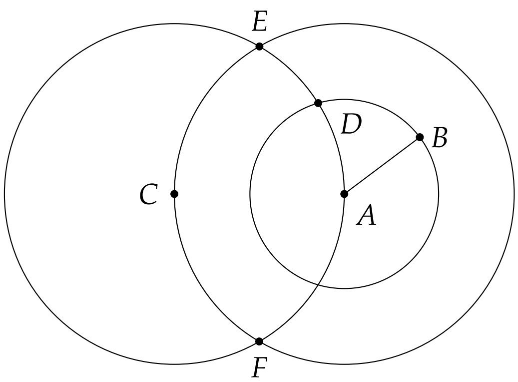
1.3 An incorrect construction for copying a line segment

Construction([7]):

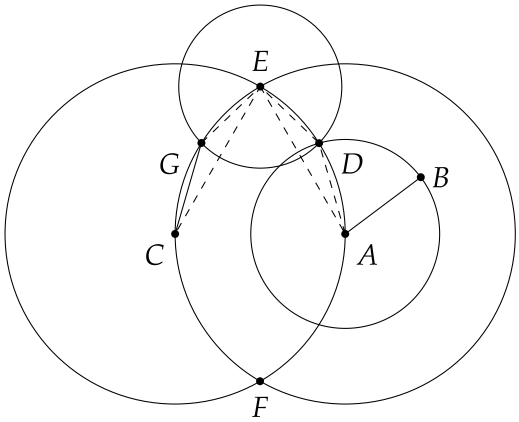
Construct a circle centered at A with radius AB:



Construct a circle centered at A with radius AC and a circle centered at C with radius $AC = CA$. Label the intersections of the two circles E, F. Label the intersection of the circle centered at C and the circle centered at A with radius AB by D:



Construct a circle centered at E with radius ED. Label the intersection of this circle with the circle centered at A with radius AC by G:



Claim: The length of the line segment GC is equal to the length of AB.

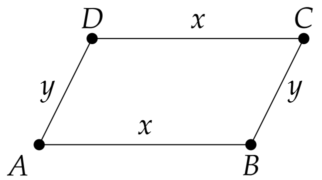
Proof: We shall show that $\triangle ADE \cong \triangle CGE$. If so, $CG = AD = AB$ because AB, AD are radii of the smaller circle centered at A. The circle centered at C has the same radius as the circle centered at A that goes through E, so we can consider that they are the "same" circle.

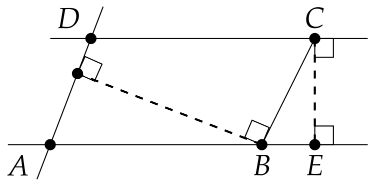
$EG = ED$ because they are radii of the circle centered at E and $EC = EA$ because they are radii of the "same" circle. $\angle GCE = \angle DAE$ because they are central angles that intercept the "same" chord and $\angle CGE = \angle ADE$ because they are inscribed angles intercepting the "same" chord. Therefore, $\angle GEC = \angle DEA$ and $\triangle GEC \cong \triangle DAE$ by SAS.

The answer: there isn't any error in the proof! The problem arises from a different source: the equality $AB = GC$ holds only when the length of AB is less that the length of AC. In contrast, Euclid's construction and proof are true, independent of the relative lengths of AB and AC, and of the position of the point C relative to the line segment AB ([10]).

1.4 A "simpler" construction for copying a line segment

Given a line segment AB and a point C, if we can build a parallelogram with these three points as its vertices, we obtain a line segment with C at one end whose length is equal to the length of AB (left diagram left):





This construction can be found in [11, pp. 207–208].

Construction (right diagram):

Construct the line segment from B to C.

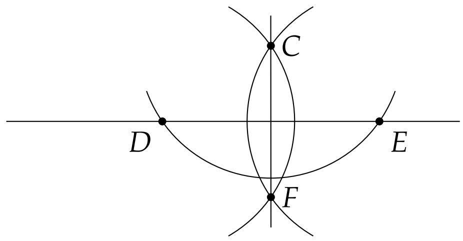
Construct an altitude from C to the line containing the line segment AB. Label the intersection by D.

Construct an altitude to the line segment CD at C. This line is parallel to AB.

Use a similar method to construct a line parallel to BC through A. Label the intersection of the two lines by D.

$AD \| BC$, $AB \| DC$ and by definition $ABCD$ is a parallelogram, so $AB = CD$ as required.

Construction with a collapsing compass: We will show that it is possible to construct an altitude through a given point with a collapsing compass. Construct a circle centered at C with a radius that is greater than the distance of C from the line. Label the intersections with the line by D, E. Construct circles centered at D, E with radii $DC = EC$:

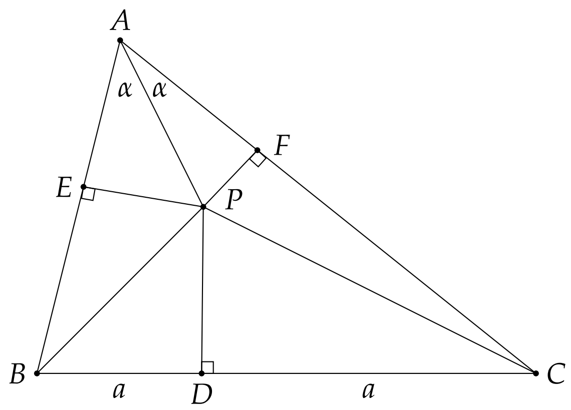


The line connecting the intersections of the circles C, F is an altitude through C.

The proof the correctness of this construction is much more difficult than Euclid's proof of his construction.

1.5 Don't trust a diagram

In Section 1.3, we saw that a diagram can lead us astray. Here is a proof that *all* triangles are isosceles!

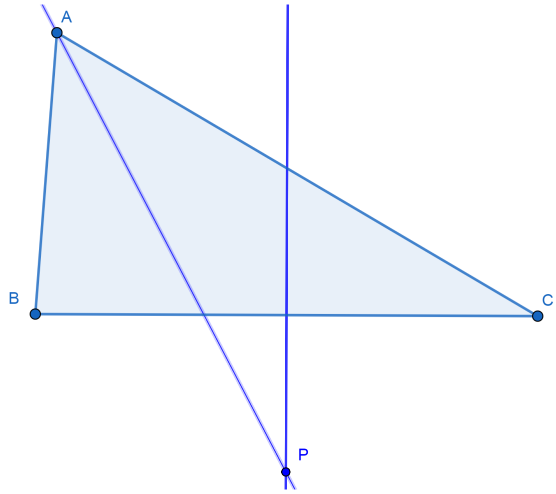


Given an arbitrary triangle $\triangle ABC$, let P be the intersection of the angle bisector of $\angle BAC$ and the perpendicular bisector BC. Label by D, E, F the intersections of the altitudes from P to the sides BC, AB, AC. $\triangle APF \cong \triangle APE$ because they are right triangles with equal angles α and a common side AP.

$\triangle DPC \cong \triangle DPB$ SAS because PD is a common side, $\angle PDB = \angle PDC = 90$ and $BD = DC = a$ because PD is the perpendicular bisector of BC. $\triangle EPB \cong \triangle EPC$ because $EP = PF$ by the first congruence and $PB = PC$ by the second congruence. By combining the equations we get that $\triangle ABC$ is isoceles:

$$AB = AE + EB = AF + FC = AC\,.$$

The problem with the proof is that the diagram is incorrect because point P is *outside* the triangle, as can be seen from the following diagram constructed using GeoGebra:



Chapter 2

How to Trisect an Angle (If You Are Willing to Cheat)

It is well known that it is impossible to trisect an arbitrary angle using a compass and a straightedge. The reason is that trisection requires the construction of cube roots, but the compass and straightedge can only construct lengths that are expressions built from the four arithmetic operators and square roots.

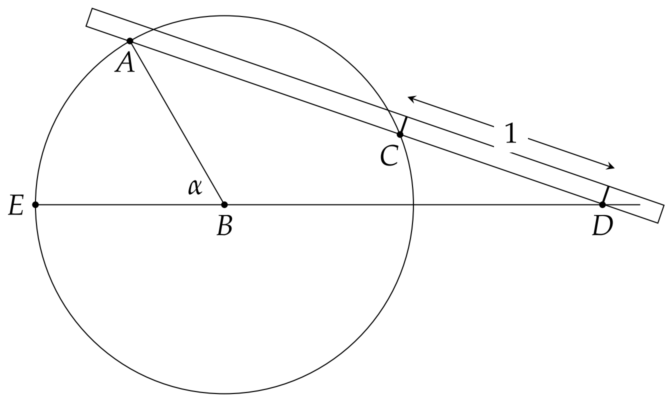
Greek mathematicians discovered that if other instruments are allowed, angles can be trisected. Section 2.1 presents a construction of Archimedes using a simple instrument called a neusis. Section 2.2 shows a more complex construction of Hippias using the quadratrix. As a bonus, Section 2.3 shows that the quadratrix can square a circle.

References:
`https://en.wikipedia.org/wiki/Angle_trisection`
`https://en.wikipedia.org/wiki/Quadratrix_of_Hippias`
`https://en.wikipedia.org/wiki/Neusis_construction`

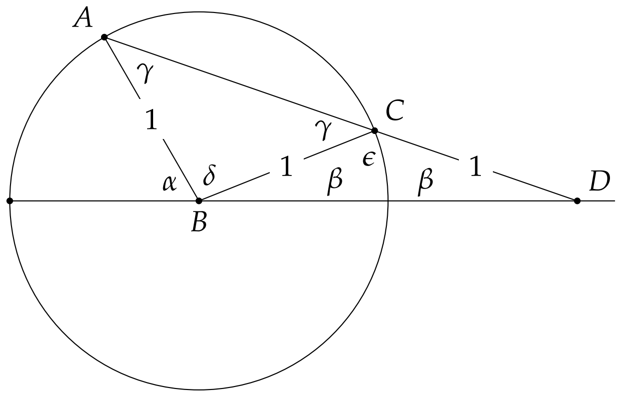
2.1 Trisection using the neusis

In geometry textbooks, constructions are performed using a "straightedge" and a compass. The term straightedge is used instead of "ruler" because a straightedge has no marks on it. The only operation it can perform is to construct a straight line between two points, while a ruler can measure distances. To trisect an angle all we need is a straightedge with two marks that are a fixed distance apart, called a *neusis*. We define the distance between the marks as 1:

Let α be an arbitrary angle $\angle ABE$ within a circle with center B and radius 1. The circle can be constructed by setting the compass to the distance between the marks on the neusis. Extend the radius EB beyond the circle. Place an edge of the neusis on A and move it until it intersects the extension of EB at D and the circle at C, using the marks so that the length of the line segment CD is 1. Draw the line AD:



Draw line BC and label the angles and line segments as shown:

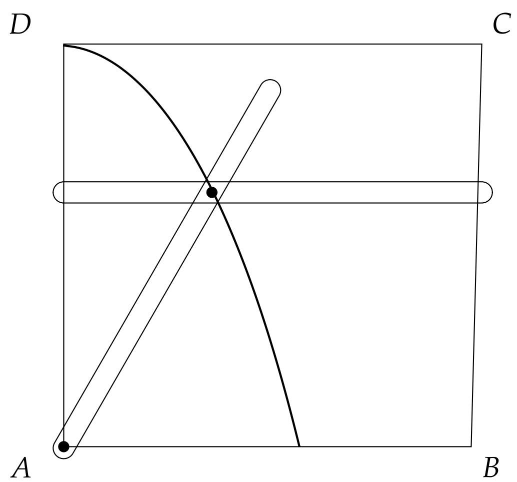


Both $\triangle ABC$ and $\triangle BCD$ are isoceles: $AB = BC$ since both are radii and $BC = CD$ by construction using the neusis. A computation (using the facts that the angles of a triangle and supplemenary angles add up to 180 radians) shows that β trisects α:

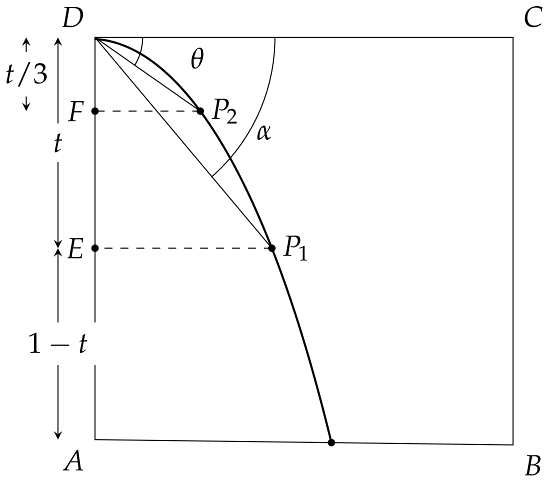
$$
\begin{aligned}
\epsilon &= \pi - 2\beta \\
\gamma &= \pi - \epsilon = 2\beta \\
\delta &= \pi - 2\gamma = \pi - 4\beta \\
\alpha &= \pi - \delta - \beta \\
&= 4\beta - \beta \\
&= 3\beta\,.
\end{aligned}
$$

2.2 Trisection using the quadratrix

The following diagram shows a *quadratrix compass*: two (unmarked) straightedges connected by a joint that constrains them to move together. One straightedge moves parallel to the x-axis from DC to AB, while the second straightedge is allowed to rotate around the origin at A until it lies horizontally along AB. The curve traced by the joint of the two straightedges is called the *quadratrix curve* or simply the *quadratrix*.



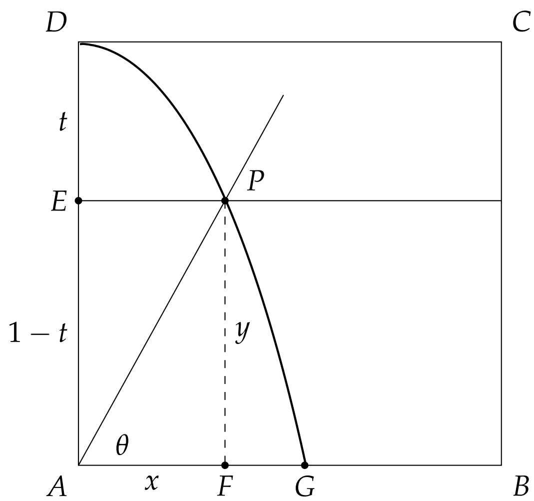
As the horizontal straightedge is moved down at a constant velocity, the other straightedge is constrained to move at a constant angular velocity. In fact, that is the definition of the quadratrix curve. As the y-coordinate of the horizontal straightedge decreases from 1 to 0, the angle of the other straightedge relative to the x-axis decreases from 90° to 0°. The following diagram shows how this can be used to trisect an arbitrary angle α:



P_1 is the intersection of the line defining the angle α with the quadratrix. This point has y-coordinate $1-t$, where t is the distance that the horizontal straightedge has moved from its initial position DC. Now trisect the *line segment* DE to obtain point F. (It is easy to trisect a line segment using Thales theorem.) Let P_2 be the intersection of a line from F parallel to DC and the quadratrix. By the equality of the velocities, we have:

$$
\begin{aligned}
\frac{\theta}{\alpha} &= \frac{t/3}{t} \\
\theta &= \alpha/3 .
\end{aligned}
$$

2.3 Squaring the circle using the quadratrix



Suppose that the horizontal straightedge has moved t down the y-axis to point E and the rotating straightedge forms an angle of θ with the x-axis. P is the intersection of the quadratrix with the horizontal straightedge, and F is the projection of P on the x-axis. What are the coordinates of quadratrix at P? Clearly, $y = PF = EA = 1 - t$. On the quadratrix, θ decreases at the same rate that t increases:

$$\begin{aligned} \frac{1-t}{1} &= \frac{\theta}{\pi/2} \\ \theta &= \frac{\pi}{2}(1-t)\,. \end{aligned}$$

Check if this makes sense: when $t = 0$, $\theta = \pi/2$ and when $t = 1$, $\theta = 0$.

The x-coordinate of P follows from trigonometry:

$$\tan\theta = \frac{y}{x}\,.$$

which gives:

$$x = \frac{y}{\tan\theta} = y\cot\theta = y\cot\frac{\pi}{2}(1-t) = y\cot\frac{\pi}{2}y\,.$$

We usually express a function as $y = f(x)$ but it can also be expressed as $x = f(y)$. Let us compute the x-coordinate of the point G, the intersection of the quadratrix with the x-axis. We can't simply plug in $y = 0$ because $\cot 0$ is not defined, but we might get lucky by computing the limit of x as y goes to 0:

$$x = y\cot\frac{\pi}{2}y = \frac{2}{\pi}\cdot\frac{\pi}{2}y\cot\frac{\pi}{2}y\,.$$

For convenience, perform a change of variable $z = \frac{\pi}{2}y$ and compute the limit:

$$\lim_{z\to 0} z\cot z = \lim_{z\to 0}\frac{z\cos z}{\sin z} = \lim_{z\to 0}\frac{\cos z}{\frac{\sin z}{z}} = \frac{\cos 0}{1} = 1\,,$$

using the well-known fact that $\lim_{z\to 0}\frac{\sin z}{z} = 1$. Therefore, as $y \to 0$:

$$x \to \frac{2}{\pi}\cdot\lim_{y\to 0}\frac{\pi}{2}y\cot\frac{\pi}{2}y = \frac{2}{\pi}\cdot 1 = \frac{2}{\pi}\,.$$

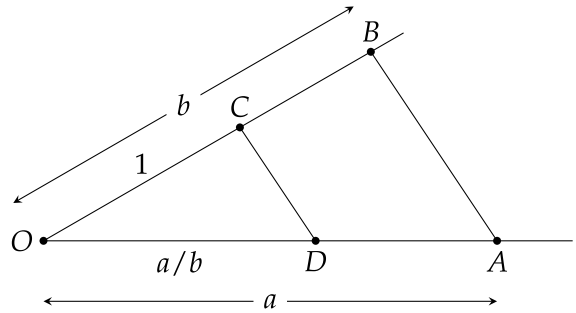
Using the quadratrix we have constructed a line segment AG whose length is $x = \frac{2}{\pi}$. With an ordinary straightedge and compass it is easy to construct a line segment of length $\sqrt{\frac{2}{x}} = \sqrt{\pi}$ and then construct a square whose area is π.

Chapter 3
How to (Almost) Square a Circle

3.1 Approximations to π

In the nineteenth century it was proved that three constructions are impossible with straightedge and compass: trisecting an angle, duplicating a cube and squaring a circle. Given a line segment defined to have length 1, the constructible numbers (lengths) are those obtainable from that line segment using the operators $+, -, \times, /, \sqrt{}$.

In the constructions in this chapter we need to divide a line segment into three parts; here we show how the division of two lengths can be constructed. Given a line segment of length 1 and line segments of lengths a, b, by similar triangles $1/b = \overline{OD}/a$ so $\overline{OD} = a/b$.



To square a circle the length $\sqrt{\pi}$ must be constructed, however, π is *transcendental*, meaning that it is not the solution of any algebraic equation.

This chapter brings three constructions of approximations to π. The following table shows the formulas of the lengths that are constructed, their approximate values, the difference between these values and the value of π, and the error in meters that results if the approximation is used to compute the circumference of the earth given that its radius is 6378 km.

Construction	Formula	Value	Difference	Error (m)
π		3.14159265359	–	–
Kochansky	$\sqrt{\frac{40}{3} - 2\sqrt{3}}$	3.14153338705	5.932×10^{-5}	756
Ramanujan 1	$\frac{355}{113}$	3.14159292035	2.667×10^{-7}	3.4
Ramanujan 2	$\left(9^2 + \frac{19^2}{22}\right)^{1/4}$	3.14159265258	1.007×10^{-9}	0.013

Kochansky's construction from 1685 can be found in [2].

Ramanujan's constructions from 1913 can be found in [8, 9].

3.2 Kochansky's construction

3.2.1 The construction

Construct three circles:

1. Construct a unit circle centered at O, let $\overline{AB}$ be a diameter and construct a tangent to the circle at A.

2. Construct a unit circle centered at A. Its intersection with the first circle is C.[1]

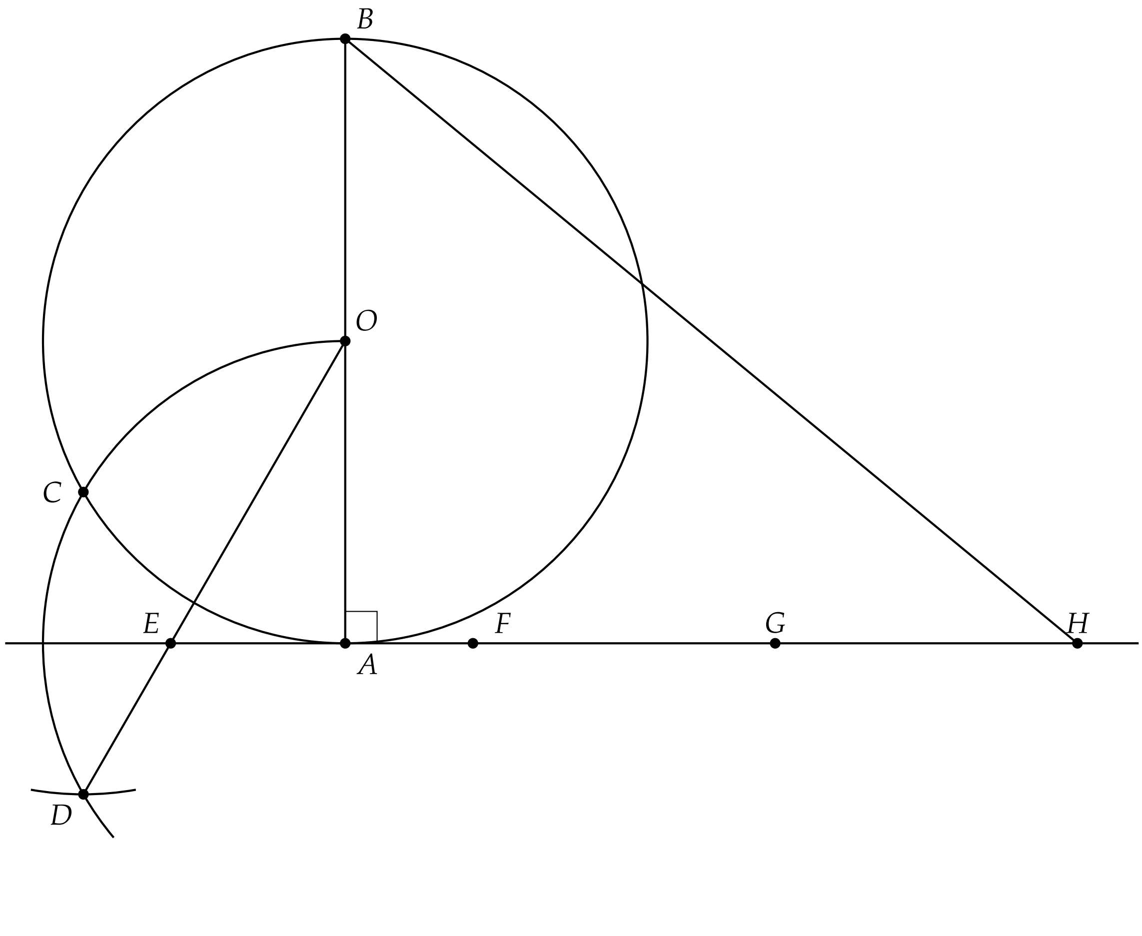
3. Construct a unit circle centered at C. Its intersection with the second circle is D.

Construct $\overline{OD}$ and denote its intersection with the tangent by E.

From E construct F, G, H, each at distance 1 from the previous point; then $\overline{AH} = 3 - \overline{EA}$.

Construct $\overline{BH}$.

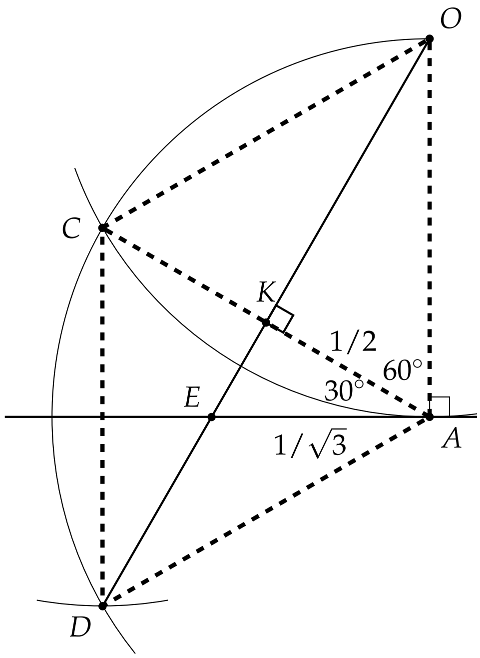
Claim: $\overline{BH} = \sqrt{\frac{40}{3} - 2\sqrt{3}} \approx \pi$.



[1]For the second and third circles, the diagram only shows the arc that intersects the previous circle.

3.2.2 The proof

Extract the following diagram from the first one. Dashed line segments have been added. Since all the circles are unit circles, it is easy to see that the length of each dashed line segment is 1. It follows that $\overline{AOCD}$ is a rhombus so its diagonals are perpendicular to and bisect each other at K, so $\overline{AK} = \frac{1}{2}$.



The diagonal $\overline{AC}$ forms two equilateral triangles $\triangle OAC, \triangle DAC$ so $\angle OAC = 60°$. Since tangent forms a right angle with the radius $\overline{OA}$, $\angle KAE = 30°$. Now:

$$\begin{aligned}
\frac{1/2}{\overline{EA}} &= \cos 30° = \frac{\sqrt{3}}{2} \\
\overline{EA} &= \frac{1}{\sqrt{3}} \\
\overline{AH} &= 3 - \overline{EA} = \left(3 - \frac{1}{\sqrt{3}}\right) = \frac{3\sqrt{3}-1}{\sqrt{3}}
\end{aligned}$$

Returning to the first diagram, we see that $\triangle ABH$ is a right triangle:

$$\begin{aligned}
\overline{BH}^2 &= \overline{OB}^2 + \overline{AH}^2 \\
&= 4 + \frac{9 \cdot 3 - 6\sqrt{3} + 1}{3} = \frac{40}{3} - 2\sqrt{3} \\
\overline{BH} &= \sqrt{\frac{40}{3} - 2\sqrt{3}} \approx 3.141533387 =\approx \pi\,.
\end{aligned}$$

3.3 Ramanujan's first construction

3.3.1 The construction

Construct a unit circle centered at O and let $\overline{PR}$ be a diameter.

H bisects $\overline{PO}$ and T trisects $\overline{RO}$.

Construct a perpendicular at T that intersects the circle at Q.

Construct a chord $\overline{RS} = \overline{QT}$.

Construct $\overline{PS}$.

Construct a line parallel to RS from T that intersects $\overline{PS}$ at N.

Construct a line parallel to $\overline{RS}$ from O that intersects $\overline{PS}$ at M.

Construct the chord $\overline{PK} = \overline{PM}$.

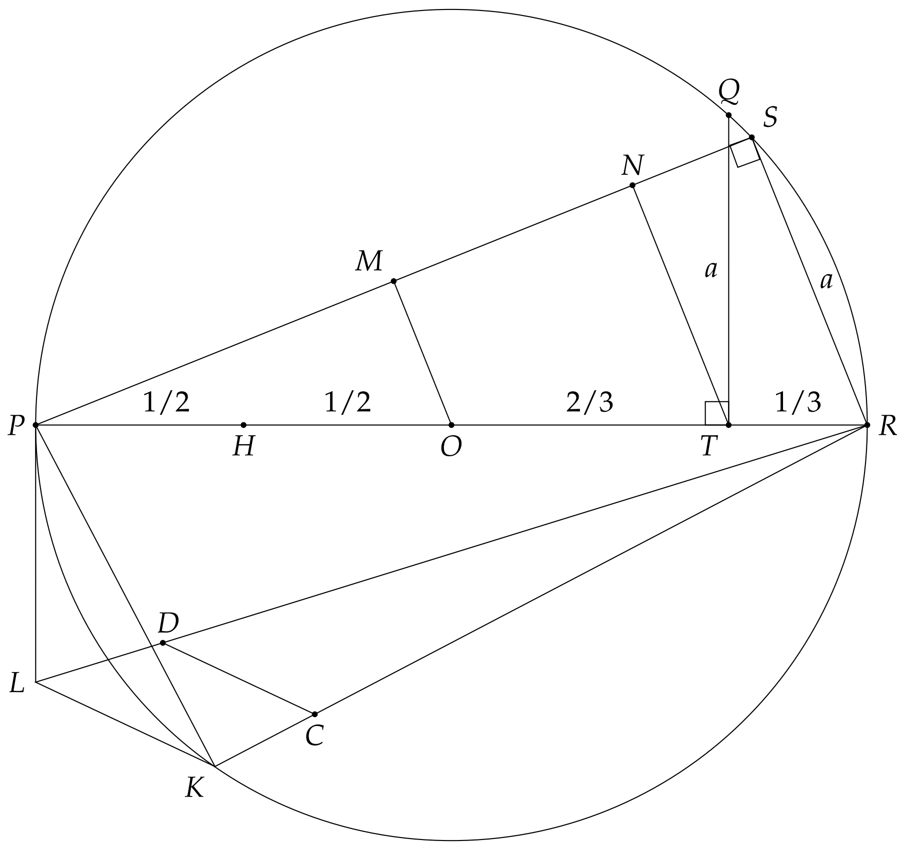
Construct the tangent at P of length $\overline{PL} = \overline{MN}$.

Connect the points K, L, R.

Find point C such that $\overline{RC}$ is equal to $\overline{RH}$.

Construct $\overline{CD}$ parallel to $\overline{KL}$ that intersects $\overline{LR}$ at D.

Claim: $\overline{RD}^2 = \dfrac{355}{113} \approx pi$.



3.3.2 The proof

By Pythagoras' theorem on $\triangle QOT$:

$$\overline{QT} = \sqrt{1^2 - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}.$$

$\triangle PSR$ is a right triangle because it subtends a diameter. By Pythagoras theorem:

$$\overline{PS} = \sqrt{2^2 - \left(\frac{\sqrt{5}}{3}\right)^2} = \sqrt{4 - \frac{5}{9}} = \frac{\sqrt{31}}{3}.$$

$\triangle MPO \sim \triangle SPR$ so:

$$\begin{aligned}
\frac{\overline{PM}}{\overline{PO}} &= \frac{\overline{PS}}{\overline{PR}} \\
\frac{\overline{PM}}{1} &= \frac{\sqrt{31}/3}{2} \\
\overline{PM} &= \frac{\sqrt{31}}{6}.
\end{aligned}$$

$\triangle NPT \sim \triangle SPR$ so:

$$\begin{aligned}
\frac{\overline{PN}}{\overline{PT}} &= \frac{\overline{PS}}{\overline{PR}} \\
\frac{\overline{PN}}{5/3} &= \frac{\sqrt{31}/3}{2} \\
\overline{PN} &= \frac{5\sqrt{31}}{18} \\
\overline{MN} &= \overline{PN} - \overline{PM} \\
&= \sqrt{31}\left(\frac{5}{18} - \frac{1}{6}\right) = \frac{\sqrt{31}}{9}.
\end{aligned}$$

$\triangle PKR$ is a right triangle because it subtends a diameter. By Pythagoras's theorem:

$$\overline{RK} = \sqrt{2^2 - \left(\frac{\sqrt{31}}{6}\right)^2} = \frac{\sqrt{113}}{6}.$$

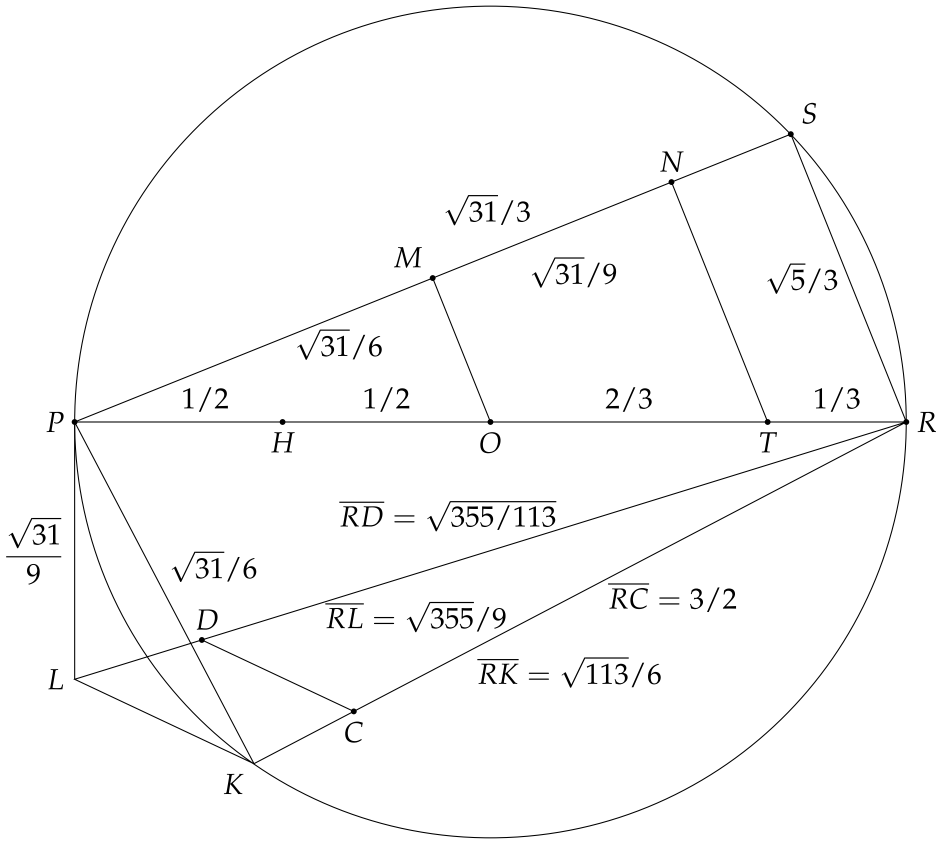
$\triangle PLR$ is a right triangle because it $\overline{PL}$ is a tangent. By Pythagoras's theorem:

$$\overline{RL} = \sqrt{2^2 + \left(\frac{\sqrt{31}}{9}\right)^2} = \frac{\sqrt{355}}{9}.$$

$\overline{RC} = \overline{RH} = \dfrac{1}{3} + \dfrac{2}{3} + \dfrac{1}{2} = \dfrac{3}{2}$. Since $\overline{CD}$ is parallel to $\overline{LK}$, by similar triangles:

$$\begin{aligned}
\frac{\overline{RD}}{\overline{RC}} &= \frac{\overline{RL}}{\overline{RK}} \\
\frac{\overline{RD}}{3/2} &= \frac{\sqrt{355}/9}{\sqrt{113}/6} \\
\overline{RD} &= \sqrt{\frac{355}{113}}.
\end{aligned}$$

Here is the construction with line segments labeled with their lengths:



The value $\dfrac{355}{113}$ could be constructed by constructing two line segments of length 355 and 113 and then using the division construction shown in Section 3.1, but that is rather tedious!

3.4 Ramanujan's second approximation

3.4.1 The construction

Construct a unit circle centered at O with diameter $\overline{AB}$, and let C be the intersection of the perpendicular at O with the circle.

Trisect $\overline{AO}$ so that $\overline{AT} = 1/3$ and $\overline{TO} = 2/3$.

Construct $\overline{BC}$ and find points M, N such that $\overline{CM} = \overline{MN} = \overline{AT} = 1/3$.

Construct $\overline{AM}$ and $\overline{AN}$ and let P be the point on $\overline{AN}$ such that $\overline{AP} = \overline{AM}$.

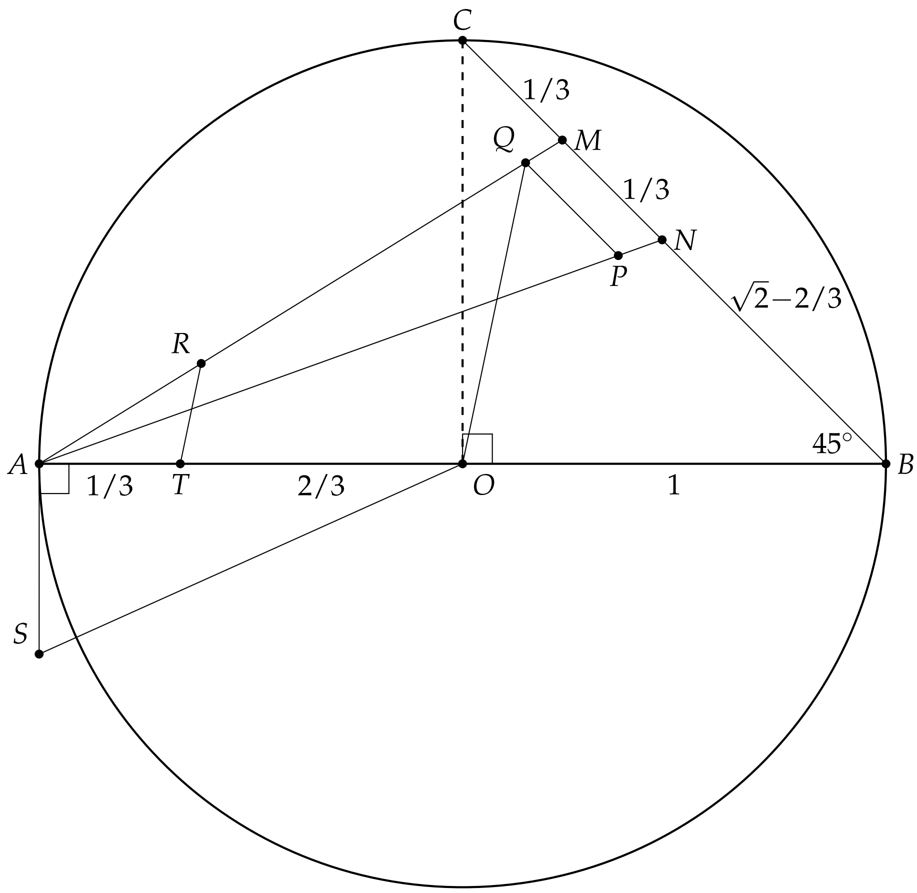
From P construct a line parallel to $\overline{MN}$ that intersects $\overline{AM}$ at Q.

Construct $\overline{OQ}$ and then from T construct a line parallel to $\overline{OQ}$ that intersects $\overline{AM}$ at R.

Construct a line segment $\overline{AS}$ tangent to A of length $\overline{AR}$.

Construct $\overline{SO}$.

Claim: $3\sqrt{\overline{SO}} = \left(9^2 + \frac{19^2}{22}\right)^{\frac{1}{4}} \approx \pi$.



3.4.2 The proof

$\triangle COB$ is a right triangle, $\overline{OB} = \overline{OC} = 1$, so by Pythagoras' theorem $\overline{CB} = \sqrt{2}$ and $\overline{NB} = \sqrt{2} - 2/3$. The triangle is isoceles so $\angle NBA = \angle MBA = 45°$.

We use the law of cosines on $\triangle NBA$ to compute $\overline{AN}$:

$$\begin{aligned}
\overline{AN}^2 &= \overline{BA}^2 + \overline{BN}^2 - 2 \cdot \overline{BA} \cdot \overline{BN} \cdot \cos \angle NBA \\
&= 2^2 + \left(\sqrt{2} - \frac{2}{3}\right)^2 - 2 \cdot 2 \cdot \left(\sqrt{2} - \frac{2}{3}\right) \cdot \frac{\sqrt{2}}{2} \\
&= \left(4 + 2 + \frac{4}{9} - 4\right) + \sqrt{2} \cdot \left(-\frac{4}{3} + \frac{4}{3}\right) = \frac{22}{9} \\
\overline{AN} &= \sqrt{\frac{22}{9}}.
\end{aligned}$$

Similarly, we use the law of cosines on $\triangle MBA$ to compute $\overline{AM}$:

$$\begin{aligned}
\overline{AM}^2 &= \overline{BA}^2 + \overline{BM}^2 - 2 \cdot \overline{BA} \cdot \overline{BM} \cdot \cos \angle MBA \\
&= 2^2 + \left(\sqrt{2} - \frac{1}{3}\right)^2 - 2 \cdot 2 \cdot \left(\sqrt{2} - \frac{1}{3}\right) \cdot \frac{\sqrt{2}}{2} \\
&= \left(4 + 2 + \frac{1}{9} - 4\right) + \sqrt{2} \cdot \left(-\frac{2}{3} + \frac{2}{3}\right) = \frac{19}{9} \\
\overline{AM} &= \sqrt{\frac{19}{9}}.
\end{aligned}$$

By construction $\overline{QP} \parallel \overline{MN}$ so $\triangle MAN \sim \triangle QAP$, and by construction $\overline{AP} = \overline{AM}$ giving:

$$\begin{aligned}
\frac{\overline{AQ}}{\overline{AM}} &= \frac{\overline{AP}}{\overline{AN}} = \frac{\overline{AM}}{\overline{AN}} \\
\overline{AQ} &= \frac{\overline{AM}^2}{\overline{AN}} = \frac{19/9}{\sqrt{22/9}} = \frac{19}{3\sqrt{22}}.
\end{aligned}$$

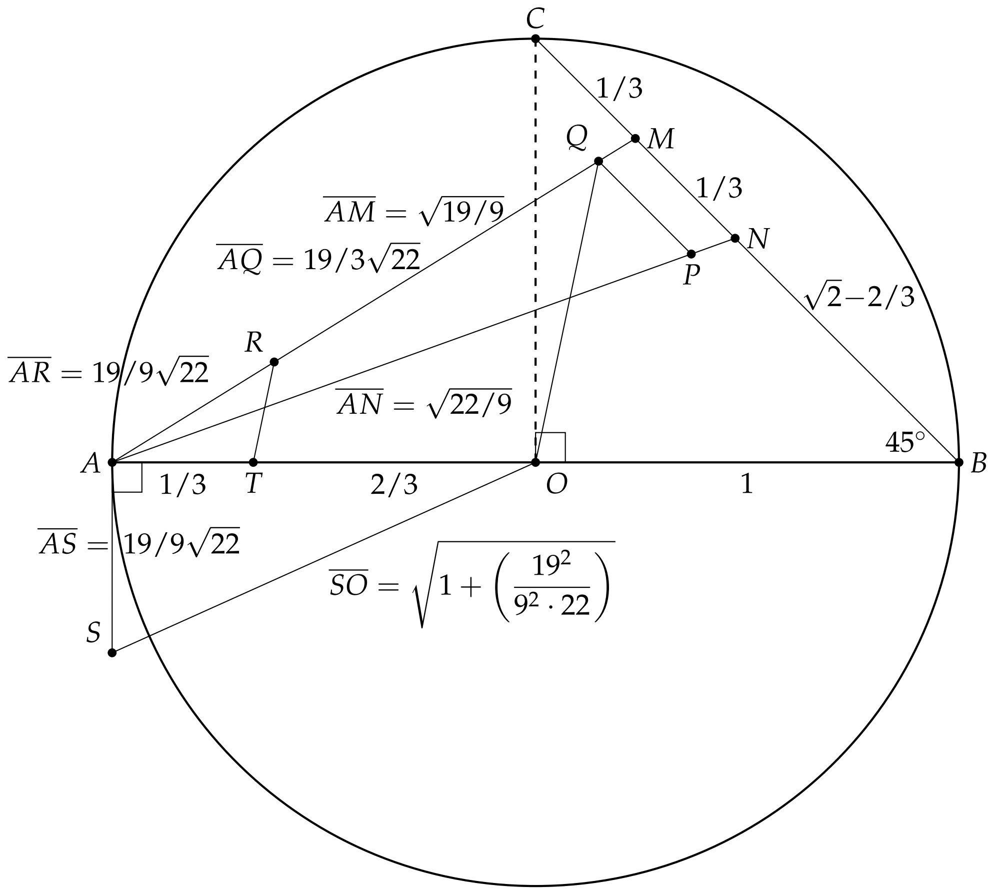
By construction $\overline{TR} \parallel \overline{OQ}$ so $\triangle RAT \sim \triangle QAO$ giving:

$$\begin{aligned}
\frac{\overline{AR}}{\overline{AQ}} &= \frac{\overline{AT}}{\overline{AO}} \\
\overline{AR} &= \overline{AQ} \cdot \frac{\overline{AT}}{\overline{AO}} = \frac{19}{3\sqrt{22}} \cdot \frac{1/3}{1} = \frac{19}{9\sqrt{22}}.
\end{aligned}$$

By construction $\overline{AS} = \overline{AR}$ and $\triangle OAS$ is a right triangle. By Pythagoras' theorem:

$$\begin{aligned}
\overline{SO} &= \sqrt{1^2 + \left(\frac{19}{9\sqrt{22}}\right)^2} \\
3\sqrt{\overline{SO}} &= 3\left(1 + \frac{19^2}{9^2 \cdot 22}\right)^{\frac{1}{4}} \\
&= \left(3^4 + \frac{3^4 \cdot 19^2}{9^2 \cdot 22}\right)^{\frac{1}{4}} \\
&= \left(9^2 + \frac{19^2}{22}\right)^{\frac{1}{4}} \approx 3.14159265262 \approx \pi\,.
\end{aligned}$$

Here is the construction with line segments labeled with their lengths:

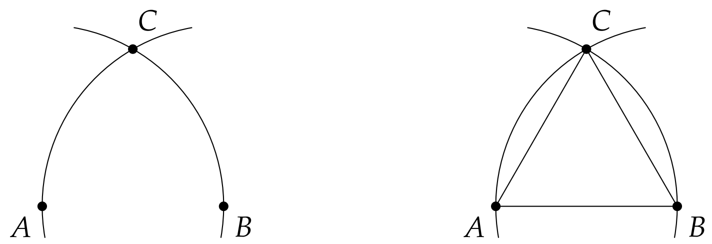


Chapter 4
A Compass is Sufficient

In 1797 the Italian mathematician Lorenzo Mascheroni proved that any construction carried out with a compass and straightedge can be carried out with the compass alone! In the twentieth century it was discovered that this theorem had been proved by the Danish mathematician Georg Mohr. The theorem is now called the Mohr-Mascheroni Theorem.

In this chapter I present a proof of the theorem based on a proof that appeared as problem 33 in [3] and reworked by Michael Woltermann [4].[1] Additional proofs can be found in [5], [6].

What does it mean to perform a construction with only a compass? The right diagram below shows the construction of an equilateral triangle using a straightedge and compass. How can we construct a triangle without the line segments AB, AC, BC? In fact, there is no need to *see* the lines. A line is defined by two points, so it is sufficient to construct the points in order to obtain a construction equivalent to the one with a straightedge (left diagram):



In the diagrams we will draw lines, but they are used only to understand the construction and the proof of its correctness. It is important that you convince yourself that the construction itself uses only a compass.

Every step of a construction with straightedge and compass consists of one of the following three operations:

- Finding the point of intersection of two straight lines.
- Finding the point of intersection of a straight line and a circle.
- Finding the point(s) of intersection of two circles.

It is clear that the third operation can be done with only a compass. We need to show that the first two operations can be done with a compass alone.

Notation:

- $C(O, A)$: the circle with center O through point A.
- $C(O, r)$: the circle with center O and radius r.
- $C(O, AB)$: the circle with center O and radius the length of line segment AB.

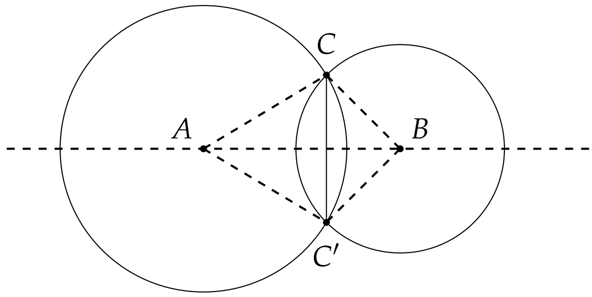
[1] I would like to thank him for giving me permission to use his work.

First we will solve four preliminary problems (Sections 4.1–4.4) and then we show the construction for finding the intersection of two lines (Section 4.5) and that of a line and a circle (Section 4.6).

4.1 Reflection of a point

Given a line AB and a point C not on AB, it is possible to build a point C' which is a reflection of C about AB. C' is a *reflection* about a line segment AB if AB (or the line containing AB is the perpendicular bisector of the line CC'.

We build a circle centered on A passing through C and circle centered on B passing through C. The intersection of the two circles is the point C' which is the reflection of C.

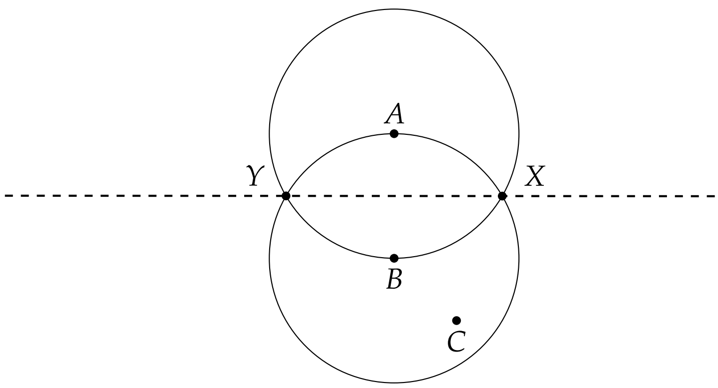


Proof: $\triangle ABC \cong \triangle ABC'$ by SSS since AC, AC' are radii of the same circle as are BC, BC', and AB is a common side. Therefore, $\angle CAB = \angle C'AB$ so AB is the angle bisector of $\angle CAC'$. But $\triangle CAC'$ is an isosceles triangle and the angle bisector AB is also the perpendicular bisector of the base of the triangle CC'. By definition C' is the reflection of C around AB.

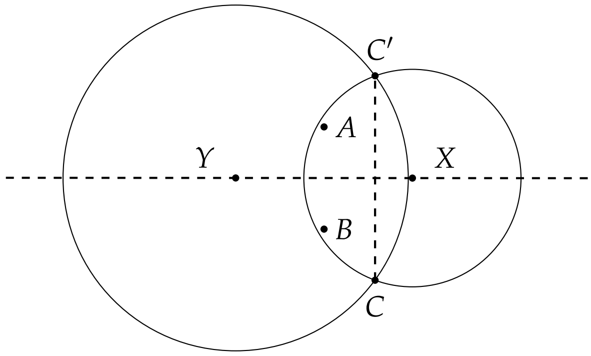
4.2 Construct a circle with a given radius

Given points A, B, C, construct$c(A, BC)$.

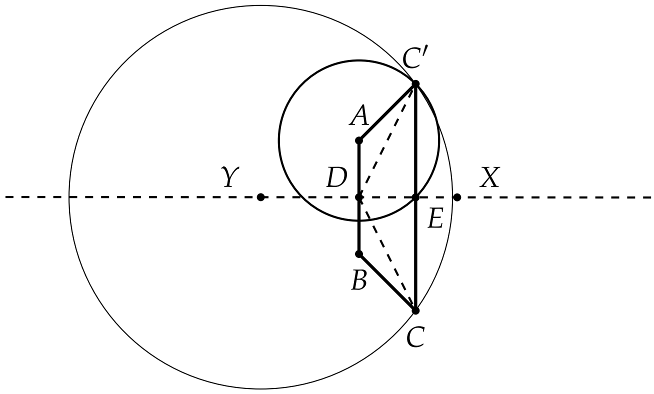
Construct $c(A, B)$ and $c(B, A)$ and let X and Y be their points of intersection.



Construct C', the reflection of C about line XY as described in Section 4.1.



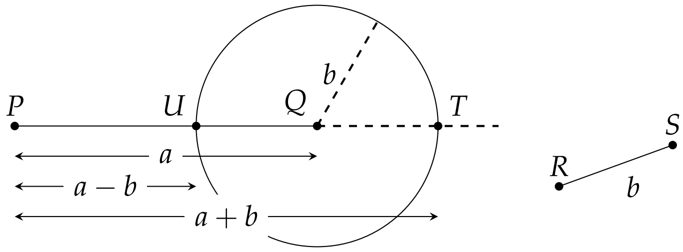
$c(A, C')$ is the desired circle.



Proof: A is the reflection of B around XY (since $\triangle YAX \cong \triangle YBX$) and C' is the reflection of C around the XY. By definition, XY is the perpendicular bisector of CC' and AB, so $C'E = EC$, $AD = DB$ and $\angle DEC = \angle DEC'(= 90°)$. $\triangle DEC \cong \triangle DEC'$ by SAS, so $DC = DC'$ and $\angle ADC' = \angle BDC$ (they are complementary to $\angle EDC'$, $\angle EDC$). Therefore, $\triangle ADC' \cong \triangle BDC$ by ASA, so $AC' = BC$.

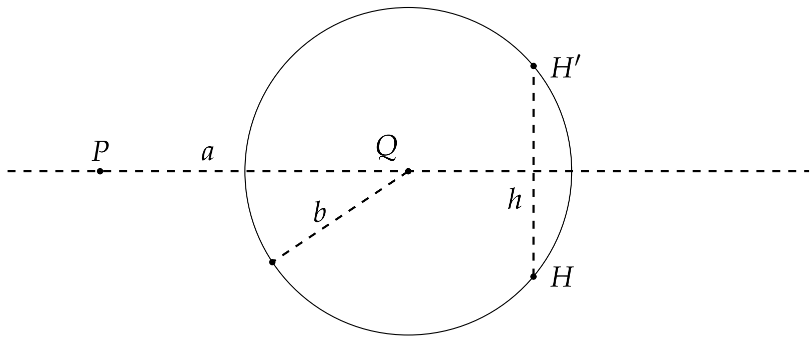
4.3 Addition and subtraction of line segments

Given line segment PQ of length a and line segment RS of length b, it is possible to construct line segments QT, QU such that $PUQT$ is a line segment, where the length of PU is $a - b$ and the length of PT is $a + b$.

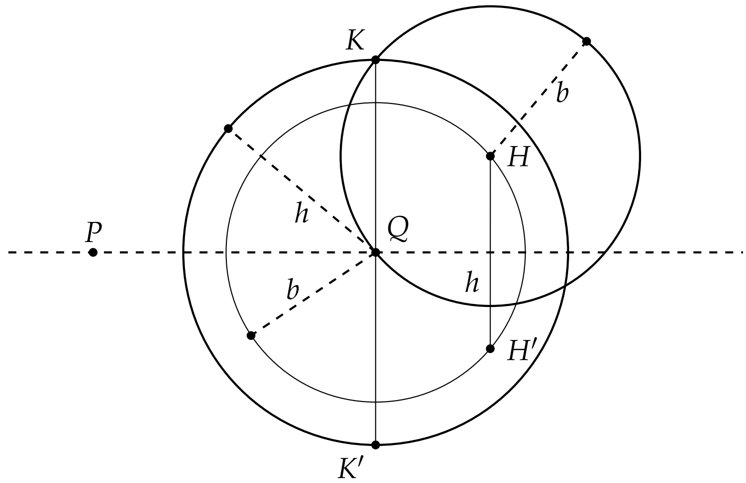


Constructing an isosceles trapezoid

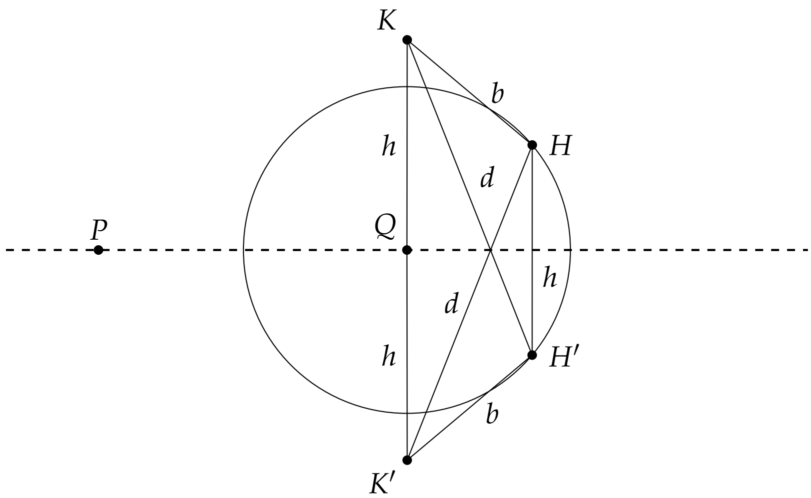
Let H be any point on $c(Q, b)$. Construct H', its reflection about PQ. h is the length of HH':



Construct the circles $C(H, b)$, $c(Q, h)$. K is the intersection of the circles and K' be the reflection of K about line PQ:



PQ is the perpendicular bisector of both HH' and KK', so these line segments are parallel. $KH = K'H' = b$ since both are radii of the circle centered on H. K', H' are reflections of K, H. Therefore, $KHH'K'$ is an isosceles trapezoid whose bases are $HH' = h$, $KK' = 2h$. Label d, the length of the diagonals $K'H = KH'$:

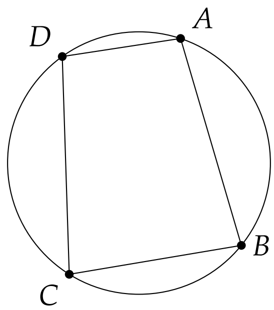


Circumscribing the trapezoid by a circle

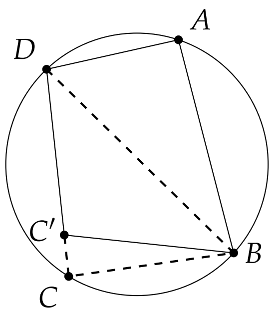
We want to proof that it is possible to circumscribe $KHH'K'$ by a circle. We prove that is the opposite angles of a quadrilateral are supplementary, then the trapezoid can be circumscribed by a circle, and we proof that in an isosceles trapezoid the opposite angles are supplementary.

Geometry textbooks give the simple proof that the opposite angles of a quadrilateral circumscribed by a circle are supplementary, but it is hard to find a proof of the converse, so I present both proofs here.

If a quadrilateral can be circumscribed by a circle then the opposite angles are supplementary: An inscribed angle equals half the subtended arc, so $\angle DAB$ is half of the arc DCB and $\angle DCB$ is half of the arc DAB. The two arcs subtend the entire circumference of the circle, so their sum is 360°. Therefore, $\angle DAB + \angle DCB = \frac{1}{2} \cdot 360° = 180°$, and similarly $\angle ADC + \angle AABC = 180°$



Quadrilateral whose opposite angles are supplementary can be circumscribed by a circle: Any triangle can be circumscribed by a circle. Circumscribed $\triangle DAB$ by a circle and suppose that C' is a point such that $\angle DAB + \angle DC'B = 180°$, but C' is *not* on the circumference of the circle. Without loss of generality, let C' be within the circle:

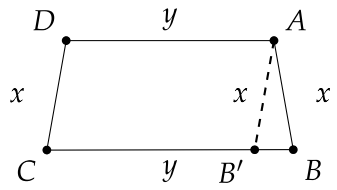


Construct a ray that extends DC' and let C be its intersection with the circle. $ABCD$ is circumscribed by a circle so:

$$\begin{aligned}
\angle DAB + \angle DCB &= 180° \\
\angle DAB + \angle DCB &= \angle DAB + \angle DC'B \\
\angle DCB &= \angle DC'B\,,
\end{aligned}$$

which is impossible if C is on the circle and C' is inside the circle.

Finally, we show that the opposite angles of an isosceles trapezoid are supplementary.



Construct the line AB' parallel to CD. $AB'CD$ is a parallelogram and $\triangle ABB'$ is an isosceles triangle, so $\angle C = \angle ABB' = \angle AB'B = \angle B$. Similarly, $\angle A = \angle D$. Since the sum of the internal angles of any quadrilateral is equal to 360°:

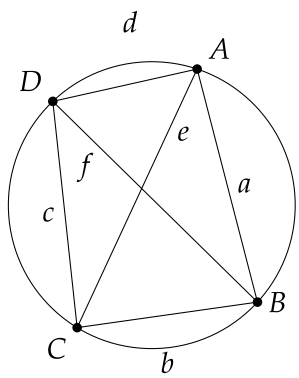
$$\begin{aligned}\angle A + \angle B + \angle C + \angle D &= 360^\circ\\ 2\angle A + 2\angle C &= 360^\circ\\ \angle A + \angle C &= 180^\circ\,.\end{aligned}$$

and similarly $\angle B + \angle D = 180^\circ$.

Ptolemy's theorem

We will use Ptolemy's theorem, an equation which relates the lengths of the diagonals and the lengths of the sides of a quadrilateral that is circumscribed by a circle:

$$ef = ac + bd\,.$$



There is a geometric proof of the theorem (see Wikipedia), but I will present a simple trigonometric proof. The law of cosines for the four triangles $\triangle ABC$, $\triangle ADC$, $\triangle DAB$, $\triangle DCB$ gives the following equations:

$$\begin{aligned}e^2 &= a^2 + b^2 - 2ab\cos\angle B\\ e^2 &= c^2 + d^2 - 2cd\cos\angle D\\ f^2 &= a^2 + d^2 - 2ad\cos\angle A\\ f^2 &= b^2 + c^2 - 2bc\cos\angle C\,.\end{aligned}$$

$\angle C = 180° - \angle A$ and $\angle D = 180° - \angle B$ because they are opposite angles of a quadrilateral circumscribed by a circle, so:

$$\begin{aligned}\cos \angle D &= -\cos \angle B\\ \cos \angle C &= -\cos \angle A\,.\end{aligned}$$

We can eliminate the cosine term from the first two equations and from the last two equations. After some messy arithmetic, we get:

$$\begin{aligned}e^2 &= \frac{(ac+bd)(ad+bc)}{(ab+cd)}\\ f^2 &= \frac{(ab+cd)(ac+bd)}{(ad+bc)}\,.\end{aligned}$$

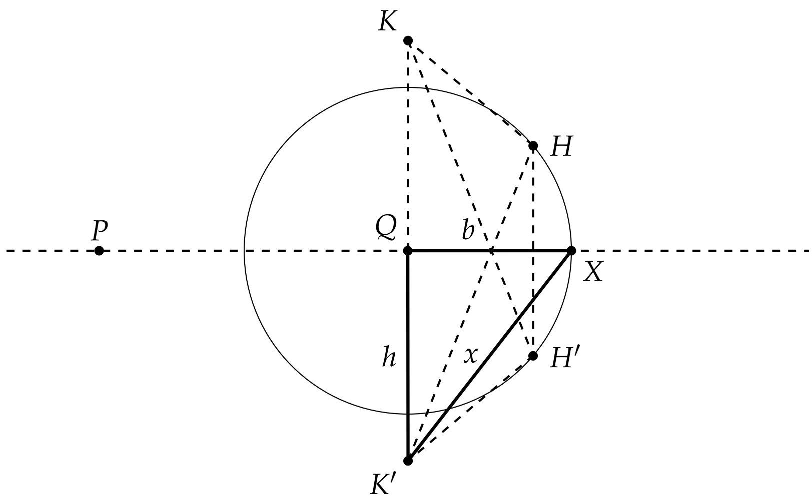
Multiply the two equations and simplify to get Ptolemy's theorem:

$$\begin{aligned}e^2 \cdot f^2 &= (ac+bd)^2\\ ef &= (ac+bd)\,.\end{aligned}$$

Using Ptolemy's theorem

For the construction on page 29, the diagonals are of length d, the legs are of length b, and the bases are of lengths h and $2h$, so Ptolemy's theorem gives $d \cdot d = b \cdot b + h \cdot 2h$ or $d^2 = b^2 + 2h^2$.

Let X be the point on line PQ that extends PQ by b. (We will eventually construct X; now we're just imagining it.) Define $x = K'X$. Since $\triangle QK'X$ is a right triangle, $x^2 = b^2 + h^2$:

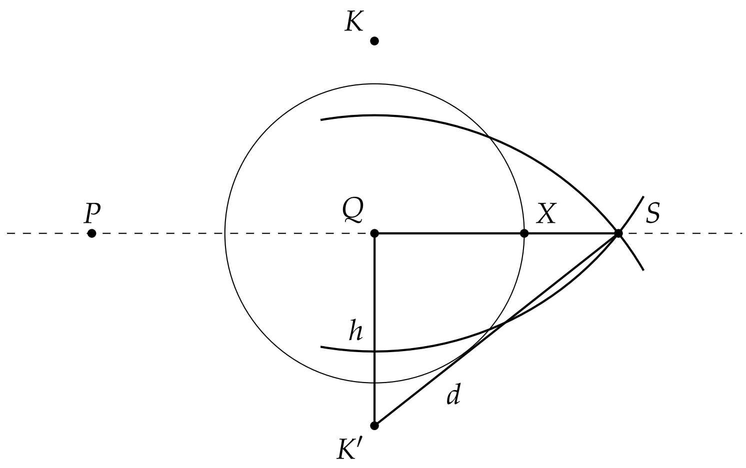


By Ptolemy's theorem:

$$\begin{aligned}d^2 &= b^2 + 2h^2\\ &= (x^2 - h^2) + 2h^2\\ &= x^2 + h^2\,.\end{aligned}$$

Don't look for a right triangle in the diagram. We are claiming that *it is possible to construct* the triangle with sides x, h, d.

Let us construct the point S as the intersection of the circles $c(K,d), c(K',d)$:

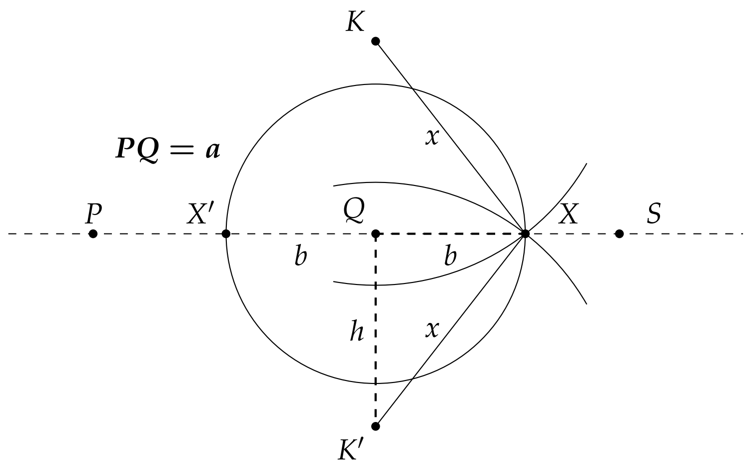


We obtain a right triangle $\triangle QSK'$. By Pythagoras' theorem $QS^2 + h^2 = d^2$, so:

$$QS^2 = d^2 - h^2 = x^2\,,$$

and $QS = x$.

It is possible to construct the point X as the intersection of the circles $c(K, x), c(K', x)$:



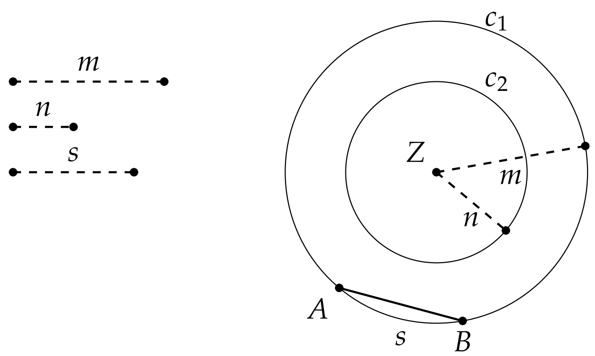
Recall that we want to extend PQ of length a by a length b, or decrease its length by b. Since the length of QX is $\sqrt{x^2 - h^2} = b$, the length of PX is $a + b$ and the length of PX' is $a - b$.

4.4 Construction of a line segment whose length is defined relative to three other line segments

Given line segments of length n, m, s it is possible to construct a line segment of length $x = \dfrac{n}{m}s$.

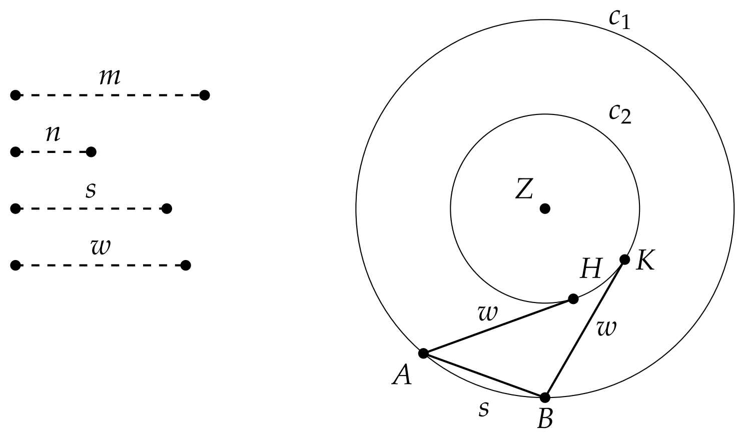
Construct two concentric circles $c_1 = c(Z, m)$ and $c_2 = c(Z, n)$, and chord $AB = s$ on c_1. (A chord can be constructed using only a compass as shown in Section 4.2.)

We assume that $m > n$. If not, exchange the notation.

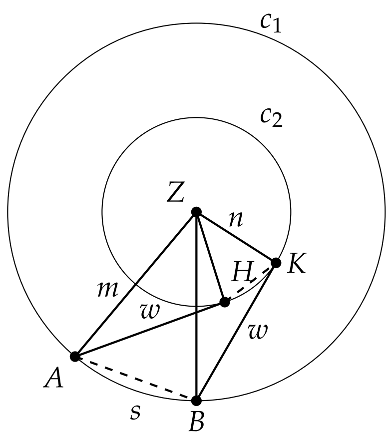


We also assume that s does not intersect c_2. If not, use the construction in Section 4.3 to multiply m, n by a number k so that the chord does not intersect the circle. Note that this does not change the value that we are trying to construct since $x = \dfrac{kn}{km}s = \dfrac{n}{m}s$.

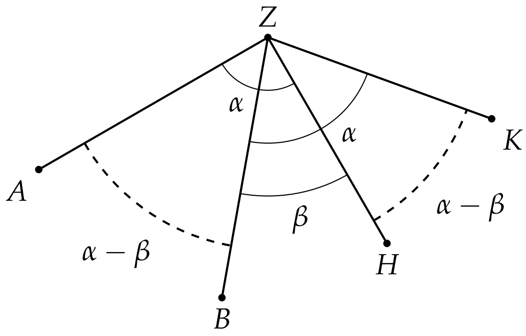
Choose any point H on circle c_2. Label the length of AH by w. Construct point K on c_2 such that the length of BK is w.



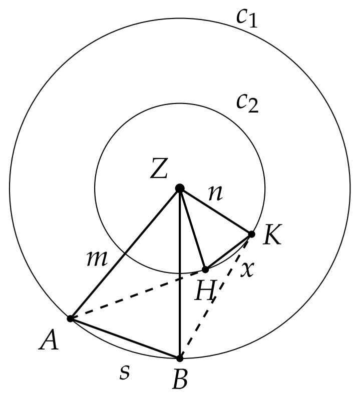
$\triangle AHZ \cong \triangle BZK$ by SSS: $ZA = ZB = m$, the radius of c_1, $ZH = ZK = n$, the radius of c_2, and $AH = BK = w$ by construction.



From $\triangle AHZ \cong \triangle BZK$, we get $\angle AZB = \angle HZK$. It is difficult to see this equality from the diagram, but the following diagram clarifies the relation among the angles. Define $\alpha = \angle AZH = \angle BZK$ and $\beta = \angle BZH$. It is easy to see that $\angle AZB = \angle HZK = \alpha - \beta$.



$\triangle ZAB \sim \triangle ZHK$ by SAS, since both are isosceles triangles and we have shown that they have the same vertex angle.



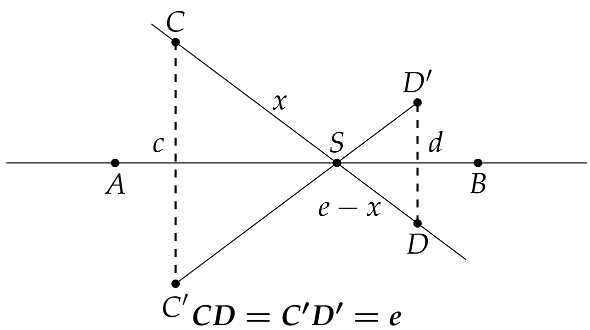
Label HK by x. Then:

$$\begin{aligned} \frac{m}{s} &= \frac{n}{x} \\ x &= \frac{n}{m}s\,. \end{aligned}$$

4.5 Finding the intersection of two lines

Given two lines containing the line segments AB, CD, it is possible to construct their intersection using only a compass.

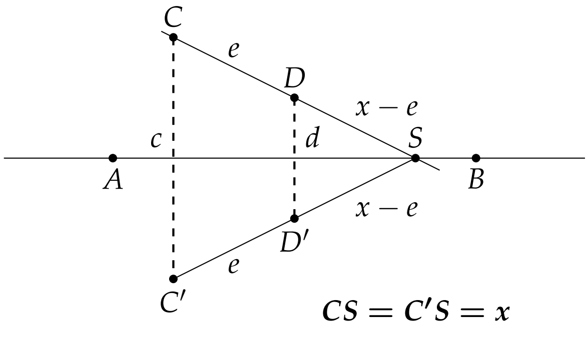
S, the point of intersection of the two lines, lies on AB, because $\triangle CZS \cong \triangle C'ZS$ by SAS: $CZ = C'Z$, $\angle CZS = \angle C'ZS = 90^\circ$ and ZS is a common side. Therefore, $C'S = CS$ and similarly $D'S = DS$.



$\boldsymbol{CD = C'D' = e}$

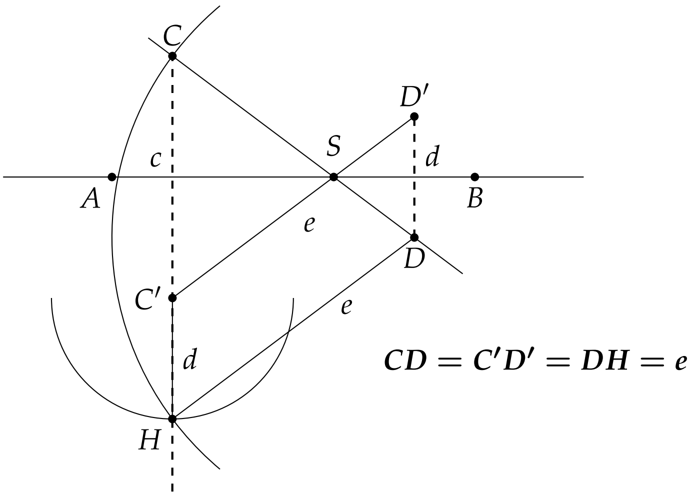
Label $x = CS, c = CC', d = DD', e = CD$. $\triangle CSC' \sim \triangle DSD'$ are similar so $\dfrac{x}{e-x} = \dfrac{c}{d}$. Solving the equation for x gives $x = \dfrac{c}{c+d}e$.

If C, D are on the same side of AB:



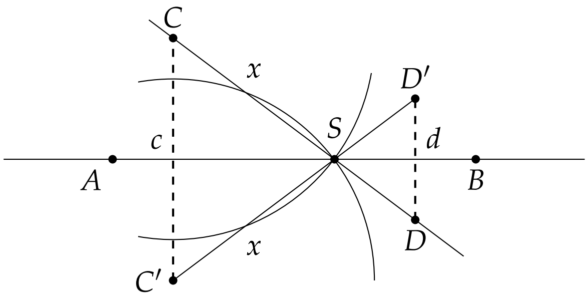
$\triangle CSC' \sim \triangle DSD'$ gives $\dfrac{x}{x-e} = \dfrac{c}{d}$. Solving for x gives $x = \dfrac{c}{c-d}e$.

Construct the circles $c(C', d), c(D, e)$ and label their intersection by H. The sum of the line segments $CC', C'H$ is $c+d$. We have to show that H is on the extension of CC' so that CH is a line segment of length $c+d$. ($CH = c - d$ in case D is on the same side of AB as C.)



From the definition of H as the intersection of $c(C', d)$,$c(D, e)$, we get $DH = e$, $C'H = d$, but $C'D' = e$, $D'D = d$, so the quadrilateral $C'D'DH$ is a parallelogram, since the lengths of both pairs of opposite sides are equal. By construction, the line segment DD' is parallel to CC', so $C'H$ is parallel to DD' is also parallel to CC'. Since one of its end points is C', it must be on the line containing CC'.

The lengths c, d, e are given and we proved in Section 4.3 that a line segment of length $c+d$ can be construction and in Section 4.4 that a line segment of length $x = \dfrac{c}{c+d}e$ can be constructed. S is the intersection of $c(C', x)$ and $c(C, x)$.

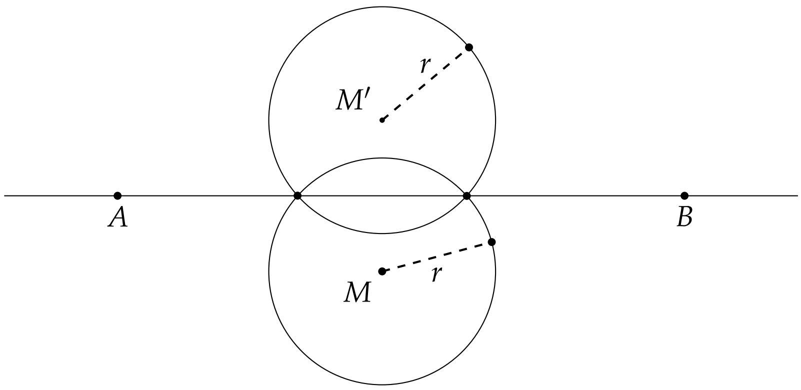


$CD = C'D' = DH = e$

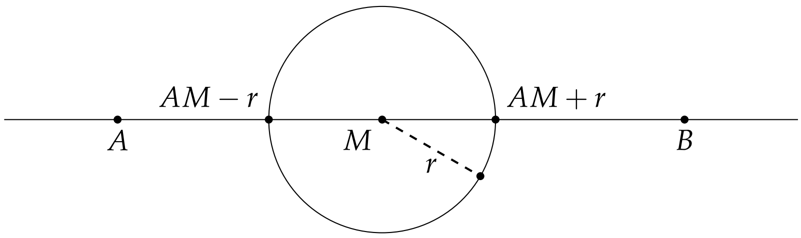
4.6 Finding the intersection of a line and a circle

Given a circle $k = C(M, r)$ and a line AB, it is possible to construct their intersections using only a compass.

Construct M', be the reflection of M about AB and the circle $k' = c(M', r)$. The points of intersection of k, k' are the points of intersection of the line AB and the circle k.



This construction cannot be done if M is on the line AB. In this case, extend and shorten AM by length r as described in Section 4.3. The end points of the extended and shortened segments are the intersections of k with AB.



Chapter 5
A Straightedge (with Something Extra) is Sufficient

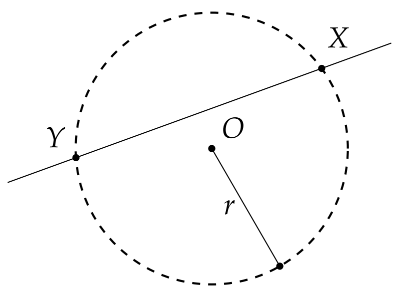
Can every construction with straightedge and compass be done with only a straightedge? The answer is no. In 1822 the French mathematician Jean-Victor Poncelet conjectured that a straightedge only is sufficient, provided that one circle exists in the plane. This theorem was proved in 1833 by the Swiss mathematician Jakob Steiner. In this chapter I present a proof of the theorem based on the one appearing as problem 34 in [3] as reworked by Michael Woltermann [4].[1]

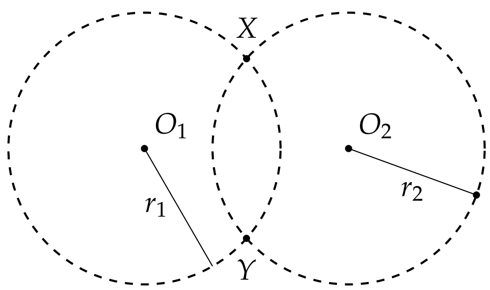
Every step of a construction with straightedge and compass can is one of the following three operations:

- Finding the point of intersection of two straight lines.
- Finding the point(s) of intersection of a straight line and a circle.
- Finding the point(s) of intersection of two circles.

It is clear that the first operation can be performed with a straightedge only. For the other two operations, we have to find an equivalent construction with straightedge alone.

What does it mean to perform a construction with straight-edge alone? A circle is defined by a point O, its center, and a line segment whose length is the radius r, one of whose endpoints is the center. If we can construct the points X, Y labeled in the following diagram, we can claim to have successfully constructed the intersection of a given circle with a given line and of two given circles. The circles drawn with dashed lines in the diagram do not actually appear in the construction. In this chapter, the single existing circle is drawn with a regular line, and the dashed circles are only used to help understand a construction and its proof.





First we present five necessary auxiliary constructions (Sections 5.1–5.5), and then show how to find the intersection of a line with a circle (Section 5.6) and of two circles (Section 5.7).

[1] I would like to thank him for giving me permission to use his work.

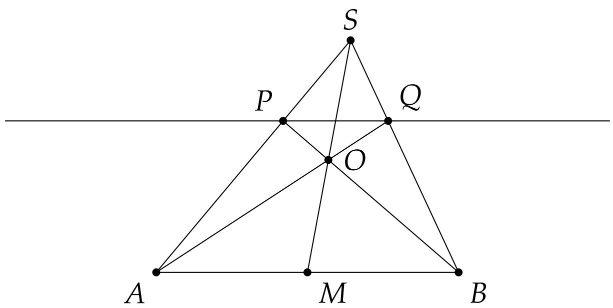
5.1 Constructing a line parallel to a given line

Given a line l defined by two points A, B and a point P not on the line, it is possible to construct a line through P that is parallel to AB.

There are two cases:

- A "directed line": The midpoint M of AB is given.
- Any other line.

Case 1, directed line: Construct a ray that extends AP and choose any point S on the ray beyond P. Construct the line BP, SM, SB. Label by O the intersection of BP with SM. Construct a ray that extends AO and label by Q the intersection of the ray AO with SB.



PQ **is parallel to** AB**.**

Proof: We will use Ceva's theorem that we proof later.

Theorem (Ceva): Given three line segments from the vertices of a triangle to the opposite edges that intersect in a point (as in the diagram, but M is not necessarily the midpoint of the side), the lengths of the segments satisfy:

$$\frac{AM}{MB} \cdot \frac{BQ}{QS} \cdot \frac{SP}{PA} = 1\,.$$

In the construction above, M is the midpoint of AB, so $\dfrac{AM}{MB} = 1$. The first factor in the multiplication cancels and we get the equation:

$$\frac{BQ}{QS} = \frac{PA}{SP} = \frac{AP}{PS}\,. \tag{5.1}$$

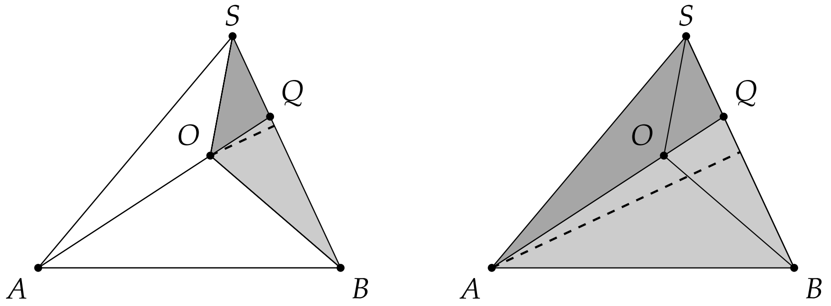
We will prove that $\triangle ABS \sim \triangle PQS$, so that PQ is parallel to AB because $\angle ABS = \angle PQS$). The proof is as follows:

$$\begin{aligned}
BS &= BQ + QS \\
\frac{BS}{QS} &= \frac{BQ}{QS} + \frac{QS}{QS} = \frac{BQ}{QS} + 1 \\
AS &= AP + PS
\end{aligned}$$

$$\frac{AS}{PS} = \frac{AP}{PS} + \frac{PS}{PS} = \frac{AP}{PS} + 1$$
$$\frac{BS}{QS} = \frac{BQ}{QS} + 1 = \frac{AP}{PS} + 1 = \frac{AS}{PS},$$

where the last equation is obtained from Equation 5.1.

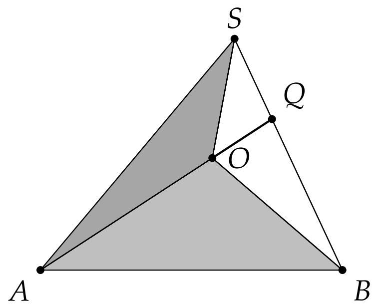
Proof of Ceva's theorem: Examine the following diagrams:



If the altitudes of two triangles are equal, their areas are proportional to the bases. In both diagrams, the altitudes of the gray triangles are equal, so:[2]

$$\frac{\triangle BQO}{\triangle SQO} = \frac{BQ}{QS}, \qquad \frac{\triangle BQA}{\triangle SQA} = \frac{BQ}{QS}.$$

By subtracting the areas of the indicated triangles, we get the proportion between the gray triangles:



$$\frac{BQ}{QS} = \frac{\triangle BQA - \triangle BQO}{\triangle SQA - \triangle SQO} = \frac{\triangle BOA}{\triangle SOA}.$$

This might look strange at first. We explain it using a simpler notation:

$$\frac{c}{d} = \frac{a}{b}$$
$$\frac{e}{f} = \frac{a}{b}$$
$$c - e = \frac{ad}{b} - \frac{af}{b} = \frac{a}{b}(d - f)$$
$$\frac{c-e}{d-f} = \frac{a}{b}.$$

[2]We use the name of a triangle as a shortcut for its area

Similarly, we can prove:

$$\frac{AM}{MB} = \frac{\triangle AOS}{\triangle BOS}\,, \qquad \frac{SP}{PA} = \frac{\triangle SOB}{\triangle AOB}\,,$$

so:

$$\frac{AM}{MB}\frac{BQ}{QS}\frac{SP}{PA} = \frac{\triangle AOS}{\triangle BOS}\frac{\triangle BOA}{\triangle SOA}\frac{\triangle SOB}{\triangle AOB} = 1\,,$$

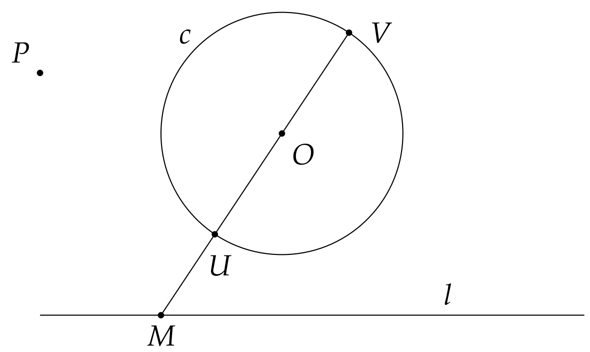
because the order of the vertices in a triangle makes no difference:

$$\triangle AOS = \triangle SOA,\ \triangle BOA = \triangle AOB,\ \triangle SOB = \triangle BOS\,.$$

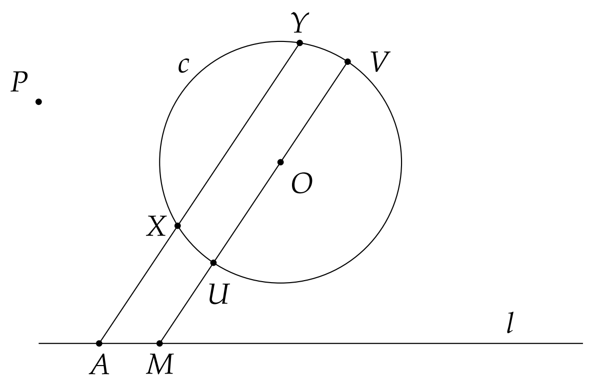
End of the proof of Ceva's theorem

Case 2, any other line: Label the line by l and the existing circle, which we will call the **fixed circle**, by c, where the center of c is O and its radius is r. P is the point not on the line through which it is required to construct a line parallel to l. Convince yourself that the construction does not depend on the location of the center of the circle or its radius.

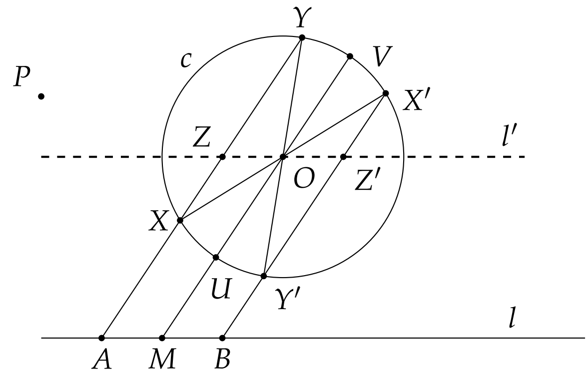
Choose M, any point on l, and construct a ray extending MO that intersects the circle at U, V.



This line is a **directed line** because O, the center of the circle, bisects the diameter UV. Choose a point A on l and use the construction for a directed line to construct a line parallel to UV, which intersects the circle at X, Y.



Construct the diameters XX' and YY'. Construct the ray from $X'Y'$ and label by B its intersection with l.

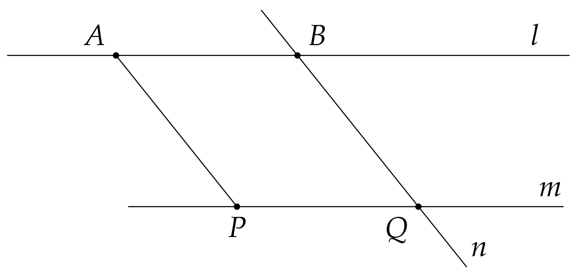


l is a directed line because M is the bisector of AB, so a line can be constructed through P parallel to l.

Proof: OX, OX', OY, OY' are all radii of the circle and $\angle XOY = \angle X'OY'$ since they are opposite angles. $\triangle XOY \cong \triangle X'OY'$ by SAS. Define (not construct!) l' to be a line through O parallel to l that intersects XY at Z and $X'Y'$ at Z'. $\angle XOZ = \angle X'OZ'$ because they are vertical angles, so $\triangle XOZ \cong \triangle X'OZ'$ by ASA and $ZO = OZ'$. $AMOZ$ and $BMOZ'$ are parallelograms (quadrilaterals with opposite sides parallel), so $AM = ZO = OZ' = MB$.

Corollary: Given a line segment AB and a point P not on the line. It is possible to construct a line through P that is parallel to AB and whose length is equal to the length of AB. In other words, it is possible to copy AB parallel to itself with P as one of its endpoints.

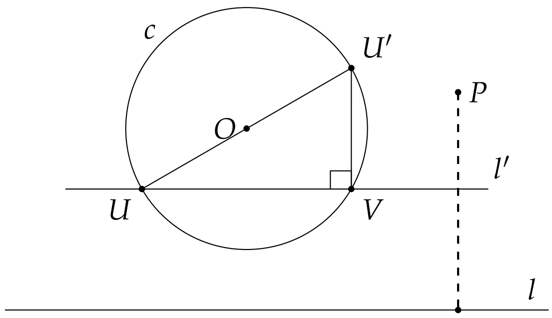
Proof: We have proved that it is possible to construct a line m through P parallel to AB and a line n through B to parallel to AP. The quadrilateral $ABQP$ is a parallelogram so opposite sides are equal $AB = PQ$.



5.2 Construction of a perpendicular to a given line

Construct a perpendicular line segment through a point P to a given line l (P is not on l).

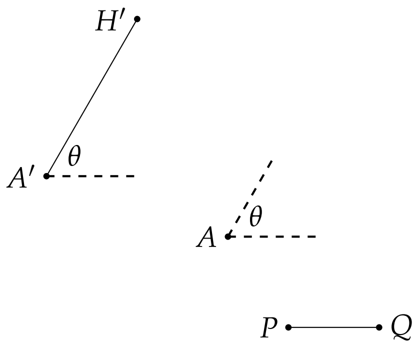
Construct (Section 5.1) a line l' parallel to l that intersects the **fixed circle** at U, V. Construct the diameter UOU' and chord VU'.



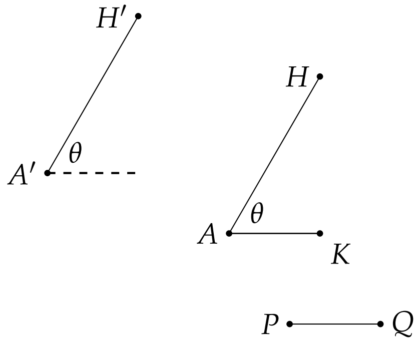
$\angle UVU'$ is an angle that subtends a semicircle so it is a right angle. Therefore, VU' is perpendicular to UV and l. Construct (Section 5.1) the parallel to VU' through P.

5.3 Copying a line segment in a given direction

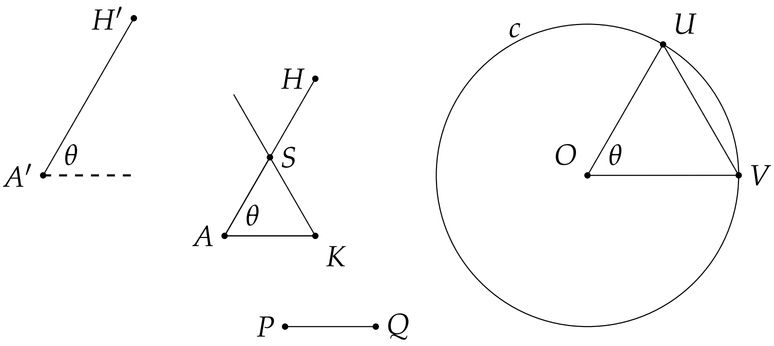
The corollary at the end of Section 5.1 shows that it is possible to copy a line segment parallel to itself. Here we show that it is possible to copy a line segment in the direction of another line. The meaning of "direction" is that the line defined by two points A', H' defines an angle θ relative to some axis. The task is to copy the PQ to AS so that AS will have the same angle θ relative to the same axis. In the diagram PQ is on the x-axis but that is of no importance.



By Section 5.1 it is possible to construct a line segment AH so that $AH \| A'H'$ and to construct a line segment AK so that $AK \| PQ$.



$\angle HAK = \theta$ so it remains to find a point S on AH so that $AS = AK$.



Construct two radii OU, OV of the fixed circle which are parallel to AH, AK, respectively, and construct a ray through K parallel to UV. Label its intersection with AH by S.

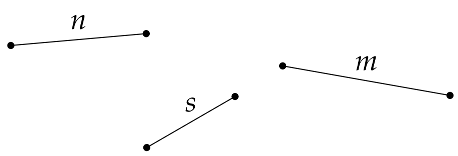
$AS = PQ$

Proof: By construction, $AH \| OU$ and $AK \| OV$, so $\angle SAK = \theta = \angle UOV$. $SK \| UV$ and $\triangle SAK \sim \triangle UOV$ by AAA, $\triangle UOV$ is isosceles, because OU, OV are radii of the same circle. Therefore, $\triangle SAK$ is isosceles and $AS = AK = PQ$.

5.4 Constructing a line segment whose length is relative to three other line segments

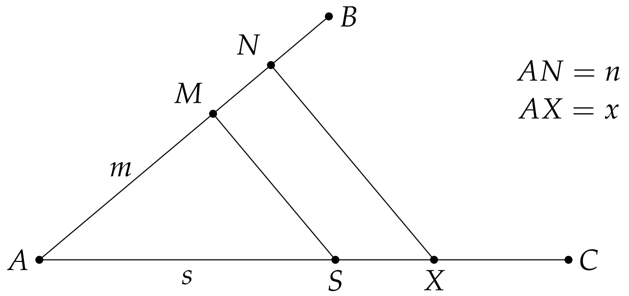
Given line segments of lengths n, m, s, it is possible to construct a segment of length $x = \frac{n}{m}s$.

The three line segments are located at arbitrary positions and directions in the plane.



Choose a point A and construct two rays AB, AC. By the construction in Section 5.3 it is possible to construct points M, N, S such that $AM = m$, $AN = n$, $AS = s$. Construct a line through N parallel to MS which intersects AC at X and label its length by x. $\triangle MAS \sim \triangle NAX$ by AAA, so:

$$\frac{m}{n} = \frac{s}{x} \qquad x = \frac{n}{m}s\,.$$



5.5 Constructing a square root

Given line segments of lengths a, b it is possible to construct a segment of length $\sqrt{ab}$.

We want to express $x = \sqrt{ab}$ in the form $\dfrac{n}{m}s$ in order to use the result of Section 5.4.

- For n we use d, the diameter of the fixed circle.
- For m we use $t = a + b$ which can be constructed from the given lengths a, b as shown in Section 5.3.
- We define $s = \sqrt{hk}$ where h, k are defined as expressions on the lengths a, b, t, d, and we will show how it is possible to construct a line segment of length $\sqrt{ab}$.

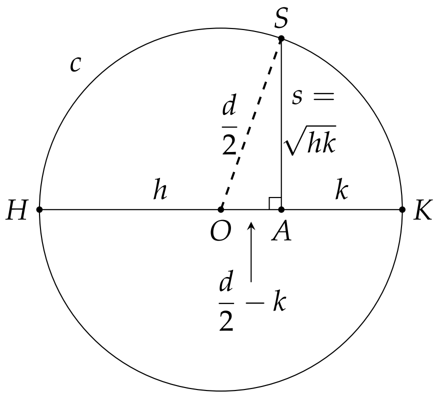
Define $h = \dfrac{d}{t}a$, $k = \dfrac{d}{t}b$ and and compute:

$$x = \sqrt{ab} = \sqrt{\frac{th}{d}\frac{tk}{d}} = \sqrt{\left(\frac{t}{d}\right)^2 hk} = \frac{t}{d}hk = \frac{t}{d}s\,.$$

We also compute:

$$h + k = \frac{d}{t}a + \frac{d}{t}b = \frac{d(a+b)}{t} = \frac{dt}{t} = d\,.$$

By Section 5.3 we can construct $HA = h$ on diameter HK of the fixed circle. From $h + k = d$ we have $AK = k$:



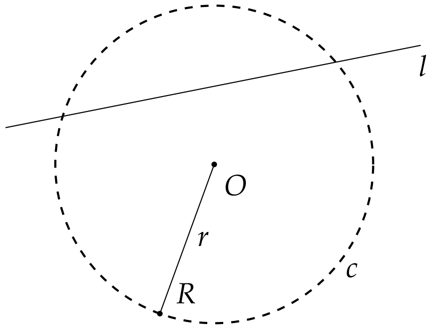
By Section 5.2 we can construct a perpendicular to HK at A and label the intersection of this line with the circle by S. $OS = OK = \dfrac{d}{2}$ because they are radii of the circle and $OA = \dfrac{d}{2} - k$. By Pythagoras's theorem:

$$\begin{aligned}
s^2 = SA^2 &= \left(\frac{d}{2}\right)^2 - \left(\frac{d}{2} - k\right)^2 \\
&= \left(\frac{d}{2}\right)^2 - \left(\frac{d}{2}\right)^2 + 2\frac{dk}{2} - k^2 \\
&= k(d-k) = kh \\
s &= SA = \sqrt{hk}\,.
\end{aligned}$$

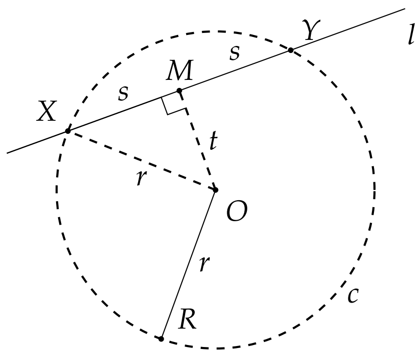
Now $x = \dfrac{t}{d}s$ can be constructed by Section 5.4.

5.6 Constructing the points of intersection of a line and a circle

Given a line l and a circle c centered on O with radius r. It is possible to construct the points of intersection of line with the circle.



By Section 5.2 it is possible to construct a perpendicular from the center of the circle O to the line l. Label the intersection of l with the perpendicular by M. M bisects the chord XY, where X, Y are the intersections of the line with the circle. $2s$ is the length of the chord XY¿ Note that s, X, Y in the diagram are just definitions: we haven't constructed them yet.



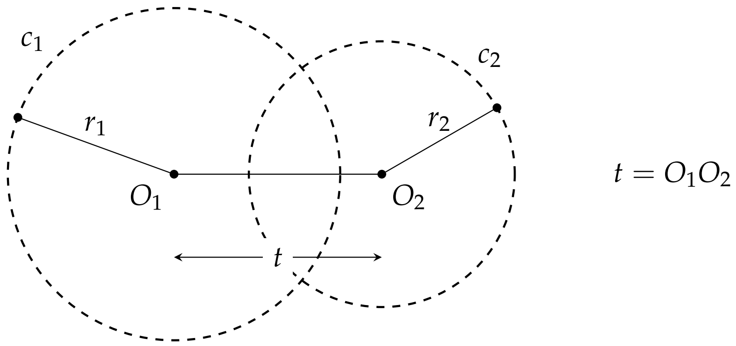
$\triangle OMX$ is a right triangle, so $s^2 = r^2 - t^2 = (r+t)(r-t)$. r is given as the radius of the circle and t is defined s the length of OM, the line segment between O and M. By Section 5.3 it is possible to construct a line segment of length t from O in the two directions OR and RO. The result is two line segments of length $r+t, r-t$.

Section 5.5 shows how to construct a line segment of length $s = \sqrt{(r+t)(r-t)}$. By Section 5.3 it is possible to construct line segments of length s from M along the given line l in both directions. The other endpoints of these segments are the points of intersection of l and c.

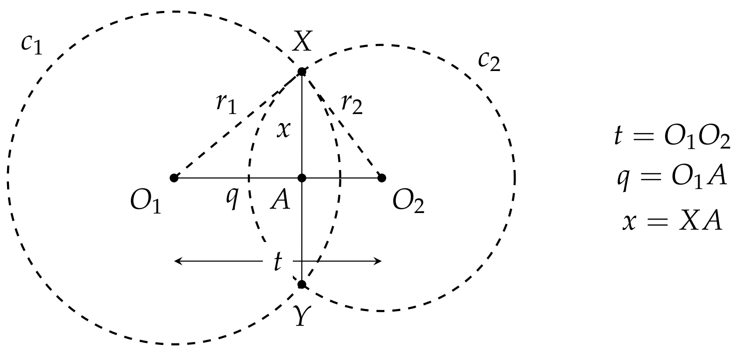
5.7 Constructing the points of intersection of two circles

Given two circles with centers O_1, O_2 with radii r_1, r_2, it is possible to construct their points of intersection X, Y.

With a straightedge it is possible to construct a line segment O_1O_2 that connects the two centers. Label its length t.



Label by A be the point of intersection of O_1O_2 and XY, and label the lengths $q = O_1A$, $x = XA$.



Note that we have not constructed A, but if we succeed in constructing the lengths q, x, by Section 5.3 we can construct A at length q from O_1 in the direction O_1O_2. By Section 5.2 perpendicular to O_1O_2 at A can be constructed and again by Section 5.3 it is possible to construct line segments of length x from A is both directions along the perpendicular. The other endpoints X, Y of the these sections are the points of intersection of the circles.

Constructing the length q: Define $d = \sqrt{r_1^2 + t^2}$, the hypotenuse of a right triangle. It can be constructed from r_1, t, the known lengths of the other two sides. On any line construct a line segment RS of length r_1, then a perpendicular to RS throught R, and finally, a line segment RT of length t through R on the perpendicular. The length of the hypotenuse ST is d. This right triangle can be constructed anywhere in the plane, not necessarily near the circles.

By the law of cosines for $\triangle O_1O_2X$:

$$\begin{aligned} r_2^2 &= r_1^2 + t^2 - 2r_1 t \cos \angle XO_1O_2 \\ &= r_1^2 + t^2 - 2tq \\ q &= \frac{(d + r_2)(d - r_2)}{2t} . \end{aligned}$$

By Section 5.3 these lengths can be constructed and by Section 5.4 q can be constructed from $d + r_2, d - r_2, 2t$.

Constructing the length x: $\triangle AO_1X$ is a right triangle, so $x = \sqrt{r_1^2 - q^2} = \sqrt{(r_1 + q)(r_1 - q)}$. By Section 5.3 $h = r_1 + q, k = r_1 - q$ can be constructed, as can $x = \sqrt{hk}$ by Section 5.5.

Chapter 6

Are Triangles with the Equal Area and Perimeter Congruent?

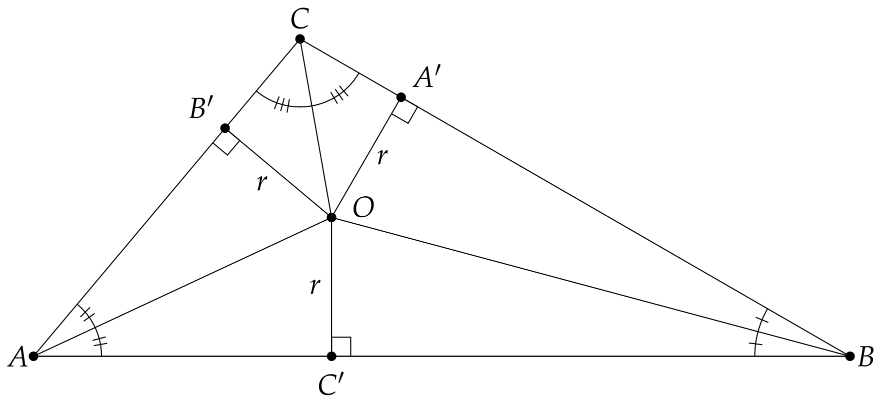
Are triangles with the equal area and equal perimeter congruent? Not necessarily: the triangles with sides $(17, 25, 28)$ and $(20, 21, 27)$ both have perimeter 70 and area 210. Barabash [1] shows that given a equilateral triangle, there are non-congruent triangles with the same area and perimeter; however, her proof is not constructive. This document (based on [2]) shows that given a triangle with rational sides, it is possible to construct a non-congruent triangle with *rational* sides and the same area and perimeter.

As a bonus, an elegant proof of Heron's formula is obtained.

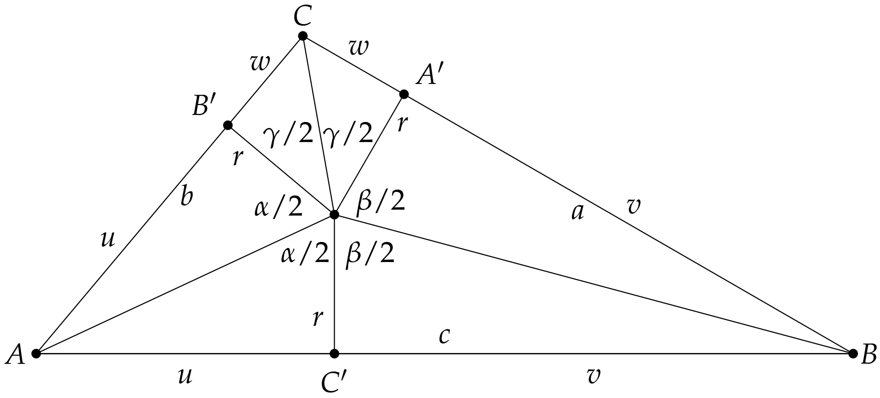
6.1 From triangles to elliptic curves

The following diagram shows O, the *incenter* of an arbitrary triangle $\triangle ABC$, which is the intersection of the bisectors of the three angles. Drop altitudes from O to the sides. The altitudes have length r, the raidus of the inscribed circle. The altitudes and angle bisectors create three pairs of congruent *right* triangles:

$$\triangle AOB' \cong \triangle AOC', \quad \triangle BOA' \cong \triangle BOC', \quad \triangle COA' \cong \triangle COB'.$$



The following diagram shows the sides a, b, c divided into segments u, v, w, and the angles $\alpha/2, \beta/2, \gamma/2$:



The area of $\triangle ABC$ is the sum of the areas of $\triangle AOC, \triangle BOC, \triangle AOB$:

$$A = \frac{1}{2}(w+v)r + \frac{1}{2}(v+u)r + \frac{1}{2}(u+w)r = \frac{1}{2} \cdot 2(u+v+w)r = rs\,, \tag{6.1}$$

where the semi-perimeter is: $s = u + v + w$. The lengths of u, v, w can be computed from the angles and r:

$$\tan\frac{\alpha}{2} = \frac{u}{r} \tag{6.2}$$

$$\tan\frac{\beta}{2} = \frac{v}{r} \tag{6.3}$$

$$\tan\frac{\gamma}{2} = \frac{w}{r}\,. \tag{6.4}$$

s can now be expressed in terms of the tangents:

$$s = u + v + w = r\tan\frac{\alpha}{2} + r\tan\frac{\beta}{2} + r\tan\frac{\gamma}{2} = r\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right)\,,$$

and by Equation 6.1 the area is:

$$A = rs = r^2\left(\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2}\right)\,. \tag{6.5}$$

From $A = rs$ we have $r = A/s$, so Equation 6.5 can be written as:

$$\tan\frac{\alpha}{2} + \tan\frac{\beta}{2} + \tan\frac{\gamma}{2} = \frac{A}{r^2} = \frac{A}{(A/s)^2} = \frac{s^2}{A}\,. \tag{6.6}$$

Since the sum of the angles α, β, γ is 2π:

$$\gamma/2 = \pi - (\alpha/2 + \beta/2) \tag{6.7}$$

$$\tan\gamma/2 = \tan(\pi - (\alpha/2 + \beta/2)) \tag{6.8}$$

$$= -\tan(\alpha/2 + \beta/2) \tag{6.9}$$

$$= \frac{\tan\alpha/2 + \tan\beta/2}{\tan\alpha/2\,\tan\beta/2 - 1}\,. \tag{6.10}$$

Here is proof of the formula for the tangent of the sum of two angles:

$$\tan(\theta+\phi) = \frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} \tag{6.11}$$

$$= \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi} \tag{6.12}$$

$$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}\,, \tag{6.13}$$

where Equation 6.13 is obtained by dividing by $\cos\theta\cos\phi$.

Let us simplify the notation by defining variables for the tangents:

$$\begin{aligned} x &= \tan\frac{\alpha}{2} \\ y &= \tan\frac{\beta}{2} \\ z &= \tan\frac{\gamma}{2}\,. \end{aligned}$$

By Equation 6.10 we can replace $z = \tan\gamma/2$ by an expression in x, y:

$$z = \frac{x+y}{xy-1}\,. \tag{6.14}$$

With this notation, Equation 6.6 becomes:

$$x + y + \frac{x+y}{xy-1} = \frac{s^2}{A}\,. \tag{6.15}$$

Given fixed values of A and s, are there multiple solutions to Equation 6.15?

For the right triangle $(3, 4, 5)$:

$$\frac{s^2}{A} = \frac{\left(\frac{1}{2}(3+4+5)\right)^2}{\frac{1}{2}\cdot 3\cdot 4} = \frac{6^2}{6} = 6\,. \tag{6.16}$$

If there is another solution Equation 6.15, it can be written as:

$$x^2y + xy^2 - 6xy + 6 = 0\,. \tag{6.17}$$

This is an equation for an *elliptic curve*. Elliptic curves were used by Andrew Wiles' in his proof of Fermat's last theorem. They have also been used in public-key cryptography.

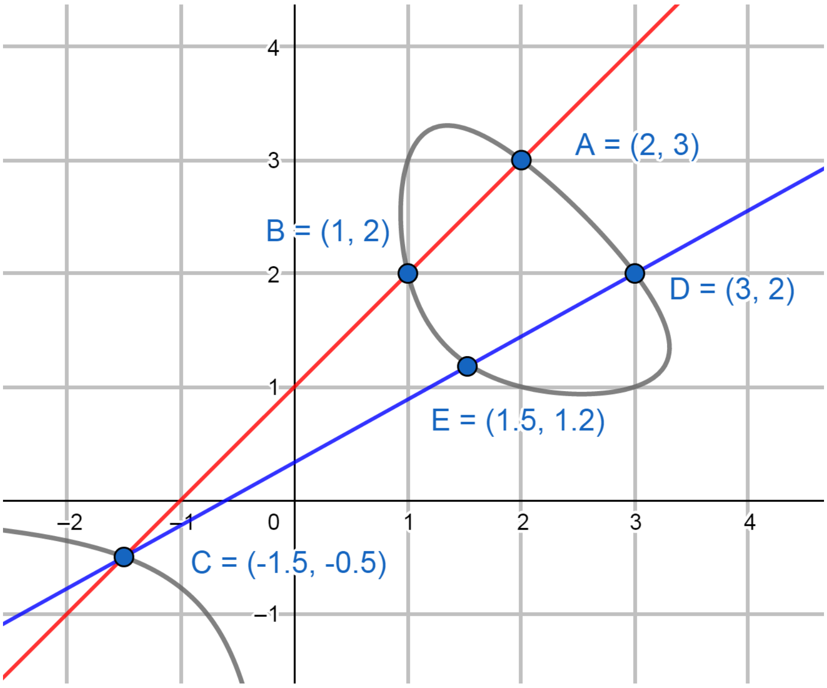
6.2 Solving the equation for the elliptic curve

A portion of the graph of Equation 6.17 is shown in the diagram below. Any point on the closed curve in the first quadrant is a solution to the equation because the lengths of the sides of the triangle must be positive. A, B, D correspond to the triangle $(3, 4, 5)$ as we shall show below. To find additional (*rational*) solutions, the *method of two secants* is used.[1]

Draw a secant through the points $A = (2, 3)$ and $B = (1, 2)$. It intersects the curve at $C = (-1.5, -0.5)$, but this does not give a solution because the values are negative. If we draw a second secant from C to $D = (3, 2)$, the intersection with the curve at E does give a new solution.[2]

[1]McCallum [2] notes that there are an infinite number of rational solutions.

[2]$(1.5, 1.2)$ is an approximation displayed by GeoGebra. We will compute the exact coordinates of E below.



The equation of the (red) line through A, B is $y = x + 1$. Substitute for y in Equation 6.17:

$$x^2(x+1) + x(x+1)^2 - 6x(x+1) + 6 = 0$$

and simplify:

$$2x^3 - 3x^2 - 5x + 6 = 0\,.$$

From A, B, we know two roots $x = 2, x = 1$, so we can factor the cubic polynomial as:

$$(x-2)(x-1)(ax+b) = 0\,,$$

where only the third root is unknown. Multiply the factors and we immediately see that a, the coefficient of the cubic term x^3, must be 2, and $2b$, the constant term, must be 6. Therefore, the third factor is $2x + 3$ which gives the third root $x = -\frac{3}{2}$ and $y = x + 1 = -\dfrac{1}{2}$. This is the point $C = (-\dfrac{3}{2}, -\dfrac{1}{2})$ in the graph.

The equation of the second secant (blue) is:

$$y = \frac{5}{9}x + \frac{1}{3}\,. \tag{6.18}$$

Substitute for y in Equation 6.17:

$$x^2\left(\frac{5}{9}x + \frac{1}{3}\right) + x\left(\frac{5}{9}x + \frac{1}{3}\right)^2 - 6x\left(\frac{5}{9}x + \frac{1}{3}\right) + 6 = 0\,,$$

and simplify:

$$\frac{70}{81}x^3 - \frac{71}{27}x^2 - \frac{17}{9}x + 6 = 0\,.$$

Again, we have two roots $x = 3, x = -\frac{3}{2}$, so we can factor the cubic polynomial as:

$$(x-3)(x+\frac{3}{2})(ax+b) = 0\,.$$

Equating the coefficient of the cubic term and equating the constant term give:

$$\frac{70}{81}x - \frac{4}{3} = 0\,,$$

so:

$$x = \frac{81}{70}\cdot\frac{4}{3} = \frac{27\cdot 4}{70} = \frac{54}{35}\,.$$

y can be computed from Equation 6.18 and the coordinates of E are:

$$\left(\frac{54}{35}, \frac{25}{21}\right)\,.$$

Finally, compute z from Equation 6.14:

$$z = \frac{x+y}{xy-1} = \left(\frac{54}{35}+\frac{25}{21}\right)\,/\,\left(\frac{54}{35}\frac{25}{21}-1\right) = \frac{2009}{615} = \frac{49}{15}\,.$$

6.3 From solutions to the elliptic curve to triangles

From x, y, z, a, b, c, the sides of the triangle $\triangle ABC$ can be computed:

$$\begin{aligned}
a &= w+v = r(z+y) = (z+y)\\
b &= u+w = r(x+z) = (x+z)\\
c &= u+v = r(x+y) = (x+y)\,,
\end{aligned}$$

since $r = \frac{A}{s} = \frac{6}{6} = 1$.

For the solution $A = (2,3)$ of the elliptic curve, the value of z is:

$$z = \frac{x+y}{xy-1} = \frac{2+3}{2\cdot 3-1} = 1\,,$$

and the sides of the triangle are:

$$\begin{aligned}
a &= z+y = 1+3 = 4\\
b &= x+z = 2+1 = 3\\
c &= x+y = 2+3 = 5\,,
\end{aligned}$$

the right triangle with $s = A = 6$. Computing the sides corresponding to B and D gives the same triangle.

For E:

$$\begin{aligned}
a &= z+y=\frac{49}{15}+\frac{25}{21}=\frac{156}{35}\\
b &= x+z=\frac{54}{35}+\frac{49}{15}=\frac{101}{21}\\
c &= x+y=\frac{54}{35}+\frac{25}{21}=\frac{41}{15}\,.
\end{aligned}$$

Let us check the result. The semi-perimeter is:

$$s=\frac{1}{2}\left(\frac{156}{35}+\frac{101}{21}+\frac{41}{15}\right)=\frac{1}{2}\left(\frac{468+505+287}{105}\right)=\frac{1}{2}\left(\frac{1260}{105}\right)=6\,,$$

and the area can be computed using Heron's formula:

$$\begin{aligned}
A &= \sqrt{s(s-a)(s-b)(s-c)}\\
&= \sqrt{6\left(6-\frac{156}{35}\right)\left(6-\frac{101}{21}\right)\left(6-\frac{41}{15}\right)}\\
&= \sqrt{6\cdot\frac{54}{35}\cdot\frac{25}{21}\cdot\frac{49}{15}}\\
&= \sqrt{\frac{396900}{11025}}\\
&= \sqrt{36}=6\,.
\end{aligned}$$

6.4 A proof of Heron's formula

The triple tangent formula states that if $\phi+\theta+\psi=\pi$ then:

$$\tan\phi+\tan\theta+\tan\psi=\tan\phi\tan\theta\tan\psi\,. \tag{6.19}$$

The proof follows immediately from Equation 6.13:

$$\begin{aligned}
\tan\psi &= \tan(\pi-(\phi+\theta))\\
&= -\tan(\phi+\theta)\\
&= \frac{\tan\phi+\tan\theta}{\tan\phi\tan\theta-1}\\
\tan\phi\tan\theta\tan\psi-\tan\psi &= \tan\phi+\tan\theta\\
\tan\phi\tan\theta\tan\psi &= \tan\phi+\tan\theta+\tan\psi\,.
\end{aligned}$$

From Equations 6.2–6.5, and $r=A/s$:

$$\begin{aligned}
A &= r^2\left(\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}\right)\\
&= r^2\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= r^2 \left(\frac{u}{r}\frac{v}{r}\frac{w}{r}\right) \\
&= \frac{u\,v\,w}{r} \\
&= \frac{s}{A}\,u\,v\,w \\
A^2 &= s\,u\,v\,w\,.
\end{aligned}
$$

$s = u + v + w$, so:

$$
\begin{aligned}
s - a &= (u + v + w) - (w + v) = u \\
s - b &= (u + v + w) - (u + w) = v \\
s - c &= (u + v + w) - (u + v) = w\,,
\end{aligned}
$$

and Heron's formula follows:

$$
\begin{aligned}
A^2 &= s\,u\,v\,w \\
&= s(s-a)(s-b)(s-c) \\
A &= \sqrt{s(s-a)(s-b)(s-c)}\,.
\end{aligned}
$$

Bibliography

[1] Marita Barabash. A non-visual counterexample in elementary geometry. *The College Mathematics Journal*, 36(5), 2005.

[2] Benjamin Bold. *Famous Problems of Mathematics: A History of Constructions with Straight Edge and Compass*. Van Nostrand, 1969.

[3] Heinrich Dörrie. *100 Problems of Elementary Mathematics: Their History and Solution*. Dover, 1965.

[4] Heinrich Dörrie. 100 problems of elementary mathematics: Their history and solution. Newly reworked by Michael Woltermann. `http://www2.washjeff.edu/users/mwoltermann/Dorrie/DorrieContents.htm`, 2010.

[5] Norbert Hungerbühler. A short elementary proof of the Mohr-Mascheroni theorem. *American Mathematical Monthly*, 101(8):784–787, 1994.

[6] Clare Ziskin Moshe Stopel, editor. *Geometric Constructions: Classic, Challenging and Computer Problems*. Shaanan, 2015. (in Hebrew).

[7] Timothy Peil. The rusty compass theorem. `http://web.mnstate.edu/peil/geometry/C2EuclidNonEuclid/1Compass.htm`.

[8] Ramanujan. Squaring the circle. *Journal of the Indian Mathematical Society*, page 13, 1913. `http://ramanujan.sirinudi.org/Volumes/published/ram05.pdf`.

[9] Ramanujan. Modular equations and approximations to π. *Quarterly Journal Mathematics*, XLV:350–372, 1914.

[10] Godfried Toussaint. A new look at Euclid's second proposition. *The Mathematical Intelligencer*, 15(3):12–23, 1993.

[11] Edward C. Wallace and Stephen F. West. *Roads to Geometry (Third Edition)*. Pearson, 2003.