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习题1.1

(A)

1.

$$\text{依题意 } \begin{cases} a+2b=1 \\ a-b=2 \end{cases}, \text{ 则 } a = \frac{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{5}{3}, \quad b = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}} = -\frac{1}{3}.$$

2.

$$(1) \quad A + 3B = \begin{pmatrix} 6 & -1 \\ 3 & 0 \\ 2 & 3 \end{pmatrix} + 3 \begin{pmatrix} 1 & -1 \\ 0 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 9 & -4 \\ 3 & 6 \\ 17 & 12 \end{pmatrix}.$$

$$(2) \quad A^T = \begin{pmatrix} 6 & 3 & 2 \\ -1 & 0 & 3 \end{pmatrix}, B^T = \begin{pmatrix} 1 & 0 & 5 \\ -1 & 2 & 3 \end{pmatrix}, \text{ 则 } A^T - 2B^T = \begin{pmatrix} 4 & 3 & -8 \\ 1 & -4 & -3 \end{pmatrix}.$$

3.

$$\text{由 } \begin{pmatrix} 2 & 3 & -1 \\ -1 & 4 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 \\ 7 & 3 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 16 & 6 \\ 31 & 11 \end{pmatrix}, \text{ 故有 } \begin{cases} y_1 = 16t_1 + 6t_2 \\ y_2 = 31t_1 + 11t_2 \end{cases}.$$

4.

$$\begin{cases} x_1 = 0 \\ y_1 = y \end{cases}, \text{ 几何意义为在 } y \text{ 轴上的投影.}$$

5.

$$(1) \quad \begin{pmatrix} 3 & 2 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 2 & -2 \\ -2 & -10 & 6 \end{pmatrix}.$$

$$(2) \quad \begin{pmatrix} 6 & 0 & 8 & -3 \end{pmatrix} \begin{pmatrix} 0.5 \\ -2 \\ 2.5 \\ -1 \end{pmatrix} = 6 \times 0.5 + 8 \times 2.5 + 3 = 26$$

$$(3) \begin{pmatrix} 0.5 \\ -2 \\ 2.5 \\ 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 8 & -3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 4 & -1.5 \\ -12 & 0 & -16 & 6 \\ 15 & 0 & 20 & -7.5 \\ 6 & 0 & -8 & 3 \end{pmatrix}$$

$$(4) \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 3 & -1 \end{pmatrix} \begin{pmatrix} 3 & -1 & 0 \\ 0 & 4 & -1 \\ 3 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -8 & 1 \\ 0 & 8 & -1 \\ -6 & 16 & -3 \end{pmatrix}$$

$$(5) \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 3 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + x_2^2 - x_3^2 - 2x_1x_2 + 2x_1x_3 + 6x_2x_3.$$

$$(6) \begin{pmatrix} 3 & 2 & 1 \\ -1 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + 2x_2 + x_3 \\ -x_1 - 2x_2 - 3x_3 \end{pmatrix}.$$

6.

$$AB = \begin{pmatrix} 5 & -1 & -1 \\ 4 & 8 & 2 \\ -3 & -1 & 11 \end{pmatrix}, \quad BA = \begin{pmatrix} 2 & 1 & -3 \\ 2 & 11 & -1 \\ 8 & -1 & 11 \end{pmatrix}, \quad AB - BA = \begin{pmatrix} 3 & -2 & 2 \\ 2 & -3 & 3 \\ -11 & 0 & 0 \end{pmatrix},$$

此题说明 $AB \neq BA$.

7.

性质2 证明: 设 $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{n \times p}$, $C = (c_{ij})_{p \times q}$, 则

$$(A+B)C = (a_{ij} + b_{ij})_{m \times n} (c_{ij})_{n \times q} = \left(\sum_{k=1}^n (a_{ik} + b_{ik}) c_{kj} \right)_{m \times q} = AC + BC.$$

同理可证 $C(A+B) = CA + CB$.性质3 证明: 设 $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{n \times p}$, 则

$$\lambda(AB) = \lambda \left(\sum_{k=1}^n a_{ik} b_{kj} \right)_{m \times p} = \left(\sum_{k=1}^n (\lambda a_{ik}) b_{kj} \right)_{m \times p} = (\lambda A)B = A(\lambda B).$$

性质4 证明: 设 $A = (a_{ij})_{m \times n}$, 则

$$E_m A_{m \times n} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

其中元素 $c_{ij} = 1 \cdot a_{ij} = a_{ij}$, 故 $E_m A_{m \times n} = A_{m \times n}$, 其余同理可证.

8.

$$(1) A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix}.$$

$$(2) A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

$$(3) A = \begin{pmatrix} -1 & 1 \\ -2 & 2 \end{pmatrix}.$$

9.

$$(1) \text{ 记 } A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\text{则 } A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -E.$$

$$\text{当 } n = 2k \text{ 时, } A^n = A^{2k} = (-E)^k = (-1)^k E;$$

$$\text{当 } n = 2k + 1 \text{ 时, } A^n = A^{2k+1} = (-E)^k A = (-1)^k A.$$

$$(2) A^2 = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}^2 = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix}.$$

$$\text{假设 } n = k \text{ 时, } A^k = \begin{pmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{pmatrix},$$

则当 $n = k + 1$ 时,

$$A^{k+1} = \begin{pmatrix} \cos k\alpha & \sin k\alpha \\ -\sin k\alpha & \cos k\alpha \end{pmatrix} \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} = \begin{pmatrix} \cos(k+1)\alpha & \sin(k+1)\alpha \\ -\sin(k+1)\alpha & \cos(k+1)\alpha \end{pmatrix}.$$

$$(3) A = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ 又 } \alpha^T \alpha = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 2,$$

$$\text{则 } A^n = (\alpha \alpha^T)^n = \alpha (\alpha^T \alpha) (\alpha^T \alpha) \cdots (\alpha^T \alpha) \alpha^T = 2^{n-1} \alpha \alpha^T = 2^{n-1} A = 2^{n-1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

10.

证明: 令 $A = (a_{ij})_{m \times n}$, 则 $a_{ij} \in \mathbf{R}$, $A^T = A$, 由 $A^2 = 0, \Rightarrow AA^T = 0$

$$\text{即 } (a_{ij})_{m \times n} (a_{ji})_{n \times m} = \left(\sum_{k=1}^n a_{ik} a_{ki} \right)_{m \times m} = 0.$$

$$\text{记 } c_{ij} = \sum_{k=1}^n a_{ik} a_{ki},$$

$$\text{考虑 } c_{ii} = \sum_{k=1}^n a_{ik}^2 = 0 \Rightarrow a_{ik} = 0,$$

故 $\forall i, j$, s.t. $a_{ij} = 0$. 故 $A = 0$.

11.

$$f(A) = A^2 + A + E = \begin{pmatrix} 8 & -1 & -1 \\ 11 & 8 & 2 \\ 8 & -1 & 11 \end{pmatrix} + \begin{pmatrix} 3 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & -1 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 12 & -2 & 0 \\ 13 & 10 & 5 \\ 9 & 2 & 11 \end{pmatrix},$$

$$f(B) = B^n + B + E = \begin{pmatrix} 3^n & & \\ & 2^n & \\ & & (-1)^n \end{pmatrix} + \begin{pmatrix} 3 & & \\ & 2 & \\ & & -1 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 3^n + 4 & & \\ & 2^n + 3 & \\ & & (-1)^n \end{pmatrix}.$$

12.

证明: 依题意, $A^H = A$, 即 $(\overline{A})^T = A$, 即 $(\overline{a_{ji}})_{n \times n} = (a_{ij})_{n \times n}$,

则 $a_{ij} = \overline{a_{ji}}$.

又当 $i = j$ 时, $a_{ii} = \overline{a_{ii}}$, 则 a_{ii} 是实数.

(B)

1.

与 A 可交换的 B 必是二阶方阵, 设 $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\text{则 } AB = BA \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

$$\Rightarrow \begin{cases} a+c=a \\ b+d=a+b \\ c+d=d \end{cases} \Rightarrow \begin{cases} c=0 \\ a=d \end{cases}$$

故 $B = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, a, b 为实数.

2.

证明: $\because A, B$ 可换, 即 $AB = BA$, 又 A, B 满足

$$(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2$$

设当 $n = k$ 时, $(A+B)^n = \sum_{i=0}^n C_n^i A^{n-i} B^i$,

则当 $n = k+1$ 时,

$$\begin{aligned} (A+B)^{n+1} &= \left(\sum_{i=0}^n C_n^i A^{n-i} B^i \right) (A+B) \\ &= C_n^0 A^{n+1} + (C_n^1 + C_n^1) A^n B + \cdots + (C_n^{r-1} + C_n^r) A^{n-r+1} B^r + \cdots \\ &= \sum_{i=0}^{n+1} C_{n+1}^i A^{n-i} B^i \end{aligned}$$

证毕.

3.

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{pmatrix} = \lambda E + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{记 } B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ 则 } B^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B^3 = 0 \ (n \geq 3),$$

$$\text{则 } A^n = (\lambda E + B)^n = \lambda^n E + n\lambda^{n-1}B + \frac{n(n-1)}{2}\lambda^{n-2}B^2 = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ & \lambda^n & n\lambda^{n-1} \\ & & \lambda^n \end{pmatrix},$$

$$\begin{aligned} \text{故 } P_n(A) &= \begin{pmatrix} \lambda^n + \lambda + 1 & n\lambda^{n-1} + 1 & \frac{n(n-1)}{2}\lambda^{n-2} \\ & \lambda^n + \lambda + 1 & n\lambda^{n-1} + 1 \\ & & \lambda^n + \lambda + 1 \end{pmatrix}, & P_5(A) &= \\ & \begin{pmatrix} \lambda^5 + \lambda + 1 & 5\lambda^4 + 1 & 10\lambda^3 \\ & \lambda^5 + \lambda + 1 & 5\lambda^4 + 1 \\ & & \lambda^5 + \lambda + 1 \end{pmatrix}. \end{aligned}$$

4.

$$\text{证明: 令 } A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ b_{21} & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

$$\text{记 } C = AB, \text{ 则 } c_{ij} = \sum_{k=1}^n a_{ik}b_{kj},$$

由假设, 当 $i < j$ 时 $a_{ij} = b_{ij} = 0$, 故 c_{ij} 中各项因子都含 0, 则 $c_{ij} = 0$ ($i < j$), 故 C 为下三角矩阵.

5.

$$\text{证明: 令 } A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & a_{(n-1)n} \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} & 0 & \cdots & 0 \\ b_{21} & b_{22} & b_{23} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_{n1} & b_{n2} & \cdots & \cdots & b_{nn} \end{pmatrix},$$

$$\text{记 } C = AB, \text{ 则 } c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj}.$$

当 $j > i + 1$ 时, $a_{ik} = b_{kj} = 0$, 则 c_{ij} 中各项都含因子 0, 所以 $c_{ij} = 0$ ($j > i + 1$).

故 n 阶下 Hessenberg 矩阵乘积仍是下 Hessenberg 矩阵.

同理可证 n 阶上 Hessenberg 矩阵乘积仍是上 Hessenberg 矩阵.

6.

(1) 由 $(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T$, 知 $A + A^T$ 为对称矩阵.

由 $(A - A^T)^T = A^T + (-A^T)^T = A^T - A = -(A - A^T)$, 知 $A - A^T$ 为反对称矩阵.

(2) 由 (1) 知 $A = \frac{1}{2} [(A + A^T) + (A - A^T)]$.

7.

$$\text{记 } A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}, A_1 = \begin{pmatrix} -1 & -4 \\ 1 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}, A_3 = \begin{pmatrix} -1 & 4 \\ -1 & -1 \end{pmatrix},$$

$$\text{则 } A = \frac{1}{3} A_1 A_2 A_3, \frac{1}{3} A_3 A_1 = E,$$

$$\begin{aligned}
 \text{则 } A^{11} &= \frac{1}{3}A_1A_2A_3 \cdot \frac{1}{3}A_1A_2A_3 \cdots A_3 \frac{1}{3}A_1A_2A_3 \\
 &= \frac{1}{3}A_1(A_2)^{11}A_3 \\
 &= \frac{1}{3} \begin{pmatrix} -1 & -4 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & \\ & 2^{11} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix} \\
 &= \frac{1}{3} \begin{pmatrix} 1 & -2^{13} \\ -1 & 2^{11} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix} \\
 &= \frac{1}{3} \begin{pmatrix} 1+2^{13} & 4+2^{13} \\ -1-2^{11} & -4-2^{11} \end{pmatrix}
 \end{aligned}$$

(C)

1.

证明: 记 $C = AB = (c_{ij})_{m \times m}$, $D = BA = (d_{ij})_{n \times n}$, 则 $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$, $d_{ij} = \sum_{k=1}^m b_{ik}a_{kj}$, 则

$$\text{tr}(AB) = \sum_{i=1}^m c_{ii} = \sum_{i=1}^m \sum_{k=1}^n a_{ik}b_{ki} = \sum_{k=1}^n \sum_{i=1}^m b_{ki}a_{ik} = \sum_{k=1}^n d_{kk} = \text{tr}(BA).$$

2.

证明: 因为 A 与任意 n 阶方阵可交换, 取 $B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, 则考虑 AB 与 BA 的第一行第二列

元素有 $a_{11} = a_{22}$. 同理依次取 B 为 $\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, 可

以得到 $a_{11} = a_{22} = \cdots = a_{nn}$. 又取 $B = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$, 可得第一行中除 a_{11} 外均为 0. 依次取 B

为 $\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$, 可得除主对角线外均为 0. 故 A 为 n 阶数量矩阵.

3.

证明: 用反证法. 假设 $\exists a_{ij} \neq 0$, 则考虑 $C = A^2$, $c_{ii} = \sum_{k=1}^n a_{ik}a_{ki} = a_{i1}a_{1i} + a_{i2}a_{2i} + \cdots + a_{ii}^2 + a_{i,i+1}a_{i+1,i} + \cdots + a_{in}a_{ni}$. 因为 A 为上三角矩阵, 则当 $i > j$ 时 $a_{ij} = 0$, 故 $c_{ii} = a_{ii}^2 > 0$, 则不可能 $\exists k$, 使 $A^k = 0$. 故矛盾. 则 A 的主对角元素全为 0.

习题1.2

(A)

1.

$$\begin{aligned}
 AB &= \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 3 & 2 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right) \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ \hline 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{array} \right) \quad BA = \left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ \hline 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{array} \right) \left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 3 & 2 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array} \right) \\
 &= \left(\begin{array}{cccc} 6 & 0 & 3 & 0 \\ 0 & 6 & 0 & 3 \\ \hline 6 & 3 & 0 & 0 \\ -9 & 3 & 0 & 0 \end{array} \right) \quad AB = \left(\begin{array}{cc|cc} 5 & 0 & 2 & 1 \\ 0 & 5 & -3 & 1 \\ \hline 10 & -1 & 1 & 0 \\ 3 & 11 & 0 & 1 \end{array} \right) \\
 BA &= \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & 2 \\ -4 & 4 & -1 & 0 \\ -12 & -8 & 0 & -1 \end{pmatrix}. \quad A\beta = (\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4) \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix} = \alpha_1 + 2\alpha_2 - \alpha_4.
 \end{aligned}$$

2.

$$\begin{aligned}
 A^4 &= \begin{pmatrix} 4 & 3 & & \\ -3 & 1 & & \\ & & 3 & 0 \\ & & 3 & 3 \end{pmatrix}^4 = \begin{pmatrix} \begin{pmatrix} 4 & 3 \\ -3 & 1 \end{pmatrix}^4 & & \\ & \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}^4 & \\ & & & \end{pmatrix} = \begin{pmatrix} -176 & -15 & & \\ 15 & -161 & & \\ & & 81 & 0 \\ & & 324 & 81 \end{pmatrix}, \quad B^2 = \\
 \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} &= \begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{假设 } n = k \text{ 时, } B^k = \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{则当 } n = k + 1 \text{ 时,} \\
 B^{k+1} &= \begin{pmatrix} 2^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

$$\text{故 } B^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

3.

$$AB^T = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -7 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 6 \\ -2 & \frac{4}{3} & -3 \\ 4 & 20 & 0 \\ 0 & 5 & -2 \\ 0 & -1 & 3 \end{pmatrix}$$

4.

证明:

$$MM^T = \begin{pmatrix} E_m & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & A^T \end{pmatrix} = \begin{pmatrix} E_m & 0 \\ 0 & AA^T \end{pmatrix} = E$$

$$M^T M = \begin{pmatrix} E_m & 0 \\ 0 & A^T \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} E_m & 0 \\ 0 & A^T A \end{pmatrix} = E$$

故 $\begin{pmatrix} E_m & 0 \\ 0 & A \end{pmatrix}$ 为正交矩阵.

(B)

1.

(1) 证明: (1) $A = \begin{pmatrix} O & E_{n-1} \\ I & O \end{pmatrix}$ 则 $k=1$ 时成立

$$\text{假设 } k=m \text{ 时 } A^m = \begin{pmatrix} O & E_{n-m} \\ E_m & O \end{pmatrix}$$

$$\text{则 } A^{m+1} = \begin{pmatrix} O & E_{n-m} \\ E_m & O \end{pmatrix} \begin{pmatrix} O & E_{n-1} \\ I & O \end{pmatrix}$$

$$\text{记 } A_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{m \times m} \quad A_2 = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{m \times (n-m)}$$

$$A_3 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{(n-m) \times m} \quad A_4 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{(n-m) \times (n-m)}$$

$$\text{则 } A^{m+1} = \begin{pmatrix} O & E_{n-m} \\ E_m & O \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} = \begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} O & E_{n-m-1} \\ E_{m+1} & O \end{pmatrix}$$

$$\text{故由数学归纳法知 } A^k = \begin{pmatrix} O & E_{n-k} \\ E_k & O \end{pmatrix}, k = 1, 2, \dots, n-1$$

(2) 在(1)中使用数学归纳法时并没有严格限制 $k < n$, 故当 $k = n$ 时 $A^n = A^{n-1} \cdot A = E$.

习题1.3

(A)

1.

$$(1) \text{ 由于 } \left(\begin{array}{cc|c} 3 & -1 & 1 \\ -2 & 1 & 1 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & -\frac{1}{3} & \frac{1}{3} \\ -1 & \frac{1}{2} & \frac{1}{2} \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & -\frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{6} & \frac{1}{3} \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array} \right),$$

$$\text{故 } \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}.$$

$$(2) \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}_3^{-1} = \begin{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^{-1} \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^* \\ \frac{1}{3} \end{pmatrix} =$$

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot \frac{1}{3}.$$

$$(3) \begin{pmatrix} 5 & 2 \\ 2 & 1 \\ & 1 & 8 \\ & 1 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} \left(\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \right)^{-1} & \\ & \begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}^{-1} \end{pmatrix} = \begin{pmatrix} \left(\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix} \right)^* & \\ & \begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}^* \end{pmatrix} =$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 5 \\ & 9 & -8 \\ & -1 & 1 \end{pmatrix}.$$

$$(4) \begin{pmatrix} 2 & \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}.$$

2.

$$(1) \text{ 由 } \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 3 & 4 & -2 & 2 \\ 5 & -4 & 1 & 3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & -14 & 6 & -2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & 5 \end{array} \right) \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -5 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array} \right), \text{ 故 } x = \begin{pmatrix} 0 \\ -2 \\ -5 \end{pmatrix}.$$

$$(2) X = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & -2 \\ 5 & -4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ -\frac{13}{2} & 3 & -\frac{1}{2} \\ -16 & 7 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} =$$

$$\begin{pmatrix} 5 & 2 \\ \frac{31}{2} & \frac{11}{2} \\ 37 & 13 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -11 \\ \frac{51}{2} & -\frac{71}{2} \\ 61 & -85 \end{pmatrix}.$$

3.

由 $|M| = \begin{vmatrix} 0 & A \\ C & B \end{vmatrix} = (-1)^{nm}|A| \cdot |C|$, $\because A, C$ 可逆, 则 $|A| \neq 0, |C| \neq 0$, 故 $|M| \neq 0$, 则 M 可逆. 设

$$M^{-1} = \begin{pmatrix} X & Y \\ Z & H \end{pmatrix}, \text{ 则 } M \cdot M^{-1} = \begin{pmatrix} 0 & A \\ C & B \end{pmatrix} \begin{pmatrix} X & Y \\ Z & H \end{pmatrix} = \begin{pmatrix} AZ & AH \\ CX + BZ & CY + BH \end{pmatrix} = \begin{pmatrix} E_1 & \\ & E_2 \end{pmatrix}, \text{ 则}$$

$$\begin{cases} AZ = E_1 \\ AH = 0 \\ CX + BZ = 0 \\ CY + BH = 0 \end{cases} \Rightarrow \begin{cases} X = -C^{-1}BA^{-1} \\ Y = C^{-1} \\ Z = A^{-1} \\ H = 0 \end{cases} \quad \text{故 } M^{-1} = \begin{pmatrix} -C^{-1}BA^{-1} & C^{-1} \\ A^{-1} & 0 \end{pmatrix}.$$

4.

因为 $|A| = 0$, 故 A 不可逆.

5.

由 $AB + E = A^2 + B$, 得 $(A - E)B = A^2 - E = (A - E)(A + E)$, $\therefore |A - E| = \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} \neq 0$, 故

$$B = A + E = \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}.$$

(B)

1.

由 $A^2 - 4A - E = 0 \Rightarrow A(A - 4E) = E$ 故 $A^{-1} = A - 4E$ 又由 $A^2 - 4A - E = 0 \Rightarrow 4A^2 + A - 17A - \frac{17}{4}E = -\frac{E}{4} \Rightarrow (4A + E)(17E - 4A) = E$ 故 $(4A + E)^{-1} = 17E - 4A$ 也可由 $A^2 = 4A + E \Rightarrow (4A + E)^{-1} = (A^2)^{-1} = (A^{-1})^2 = (A - 4E)^2$

2.

由 $A^2 = A$ 得 $A^2 + A - 2A - 2E = 2E \Rightarrow (A + E)(A - 2E) = -2E$ 故 $(A + E)^{-1} = \frac{1}{2}(2E - A)$.

3.

由 $A^{-1}BA = 6A + BA$ 得 $(A^{-1} - E)BA = 6A$ 故 $B = 6(A^{-1} - E)^{-1} = \begin{pmatrix} 6 & & \\ & 2 & \\ & & \frac{3}{2} \end{pmatrix}$

4.

由 $E^k - A^k = E$ 得 $(E - A)(E^{k-1} + E^{k-2}A + E^{k-3}A^2 + \cdots + A^{k-1}) = E \Rightarrow (E - A)^{-1} = E + A + A^2 + \cdots + A^{k-1}$

5.

$$\text{设 } \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} X & Y \\ Z & H \end{pmatrix} = \begin{pmatrix} E & O \\ O & E \end{pmatrix} \text{ 则 } \begin{cases} AX + BZ = E \\ AY + BH = O \\ BX + AZ = O \\ BY + AH = E \end{cases} \Rightarrow \begin{cases} X = \frac{1}{2}[(A+B)^{-1} + (A-B)^{-1}] \\ Y = \frac{1}{2}[(A+B)^{-1} - (A-B)^{-1}] \\ Z = \frac{1}{2}[(A+B)^{-1} - (A-B)^{-1}] \\ H = \frac{1}{2}[(A+B)^{-1} + (A-B)^{-1}] \end{cases}$$

$$\text{故 } \begin{pmatrix} A & B \\ B & A \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} (A+B)^{-1} + (A-B)^{-1} & (A+B)^{-1} - (A-B)^{-1} \\ (A+B)^{-1} - (A-B)^{-1} & (A+B)^{-1} + (A-B)^{-1} \end{pmatrix}$$

(C)

1.

$$\text{证明: 设 } A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, A^{-1} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \text{ 由于 } |A| \neq 0, \text{ 可得 } a_{ii} \neq$$

$$0 (i = 1, 2, \dots, n). \because AA^{-1} = E, \text{ 比较第1行的元素, 可得 } \begin{cases} a_{11}b_{11} = 1 \\ a_{11}b_{12} = 0 \\ a_{11}b_{13} = 0 \\ \vdots \\ a_{11}b_{1n} = 0 \end{cases} \text{ 从而可得 } b_{12} = 0, b_{13} = 0, \dots, b_{1n} = 0.$$

同理可以比较 AA^{-1} 和 E 的其他行, 得 $b_{ij} = 0 (i < j)$. 可见 A^{-1} 是下三角矩阵.

2.

证明: 假设 $AB - BA = E$, 则考虑主对角元素之和

$$\begin{aligned} & \sum_{i=1}^n \left[\sum_{k=1}^n a_{ik}b_{ki} - \sum_{k=1}^n b_{ik}a_{ki} \right] \\ &= \sum_{i=1}^n \sum_{k=1}^n a_{ik}b_{ki} - \sum_{i=1}^n \sum_{k=1}^n b_{ik}a_{ki} \\ &= \sum_{k=1}^n \sum_{i=1}^n a_{ik}b_{ki} - \sum_{i=1}^n \sum_{k=1}^n b_{ik}a_{ki} \\ &= 0 \neq n, \text{ 矛盾! 故 } AB - BA \neq E. \end{aligned}$$

习题1.4

(A)

1.

$$(1) \begin{pmatrix} 1 & -2 & 1 & 1 \\ 2 & -1 & 5 & 0 \\ 0 & 3 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & \frac{17}{3} \\ & 1 & & \frac{9}{3} \\ & & 1 & -2 \end{pmatrix}, \text{ 故 } x = \begin{pmatrix} \frac{17}{3} \\ \frac{9}{3} \\ -2 \end{pmatrix}.$$

$$(2) \begin{pmatrix} 1 & -1 & 1 & 1 \\ 3 & 0 & 4 & 5 \\ 1 & 2 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \frac{4}{3} & \\ & 1 & \frac{1}{3} & \\ & & & 1 \end{pmatrix} \text{ 无解.}$$

$$(3) \begin{pmatrix} 3 & 4 & -5 & 7 \\ 2 & -3 & 3 & -2 \\ 4 & 11 & -13 & 16 \\ 7 & -2 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{3}{17} & \frac{13}{17} \\ & 1 & -\frac{19}{17} \\ & & 0 \\ & & & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = \frac{3}{17}x_3 - \frac{13}{17}x_4 \\ x_2 = \frac{19}{17}x_3 - \frac{20}{17}x_4 \end{cases} \quad x_3, x_4 \text{ 为自由未知}$$

量, 令 $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ 取 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \eta_1 = \begin{pmatrix} \frac{3}{17} \\ \frac{19}{17} \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -\frac{13}{17} \\ -\frac{20}{17} \\ 0 \\ 1 \end{pmatrix}$ 则 $x = p \begin{pmatrix} \frac{3}{17} \\ \frac{19}{17} \\ 1 \\ 0 \end{pmatrix} + q \begin{pmatrix} -\frac{13}{17} \\ -\frac{20}{17} \\ 0 \\ 1 \end{pmatrix}.$

$$(4) \begin{pmatrix} 3 & -2 & 1 & -3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 1 & 4 & -3 & 5 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{7} & -\frac{1}{7} & \frac{6}{7} \\ & 1 & -\frac{5}{7} & \frac{9}{7} \\ & & 0 & 0 \end{pmatrix} \text{ 令自由未知量 } x_3, x_4 \text{ 为 } 0, \text{ 得特解 } \eta^* =$$

$$\begin{pmatrix} \frac{6}{7} \\ -\frac{5}{7} \\ 0 \\ 0 \end{pmatrix}, \text{ 令 } \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \text{ 为 } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ 可得基础解系 } \eta_1 = \begin{pmatrix} \frac{1}{7} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} \frac{1}{7} \\ -\frac{9}{7} \\ 0 \\ 1 \end{pmatrix}, \text{ 则 } x = p \begin{pmatrix} \frac{1}{7} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix} +$$

$$q \begin{pmatrix} \frac{1}{7} \\ -\frac{9}{7} \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{6}{7} \\ -\frac{5}{7} \\ 0 \\ 0 \end{pmatrix}.$$

2.

$$(1) \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} 4 & -3 \\ -1 & 2 \end{vmatrix}} \cdot \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}^* = \frac{1}{5} \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

$$(2) \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$(3) \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & -2 & -6 \end{pmatrix}^{-1} = \begin{pmatrix} 22 & -6 & -26 & 17 \\ -17 & 5 & 20 & -13 \\ -1 & 0 & 2 & -1 \\ 4 & -1 & -5 & 3 \end{pmatrix}.$$

$$(4) \begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{n-1}} \\ \frac{1}{a_n} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

$$(5) \text{ 由 } \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}, \text{ 多次使用该公式有 } \begin{pmatrix} 0 & 0 & \cdots & 0 & b_1 \\ 0 & 0 & \cdots & b_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b_{n-1} & \cdots & 0 & 0 \\ b_n & 0 & \cdots & 0 & 0 \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{1}{b_n} \\ 0 & 0 & \cdots & \frac{1}{b_{n-1}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{b_2} & \cdots & 0 & 0 \\ \frac{1}{b_1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

3.

$$(1) \text{ 由 } \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix}.$$

$$(2) X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{1}{2} \end{pmatrix}.$$

$$(3) X = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 2 & 1 \end{pmatrix} \text{ 则由于 } \begin{pmatrix} 1 & 2 & 3 & 1 & 3 \\ 3 & 2 & -4 & 0 & -2 \\ 2 & -1 & 0 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 13 & 11 \\ 1 & \frac{13}{4} & \frac{3}{4} & \frac{11}{4} \\ 1 & \frac{15}{41} & \frac{35}{41} \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & & \frac{32}{41} & \frac{20}{41} \\ & 1 & -\frac{18}{41} & -\frac{1}{41} \\ & & \frac{15}{41} & \frac{35}{41} \end{pmatrix}, \text{ 故 } X = \begin{pmatrix} \frac{32}{41} & \frac{20}{41} \\ -\frac{18}{41} & -\frac{1}{41} \\ \frac{15}{41} & \frac{35}{41} \end{pmatrix}.$$

4.

$$(1) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & -3 \\ 3 & 2 & -5 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{11}{3} & -\frac{7}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & -\frac{5}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}.$$

$$(2) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 4 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} & \frac{7}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \\ -2 \end{pmatrix}.$$

5.

$$(1) \begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$(2) \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 5 \\ 6 & -3 & 2 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 8 \\ 0 & -15 & 8 \\ 0 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & -52 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

6.

$$\text{由 } PAQ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ 可取 } P = E_3(3, 2(2))E_3(3, 1(1))E_3(2, 1(-2)) \text{ 取 } Q = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix}.$$

7.

证明: 由题设 P, Q 是可逆矩阵, 故 P, Q 是若干个初等矩阵的乘积. 用 P 左乘 A 或用 Q 右乘 A , 即对 A 作若干次初等行变换或初等列变换, 初等变换不改变矩阵的秩, 故 $r(A) = r(PA) = r(AQ) = r(PAQ)$.

8.

$$(1) \text{ 由于 } \left(\begin{array}{cc|c} 0 & A & E_n \\ B & 0 & E_m \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{cc|c} B & 0 & 0 & E_m \\ 0 & A & E_n & 0 \end{array} \right) \xrightarrow[B^{-1}r_1]{A^{-1}r_2} \left(\begin{array}{cc|c} E_m & 0 & 0 & B^{-1} \\ 0 & E_n & A^{-1} & 0 \end{array} \right)$$

$$\text{故 } \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}.$$

$$(2) \text{ 由于 } \left(\begin{array}{cc|c} 0 & A & E_n \\ B & C & E_m \end{array} \right) \xrightarrow{r_1 \leftrightarrow r_2} \left(\begin{array}{cc|c} B & C & 0 & E \\ 0 & A & E & 0 \end{array} \right) \xrightarrow{A^{-1}r_2} \left(\begin{array}{cc|c} B & C & 0 & E \\ 0 & E & A^{-1} & 0 \end{array} \right)$$

$$\xrightarrow{r_1 - Cr_2} \left(\begin{array}{cc|c} B & 0 & CA^{-1} & E \\ 0 & E & A^{-1} & 0 \end{array} \right) \xrightarrow{B^{-1}r_1} \left(\begin{array}{cc|c} E & 0 & B^{-1}CA^{-1} & B^{-1} \\ 0 & E & A^{-1} & 0 \end{array} \right) \text{ 故 } \begin{pmatrix} 0 & A \\ B & C \end{pmatrix}^{-1} =$$

$$\begin{pmatrix} B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & 0 \end{pmatrix}.$$

(B)

1.

$$\text{由于 } \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & \lambda \\ \lambda & 1 & 1 & \lambda^2 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda & \lambda-1 & \lambda-1 \\ 0 & 1-\lambda^2 & 1-\lambda & \lambda^2-\lambda \end{array} \right) \text{ 故当 } \lambda = 1 \text{ 时有无穷解. 当 } \lambda \neq 1$$

$$\text{时 } \rightarrow \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \lambda+2 & 1 \end{array} \right) \text{ 则当 } \lambda \neq 2 \text{ 且 } \lambda \neq 1 \text{ 时有唯一解当 } \lambda = 2 \text{ 时无解. } \lambda = 1 \text{ 时 } \rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \text{ 取 } \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \text{ 为 } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ 可得特解 } \eta^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ 取 } \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \text{ 为 } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ 可得基础解系}$$

$$\eta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ 故 } x = p \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + q \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

2.

$$\begin{aligned} \text{由 } XA = A + 3X \Rightarrow X = A(A - 3E)^{-1} &= \begin{pmatrix} 4 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 0 \\ 1 & 2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

3.

$$\begin{aligned} A &= PBP^{-1}, \text{ 则 } A^5 = PBP^{-1}PBP^{-1}PBP^{-1}PBP^{-1}PBP^{-1} = PB^5P^{-1} \because B^2 = \\ &\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -4 & 1 & 1 \end{pmatrix} \text{ 则 } A = PBP^{-1} = \\ &\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 6 & -1 & -1 \end{pmatrix} A^5 = PB^5P^{-1} = PBP^{-1} = A = \\ &\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 6 & -1 & -1 \end{pmatrix}. \end{aligned}$$

4.

依题意, 考虑 $\begin{vmatrix} 1 & x \\ y & 1 \end{vmatrix} = 1 - xy \neq 0$ 知 $x, y \in \mathbb{R}, y \neq 0$.

5.

$$\begin{aligned} (1) \text{ 由 } &\begin{pmatrix} 1 & a & a^2 & \cdots & a^{n-1} & 1 \\ 1 & a & a^2 & \cdots & a^{n-2} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a & a^2 & \cdots & a^{n-2} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & & 1 & -a \\ & 1 & & & & 1 & -a \\ & & \ddots & & & & \ddots & -a \\ & & & 1 & & & & 1 \end{pmatrix} \text{ 知} \\ &\begin{pmatrix} 1 & a & a^2 & \cdots & a^{n-1} \\ 1 & a & a^2 & \cdots & a^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a & a^2 & \cdots & a^{n-2} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & & & \\ & 1 & -a & & \\ & & \ddots & \ddots & \\ & & & 1 & -a \\ & & & & 1 \end{pmatrix}. \end{aligned}$$

$$\begin{aligned}
 (2) \text{ 由 } & \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} n-1 & n-1 & n-1 & \cdots & n-1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \rightarrow \\
 & \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -\frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 0 & 0 & 1 & \cdots & 0 & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & \frac{n-2}{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & & -\frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ & 1 & & & \frac{1}{n-1} & -\frac{n-2}{n-1} & \cdots & \frac{1}{n-1} \\ & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ & & & 1 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & -\frac{n-2}{n-1} \end{pmatrix} \\
 \text{故 } & \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & -\frac{n-2}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \cdots & \frac{1}{n-1} & -\frac{n-2}{n-1} \end{pmatrix}.
 \end{aligned}$$

6.

依题意 $r(A) = 2 < n = 3$ 则 $\begin{vmatrix} 1 & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b+c \end{vmatrix} = 0 \Rightarrow a = 0$ 或 $b+c = 0$ ①当 $a = 0$, 则由 $r(A) = 2$ 知

$b+c \neq 0$ ②当 $b+c = 0$, 则由 $r(A) = 2$ 知 $a \neq 0$ A 的相抵标准形为 $I_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

7.

证明: 充分性: $\because P, Q$ 可逆, $\therefore P, Q$ 可分解为若干个基本初等矩阵的积即 A 可以经过若干次初等变换得到 B , $\therefore A \sim B$ 必要性: $\because A \sim B$, $\therefore A$ 经过若干次初等变换可以得到 B 即 $PAQ = B$ (P, Q 可逆)

8.

$$\begin{aligned}
 A &= \begin{pmatrix} a & 1 & \cdots & 1 \\ 1 & a & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & a \end{pmatrix}_{n \times n} \quad \text{则 } a = 1 \text{ 时 } \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ 此时 } I = \\
 \begin{pmatrix} 1 & 0 \\ 0 & O_{n-1} \end{pmatrix} \text{ 当 } a = 1 - n \text{ 时, } \begin{pmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-n \end{pmatrix} \rightarrow \begin{pmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \rightarrow \\
 \begin{pmatrix} -n & -n & \cdots & -n & 1 \\ & -n & \cdots & -n & 1 \\ & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix} \text{ 故 } I = \begin{pmatrix} E_{n-1} & 0 \\ 0 & 0 \end{pmatrix} \text{ 当 } a \neq 1, 1-n \text{ 时则} \\
 \begin{pmatrix} a-1 & & & \\ & a-1 & & \\ & & \ddots & \\ & & & a \end{pmatrix} \rightarrow \begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = E_n \text{ 故此时 } I = E_n. \text{ 综上,} \\
 r(A) = \begin{cases} 1 & a = 1 \\ n-1 & a = 1-n \\ n & a \neq 1, 1-n \end{cases}
 \end{aligned}$$

9.

由 $\begin{pmatrix} A_{n-1} & \beta \\ \alpha^T & a_{nn} \end{pmatrix}$ 进行初等行变换

$$\begin{aligned}
 &\left(\begin{array}{cc|cc} A_{n-1} & \beta & E & 0 \\ \alpha^T & a_{nn} & 0 & E \end{array} \right) \rightarrow \left(\begin{array}{cc|cc} E & A_{n-1}^{-1}\beta & A_{n-1}^{-1} & 0 \\ \alpha^T & a_{nn} & 0 & E \end{array} \right) \\
 &\rightarrow \left(\begin{array}{cc|cc} E & A_{n-1}^{-1}\beta & A_{n-1}^{-1} & 0 \\ 0 & a_{nn} - \alpha^T A_{n-1}^{-1}\beta & -\alpha^T A_{n-1}^{-1} & E \end{array} \right) \\
 &\rightarrow \left(\begin{array}{cc|cc} E & 0 & A_{n-1}^{-1}(E_{n-1} + \beta t^{-1} \alpha^T A_{n-1}^{-1}) & -A_{n-1}^{-1}\beta t^{-1} \\ 0 & E & -t^{-1} \alpha^T A_{n-1}^{-1} & t^{-1} \end{array} \right)
 \end{aligned}$$

其中 $t = a_{nn} - \alpha^T A_{n-1}^{-1} \beta$ 令 $A_{n-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$, $\alpha^T = (2 \ 3 \ 3)$, $a_{nn} = 2$ 则

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 1 & -1 & 2 & 1 \\ -3 & -12 & -11 & 7 \\ 5 & 20 & 15 & -10 \\ -4 & -11 & -8 & 6 \end{pmatrix}.$$

10.

证明: 假设 A 为满秩矩阵, 则 $|A| \neq 0$, 这与 A 不可逆矛盾。故 A 为降秩矩阵。

(C)

1.

(1) 考虑第1行第1列元素, 则 $q_1 = b_1$ 考虑第1行第2列元素, 则 $q_1 r_1 = c_1 \Rightarrow r_1 = \frac{c_1}{q_1}$ 考虑第 i 行第 $i-1$ 列元素 ($i = 2, \dots, n$), 则 $p_i = a_i$ 考虑第 i 行第 i 列元素 ($i = 2, \dots, n$), 则 $p_i r_{i-1} + q_i = b_i \Rightarrow q_i = b_i - p_i r_{i-1}$ 考虑第 i 行第 $i+1$ 列元素 ($i = 2, \dots, n-1$), 则 $q_i r_i = c_i \Rightarrow r_i = \frac{c_i}{q_i}$

$$(2) \text{ 由 } \left(\begin{array}{ccc|c} 1 & 2 & & 6 \\ 2 & 1 & 1 & 8 \\ & 1 & 2 & 8 \\ 1 & 2 & & 6 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & & & 2 \\ & 1 & & 2 \\ & & 1 & 2 \\ & & & 2 \end{array} \right) \text{ 则 } x = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}.$$

2.

$$f(A) = (A - E)^n = \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \\ & & & 1 \end{pmatrix}^n = \left[\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix} + E \right]^n \text{ 记 } B = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix}$$

$$\text{则 } B^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & 0 & 0 & 1 \\ & & \ddots & \ddots & \ddots \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}, B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & \\ & 0 & 0 & 0 & 1 \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix} B^n = O \ (n \geq 4), \text{ 则 } f(A) =$$

$$(B+E)^n = B^n + C_n^1 B^{n-1} + C_n^2 B^{n-2} + \cdots + C_n^{n-2} B^2 + C_n^{n-1} B + E = E + C_n^1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} +$$

$$C_n^2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} + C_n^3 \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 1 & C_n^1 & C_n^2 & C_n^3 \\ & 1 & C_n^1 & C_n^2 \\ & & 1 & C_n^1 \\ & & & 1 \end{pmatrix}.$$

习题2.1

(A)

1.

$$(1) \begin{vmatrix} 4 & -3 \\ -7 & 6 \end{vmatrix} = 24 - 21 = 3.$$

$$(2) \begin{vmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{vmatrix} = \cos^2 \alpha + \sin^2 \alpha = 1.$$

$$(3) \begin{vmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 7 & 0 & 9 \end{vmatrix} = 5 \times 9 + 2 \times 6 \times 7 - 3 \times 5 \times 7 + 4 \times 2 \times 9 = 96.$$

$$(4) \begin{vmatrix} x & 1 & -1 \\ -1 & x & 1 \\ 1 & -1 & x \end{vmatrix} = x^3 + 3x.$$

$$(5) \begin{vmatrix} 0 & 0 & a_1 \\ 0 & a_2 & 0 \\ a_3 & 0 & 0 \end{vmatrix} = -a_1 a_2 a_3.$$

2.

$$(1) \text{ 由 } \begin{pmatrix} 3 & -2 & 1 \\ -7 & 5 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 5 \\ & 1 & 7 \end{pmatrix} \text{ 故 } \begin{cases} x = 5 \\ y = 7 \end{cases}.$$

$$(2) \text{ 由 } \begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 7 \\ & 1 & -3 \end{pmatrix} \text{ 故 } \begin{cases} x = 7 \\ y = -3 \end{cases}.$$

$$(3) \text{ 由 } \begin{pmatrix} 2 & -3 & -3 & 0 \\ 1 & 4 & 6 & -1 \\ 3 & -1 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & -\frac{9}{4} & \\ & 1 & -\frac{41}{8} & \\ & & 1 & \frac{29}{8} \end{pmatrix} \text{ 故 } \begin{cases} x = -\frac{9}{4} \\ y = -\frac{41}{8} \\ z = \frac{29}{8} \end{cases}.$$

$$(4) \text{ 由 } \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 5 & -1 \\ 2 & 0 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \frac{22}{13} & \\ & 1 & -\frac{5}{13} & \\ & & 1 & -\frac{5}{13} \end{pmatrix} \text{ 故 } \begin{cases} x = \frac{22}{13} \\ y = -\frac{5}{13} \\ z = -\frac{5}{13} \end{cases}.$$

3.

(1) $\tau(4357261) = 1 + 4 + 1 + 1 + 6 = 12$. 偶排列.(2) $\tau(217986354) = 1 + 1 + 3 + 4 + 4 + 4 + 5 = 18$. 偶排列.

4.

因为对于元素 x_1, x_2, \dots, x_n 中任何两个不同的 x_i 与 x_j , 在 x_1, x_2, \dots, x_n 与 x_n, x_{n-1}, \dots, x_1 中必有且只有一个构成逆序, 所以这两个排列的逆序数之和应等于从 n 个元素中任取两个元素的组合数 $C_n^2 = \frac{n(n-1)}{2}$ 故 $x_n \cdots x_2 x_1$ 的逆序数为 $\frac{n(n-1)}{2} - k$.

5.

$$-a_{13}a_{21}a_{34}a_{42}, a_{14}a_{21}a_{33}a_{42}.$$

6.

$$-a_{11}a_{23}a_{32}a_{44}, -a_{12}a_{23}a_{34}a_{41}, -a_{14}a_{23}a_{31}a_{42}.$$

7.

$$(1) \begin{vmatrix} 1 & 1 & & \\ 2 & -1 & & \\ & & 3 & 0 \\ & & 4 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 \\ 4 & 4 \end{vmatrix} = (-1-2) \times 12 = -36.$$

$$(2) \begin{vmatrix} 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 2 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} n & n-1 & \cdots & 2 \end{vmatrix} = (-1)^{n-1} n!.$$

$$(3) \begin{vmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{vmatrix} = (-1)^{2 \times 2} \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = 24.$$

(B)

1.

$$f(x) = \begin{vmatrix} x & x & 1 & 2 & 3 \\ 1 & x & 0 & 2 & 4 \\ 2 & 5 & x & 1 & 2 \\ 1 & 3 & -4 & x & 0 \\ 2 & 6 & 4 & 1 & x \end{vmatrix} = \begin{vmatrix} x-1 & 0 & 1 & 0 & -1 \\ 1 & x & 0 & 2 & 4 \\ 2 & 5 & x & 1 & 2 \\ 1 & 3 & -4 & x & 0 \\ 2 & 6 & 4 & 1 & x \end{vmatrix}$$

则会出现 x^4 项, 只能取主对角线元素 $(x -$ 1) $x^4 \Rightarrow x^4$ 的系数为 -1 .

2.

$$\text{由 } D_n = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix} = 0, \text{ 反行列式定义知 } D \text{ 为偶排列个数减奇排列个数所以奇偶排列各半.}$$

3.

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & & & \\ d_1 & d_2 & & & \\ e_1 & e_2 & & & \end{vmatrix} = 0 \text{ 的一般项可表示为 } (-1)^{N(j_1, j_2, j_3, j_4, j_5)} a_{1j_1} a_{2j_2} a_{3j_3} a_{4j_4} a_{5j_5} \text{ 记 } a_{1i} = a_i, a_{2i} =$$

$b_i, a_{3i} = c_i (i < 3), a_{4i} = d_i (i < 3), a_{5i} = e_i (i < 3), a_{mn} = 0 (m > 2, n > 2)$ 则一般项的列下标 j_3, j_4, j_5 只能在 $1, 2, 3, 4, 5$ 中取3个不同值, 故 j_3, j_4, j_5 必在 $3, 4, 5$ 中取一个数, 从而至少有一项都包含至少一个 0 因子, 故任意一项必为 0. 从而该行列式为 0.

习题2.2

(A)

1.

$$(1) \begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & -1 & 6 & 7 \\ 41 & -7 & -9 & 0 \\ -9 & 21 & 32 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 10 \\ 3 & -1 & 6 & 15 \\ 41 & -7 & -9 & 25 \\ -9 & 21 & 32 & 45 \end{vmatrix} = 5 \begin{vmatrix} 1 & 2 & 3 & 2 \\ 3 & -1 & 6 & 3 \\ 41 & -7 & -9 & 5 \\ -9 & 21 & 32 & 9 \end{vmatrix} \text{ 故行列式是5的整数倍.}$$

(2) 由 $|\alpha_1, \alpha_2, \beta_2, \alpha_3| = n \Rightarrow |\alpha_1, \alpha_2, \alpha_3, \beta_2| = -n$, 故 $|\alpha_1, \alpha_2, \alpha_3, (\beta_1 - \beta_2)| = m - (-n) = m + n$, 则 $|\alpha_3, \alpha_2, \alpha_1, (\beta_1 - \beta_2)| = -(m + n)$.

2.

$$(1) \begin{vmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 3 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

$$(2) \begin{vmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 0 & 1 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 & -3 \\ 1 & 2 & -2 & -12 \\ 10 & 5 & -3 & -35 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 1 & -3 \\ 1 & -2 & -12 \\ 10 & -3 & -35 \end{vmatrix} = 0.$$

$$(3) \begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix} = 0.$$

$$(4) \begin{vmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{vmatrix} = \begin{vmatrix} 2(a+b) & b & a+b \\ 2(a+b) & a+b & a \\ 2(a+b) & a & b \end{vmatrix} = 2(a+b) \begin{vmatrix} 1 & b & a+b \\ 1 & a+b & a \\ 1 & a & b \end{vmatrix} = 2(a+b) \begin{vmatrix} a & -b \\ a-b & -a \end{vmatrix} = 2(a+b)(ab - a^2 - b^2) = -2(a^3 + b^3).$$

3.

$$\begin{vmatrix} a_1 + b_1 & a_1 + b_2 & \cdots & a_1 + b_n \\ a_2 + b_1 & a_2 + b_2 & \cdots & a_2 + b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n + b_1 & a_n + b_2 & \cdots & a_n + b_n \end{vmatrix} = \begin{vmatrix} a_1 + b_1 & a_1 + b_2 & \cdots & a_1 + b_n \\ a_2 - a_1 & a_2 - a_1 & \cdots & a_2 - a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_n - a_1 & a_n - a_1 & \cdots & a_n - a_1 \end{vmatrix} \quad \text{故 } n = 1 \text{ 时, 值为 } a_1 + b_1,$$

$$n = 2 \text{ 时, } \begin{vmatrix} a_1 + b_1 & a_1 + b_2 \\ a_2 - a_1 & a_2 - a_1 \end{vmatrix} = (a_1 - a_2)(b_2 - b_1) \text{ 当 } n \geq 3 \text{ 时, 第2行与第} n \text{行成比例, 值为0.}$$

4.

对 A 作 $r_i + kr_k$ 变换, 相当于 A 加上一个第 i 行为 kr_k 的行列式, 该行列式的第 i 行与第 k 行成比例, 值为0. 同理作 $c_i + kc_k$ 变换也相当于 A 加上一个值为0的行列式故 $A \xrightarrow[r_i + kr_k]{c_i + kc_k} B$ 后, $|B| = |A|$.

5.

$$AA^T = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix} \begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix} = \begin{pmatrix} a^2 + b^2 + c^2 + d^2 & 0 & 0 & 0 \\ 0 & a^2 + b^2 + c^2 + d^2 & 0 & 0 \\ 0 & 0 & \sum a^2 & 0 \\ 0 & 0 & 0 & \sum a^2 \end{pmatrix}$$

$$\text{故 } |AA^T| = (a^2 + b^2 + c^2 + d^2)^4. \quad \text{又} \quad \begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = - \begin{vmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{vmatrix} =$$

$$-\frac{1}{abcd} \begin{vmatrix} a^2 & ba & ca & da \\ b^2 & -ab & -db & cb \\ c^2 & dc & -ac & bc \\ d^2 & -cd & bd & -ad \end{vmatrix} = -\frac{1}{abcd} \begin{vmatrix} a^2 & ba & ca & da \\ b^2 & -ab & -db & cb \\ c^2 & dc & -ac & bc \\ a^2 + b^2 + c^2 + d^2 & 0 & 0 & 0 \end{vmatrix} = (-1)^{1+4} \cdot (-1) \cdot$$

$$\frac{a^2 + b^2 + c^2 + d^2}{abcd} \begin{vmatrix} ba & ca & da \\ -ab & -db & cb \\ dc & -ac & bc \end{vmatrix} = \frac{a^2 + b^2 + c^2 + d^2}{d} \begin{vmatrix} b & c & d \\ -a & -d & c \\ d & -a & -b \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2.$$

(B)

1.

$$(1) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & n \end{vmatrix} = \begin{vmatrix} \sum_{i=1}^n \frac{1}{i} & 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{vmatrix} = n! \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right).$$

$$(2) \begin{vmatrix} a_1 & 1 & 1 & \cdots & 1 \\ 1 & a_2 & 0 & \cdots & 0 \\ 1 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_1 - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} & 1 & 1 & \cdots & 1 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = a_2 a_3 \cdots a_n \left(a_1 - \sum_{i=2}^n \frac{1}{a_i} \right).$$

2.

$$\text{证明: } \begin{vmatrix} n & n-1 & \cdots & 3 & 2 & 1 \\ n & n-1 & \cdots & 3 & 2 & 2 \\ n & n-1 & \cdots & 3 & 3 & 3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & n-1 & \cdots & n-1 & n-1 & n-1 \\ n & n & \cdots & n & n & n \end{vmatrix} = \begin{vmatrix} -1 & & & & & \\ -1 & -1 & & & & \\ -1 & -1 & -1 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ -1 & -1 & -1 & \cdots & -1 & \\ n & n & n & \cdots & n & \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} \cdot (-1)^{n-1} \cdot n$$

$$= (-1)^{\frac{(n-1)(n+2)}{2}} \cdot n.$$

3.

证明: 若有两行成比例, 则将比例常数提出后, 有两行相同, 则交换这两行, 有 $|A| = -|A|$, 得 $|A| = 0$.

4.

(1) 证明: 在等式左端的 $k+t$ 阶行列式中, 取定前 k 行, 由这 k 行元素构成的 k 阶子式中, 只

$$\text{有取前 } k \text{ 列时该子式不为 } 0, \text{ 根据拉普拉斯定理, 左边} = \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & \cdots & b_{1t} \\ \vdots & & \vdots \\ b_{t1} & \cdots & b_{tt} \end{vmatrix}.$$

$$(-1)^{(1+2+\cdots+k)+(1+2+\cdots+k)} = \text{右边}.$$

(2) 证明: 同(1), 由拉普拉斯定理左边 =
$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & \cdots & b_{1t} \\ \vdots & & \vdots \\ b_{t1} & \cdots & b_{tt} \end{vmatrix}.$$

$$(-1)^{[(t+1)+(t+2)+\cdots+k]+[1+2+\cdots+k]} = (-1)^{tk+2(1+2+\cdots+k)}$$

边.

5.

$$f(x) = \begin{vmatrix} 2 & 1 & 2 & 3 \\ 2 & 5-x^2 & 2 & 3 \\ 10 & 5 & 2 & 1 \\ 10 & 5 & 2 & 2-x^2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 & 3 \\ 4-x^2 & & & \\ -8 & -14 & & \\ -8 & -13-x^2 & & \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 & 3 \\ 4-x^2 & & -8 & -14 \\ & -8 & & 1-x^2 \end{vmatrix} = -16(1-x^2)(4-x^2).$$

则零点为 $x_{1,2} = \pm 1, x_{3,4} = \pm 2$.

习题2.3

(A)

1.

$$(1) A_{11} = \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} = 5, A_{12} = -\begin{vmatrix} -3 & 0 \\ 1 & 5 \end{vmatrix} = 15, A_{13} = \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} = -7, A_{21} = -\begin{vmatrix} 0 & 0 \\ 2 & 5 \end{vmatrix} = 0, A_{22} = \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} = 10, A_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = -4, A_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0, A_{32} = -\begin{vmatrix} 2 & 0 \\ -3 & 0 \end{vmatrix} = 0, A_{33} = \begin{vmatrix} 2 & 0 \\ -3 & 1 \end{vmatrix} = 2.$$

$$(2) A_{11} = \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -1, A_{12} = 1, A_{13} = 0, A_{21} = 2, A_{22} = -2, A_{23} = 0, A_{31} = 0, A_{32} = 0, A_{33} = 0.$$

2.

$$(1) \begin{vmatrix} 2 & -1 & 3 & 1 & 0 \\ 1 & 2 & -1 & 4 & 3 \\ 0 & -1 & -3 & 2 & 3 \\ 4 & 5 & 0 & 3 & 1 \\ 1 & -1 & 2 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & -1 & 3 & 1 & 0 \\ -11 & -13 & -1 & -5 & 0 \\ -12 & -16 & -3 & -7 & 0 \\ 4 & 5 & 0 & 3 & 1 \\ -11 & -16 & 2 & -11 & 0 \end{vmatrix} = -\begin{vmatrix} 2 & -1 & 3 & 1 \\ -11 & -13 & -1 & -5 \\ -12 & -16 & -3 & -7 \\ -11 & -16 & 2 & -11 \end{vmatrix}$$

$$= -\begin{vmatrix} 2 & 0 & 3 & 1 \\ -11 & -18 & -1 & -5 \\ -12 & -23 & -3 & -7 \\ -11 & -27 & 2 & -11 \end{vmatrix} = -\begin{vmatrix} 0 & 0 & 0 & 1 \\ -1 & -18 & 14 & -5 \\ 2 & -23 & 18 & -7 \\ 11 & -27 & 35 & -11 \end{vmatrix} = \begin{vmatrix} -1 & -18 & 14 \\ 2 & -23 & 18 \\ 11 & -27 & 35 \end{vmatrix} = \begin{vmatrix} -1 & -18 & 14 \\ 0 & -59 & 46 \\ 0 & -225 & 189 \end{vmatrix} =$$

$$-\begin{vmatrix} -59 & 46 \\ -225 & 189 \end{vmatrix} = 11151 - 10350 = 801.$$

$$(2) \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

$$(3) \begin{vmatrix} \cos \alpha & 1 & & & \\ 1 & 2 \cos \alpha & 1 & & \\ & 1 & 2 \cos \alpha & 1 & \\ & & 1 & 2 \cos \alpha & 1 \\ & & & 1 & 2 \cos \alpha \end{vmatrix} = \begin{vmatrix} 1 & \cos \alpha & & & \\ \cos \alpha & 1 & & & \\ & & 1 & \cos \alpha & \\ & & \cos \alpha & 1 & \\ & & & & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & \cos \alpha & & & \\ \cos \alpha & 1 & & & \\ & & 1 - 2 \cos^2 \alpha & & \\ & & & 1 - 2 \cos^2 \alpha & \\ & & & & 2 \cos \alpha \end{vmatrix} = \begin{vmatrix} 1 - 2 \cos^2 \alpha & 1 \\ -\cos \alpha & 2 \cos \alpha \end{vmatrix} = 1 - 2 \cos^2 \alpha -$$

$$2 \cos^2 \alpha + (1 - 2 \cos^2 \alpha) 4 \cos^2 \alpha = 1 - 2 \sin^2 2\alpha = \cos 4\alpha.$$

$$(4) \text{ 设 } D_n = \begin{vmatrix} a+b & ab & & & \\ 1 & a+b & ab & & \\ & 1 & a+b & \ddots & \\ & & \ddots & \ddots & ab \\ & & & 1 & a+b \end{vmatrix}_n \text{ 则 } D_n = (a+b) \cdot D_{n-1} + ab \cdot \begin{vmatrix} 1 & ab & & & \\ a+b & ab & & & \\ & \ddots & \ddots & & \\ & & & a+b \end{vmatrix}_{n-1}$$

$$= (a+b)D_{n-1} + abD_{n-2}, \text{ 又 } D_1 = a+b, D_2 = a^2 + ab + b^2 \Rightarrow D_n - aD_{n-1} = b(D_{n-1} -$$

$$aD_{n-2}) = b^2(D_{n-2} - aD_{n-3}) = b^{n-2}(D_2 - aD_1) = b^{n-2}(a^2 + ab + b^2 - a^2 - ab) = b^n \text{ 则 } \frac{D_n}{a^n} -$$

$$\frac{D_{n-1}}{a^{n-1}} = \left(\frac{b}{a}\right)^n \frac{D_n}{a^n} - \frac{D_1}{a} = \left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\right)^3 + \cdots + \left(\frac{b}{a}\right)^n = \left(\frac{b}{a}\right)^2 \frac{1 - \left(\frac{b}{a}\right)^{n-1}}{1 - \frac{b}{a}} \Rightarrow D_n =$$

$$a^n \left[\frac{a+b}{a} + \frac{b^2}{a} \cdot \frac{1 - \left(\frac{b}{a}\right)^{n-1}}{a-b} \right] = \frac{(a^2 - b^2)a^{n-1}}{a-b} + \frac{a^{n-1} - b^{n-1}}{a-b} b^2 = \frac{a^{n+1} - b^{n+1}}{a-b}.$$

3.

$$(1) \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}^{-1} = \frac{1}{|2 \ 1|} \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}^* = \frac{1}{2} \begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}.$$

$$(2) \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{vmatrix}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}^* = \frac{1}{6} \begin{pmatrix} 2 & 1 & 4 \\ 2 & 1 & -2 \\ -2 & 2 & 2 \end{pmatrix}.$$

4.

(1) 证明: 按第一行展开, 则 x 只能出现在第一行元素中故 $P(x)$ 的最高次数项为

$$\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-2} \end{vmatrix} (-1)^{n+1} x^{n-1} \text{ 故 } P(x) \text{ 的次数不超过 } n-1.$$

(2) $\because P(x)$ 的次数不超过 $n-1$. 注意到 x 为 a_1, a_2, \dots, a_{n-1} 时 $P(x)$ 为 0 故 $P(x)$ 的根为 $\{a_1, a_2, \dots, a_{n-1}\}$ 中的不重复元素 1) 若 $a_i \neq a_j$ ($i \neq j$), 则 $P(x)$ 的根为 a_1, a_2, \dots, a_{n-1} 2) 若 $a_i = a_j$, 则 $P(x) \equiv 0$, 此时 $P(x)$ 的根为 $x \in \mathbb{R}$ 由题设条件, a_1, a_2, \dots, a_n 互异故 $P(x)$ 的根为 a_1, a_2, \dots, a_{n-1} .

$$(3) \text{ 由于 } \begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-2} \end{vmatrix} = \prod_{1 \leq j < i \leq n-1} (a_i - a_j) \text{ 则 } x^{n-1} \text{ 的系数为 } (-1)^{n+1} \prod_{1 \leq j < i \leq n-1} (a_i - a_j).$$

(B)

1.

$$(1) \text{ 在 } D_{2n} = \begin{vmatrix} a & & & b \\ & \ddots & & \\ & & a & b \\ & & b & a \\ b & & & a \end{vmatrix}_{2n}$$

中, 取定 n 和 $n+1$ 行, 由这两行元素组成的所有 2 阶子式中, 只有

取第 n 和 $n+1$ 列时子式不为 0. 故由拉普拉斯展开定理, 得

$$D_{2n} = \begin{vmatrix} a & b \\ b & a \end{vmatrix} \cdot (-1)^{n+(n+1)+n+(n+1)} \begin{vmatrix} a & & & b \\ & \ddots & & \\ & & a & b \\ & & b & a \\ b & & & a \end{vmatrix}_{2n-2}$$

$$= (a^2 - b^2)D_{2n-2}$$

$$\text{又 } D_2 = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2 - b^2. \text{ 故 } D_{2n} = (a^2 - b^2)^{n-1} D_2 = (a^2 - b^2)^n.$$

$$(2) \text{ 将 } D_n = \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & x & -1 & \cdots & 0 \\ a_3 & 0 & x & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ a_n & 0 & \cdots & 0 & x \end{vmatrix} \text{ 按最后一行展开}$$

$$D_n = x(-1)^{n+n} \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & x & -1 & \cdots & 0 \\ a_3 & 0 & x & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ a_{n-1} & 0 & \cdots & 0 & x \end{vmatrix} + (-1)^{n+1} a_n \begin{vmatrix} -1 & -1 & & & \\ x & -1 & & & \\ & x & \ddots & & \\ & & \ddots & -1 & \end{vmatrix}$$

$$= xD_{n-1} + a_n$$

$$= x(xD_{n-2} + a_{n-1}) + a_n$$

$$= x^2 D_{n-2} + a_{n-1}x + a_n$$

...

$$= a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2} x^2 + a_{n-1} x + a_n$$

$$= \sum_{i=1}^n a_i x^{n-i}$$

2.

由 $A^*BA = 2BA - 8E$ 得 $(2E - A^*)BA = 8E$ 即 $B = 8(2E - A^*)^{-1}A^{-1} = 8(A(2E - A^*))^{-1}$

$$= 8(2A - |A|E)^{-1} = 8(2A + 2E)^{-1} = 4 \begin{pmatrix} 1 & 2 & -2 \\ -2 & 4 & \\ & & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 4 & -6 \\ -4 & 8 & \\ & & 2 \end{pmatrix}.$$

3.

$$|2A^*B^{-1}| = 2^n |A| |A^{-1}B^{-1}| = \frac{2^n |A|^n}{|A||B|} = -\frac{4^n}{b}.$$

4.

$$(1) |A^*| = ||A|A^{-1}| = |A|^n|A^{-1}| = |A|^{n-1}.$$

$$(2) (A^*)^* = |A^*|(A^*)^{-1} = |A|^{n-1} \cdot \frac{A}{|A|} = |A|^{n-2}A.$$

$$(3) (kA)^* = |kA|(kA)^{-1} = k^{n-1}|A|A^{-1} = k^{n-1}A^*.$$

5.

$$\text{由 } \begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & & 1 & -1 \\ & 1 & & & 1 & -1 \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 1 & -1 \\ & & & & 1 & 1 \end{pmatrix} \text{ 故 } A^* = 2A^{-1} = \begin{pmatrix} 1 & -2 & & & & \\ & 2 & -2 & & & \\ & & 2 & -2 & & \\ & & & \ddots & -2 & \\ 0 & & & & & 2 \end{pmatrix},$$

$$\text{故 } \sum_{i,j=1}^n A_{ij} = 1.$$

(C)

1.

(1)

$$\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix} = \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ y_1(x_2-x_1) & y_2(x_2-x_1) & \cdots & y_n(x_2-x_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(x_n-x_1) & y_2(x_n-x_1) & \cdots & y_n(x_n-x_1) \end{vmatrix}$$

则当 $n=1$ 时, 值为 $1+x_1y_1$

$$\text{当 } n=2 \text{ 时, } \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 \\ 1+x_2y_1 & 1+x_2y_2 \end{vmatrix} = (x_2-x_1)(y_2-y_1)$$

当 $n \geq 3$ 时, 第2行与第 n 行成比例, 值为0.

$$(2) \text{ 记 } A = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{k-1} & x_1^{k+1} & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^{k-1} & x_2^{k+1} & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{k-1} & x_n^{k+1} & \cdots & x_n^n \end{vmatrix}$$

$$\text{考虑多项式 } f(t) = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{k-1} & x_1^k & x_1^{k+1} & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^{k-1} & x_2^k & x_2^{k+1} & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{k-1} & x_n^k & x_n^{k+1} & \cdots & x_n^n \\ 1 & t & \cdots & t^{k-1} & t^k & t^{k+1} & \cdots & t^n \end{vmatrix}$$

$$\text{则 } f(t) = (x_2 - x_1)(x_3 - x_1) \cdots (t - x_1)(x_3 - x_2) \cdots (t - x_2) \cdots (t - x_n)$$

$$= \prod_{k=1}^n (t - x_k) \prod_{1 \leq j < i \leq n} (x_i - x_j)$$

该行列式中第 $n+1$ 行第 $k+1$ 列元素 t^k 的代数余子式为

$$t^k (-1)^{n+1+k+1} \begin{vmatrix} 1 & x_1 & \cdots & x_1^{k-1} & x_1^{k+1} & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^{k-1} & x_2^{k+1} & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{k-1} & x_n^{k+1} & \cdots & x_n^n \end{vmatrix}$$

$$\text{考察 } f(t) \text{ 中 } t^k \text{ 的系数为 } (-1)^{n-k} \cdot \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n} \prod_{k=1}^{n-k} x_{i_k} \right) \cdot \prod_{1 \leq j < i \leq n} (x_i - x_j)$$

$$\text{故 } A = \prod_{1 \leq j < i \leq n} (x_i - x_j) \left(\sum_{1 \leq i_1 < i_2 < \cdots < i_{n-k} \leq n} \prod_{k=1}^{n-k} x_{i_k} \right).$$

- (3) 在该行列式中取定第 n 和第 $n+1$ 行, 由这两行元素组成的所有 2 阶子式中, 只有取第 n 和 $n+1$ 列时的子式不为 0.

由拉普拉斯展开定理, 得

$$\begin{vmatrix} a_1 & \cdots & b_{2n} \\ \vdots & \ddots & \vdots \\ a_n & b_{n+1} & \cdots \\ b_n & a_{n+1} & \cdots \\ \vdots & \ddots & \vdots \\ b_1 & \cdots & a_{2n} \end{vmatrix} = \begin{vmatrix} a_n & b_{n+1} \\ b_n & a_{n+1} \end{vmatrix} \times (-1)^{n+(n+1)+n+(n+1)} \begin{vmatrix} a_1 & \cdots & b_{2n} \\ \vdots & \ddots & \vdots \\ a_{n-1} & b_{n+2} & \cdots \\ b_{n-1} & a_{n+2} & \cdots \\ \vdots & \ddots & \vdots \\ b_1 & \cdots & a_{2n} \end{vmatrix}$$

$$= \begin{vmatrix} a_n & b_{n+1} \\ b_n & a_{n+1} \end{vmatrix} \cdot \begin{vmatrix} a_{n-1} & b_{n+2} \\ b_{n-1} & a_{n+2} \end{vmatrix} \cdots \begin{vmatrix} a_1 & b_{2n} \\ b_1 & a_{2n} \end{vmatrix}$$

$$= \prod_{i=1}^n (a_i a_{2n+1-i} - b_{2n+1-i} b_i)$$

2.

证明: 行列式的每一个元素都是两项和 $a_{ij} + x$ ($i, j = 1, 2, \dots, n$), 这样行列式的每一列都可以看作由两个子列所组成, 第1子列元素为 a_{ij} , 第2子列元素为 x .

则该行列式可以拆成 2^n 个 n 阶行列式之和, 其中包含2列及以上的元素皆为 x 的 n 阶行列式值为0, 于是

$$\begin{aligned}
 & \begin{vmatrix} a_{11} + x & a_{12} + x & \cdots & a_{1n} + x \\ a_{21} + x & a_{22} + x & \cdots & a_{2n} + x \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + x & a_{n2} + x & \cdots & a_{nn} + x \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} x & a_{12} & \cdots & a_{1n} \\ x & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & \cdots & a_{1,n-1} & x \\ a_{21} & \cdots & a_{2,n-1} & x \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} & x \end{vmatrix} \\
 & = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + x \sum_{i=1}^n A_{i1} + x \sum_{i=1}^n A_{i2} + \cdots + x \sum_{i=1}^n A_{in} \\
 & = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + x \sum_{j=1}^n \sum_{i=1}^n A_{ij} \\
 & = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + x \sum_{i=1}^n \sum_{j=1}^n A_{ij}
 \end{aligned}$$

习题2.4

(A)

1.

$$(1) x_1 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 12 \end{vmatrix}} = \frac{9}{1} = 9$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 2 & 0 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 12 \end{vmatrix}} = 6 \quad x_3 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 12 \end{vmatrix}} = -2$$

$$\text{故 } x = \begin{pmatrix} 9 \\ 6 \\ -2 \end{pmatrix}$$

$$(2) D = \begin{vmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 1 & 11 \\ 2 & -3 & -1 & 4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -80 \quad D_1 = \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 11 \\ 0 & -3 & -1 & 4 \\ 0 & 1 & 1 & 1 \end{vmatrix} = 50 \quad D_3 = \begin{vmatrix} 1 & 2 & 1 & 4 \\ 3 & 1 & 0 & 11 \\ 2 & -3 & 1 & 4 \\ 1 & 1 & 0 & 1 \end{vmatrix} = +50$$

$$D_2 = \begin{vmatrix} 1 & 1 & -1 & 4 \\ 3 & 0 & 1 & 11 \\ 2 & 1 & -1 & 4 \\ 1 & 0 & 1 & 1 \end{vmatrix} = -10 \quad D_4 = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 1 & 0 \\ 2 & -3 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 10$$

$$\text{则 } x_1 = \frac{D_1}{D} = \frac{5}{8} \quad x_2 = \frac{D_2}{D} = \frac{1}{8} \quad x_3 = \frac{D_3}{D} = -\frac{5}{8} \quad x_4 = \frac{D_4}{D} = -\frac{1}{8}$$

$$(3) D = \begin{vmatrix} 2 & 1 & -5 & 1 \\ 1 & -3 & 0 & -6 \\ 0 & 2 & -1 & 2 \\ 1 & 4 & -7 & 0 \end{vmatrix} = 27 \quad D_1 = \begin{vmatrix} 1 & 1 & -5 & 1 \\ 2 & -3 & 0 & -6 \\ -1 & 2 & -1 & 2 \\ 0 & 4 & -7 & 0 \end{vmatrix} = 27$$

$$D_2 = \begin{vmatrix} 2 & 1 & -5 & 1 \\ 1 & 0 & 0 & -6 \\ 0 & 0 & -1 & 2 \\ 1 & -1 & -7 & 0 \end{vmatrix} = 15 \quad D_3 = \begin{vmatrix} 2 & 1 & 1 & 1 \\ 1 & -3 & 2 & -6 \\ 0 & 2 & 0 & 2 \\ 1 & 4 & -1 & 0 \end{vmatrix} = 6 \quad D_4 = \begin{vmatrix} 2 & 1 & -5 & 2 \\ 1 & -3 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 1 & 4 & -7 & -1 \end{vmatrix} = -12$$

$$\text{故 } x_1 = \frac{D_1}{D} = 1 \quad x_2 = \frac{D_2}{D} = \frac{5}{9} \quad x_3 = \frac{D_3}{D} = \frac{2}{9} \quad x_4 = \frac{D_4}{D} = -\frac{4}{9}$$

2.

$$\text{依题意, } \begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 2a & 1 \end{vmatrix} = 0 \text{ 即 } a(1-a) = 0 \text{ 时有非零解.}$$

∴ 当 $a = 0$ 或 $a = 1$ 时有非零解.

(B)

1.

证明: 设 $P(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$

则由 $P(x_i) = y_i$, 可知 a_0, a_1, \dots, a_{n-1} 为该方程组的解.

$$\text{由于 } D = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{n-1} \\ 1 & x_2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_i - x_j) \neq 0$$

故该方程组有唯一解.

即存在唯一的次数小于 n 的多项式 $P(x)$, 使 $P(x_i) = y_i$.

2.

$$\text{由 } \left(\begin{array}{ccc|c} 1 & 1 & b & 4 \\ -1 & b & 1 & b^2 \\ 1 & -1 & 2 & -4 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & b & 4 \\ 0 & b+1 & b+1 & b^2+4 \\ 0 & -2 & 2-b & -8 \end{array} \right)$$

则当 $b = -1$ 时, 方程组无解.

$$b \neq -1 \text{ 时 } \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & b & 4 \\ 0 & 1 & 1 & \frac{b^2+4}{b+1} \\ 0 & 0 & 4-b & \frac{2b(b-4)}{b+1} \end{array} \right)$$

故当 $b \neq -1, 4$ 时, 方程组有唯一解.

$$b = 4 \text{ 时, 方程组有无穷解, 此时 } \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

取自由未知量 x_3 为1, 得基础解系 $\eta_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$

取 $x_3 = 0$, 得特解 $\eta^* = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$

故 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$, k 为任意常数.

(C)

1.

设圆上的点为 (x, y) , 则

设圆方程为 $A(x^2 + y^2) + Dx + Ey + F = 0$

$$\text{则} \begin{cases} A(x^2 + y^2) + Dx + Ey + F = 0 \\ A(x_1^2 + y_1^2) + Dx_1 + Ey_1 + F = 0 \\ A(x_2^2 + y_2^2) + Dx_2 + Ey_2 + F = 0 \\ A(x_3^2 + y_3^2) + Dx_3 + Ey_3 + F = 0 \end{cases}$$

由该方程组为非零解, 则

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

此即过这三点的圆方程.

2.

设平面方程为 $Ax + By + Cz + D = 0$

$$\text{则} \begin{cases} Ax + By + Cz + D = 0 \\ Ax_0 + By_0 + Cz_0 + D = 0 \end{cases}$$

由于该平面与 π_1, π_2 垂直, 则

$$(A, B, C) \cdot (a_1, b_1, c_1) = 0 \quad (A, B, C) \cdot (a_2, b_2, c_2) = 0$$

$$\text{即} \begin{cases} Aa_1 + Bb_1 + Cc_1 = 0 \\ Aa_2 + Bb_2 + Cc_2 = 0 \\ Ax_0 + By_0 + Cz_0 + D = 0 \end{cases}$$

该方程组有非零解, 则

$$\begin{vmatrix} x & y & z & 1 \\ a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ x_0 & y_0 & z_0 & 1 \end{vmatrix} = 0$$

习题3.1

(A)

1.

(1) $\alpha_1 + 2\alpha_2 - \alpha_3 = (3, 8, -8)^T$

(2) $(\alpha_1 + \alpha_2) + 2(\alpha_2 + \alpha_3) - 3(\alpha_3 + \alpha_1) = (3, 12, -19)^T$

(3) $(\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3) + (\alpha_3 - \alpha_1) = (0, 0, 0)^T$

2.

依题意, 有 $\gamma = \frac{5}{3}\beta - \frac{2}{3}\alpha = \frac{1}{3}(4, 10, 16)^T$

3.

$\gamma = \frac{3}{2}\beta - \frac{1}{2}\alpha = (1, 2, 3, 4)^T$

4.

(1) 依题意,

$$\begin{aligned} V_1 &= \{x_1\varepsilon_1 + x_2\varepsilon_2 \mid x_1, x_2 \in \mathbf{R}\} \\ &= \{(x_1, x_2, 0, 0)^T \mid x_1, x_2 \in \mathbf{R}\} \end{aligned}$$

(2) 对任意向量 $\alpha = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, 总有 $\alpha = a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 + d\varepsilon_4$,

故 $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ 生成的子空间为 \mathbf{R}^4

5.

$V = \{k_1\alpha + k_2\beta = (k_1, k_1, k_2)^T \mid k_1, k_2 \in \mathbf{R}\}$, 几何意义为过 z 轴的平面.

6.

(1) 不是. 设 $f(x) \in V$, 则 $f(x) - f(x) = 0 \notin V$

$\therefore V$ 不能构成线性空间

(2) 是. 首先, V 中的矩阵有加法运算, 且满足交换律, 结合律.

V 中有零矩阵 O , 使得对任意 $A \in V$, 有 $A + O = A$.

对 $\forall A \in V, \exists -A \in V$, 使 $A + (-A) = 0$, 从而性质 (1) \sim (4) 满足.

由矩阵乘法定义易验证性质 (5) \sim (8) 也满足.

故 V 是线性空间.

(B)

1.

证明: $\because V_1 \cap V_2 \subset V_1, V_1 \cap V_2 \subset V_2$, 则:

$$(i) \quad \alpha + \beta = \beta + \alpha$$

$$(ii) \quad (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$$

$$(iii) \quad \text{又由于 } 0 \in V_1, 0 \in V_2, \text{ 故 } 0 \in V_1 \cap V_2$$

对任何 $\alpha \in V_1 \cap V_2$, 有 $\alpha + 0 = \alpha$

$$(iv) \quad \text{对 } \alpha \in V_1 \cap V_2, \text{ 一定有 } \beta \in V_1 \cap V_2, \text{ 使 } \alpha + \beta = 0$$

$$(v) \quad 1\alpha = \alpha$$

$$(vi) \quad \lambda(\mu\alpha) = (\lambda\mu)\alpha$$

$$(vii) \quad (\lambda + \mu)\alpha = \lambda\alpha + \mu\alpha$$

$$(viii) \quad \lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$$

故 $V_1 \cap V_2$ 也是 \mathbf{R} 上的线性空间

2.

证明同1, 易验证满足八条性质.

3.

可以.

$$(1) \quad a \oplus b = ab = ba = b \oplus a$$

$$(2) \quad (a \oplus b) \oplus c = ab \oplus c = abc = a \oplus (b \oplus c)$$

$$(3) \quad a \oplus 1 = a \cdot 1 = a \quad (\text{存在零元为} 1)$$

$$(4) \quad a \oplus \frac{1}{a} = a \cdot \frac{1}{a} = 1 \quad (\text{存在负元素为} \frac{1}{a})$$

$$(5) \quad 1 \odot a = a^1 = a$$

$$(6) \quad k \odot (l \odot a) = k \odot a^l = a^{lk} = a^{kl} = (kl) \odot a$$

$$(7) (k+l) \odot a = a^{k+l} = a^k \cdot a^l = a^k \oplus a^l = k \odot a \oplus l \odot a$$

$$(8) k \odot (a \oplus b) = k \odot ab = k \odot a \oplus k \odot b$$

故可以构成线性空间

4.

(1) 证明: 设 $0_1, 0_2$ 为两个零元素, 则

$$\alpha + 0_1 = \alpha, \quad \alpha + 0_2 = \alpha$$

于是

$$\begin{cases} 0_1 + 0_2 = 0_1 \\ 0_2 + 0_1 = 0_2 \end{cases} \Rightarrow 0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$$

故零元素唯一

(2) 证明: 设 α 有两个负元素 β, γ , 则 $\alpha + \beta = 0, \alpha + \gamma = 0$

$$\text{则 } \beta = \beta + 0 = \beta + (\alpha + \gamma) = (\alpha + \beta) + \gamma = 0 + \gamma = \gamma$$

(3) 证明: $\alpha + 0\alpha = 1\alpha + 0\alpha = (1+0)\alpha = 1\alpha = \alpha$, 所以 $0\alpha = 0$,

$$\alpha + (-1)\alpha = [1 + (-1)]\alpha = 0\alpha = 0$$

$$\text{故 } (-1)\alpha = -\alpha. \quad k0 = k[\alpha + (-1)\alpha] = k\alpha + (-k)\alpha = [k + (-k)]\alpha = 0\alpha = 0$$

(4) 证明: 若 $k \neq 0$, 则 $\frac{1}{k}(k\alpha) = \frac{1}{k} \cdot 0 = 0$ 而 $\frac{1}{k}(k\alpha) = (\frac{1}{k} \cdot k)\alpha = 1\alpha = \alpha$ 故 $\alpha = 0$

习题3.2

(A)

1.

$$(1) \left(\begin{array}{cccc|c} 1 & -2 & 3 & -1 & 1 \\ 3 & -1 & 5 & -3 & 2 \\ 2 & 1 & 2 & -2 & 3 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & \frac{7}{5} & -1 & 0 \\ & 1 & -\frac{9}{5} & 0 & 0 \\ & & & & -1 \end{array} \right)$$

故不能表示.

$$(2) \left(\begin{array}{cccc|c} 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 & 2 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ & & 1 & 0 & 2 \\ & & & 1 & -1 \end{array} \right) \text{ 故可以表示.}$$

$$\beta = \alpha_1 + 2\alpha_3 - \alpha_4$$

2.

(1) 有零向量 α_3^T , 故可取 $k_1 = k_2 = 0, k_3 = 1$ 使 $k_1\alpha_1^T + k_2\alpha_2^T + k_3\alpha_3^T = 0$. 故线性相关.

(2) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 3 \\ 3 & 6 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, 故 $\alpha_3 + \alpha_1 - \alpha_2 = 0$ 线性相关

(3) $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 6 \\ 3 & 6 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, 故线性无关.

3.

(1) 错误. 如 $\alpha_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(2) 错误. 如 $\alpha_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(3) 正确. 若 $\alpha_1, \alpha_2, \dots, \alpha_s$ 线性相关, 则存在不全为 0 的 k_1, k_2, \dots, k_s , 使

$$k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s = 0$$

不妨设 $k_i \neq 0$, 则 $\alpha_i = -\frac{1}{k_i}(k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s)$.

即 α_i 可以由其余的向量线性表示, 矛盾!

故向量组线性无关.

(4) 正确. 为(3)的逆否命题

(5) 错误. 如 $\alpha_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(6) 正确. 依题意, 存在不全为 0 的 k_1, k_2, \dots, k_s 使 $k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s = 0$

则 $k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s + 0\beta_1 + 0\beta_2 + \dots + 0\beta_t = 0$

故整体组相关

(7) 错误. 如 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \beta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

4.

$A\alpha = \begin{pmatrix} a \\ 2a+3 \\ 3a+4 \end{pmatrix}$ 与 $\alpha = \begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix}$ 线相关, 则对应元素成比例, 有

$$\frac{a}{a} = \frac{2a+3}{1} = \frac{3a+4}{1}$$

得 $a = -1$

5.

设 $k_1(\alpha_1 + \alpha_2) + k_2(\alpha_2 + \alpha_3) + k_3(\alpha_3 + \alpha_1) = 0$

则整理得 $(k_1 + k_3)\alpha_1 + (k_1 + k_2)\alpha_2 + (k_2 + k_3)\alpha_3 = 0$

由于 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, 则

$$\begin{cases} k_1 + k_3 = 0 \\ k_1 + k_2 = 0 \\ k_2 + k_3 = 0 \end{cases} \Rightarrow \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

故 $\alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_1$ 线性无关.

又 $(\alpha_1 - \alpha_2) + (\alpha_2 - \alpha_3) + (\alpha_3 - \alpha_1) = \alpha_1 - \alpha_2 + \alpha_2 - \alpha_3 + \alpha_3 - \alpha_1 = 0$,

故取 k'_1, k'_2, k'_3 为 $1, 1, 1$,

可知 $\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \alpha_3 - \alpha_1$ 线性相关.

(B)

1.

(1) 相关. 因为4个3维向量必然线性相关.

(2) 无关. 考虑 $\beta_1^T = (1, 0, 0)$, $\beta_2^T = (0, 2, 3)$, $\beta_3^T = (0, 0, 4)$ 则

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{vmatrix} = 8 \neq 0$$

故 $\beta_1^T, \beta_2^T, \beta_3^T$ 线性无关. 则其延伸组 $\alpha_1^T, \alpha_2^T, \alpha_3^T$ 必线性无关.

2.

充分性: 若 α_i 可以由 $\alpha_1, \alpha_2, \dots, \alpha_{i-1}$ 线性表示, 则

$$\alpha_i = k_1\alpha_1 + k_2\alpha_2 + \dots + k_{i-1}\alpha_{i-1}$$

即

$$k_1\alpha_1 + k_2\alpha_2 + \dots + k_{i-1}\alpha_{i-1} - \alpha_i = 0$$

取 $k_i = -1, k_{j+1} = 0 (j \leq i-1)$, 可知 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性相关.

必要性: 由 $\alpha_1, \alpha_2, \dots, \alpha_m$ ($\alpha_i \neq 0$) 线性相关, 则存在不全为0的 k_1, k_2, \dots, k_m , 使

$$k_1\alpha_1 + k_2\alpha_2 + \dots + k_m\alpha_m = 0$$

则从 k_m 开始往前, 必有一个 k_i ($i \neq 1$) 使 $k_i \neq 0$ (否则 $k_1\alpha_1 = 0$, 又 $\alpha_1 \neq 0$, 矛盾). 故有

$$\begin{aligned} k_1\alpha_1 + k_2\alpha_2 + \dots + k_i\alpha_i &= 0 \\ \Rightarrow \alpha_i &= -\frac{1}{k_i}(k_1\alpha_1 + k_2\alpha_2 + \dots + k_{i-1}\alpha_{i-1}) \end{aligned}$$

3.

证明: 据已知有

$$\begin{aligned} r(\alpha_1, \alpha_2, \dots, \alpha_r) &= r(\alpha_1, \alpha_2, \dots, \alpha_r, \beta) \\ r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}) + 1 &= r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}, \beta) \end{aligned}$$

考察

$$(\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \dots \quad \alpha_{r-1}) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r-1} \end{pmatrix} = \alpha_r$$

$$\begin{aligned} r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}, \alpha_r) &\leq r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}) + 1 \\ &= r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}, \beta) \\ &\leq r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}, \alpha_r, \beta) \\ &= r(\alpha_1, \alpha_2, \dots, \alpha_r). \end{aligned}$$

故 $r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}) + 1 = r(\alpha_1, \alpha_2, \dots, \alpha_r)$, 即 α_r 不能由 $\alpha_1, \dots, \alpha_{r-1}$ 表出.

由于 $r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}, \beta) = r(\alpha_1, \alpha_2, \dots, \alpha_r, \beta) = r(\alpha_1, \alpha_2, \dots, \alpha_{r-1}, \beta, \alpha_r)$

故 α_r 可以由 $\alpha_1, \alpha_2, \dots, \alpha_{r-1}, \beta$ 线性表出.

(C)

1.

证明: 设 $k_1\alpha + k_2A\alpha + \dots + k_mA^{m-1}\alpha = 0$ 则

$$A^{m-1}(k_1\alpha + k_2A\alpha + \dots + k_mA^{m-1}\alpha) = 0$$

$$\Rightarrow k_1A^{m-1}\alpha = 0$$

又 $A^{m-1}\alpha \neq 0$

$\therefore k_1 = 0$ 同理依次对假设式乘 $A^{m-2}, A^{m-3}, \dots, A, 1$, 有

$$k_2 = 0, k_3 = 0, \dots, k_m = 0$$

故 $\alpha, A\alpha, \dots, A^{m-1}\alpha$ 线性无关.

2.

证明: 设

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0 \quad (1)$$

则

$$k_1A\alpha_1 + k_2A\alpha_2 + k_3A\alpha_3 = 0$$

即

$$(k_1 + k_2)\alpha_1 + (k_1 + k_3)\alpha_2 + k_3\alpha_3 = 0 \quad (2)$$

(2) - (1) 得

$$k_2\alpha_1 + k_3\alpha_2 = 0 \quad (3)$$

则

$$k_2A\alpha_1 + k_3A\alpha_2 = 0$$

即

$$k_2\alpha_1 + k_3\alpha_1 + k_3\alpha_2 = 0 \quad (4)$$

(4) - (3) $\Rightarrow k_3\alpha_1 = 0$, 又 $\alpha_1 \neq 0$ $\therefore k_3 = 0$ 将 k_3 往(3)、(1)回代可得

$$k_2 = 0, k_1 = 0$$

故 $\alpha_1, \alpha_2, \alpha_3$ 线性无关.

3.

证明: 设 $k_1\beta_1 + k_2\beta_2 + \cdots + k_s\beta_s = 0$ 可得 $(k_1 + k_2)\alpha_1 + (k_1 + k_2)\alpha_2 + (k_2 + k_3)\alpha_3 + \cdots + (k_{s-1} + k_s)\alpha_s = 0$ 由于 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性无关, 故

$$\begin{cases} k_1 + 0 + \cdots + k_s = 0 \\ k_1 + k_2 + \cdots + 0 = 0 \\ 0 + k_2 + k_3 + \cdots + 0 = 0 \\ \cdots \\ 0 + \cdots + k_{s-1} + k_s = 0 \end{cases}$$

其系数行列式

$$|A| = \begin{vmatrix} 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 - (-1)^{s-1} \end{vmatrix} = 1 - (-1)^{s-1}$$

则当 s 为奇数时 $|A| = 2$, 向量组线性无关;

当 s 为偶数时 $|A| = 0$, 向量组线性相关.

习题3.3

(A)

1.

(1) 由 $\begin{pmatrix} 1 & 0 & 2 & 0 \\ & 1 & -3 & 0 \\ & & & 0 \end{pmatrix}$ 知 α_1, α_2 为一个极大线性无关组. 秩为2.

(2) 由 $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & -\frac{5}{2} \\ & 1 & \frac{15}{4} \\ & & 1 \end{pmatrix}$ 知 $\alpha_1, \alpha_2, \alpha_4$ 为一个极大线性无关组. 秩为3.

2.

(1) 不正确. 不一定是极大线性无关组

(2) 正确. 若秩大于等于 r , 则存在 r 个线性无关的向量, 与已知矛盾. 故命题正确.

(3) 错误. 如 $(1, 0)^T, (0, 1)^T, (2, 0)^T$.

(4) 正确. 由秩的定义, 存在 r 个线性无关的向量, 故其中的 $r - 1$ 个向量必线性无关

3.

依题意, $r(\alpha_1, \alpha_2, \cdots, \alpha_r) = r(\alpha_1, \alpha_2, \cdots, \alpha_{r+1}) = r$

则向量组A的秩为 r .

又 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 是 r 个线性无关的向量,

设 α_j ($j = 1, 2, \cdots, n$) 为向量组 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 中的任一向量

则 $\alpha_1, \alpha_2, \cdots, \alpha_r, \alpha_j$ 线性相关

故 A中任一向量都可以由 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 线性表示

从而 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 是极大线性无关组.

4.

过渡矩阵为

$$\begin{aligned}
 Q &= (\alpha_1, \alpha_2, \alpha_3)^{-1}(\beta_1, \beta_2, \beta_3) \\
 &= \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -2 & 2 & 1 \\ 1 & -1 & -2 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -2 & 1 & 0 \end{pmatrix}
 \end{aligned}$$

在基 $(\alpha_1, \alpha_2, \alpha_3)$ 下: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (\alpha_1, \alpha_2, \alpha_3)^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

在基 $(\beta_1, \beta_2, \beta_3)$ 下: $\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = Q^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

5.

$$\begin{aligned}
 (1) \quad (\alpha, \beta) &= 1 \times (-2) + 2 \times 1 + 1 \times \frac{1}{2} = \frac{1}{2} \\
 (\alpha, \gamma) &= 1 \times 2 + 2 \times (-2) + 1 \times 2 = 0 \\
 (\beta, \gamma) &= 2 \times (-2) - 2 \times 1 + 2 \times \frac{1}{2} = -5
 \end{aligned}$$

$$(2) \quad \frac{1}{|\alpha|} \alpha = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$(3) \quad \theta = \arccos \frac{|(\alpha, \beta)|}{|\alpha||\beta|} = \arccos \frac{1}{3\sqrt{14}}$$

6.

$$\begin{aligned}
 (1) \quad &\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \neq E \\
 &\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 8 & 0 \\ 8 & 9 & 4 \\ 0 & 4 & 9 \end{pmatrix} \neq E
 \end{aligned}$$

$$\begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 9 & & \\ & 9 & \\ & & 9 \end{pmatrix} \neq E$$

$$\frac{1}{3} \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = E$$

$$\text{故 } \frac{1}{3} \begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix} \text{ 是正交矩阵.}$$

7.

$$\beta_1 = \alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

(B)

1.

$$\text{设 } \alpha_3 = (x, y, z)^T, \text{ 则 } \begin{cases} x + 2y + 2z = 0 \\ -2x + y = 0 \end{cases}$$

$$\text{由于 } \begin{pmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \frac{2}{5} \\ & 1 & \frac{4}{5} \end{pmatrix},$$

$$\text{故可取 } x = -\frac{2}{5}z, y = -\frac{4}{5}z$$

$$\text{又由 } x^2 + y^2 + z^2 = 1$$

$$\Rightarrow \alpha_3 = \pm \frac{1}{3\sqrt{5}} \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$$

2.

因 $\alpha_1, \alpha_2, \dots, \alpha_s$ 中有 r_1 个向量线性无关,

$\beta_1, \beta_2, \dots, \beta_t$ 中有 r_2 个向量线性无关,

则整体组中的线性无关的向量个数一定大于 r_1 与 r_2 中的较大者.

又因 $\forall \alpha_1, \alpha_2, \dots, \alpha_{r_1}, \alpha_{r_1+1}$ 线性相关,

则 $\alpha_1, \alpha_2, \dots, \alpha_{r_1}, \alpha_{r_1+1}, \beta_1, \beta_2, \dots, \beta_{r_2}$ 也必然线性相关,

故秩 $r < r_1 + r_2 + 1$,

综上, $\max\{r_1, r_2\} \leq r_3 \leq r_1 + r_2$.

3.

设 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\beta = 0$

则 $k_4\beta^2 = 0$, 从而有 $k_4 = 0$.

又 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, $\therefore k_1 = k_2 = k_3 = 0$

故 $\alpha_1, \alpha_2, \alpha_3, \beta$ 线性无关.

4.

依题意, 有 $(\alpha_1, \alpha_2, \dots, \alpha_m) \cdot \beta = 0$

设 $k_1\alpha_1 + k_2\alpha_2 + \dots + k_m\alpha_m + k_{m+1}\beta = 0$

$$\Rightarrow k_1\alpha_1\beta + k_2\alpha_2\beta + \dots + k_m\alpha_m\beta + k_{m+1}\beta^2 = 0$$

$$\Rightarrow k_{m+1}\beta^2 = 0$$

$$\Rightarrow k_{m+1} = 0$$

又 $\alpha_1, \alpha_2, \dots, \alpha_m$ 线性无关, 则 $k_1 = k_2 = \dots = k_m = 0$

从而 $\alpha_1, \alpha_2, \dots, \alpha_m, \beta$ 线性无关.

(C)

1.

证明: **必要性:** 设 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性无关, α 是任意一个 n 维向量.

$\therefore n+1$ 个 n 维向量必线性相关,

故 α 可以由 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性表示.

充分性: 设任何一个 n 维向量可由 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性表示.

故 $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, \dots, 0)$, \dots , $e_n = (0, 0, \dots, 1)$ 能被 $\alpha_1, \alpha_2, \dots, \alpha_n$ 表示.

又 e_1, e_2, \dots, e_n 这组向量可以表示任何 n 维向量, 故 e_1, e_2, \dots, e_n 与 $\alpha_1, \alpha_2, \dots, \alpha_n$ 等价,

所以 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性无关.

2.

证明: 设 $\alpha_1, \alpha_2, \dots, \alpha_s$ 是某一向量组中的线性无关部分组.

在向量组考虑向量 β , 若 β 可由 $\alpha_1, \alpha_2, \dots, \alpha_s$ 表示, 则放弃此向量,

否则将 β 添加至 $\alpha_1, \alpha_2, \dots, \alpha_s$, 即 $\alpha_{s+1} = \beta$.

如此遍历下去, 遍历整个原向量组, 使得扩充的部分组 $\alpha_1, \alpha_2, \dots, \alpha_r$ 满足:

(1) 线性无关,

(2) 原向量组中任一向量都可以由此部分组表示,
则该部分组即扩充为了一个极大线性无关组.

3.

证明: 当 $s \leq m - r$ 时,

$$\because r(\alpha_{k_1}, \alpha_{k_2}, \dots, \alpha_{k_s}) > 0 \geq r + s - m$$

此时成立.

当 $s > m - r$ 时, 则设 $\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_r}$ 为一个极大线性无关组.

则 s 个向量中必有 $s + r - m$ 个向量 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{(s+r-m)}}$ 属于极大线性无关组, 故

$$\begin{aligned} r(\alpha_{k_1}, \alpha_{k_2}, \dots, \alpha_{k_s}) &= r(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{(s+r-m)}} + \alpha_{k_i} + \dots) \\ &> r(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{(s+r-m)}}) = s + r - m \end{aligned}$$

综上, 向量组B的秩大于等于 $r - (m - s)$.

习题3.4

1.

基可取 $\alpha_1 = 1, \alpha_2 = i$

若 $k_1\alpha_1 + k_2\alpha_2 = 0$ 则 $k_1 = 0, k_2 = 0$, 故 α_1, α_2 是该空间的一组基.

维数为2.

2.

$$\text{设 } \alpha_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\text{则若 } k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0, \text{ 即 } \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

则 $k_1 = k_2 = k_3 = 0$, 从而 $\alpha_1, \alpha_2, \alpha_3$ 线性无关.

所以 $\alpha_1, \alpha_2, \alpha_3$ 是该空间的一组基.

维数为3.

3.

定义A的一个部分组 $\alpha_1, \alpha_2, \dots, \alpha_r$ 满足:

(1) 向量组 $\alpha_1, \alpha_2, \dots, \alpha_r$ 线性无关

(2) 向量组A的任意向量可以由 $\alpha_1, \alpha_2, \dots, \alpha_r$ 线性表示,

则称 $\alpha_1, \alpha_2, \dots, \alpha_r$ 为A的一个极大线性无关组.

极大线性无关组所含个数称为A的秩.

4.

假设对于一向量有

$$A = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \quad (1)$$

$$A = y_1 A_1 + y_2 A_2 + \cdots + y_n A_n \quad (2)$$

则 (1) - (2) 有

$$(x_1 - y_1)A_1 + (x_2 - y_2)A_2 + \cdots + (x_n - y_n)A_n = 0$$

由于 A_1, A_2, \cdots, A_n 线性无关, 故 $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, 故坐标唯一.

5.

若有 m 阶矩阵 $\mathbf{P} = (p_{ij})$, 使

$$(\beta_1, \beta_2, \cdots, \beta_m) = (\alpha_1, \alpha_2, \cdots, \alpha_m) \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix}$$

则称 \mathbf{P} 为基 $\alpha_1, \alpha_2, \cdots, \alpha_m$ 到基 $\beta_1, \beta_2, \cdots, \beta_m$ 的过渡矩阵.

若 α 在基 $\alpha_1, \alpha_2, \cdots, \alpha_m$ 下坐标为 x_1, x_2, \cdots, x_m , 在基 $\beta_1, \beta_2, \cdots, \beta_m$ 下坐标为 y_1, \cdots, y_m . 则

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

习题3.5

(A)

1.

$$(1) \begin{pmatrix} 1 & 2 & -3 & 4 \\ 2 & 4 & -6 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

故秩 $r = 1$

$$(2) \begin{pmatrix} 2 & -1 & 3 & -2 & 4 \\ 4 & -2 & 5 & 1 & 7 \\ 2 & -1 & 1 & 8 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} & 0 & \frac{13}{2} & \frac{1}{2} \\ & 1 & -5 & 1 & \end{pmatrix},$$

故秩 $r = 2$

$$(3) \begin{pmatrix} 3 & 2 & -1 & -3 & -1 \\ 2 & -1 & 3 & 1 & -3 \\ 2 & 0 & 5 & 1 & 8 \\ 5 & 1 & 2 & -2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -\frac{2}{5} & -\frac{3}{5} \\ & 1 & 0 & -\frac{18}{25} & \frac{27}{5} \\ & & 1 & \frac{9}{25} & \frac{14}{5} \end{pmatrix}$$

故秩 $r = 3$

2.

$$(1) \begin{pmatrix} 1 & 3 & 0 & 2 & 2 \\ -1 & 0 & 3 & 1 & -2 \\ 2 & 7 & 1 & 5 & 4 \\ 4 & 4 & -8 & 6 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -3 & 0 & 2 \\ & 1 & 1 & 0 & 0 \\ & & & 1 & 0 \end{pmatrix} \text{ 秩为 } 3$$

则极大线性无关组可取 $\alpha_1, \alpha_2, \alpha_4$, 且有

$$\alpha_3 = -3\alpha_1 + \alpha_2, \quad \alpha_5 = 2\alpha_1$$

$$(2) \begin{pmatrix} 1 & 2 & 3 & 1 & 3 \\ 1 & 3 & 4 & 2 & 2 \\ 2 & 7 & 9 & 3 & 7 \\ 3 & 7 & 10 & 2 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 & 3 \\ & 1 & 1 & 0 & 1 \\ & & & 1 & -2 \end{pmatrix}$$

则秩为 3. $\alpha_1, \alpha_2, \alpha_4$ 是一个极大无关组, 且

$$\alpha_3 = \alpha_1 + \alpha_2, \quad \alpha_5 = 3\alpha_1 + \alpha_2 - 2\alpha_4$$

3.

$$(1) \text{ 由 } \begin{vmatrix} 1 & -1 & -2 & 3 \\ 1 & -3 & -5 & 2 \\ 1 & 1 & a & 4 \\ 1 & 7 & 10 & 7 \end{vmatrix} = 0 \Rightarrow a = 1, \text{ 又由 } \begin{vmatrix} 1 & -1 & 3 & 0 \\ 1 & -3 & 2 & -1 \\ 1 & 1 & 4 & 1 \\ 1 & 7 & 7 & b \end{vmatrix} = 0 \Rightarrow b = 4$$

$$(2) \begin{pmatrix} 1 & -1 & -2 & 3 & 0 \\ 1 & -3 & -5 & 2 & -1 \\ 1 & 1 & 1 & 4 & 1 \\ 1 & 7 & 10 & 7 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{7}{2} & \frac{1}{2} \\ & 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

故 α_1, α_2 是一个极大无关组,

$$\alpha_3 = -\frac{1}{2}\alpha_1 + \frac{3}{2}\alpha_2, \quad \alpha_4 = \frac{7}{2}\alpha_1 + \frac{1}{2}\alpha_2, \quad \alpha_5 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2.$$

4.

$$(1) \text{ 由 } \begin{pmatrix} 1 & 3 & a & 7 \\ -1 & 2 & 3 & 8 \\ 2 & -1 & 0 & b \\ 0 & 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & a & 7 \\ & 1 & \frac{3+a}{5} & 3 \\ & & 14-2a & 7+b \\ & & \frac{7-a}{5} & 0 \end{pmatrix}$$

以及秩为 2 知

$$\begin{cases} 14-2a=0 \\ 7+b=0 \\ \frac{7-a}{5}=0 \end{cases} \Rightarrow \begin{cases} a=7 \\ b=-7 \end{cases}$$

$$(2) \text{ 由 } \begin{pmatrix} 1 & 3 & 7 & 7 \\ -1 & 2 & 3 & 8 \\ 2 & -1 & 0 & -7 \\ 0 & 1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & -2 \\ & 1 & 2 & 3 \end{pmatrix}$$

则 α_1, α_2 是一个极大无关组, $\alpha_3 = \alpha_1 + 2\alpha_2, \alpha_4 = -2\alpha_1 + 3\alpha_2$.(3) 由于任何两列都线性无关, 故有 $C_4^2 = 6$ 个极大无关组.

5.

$$(1) \text{ 错误. 如 } A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

(2) 正确. 若 A 的所有 $r-1$ 阶子式均为 0, 则秩 $r^* < r-1$, 矛盾.

(3) 正确. 若存在 $r+1$ 阶子式不为零, 则秩 $r^* \geq r+1$, 矛盾.

(4) 错误. 如 $A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

(5) 错误. 如 $A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

(6) 正确. 若存在 r 阶子式不为零, 则秩 $r^* \geq r$ 的逆否命题即该命题.

或者用反证法: 若秩 $r^* \geq r$, 则存在一个 r 阶子式不为零, 矛盾!

6.

由 $r(AB) = m \leq \min\{r(A), r(B)\} \leq r(B)$

知 B 的列向量组的秩 $r(\beta_1, \beta_2, \dots, \beta_m) \geq m$

故 B 的列向量组线性无关.

7.

依题意, $(\alpha_1, \alpha_2, \dots, \alpha_n)x = 0$ 只有零解, 则 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性无关.

故 $r(A) = r(\alpha_1, \alpha_2, \dots, \alpha_n) = n$

(B)

1.

$$A = \begin{pmatrix} 1 & \lambda & -1 & 2 \\ 2 & -1 & \lambda & 5 \\ 1 & 10 & -6 & 1 \end{pmatrix} \xrightarrow{\text{初等行变换}} \begin{pmatrix} 1 & \lambda & -1 & 2 \\ -1-2\lambda & \lambda+2 & 1 \\ 9-3\lambda & \lambda-3 & 0 \end{pmatrix}$$

故当 $\lambda = 3$ 时, $r(A) = 2$

当 $\lambda \neq 3$ 时, $r(A) = 3$.

2.

由于 A 是秩为 1 的 $m \times n$ 矩阵, 则存在可逆矩阵 P, Q , 使

$$A = P \cdot \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}_{m \times n} \cdot Q$$

$$\text{则 } A = P \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m \times 1} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}_{1 \times n} \cdot Q$$

$$\text{故取 } \alpha = P \cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{m \times 1} \quad \beta^T = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}_{1 \times n} Q \text{ 即证.}$$

3.

必要性: 若 $r(A) = r(B)$, 则

$$r(\alpha_1, \alpha_2, \dots, \alpha_n) = r(\alpha_1, \alpha_2, \dots, \alpha_{s-1}, \alpha_{s+1}, \dots, \alpha_n) \leq n-1$$

故 $\alpha_1, \alpha_2, \dots, \alpha_n$ 线性相关,

即所划去的行可用其余的行线性表示.

充分性: 若所划去的行可由其余的行线性表示, 则 $r(B) \leq r(A)$

又

$$r(\alpha_1, \alpha_2, \dots, \alpha_n) \geq r(\alpha_1, \alpha_2, \dots, \alpha_{s-1}, \alpha_{s+1}, \dots, \alpha_n)$$

则 B 中的极大无关组也是 A 中的极大无关组,

故 $r(B) = r(A)$.

4.

记 $r(A) = r$. 若 $s < m - r$, 则 $r(B) \geq 0 > s + r - m$, 成立.

若 $s \geq m - r$, 设 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 是 A 中行向量组的一个极大无关组,

则 B 中必可取到 $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ 中的 $s + r - m$ 个元素 $\alpha_{k_1}, \alpha_{k_2}, \dots, \alpha_{k_{(s+r-m)}}$

故 $r(B) \geq r(\alpha_{k_1}, \alpha_{k_2}, \dots, \alpha_{k_{(s+r-m)}}) = s + r - m$

综上, $r(B) \geq r(A) + s - m$.

(C)

1.

$$\text{证明: 由 } r \begin{pmatrix} A & O \\ E & B \end{pmatrix} \geq r \begin{pmatrix} A & O \\ O & B \end{pmatrix} = r(A) + r(B)$$

$$\text{又 } \begin{pmatrix} A & O \\ E & B \end{pmatrix} \xrightarrow{r_1 - Ar_2} \begin{pmatrix} O & -AB \\ E & B \end{pmatrix} \xrightarrow{c_2 + Bc_1} \begin{pmatrix} O & -AB \\ E & O \end{pmatrix} \rightarrow \begin{pmatrix} AB & O \\ O & E \end{pmatrix}$$

$$\text{故 } r(AB) + r(E) = r \begin{pmatrix} AB & O \\ O & E \end{pmatrix} = r \begin{pmatrix} A & O \\ E & B \end{pmatrix} \geq r(A) + r(B)$$

即 $r(AB) \geq r(A) + r(B) - n$.

2.

必要性: 设 $\alpha_1, \alpha_2, \dots, \alpha_r$ 线性无关.

$$(\beta_1, \beta_2, \dots, \beta_r) = (\alpha_1, \alpha_2, \dots, \alpha_r)C$$

且 $\beta_1, \beta_2, \dots, \beta_r$ 线性无关, 则

$$(\beta_1, \beta_2, \dots, \beta_r) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = (\alpha_1, \alpha_2, \dots, \alpha_r)C \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = 0, \text{ 当且仅当 } C \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = 0 \text{ 且 } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = 0$$

即齐次线性方程组 $CX = 0$ 只有零解, 于是 $|C| \neq 0$

充分性: 若 $|C| \neq 0$, 则

$$(\alpha_1, \alpha_2, \dots, \alpha_r) = (\beta_1, \beta_2, \dots, \beta_r)C^{-1}$$

即 $\alpha_1, \alpha_2, \dots, \alpha_r$ 可以由 $\beta_1, \beta_2, \dots, \beta_r$ 线性表示.

故 $\alpha_1, \alpha_2, \dots, \alpha_r$ 与 $\beta_1, \beta_2, \dots, \beta_r$ 等价, 于是

$$r(\alpha_1, \alpha_2, \dots, \alpha_r) = r(\beta_1, \beta_2, \dots, \beta_r) = r$$

故 $\beta_1, \beta_2, \dots, \beta_r$ 线性无关.

3.

$$\text{令 } K = \begin{pmatrix} a_{11} & \cdots & a_{1r} & \cdots & a_{1s} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nr} & \cdots & a_{ns} \end{pmatrix}, \text{ 设 } r(K) = r \leq \min(n, s)$$

不失一般性, 不妨设 K 前 r 列是极大线性无关组, 则

$$\begin{cases} \beta_1 = a_{11}\alpha_1 + \cdots + a_{n1}\alpha_n \\ \cdots \\ \beta_r = a_{1r}\alpha_1 + \cdots + a_{nr}\alpha_n \\ \cdots \\ \beta_s = a_{1s}\alpha_1 + \cdots + a_{ns}\alpha_n \end{cases}$$

下面证明 $\beta_1, \beta_2, \dots, \beta_r$ 是极大线性无关组.

$$\text{设 } k_1\beta_1 + k_2\beta_2 + \cdots + k_r\beta_r = 0 \Rightarrow (k_1a_{11} + k_2a_{12} + \cdots + k_ra_{1r})\alpha_1 + \cdots + (k_1a_{n1} + \cdots + k_ra_{nr})\alpha_n = 0$$

$$\text{则 } \begin{cases} a_{11}k_1 + a_{12}k_2 + \cdots + a_{1r}k_r = 0 \\ \cdots \\ a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nr}k_r = 0 \end{cases}$$

该方程组系数矩阵秩为 r , 故只有零点 $k_1 = k_2 = \cdots = k_r = 0$,

故 $\beta_1, \beta_2, \dots, \beta_r$ 线性无关.

其次, 任意添加一个向量 β_j 后, 方程组的秩 $r < r + 1$, 则有非零解, 即线性相关.

故向量组 $\beta_1, \beta_2, \dots, \beta_s$ 的秩等于 K 的秩,

故 β_1, \dots, β_s 线性无关 $\Leftrightarrow r(K) = s$.

习题4.1

(A)

1.

$$(1) \text{ 由于 } \begin{pmatrix} 3 & 4 & -7 & 1 \\ 2 & 1 & -6 & 0 \\ -1 & 2 & 5 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{17}{5} & -\frac{1}{5} \\ & 1 & \frac{4}{5} & \frac{2}{5} \\ & & & \end{pmatrix}$$

故取 x_3, x_4 为自由未知量. 令 $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 得基础解系

$$\eta_1 = \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 1 \\ 0 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{pmatrix}$$

$$\text{故 } x = c_1 \eta_1 + c_2 \eta_2 = c_1 \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{pmatrix}$$

(2) 由于 $\begin{pmatrix} 2 & 3 & 1 & 0 \\ -5 & 7 & 0 & 1 \end{pmatrix}$ 则取 x_3, x_4 为自由未知量.

$$\text{令 } \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ 为 } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ 得 } \eta_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} 0 \\ 1 \\ -3 \\ -7 \end{pmatrix}$$

$$\text{故 } x = c_1 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \\ -7 \end{pmatrix}$$

(3) 由于 $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & -1 \\ 5 & 6 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -2 \\ & 1 & 1 & \frac{3}{2} \end{pmatrix}$ 则取 x_3, x_4 为自由未知量

$$\text{令 } \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} \text{ 为 } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \text{ 得 } \eta_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 2 \end{pmatrix}$$

$$\text{故 } x = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ -3 \\ 0 \\ 2 \end{pmatrix}$$

(4) 由于 $\begin{pmatrix} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 2 & -1 & -1 \\ -1 & 2 & -1 & -2 & -3 \\ 2 & -4 & 3 & 1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & & & -3 \\ & & 1 & & 2 \\ & & & 1 & 2 \end{pmatrix}$ 故取 x_2, x_5 为自由未知量

$$\text{令 } \begin{pmatrix} x_2 \\ x_5 \end{pmatrix} \text{ 为 } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ 得 } \eta_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{故 } x = c_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}$$

(5) 由于系数矩阵为 $\begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & \end{pmatrix}$, 则取 x_2, x_3, x_4 为自由未知量.

$$\text{则令 } \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} \text{ 为 } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ 得 } \eta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{故 } x = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$(6) \text{ 同(5), 有 } x = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_{n-1} \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

2.

$$(1) \text{ 对 I, 有 } \begin{pmatrix} 1 & 2 & 3 & -1 \\ 3 & 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -\frac{1}{2} \end{pmatrix} \Rightarrow x_{\text{I}} = a_1 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$$

$$\text{对 II, 有 } \begin{pmatrix} 2 & 3 & 1 & 1 \\ 2 & 2 & 2 & -1 \\ 5 & 5 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & -\frac{5}{6} \\ & 1 & \frac{7}{6} \\ & & 1 & -\frac{5}{6} \end{pmatrix} \Rightarrow x_{\text{II}} = b \begin{pmatrix} -5 \\ -7 \\ 5 \\ 6 \end{pmatrix}$$

$$\text{故取 } a_1 = 5a_0, a_2 = 3a_0 \text{ 时 } x_{\text{I}} = a_0 \begin{pmatrix} 5 \\ -7 \\ 5 \\ 6 \end{pmatrix} \text{ 为公共解.}$$

$$\text{故全部非空公共解为 } x = c \begin{pmatrix} 5 \\ -7 \\ 5 \\ 6 \end{pmatrix}$$

3.

依题意, 系数行列式为0. 即

$$\begin{vmatrix} 1 & 1 & a \\ -1 & a & 1 \\ 1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & a \\ 0 & a+1 & a+1 \\ 0 & -2 & 2-a \end{vmatrix} = (a+1) \begin{vmatrix} 1 & 1 & a \\ & 1 & 1 \\ & & 4-a \end{vmatrix} = (a+1)(4-a) = 0$$

$$\text{当 } a = -1 \text{ 时 } \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} \\ & 1 & -\frac{3}{2} \end{pmatrix}$$

$$\text{则通解为 } c \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$$

$$\text{当 } a = 4 \text{ 时 } \begin{pmatrix} 1 & 1 & 4 \\ -1 & 4 & 1 \\ 1 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 \\ & 1 & 1 \end{pmatrix}$$

$$\text{则通解为 } c \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

4.

依题意, 系数行列式为 0. 即

$$\begin{vmatrix} a & -2 & 3 \\ 1 & a+2 & 3 \\ 2 & 1 & -1 \end{vmatrix} = \begin{vmatrix} a+6 & 0 & 0 \\ 7 & a+5 & 0 \\ 2 & 1 & -1 \end{vmatrix} = -(a+5)(a+6) = 0$$

当 $a = -5$ 时,

$$\text{由 } \begin{pmatrix} -5 & -3 & 3 \\ 1 & -3 & 3 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \text{ 知, 通解为 } c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

当 $a = -6$ 时,

$$\text{由 } \begin{pmatrix} -6 & -3 & 3 \\ 1 & -4 & 3 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{7}{4} \\ 0 & 0 & 0 \end{pmatrix} \text{ 知, 通解为 } c \begin{pmatrix} 1 \\ 7 \\ 9 \end{pmatrix}$$

5.

$$\begin{pmatrix} \xi_1 \\ \xi_1 + 2\xi_2 \\ \xi_1 + 2\xi_2 + 3\xi_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

$$\text{由于 } \begin{vmatrix} 1 & & \\ 1 & 2 & \\ 1 & 2 & 3 \end{vmatrix} = 6 \neq 0. \text{ 故 } r \begin{pmatrix} \xi_1 \\ \xi_1 + 2\xi_2 \\ \xi_1 + 2\xi_2 + 3\xi_3 \end{pmatrix} = r \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$$

故 $\xi_1, \xi_1 + 2\xi_2, \xi_1 + 2\xi_2 + 3\xi_3$ 也是 $Ax = 0$ 的基础解系.

6.

显然 $\beta, \alpha_1, \alpha_2, \dots, \alpha_s$ 与 $\beta, \beta + \alpha_1, \dots, \beta + \alpha_s$ 可以互相线性表示, 故

$$r(\beta, \alpha_1, \alpha_2, \dots, \alpha_s) = r(\beta, \beta + \alpha_1, \dots, \beta + \alpha_s)$$

又 β 不是 $Ax = 0$ 的解, 故 $\beta, \alpha_1, \dots, \alpha_s$ 线性无关,

$$\therefore r(\beta, \alpha_1, \dots, \alpha_s) = s + 1$$

$$\therefore r(\beta, \beta + \alpha_1, \dots, \beta + \alpha_s) = s + 1$$

$$\text{故 } r(\beta + \alpha_1, \dots, \beta + \alpha_s) = s$$

即 $\beta + \alpha_1, \beta + \alpha_2, \dots, \beta + \alpha_s$ 线性无关.

7.

$$\text{由于 } r \begin{pmatrix} A & O \\ O & B \end{pmatrix} = r(A) + r(B)$$

$$\text{又 } r \begin{pmatrix} A & O \\ O & B \end{pmatrix} \leq r \begin{pmatrix} A & O \\ E & B \end{pmatrix} \text{ 且 } \begin{pmatrix} A & O \\ E & B \end{pmatrix} \xrightarrow{c_2 - c_1 B} \begin{pmatrix} A & -AB \\ E & O \end{pmatrix} \xrightarrow{r_1 - Ar_2} \begin{pmatrix} O & -AB \\ E & O \end{pmatrix}$$

$$\text{故 } r(AB) + r(E) = r \begin{pmatrix} O & -AB \\ E & O \end{pmatrix} = r \begin{pmatrix} A & O \\ E & B \end{pmatrix} \geq r \begin{pmatrix} A & O \\ O & B \end{pmatrix} = r(A) + r(B)$$

$\therefore AB = O$, 则 $r(AB) = 0$, 故 $r(A) + r(B) \leq n$.

8.

证明: 当 $r(A) = n$ 时, $|A| \neq 0$, 则 $|A^*| = |A|^{n-1} \neq 0$, 故 $r(A^*) = n$.

当 $r(A) = n - 1$ 时, 则 A 中至少有一个 $n - 1$ 阶子式不为0, 故 A^* 中至少有1个不为0的元素, 故 $r(A^*) \geq 1$

又 $AA^* = |A|E = O$, 故 $r(A) + r(A^*) \leq n$ 即 $r(A^*) \leq 1$

故 $r(A^*) = 1$

当 $r(A) < n - 1$ 时, A 的每个 $n - 1$ 阶子式均为零, 故 $A^* = O$

故 $r(A^*) = 0$

9.

证明: 假设 A 不可逆

则由 $AB = AC \Rightarrow A(B - C) = O$

则 $(B - C)^T A^T = O$, 即 A^T 是 $(B - C)^T x = O$ 的解.

由于 A 不可逆, 则 $r(A^T) < n$

又 $r(A^T) = n - r(B - C)$

故 $r(B - C) > 0$ 则 $B \neq C$, 矛盾!

故 A 可逆

(B)

1.

设所求齐次方程组为 $Ax = O$

由 $n - r(A) = 2$, 且 $n = 4$, 知 $r(A) = 2$

又 $A(\alpha_1, \alpha_2) = O$, 则 $\begin{pmatrix} \alpha_1^T \\ \alpha_2^T \end{pmatrix} A^T = O$

考虑 $Bx = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \end{pmatrix} x = O$, 由 $n - r(B) = 4 - 2 = 2$, 知 $Bx = O$ 的基础解系是 A^T 的列向量

$$B = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \end{pmatrix} = \begin{pmatrix} 2 & 1 & -5 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -6 & -1 \\ 0 & 1 & 7 & 2 \end{pmatrix}$$

则基础解系为 $\begin{pmatrix} 6 \\ -7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$

故 A 可取 $\begin{pmatrix} 6 & -7 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}$

则所求齐次方程组为 $\begin{cases} 6x_1 - 7x_2 + x_3 = 0 \\ x_1 - 2x_2 + x_4 = 0 \end{cases}$

2.

由 $\alpha_4 = \alpha_1 + 2\alpha_2 - \alpha_3$ 知 $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \end{pmatrix} = 0$

即 $\begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \end{pmatrix}$ 是一组解.

又由于 $\alpha_2, \alpha_3, \alpha_4$ 线性无关, 则 $r(A) = 3$

故基础解系的解的个数为 $n - r(A) = 1$

从而 $Ax = O$ 的通解为 $c \begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \end{pmatrix}$

3.

基础解系中向量个数为 $n - r(A) = 1$ 又由于 A 的各行元素之和均为0, 则

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0,$$

故 $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ 是特解

$Ax = 0$ 的通解为 $c \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

4.

(1) 考虑

$$A = \begin{vmatrix} a_1 + b & a_2 & \cdots & a_n \\ a_1 & a_2 + b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n + b \end{vmatrix}$$

当 $b = 0$ 时, 行列式所有列对应成比例, $A = 0$ 当 $b \neq 0$ 时

$$A = \begin{vmatrix} a_1 + b & a_2 & \cdots & a_n \\ a_1 & a_2 + b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n + b \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & a_1 + b & a_2 & \cdots & a_n \\ 0 & a_1 & a_2 + b & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_1 & a_2 & \cdots & a_n + b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & b & 0 & \cdots & 0 \\ -1 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & b \end{vmatrix} = \begin{vmatrix} 1 & \frac{a_1}{b} & \frac{a_2}{b} & \cdots & \frac{a_n}{b} \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1 + \frac{\sum_{i=1}^n a_i}{b}$$

则当 $\sum_{i=1}^n a_i = -b$ 时, $A = 0$,

综上, 当 $\sum_{i=1}^n a_i \neq -b$ 且 $b \neq 0$ 时, 方程组仅有零解.

(2) 当 $b = 0$ 时, 则

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \rightarrow \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

则不妨 $a_1 \neq 0$, 通解为

$$c_1 \begin{pmatrix} -a_2 \\ a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -a_3 \\ 0 \\ a_1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_{n-1} \begin{pmatrix} -a_n \\ 0 \\ 0 \\ \vdots \\ a_1 \end{pmatrix}$$

当 $\sum_{i=1}^n a_i = -b$ 时, 则

$$\begin{pmatrix} a_1 + b & a_2 & \cdots & a_n \\ a_1 & a_2 + b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n + b \end{pmatrix} \rightarrow \begin{pmatrix} a & a_2 & \cdots & a_n \\ -b & b & & \\ -b & & b & \\ \vdots & & & \ddots \\ -b & & & & b \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 1 & & & -1 \\ & 1 & & -1 \\ & & \ddots & \vdots \\ & & & 1 & -1 \\ & & & & 0 \end{pmatrix}$$

则通解为 $c \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

(C)

1.

(1) 由 $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

则 x_3, x_4 为自由未知量, 取 $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

则 $\eta_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} \alpha_1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$ 则 $x = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1\alpha_1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

(2) 依题意, 由 $\begin{pmatrix} -c_2 \\ c_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -k_2 \\ k_1 + 2k_2 \\ k_1 + 2k_2 \\ k_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 = c_2 \\ c_1 = k_2 \\ k_1 = -k_2 \end{cases}$

则 I 与 II 有公共解 $c \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

2.

记 $A = (\alpha_1, \alpha_2, \dots, \alpha_n) \quad B = (\beta_1, \beta_2, \dots, \beta_n)$

若 $(\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n)x = 0$ 有解,

则即非零公共解.

由于 $(A \ B)$ 的列向量组由 A, B 的列向量扩充而成

故 $(A \ B)$ 的列向量组可以由 A 的列极大线性无关组, B 的列极大无关组线性表示, 则

$$r(A \ B) \leq r(A) + r(B) < n$$

从而 $(A \ B)x = 0$ 有非零解,

即 $Ax = 0$ 与 $Bx = 0$ 有非零公共解.

习题4.2

(A)

1.

$$(1) \text{ 由 } \left(\begin{array}{cccc|c} 2 & 1 & 3 & 3 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & -2 & 4 & 1 & 4 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array} \right)$$

$$\text{则取自由未知量为 } 0, \text{ 得特解 } \eta^* = \frac{1}{2} \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{又基础解系为 } \eta = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \text{ 故解为 } x = \frac{1}{2} \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$(2) \text{ 由 } \left(\begin{array}{ccc|c} 1 & 1 & -1 & -1 \\ 2 & -5 & 3 & 2 \\ 7 & -7 & 2 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{3}{7} \\ 0 & 1 & 0 & -\frac{4}{7} \\ 0 & 0 & 1 & 0 \end{array} \right)$$

$$\text{故解为 } x = \frac{1}{7} \begin{pmatrix} -3 \\ -4 \\ 0 \end{pmatrix}$$

$$(3) \text{ 由 } \left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ -1 & 3 & 0 & 1 \\ 2 & 1 & 7 & 5 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

自由未知量为 x_3

$$\text{取 } x_3 = 0, \text{ 得 } \eta^* = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \text{ 取 } x_3 = 1, \text{ 得 } \eta = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

$$\text{则 } x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$$

$$(4) \text{ 由 } \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 2 \\ 1 & -3 & 0 & 1 & -1 \\ 2 & 1 & 7 & 2 & 5 \\ 4 & 2 & 14 & 0 & 6 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

自由未知量为 x_3

$$\text{取 } x_3 = 0, \text{ 得 } \eta^* = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{取 } x_3 = 1, \text{ 得 } \eta = \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{故 } x = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

2.

$$\text{由 } \left(\begin{array}{ccccc} 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & a & -a & -1 \\ 1 & 2 & 0 & 3 & 1 \end{array} \right) \rightarrow \left(\begin{array}{ccccc} 1 & 3 & 2 & 1 & 1 \\ 0 & 1 & a & -a & -1 \\ 0 & 0 & 2-2a & 2-2a & 0 \end{array} \right)$$

则当 $a = 2$ 时, 无解.

$$\text{当 } a \neq 2 \text{ 时, 通解为 } \frac{1}{a-2} \begin{pmatrix} 7a-10 \\ 2-2a \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

3.

系数行列式

$$A = \begin{vmatrix} 1 & 1 & a \\ -1 & a & 1 \\ 1 & -1 & 2 \end{vmatrix} = (a+1) \begin{vmatrix} 1 & 1 & a \\ 1 & 1 & 1 \\ 4-a & & \end{vmatrix} = (a+1)(4-a)$$

则当 $a \neq 4$ 且 $a \neq -1$ 时, 方程组为唯一解.

当 $a = 4$ 时, 方程组有无穷多解, 由于

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 4 \\ -1 & 4 & 1 & 16 \\ 1 & -1 & 2 & -4 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

故通解为 $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + c \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$

当 $a = -1$ 时, 由于 $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 4 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 2 & -4 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$

故为方程组无解.

4.

由 $\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 3 & 2 & 1 & 1 & -3 & a \\ 5 & 4 & 3 & 3 & -1 & b \end{array} \right) \rightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 2 & 6 & 3 \\ & & & & & a \\ & & & & & a-2 \end{array} \right)$

则由方程组有解得 $\begin{cases} a = 0 \\ b = 2 \end{cases}$

则 $\rightarrow \left(\begin{array}{ccccc|c} 1 & 0 & -1 & -1 & -5 & -2 \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$ 自由未知量为 x_3, x_4, x_5 ,

取 $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

得 $\eta^* = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

取 $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

得基础解系 $\eta_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

$$\text{则 } x = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

5.

考虑 $A = (\alpha_1, \alpha_2, \alpha_3)$, 研究 $Ax = \beta$ 的解.(1) 即 $Ax = \beta$ 有唯一解, 则 $|A| \neq 0$

$$\text{即 } \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ a & -2 & 1 \\ 10 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -2 - \frac{a}{2} & 1 - \frac{a}{2} \\ 0 & 0 & -1 \end{vmatrix} = (2 + \frac{a}{2}) \neq 0$$

则 $a \neq -4$. 此时 β 由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示, 且表示法唯一.(2) 即 $Ax = \beta$ 无解, 则 $|A| = 0 \Rightarrow a = -4$

$$\text{由于 } \left(\begin{array}{ccc|c} -4 & -2 & 1 & 1 \\ 2 & 1 & 1 & b \\ 10 & 5 & 4 & c \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{b}{2} \\ 0 & 0 & 3 & 1+2b \\ 0 & 0 & -1 & c-5b \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{b}{2} \\ 0 & 0 & -1 & c-5b \\ 0 & 0 & 0 & 3c-13b+1 \end{array} \right)$$

故当 $3c - 13b + 1 \neq 0$, β 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示. (此时 $a = -4$)(3) 即 $Ax = \beta$ 有无穷多解, 则 $a = -4$ 且 $3c - 13b + 1 = 0$

$$\text{则 } \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{b}{2} \\ 0 & 0 & -1 & c-5b \\ 0 & 0 & 0 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & \frac{b+1}{6} \\ 0 & 0 & 1 & \frac{1+2b}{3} \\ 0 & 0 & 0 & 0 \end{array} \right)$$

自由未知量为 x_2 .

$$\text{令 } x_2 = 0 \Rightarrow \eta^* = \begin{pmatrix} \frac{b-1}{6} \\ 0 \\ \frac{1+2b}{3} \end{pmatrix}$$

$$\text{令 } x_2 = 1 \Rightarrow \eta = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$$

$$\text{则 } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \eta^* + k\eta = \begin{pmatrix} \frac{b-1+3k}{6} \\ k \\ \frac{1+2b}{3} \end{pmatrix}$$

$$\text{故 } \beta = \frac{b-1+3k}{6}\alpha_1 + k\alpha_2 + \frac{2b+1}{3}\alpha_3$$

6. C

A 为特解 + 通解,

B 中 $\frac{1}{2}\eta_1 + k_1\xi_1$ 是 $Ax = \frac{1}{2}\beta$ 的解, $\frac{1}{2}\eta_2 + k_2(\xi_1 + \xi_2)$ 是 $Ax + \frac{1}{2}\beta$ 的解.

故 $k_1\xi_1 + k_2(\xi_1 + \eta_2) + k_2(\eta_1 + \eta_2)$ 是 $Ax = \beta$ 的解.

C 应是 $Ax = 2\beta$ 的解, D 同 B 理可证.

7.

$$\text{由 } \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 3 & a+2 & 3 \\ 1 & a & -2 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & a & 1 \\ 0 & a-2 & -3 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & a & 1 \\ 0 & 0 & (a-3)(a+1) & a-3 \end{array} \right)$$

则 $a = -1$ 时无解.

8.

$$\text{证明: } \lambda_1\eta_1 + \lambda_2\eta_2 + \cdots + \lambda_s\eta_s = (1 - \lambda_2 - \lambda_3 - \cdots - \lambda_s)\eta_1 + \lambda_2\eta_2 + \cdots + \lambda_s\eta_s = \eta_1 + \sum_{i=2}^s \lambda_i(\eta_i - \eta_1)$$

因为 $\eta_i - \eta_1$ 是导出组的解, 故 $\sum_{i=2}^s \lambda_i(\eta_i - \eta_1)$ 也是导出组的解.

又 η_1 是原方程一个特解, 故 $\lambda_1\eta_1 + \lambda_2\eta_2 + \cdots + \lambda_s\eta_s$ 也是 $Ax = \beta$ 的解.

9.

$$(1) \text{ 设 } k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r} + k_{n-r+1}\eta = 0$$

$$\text{则 } k_1A\xi_1 + k_2A\xi_2 + \cdots + k_{n-r}A\xi_{n-r} + k_{n-r+1}A\eta = 0$$

$$\text{即 } k_{n-r+1}\beta = 0$$

$$\text{故 } k_{n-r+1} = 0$$

则 $\xi_1, \xi_2, \dots, \xi_{n-r}, \eta$ 线性无关.

$$(2) \text{ 设 } k_1(\xi_1 + \eta) + k_2(\xi_2 + \eta) + \cdots + k_{n-r}(\xi_{n-r} + \eta) + k_{n-r+1}\eta = 0$$

$$\text{即 } k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r} + (k_1 + k_2 + \cdots + k_{n-r+1})\eta = 0$$

$$\text{则由(1)知 } k_1 + k_2 + \cdots + k_{n-r+1} = 0,$$

$$\text{则 } k_1\xi_1 + k_2\xi_2 + \cdots + k_{n-r}\xi_{n-r} = 0 \text{ 又 } \because \xi_1, \xi_2, \dots, \xi_{n-r} \text{ 线性无关}$$

$$\text{故 } k_1 = k_2 = \cdots = k_{n-r} = 0$$

$$\Rightarrow k_{n-r+1} = 0$$

故 $\xi_1 + \eta, \xi_2 + \eta, \dots, \xi_{n-r} + \eta, \eta$ 线性无关.

(B)

1.

$$\text{由于 } \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 2 & 1 & -6 & 4 & -1 \\ 3 & 2 & a & 7 & -1 \\ 1 & -1 & -b & -2 & b \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 0 \\ 0 & -1 & -2 & -2 & -1 \\ 0 & -1 & a+6 & -2 & -1 \\ 0 & -2 & -4 & -4 & b \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & -4 & 1 & -1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & a+8 & 0 & 0 \\ 0 & 0 & 0 & 0 & b+2 \end{array} \right)$$

故当 $b = -2$ 时方程组有解.(1) 当 $b = -2$ 且 $a = -8$ 时, 自由未知量为 x_1, x_4 ,

$$\text{取 } \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ 得特解 } \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{取 } \begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ 得基础解系}$$

$$\eta_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \eta_2 = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{故此时 } x = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

2.

(1) X 应为 3 行 2 列矩阵, 才满足矩阵乘法.

$$(2) \text{ 由 } \left(\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 4 & s \\ 1 & 0 & 1 & 0 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & s-1 \\ 0 & 1 & 2 & 2-s \\ 0 & 0 & 0 & 1-s \end{array} \right) \text{ 知当 } s \leq 1 \text{ 时, } \begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ s \\ 0 \end{pmatrix} \text{ 有解.}$$

此时 x_3 为自由未知量,

$$\text{取 } x_3 = 0, \text{ 得 } \eta^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ 又基础解系 } \eta = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

故通解为 $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

由 $\left(\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 1 \\ 1 & 0 & 1 & t \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & t+1 \end{array} \right)$ 知当 $t = -1$ 时, $\begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ t \end{pmatrix}$ 有解.

此时 x_3 为自由未知量, 取 $x_3 = 0$, 可得 $\eta^* = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$, 又基础解系 $\eta = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

故通解为 $x = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

(3) $\begin{pmatrix} 0 & -1 \\ 1 & 3 \\ 0 & 0 \end{pmatrix}$

3.

(1) 由 $\left(\begin{array}{cccc|c} 1 & 1 & 0 & -2 & -6 \\ 4 & -1 & -1 & -1 & 1 \\ 3 & -1 & -1 & 0 & 3 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & -2 & -5 \end{array} \right)$

取 x_4 为自由未知量, 令 $x_4 = 0$, 得特解 $\eta^* = \begin{pmatrix} -2 \\ -4 \\ -5 \\ 0 \end{pmatrix}$

令 $x_4 = 1$, 得基础解系 $\eta = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$

则通解为 $\begin{pmatrix} -2 \\ -4 \\ -5 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$

(2) 由

$$\left(\begin{array}{cccc|c} 1 & m & -1 & -1 & -5 \\ 0 & n & -1 & -2 & -11 \\ 0 & 0 & 1 & -2 & -t+1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 + \frac{4m}{n} & -7 + \frac{24m}{n} \\ 0 & 1 & 0 & -\frac{4}{n} & -\frac{12}{n} + t \\ 0 & 0 & 1 & -2 & -t+1 \end{array} \right)$$

则

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 + \frac{4m}{n} & -7 + \frac{24m}{n} \\ 0 & 1 & 0 & -\frac{4}{n} & -\frac{12}{n} + t \\ 0 & 0 & 1 & -2 & -t+1 \end{array} \right) = \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & -2 & -5 \end{array} \right)$$

$$\Rightarrow m = 2, n = 4, t = 6.$$

4.

$$A = \alpha\beta^T = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 2 & 1 & 0 \\ 1 & \frac{1}{2} & 0 \end{pmatrix}$$

$$B = \beta^T\alpha = \begin{pmatrix} 1 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 2$$

$$\text{则 } 2B^2A^2X = A^4X + B^4X + Y$$

$$\Leftrightarrow 8A^2X - A^4X - 16X = Y$$

$$\text{又 } A^2 = \alpha\beta^T\alpha\beta^T = 2\alpha\beta^T = 2A, \text{ 故 } A^4 = 2^3A = 8A$$

则 $(8A - 16E)X = Y$, 故 X 是该方程组的解.

$$\text{又 } 8A - 16E = \begin{pmatrix} -8 & 4 & 0 \\ 16 & -8 & 0 \\ 8 & 4 & -16 \end{pmatrix}$$

$$\text{由 } \left(\begin{array}{ccc|c} -8 & 4 & 0 & 0 \\ 16 & -8 & 0 & 0 \\ 8 & 4 & -16 & 8 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & \frac{1}{2} \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right), \text{ 自由未知量为 } x_3.$$

$$\text{取 } x_3 = 0, \text{ 则 } \eta^* = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}, \text{ 取 } x_3 = 1, \text{ 得基础解系 } \eta = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\text{则 } X = \begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

5.

$$A = (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 2 & 1 & 1 \\ 2 & 3 & a+2 & a+3 & a+6 & a+4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 & 1 & 1 \\ 0 & 0 & a-1 & a-1 & a+1 & a-1 \end{pmatrix}$$

若 $a = -1$, 则 β_1, β_3 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示, 此时向量组 I 与向量组 II 不等价;

若 $a \neq -1$, 则 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, $\beta_1, \beta_2, \beta_3$ 线性无关, 此时 I 与 II 等价.

6.

依题意, $\eta_{n-r+1} - \eta_1, \eta_{n-r} - \eta_1, \dots, \eta_s - \eta_1$ 是 $Ax = 0$ 的 r 个线性无关解

则 $Ax = \beta$ 的通解为

$$x = \eta_1 + k_1(\eta_{n-r+1} - \eta_1) + \dots + k_r(\eta_s - \eta_1)$$

$$= \left(1 - \sum_{i=1}^r k_i\right) \eta_1 + k_1 \eta_2 + k_2 \eta_3 + \dots + k_r \eta_{n-r+1}$$

记 $\lambda_1 = 1 - \sum_{i=1}^r k_i$, $\lambda_i = k_{r+2-i}$, 则

$$\sum_{i=1}^{n-r+1} \lambda_i = 1$$

故 $Ax = \beta$ 的通解为 $\lambda_1 \eta_1 + \lambda_2 \eta_2 + \dots + \lambda_{n-r+1} \eta_{n-r+1}$, 其中 λ_i 满足 $\sum_{i=1}^{n-r+1} \lambda_i = 1$.

习题5.1

(A)

1.

(1) A 的特征多项式为

$$f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

$\therefore A$ 有两个特征值 $\lambda_1 = 1, \lambda_2 = -1$

对于特征值 $\lambda_1 = 1$, 解 $(\lambda_1 E - A)x = 0$, 即 $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

可得一个特征向量为 $(1, 1)^T$. (全部为 $k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$)

对于特征值 $\lambda_2 = -1$, 解 $(\lambda_2 E - A)x = 0$, 即 $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

得一个特征向量为 $(1, -1)^T$. (全部为 $k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$)

$$(2) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 11 & -25 \\ 4 & \lambda + 9 \end{vmatrix} = (\lambda - 1)^2,$$

特征值为 1, 1

特征值 1 的特征向量由 $\begin{pmatrix} -10 & -25 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

得 $k(-5, 2)^T$.

$$(3) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda + 2 & -1 \\ -5 & \lambda - 2 \end{vmatrix} = (\lambda - 3)(\lambda + 3),$$

特征值为 3, -3

特征值 -3 的特征向量为 $k_1(1, -1)^T$

特征值 3 的特征向量为 $k_2(1, 5)^T$

$$(4) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda + 1 & -2 \\ -8 & \lambda + 1 \end{vmatrix} = (\lambda + 5)(\lambda - 3),$$

特征值为 -5, 3

特征值为 -5 的特征向量为 $k_1(1, -2)^T$

特征值为 3 的特征向量为 $k_2(1, 2)^T$

2.

$$(1) A \text{ 的特征多项式为 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 3 & -6 & -6 \\ 0 & \lambda - 2 & 0 \\ 3 & 12 & \lambda + 6 \end{vmatrix} = (\lambda + 3)(\lambda - 2)\lambda,$$

特征值 -3, 0, 2.

特征值 -3 的特征向量为 $k_1(1, 0, -1)^T$

特征值 0 的特征向量为 $k_2(-2, 0, 1)^T$

特征值 2 的特征向量为 $k_3(2, -5, 3)^T$

$$(2) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 3 & 2 & 0 \\ 1 & \lambda - 3 & 1 \\ 5 & -7 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda - 2)^2,$$

特征值 1, 2, 2

特征值 1 的特征向量为 $k_1(1, 1, 1)^T$

特征值 2 的特征向量为 $k_2(-2, -1, 1)^T$

$$(3) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{vmatrix} = \lambda^2(\lambda - 3),$$

特征值 0, 0, 3

特征值 0 的特征向量为 $k_1(-1, 1, 0)^T + k_2(-1, 0, 1)^T$

特征值 3 的特征向量为 $k_3(1, 1, 1)^T$

$$(4) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 4 & 2 & 1 \\ -5 & \lambda + 2 & 1 \\ 2 & -1 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3,$$

特征值 1, 1, 1

特征值 1 的向量为 $k(-1, -2, 1)^T$

$$(5) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 2 & 8 & 4 \\ 2 & \lambda - 2 & -2 \\ -7 & 14 & \lambda + 9 \end{vmatrix} = (\lambda + 1)(\lambda + 2)^2,$$

特征值 -1, -2, -2

特征值 -1 的特征向量为 $k_1(-4, 2, 7)^T$

特征值 -2 的特征向量为 $k_2(2, 1, 0)^T + k_3(1, 0, 1)^T$

$$(6) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 23 & -10 & -16 \\ 8 & \lambda + 2 & 6 \\ 27 & 11 & \lambda + 19 \end{vmatrix} = (\lambda - 1)^2\lambda,$$

特征值 0, 1, 1

特征值 0 的特征向量为 $k_1(-14, 5, 17)^T$

特征值 1 的特征向量为 $k_2(-6, 2, 7)^T$.

3.

$$(1) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda-1 & -1 & 0 & 0 \\ 0 & \lambda-1 & -1 & 0 \\ 0 & 0 & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^4,$$

特征值 1, 1, 1, 1

特征值 1 的特征向量为 $k_1(0, 0, 0, 1)^T + k_2(1, 0, 0, 0)^T$

$$(2) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda-1 & -1 & -1 & -1 \\ -1 & \lambda-1 & 1 & 1 \\ -1 & 1 & \lambda-1 & 1 \\ -1 & 1 & 1 & \lambda-1 \end{vmatrix} = (\lambda+2)(\lambda-2)^3,$$

特征值 -2, 2, 2, 2

特征值为 -2 的特征向量为 $k_1(-1, 1, 1, 1)^T$ 特征值为 2 的特征向量为 $k_2(1, 1, 0, 0)^T + k_3(1, 0, 1, 0)^T + k_4(1, 0, 0, 1)^T$

4.

证明: 由 $A\alpha = \lambda\alpha$, 左乘 A^{-1} , 有

$$\begin{aligned} A^{-1}(A\alpha) &= A^{-1}(\lambda\alpha) \\ \implies \lambda A^{-1}\alpha &= \alpha \Rightarrow A^{-1}\alpha = \frac{1}{\lambda}\alpha \end{aligned}$$

则 $\frac{1}{\lambda}$ 为 A^{-1} 的特征值.

5.

(1) 证明: $\because A^2\alpha = A(A\alpha) = A(\lambda\alpha) = \lambda(A\alpha) = \lambda^2\alpha$ 设当 $n = k$ 时, $A^k\alpha = \lambda^k\alpha$ 当 $n = k + 1$ 时, $A^{k+1}\alpha = A(A^k\alpha) = A(\lambda^k\alpha) = \lambda^k A\alpha = \lambda^{k+1}\alpha$ 故 $A^n\alpha = \lambda^n\alpha$. $\because A^n = O$, 则 $\lambda^n\alpha = 0$ 又 $\alpha \neq 0$, $\therefore \lambda^n = 0$ 从而 $\lambda = 0$ (2) 证明: 设 $A\alpha = \lambda\alpha$, 同时左乘 A , 有

$$A^2\alpha = \lambda A\alpha = \lambda^2\alpha$$

又 $A^2 = A$, 故 $\lambda\alpha = A\alpha = A^2\alpha = \lambda^2\alpha$ $\Rightarrow \lambda(\lambda - 1)\alpha = 0$ 又 $\alpha \neq 0$ $\Rightarrow \lambda = 0$ 或 $\lambda = 1$

(3) 证明: 设 $A\alpha = \lambda\alpha$, 同时左乘 A , 有

$$A^2\alpha = \lambda A\alpha = \lambda^2\alpha$$

$$\text{又 } \alpha = E\alpha = A^2\alpha$$

$$\Rightarrow (\lambda^2 - 1)\alpha = 0 \text{ 又 } \alpha \neq 0$$

$$\Rightarrow \lambda = 1 \text{ 或 } -1$$

6.

证明: 设 $A\alpha = \lambda\alpha$

$$\text{则 } (kE + A)\alpha = k\alpha + A\alpha = k\alpha + \lambda\alpha = (k + \lambda)\alpha$$

$\therefore k + \lambda$ 是 $kE + A$ 的特征值

7.

$$f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{vmatrix} = \lambda^2(\lambda - 3),$$

特征值为 $0, 0, 3$

特征值 0 的特征向量为 $k_1(-1, 1, 0)^T + l(-1, 0, 1)^T$

特征值 3 的特征向量为 $k_2(1, 1, 1)^T$

故特征值 0 的特征子空间为 $\{k\alpha + l\beta \mid \alpha = (-1, 1, 0)^T, \beta = (-1, 0, 1)^T, k, l \in \mathbf{R}\}$

特征值 3 的特征子空间为 $\{k\alpha \mid \alpha = (1, 1, 1)^T, k \in \mathbf{R}\}$.

8.

依题意, $A\alpha = \lambda\alpha$ 有一个特征向量为 $(1, -2, 1)^T$.

则

$$\begin{pmatrix} k & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & k \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} k-2 \\ -2 \\ k-2 \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ -2 \\ \lambda_0 \end{pmatrix}$$

则 $\lambda_0 = k - 2$, 则

$$|\lambda_0 E - A| = \begin{vmatrix} -2 & -1 & 0 \\ -1 & k-4 & -1 \\ 0 & -1 & -2 \end{vmatrix} = 0 \text{ 得 } k = 3$$

$$\begin{aligned} \text{则 } f_A(\lambda) &= |\lambda E - A| = \begin{vmatrix} \lambda - 3 & -1 & 0 \\ -1 & \lambda - 2 & -1 \\ 0 & -1 & \lambda - 3 \end{vmatrix} = (\lambda - 3) \begin{vmatrix} \lambda - 2 & 1 \\ -1 & \lambda - 3 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & \lambda - 3 \end{vmatrix} \\ &= (\lambda - 3)(\lambda - 1)(\lambda - 4) \end{aligned}$$

故 A 的特征值为 3, 1, 4.

9.

$\because A$ 可逆, 则 $|A| \neq 0, \therefore \lambda \neq 0$

则由 $A^{-1}\alpha = \lambda\alpha \Rightarrow A\alpha = \frac{1}{\lambda}\alpha$

故 α 也是 A 的特征向量, 则

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3+k \\ 2k+2 \\ 3+k \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda k \\ \lambda \end{pmatrix} \Rightarrow \begin{cases} 3+k = \lambda \\ 2k+2 = \lambda k \\ 3+k = \lambda \end{cases}$$

得 $k = 1$ 或 -2 .

10.

(1) 由第5题(1)的证明知由 $A\alpha = \lambda\alpha$ 可得 $A^m\alpha = \lambda^m\alpha$

$$\begin{aligned} \text{则 } f(A)\alpha &= (A^3 - 2A^2 - A + 2E)\alpha \\ &= A^3\alpha - 2A^2\alpha - A\alpha + 2\alpha \\ &= (\lambda^3 - 2\lambda^2 - \lambda + 2)\alpha \\ &= f(\lambda)\alpha \end{aligned}$$

故 $f(\lambda)$ 是 $f(A)$ 的特征值.

(2) 由10.(1)的公式可得

$$1.(1) 0, 0 \quad (2) 0, 0 \quad (3) -40, 8 \quad (4) -168, 8$$

11.

证明: 同10.(1)有 $\varphi(A)\alpha = (a_m A^m + a_{m-1} A^{m-1} + \cdots + a_0 E)\alpha = (a_m \lambda^m + \cdots + a_0)\alpha = \varphi(\lambda)\alpha$
即 $\varphi(\lambda)$ 是 $\varphi(A)$ 的特征值.

(B)

1.

证明: 设 $A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$

$$\text{则 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - a & -b \\ -b & \lambda - a \end{vmatrix} = (\lambda - a - b) \begin{vmatrix} 1 & -b \\ 1 & \lambda - a \end{vmatrix} = (\lambda - a - b)(\lambda - a + b)$$

故 A 一定有两个实特征值 $a + b, a - b$.

2.

证明: $\because |\lambda E - A^T| = |(\lambda E - A)^T| = |\lambda E - A|,$

$\therefore A$ 与 A^T 有相同的特征多项式

3.4.

只考虑 n 维向量. 首先, 由于

$$r(\alpha\beta^T) \leq \min\{r(\alpha), r(\beta^T)\} = 1$$

故矩阵 $\alpha\beta^T$ 必有 $\lambda = 0$ 的特征值. 记 $A = \alpha\beta^T$

则 $\lambda = 0$ 的 $Ax = 0$ 所含的向量个数

$$n - r(A) \geq n - 1$$

则 λ 的为0的代数重数至少有 $n - 1$ 个,

故 A 最多只有1个不为0的特征值. 记 $k = \beta^T\alpha$

又

$$(\alpha\beta^T)\alpha = \alpha \cdot k = k\alpha$$

则 $k = \beta^T\alpha$ 是一个特征值.

则 A 的特征多项式为 $\lambda^{n-1}(\lambda - \beta^T\alpha)$

特征值为 $\beta^T\alpha, 0$ ($n - 1$ 重)

5.

$$(1) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 4 & 1 & 1 \\ 12 & \lambda - 1 & -5 \\ -4 & 2 & \lambda \end{vmatrix} = \lambda(\lambda - 2)(\lambda - 3)$$

(2) 特征值为 $0, 2, 3$.

特征值 0 的特征向量为 $k_1(1, 2, 2)^T$

特征值 2 的特征向量为 $k_2(3, -1, 7)^T$

特征值 3 的特征向量为 $k_3(-1, 1, -2)^T$

(3) 由A组5.(1)的结论, 有 A^5 的特征值为 λ^5 , $A + 2E$ 的特征值为 $\lambda + 2$

故 A^5 的特征值为 $0, 32, 243$

$A + 2E$ 的特征值为 $2, 4, 5$

6.

证明: 由定理5.1知 $m = 2$ 时成立.

假设当 $m = k$ 时成立, 则立 $m = k + 1$ 时, 对于

$$k_1x_1 + k_2x_2 + \cdots + k_kx_k + k_{k+1}x_{k+1} = 0 \quad (1)$$

对(1)用 A 左乘两端有

$$k_1\lambda_1x_1 + k_2\lambda_2x_2 + \cdots + k_k\lambda_kx_k + k_{k+1}\lambda_{k+1}x_{k+1} = 0 \quad (2)$$

(1) 与 (2) 消去 x_{k+1} , 有

$$k_1(\lambda_{k+1} - \lambda_1)x_1 + k_2(\lambda_{k+1} - \lambda_2)x_2 + \cdots + k_{k+1}(\lambda_{k+1} - \lambda_k)x_k = 0$$

由归纳假设知 x_1, x_2, \dots, x_k 线性无关. 故

$$k_1 = k_2 = \cdots = k_k = 0$$

则 $k_{k+1}x_{k+1} = 0 \Rightarrow k_{k+1} = 0$

从而 $x_1, x_2, \dots, x_k, x_{k+1}$ 线性无关.

综上, 由数学归纳法知定理5.2成立.

7.

$m = 2$ 时, 设 λ_1, λ_2 是两个不同的特征值, 设

$$k_1x_{11} + k_2x_{12} + \cdots + k_{k_1}x_{1k_1} + l_1x_{21} + \cdots + l_{k_2}x_{2k_2} = 0 \quad (1)$$

对 (1) 两边左乘 A , 有

$$k_1Ax_{11} + k_2Ax_{12} + \cdots + k_{k_1}Ax_{1k_1} + l_1Ax_{21} + \cdots + l_{k_2}Ax_{2k_2} = 0$$

由于 $x_{1i}(i = 1, 2, \dots, k_1)$ 和 $x_{2j}(j = 1, 2, \dots, k_2)$ 分别是 A 的属于 λ_1, λ_2 的特征向量, 则

$$k_1\lambda_1x_{11} + k_2\lambda_1x_{12} + \cdots + k_{k_1}\lambda_1x_{1k_1} + l_1\lambda_2x_{21} + \cdots + l_{k_2}\lambda_2x_{2k_2} = 0 \quad (2)$$

(2) $-\lambda_2$ (1), 有

$$k_1(\lambda_1 - \lambda_2)x_{11} + k_2(\lambda_1 - \lambda_2)x_{12} + \cdots + k_{k_1}(\lambda_1 - \lambda_2)x_{1k_1} = 0$$

由于 $x_{11}, x_{12}, \dots, x_{1k_1}$ 线性无关, 且 $\lambda_1 - \lambda_2 \neq 0$, 从而

$$k_1 = k_2 = \cdots = k_{k_1} = 0$$

再代入 (1) 有

$$l_1x_{21} + l_2x_{22} + \cdots + l_{k_2}x_{2k_2} = 0$$

由于 $x_{21}, x_{22}, \dots, x_{2k_2}$ 线性无关, 则 $l_1 = l_2 = \cdots = l_{k_2} = 0$

因此向量组 $x_{11}, x_{12}, \dots, x_{1k_1}, x_{21}, \dots, x_{2k_2}$ 线性无关. $m = 2$ 时成立.

假设 $m = n$ 时结论成立, 则立 $m = n + 1$ 时, 设

$$k_{11}x_{11} + k_{12}x_{12} + \cdots + k_{1k_1}x_{1k_1} + k_{21}x_{21} + \cdots + k_{2k_2}x_{2k_2} + \cdots + k_{n1}x_{n1} + \cdots + k_{nk_n}x_{nk_n} + \cdots + k_{(n+1)k_{n+1}}x_{(n+1)k_{n+1}} = 0$$

同左乘 A 后有

$$k_{11}\lambda_1x_{11} + k_{12}\lambda_1x_{12} + \cdots + k_{1k_1}\lambda_1x_{1k_1} + \cdots + k_{(n+1)1}\lambda_{n+1}x_{(n+1)1} + \cdots + k_{(n+1)k_{n+1}}\lambda_{n+1}x_{(n+1)k_{n+1}} = 0$$

消去 $x_{(n+1)1}, x_{(n+1)2}, \dots, x_{(n+1)k_{n+1}}$, 有

$$k_{11}(\lambda_{n+1} - \lambda_1)x_{11} + k_{12}(\lambda_{n+1} - \lambda_1)x_{12} + \cdots + k_{nn}(\lambda_{n+1} - \lambda_n)x_{nk_n} = 0$$

由归纳假设知 $k_{11} = k_{12} = \cdots = k_{nk_n} = 0$, 往回代入, 有

$$k_{(n+1)1}x_{(n+1)1} + k_{(n+1)2}x_{(n+1)2} + \cdots + k_{(n+1)k_{n+1}}x_{(n+1)k_{n+1}} = 0$$

从而 $k_{(n+1)1} = k_{(n+1)2} = \cdots = k_{(n+1)k_{n+1}} = 0$

综上, 结论成立.

8.

依题意, 设 $f(t) = t^2 + at + b$

则有 $f(\lambda) = \lambda^2 + a\lambda + b = 0$

考虑 $f(A)\alpha = A^2\alpha + aA\alpha + b\alpha$

$$= \lambda^2\alpha + a\lambda\alpha + b\alpha$$

$$= (\lambda^2 + a\lambda + b)\alpha$$

$$= 0$$

又 $\alpha \neq 0$,

则 $f(A) = 0$

9.

$$\text{证明: } f_C(\lambda) = |\lambda E_{m+n} - C| = \begin{vmatrix} \lambda E_m - A & D \\ 0 & \lambda E_n - B \end{vmatrix} = |\lambda E_m - A| \cdot |\lambda E_n - B| = f_A(\lambda)f_B(\lambda)$$

10.

$$(1) f_A(\lambda) = \begin{vmatrix} \lambda-1 & -2 & 1 & -2 \\ -2 & \lambda-4 & -2 & 1 \\ & & \lambda-1 & -2 \\ & & 1 & \lambda+2 \end{vmatrix} = \begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-4 \end{vmatrix} \cdot \begin{vmatrix} \lambda-1 & -2 \\ 1 & \lambda+2 \end{vmatrix} = (\lambda-5)(\lambda+1)\lambda^2$$

故特征值为 $-1, 5, 0, 0$.

特征值为 -1 的特征向量是 $k_1(-7, 4, -2, 2)^T$

特征值为 5 的特征向量是 $k_2(1, 2, 0, 0)^T$

特征值为 0 的特征向量是 $k_3(-2, 1, 0, 0)^T$

$$(2) f_A(\lambda) = \begin{vmatrix} \lambda-3 & 1 & & \\ -5 & \lambda-3 & & \\ & & \lambda-1 & 1 \\ & & -1 & \lambda-3 \end{vmatrix} = \begin{vmatrix} \lambda-3 & 1 \\ -5 & \lambda-3 \end{vmatrix} \cdot \begin{vmatrix} \lambda-1 & 1 \\ -1 & \lambda-3 \end{vmatrix} = (\lambda-2)^2(\lambda+2)$$

故特征值为 $2, 2, -2$.

特征值为 -2 的特征向量是 $k_1(1, 5, 0)^T$

特征值为 2 的特征向量是 $k_2(0, 0, -1, 1)^T + k_3(1, 1, 0, 0)^T$

11.

(1)

$$\begin{aligned}
 f_A(\lambda) &= \begin{vmatrix} \lambda-1 & 1 & \cdots & 1 \\ 1 & \lambda-1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & \lambda-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & \lambda-1 & 1 & \cdots & 1 \\ 0 & 1 & \lambda-1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 1 & \cdots & \lambda-1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & \lambda-2 & & & \\ -1 & & \lambda-2 & & \\ \vdots & & & \ddots & \\ -1 & & & & \lambda-2 \end{vmatrix} \\
 &= (\lambda-2) \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -1 & \lambda-2 & & & & \\ -1 & & \lambda-2 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 1 & -1 \\ -1 & & & & \lambda-2 & \end{vmatrix} = \begin{vmatrix} \lambda-2 & 1 & 1 & \cdots & 1 & 1 \\ -\lambda+2 & \lambda-2 & & & & \\ -\lambda+2 & & \lambda-2 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 1 & -1 \\ -\lambda & & & & \lambda-2 & \end{vmatrix} \\
 &= \begin{vmatrix} \lambda+n-2 & 1 & \cdots & 1 & 1 \\ & \lambda-2 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \lambda-2 \end{vmatrix} = (\lambda+n-2)(\lambda-2)^{n-1}
 \end{aligned}$$

则 A 的特征值为 2 和 $2-n$. 又 $n-r(2E-A)=n-1$, $n-r((2-n)E-A)=1$.

$$(2) \text{ 当 } \lambda \text{ 为 } 2 \text{ 时, 特征向量是 } k_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + k_{n-1} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix};$$

$$\text{当 } \lambda \text{ 为 } 2-n \text{ 时, 特征向量是 } k_n \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

习题5.2

(A)

1.

(1) 不能. 因 $r(E - A) = r \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 1$, 说明 $(E - A)x = 0$ 的基础解系中只有一个解向量, 即 $\lambda = 1$

(二重根) 只有一个特征向量, 所以 $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ 不能对角化.

(2) 可以. 如 $\begin{pmatrix} \frac{-11+\sqrt{97}}{2} & \\ & \frac{-11-\sqrt{97}}{2} \end{pmatrix}$

2.

(1) 由 $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 2 & -1 & 1 \\ -1 & \lambda - 2 & -1 \\ 1 & -1 & \lambda \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)$, 则特征值为 $1, 1, 2$

又 $\lambda = 1$ 时, 由 $\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 故基础解系 $\alpha_1 = (1, -1, 0)^T$, $\alpha_2 = (1, 0, 1)^T$

$\lambda = 2$ 时, 由 $\begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ 故基础解系 $\alpha_3 = (1, 1, 1)^T$

故 $P = (\alpha_1, \alpha_2, \alpha_3) = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $P^{-1}AP = \text{diag}(1, 1, 2)$

(2) 由 $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -2 & -3 \\ -2 & \lambda - 1 & -3 \\ -3 & -3 & \lambda - 6 \end{vmatrix} = \lambda(\lambda + 1)(\lambda - 9)$, 则特征值为 $0, -1, 9$

$\lambda = 0$ 时, 由 $\begin{pmatrix} -1 & -2 & -3 \\ -2 & -1 & -3 \\ -3 & -3 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, 故基础解系 $\alpha_1 = (1, 1, -1)^T$

$$\lambda = -1 \text{ 时, 由 } \begin{pmatrix} -2 & -2 & -3 \\ -2 & -2 & -3 \\ -3 & -3 & -7 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & \\ & & 1 \end{pmatrix}, \text{ 故基础解系 } \alpha_2 = (-1, 1, 0)^T$$

$$\lambda = 9 \text{ 时, 由 } \begin{pmatrix} 8 & -2 & -3 \\ -2 & 8 & -3 \\ -3 & -3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & -\frac{1}{2} \\ & 1 & -\frac{1}{2} \\ & & 3 \end{pmatrix}, \text{ 故基础解系 } \alpha_3 = (1, 1, 2)^T$$

$$\text{故 } P = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & 0 & 2 \end{pmatrix}, P^{-1}AP = \text{diag}(0, -1, 9)$$

$$(3) \text{ 由 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda & 2 & 2 \\ -2 & \lambda + 4 & 2 \\ 2 & -2 & \lambda \end{vmatrix} = \lambda(\lambda + 2)^2, \text{ 特征值为 } 0, -2, -2$$

$$\lambda = 0 \text{ 时, 由 } \begin{pmatrix} 0 & 2 & 2 \\ -2 & 4 & 2 \\ 2 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & 1 \\ & 1 & 1 \\ & & 0 \end{pmatrix} \text{ 得基础解系 } \alpha_1 = (1, 1, -1)^T$$

$$\lambda = 2 \text{ 时, 由 } \begin{pmatrix} -2 & 2 & 2 \\ -2 & 2 & 2 \\ 2 & -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 \\ & & \\ & & \end{pmatrix} \text{ 得 } \alpha_2 = (1, 1, 0)^T, \alpha_3 = (1, 0, 1)^T$$

$$\text{故 } P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, P^{-1}AP = \text{diag}(0, -2, -2)$$

$$(4) f_A(\lambda) = \begin{vmatrix} \lambda - 8 & 2 & 1 \\ 2 & \lambda - 5 & 2 \\ 3 & 6 & \lambda - 6 \end{vmatrix} = (\lambda - 1)(\lambda - 9)^2, \text{ 特征值为 } 1, 9, 9$$

$$\lambda = 1 \text{ 时, 由 } \begin{pmatrix} -7 & 2 & 1 \\ 2 & -4 & 2 \\ 3 & 6 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & -\frac{1}{3} \\ & 1 & -\frac{2}{3} \\ & & \end{pmatrix}, \text{ 得基础解系 } \alpha = (1, 2, 3)^T$$

$$\lambda = 9 \text{ 时, 由 } \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 6 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 \\ & & \\ & & \end{pmatrix}, \text{ 得基础解系 } \alpha_2 = (-2, 1, 0)^T, \alpha_3 = (-1, 0, 1)^T$$

$$\text{则 } P = \begin{pmatrix} 1 & -2 & -1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}, P^{-1}AP = \text{diag}(1, 9, 9)$$

3.

$$(1) f_A(\lambda) = \begin{vmatrix} \lambda - 7 & 12 & -6 \\ -10 & \lambda + 19 & -10 \\ -12 & 24 & \lambda - 13 \end{vmatrix} = (\lambda + 1)(\lambda - 1)^2, \text{ 特征值为 } -1, 1, 1$$

$$\lambda = -1 \text{ 时, 由 } \begin{pmatrix} -8 & 12 & -6 \\ -10 & 18 & -10 \\ -12 & 24 & -14 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} \\ & 1 & -\frac{5}{6} \end{pmatrix}, \text{ 得 } \alpha_1 = (3, 5, 6)^T$$

$$\lambda = 1 \text{ 时, 由 } \begin{pmatrix} -6 & 12 & -6 \\ -10 & 20 & -10 \\ -12 & 24 & -12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 1 \\ & & \end{pmatrix}, \text{ 得 } \alpha_2 = (2, 1, 0)^T, \alpha_3 = (-1, 0, 1)^T$$

$$\text{则 } P = \begin{pmatrix} 3 & 2 & -1 \\ 5 & 1 & 0 \\ 6 & 0 & 1 \end{pmatrix}, P^{-1}AP = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \Rightarrow A = P \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} P^{-1}$$

$$\text{故 } A^n = P \cdot \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}^n P^{-1}$$

$$\text{故 } n \text{ 为奇数时 } A^n = P \cdot \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} P^{-1} = A$$

$$n \text{ 为偶数时 } A^n = P \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} P^{-1} = PP^{-1} = E$$

$$(2) f_A(\lambda) = \begin{vmatrix} \lambda - 3 & 0 & 0 \\ -9 & \lambda - 6 & -5 \\ 12 & 6 & \lambda + 5 \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 3), \text{ 特征值为 } 0, 1, 3$$

$$\lambda = 0 \text{ 时, 由 } \begin{pmatrix} -3 & 0 & 0 \\ -9 & -6 & -5 \\ 12 & 6 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \frac{5}{6} \end{pmatrix}, \text{ 基础解系 } \alpha_1 = (0, -5, 6)^T$$

$$\lambda = 1 \text{ 时, 由 } \begin{pmatrix} -2 & 0 & 0 \\ -9 & -5 & -5 \\ 12 & 6 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & \end{pmatrix}, \text{ 基础解系 } \alpha_2 = (0, -1, 1)^T$$

$$\lambda = 3 \text{ 时, 由 } \begin{pmatrix} 0 & 0 & 0 \\ -9 & -3 & -5 \\ 12 & 6 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \frac{1}{3} \\ & 1 & \frac{2}{3} \\ & & \end{pmatrix}, \alpha_3 = (-1, -2, 3)^T,$$

$$\text{则 } P = \begin{pmatrix} 0 & 0 & -1 \\ -5 & -1 & -2 \\ 6 & 1 & 3 \end{pmatrix}$$

$$P^{-1}AP = \text{diag}(0, 1, 3),$$

$$\text{则 } A = P \text{diag}(0, 1, 3)P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ -5 & -1 & -2 \\ 6 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -3 & -6 & -5 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \text{故 } A^n &= P \begin{pmatrix} 0 & & \\ & 1 & \\ & & 3^n \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ -5 & -1 & -2 \\ 6 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 1 & \\ & & 3^n \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -3 & -6 & -5 \\ -1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3^n & 0 & 0 \\ 2 \times 3^n - 27 & 6 & 5 \\ 2 \times 3^n + 27 & -6 & -5 \end{pmatrix} \end{aligned}$$

$$(3) f_A(\lambda) = \begin{vmatrix} \lambda - 122 & 100 \\ -150 & \lambda + 123 \end{vmatrix} = (\lambda - 2)(\lambda + 3), \text{ 特征值为 } 2, -3$$

$$\lambda = 2 \text{ 时, 由 } \begin{pmatrix} -120 & 100 \\ -150 & 125 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{5}{6} \\ 0 & 0 \end{pmatrix}, \text{ 得 } \alpha_1 = (5, 6)^T$$

$$\lambda = -3 \text{ 时, 由 } \begin{pmatrix} -125 & 100 \\ -150 & 120 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{4}{5} \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_2 = (4, 5)^T$$

$$\text{故 } P = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}, P^{-1}AP = \begin{pmatrix} 2 & \\ & -3 \end{pmatrix}$$

$$\text{故 } A = P \begin{pmatrix} 2 & \\ & -3 \end{pmatrix} P^{-1}$$

$$\begin{aligned} \text{则 } A^n &= P \begin{pmatrix} 2^n & \\ & (-3)^n \end{pmatrix} P^{-1} = \begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 2^n & \\ & (-3)^n \end{pmatrix} \begin{pmatrix} 5 & -4 \\ -6 & 5 \end{pmatrix} \\ &= \begin{pmatrix} -24 \times (-3)^n + 25 \times 2^n & 20 \times (-3)^n - 20 \times 2^n \\ -30 \times (-3)^n + 30 \times 2^n & 25 \times (-3)^n - 24 \times 2^n \end{pmatrix} \end{aligned}$$

$$(4) f_A(\lambda) = \begin{vmatrix} \lambda - 10 & -4 \\ 24 & \lambda + 10 \end{vmatrix} = (\lambda - 2)(\lambda + 2), \text{ 特征值为 } 2, -2$$

$$\text{则 } A = P \begin{pmatrix} 2 & \\ & -2 \end{pmatrix} P^{-1} = 2P \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} P^{-1}$$

$$\text{则当 } n \text{ 为奇数时, } A^n = 2^n P \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} P^{-1} = 2^n A$$

$$n \text{ 为偶数时, } A^n = 2^n P \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} P^{-1} = 2^n PP^{-1} = 2^n E$$

4.

证明: 依题意, $P^{-1}AP = B$ 则 $(P^{-1}AP)^k = B^k$ 即 $B^k = P^{-1}AP P^{-1}AP \cdots P^{-1}AP = P^{-1}A^k P$, 则 B^k 与 A^k 也相似.

5.

证明: $\because A \sim B, C \sim D$, 则存在可逆矩阵 P_1, P_2 , 使

$$B = P_1^{-1}AP_1 \quad D = P_2^{-1}CP_2$$

$$\begin{aligned} \text{则 } \begin{pmatrix} B & \\ & D \end{pmatrix} &= \begin{pmatrix} P_1^{-1}AP_1 & \\ & P_2^{-1}CP_2 \end{pmatrix} = \begin{pmatrix} P_1^{-1} & \\ & P_2^{-1} \end{pmatrix} \begin{pmatrix} A & \\ & C \end{pmatrix} \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix} \\ &= \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}^{-1} \begin{pmatrix} A & C \end{pmatrix} \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix} \end{aligned}$$

$$\text{故 } \begin{pmatrix} A & \\ & C \end{pmatrix} \sim \begin{pmatrix} B & \\ & D \end{pmatrix}$$

6.

证明:

(1) 设 A 可对角化为 B , 则 $B = P^{-1}AP$

$$\text{故 } (P^{-1}AP)^T = B^T = B = P^{-1}AP$$

$$\text{即 } P^T A^T (P^T)^T = P^{-1}AP$$

$$\Rightarrow A^T = (PP^T)^{-1}APP^T, \text{ 从而 } A^T \sim A$$

(2) 取 $Y = PP^T$, 由(1)知 $A^T = Y^{-1}AY$

$$\text{故 } AY - YA^T = 0$$

7.

$$f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & & & \\ -a & \lambda - 1 & & \\ -2 & -b & \lambda - 2 & \\ -2 & -3 & -c & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)^2, \text{ 特征值 } 1, 1, 2, 2$$

由于 A 可对角化, 故 $(E - A)x = 0$ 与 $(2E - A)x = 0$ 的基础解系均有2个解向量, 则 $r(E - A) = 2, r(2E - A) = 2$.

$$\text{对 } E - A = \begin{pmatrix} 0 & & & \\ -a & 0 & & \\ -2 & -b & -1 & \\ -2 & -3 & -c & -1 \end{pmatrix} \text{ 与 } 2E - A = \begin{pmatrix} 1 & & & \\ -a & 1 & & \\ -2 & -b & 0 & \\ -2 & -3 & -c & 0 \end{pmatrix},$$

由于它们秩为 2, 故 3 阶子式的行列式 $\begin{vmatrix} -a & & \\ -2 & -b & -1 \\ -2 & -3 & -c \end{vmatrix}$ 与 $\begin{vmatrix} -a & 1 & \\ -2 & -b & \\ -2 & -3 & -c \end{vmatrix}$ 均为 0

$$\Rightarrow a(bc - 3) = 0 \text{ 且 } c(ab - 2) = 0$$

当 $a = 0$ 时, 则 $-2c = 0 \Rightarrow c = 0$, 此时 $r(E - A) = 2, r(2E - A) = 2$, 可对角化

当 $a \neq 0$ 时, 则由 $a(bc - 3) = 0 \Rightarrow bc = 3$ 又 $c(ab - 2) = 0$, 则 $ab = 2$

$$\text{但由 } \begin{pmatrix} 1 & & & \\ -a & 1 & & \\ -2 & -b & 0 & \\ -2 & -3 & -c & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -c & \\ & & & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1-c & \\ 0 & 1 & -c & \\ 0 & 0 & 0 & \end{pmatrix} \text{ 则 } c = 0, \text{ 矛盾, 故 } a = 0.$$

综上, $a = c = 0, b \in \mathbf{R}$ 时, A 可对角化.

8.

$$\text{由于 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & & & & \\ 1 & \lambda - 2 & & & \\ 1 & 2 & \lambda - 3 & & \\ \vdots & \vdots & \vdots & \ddots & \\ 1 & 2 & 3 & \cdots & \lambda - n \end{vmatrix} = (\lambda - 1)(\lambda - 2) \cdots (\lambda - n),$$

故 A 有 n 个特征根, 则共有 n 个无关特征向量, 可以对角化.

则其相似标准形为 $\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & \ddots \\ & & & & n \end{pmatrix}$

(B)

1.

设一组二维向量 $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$, 则

$$\begin{aligned} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} &= \begin{pmatrix} 3x_{n-1} - 2x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} x_{n-2} \\ x_{n-3} \end{pmatrix} \\ &\dots \\ &= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^{n-2} \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}, \quad \text{记 } A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

又由 $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 3 & 2 \\ -1 & \lambda \end{vmatrix} = (\lambda - 2)(\lambda - 1)$, 知有特征值 2, 1. 则

$\lambda = 2$ 时, 由 $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_1 = (2, 1)^T$

$\lambda = 1$ 时, 由 $\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_2 = (1, 1)^T$

$$\text{故 } \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 2 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}$$

$$\text{则 } \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^{n-2} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 2 \end{pmatrix}^{n-2} \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{n-1} - 1 & 2 - 2^{n-1} \\ -1 + 2^{n-2} & -2^{n-2} + 2 \end{pmatrix}$$

$$\text{于是 } \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 2^{n-1} - 1 & 2 - 2^{n-1} \\ -1 + 2^{n-2} & -2^{n-2} + 2 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \times 2^{n-1} - 2 \\ 3 \times 2^{n-2} - 2 \end{pmatrix}$$

故 $x_n = 3 \times 2^{n-1} - 2$

2.

$$\text{由 } \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 7x_{n-1} - 12x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 7 & -12 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}, \text{ 记 } A = \begin{pmatrix} 7 & -12 \\ 1 & 0 \end{pmatrix}$$

$$\text{由 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 7 & 12 \\ -1 & \lambda \end{vmatrix} = (\lambda - 5)(\lambda - 2), \text{ 得 } \lambda \text{ 为 } 2, 5. \text{ 则}$$

$$2E - A = \begin{pmatrix} -5 & 12 \\ -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -5 & 2 \\ 0 & 0 \end{pmatrix} \text{ 则 } \alpha_1 = (2, 5)^T$$

$$5E - A = \begin{pmatrix} -2 & 12 \\ -1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix} \text{ 则 } \alpha_2 = (1, 1)^T, \text{ 故 } P = \begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix}, P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ -5 & 2 \end{pmatrix}$$

$$\text{则 } A = P \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} P^{-1}, \text{ 则 } A^{n-2} = P \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}^{n-2} P^{-1} \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\text{故 } \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}^{n-2} \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} \\ \frac{5}{3} & -\frac{2}{3} \end{pmatrix} \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 11 \times 3^{n-1} - 9 \times 4^{n-1} \\ 11 \times 3^{n-2} - 9 \times 4^{n-2} \end{pmatrix}$$

$$\text{因此 } x_n = 11 \times 3^{n-1} - 9 \times 4^{n-1}$$

3.

由

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} &= \begin{pmatrix} 5x_{n-1} - 3y_{n-1} + 2z_{n-1} \\ 6x_{n-1} - 4y_{n-1} + 4z_{n-1} \\ 4x_{n-1} - 4y_{n-1} + 5z_{n-1} \end{pmatrix} = \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}^2 \begin{pmatrix} x_{n-2} \\ y_{n-2} \\ z_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}^{n-1} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \end{aligned}$$

$$\text{设 } A = \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}$$

$$\text{则 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 5 & 3 & -2 \\ -6 & \lambda + 4 & -4 \\ -4 & 4 & \lambda - 5 \end{vmatrix} = (\lambda - 1)(\lambda - 2)(\lambda - 3)$$

$$\text{又 } E - A = \begin{pmatrix} -4 & 3 & -2 \\ -6 & 5 & -4 \\ -4 & 4 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ & 1 & -2 \end{pmatrix} \text{ 得 } \alpha_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$2E - A = \begin{pmatrix} -3 & 3 & -2 \\ -6 & 6 & -4 \\ -4 & 4 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ & & 1 \end{pmatrix} \text{ 得 } \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$3E - A = \begin{pmatrix} -2 & 3 & -2 \\ -6 & 7 & -4 \\ -4 & 4 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} \\ & 1 & -1 \end{pmatrix} \text{ 得 } \alpha_3 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \text{ 故 } P = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\text{则 } A = P \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix} P^{-1},$$

$$\text{故 } A^{n-1} = P \begin{pmatrix} 1 & & \\ & 2^{n-1} & \\ & & 3^{n-1} \end{pmatrix} P^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} P \begin{pmatrix} 1 & & \\ & 2^{n-1} & \\ & & 3^{n-1} \end{pmatrix} \begin{pmatrix} -2 & 2 & -1 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$\text{则 } \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = A^{n-1} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 3 - 2 \times 3^{n-1} \\ 6 - 4 \times 3^{n-1} \\ 3 - 4 \times 3^{n-1} \end{pmatrix}$$

4.

$$\text{证明: 由 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & -1 & \cdots & -1 \\ & \lambda - 1 & -1 & \cdots & -1 \\ & & \lambda - 1 & \cdots & -1 \\ & & & \ddots & \vdots \\ & & & & \lambda - 1 \end{vmatrix} = (\lambda - 1)^n, \text{ 则特征值为 } 1 \text{ (n重根)}$$

又 $r(E - A) = n - 1$, 故 A 只有1个线性无关的特征向量, 则 A 不能与对角阵相似.

5.

(1) 依题意, $tr(A) = tr(B)$ 且 $|A| = |B|$

$$\text{则 } \begin{cases} 1 + 4 + a = 2 + 2 + b \\ \begin{vmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{vmatrix} = \begin{vmatrix} 2 \\ & 2 \\ & & b \end{vmatrix} \end{cases}$$

$$\text{得} \begin{cases} a = 5 \\ b = 6 \end{cases}$$

(2) 由(1)知 A 的特征值为 2, 2, 6, 则

$$2E - A = \begin{pmatrix} 1 & 1 & -1 \\ -2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ & & \\ & & \end{pmatrix} \text{得 } \alpha_1 = (-1, 1, 0)^T, \alpha_2 = (1, 0, 1)^T$$

$$6E - A = \begin{pmatrix} 5 & 1 & -1 \\ -2 & 2 & 2 \\ 3 & 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & -\frac{1}{3} \\ & 1 & \frac{1}{3} \\ & & 0 \end{pmatrix} \text{得 } \alpha_3 = (1, -2, 3)^T$$

$$\text{故 } P = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$$

6.

证明: 设 $\lambda_1, \lambda_2, \dots, \lambda_n$ 为 A 的 n 个不同的特征值, 则存在可逆矩阵 P 使

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix} = \Lambda_1$$

记 $P = (\beta_1, \beta_2, \dots, \beta_n)$, 则 p_i ($i = 1, 2, \dots, n$) 也是 B 的特征向量, 则记对应特征值 μ_i

$$Bp_i = \mu_i p_i$$

$$\text{则有 } P^{-1}BP = \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \ddots \\ & & & \mu_n \end{pmatrix} = \Lambda_2$$

于是 $P^{-1}ABP = (P^{-1}AP)(P^{-1}BP) = \Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 = (P^{-1}BP)(P^{-1}AP) = P^{-1}BAP$

于是 $AB = BA$

7.

由于 $A \sim B$, 若 $r(A) = n$, 则由于 $A \sim B$, 则 $|A| = |B|$, 且 $A^{-1} \sim B^{-1}$.

于是 $P^{-1}((nA^{-1})P) = |B|B^T$, 即 A^* 与 B^* 相似.

若 $r(A) < n$, 则 A, B 不可逆, 必 $\exists \delta > 0$, 当 $t \in (0, \delta)$, 使 $|tE + A| \neq 0$, $|tE + B| \neq 0$

记 $A_t = tE + A$, $B_t = tE + B$, 则 $B_t = tE + B = tE + P^{-1}AP = P^{-1}(tE + A)P = P^{-1}A_tP$

则由 $r(A) = n$ 的证明知 $B_t^* \sim A_t^*$ 即 $(tE + B)^* = P^{-1}(tE + A)^*P$

上式两端矩阵均为 t 的多项式, 由于当 $t \in (0, \delta)$ 时, 对应元素相似, 则取 $t \rightarrow 0$, 有 A^* 与 B^* 相似.

8.

证明: 设 λ 是 A 的任一特征值, α 是 λ 的特征向量

则 $A\alpha = \lambda\alpha$, 左乘 A , 为

$$A^2\alpha = \lambda A\alpha = \lambda^2\alpha, \text{ 又 } A^2 = A$$

则 $(\lambda^2 - \lambda)\alpha = 0$ 又 $\alpha \neq 0$, $\therefore \lambda^2 - \lambda = 0$

于是 A 的特征值只能是0或1

又秩 $r(A) = r$, 故特征值为0的特征向量有 $n - r$ 个

$$\text{则 } A \text{ 的相似标准形为 } \begin{pmatrix} \overbrace{1 \cdots 1}^{r \text{ 个}} & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \\ & & & & \ddots \\ & & & & & 0 \end{pmatrix}_n = \begin{pmatrix} E_r & \\ & O \end{pmatrix}$$

9.

设 $A\alpha = \lambda\alpha$, 则 $A^2\alpha = \lambda^2\alpha \Rightarrow \lambda = \pm 1$

故仿8知 $A \sim \begin{pmatrix} E_r & \\ & -E_{n-r} \end{pmatrix}$

10.

反证法. 假设 A 可对角化, 设

$$A \sim \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \Lambda, \text{ 即设 } A = P\Lambda P^{-1}$$

由 $A^k = O$ 得

$$A^k = (P\Lambda P^{-1})(P\Lambda P^{-1}) \cdots (P\Lambda P^{-1}) = O$$

$$\Rightarrow P\Lambda^k P^{-1} = O \Rightarrow \Lambda^k = O$$

则 $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, 则 $A = O$, 与题设矛盾.

11.

依题意, 设 $\Lambda = \begin{pmatrix} a & \\ & b \end{pmatrix}$

$$\text{则由 } \begin{cases} |A| = |\Lambda| \\ \text{tr}(A) = \text{tr}(\Lambda) \end{cases}$$

$$\text{得 } \begin{cases} a+b=6 \\ ab=8 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=4 \end{cases} \quad \text{或} \quad \begin{cases} a=4 \\ b=2 \end{cases}$$

$$\text{从而 } \Lambda \text{ 为 } \begin{pmatrix} 2 & \\ & 4 \end{pmatrix} \text{ 或 } \begin{pmatrix} 4 & \\ & 2 \end{pmatrix}$$

(C)

1.

$$(1) \text{ 设 } X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}, \text{ 则 } X' = AX$$

$$\text{又由 } f_A(\lambda) = \begin{vmatrix} \lambda-1 & -2 \\ -4 & \lambda-8 \end{vmatrix} = \lambda^2 - 9\lambda = \lambda(\lambda-9) \text{ 得特征值为 } 0, 9.$$

$$\text{又 } \lambda=0 \text{ 时, 由 } \begin{pmatrix} -1 & -2 \\ -4 & -8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_1 = (-2, 1)^T$$

$$\text{又 } 9E - A = \begin{pmatrix} 8 & -2 \\ -4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_2 = (1, 4)^T, \text{ 则 } P = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix}, \Lambda = \begin{pmatrix} 0 & \\ & 9 \end{pmatrix}$$

$$\text{故由 } X = PY \text{ 可得 } Y' = \begin{pmatrix} 0 & \\ & 9 \end{pmatrix} Y \Rightarrow \begin{cases} \frac{dy_1}{dt} = 0 \\ \frac{dy_2}{dt} = 9y_2 \end{cases} \Rightarrow Y = \begin{pmatrix} c_1 \\ c_2 e^{9t} \end{pmatrix}$$

$$\text{则 } X = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 e^{9t} \end{pmatrix} = \begin{pmatrix} -2c_1 + c_2 e^{9t} \\ c_1 + 4c_2 e^{9t} \end{pmatrix}$$

$$(2) \text{ 设 } X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \text{ 则 } X' = AX$$

$$\text{由 } f_A(\lambda) = \begin{vmatrix} \lambda-2 & 1 \\ 1 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-3) \text{ 得特征值 } 1, 3$$

$$\text{则 } E - A = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_1 = (1, 1)^T$$

$$3E - A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_2 = (1, -1)^T, \text{ 则 } P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \Lambda = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix}$$

$$\text{由 } X = PY \text{ 可得 } Y' = \begin{pmatrix} 1 & \\ & 3 \end{pmatrix} Y \Rightarrow \begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = 3y_2 \end{cases} \Rightarrow Y = \begin{pmatrix} c_1 e^t \\ c_2 e^{3t} \end{pmatrix}$$

$$\text{故 } X = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^t \\ c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} c_1 e^t + c_2 e^{3t} \\ c_1 e^t - c_2 e^{3t} \end{pmatrix}$$

$$(3) \text{ 设 } X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, A = \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \text{ 则 } X' = AX$$

$$\text{由 } f_A(\lambda) = \begin{vmatrix} \lambda - 1 & 2 \\ -1 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 3) \text{ 得特征值 } 2, 3.$$

$$\text{则 } 2E - A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_1 = (-2, 1)^T$$

$$3E - A = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_2 = (1, -1)^T, \text{ 则 } P = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}, \Lambda = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix}$$

$$\text{则由 } X = PY \Rightarrow Y' = \begin{pmatrix} 2 & \\ & 3 \end{pmatrix} Y \Rightarrow \begin{cases} \frac{dy_1}{dt} = y_1 \\ \frac{dy_2}{dt} = y_2 \end{cases} \Rightarrow Y = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{pmatrix}$$

$$\Rightarrow X = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{pmatrix} = \begin{pmatrix} -2c_1 e^{2t} + c_2 e^{3t} \\ c_1 e^{2t} - c_2 e^{3t} \end{pmatrix}$$

$$\text{又 } x_1(0) = x_2(0) = 1 \text{ 得 } \begin{cases} c_1 = -2 \\ c_2 = -3 \end{cases}$$

$$\text{故 } X = \begin{pmatrix} 4e^{2t} - 3e^{3t} \\ -2e^{2t} + 3e^{3t} \end{pmatrix}$$

$$(4) \text{ 记 } X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}, A = \begin{pmatrix} 3 & 1 & 1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{pmatrix}, \text{ 则 } X' = AX$$

$$\text{由 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 3 & -1 & -1 \\ 12 & \lambda & -5 \\ -4 & 2 & \lambda + 1 \end{vmatrix} = \lambda^3 - 2\lambda^2 + 5\lambda - 62, \text{ 考察函数图像, 此方程有且只}$$

有一个实根. 故该微分方程组解的形式过于复杂, 但计算方法与(3)相同. 此处怀疑是题目未设置好. 但教材也没给出答案, 故不作过多讨论.

(5) 记 $X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ -2 & 1 & 1 \end{pmatrix}$, 则 $X' = AX$

由 $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ 0 & \lambda - 3 & -3 \\ -2 & -1 & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 2)(\lambda - 3)$ 故特征值为 0, 2, 3.

则 $0E - A = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -3 & -3 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & \end{pmatrix}$ 则 $\alpha_1 = (0, -1, 1)^T$

$2E - A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & -3 \\ -2 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ & 1 & 3 \\ & & \end{pmatrix}$ 则 $\alpha_2 = (2, 3, -1)^T$

$3E - A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & -3 \\ 2 & -1 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} \\ & & 1 \\ & & \end{pmatrix}$ 则 $\alpha_3 = (1, 2, 0)^T$

故 $P = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -1 & 0 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & & \\ & 2 & \\ & & 3 \end{pmatrix}$, 则由 $X = PY$

有 $Y' = \begin{pmatrix} 0 & & \\ & 2 & \\ & & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \Rightarrow \begin{cases} \frac{dy_1}{dt} = 0 \\ \frac{dy_2}{dt} = 2y_2 \\ \frac{dy_3}{dt} = 3y_3 \end{cases} \Rightarrow Y = \begin{pmatrix} c_1 \\ c_2 e^{2t} \\ c_3 e^{3t} \end{pmatrix}$

故 $X = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 3 & 2 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 e^{2t} \\ c_3 e^{3t} \end{pmatrix} = \begin{pmatrix} 2c_2 e^{2t} + c_3 e^{3t} \\ -c_1 + 3c_2 e^{2t} + 2c_3 e^{3t} \\ c_1 - c_2 e^{2t} \end{pmatrix}$ 又 $\begin{cases} x_1(0) = 3 \\ x_2(0) = 3 \\ x_3(0) = 1 \end{cases}$

得 $\begin{cases} 2c_2 + c_3 = 3 \\ -c_1 + 3c_2 + 2c_3 = 3 \\ c_1 - c_2 = 1 \end{cases} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ 故

$X = \begin{pmatrix} 2e^{2t} + e^{3t} \\ -2 + 3e^{2t} + 2e^{3t} \\ 2 - e^{2t} \end{pmatrix}$

2.

(1) 学习过高等数学(微积分/工科数学分析)可直接求 λ 用公式.

这里先使用线性代数方法示范.

令 $x_1(t) = x(t)$, $x_2(t) = x'(t)$

$$\text{则 } \begin{cases} x_1'(t) = x_2(t) \\ x_2'(t) = -3x_1(t) - 4x_2(t) \end{cases} \quad \text{记 } X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}$$

则 $X' = AX$, 则由 $f_A(\lambda) = |\lambda E - A| = (\lambda + 1)(\lambda + 3) \Rightarrow \lambda$ 为 $-1, -3$

$$\text{则由 } X = PY \Rightarrow Y' = \begin{pmatrix} -1 & \\ & -3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow Y = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-3t} \end{pmatrix}$$

$$\text{又由 } -3\lambda - A = \begin{pmatrix} -3 & -1 \\ 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_1 = (-1, 3)^T$$

$$\text{由 } -\lambda - A = \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_2 = (-1, 1)^T, \text{ 则 } P = \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix}$$

$$\text{故 } X = \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-3t} \end{pmatrix} = \begin{pmatrix} -c_1 e^{-t} - c_2 e^{-3t} \\ 3c_1 e^{-t} + c_2 e^{-3t} \end{pmatrix}$$

$$\text{故 } x = c_1 e^{-t} + c_2 e^{-3t}$$

(2) 由 $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2, 3$

$$\text{则 } x = c_1 e^{2t} + c_2 e^{3t}$$

(3) 由 $\lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda = 6, 1$

$$\text{则 } x = c_1 e^{6t} + c_2 e^t \text{ 又 } \begin{cases} x(0) = 1 \\ x'(0) = 1 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ 6c_1 + c_2 = 1 \end{cases}$$

$$\text{得 } x = e^t$$

(4) 由 $\lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda = 2, 4$ 又 $D^2 x - 6Dx + 8x = 8t + 10$

$$\text{得 } x = \frac{8t + 10}{(D - 2)(D - 4)} \text{ 由 } L(D) = D^2 - 6D + 8 \text{ 且 } L(0) \neq 0$$

$$\text{得 } x = \frac{8t + 10}{D^2 - 6D + 8} = \left(\frac{1}{8} + \frac{3}{32} D \right) (8t + 10) = t + 2$$

$$\text{故特解为 } x^* = t + 2$$

$$\text{故 } x = c_1 e^{2t} + c_2 e^{4t} + t + 2$$

3.

(1) 同2. 先用线性代数思想示范, 然后回归高数公式解题.

$$\text{令 } x_1(t) = x(t), x_2(t) = x'(t)$$

$$\text{则记 } X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, X' = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} x_2(t) \\ -4x_1(t) \end{pmatrix} = AX$$

$$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \text{ 则 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda & -1 \\ 4 & \lambda \end{vmatrix} = \lambda^2 + 4 \text{ 得 } \lambda \text{ 为 } \pm 2i$$

$$\lambda = 2i, 2iE - A = \begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix} \rightarrow \begin{pmatrix} 2i & -1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_1 = (1, 2i)^T$$

$$\lambda = -2i, -2iE - A = \begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \rightarrow \begin{pmatrix} 2i & 1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_2 = (-1, 2i)^T \text{ 则 } P = \begin{pmatrix} 1 & -1 \\ 2i & 2i \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 2i & \\ & -2i \end{pmatrix}, \text{ 则由 } X = PY \text{ 得 } Y' = \begin{pmatrix} 2i & \\ & -2i \end{pmatrix} Y \Rightarrow Y = \begin{pmatrix} c_1 e^{2it} \\ c_2 e^{-2it} \end{pmatrix}$$

$$\text{故 } X = \begin{pmatrix} 1 & -1 \\ 2i & 2i \end{pmatrix} \begin{pmatrix} c_1 e^{2it} \\ c_2 e^{-2it} \end{pmatrix} = \begin{pmatrix} c_1 e^{2it} - c_2 e^{-2it} \\ 2ic_1 e^{2it} + 2ic_2 e^{-2it} \end{pmatrix}$$

$$\text{又由欧拉公式, 则 } x = c_1 e^{2it} - c_2 e^{-2it} = (c_1 + c_2) \cos 2t + (c_1 - c_2)i \sin 2t$$

$$\text{故 } x = C_1 \cos 2t + C_2 \sin 2t$$

(2) 由 $\lambda^2 - 6\lambda + 10 = 0 \Rightarrow \lambda = 3 \pm i$

$$\text{故 } x = e^{3t}(C_1 \cos t + C_2 \sin t)$$

$$(3) \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ 记 } X' = AX$$

$$\text{则 } |\lambda E - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 \text{ 得 } \lambda \text{ 为 } \pm i$$

$$iE - A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \rightarrow \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_1 = (-1, i)^T$$

$$-iE - A = \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} \rightarrow \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_2 = (1, i)^T, \text{ 则 } P = \begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}, \Lambda = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

$$\text{则 } Y' = \Lambda Y = \begin{pmatrix} i & \\ & -i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow Y = \begin{pmatrix} c_1 e^{it} \\ c_2 e^{-it} \end{pmatrix}$$

$$\text{则 } X = \begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix} \begin{pmatrix} c_1 e^{it} \\ c_2 e^{-it} \end{pmatrix} = \begin{pmatrix} -c_1 e^{it} + c_2 e^{-it} \\ i c_1 e^{it} + i c_2 e^{-it} \end{pmatrix} = \begin{pmatrix} (c_2 - c_1) \cos t - (c_1 + c_2) i \sin t \\ (c_1 + c_2) \cos t - (c_1 - c_2) i \sin t \end{pmatrix}$$

$$\text{故 } x_1 = C_1 \cos t + C_2 \sin t, x_2 = -C_2 \cos t + C_1 \sin t$$

$$(4) \text{ 记 } X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \text{ 则 } X' = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -8x_1 - 4x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\text{则由 } f_A(\lambda) = \begin{vmatrix} \lambda & -1 \\ 8 & \lambda + 4 \end{vmatrix} = \lambda^2 + 4\lambda + 8 \text{ 得 } \lambda \text{ 为 } -2 \pm 2i$$

$$(-2 + 2i)E - A = \begin{pmatrix} -2 + 2i & -1 \\ 8 & 2 + 2i \end{pmatrix} \rightarrow \begin{pmatrix} -2 + 2i & -1 \\ 0 & 0 \end{pmatrix} \text{ 则 } \alpha_1 = (1, -2 + 2i)^T$$

$$(-2 - 2i)E - A = \begin{pmatrix} -2 - 2i & -1 \\ 8 & 2 - 2i \end{pmatrix} \rightarrow \begin{pmatrix} 2 + 2i & 1 \\ 0 & 0 \end{pmatrix} \text{ 则 } \alpha_2 = (1, -2 - 2i)^T$$

$$\text{故 } P = \begin{pmatrix} 1 & 1 \\ -2 + 2i & -2 - 2i \end{pmatrix}, \Lambda = \begin{pmatrix} -2 + 2i & \\ & -2 - 2i \end{pmatrix}$$

$$\text{由 } X = PY \text{ 得 } Y' = \begin{pmatrix} -2 + 2i & \\ & -2 - 2i \end{pmatrix} Y \Rightarrow Y = \begin{pmatrix} c_1 e^{-2+2it} \\ c_2 e^{-2-2it} \end{pmatrix}$$

$$\text{则 } X = \begin{pmatrix} 1 & 1 \\ -2 + 2i & -2 - 2i \end{pmatrix} \begin{pmatrix} c_1 e^{-2+2it} \\ c_2 e^{-2-2it} \end{pmatrix} = \begin{pmatrix} c_1 e^{-2+2it} + c_2 e^{-2-2it} \\ (-2 + 2i c_1) e^{-2+2it} + (-2 - 2i c_2) e^{-2-2it} \end{pmatrix}$$

$$\text{即 } \begin{cases} x_1(t) = e^{-2t} (C_1 \cos 2x + C_2 \sin 2x) \\ x_2(t) = e^{-2t} [C_2(-2C_2 - 2C_1) \cos 2x - 2(C_1 + C_2) \sin 2x] \end{cases}$$

习题5.3

(A)

1.

$$(1) f_A(\lambda) = \begin{vmatrix} \lambda - 1 & 2 \\ 2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)(\lambda - 3), \text{ 得特征值为 } -1, 3$$

$$3E - A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ 得基础解系 } \alpha_1 = (1, -1)^T$$

$$-E - A = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \text{ 得 } \alpha_2 = (1, 1)^T \text{ 单位化 } \begin{cases} p_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)^T \\ p_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T \end{cases}$$

故 $Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, $Q^{-1}AQ = \text{diag}(3, -1)$.

(2) $f_A(\lambda) = \begin{vmatrix} \lambda & 6 & -6 \\ 6 & \lambda+3 & 0 \\ -6 & 0 & \lambda-3 \end{vmatrix} = \lambda(\lambda-9)(\lambda+9)$, 特征值为 $-9, 0, 9$

$0E - A = \begin{pmatrix} 0 & 6 & -6 \\ 6 & 3 & 0 \\ -6 & 0 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{2} \\ & 1 & -1 \end{pmatrix}$ 得 $\alpha_1 = (-1, 1, 2)^T$

$9E - A = \begin{pmatrix} 9 & 6 & -6 \\ 6 & 12 & 0 \\ -6 & 0 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ & 1 & \frac{1}{2} \end{pmatrix}$ 得 $\alpha_2 = (2, -1, 2)^T$

$-9E - A = \begin{pmatrix} -9 & 6 & -6 \\ 6 & -6 & 0 \\ -6 & 0 & -12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} \\ & 1 & -1 \end{pmatrix}$ 得 $\alpha_3 = (-2, -2, 1)^T$

则 $p_1 = \left(-\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)^T$, $p_2 = \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)^T$, $p_3 = \left(-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)^T$

故 $Q = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$, $Q^T A Q = \text{diag}(0, 9, -9)$.

(3) $f_A(\lambda) = \begin{vmatrix} \lambda-4 & -2 & 2 \\ -2 & \lambda-3 & 2 \\ 2 & 2 & \lambda-2 \end{vmatrix} = \lambda(\lambda-3)(\lambda-6)$, 特征值为 $0, 3, 6$

$6E - A = \begin{pmatrix} 2 & -2 & 2 \\ -2 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ & 1 & 2 \end{pmatrix}$ 得 $\alpha_1 = (2, 2, -1)^T$

$3E - A = \begin{pmatrix} -1 & -2 & 2 \\ -2 & 0 & 2 \\ 2 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 \\ & 1 & \frac{1}{2} \end{pmatrix}$ 得 $\alpha_2 = (2, -1, 2)^T$

$0E - A = \begin{pmatrix} -4 & -2 & 2 \\ -2 & -3 & 2 \\ 2 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} \\ & 1 & -1 \end{pmatrix}$ 得 $\alpha_3 = (-1, 2, 2)^T$

$$\text{则 } p_1 = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}\right)^T, p_2 = \left(-\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)^T, p_3 = \left(\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}\right)^T$$

$$\text{则 } Q = \begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \end{pmatrix}, Q^T A Q = \text{diag}(6, 3, 0)$$

$$(4) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 2 & 2 & 2 \\ 2 & \lambda - 5 & -4 \\ 2 & -4 & \lambda - 5 \end{vmatrix} = (\lambda - 1)^2(\lambda - 10), \text{ 特征值为 } 1, 1, 10$$

$$\text{由 } E - A = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -4 & -4 \\ 2 & -4 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -2 \\ & & \\ & & \end{pmatrix} \text{ 则 } \alpha_1 = (2, 1, 0)^T, \alpha_2 = (2, 0, 1)^T$$

$$10E - A = \begin{pmatrix} 8 & 2 & 2 \\ 2 & 5 & -4 \\ 2 & -4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & \frac{1}{2} \end{pmatrix} \text{ 则 } \alpha_3 = (1, -2, -2)^T$$

$$\text{由 } \alpha_1, \alpha_2, \alpha_3 \text{ 正交单位化, 有 } p_1 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)^T, p_2 = \left(\frac{2}{3\sqrt{5}}, -\frac{4}{3\sqrt{5}}, \frac{\sqrt{5}}{3}\right)^T, p_3 = \left(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)^T$$

$$\text{则 } Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & -\frac{1}{3} \\ \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3} \end{pmatrix}, Q^{-1} A Q = \text{diag}(1, 1, 10)$$

$$(5) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & -2 & -2 \\ -2 & \lambda - 1 & -2 \\ -2 & -2 & \lambda - 1 \end{vmatrix} = (\lambda + 1)^2(\lambda - 5), \text{ 特征值为 } 5, -1, -1$$

$$5E - A = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & \end{pmatrix} \text{ 则 } \alpha_1 = (1, 1, 1)^T$$

$$-E - A = \begin{pmatrix} -2 & -2 & -2 \\ -2 & -2 & -2 \\ -2 & -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix} \text{ 则 } \alpha_2 = (1, -1, 0)^T, \alpha_3 = (1, 0, -1)^T$$

$$\text{正交单位化, 有 } p_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T, p_2 = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)^T, p_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)^T$$

$$\text{则 } Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \end{pmatrix}, Q^{-1}AQ = \text{diag}(5, -1, -1)$$

$$(6) f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & & 1 \\ & \lambda - 1 & \\ 1 & & \lambda - 1 \end{vmatrix} = \lambda(\lambda - 1)(\lambda - 2), \text{ 特征值为 } 0, 1, 2$$

$$\text{又由 } 0E - A = \begin{pmatrix} -1 & & 1 \\ & -1 & \\ 1 & & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{pmatrix} \text{ 得 } \alpha_1 = (1, 0, 1)^T$$

$$E - A = \begin{pmatrix} 0 & & 1 \\ & 0 & \\ 1 & & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ & & 1 \\ & & \end{pmatrix} \text{ 得 } \alpha_2 = (0, 1, 0)^T$$

$$2E - A = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ 1 & & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & 1 \\ & 1 & \\ & & 1 \end{pmatrix} \text{ 得 } \alpha_3 = (1, 0, -1)^T$$

$$\text{单位化, 得 } p_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)^T, p_2 = (0, 1, 0)^T, p_3 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)^T$$

$$\text{则 } Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}, Q^{-1}AQ = \text{diag}(0, 1, 2)$$

$$(7) f_A(\lambda) = \begin{vmatrix} \lambda & -1 & -1 & 1 \\ -1 & \lambda & 1 & -1 \\ -1 & 1 & \lambda & -1 \\ 1 & -1 & -1 & \lambda \end{vmatrix} = (\lambda + 3)(\lambda - 1)^3, \text{ 特征值为 } 1, 1, 1, -3$$

$$E - A = \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & -1 & 1 \\ & & & \\ & & & \\ & & & \end{pmatrix} \text{ 得 } \alpha_1 = (1, 1, 0, 0)^T, \alpha_2 = (1, 0, 1, 0)^T,$$

$$\alpha_3 = (1, 0, 0, -1)^T$$

$$-3E - A = \begin{pmatrix} -3 & -1 & -1 & 1 \\ -1 & -3 & 1 & -1 \\ -1 & 1 & -3 & -1 \\ 1 & -1 & -1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & -1 \\ & 1 & & 1 \\ & & 1 & 1 \\ & & & 1 \end{pmatrix} \text{ 得 } \alpha_4 = (1, -1, -1, 1)^T$$

$$\text{正交单位化为 } p_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right)^T, p_2 = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0 \right)^T,$$

$$p_3 = \left(-\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2} \right)^T, p_4 = \left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right)^T$$

$$\text{则 } Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{2\sqrt{3}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2} \\ 0 & \frac{1}{\sqrt{6}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, Q^{-1}AQ = \text{diag}(1, 1, 1, -3)$$

$$(8) f_A(\lambda) = \begin{vmatrix} \lambda & -1 & & \\ -1 & \lambda & & \\ & & \lambda & -1 \\ & & -1 & \lambda \end{vmatrix} = (\lambda - 1)^2(\lambda + 1)^2, \text{ 则特征值为 } -1, -1, 1, 1$$

$$-E - A = \begin{pmatrix} -1 & -1 & & \\ -1 & -1 & & \\ & & -1 & -1 \\ & & -1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & & \\ & 1 & & -1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \text{ 则 } \begin{cases} \alpha_1 = (-1, 1, 0, 0)^T, \\ \alpha_2 = (0, 0, -1, 1)^T, \end{cases}$$

$$E - A = \begin{pmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & 1 & -1 \\ & & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & & \\ & 1 & & -1 \\ & & 1 & -1 \\ & & & 1 \end{pmatrix} \text{ 则 } \begin{cases} \alpha_3 = (1, 1, 0, 0)^T, \\ \alpha_4 = (0, 0, 1, 1)^T, \end{cases}$$

$$\text{故 } p_1 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right)^T, p_2 = \left(0, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T,$$

$$p_3 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right)^T, p_4 = \left(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)^T$$

$$\text{则 } Q = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, Q^{-1}AQ = \text{diag}(-1, -1, 1, 1)$$

2.

证明: 依题意, 存在 Q , 使 $Q^{-1}AQ = \text{diag}(a_1, a_2, \dots, a_n)$

由于 $r(A) = r$, 则 a_1, a_2, \dots, a_n 中有 r 个非零, 不妨设为前 r 个, 则

$$A = Q \cdot \text{diag}(a_1, a_2, \dots, a_r, 0, \dots, 0) \cdot Q^{-1}$$

令 $B_1 = Q^{-1} \text{diag}(a_1, 0, \dots, 0, \dots, 0)Q$, $B_2 = Q^{-1} \text{diag}(0, a_2, 0, \dots, 0)Q$, \dots , $B_r = Q^{-1} \text{diag}(0, \dots, a_r, 0)Q$

则 $Q(B_1 + B_2 + \dots + B_r)Q = \text{diag}(a_1, a_2, \dots, a_r, 0, \dots, 0) = Q^{-1}AQ$

故 $A = B_1 + B_2 + \dots + B_r$ 又每个 B_i ($i = 1, 2, \dots, r$) 都为对称阵且秩为1
证毕.

3.

$$A = \alpha^T \alpha = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} (1 \ -2 \ 3) = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{pmatrix}$$

$$\text{则由 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & 2 & -3 \\ 2 & \lambda - 4 & 6 \\ -3 & 6 & \lambda - 9 \end{vmatrix} = \lambda^2(\lambda - 14) \text{ 故特征值为 } 0, 14, 0$$

$$\text{又 } 0E - A = \begin{pmatrix} -1 & 2 & -3 \\ 2 & -4 & 6 \\ -3 & 6 & -9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & 3 \\ & & \\ & & \end{pmatrix} \text{ 得 } \alpha_1 = (2, 1, 0)^T, \alpha_2 = (-3, 0, 1)^T$$

$$14E - A = \begin{pmatrix} 13 & 2 & -3 \\ 2 & 10 & 6 \\ -3 & 6 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\frac{1}{3} \\ & 1 & \frac{2}{3} \\ & & \end{pmatrix} \text{ 得 } \alpha_3 = (1, -2, 3)^T$$

$$\text{单位正交化 } \rightarrow p_1 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0 \right)^T, p_2 = \left(-\frac{3\sqrt{70}}{70}, \frac{3\sqrt{70}}{35}, \frac{\sqrt{70}}{14} \right)^T, p_3 = \left(\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)^T$$

$$\text{故 } Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{3\sqrt{70}}{70} & \frac{1}{\sqrt{14}} \\ \frac{1}{\sqrt{5}} & \frac{3\sqrt{70}}{35} & -\frac{2}{\sqrt{14}} \\ 0 & \frac{\sqrt{70}}{14} & \frac{3}{\sqrt{14}} \end{pmatrix}$$

4.

证明: 由于 $Ax = \lambda x$, 则

$$(A\bar{x})^T x = \bar{x}^T A^T x = -\bar{x}^T (Ax) = -\bar{x}^T (\lambda x) = -\lambda(\bar{x}^T x)$$

$$(A\bar{x})^T x = (\overline{Ax})^T x = (\overline{\lambda x})^T x = (\overline{\lambda x})^T x = \bar{\lambda}(\bar{x}^T x)$$

则 $(\lambda + \bar{\lambda})\bar{x}^T x = 0$, 又 $x \neq 0$, 于是 $\bar{x}^T x \neq 0$

故 $\lambda + \bar{\lambda} = 0$ 则 λ 是纯虚数.

习题6.1

(A)

1.

$$(1) \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \text{ 由 } \begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \text{ 知秩为2}$$

$$(2) \begin{pmatrix} 1 & -2 & -4 \\ -2 & -2 & 3 \\ -4 & 3 & 3 \end{pmatrix} \text{ 由 } \begin{pmatrix} 1 & -2 & -4 \\ -2 & -2 & 3 \\ -4 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \text{ 知秩为3.}$$

$$(3) \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \text{ 由 } \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ & & \\ & & \end{pmatrix} \text{ 知秩为1}$$

$$(4) \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \text{ 由 } \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & & \\ & & & \end{pmatrix} \text{ 知秩为2}$$

$$(5) \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \text{ 由 } \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \text{ 知秩为4.}$$

2.

$$(1) 3x_2^2 - 4x_1x_2$$

$$(2) 7x_1^2 + 5x_2^2 + 8x_1x_2$$

$$(3) -x_1^2 + 2x_2^2 - 3x_3^2 + 8x_1x_2 + 12x_1x_3 - 10x_2x_3$$

$$(4) -6x_1^2 + 7x_2^2 - 2x_3^2 + 6x_1x_2$$

$$(5) x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2 + 4x_1x_2 - 2x_3x_4$$

$$(6) x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4$$

(B)

1.

证明: 必要性: 由 $A = O$, 则必有 $\alpha^T A \alpha = 0$

充分性: 设 $A = (a_{ij})_{n \times n}$, 则 $X = (x_1, x_2, \dots, x_n)^T$,

由 $X^T A X = 0$, 取 X 为 $\varepsilon_1 = (1, 0, 0, \dots, 0)^T$, $\varepsilon_2 = (0, 1, \dots, 0)^T$, ..., $\varepsilon_n = (0, 0, \dots, 1)^T$.

代入 $X^T A X = 0 \Rightarrow a_{11} = 0, a_{22} = 0, \dots, a_{nn} = 0$.

又取 X 为 $\varepsilon_i + \varepsilon_j = (0, 0, \dots, 1, 0, \dots, 1, \dots, 0)^T$,

代入 $X^T A X = 0$, 可以得到 $a_{ij} = 0$ ($i \neq j, i, j = 1, 2, \dots, n$).

则 $A = O$.

2.

证明: 必要性: 由 $A^T = -A$, 则 $(X^T A X)^T = X^T A^T X = -X^T A X$,

又 $X^T A X$ 是一个数, 于是 $(X^T A X)^T = X^T A X$,

则 $X^T A X = 0$.

充分性: 设 $X^T A X = 0$, 取 X 为 $\varepsilon_1 = (1, 0, \dots, 0)^T$, $\varepsilon_2 = (0, 1, \dots, 0)^T$, ..., $\varepsilon_n = (0, 0, \dots, 1)^T$,

代入 $X^T A X = 0 \Rightarrow a_{11} = 0, a_{22} = 0, \dots, a_{nn} = 0$. 又取 $X = \varepsilon_i + \varepsilon_j$ ($i \neq j$),

代入 $X^T A X = 0$ 有 $X^T A X = (\varepsilon_i + \varepsilon_j)^T A (\varepsilon_i + \varepsilon_j) = a_{ij} + a_{ji} = 0$ 故 $a_{ij} = -a_{ji}$, 从而 A 为反对称矩阵.

习题6.2

(A)

1.

$$(1) A = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix},$$

$$\text{则 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 2 & 2 & 0 \\ 2 & \lambda - 1 & 2 \\ 0 & 2 & \lambda \end{vmatrix} = (\lambda + 2)(\lambda - 1)(\lambda - 4),$$

$$\lambda = -2 \text{ 时, } \begin{pmatrix} -4 & 2 & 0 \\ 2 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \alpha_1 = (1, 2, 2)^T;$$

$$\lambda = 1 \text{ 时, } \begin{pmatrix} -1 & 2 & 0 \\ 2 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}, \alpha_2 = (2, 1, -2)^T;$$

$$\lambda = 4 \text{ 时, } \begin{pmatrix} 2 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \alpha_3 = (2, -2, 1)^T;$$

$$\text{正交单位化, 有 } p_1 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)^T, p_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)^T, p_3 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)^T,$$

$$\text{标准形 } y_1^2 - 2y_2^2 + 4y_3^2, X = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix} Y.$$

$$(2) A = \begin{pmatrix} 1 & -2 & -4 \\ -2 & 4 & -2 \\ -4 & -2 & 1 \end{pmatrix},$$

$$\text{则 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & 2 & 4 \\ 2 & \lambda - 4 & 2 \\ 4 & 2 & \lambda - 1 \end{vmatrix} = (\lambda + 4)(\lambda - 5)^2,$$

$$5E - A = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ & & \end{pmatrix},$$

$$\text{得 } \alpha_1 = (1, -2, 0)^T, \alpha_2 = (-1, 0, 1)^T;$$

$$-4E - A = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & -1 \\ & 1 & -\frac{1}{2} \\ & & \end{pmatrix},$$

$$\text{得 } \alpha_3 = (2, 1, 2)^T;$$

正交单位化, 有

$$p_1 = \left(\frac{1}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right)^T, p_2 = \left(\frac{\sqrt{5}}{6}, -\frac{2\sqrt{5}}{3}, -\frac{\sqrt{5}}{6}\right)^T, p_3 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)^T,$$

$$\text{标准形 } 5y_1^2 + 5y_2^2 - 4y_3^2, X = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{5}}{6} & \frac{2}{3} \\ 0 & -\frac{2\sqrt{5}}{3} & \frac{1}{3} \\ -\frac{1}{\sqrt{5}} & -\frac{\sqrt{5}}{6} & \frac{2}{3} \end{pmatrix} Y.$$

$$(3) A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

$$\text{则 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = (\lambda + \frac{1}{2})^2(\lambda - 1),$$

$$-\frac{1}{2}E - A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix},$$

$$\text{得 } \alpha_1 = (1, -1, 0)^T, \alpha_2 = (1, 0, -1)^T;$$

$$E - A = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix},$$

$$\text{得 } \alpha_3 = (1, 1, 1)^T$$

$$\text{则 } p_1 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)^T, p_2 = \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right),$$

$$p_3 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right),$$

$$\text{标准形 } -\frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 + y_3^2, X = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} Y.$$

$$(4) A = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix},$$

$$\text{则 } f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda & 0 & 0 & -\frac{1}{2} \\ 0 & \lambda & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \lambda & 0 \\ -\frac{1}{2} & 0 & 0 & \lambda \end{vmatrix} = (\lambda - \frac{1}{2})^2(\lambda + \frac{1}{2})^2,$$

$$\frac{1}{2}E - A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\text{得 } \alpha_1 = (1, 0, 0, 1)^T, \alpha_2 = (0, 1, 1, 0)^T;$$

$$-\frac{1}{2}E - A = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

得 $\alpha_3 = (1, 0, 0, -1)^T$, $\alpha_4 = (0, 1, -1, 0)^T$;

$$\text{标准形 } \frac{1}{2}y_1^2 + \frac{1}{2}y_2^2 - \frac{1}{2}y_3^2 - \frac{1}{2}y_4^2, X = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} Y$$

2.

(1)

$$\begin{aligned} & x_1^2 + 2x_2^2 - x_3^2 + 2x_1x_2 - 2x_3x_1 \\ &= [x_1^2 + 2x_1(x_2 - x_3) + (x_2 - x_3)^2] - (x_2 - x_3)^2 + 2x_2^2 - x_3^2 \\ &= (x_1 + x_2 - x_3)^2 + x_2^2 + 2x_2x_3 - 2x_3^2 \\ &= (x_1 + x_2 - x_3)^2 + (x_2 + x_3)^2 - 3x_3^2 \end{aligned}$$

$$\text{则 } \begin{cases} y_1 = x_1 + x_2 - x_3 \\ y_2 = x_2 + x_3 \\ y_3 = x_3 \end{cases},$$

$$\text{得 } X = \begin{pmatrix} 1 & -1 & 2 \\ & 1 & -1 \\ & & 1 \end{pmatrix} Y, \text{ 化为 } y_1^2 + y_2^2 - 3y_3^2.$$

$$(2) \text{ 令 } \begin{cases} x_1 = y_1 + y_2 \\ x_2 = y_1 - y_2 \\ x_3 = y_3 \end{cases}$$

$$\text{则 } f = 2y_1^2 - 2y_2^2 + 4y_1y_3 = 2(y_1^2 + 2y_1y_3 + y_3^2) - 2y_2^2 - 2y_3^2 = 2(y_1 + y_3)^2 - 2y_2^2 - 2y_3^2,$$

$$\text{则 } \begin{cases} z_1 = y_1 + y_3 \\ z_2 = y_2 \\ z_3 = y_3 \end{cases} \Rightarrow \begin{cases} y_1 = z_1 - z_3 \\ y_2 = z_2 \\ y_3 = z_3 \end{cases}$$

故 $X = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} Z$, 化为 $2z_1^2 - 2z_2^2 - 2z_3^2$.

(3)

$$\begin{aligned} & x_1^2 - x_2^2 + 2x_1x_2 + 4x_3x_1 \\ &= [x_1^2 + 2(x_2 + 2x_3)x_1 + (x_2 + 2x_3)^2] - (x_2 + 2x_3)^2 - x_2^2 \\ &= (x_1 + x_2 + 2x_3)^2 - x_2^2 - (x_2 + 2x_3)^2 \end{aligned}$$

$$\text{令} \begin{cases} y_1 = x_1 + x_2 + 2x_3 \\ y_2 = x_2 \\ y_3 = x_2 + 2x_3 \end{cases}$$

得 $X = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} Y$, 化为 $y_1^2 - y_2^2 - y_3^2$.

(4) 令 $X = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} Y$, 则 $x_1x_4 + x_2x_3 = y_1^2 - y_4^2 + y_2^2 - y_3^2$

3.

	秩	正惯性指数	负惯性指数	符号差
1.				
(1)	3	2	1	1
(2)	3	2	1	1
(3)	3	1	2	-1
(4)	4	2	2	0
2.				
(1)	3	2	1	1
(2)	3	1	2	-1
(3)	3	1	2	-1
(4)	4	2	2	0

4.

依题意 $A \sim \begin{pmatrix} 9 & & \\ & 9 & \\ & & 0 \end{pmatrix}$ 且 $A\beta_3 = 0$

考虑 $PP^T = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} (1 \ 2 \ 2) = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 2 \\ & & \\ & & \end{pmatrix}$

由于不同的特征值的特征向量正交. 故 $PP^T\alpha = 0$ 的解即其他特征向量

则可取 $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & & \\ & 9 & \\ & & 9 \end{pmatrix}$

故 $A = Q\Lambda Q^{-1} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 9 & \\ & & 9 \end{pmatrix} \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & -\frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & -\frac{5}{9} \end{pmatrix} = \begin{pmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$

习题6.3

(A)

1.

(1) 由 $A = \begin{pmatrix} 1 & -4 & 1 \\ -4 & 1 & -2 \\ 1 & -2 & 2 \end{pmatrix}$, $f_A(\lambda) = \begin{vmatrix} \lambda-1 & 4 & -1 \\ 4 & \lambda-1 & 2 \\ -1 & 2 & \lambda-2 \end{vmatrix}$,

由于 $\begin{vmatrix} 1 & -4 \\ -4 & 1 \end{vmatrix} = -7 < 0$, 故 A 不是正定的.

(2) $A = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 6 \end{pmatrix}$, 又 $|7| > 0$, $\begin{vmatrix} 7 & -2 \\ -2 & 8 \end{vmatrix} = 52 > 0$, $\begin{vmatrix} 7 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 6 \end{vmatrix} = 284 > 0$, 故 A 是正定的.

(3) $A = \begin{pmatrix} 99 & -6 & 24 \\ -6 & 130 & -30 \\ 24 & -30 & 70 \end{pmatrix}$, 又 $|99| > 0$, $\begin{vmatrix} 99 & -6 \\ -6 & 130 \end{vmatrix} > 0$, $\begin{vmatrix} 99 & -6 & 24 \\ -6 & 130 & -30 \\ 24 & -30 & 70 \end{vmatrix} = 743040 > 0$, 故 A 是正定的.

$$(4) A = \begin{pmatrix} 10 & 4 & 12 \\ 4 & 2 & -14 \\ 12 & -14 & 1 \end{pmatrix}, \text{ 又 } |A| = -3588 < 0, \text{ 不是正定的}$$

2.

证明: 依题意, X^TAX , X^TBX 是正定二次型, 则

$$X^T(A+B)X = X^TAX + X^TBX,$$

由于 $X^TAX > 0$, $X^TBX > 0$, 故 $X^T(A+B)X > 0$,

故 $X^T(A+B)X$ 也是正定的,

从而 $A+B$ 也是正定矩阵.

3.

$$(1) A = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & t \end{pmatrix}, \text{ 则 } |A| > 1, \begin{vmatrix} 4 & -1 \\ -1 & 1 \end{vmatrix} = 3 > 0, \begin{vmatrix} 4 & -1 & 2 \\ -1 & 1 & -1 \\ 2 & -1 & t \end{vmatrix} > 0,$$

$$\text{得 } t > \frac{4}{3}.$$

$$(2) A = \begin{pmatrix} 2 & t & -3 \\ t & 3 & 1 \\ -3 & 1 & 4 \end{pmatrix}, \text{ 由 } \begin{vmatrix} 2 & t \\ t & 3 \end{vmatrix} > 0 \text{ 得 } t^2 < 6, \text{ 又 } \begin{vmatrix} 2 & t & -3 \\ t & 3 & 1 \\ -3 & 1 & 4 \end{vmatrix} > 0,$$

$$\text{得 } -4t^2 - 72t - 49 > 0,$$

$$\text{但由 } t > -\sqrt{6},$$

$$\text{则 } 4t^2 + 72t + 49 > 193 - 72\sqrt{6} > 16 > 0, \text{ 故它不可能正定}$$

4.

证明: 由于 $A^T = A$, 则 $A = A^T A^{-1} A$, 即 A^{-1} 合同于 A ,

由于 A 是正定的, 故 A^{-1} 也正定.

5.

证明: 由于 $f(x_1, x_2, \dots, x_n) = X^TAX = \sum_{i=1}^n \sum_{j=1}^n a_{ij}x_i x_j$ 正定,

取 $X_i = \varepsilon_i^T = (0, \dots, 0, 1, 0, \dots, 0)^T$ (其中第 i 个分量 $x_i = 1$),

则 $X_i^TAX_i = a_{ii}x_i^2 = a_{ii} > 0$.