# 目 录

# 习题1.1

## (A)

1.

依题意 $\begin{cases} a+2b=1 \\ a-b=2 \end{cases}$，则 $a=\dfrac{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}=\dfrac{5}{3}$， $b=\dfrac{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix}}=-\dfrac{1}{3}$.

2.

(1) $A+3B=\begin{pmatrix} 6 & -1 \\ 3 & 0 \\ 2 & 3 \end{pmatrix}+3\begin{pmatrix} 1 & -1 \\ 0 & 5 \\ 2 & 3 \end{pmatrix}=\begin{pmatrix} 9 & -4 \\ 3 & 6 \\ 17 & 12 \end{pmatrix}$.

(2) $A^{\mathrm{T}}=\begin{pmatrix} 6 & 3 & 2 \\ -1 & 0 & 3 \end{pmatrix}$, $B^{\mathrm{T}}=\begin{pmatrix} 1 & 0 & 5 \\ -1 & 2 & 3 \end{pmatrix}$, 则 $A^T-2B^T=\begin{pmatrix} 4 & 3 & -8 \\ 1 & -4 & -3 \end{pmatrix}$.

3.

由 $\begin{pmatrix} 2 & 3 & -1 \\ -1 & 4 & 0 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ 7 & 3 \\ -1 & 5 \end{pmatrix}=\begin{pmatrix} 16 & 6 \\ 31 & 11 \end{pmatrix}$, 故有 $\begin{cases} y_1=16t_1+6t_2 \\ y_2=31t_1+11t_2 \end{cases}$.

4.

$\begin{cases} x_1=0 \\ y_1=y \end{cases}$，几何意义为在 $y$ 轴上的投影.

5.

(1) $\begin{pmatrix} 3 & 2 & 1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 3 & -1 \end{pmatrix}=\begin{pmatrix} 6 & 2 & -2 \\ -2 & -10 & 6 \end{pmatrix}$.

(2) $\begin{pmatrix} 6 & 0 & 8 & -3 \end{pmatrix}\begin{pmatrix} 0.5 \\ -2 \\ 2.5 \\ -1 \end{pmatrix} = 6 \times 0.5 + 8 \times 2.5 + 3 = 26$

(3) $\begin{pmatrix} 0.5 \\ -2 \\ 2.5 \\ 1 \end{pmatrix}\begin{pmatrix} 6 & 0 & 8 & -3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 4 & -1.5 \\ -12 & 0 & -16 & 6 \\ 15 & 0 & 20 & -7.5 \\ 6 & 0 & -8 & 3 \end{pmatrix}$

(4) $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -2 \\ -1 & 3 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 & 0 \\ 0 & 4 & -1 \\ 3 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -8 & 1 \\ 0 & 8 & -1 \\ -6 & 16 & -3 \end{pmatrix}$

(5) $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 3 \\ 1 & 3 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + x_2^2 - x_3^2 - 2x_1x_2 + 2x_1x_3 + 6x_2x_3.$

(6) $\begin{pmatrix} 3 & 2 & 1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 + 2x_2 + x_3 \\ -x_1 - 2x_2 - 3x_3 \end{pmatrix}.$

6.

$$AB = \begin{pmatrix} 5 & -1 & -1 \\ 4 & 8 & 2 \\ -3 & -1 & 11 \end{pmatrix}, \quad BA = \begin{pmatrix} 2 & 1 & -3 \\ 2 & 11 & -1 \\ 8 & -1 & 11 \end{pmatrix}, \quad AB - BA = \begin{pmatrix} 3 & -2 & 2 \\ 2 & -3 & 3 \\ -11 & 0 & 0 \end{pmatrix},$$

此题说明 $AB \neq BA$.

7.

**性质2 证明**: 设 $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$, $C = (c_{ij})_{p\times q}$, 则

$$(A+B)C = (a_{ij} + b_{ij})_{m\times n}(c_{ij})_{n\times q} = \left(\sum_{k=1}^{n}(a_{ik} + b_{ik})c_{kj}\right)_{m\times q} = AC + BC.$$

同理可证 $C(A+B) = CA + CB$.

**性质3 证明**: 设 $A=(a_{ij})_{m\times n}$, $B=(b_{ij})_{n\times p}$, 则

$$\lambda(AB)=\lambda\left(\sum_{k=1}^{n}a_{ik}b_{kj}\right)_{m\times p}=\left(\sum_{k=1}^{n}(\lambda a_{ik})b_{kj}\right)_{m\times p}=(\lambda A)B=A(\lambda B).$$

**性质4 证明**: 设 $=(a_{ij})_{m\times n}$, 则

$$E_mA_{m\times n}=\begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix},$$

其中元素 $c_{ij}=1\cdot a_{ij}=a_{ij}$, 故 $E_mA_{m\times n}=A_{m\times n}$, 其余同理可证.

8.

(1) $A=\begin{pmatrix}2 & -1\\ 2 & -1\end{pmatrix}$, $\quad B=\begin{pmatrix}-1 & -1\\ 2 & 2\end{pmatrix}$.

(2) $A=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$.

(3) $A=\begin{pmatrix}-1 & 1\\ -2 & 2\end{pmatrix}$.

9.

(1) 记 $A=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$,

则 $A^2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}=-E$.

当 $n=2k$ 时, $A^n=A^{2k}=(-E)^k=(-1)^kE$;

当 $n=2k+1$ 时, $A^n=A^{2k+1}=(-E)^kA=(-1)^kA$.

(2) $A^2=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}^2=\begin{pmatrix}\cos 2\alpha & \sin 2\alpha\\ -\sin 2\alpha & \cos 2\alpha\end{pmatrix}$.

假设 $n=k$ 时, $A^k=\begin{pmatrix}\cos k\alpha & \sin k\alpha\\ -\sin k\alpha & \cos k\alpha\end{pmatrix}$,

则当 $n=k+1$ 时,

$$A^{k+1}=\begin{pmatrix}\cos k\alpha & \sin k\alpha\\ -\sin k\alpha & \cos k\alpha\end{pmatrix}\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}=\begin{pmatrix}\cos(k+1)\alpha & \sin(k+1)\alpha\\ -\sin(k+1)\alpha & \cos(k+1)\alpha\end{pmatrix}.$$

(3) $A=\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1&1&0\end{pmatrix}=\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$, 又 $\alpha^{\mathrm{T}}\alpha=\begin{pmatrix}1&1&0\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=2$,

则 $A^n=(\alpha\alpha^{\mathrm{T}})^n=\alpha(\alpha^{\mathrm{T}}\alpha)(\alpha^{\mathrm{T}}\alpha)\cdots(\alpha^{\mathrm{T}}\alpha)\alpha^{\mathrm{T}}=2^{n-1}\alpha\alpha^{\mathrm{T}}=2^{n-1}A=2^{n-1}\begin{pmatrix}1&1&0\\1&1&0\\0&0&0\end{pmatrix}$.

10.

证明: 令 $A=(a_{ij})_{m\times n}$, 则 $a_{ij}\in\mathbf{R}$, $A^{\mathrm{T}}=A$, 由 $A^2=0,\Rightarrow AA^{\mathrm{T}}=0$

即 $(a_{ij})_{m\times n}(a_{ji})_{n\times m}=\left(\sum_{k=1}^{n}a_{ik}a_{ki}\right)_{m\times m}=0$.

记 $c_{ij}=\sum_{k=1}^{n}a_{ik}a_{ki}$,

考虑 $c_{ii}=\sum_{k=1}^{n}a_{ik}^2=0\Rightarrow a_{ik}=0$,

故 $\forall i,j$, s.t. $a_{ij}=0$. 故 $A=0$.

11.

$$f(A)=A^2+A+E=\begin{pmatrix}8&-1&-1\\11&8&2\\8&-1&11\end{pmatrix}+\begin{pmatrix}3&-1&1\\2&1&3\\1&3&-1\end{pmatrix}+\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}=\begin{pmatrix}12&-2&0\\13&10&5\\9&2&11\end{pmatrix},$$

$$f(B)=B^n+B+E=\begin{pmatrix}3^n&&\\&2^n&\\&&(-1)^n\end{pmatrix}+\begin{pmatrix}3&&\\&2&\\&&-1\end{pmatrix}+\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}=\begin{pmatrix}3^n+4&&\\&2^n+3&\\&&(-1)^n\end{pmatrix}.$$

12.

证明: 依题意, $A^H=A$, 即 $(\overline{A})^{\mathrm{T}}=A$, 即 $(\overline{a_{ji}})_{n\times n}=(a_{ij})_{n\times n}$,

则 $a_{ij}=\overline{a_{ji}}$.

又当 $i=j$ 时, $a_{ii}=\overline{a_{ii}}$, 则 $a_{ii}$ 是实数.

## (B)

1.

与 $A$ 可交换的 $B$ 必是二阶方阵, 设 $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

则 $AB=BA \Rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}=\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

$$\Rightarrow \begin{cases} a+c=a \\ b+d=a+b \\ c+d=d \end{cases} \Rightarrow \begin{cases} c=0 \\ a=d \end{cases}$$

故 $B=\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$, $a,b$ 为实数.

2.

证明: $\because A,B$ 可换, 即 $AB=BA$, 又 $A,B$ 满足

$$(A+B)^2=(A+B)(A+B)=A^2+AB+BA+B^2=A^2+2AB+B^2$$

设当 $n=k$ 时, $(A+B)^n=\sum_{i=0}^{n} C_n^i A^{n-i}B^i$,

则当 $n=k+1$ 时,

$$\begin{aligned}(A+B)^{n+1}&=\left(\sum_{i=0}^{n} C_n^i A^{n-i}B^i\right)(A+B)\\&=C_n^0A^{n+1}+(C_k^n+C_n^1)A^nB+\cdots+(C_n^{r-1}+C_n^r)A^{n-r+1}B^r+\cdots\\&=\sum_{i=0}^{n+1} C_{n+1}^i A^{n-i}B^i\end{aligned}$$

证毕.

3.

$$A=\begin{pmatrix}\lambda & 1 & 0\\ & \lambda & 1\\ & & \lambda\end{pmatrix}=\lambda E+\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix},$$

记 $B=\begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{pmatrix}$, 则 $B^2=\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, $B^3=0\ (n\geq 3)$,

则 $A^n=(\lambda E+B)^n=\lambda^n E+n\lambda^{n-1}B+\dfrac{n(n-1)}{2}\lambda^{n-2}B^2=\begin{pmatrix}\lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2}\\ & \lambda^n & n\lambda^{n-1}\\ & & \lambda^n\end{pmatrix}$,

故 $P_n(A)=\begin{pmatrix}\lambda^n+\lambda+1 & n\lambda^{n-1}+1 & \frac{n(n-1)}{2}\lambda^{n-2}\\ & \lambda^n+\lambda+1 & n\lambda^{n-1}+1\\ & & \lambda^n+\lambda+1\end{pmatrix}$, $P_5(A)=\begin{pmatrix}\lambda^5+\lambda+1 & 5\lambda^4+1 & 10\lambda^3\\ & \lambda^5+\lambda+1 & 5\lambda^4+1\\ & & \lambda^5+\lambda+1\end{pmatrix}$.

4.

证明: 令 $A=\begin{pmatrix}a_{11} & 0 & \cdots & 0\\ a_{21} & a_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$, $B=\begin{pmatrix}b_{11} & 0 & \cdots & 0\\ b_{21} & b_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{nn}\end{pmatrix}$,

记 $C=AB$, 则 $c_{ij}=\sum\limits_{k=1}^{n}a_{ik}b_{kj}$,

由假设, 当 $i<j$ 时 $a_{ij}=b_{ij}=0$, 故 $c_{ij}$ 中各项因子都含 0, 则 $c_{ij}=0\ (i<j)$, 故 $C$ 为下三角矩阵.

5.

证明: 令 $A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 \\ a_{21} & a_{22} & a_{23} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & a_{(n-1)n} \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & 0 & \cdots & 0 \\ b_{21} & b_{22} & b_{23} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ b_{n1} & b_{n2} & \cdots & \cdots & b_{nn} \end{pmatrix}$,

记 $C = AB$, 则 $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum\limits_{k=1}^{n} a_{ik}b_{kj}$.

当 $j > i+1$ 时, $a_{ik} = b_{kj} = 0$, 则 $c_{ij}$ 中各项都含因子 0, 所以 $c_{ij} = 0\ (j > i+1)$.

故 $n$ 阶下 Hessenberg 矩阵乘积仍是下 Hessenberg 矩阵.

同理可证 $n$ 阶上 Hessenberg 矩阵乘积仍是上 Hessenberg 矩阵.

6.

(1) 由 $(A + A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}} + (A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}} + A = A + A^{\mathrm{T}}$, 知 $A + A^{\mathrm{T}}$ 为对称矩阵.

由 $(A - A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}} + (-A^{\mathrm{T}})^{\mathrm{T}} = A^{\mathrm{T}} - A = -(A - A^{\mathrm{T}})$, 知 $A - A^{\mathrm{T}}$ 为反对称矩阵.

(2) 由 (1) 知 $A = \frac{1}{2}\left[(A + A^{\mathrm{T}}) + (A - A^{\mathrm{T}})\right]$.

7.

记 $A = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$, $A_1 = \begin{pmatrix} -1 & -4 \\ 1 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, $A_3 = \begin{pmatrix} -1 & 4 \\ -1 & -1 \end{pmatrix}$,

则 $A = \frac{1}{3}A_1A_2A_3$, $\frac{1}{3}A_3A_1 = E$,

$$
\begin{aligned}
\text{则 } A^{11} &= \frac{1}{3}A_1A_2A_3 \cdot \frac{1}{3}A_1A_2A_3 \cdots A_3\frac{1}{3}A_1A_2A_3 \\
&= \frac{1}{3}A_1(A_2)^{11}A_3 \\
&= \frac{1}{3}\begin{pmatrix} -1 & -4 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & \\ & 2^{11} \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 & -2^{13} \\ -1 & 2^{11} \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -1 & -1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1+2^{13} & 4+2^{13} \\ -1-2^{11} & -4-2^{11} \end{pmatrix}
\end{aligned}
$$

## (C)

1.

证明: 记 $C=AB=(c_{ij})_{m\times m}$, $D=BA=(d_{ij})_{n\times n}$, 则 $c_{ij}=\sum\limits_{k=1}^{n}a_{ik}b_{kj}$, $d_{ij}=\sum\limits_{k=1}^{m}b_{ik}a_{kj}$, 则 $\operatorname{tr}(AB)=\sum\limits_{i=1}^{m}c_{ii}=\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{n}a_{ik}b_{ki}=\sum\limits_{k=1}^{n}\sum\limits_{i=1}^{m}b_{ki}a_{ik}=\sum\limits_{k=1}^{n}d_{kk}=\operatorname{tr}(BA)$.

2.

证明: 因为 $A$ 与任意 $n$ 阶方阵可交换, 取 $B=\begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix}$, 则考虑 $AB$ 与 $BA$ 的第一行第二列元素有 $a_{11}=a_{22}$. 同理依次取 $B$ 为 $\begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix}$, $\begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix}$, $\cdots$, $\begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix}$, 可以得到 $a_{11}=a_{22}=\cdots=a_{nn}$. 又取 $B=\begin{pmatrix}0&0&\cdots&0\\0&0&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&0\end{pmatrix}$, 可得第一行中除 $a_{11}$ 外均为 0. 依次取 $B$ 为 $\begin{pmatrix}0\\1\end{pmatrix}$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}$, $\cdots$, $\begin{pmatrix}0&\cdots&0\\\vdots&\ddots&\vdots\\0&\cdots&1\end{pmatrix}$, 可得除主对角线外均为 0. 故 $A$ 为 $n$ 阶数量矩阵.

3.

证明: 用反证法. 假设 $\exists a_{ij}\neq 0$, 则考虑 $C=A^2$, $c_{ii}=\sum\limits_{k=1}^{n}a_{ik}a_{ki}=a_{i1}a_{1i}+a_{i2}a_{2i}+\cdots+a_{ii}^2+a_{i,i+1}a_{i+1,i}+\cdots+a_{in}a_{ni}$. 因为 $A$ 为上三角矩阵, 则当 $i>j$ 时 $a_{ij}=0$, 故 $c_{ii}=a_{ii}^2>0$, 则不可能 $\exists k$, 使 $A^k=0$. 故矛盾. 则 $A$ 的主对角元素全为 0.

# 习题1.2

## (A)

1.

$$AB=\left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 3 & 2 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right)\left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ \hline 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{array}\right)=\begin{pmatrix} 6 & 0 & 3 & 0 \\ 0 & 6 & 0 & 3 \\ 6 & 3 & 0 & 0 \\ -9 & 3 & 0 & 0 \end{pmatrix}$$

$$BA=\left(\begin{array}{cc|cc} 2 & 1 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ \hline 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{array}\right)\left(\begin{array}{cc|cc} 1 & -1 & 0 & 0 \\ 3 & 2 & 0 & 1 \\ \hline 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right)=\left(\begin{array}{cc|cc} 5 & 0 & 2 & 1 \\ 0 & 5 & -3 & 1 \\ \hline 10 & -1 & 1 & 0 \\ 3 & 11 & 0 & 1 \end{array}\right)$$

$AB -$

$$BA=\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 3 & 2 \\ -4 & 4 & -1 & 0 \\ -12 & -8 & 0 & -1 \end{pmatrix}. \quad A\beta=\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}=\alpha_1+2\alpha_2-\alpha_4.$$

2.

$$A^4=\begin{pmatrix} 4 & 3 & & \\ -3 & 1 & & \\ & & 3 & 0 \\ & & 3 & 3 \end{pmatrix}^4=\begin{pmatrix} \begin{pmatrix} 4 & 3 \\ -3 & 1 \end{pmatrix}^4 & \\ & \begin{pmatrix} 3 & 0 \\ 3 & 3 \end{pmatrix}^4 \end{pmatrix}=\begin{pmatrix} -176 & -15 & & \\ 15 & -161 & & \\ & & 81 & 0 \\ & & 324 & 81 \end{pmatrix},\quad B^2=$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}=\begin{pmatrix} 2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$，假设 $n=k$ 时，$B^k=\begin{pmatrix} 2^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$，则当 $n=k+1$ 时，

$$B^{k+1}=\begin{pmatrix} 2^k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}=\begin{pmatrix} 2^{k+1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

故 $B^n = \begin{pmatrix} 2^n & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

3.

$$AB^{\mathrm{T}} = \left(\begin{array}{ccc:cc} 2 & -1 & 0 & 0 & 0 \\ -1 & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & -7 & 3 \end{array}\right) \left(\begin{array}{c:cc} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 2 & 5 & 0 \\ \hdashline 0 & 2 & 0 \\ 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{c:cc} 4 & -2 & 6 \\ -2 & \frac{4}{3} & -3 \\ 4 & 20 & 0 \\ \hdashline 0 & 5 & -2 \\ 0 & -1 & 3 \end{array}\right)$$

4.

证明:

$$MM^{\mathrm{T}} = \begin{pmatrix} E_m & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & A^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} E_m & 0 \\ 0 & AA^{\mathrm{T}} \end{pmatrix} = E$$

$$M^{\mathrm{T}}M = \begin{pmatrix} E_m & 0 \\ 0 & A^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} E_m & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} E_m & 0 \\ 0 & A^{\mathrm{T}}A \end{pmatrix} = E$$

故 $\begin{pmatrix} E_m & 0 \\ 0 & A \end{pmatrix}$ 为正交矩阵.

## (B)

1.

(1) 证明: (1) $A=\begin{pmatrix} O & E_{n-1} \\ I & O \end{pmatrix}$ 则 $k=1$ 时成立

假设 $k=m$ 时 $A^m=\begin{pmatrix} O & E_{n-m} \\ E_m & O \end{pmatrix}$

则 $A^{m+1}=\begin{pmatrix} O & E_{n-m} \\ E_m & O \end{pmatrix}\begin{pmatrix} O & E_{n-1} \\ I & O \end{pmatrix}$

记 $A_1=\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{m\times m}$ $\quad A_2=\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{m\times(n-m)}$

$A_3=\begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}_{(n-m)\times m}$ $\quad A_4=\begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}_{(n-m)\times(n-m)}$

则 $A^{m+1}=\begin{pmatrix} O & E_{n-m} \\ E_m & O \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}=\begin{pmatrix} A_3 & A_4 \\ A_1 & A_2 \end{pmatrix}$

$$=\left(\begin{array}{ccccccc} 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hdashline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right)$$

$$=\begin{pmatrix} O & E_{n-m-1} \\ E_{m+1} & O \end{pmatrix}$$

故由数学归纳法知 $A^k = \begin{pmatrix} O & E_{n-k} \\ E_k & O \end{pmatrix}, k = 1, 2, \ldots, n-1$

(2) 在(1)中使用数学归纳法时并没有严格限制 $k < n$, 故当 $k = n$ 时 $A^n = A^{n-1} \cdot A = E$.

# 习题1.3

## (A)

1.

(1) 由于 $\left(\begin{array}{cc:cc} 3 & -1 & 1 & \\ -2 & 1 & & 1 \end{array}\right) \to \left(\begin{array}{cc:cc} 1 & -\frac{1}{3} & \frac{1}{3} & \\ -1 & \frac{1}{2} & & \frac{1}{2} \end{array}\right) \to \left(\begin{array}{cc:cc} 1 & -\frac{1}{3} & \frac{1}{3} & \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right) \to$ $\left(\begin{array}{cc:cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}\right)$, 故 $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$.

(2) $\begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 3 \end{pmatrix}^{-1} = \begin{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{-1} & \\ & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^{*} & \\ & \frac{1}{3} \end{pmatrix} =$ $\begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & \frac{1}{3} \end{pmatrix}$.

(3) $\begin{pmatrix} 5 & 2 & & \\ 2 & 1 & & \\ & & 1 & 8 \\ & & 1 & 9 \end{pmatrix}^{-1} = \begin{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{-1} & \\ & \begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}^{-1} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}^{*} & \\ & \begin{pmatrix} 1 & 8 \\ 1 & 9 \end{pmatrix}^{*} \end{pmatrix} =$ $\begin{pmatrix} 1 & -2 & & \\ -2 & 5 & & \\ & & 9 & -8 \\ & & -1 & 1 \end{pmatrix}$.

(4) $\begin{pmatrix} 2 & \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}$.

2.

(1) 由 $\left(\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 3 & 4 & -2 & 2 \\ 5 & -4 & 1 & 3 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & -2 & 1 & -1 \\ 0 & -14 & 6 & -2 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & -1 & 5 \end{array}\right) \to$
$\left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -5 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -5 \end{array}\right)$, 故 $x = \begin{pmatrix} 0 \\ -2 \\ -5 \end{pmatrix}$.

(2) $X = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & -2 \\ 5 & -4 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 0 \\ 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 & 0 \\ -\frac{13}{2} & 3 & -\frac{1}{2} \\ -16 & 7 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 3 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} =$
$\begin{pmatrix} 5 & 2 \\ \frac{31}{2} & \frac{11}{2} \\ 37 & 13 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & -11 \\ \frac{51}{2} & -\frac{71}{2} \\ 61 & -85 \end{pmatrix}$.

3.

由 $|M| = \begin{vmatrix} 0 & A \\ C & B \end{vmatrix} = (-1)^{nm}|A| \cdot |C|$, $\because A, C$ 可逆, 则 $|A| \neq 0, |C| \neq 0$, 故 $|M| \neq 0$, 则 $M$ 可逆. 设 $M^{-1} = \begin{pmatrix} X & Y \\ Z & H \end{pmatrix}$, 则 $M \cdot M^{-1} = \begin{pmatrix} 0 & A \\ C & B \end{pmatrix} \begin{pmatrix} X & Y \\ Z & H \end{pmatrix} = \begin{pmatrix} AZ & AH \\ CX + BZ & CY + BH \end{pmatrix} = \begin{pmatrix} E_1 & \\ & E_2 \end{pmatrix}$,

则 $\begin{cases} AZ = E_1 \\ AH = 0 \\ CX + BZ = 0 \\ CY + BH = 0 \end{cases} \Rightarrow \begin{cases} X = -C^{-1}BA^{-1} \\ Y = C^{-1} \\ Z = A^{-1} \\ H = 0 \end{cases}$ 故 $M^{-1} = \begin{pmatrix} -C^{-1}BA^{-1} & C^{-1} \\ A^{-1} & 0 \end{pmatrix}$.

4.

因为 $|A| = 0$, 故 $A$ 不可逆.

5.

由 $AB+E=A^2+B$, 得 $(A-E)B=A^2-E=(A-E)(A+E)$, $\because |A-E|=\begin{vmatrix}0 & 2\\1 & 2\end{vmatrix}\neq 0$, 故 $B=A+E=\begin{pmatrix}2 & 2\\1 & 4\end{pmatrix}$.

## (B)

1.

由 $A^2-4A-E=0\Rightarrow A(A-4E)=E$ 故 $A^{-1}=A-4E$ 又由 $A^2-4A-E=0\Rightarrow 4A^2+A-17A-\dfrac{17}{4}E=-\dfrac{E}{4}\Rightarrow (4A+E)(17E-4A)=E$ 故 $(4A+E)^{-1}=17E-4A$ 也可由 $A^2=4A+E\Rightarrow (4A+E)^{-1}=(A^2)^{-1}=(A^{-1})^2=(A-4E)^2$

2.

由 $A^2=A$ 得 $A^2+A-2A-2E=2E\Rightarrow (A+E)(A-2E)=-2E$ 故 $(A+E)^{-1}=\dfrac{1}{2}(2E-A)$.

3.

由 $A^{-1}BA=6A+BA$ 得 $(A^{-1}-E)BA=6A$ 故 $B=6(A^{-1}-E)^{-1}=\begin{pmatrix}6 & & \\ & 2 & \\ & & \frac{3}{2}\end{pmatrix}$

4.

由 $E^k-A^k=E$ 得 $(E-A)(E^{k-1}+E^{k-2}A+E^{k-3}A^2+\cdots+A^{k-1})=E\Rightarrow (E-A)^{-1}=E+A+A^2+\cdots+A^{k-1}$

5.

设 $\begin{pmatrix}A & B\\B & A\end{pmatrix}\begin{pmatrix}X & Y\\Z & H\end{pmatrix}=\begin{pmatrix}E & O\\O & E\end{pmatrix}$ 则 $\begin{cases}AX+BZ=E\\AY+BH=O\\BX+AZ=O\\BY+AH=E\end{cases}\Rightarrow\begin{cases}X=\dfrac{1}{2}[(A+B)^{-1}+(A-B)^{-1}]\\Y=\dfrac{1}{2}[(A+B)^{-1}-(A-B)^{-1}]\\Z=\dfrac{1}{2}[(A+B)^{-1}-(A-B)^{-1}]\\H=\dfrac{1}{2}[(A+B)^{-1}+(A-B)^{-1}]\end{cases}$

故 $\begin{pmatrix}A & B\\B & A\end{pmatrix}^{-1}=\dfrac{1}{2}\begin{pmatrix}(A+B)^{-1}+(A-B)^{-1} & (A+B)^{-1}-(A-B)^{-1}\\(A+B)^{-1}-(A-B)^{-1} & (A+B)^{-1}+(A-B)^{-1}\end{pmatrix}$

## (C)

1.

证明：设 $A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, $A^{-1} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$ 由于 $|A| \neq 0$，可得 $a_{ii} \neq 0 (i = 1, 2, \dots, n)$. $\because AA^{-1} = E$，比较第1行的元素，可得 $\begin{cases} a_{11}b_{11} = 1 \\ a_{11}b_{12} = 0 \\ a_{11}b_{13} = 0 \\ \vdots \\ a_{11}b_{1n} = 0 \end{cases}$ 从而可得 $b_{12} = 0$, $b_{13} = 0$, $\dots$, $b_{1n} = 0$. 同理可以比较 $AA^{-1}$ 和 $E$ 的其他行，得 $b_{ij} = 0 (i < j)$. 可见 $A^{-1}$ 是下三角矩阵.

2.

证明：假设 $AB - BA = E$，则考虑主对角元素之和

$$\sum_{i=1}^{n}\left[\sum_{k=1}^{n} a_{ik}b_{ki} - \sum_{k=1}^{n} b_{ik}a_{ki}\right]$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki} - \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}a_{ki}$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}b_{ki} - \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}a_{ki}$$

$= 0 \neq n$，矛盾！故 $AB - BA \neq E$.

# 习题1.4

## (A)

1.

(1) $\left(\begin{array}{ccc:c} 1 & -2 & 1 & 1 \\ 2 & -1 & 5 & 0 \\ 0 & 3 & 1 & 2 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & & & \frac{17}{3} \\ & 1 & & \frac{9}{3} \\ & & 1 & -2 \end{array}\right)$, 故 $x = \begin{pmatrix} \frac{17}{3} \\ \frac{9}{3} \\ -2 \end{pmatrix}$.



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(2) $\left(\begin{array}{ccc:c} 1 & -1 & 1 & 1 \\ 3 & 0 & 4 & 5 \\ 1 & 2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & & \frac{4}{3} & \\ & 1 & \frac{1}{3} & \\ & & & 1 \end{array}\right)$ 无解.

(3) $\begin{pmatrix} 3 & 4 & -5 & 7 \\ 2 & -3 & 3 & -2 \\ 4 & 11 & -13 & 16 \\ 7 & -2 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & & -\frac{3}{17} & \frac{13}{17} \\ & 1 & -\frac{19}{17} & \frac{20}{17} \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 = \dfrac{3}{17}x_3 - \dfrac{13}{17}x_4 \\ x_2 = \dfrac{19}{17}x_3 - \dfrac{20}{17}x_4 \end{cases}$ $x_3, x_4$ 为自由未知量，令 $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ 取 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \eta_1 = \begin{pmatrix} \frac{3}{17} \\ \frac{19}{17} \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -\frac{13}{17} \\ -\frac{20}{17} \\ 0 \\ 1 \end{pmatrix}$ 则 $x = p\begin{pmatrix} \frac{3}{17} \\ \frac{19}{17} \\ 1 \\ 0 \end{pmatrix} + q\begin{pmatrix} -\frac{13}{17} \\ -\frac{20}{17} \\ 0 \\ 1 \end{pmatrix}$.

(4) $\left(\begin{array}{cccc:c} 3 & -2 & 1 & -3 & 4 \\ 2 & 1 & -1 & 1 & 1 \\ 1 & 4 & -3 & 5 & -2 \end{array}\right) \to \left(\begin{array}{cccc:c} 1 & & -\frac{1}{7} & -\frac{1}{7} & \frac{6}{7} \\ & 1 & -\frac{5}{7} & \frac{9}{7} & \frac{5}{7} \\ & & 0 & 0 & 0 \end{array}\right)$ 令自由未知量 $x_3, x_4$ 为 0，得特解 $\eta^* = \begin{pmatrix} \frac{6}{7} \\ -\frac{5}{7} \\ 0 \\ 0 \end{pmatrix}$，令 $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 可得基础解系 $\eta_1 = \begin{pmatrix} \frac{1}{7} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} \frac{1}{7} \\ -\frac{9}{7} \\ 0 \\ 1 \end{pmatrix}$，则 $x = p\begin{pmatrix} \frac{1}{7} \\ \frac{5}{7} \\ 1 \\ 0 \end{pmatrix} + q\begin{pmatrix} \frac{1}{7} \\ -\frac{9}{7} \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{6}{7} \\ -\frac{5}{7} \\ 0 \\ 0 \end{pmatrix}$.

2.

(1) $\begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}^{-1} = \dfrac{1}{\begin{vmatrix} 4 & -3 \\ -1 & 2 \end{vmatrix}} \cdot \begin{pmatrix} 4 & -3 \\ -1 & 2 \end{pmatrix}^* = \dfrac{1}{5}\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

(2) $\begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(3) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & -2 & -6 \end{pmatrix}^{-1} = \begin{pmatrix} 22 & -6 & -26 & 17 \\ -17 & 5 & 20 & -13 \\ -1 & 0 & 2 & -1 \\ 4 & -1 & -5 & 3 \end{pmatrix}$.

(4) $\begin{pmatrix} 0 & 0 & \cdots & 0 & a_n \\ a_1 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{n-1} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{a_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{a_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{n-1}} \\ \frac{1}{a_n} & 0 & 0 & \cdots & 0 \end{pmatrix}$.

(5) 由 $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}$，多次使用该公式有 $\begin{pmatrix} 0 & 0 & \cdots & 0 & b_1 \\ 0 & 0 & \cdots & b_2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & b_{n-1} & \cdots & 0 & 0 \\ b_n & 0 & \cdots & 0 & 0 \end{pmatrix}^{-1} =$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{1}{b_n} \\ 0 & 0 & \cdots & \frac{1}{b_{n-1}} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \frac{1}{b_2} & \cdots & 0 & 0 \\ \frac{1}{b_1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

3.

(1) 由 $\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \Rightarrow X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} \frac{9}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 2 & 1 \end{pmatrix}$.

(2) $X = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{1}{2} \end{pmatrix}$.

(3) $X = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & -4 \\ 2 & -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 \\ 0 & -2 \\ 2 & 1 \end{pmatrix}$ 则由于 $\begin{pmatrix} 1 & 2 & 3 & 1 & 3 \\ 3 & 2 & -4 & 0 & -2 \\ 2 & -1 & 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 & 13 & 11 \\ 1 & \frac{13}{4} & \frac{3}{4} & \frac{11}{4} & \\ 1 & \frac{15}{41} & \frac{35}{41} & & \end{pmatrix} \to$



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$$\begin{pmatrix} 1 & & & \frac{32}{41} & \frac{20}{41} \\ & 1 & & -\frac{18}{41} & -\frac{1}{41} \\ & & 1 & \frac{15}{41} & \frac{35}{41} \end{pmatrix}, \text{故 } X = \begin{pmatrix} \frac{32}{41} & \frac{20}{41} \\ -\frac{18}{41} & -\frac{1}{41} \\ \frac{15}{41} & \frac{35}{41} \end{pmatrix}.$$

4.

(1) $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -1 & -3 \\ 3 & 2 & -5 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{11}{3} & -\frac{7}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{7}{3} & -\frac{5}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 3 \end{pmatrix}.$$

(2) $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -1 & 4 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{5}{2} & -\frac{3}{2} & \frac{7}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 13 \\ 5 \\ -2 \end{pmatrix}.$$

5.

(1) $$\begin{pmatrix} 1 & -2 & 1 \\ 2 & -4 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(2) $$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 5 \\ 6 & -3 & 2 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & -2 & 8 \\ 0 & -15 & 8 \\ 0 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & -52 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

6.

由 $PAQ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$，可取 $P = E_3(3, 2(2))E_3(3, 1(1))E_3(2, 1(-2))$ 取 $Q = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} =$

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 2 & 1 \end{pmatrix}.$$

7.

证明：由题设 $P, Q$ 是可逆矩阵，故 $P, Q$ 是若干个初等矩阵的乘积。用 $P$ 左乘 $A$ 或用 $Q$ 右乘 $A$，即对 $A$ 作若干次初等行变换或初等列变换，初等变换不改变矩阵的秩，故 $r(A) = r(PA) = r(AQ) =$

$r(PAQ)$.

8.

(1) 由于 $\begin{pmatrix} 0 & A & | & E_n & \\ B & 0 & | & & E_m \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} B & 0 & | & 0 & E_m \\ 0 & A & | & E_n & 0 \end{pmatrix} \xrightarrow[B^{-1}r_1]{A^{-1}r_2} \begin{pmatrix} E_m & 0 & | & 0 & B^{-1} \\ 0 & E_n & | & A^{-1} & 0 \end{pmatrix}$

故 $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & B^{-1} \\ A^{-1} & 0 \end{pmatrix}$.

(2) 由于 $\begin{pmatrix} 0 & A & | & E_n & \\ B & C & | & & E_m \end{pmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{pmatrix} B & C & | & 0 & E \\ 0 & A & | & E & 0 \end{pmatrix} \xrightarrow{A^{-1}r_2} \begin{pmatrix} B & C & | & 0 & E \\ 0 & E & | & A^{-1} & 0 \end{pmatrix}$

$\xrightarrow{r_1 - Cr_2} \begin{pmatrix} B & 0 & | & CA^{-1} & E \\ 0 & E & | & A^{-1} & 0 \end{pmatrix} \xrightarrow{B^{-1}r_1} \begin{pmatrix} E & 0 & | & B^{-1}CA^{-1} & B^{-1} \\ 0 & E & | & A^{-1} & 0 \end{pmatrix}$ 故 $\begin{pmatrix} 0 & A \\ B & C \end{pmatrix}^{-1} = \begin{pmatrix} B^{-1}CA^{-1} & B^{-1} \\ A^{-1} & 0 \end{pmatrix}$.

## (B)

1.

由于 $\left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & \lambda \\ \lambda & 1 & 1 & \lambda^2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1-\lambda & \lambda-1 & \lambda-1 \\ 0 & 1-\lambda^2 & 1-\lambda & \lambda^2-\lambda \end{array}\right)$ 故当 $\lambda = 1$ 时有无穷解. 当 $\lambda \neq 1$ 时 $\to \left(\begin{array}{ccc|c} 1 & \lambda & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \lambda+2 & 1 \end{array}\right)$ 则当 $\lambda \neq 2$ 且 $\lambda \neq 1$ 时有唯一解当 $\lambda = 2$ 时无解. $\lambda = 1$ 时 $\to$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ 取 $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$ 为 $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ 可得特解 $\eta^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ 取 $\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 可得基础解系

$\eta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ 故 $x = p\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + q\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

2.

由 $XA = A+3X \Rightarrow X = A(A-3E)^{-1} = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 0 \\ 1 & 2 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & -\frac{1}{3} \end{pmatrix}$

$= \begin{pmatrix} 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{1}{4} & \frac{1}{4} \\ 1 & 0 & 0 \end{pmatrix}$.

3.

$A = PBP^{-1}$, 则 $A^5 = PBP^{-1}PBP^{-1}PBP^{-1}PBP^{-1}PBP^{-1} = PB^5P^{-1}$ $\because$ $B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = B$ $P^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -4 & 1 & 1 \end{pmatrix}$ 则 $A = PBP^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -4 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 6 & -1 & -1 \end{pmatrix}$ $A^5 = PB^5P^{-1} = PBP^{-1} = A =$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 6 & -1 & -1 \end{pmatrix}.$$

4.

依题意，考虑 $\begin{vmatrix} 1 & x \\ y & 1 \end{vmatrix} = 1 - xy \neq 0$ 知 $x, y \in \mathbb{R}$, $y \neq 0$.

5.

(1) 由 $\begin{pmatrix} 1 & a & a^2 & \cdots & a^{n-1} & 1 \\ 1 & a & a^2 & \cdots & a^{n-2} & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & a & a^2 & \cdots & a^{n-2} & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & 1 & -a & & \\ & 1 & & & 1 & -a & \\ & & \ddots & & & \ddots & -a \\ & & & 1 & & & 1 \end{pmatrix}$ 知

$$\begin{pmatrix} 1 & a & a^2 & \cdots & a^{n-1} \\ 1 & a & a^2 & \cdots & a^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a & a^2 & \cdots & a^{n-2} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -a & & & \\ & 1 & -a & & \\ & & \ddots & \ddots & \\ & & & 1 & -a \\ & & & & 1 \end{pmatrix}.$$

(2) 由 $\begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & 0 & & \\ & & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} n-1 & n-1 & n-1 & \cdots & n-1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix} \rightarrow$

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & \frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -\frac{1}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ 0 & 0 & 1 & \cdots & 0 & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{n-1} & -\frac{1}{n-1} & \cdots & \frac{n-2}{n-1} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & & -\frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ & 1 & & & \frac{1}{n-1} & -\frac{n-2}{n-1} & \cdots & \frac{1}{n-1} \\ & & \ddots & & \vdots & \vdots & \ddots & \vdots \\ & & & 1 & \frac{1}{n-1} & \cdots & \frac{1}{n-1} & -\frac{n-2}{n-1} \end{pmatrix}$$



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$$\text{故}\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{n-2}{n-1} & \frac{1}{n-1} & \cdots & \frac{1}{n-1} \\ \frac{1}{n-1} & -\frac{n-2}{n-1} & \cdots & \frac{1}{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \cdots & \frac{1}{n-1} & -\frac{n-2}{n-1} \end{pmatrix}.$$

6.

依题意 $r(A)=2<n=3$ 则 $\begin{vmatrix} 1 & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b+c \end{vmatrix}=0 \Rightarrow a=0$ 或 $b+c=0$ ①当 $a=0$, 则由 $r(A)=2$ 知 $b+c\neq 0$ ②当 $b+c=0$, 则由 $r(A)=2$ 知 $a\neq 0$ $A$ 的相抵标准形为 $I_2=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

7.

证明：充分性：$\because P,Q$ 可逆，$\therefore P,Q$ 可分解为若干个基本初等矩阵的积即 $A$ 可以经过若干次初等变换得到 $B$, $\therefore A\sim B$ 必要性：$\because A\sim B$, $\therefore A$ 经过若干次初等变换可以得到 $B$ 即 $PAQ=B$ $(P,Q$ 可逆)

8.

$A=\begin{pmatrix} a & 1 & \cdots & 1 \\ 1 & a & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & a \end{pmatrix}_{n\times n}$ 则 $a=1$ 时 $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}\to\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ 此时 $I=\begin{pmatrix} 1 & 0 \\ 0 & O_{n-1} \end{pmatrix}$ 当 $a=1-n$ 时, $\begin{pmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-n \end{pmatrix}\to\begin{pmatrix} 1-n & 1 & \cdots & 1 \\ 1 & 1-n & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}\to$ $\begin{pmatrix} -n & -n & \cdots & -n & 1 \\ & -n & \cdots & -n & 1 \\ & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}\to\begin{pmatrix} 1 & & & 0 \\ & 1 & & 0 \\ & & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$ 故 $I=\begin{pmatrix} E_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$ 当 $a\neq 1,1-n$ 时则

$$\begin{pmatrix} a-1 & & & \\ & a-1 & & \\ & & \ddots & \\ & & & a \end{pmatrix} \to \begin{pmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \to \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = E_n$$ 故此时 $I = E_n$. 综上,

$$r(A) = \begin{cases} 1 & a = 1 \\ n-1 & a = 1-n \\ n & a \neq 1, 1-n \end{cases}$$

9.

由 $\begin{pmatrix} A_{n-1} & \beta \\ \alpha^T & a_{nn} \end{pmatrix}$ 进行初等行变换

$$\begin{pmatrix} A_{n-1} & \beta & | & E & 0 \\ \alpha^T & a_{nn} & | & 0 & E \end{pmatrix} \to \begin{pmatrix} E & A_{n-1}^{-1}\beta & | & A_{n-1}^{-1} & 0 \\ \alpha^T & a_{nn} & | & 0 & E \end{pmatrix}$$

$$\to \begin{pmatrix} E & A_{n-1}^{-1}\beta & | & A_{n-1}^{-1} & 0 \\ 0 & a_{nn} - \alpha^T A_{n-1}^{-1}\beta & | & -\alpha^T A_{n-1}^{-1} & E \end{pmatrix}$$

$$\to \begin{pmatrix} E & 0 & | & A_{n-1}^{-1}(E_{n-1} + \beta t^{-1}\alpha^T A_{n-1}^{-1}) & -A_{n-1}^{-1}\beta t^{-1} \\ 0 & E & | & -t^{-1}\alpha^T A_{n-1}^{-1} & t^{-1} \end{pmatrix}$$

其中 $t = a_{nn} - \alpha^T A_{n-1}^{-1}\beta$ 令 $A_{n-1} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}$, $\beta = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$, $\alpha^T = (2\ 3\ 3)$, $a_{nn} = 2$ 则

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -1 & 2 & 1 \\ -3 & -12 & -11 & 7 \\ 5 & 20 & 15 & -10 \\ -4 & -11 & -8 & 6 \end{pmatrix}.$$

10.

证明：假设 $A$ 为满秩矩阵，则 $|A| \neq 0$, 这与 $A$ 不可逆矛盾。故 $A$ 为降秩矩阵。

## (C)

1.

(1) 考虑第1行第1列元素，则 $q_1 = b_1$ 考虑第1行第2列元素，则 $q_1 r_1 = c_1 \Rightarrow r_1 = \dfrac{c_1}{q_1}$ 考虑第$i$行第$i-1$列元素 $(i = 2, \ldots, n)$，则 $p_i = a_i$ 考虑第$i$行第$i$列元素 $(i = 2, \ldots, n)$，则 $p_i r_{i-1} + q_i = b_i \Rightarrow q_i = b_i - p_i r_{i-1}$ 考虑第$i$行第$i+1$列元素 $(i = 2, \ldots, n-1)$，则 $q_i r_i = c_i \Rightarrow r_i = \dfrac{c_i}{q_i}$

(2) 由 $\left(\begin{array}{cccc:c} 1 & 2 & & & 6 \\ 2 & 1 & 1 & & 8 \\ & 1 & 2 & & 8 \\ 1 & 2 & & & 6 \end{array}\right) \to \left(\begin{array}{cccc:c} 1 & & & & 2 \\ & 1 & & & 2 \\ & & 1 & & 2 \\ & & & & 2 \end{array}\right)$ 则 $x = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$.

2.

$$f(A) = (A - E)^n = \begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \\ & & & 1 \end{pmatrix}^n = \left[\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix} + E\right]^n$$ 记 $B = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 0 \end{pmatrix}$

则 $B^2 = \begin{pmatrix} 0 & 0 & 1 & & \\ & 0 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 0 & 0 \\ & & & & 0 \end{pmatrix}$, $B^3 = \begin{pmatrix} 0 & 0 & 0 & 1 & & \\ & 0 & 0 & 0 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots \\ & & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}$ $B^n = O\ (n \geq 4)$, 则 $f(A) =$

$$(B+E)^n = B^n + C_n^1 B^{n-1} + C_n^2 B^{n-2} + \cdots + C_n^{n-2} B^2 + C_n^{n-1} B + E = E + C_n^1 \begin{pmatrix} 0 & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} +$$

$$C_n^2 \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & 0 & 1 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} + C_n^3 \begin{pmatrix} 0 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{pmatrix} = \begin{pmatrix} 1 & C_n^1 & C_n^2 & C_n^3 \\ & 1 & C_n^1 & C_n^2 \\ & & 1 & C_n^1 \\ & & & 1 \end{pmatrix}.$$

# 习题2.1

## (A)

1.

(1) $\begin{vmatrix} 4 & -3 \\ -7 & 6 \end{vmatrix} = 24 - 21 = 3.$

(2) $\begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} = \cos^2\alpha + \sin^2\alpha = 1.$

(3) $\begin{vmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 7 & 0 & 9 \end{vmatrix} = 5 \times 9 + 2 \times 6 \times 7 - 3 \times 5 \times 7 + 4 \times 2 \times 9 = 96.$

(4) $\begin{vmatrix} x & 1 & -1 \\ -1 & x & 1 \\ 1 & -1 & x \end{vmatrix} = x^3 + 3x.$

(5) $\begin{vmatrix} 0 & 0 & a_1 \\ 0 & a_2 & 0 \\ a_3 & 0 & 0 \end{vmatrix} = -a_1a_2a_3.$

2.

(1) 由 $\begin{pmatrix} 3 & -2 & 1 \\ -7 & 5 & 10 \end{pmatrix} \to \begin{pmatrix} 1 & & 5 \\ & 1 & 7 \end{pmatrix}$ 故 $\begin{cases} x = 5 \\ y = 7 \end{cases}$.

(2) 由 $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & & 7 \\ & 1 & -3 \end{pmatrix}$ 故 $\begin{cases} x = 7 \\ y = -3 \end{cases}$.

(3) 由 $\begin{pmatrix} 2 & -3 & -3 & 0 \\ 1 & 4 & 6 & -1 \\ 3 & -1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & & & -\frac{9}{4} \\ & 1 & & -\frac{41}{8} \\ & & 1 & \frac{29}{8} \end{pmatrix}$ 故 $\begin{cases} x = -\dfrac{9}{4} \\ y = -\dfrac{41}{8} \\ z = \dfrac{29}{8} \end{cases}$.

(4) 由 $\begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 2 & 5 & -1 \\ 2 & 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & & & \frac{22}{13} \\ & 1 & & -\frac{5}{13} \\ & & 1 & -\frac{5}{13} \end{pmatrix}$ 故 $\begin{cases} x = \dfrac{22}{13} \\ y = -\dfrac{5}{13} \\ z = -\dfrac{5}{13} \end{cases}$.

3.

(1) $\tau(4357261) = 1 + 4 + 1 + 1 + 6 = 12$. 偶排列.

(2) $\tau(217986354) = 1 + 1 + 3 + 4 + 4 + 4 + 5 = 18$. 偶排列.

4.

因为对于元素 $x_1, x_2, \ldots, x_n$ 中任何两个不同的 $x_i$ 与 $x_j$，在 $x_1, x_2, \ldots, x_n$ 与 $x_n, x_{n-1}, \ldots, x_1$ 中必有且只有一个构成逆序，所以这两个排列的逆序数之和应等于从 $n$ 个元素中任取两个元素的组合数 $C_n^2 = \dfrac{n(n-1)}{2}$ 故 $x_n \cdots x_2 x_1$ 的逆序数为 $\dfrac{n(n-1)}{2} - k$.

5.

$-a_{13}a_{21}a_{34}a_{42}$, $a_{14}a_{21}a_{33}a_{42}$.

6.

$-a_{11}a_{23}a_{32}a_{44}$, $-a_{12}a_{23}a_{34}a_{41}$, $-a_{14}a_{23}a_{31}a_{42}$.

7.

(1) $\begin{vmatrix} 1 & 1 & & \\ 2 & -1 & & \\ & & 3 & 0 \\ & & 4 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 \\ 4 & 4 \end{vmatrix} = (-1-2) \times 12 = -36.$

(2) $\begin{vmatrix} 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 2 \\ 1 & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{n-1} \begin{vmatrix} n & n-1 & \cdots & 2 \end{vmatrix} = (-1)^{n-1} n!.$

(3) $\begin{vmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 2 \\ 3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{vmatrix} = (-1)^{2\times 2} \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = 24.$

## (B)

1.

$$f(x)=\begin{vmatrix} x & x & 1 & 2 & 3 \\ 1 & x & 0 & 2 & 4 \\ 2 & 5 & x & 1 & 2 \\ 1 & 3 & -4 & x & 0 \\ 2 & 6 & 4 & 1 & x \end{vmatrix}=\begin{vmatrix} x-1 & 0 & 1 & 0 & -1 \\ 1 & x & 0 & 2 & 4 \\ 2 & 5 & x & 1 & 2 \\ 1 & 3 & -4 & x & 0 \\ 2 & 6 & 4 & 1 & x \end{vmatrix}$$ 则要出现 $x^4$ 项, 只能取主对角线元素 $(x-1)x^4 \Rightarrow x^4$ 的系数为 $-1$.

2.

由 $D_n=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{vmatrix}=0$, 反行列式定义知 $D$ 为偶排列个数减奇排列个数所以奇偶排列各半.

3.

$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & & & \\ d_1 & d_2 & & & \\ e_1 & e_2 & & & \end{vmatrix}=0$ 的一般项可表示为 $(-1)^{N(j_1,j_2,j_3,j_4,j_5)}a_{1j_1}a_{2j_2}a_{3j_3}a_{4j_4}a_{5j_5}$ 记 $a_{1i}=a_i$, $a_{2i}=b_i$, $a_{3i}=c_i\ (i<3)$, $a_{4i}=d_i\ (i<3)$, $a_{5i}=e_i\ (i<3)$, $a_{mn}=0\ (m>2,\ n>2)$ 则一般项的列下标 $j_3,j_4,j_5$ 只能在 $1,2,3,4,5$ 中取3个不同值, 故 $j_3,j_4,j_5$ 必在 $3,4,5$ 中取一个数, 从而至少有一项都包含至少一个 0 因子, 故任意一项必为 0. 从而该行列式为 0.

# 习题2.2

## (A)

1.

(1) $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 3 & -1 & 6 & 7 \\ 41 & -7 & -9 & 0 \\ -9 & 21 & 32 & 1 \end{vmatrix}=\begin{vmatrix} 1 & 2 & 3 & 10 \\ 3 & -1 & 6 & 15 \\ 41 & -7 & -9 & 25 \\ -9 & 21 & 32 & 45 \end{vmatrix}=5\begin{vmatrix} 1 & 2 & 3 & 2 \\ 3 & -1 & 6 & 3 \\ 41 & -7 & -9 & 5 \\ -9 & 21 & 32 & 9 \end{vmatrix}$ 故行列式是5的整数倍.

(2) 由 $|\alpha_1,\alpha_2,\beta_2,\alpha_3| = n \Rightarrow |\alpha_1,\alpha_2,\alpha_3,\beta_2| = -n$, 故 $|\alpha_1,\alpha_2,\alpha_3,(\beta_1-\beta_2)| = m-(-n) = m+n$, 则 $|\alpha_3,\alpha_2,\alpha_1,(\beta_1-\beta_2)| = -(m+n)$.

2.

(1) $\begin{vmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ -1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 3 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix} = 0.$

(2) $\begin{vmatrix} 4 & 1 & 2 & 4 \\ 1 & 2 & 0 & 2 \\ 10 & 5 & 2 & 0 \\ 0 & 1 & 1 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 1 & 1 & -3 \\ 1 & 2 & -2 & -12 \\ 10 & 5 & -3 & -35 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 1 & -3 \\ 1 & -2 & -12 \\ 10 & -3 & -35 \end{vmatrix} = 0.$

(3) $\begin{vmatrix} a^2 & (a+1)^2 & (a+2)^2 & (a+3)^2 \\ b^2 & (b+1)^2 & (b+2)^2 & (b+3)^2 \\ c^2 & (c+1)^2 & (c+2)^2 & (c+3)^2 \\ d^2 & (d+1)^2 & (d+2)^2 & (d+3)^2 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 4a+4 & 6a+9 \\ b^2 & 2b+1 & 4b+4 & 6b+9 \\ c^2 & 2c+1 & 4c+4 & 6c+9 \\ d^2 & 2d+1 & 4d+4 & 6d+9 \end{vmatrix} = \begin{vmatrix} a^2 & 2a+1 & 2 & 6 \\ b^2 & 2b+1 & 2 & 6 \\ c^2 & 2c+1 & 2 & 6 \\ d^2 & 2d+1 & 2 & 6 \end{vmatrix} = 0.$

(4) $\begin{vmatrix} a & b & a+b \\ b & a+b & a \\ a+b & a & b \end{vmatrix} = \begin{vmatrix} 2(a+b) & b & a+b \\ 2(a+b) & a+b & a \\ 2(a+b) & a & b \end{vmatrix} = 2(a+b)\begin{vmatrix} 1 & b & a+b \\ 1 & a+b & a \\ 1 & a & b \end{vmatrix} = 2(a+b)\begin{vmatrix} a & -b \\ a-b & -a \end{vmatrix} = 2(a+b)(ab-a^2-b^2) = -2(a^3+b^3).$

3.

$\begin{vmatrix} a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ a_2+b_1 & a_2+b_2 & \cdots & a_2+b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n+b_1 & a_n+b_n & \cdots & a_n+b_n \end{vmatrix} = \begin{vmatrix} a_1+b_1 & a_1+b_2 & \cdots & a_1+b_n \\ a_2-a_1 & a_2-a_1 & \cdots & a_2-a_1 \\ \vdots & \vdots & \ddots & \vdots \\ a_n-a_1 & a_n-a_1 & \cdots & a_n-a_1 \end{vmatrix}$ 故 $n=1$ 时，值为 $a_1+b_1$，

$n=2$ 时，$\begin{vmatrix} a_1+b_1 & a_1+b_2 \\ a_2-a_1 & a_2-a_1 \end{vmatrix} = (a_1-a_2)(b_2-b_1)$ 当 $n \geq 3$ 时，第2行与第n行成比例，值为0.

4.

对 $A$ 作 $r_i+kr_k$ 变换，相当于 $A$ 加上一个第 $i$ 行为 $kr_k$ 的行列式，该行列式的第 $i$ 行与第 $k$ 行成比例，值为0. 同理作 $c_i+kc_k$ 变换也相当于 $A$ 加上一个值为0的行列式故 $A \xrightarrow[r_i+kr_k]{c_i+kc_k} B$ 后，$|B|=|A|$.

5.

$$AA^T=\begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{pmatrix}\begin{pmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{pmatrix}=\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 & 0 & 0 \\ 0 & a^2+b^2+c^2+d^2 & 0 & 0 \\ 0 & 0 & \sum a^2 & 0 \\ 0 & 0 & 0 & \sum a^2 \end{pmatrix}$$

故 $|AA^T|=(a^2+b^2+c^2+d^2)^4$. 又 $\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix}=-\begin{vmatrix} a & b & c & d \\ b & -a & -d & c \\ c & d & -a & -b \\ d & -c & b & -a \end{vmatrix}=$

$$-\frac{1}{abcd}\begin{vmatrix} a^2 & ba & ca & da \\ b^2 & -ab & -db & cb \\ c^2 & dc & -ac & bc \\ d^2 & -cd & bd & -ad \end{vmatrix}=-\frac{1}{abcd}\begin{vmatrix} a^2 & ba & ca & da \\ b^2 & -ab & -db & cb \\ c^2 & dc & -ac & bc \\ a^2+b^2+c^2+d^2 & 0 & 0 & 0 \end{vmatrix}=(-1)^{1+4}\cdot(-1)\cdot$$

$$\frac{a^2+b^2+c^2+d^2}{abcd}\begin{vmatrix} ba & ca & da \\ -ab & -db & cb \\ dc & -ac & bc \end{vmatrix}=\frac{a^2+b^2+c^2+d^2}{d}\begin{vmatrix} b & c & d \\ -a & -d & c \\ d & -a & -b \end{vmatrix}=(a^2+b^2+c^2+d^2)^2.$$

## (B)

1.

(1) $$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ -1 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & n \end{vmatrix} = \begin{vmatrix} \sum\limits_{i=1}^{n} \frac{1}{i} & 1 & 1 & 1 & \cdots & 1 \\ 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{vmatrix} = n!\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

(2) $$\begin{vmatrix} a_1 & 1 & 1 & \cdots & 1 \\ 1 & a_2 & 0 & \cdots & 0 \\ 1 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & a_n \end{vmatrix} = \begin{vmatrix} a_1 - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n} & 1 & 1 & \cdots & 1 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{vmatrix} =$$
$$a_2 a_3 \cdots a_n \left(a_1 - \sum_{i=2}^{n} \frac{1}{a_i}\right).$$

2.

证明: $$\begin{vmatrix} n & n-1 & \cdots & 3 & 2 & 1 \\ n & n-1 & \cdots & 3 & 2 & 2 \\ n & n-1 & \cdots & 3 & 3 & 3 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ n & n-1 & \cdots & n-1 & n-1 & n-1 \\ n & n & \cdots & n & n & n \end{vmatrix} = \begin{vmatrix} -1 & & & & \\ -1 & -1 & & & \\ -1 & -1 & -1 & & \\ \vdots & \vdots & \vdots & \ddots & \\ -1 & -1 & -1 & \cdots & -1 \\ n & n & n & \cdots & n \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} \cdot (-1)^{n-1} \cdot n$$
$$= (-1)^{\frac{(n-1)(n+2)}{2}} \cdot n.$$

3.

证明: 若有两行成比例, 则将比例常数提出后, 有两行相同, 则交换这两行, 有 $|A| = -|A|$, 得 $|A| = 0$.

4.

(1) 证明: 在等式左端的 $k+t$ 阶行列式中, 取定前 $k$ 行, 由这 $k$ 行元素构成的 $k$ 阶子式中, 只有取前 $k$ 列时该子式不为 0, 根据拉普拉斯定理, 左边 $= \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & \cdots & b_{1t} \\ \vdots & & \vdots \\ b_{t1} & \cdots & b_{tt} \end{vmatrix} \cdot (-1)^{(1+2+\cdots+k)+(1+2+\cdots+k)}$ =右边.

(2) 证明: 同(1), 由拉普拉斯定理左边 $= \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & \cdots & b_{1t} \\ \vdots & & \vdots \\ b_{t1} & \cdots & b_{tt} \end{vmatrix} \cdot (-1)^{[(t+1)+(t+2)+\cdots+k]+[1+2+\cdots+k]} = (-1)^{tk+2(1+2+\cdots+k)} \begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & \cdots & b_{1t} \\ \vdots & & \vdots \\ b_{t1} & \cdots & b_{tt} \end{vmatrix}$ =右边.

5.

$f(x) = \begin{vmatrix} 2 & 1 & 2 & 3 \\ 2 & 5-x^2 & 2 & 3 \\ 10 & 5 & 2 & 1 \\ 10 & 5 & 2 & 2-x^2 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 & 3 \\ 4-x^2 & & & \\ -8 & -14 & & \\ -8 & -13-x^2 & & \end{vmatrix} = \begin{vmatrix} 2 & 1 & 2 & 3 \\ 4-x^2 & & -8 & -14 \\ & -8 & & 1-x^2 \end{vmatrix} = -16(1-x^2)(4-x^2)$. 则零点为 $x_{1,2}=\pm 1$, $x_{3,4}=\pm 2$.

# 习题2.3

## (A)

1.

(1) $A_{11} = \begin{vmatrix} 1 & 0 \\ 2 & 5 \end{vmatrix} = 5$, $A_{12} = -\begin{vmatrix} -3 & 0 \\ 1 & 5 \end{vmatrix} = 15$, $A_{13} = \begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix} = -7$, $A_{21} = -\begin{vmatrix} 0 & 0 \\ 2 & 5 \end{vmatrix} = 0$, $A_{22} = \begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} = 10$, $A_{23} = -\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = -4$, $A_{31} = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$, $A_{32} = -\begin{vmatrix} 2 & 0 \\ -3 & 0 \end{vmatrix} = 0$, $A_{33} = \begin{vmatrix} 2 & 0 \\ -3 & 1 \end{vmatrix} = 2$.

(2) $A_{11} = \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = -1$, $A_{12}=1$, $A_{13}=0$, $A_{21}=2$, $A_{22}=-2$, $A_{23}=0$, $A_{31}=0$, $A_{32}=0$, $A_{33}=0$.

2.

(1) $$\begin{vmatrix}2&-1&3&1&0\\1&2&-1&4&3\\0&-1&-3&2&3\\4&5&0&3&1\\1&-1&2&-2&3\end{vmatrix}=\begin{vmatrix}2&-1&3&1&0\\-11&-13&-1&-5&0\\-12&-16&-3&-7&0\\4&5&0&3&1\\-11&-16&2&-11&0\end{vmatrix}=-\begin{vmatrix}2&-1&3&1\\-11&-13&-1&-5\\-12&-16&-3&-7\\-11&-16&2&-11\end{vmatrix}$$

$$=-\begin{vmatrix}2&0&3&1\\-11&-18&-1&-5\\-12&-23&-3&-7\\-11&-27&2&-11\end{vmatrix}=-\begin{vmatrix}0&0&0&1\\-1&-18&14&-5\\2&-23&18&-7\\11&-27&35&-11\end{vmatrix}=\begin{vmatrix}-1&-18&14\\2&-23&18\\11&-27&35\end{vmatrix}=\begin{vmatrix}-1&-18&14\\0&-59&46\\0&-225&189\end{vmatrix}=$$

$$-\begin{vmatrix}-59&46\\-225&189\end{vmatrix}=11151-10350=801.$$

(2) $$\begin{vmatrix}1&2&3&4&5\\2&3&4&5&6\\3&4&5&6&7\\4&5&6&7&8\\5&6&7&8&9\end{vmatrix}=\begin{vmatrix}1&2&3&4&5\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\\1&1&1&1&1\end{vmatrix}=0.$$

(3) $$\begin{vmatrix}\cos\alpha&1&&&\\1&2\cos\alpha&1&&\\&1&2\cos\alpha&1&\\&&1&2\cos\alpha&1\\&&&1&2\cos\alpha\end{vmatrix}=\begin{vmatrix}1&\cos\alpha&&&\\\cos\alpha&1&&&\\&&1&\cos\alpha&\\&&\cos\alpha&1&\\&&&&1\end{vmatrix}=$$

$$\begin{vmatrix}1&\cos\alpha&&&\\\cos\alpha&1&&&\\&&1-2\cos^2\alpha&&\\&&&1-2\cos^2\alpha&\\&&&&2\cos\alpha\end{vmatrix}=\begin{vmatrix}1-2\cos^2\alpha&1\\-\cos\alpha&2\cos\alpha\end{vmatrix}=1-2\cos^2\alpha-$$

$$2\cos^2\alpha+(1-2\cos^2\alpha)4\cos^2\alpha=1-2\sin^2 2\alpha=\cos 4\alpha.$$

(4) 设 $D_n = \begin{vmatrix} a+b & ab & & & \\ 1 & a+b & ab & & \\ & 1 & a+b & \ddots & \\ & & \ddots & \ddots & ab \\ & & & 1 & a+b \end{vmatrix}_n$ 则 $D_n = (a+b)\cdot D_{n-1} + ab\cdot \begin{vmatrix} 1 & ab & & \\ & a+b & ab & \\ & & \ddots & \ddots \\ & & & a+b \end{vmatrix}_{n-1}$ $= (a+b)D_{n-1} + abD_{n-2}$, 又 $D_1 = a+b$, $D_2 = a^2+ab+b^2 \Rightarrow D_n - aD_{n-1} = b(D_{n-1} - aD_{n-2}) = b^2(D_{n-2} - aD_{n-3}) = b^{n-2}(D_2 - aD_1) = b^{n-2}(a^2+ab+b^2-a^2-ab) = b^n$ 则 $\frac{D_n}{a^n} - \frac{D_{n-1}}{a^{n-1}} = \left(\frac{b}{a}\right)^n$ $\frac{D_n}{a^n} - \frac{D_1}{a} = \left(\frac{b}{a}\right)^2 + \left(\frac{b}{a}\right)^3 + \cdots + \left(\frac{b}{a}\right)^n = \left(\frac{b}{a}\right)^2 \frac{1-\left(\frac{b}{a}\right)^{n-1}}{1-\frac{b}{a}} \Rightarrow D_n = a^n\left[\frac{a+b}{a} + \frac{b^2}{a}\cdot\frac{1-\left(\frac{b}{a}\right)^{n-1}}{a-b}\right] = \frac{(a^2-b^2)a^{n-1}}{a-b} + \frac{a^{n-1}-b^{n-1}}{a-b}b^2 = \frac{a^{n+1}-b^{n+1}}{a-b}$.

3.

(1) $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}^{-1} = \frac{1}{|2\ 1|}\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}^* = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$.

(2) $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}^{-1} = \frac{1}{\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{vmatrix}}\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}^* = \frac{1}{6}\begin{pmatrix} 2 & 1 & 4 \\ 2 & 1 & -2 \\ -2 & 2 & 2 \end{pmatrix}$.

4.

(1) 证明: 按第一行展开, 则 $x$ 只能出现在第一行元素中故 $P(x)$ 的最高次数项为 $\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-2} \end{vmatrix}(-1)^{n+1}x^{n-1}$ 故 $P(x)$ 的次数不超过 $n-1$.

(2) $\because$ $P(x)$ 的次数不超过 $n-1$. 注意到 $x$ 为 $a_1, a_2, \ldots, a_{n-1}$ 时 $P(x)$ 为 0 故 $P(x)$ 的根为 $\{a_1, a_2, \ldots, a_{n-1}\}$ 中的不重复元素 1) 若 $a_i \neq a_j$ $(i \neq j)$, 则 $P(x)$ 的根为 $a_1, a_2, \ldots, a_{n-1}$ 2) 若 $a_i = a_j$, 则 $P(x) \equiv 0$, 此时 $P(x)$ 的根为 $x \in \mathbb{R}$ 由题设条件, $a_1, a_2, \ldots, a_n$ 互异故 $P(x)$ 的根为 $a_1, a_2, \ldots, a_{n-1}$.

(3) 由于 $\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-2} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{n-1} & a_{n-1}^2 & \cdots & a_{n-1}^{n-2} \end{vmatrix} = \prod\limits_{1\le j<i\le n-1}(a_i-a_j)$ 则 $x^{n-1}$ 的系数为 $(-1)^{n+1}\prod\limits_{1\le j<i\le n-1}(a_i-a_j)$.

## (B)

1.

(1) 在 $D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & & \\ & & a & b & & \\ & & b & a & & \\ & & & & & \\ b & & & & & a \end{vmatrix}_{2n}$ 中, 取定 $n$ 和 $n+1$ 行, 由这两行元素组成的所有 2 阶子式中, 只有取第 $n$ 和 $n+1$ 列时子式不为 0. 故由拉普拉斯展开定理, 得

$$\begin{aligned} D_{2n} &= \begin{vmatrix} a & b \\ b & a \end{vmatrix} \cdot (-1)^{n+(n+1)+n+(n+1)} \begin{vmatrix} a & & & & & b \\ & \ddots & & & & \\ & & a & b & & \\ & & b & a & & \\ & & & & & \\ b & & & & & a \end{vmatrix}_{2n-2} \\ &= (a^2-b^2)D_{2n-2} \end{aligned}$$

又 $D_2 = \begin{vmatrix} a & b \\ b & a \end{vmatrix} = a^2-b^2$. 故 $D_{2n} = (a^2-b^2)^{n-1}D_2 = (a^2-b^2)^n$.

(2) 将 $D_n = \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & x & -1 & \cdots & 0 \\ a_3 & 0 & x & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ a_n & 0 & \cdots & 0 & x \end{vmatrix}$ 按最后一行展开

$$D_n = x(-1)^{n+n} \begin{vmatrix} a_1 & -1 & 0 & \cdots & 0 \\ a_2 & x & -1 & \cdots & 0 \\ a_3 & 0 & x & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -1 \\ a_{n-1} & 0 & \cdots & 0 & x \end{vmatrix} + (-1)^{n+1} a_n \begin{vmatrix} -1 & -1 & & \\ x & -1 & & \\ & x & \ddots & \\ & & \ddots & -1 \end{vmatrix}$$

$$= xD_{n-1} + a_n$$

$$= x(xD_{n-2} + a_{n-1}) + a_n$$

$$= x^2 D_{n-2} + a_{n-1}x + a_n$$

$$\cdots$$

$$= a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + a_n$$

$$= \sum_{i=1}^{n} a_i x^{n-i}$$

2.

由 $A^*BA = 2BA - 8E$ 得 $(2E - A^*)BA = 8E$ 即 $B = 8(2E - A^*)^{-1}A^{-1} = 8(A(2E - A^*))^{-1}$ $= 8(2A - |A|E)^{-1} = 8(2A + 2E)^{-1} = 4\begin{pmatrix} 1 & 2 & -2 \\ -2 & 4 & \\ & & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 4 & -6 \\ -4 & 8 & \\ & & 2 \end{pmatrix}$.

3.

$|2A^*B^{-1}| = 2^n|A||A^{-1}B^{-1}| = \dfrac{2^n|A|^n}{|A||B|} = -\dfrac{4^n}{b}$.

4.

(1) $|A^*| = ||A|A^{-1}| = |A|^n|A^{-1}| = |A|^{n-1}$.

(2) $(A^*)^* = |A^*|(A^*)^{-1} = |A|^{n-1} \cdot \dfrac{A}{|A|} = |A|^{n-2}A.$

(3) $(kA)^* = |kA|(kA)^{-1} = k^{n-1}|A|A^{-1} = k^{n-1}A^*.$

5.

由 $\begin{pmatrix} 2 & 2 & 2 & \cdots & 2 & 1 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & & & & 1 & -1 \\ & 1 & & & 1 & -1 \\ & & \ddots & & \vdots & \vdots \\ & & & 1 & 1 & -1 \\ & & & & 1 & 1 \end{pmatrix}$ 故 $A^* = 2A^{-1} = \begin{pmatrix} 1 & -2 & & & \\ & 2 & -2 & & \\ & & 2 & -2 & \\ & & & \ddots & -2 \\ 0 & & & & 2 \end{pmatrix}$,

故 $\sum\limits_{i,j=1}^{n} A_{ij} = 1.$

## (C)

1.

(1)

$$\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \cdots & 1+x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ 1+x_ny_1 & 1+x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix} = \begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \cdots & 1+x_1y_n \\ y_1(x_2-x_1) & y_2(x_2-x_1) & \cdots & y_n(x_2-x_1) \\ \vdots & \vdots & \ddots & \vdots \\ y_1(x_n-x_1) & y_2(x_n-x_1) & \cdots & y_n(x_n-x_1) \end{vmatrix}$$

则当 $n=1$ 时, 值为 $1+x_1y_1$

当 $n=2$ 时, $\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 \\ 1+x_2y_1 & 1+x_2y_2 \end{vmatrix} = (x_2-x_1)(y_2-y_1)$

当 $n \geq 3$ 时, 第2行与第n行成比例, 值为0.

(2) 记 $A = \begin{vmatrix} 1 & x_1 & \cdots & x_1^{k-1} & x_1^{k+1} & \cdots & x_1^n \\ 1 & x_2 & \cdots & x_2^{k-1} & x_2^{k+1} & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{k-1} & x_n^{k+1} & \cdots & x_n^n \end{vmatrix}$

考虑多项式 $f(t)=\begin{vmatrix}1 & x_1 & \cdots & x_1^{k-1} & x_1^k & x_1^{k+1} & \cdots & x_1^n\\ 1 & x_2 & \cdots & x_2^{k-1} & x_2^k & x_2^{k+1} & \cdots & x_2^n\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_n & \cdots & x_n^{k-1} & x_n^k & x_n^{k+1} & \cdots & x_n^n\\ 1 & t & \cdots & t^{k-1} & t^k & t^{k+1} & \cdots & t^n\end{vmatrix}$

则 $f(t)=(x_2-x_1)(x_3-x_1)\cdots(t-x_1)(x_3-x_2)\cdots(t-x_2)\cdots(t-x_n)$

$=\prod\limits_{k=1}^{n}(t-x_k)\prod\limits_{1\le j<i\le n}(x_i-x_j)$

该行列式中第 $n+1$ 行第 $k+1$ 列元素 $t^k$ 的代数余子式为

$$t^k(-1)^{n+1+k+1}\begin{vmatrix}1 & x_1 & \cdots & x_1^{k-1} & x_1^{k+1} & \cdots & x_1^n\\ 1 & x_2 & \cdots & x_2^{k-1} & x_2^{k+1} & \cdots & x_2^n\\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_n & \cdots & x_n^{k-1} & x_n^{k+1} & \cdots & x_n^n\end{vmatrix}$$

考察 $f(t)$ 中 $t^k$ 的系数为 $(-1)^{n-k}\cdot\left(\sum\limits_{1\le i_1<i_2<\cdots<i_{n-k}\le n}\prod\limits_{k=1}^{n-k}x_{i_k}\right)\cdot\prod\limits_{1\le j<i\le n}(x_i-x_j)$

故 $A=\prod\limits_{1\le j<i\le n}(x_i-x_j)\left(\sum\limits_{1\le i_1<i_2<\cdots<i_{n-k}\le n}\prod\limits_{k=1}^{n-k}x_{i_k}\right)$.

(3) 在该行列式中取定第 $n$ 和第 $n+1$ 行，由这两行元素组成的所有2阶子式中，只有取第 $n$ 和 $n+1$ 列时的子式不为0.

由拉普拉斯展开定理，得

$$\begin{vmatrix}a_1 & \cdots & b_{2n}\\ \vdots & \ddots & \vdots\\ a_n & b_{n+1} & \cdots\\ b_n & a_{n+1} & \cdots\\ \vdots & \ddots & \vdots\\ b_1 & \cdots & a_{2n}\end{vmatrix}=\begin{vmatrix}a_n & b_{n+1}\\ b_n & a_{n+1}\end{vmatrix}\times(-1)^{n+(n+1)+n+(n+1)}\begin{vmatrix}a_1 & \cdots & b_{2n}\\ \vdots & \ddots & \vdots\\ a_{n-1} & b_{n+2} & \cdots\\ b_{n-1} & a_{n+2} & \cdots\\ \vdots & \ddots & \vdots\\ b_1 & \cdots & a_{2n}\end{vmatrix}$$

$=\begin{vmatrix}a_n & b_{n+1}\\ b_n & a_{n+1}\end{vmatrix}\cdot\begin{vmatrix}a_{n-1} & b_{n+2}\\ b_{n-1} & a_{n+2}\end{vmatrix}\cdots\begin{vmatrix}a_1 & b_{2n}\\ b_1 & a_{2n}\end{vmatrix}$

$$=\prod_{i=1}^{n}(a_i a_{2n+1-i}-b_{2n+1-i}b_i)$$

2.

证明: 行列式的每一个元素都是两项和 $a_{ij}+x$ $(i,j=1,2,\ldots,n)$, 这样行列式的每一列都可以看作由两个子列所组成, 第1子列元素为 $a_{ij}$, 第2子列元素为 $x$.

则该行列式可以拆成 $2^n$ 个 $n$ 阶行列式之和, 其中包含2列及以上的元素皆为 $x$ 的 $n$ 阶行列式值为0, 于是

$$\begin{vmatrix} a_{11}+x & a_{12}+x & \cdots & a_{1n}+x \\ a_{21}+x & a_{22}+x & \cdots & a_{2n}+x \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}+x & a_{n2}+x & \cdots & a_{nn}+x \end{vmatrix}=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}+\begin{vmatrix} x & a_{12} & \cdots & a_{1n} \\ x & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x & a_{n2} & \cdots & a_{nn} \end{vmatrix}+\cdots+\begin{vmatrix} a_{11} & \cdots & a_{1,n-1} & x \\ a_{21} & \cdots & a_{2,n-1} & x \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} & x \end{vmatrix}$$

$$=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}+x\sum_{i=1}^{n}A_{i1}+x\sum_{i=1}^{n}A_{i2}+\cdots+x\sum_{i=1}^{n}A_{in}$$

$$=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}+x\sum_{j=1}^{n}\sum_{i=1}^{n}A_{ij}$$

$$=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}+x\sum_{i=1}^{n}\sum_{j=1}^{n}A_{ij}$$

# 习题2.4

## (A)

1.

(1) $x_1 = \dfrac{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 12 \end{vmatrix}} = \dfrac{9}{1} = 9$

$x_2 = \dfrac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 2 & 0 & 12 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 12 \end{vmatrix}} = 6 \quad x_3 = \dfrac{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 2 & 1 & 12 \end{vmatrix}} = -2$

故 $x = \begin{pmatrix} 9 \\ 6 \\ -2 \end{pmatrix}$

(2) $D = \begin{vmatrix} 1 & 2 & -1 & 4 \\ 3 & 1 & 1 & 11 \\ 2 & -3 & -1 & 4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = -80 \quad D_1 = \begin{vmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 1 & 11 \\ 0 & -3 & -1 & 4 \\ 0 & 1 & 1 & 1 \end{vmatrix} = 50 \quad D_3 = \begin{vmatrix} 1 & 2 & 1 & 4 \\ 3 & 1 & 0 & 11 \\ 2 & -3 & 1 & 4 \\ 1 & 1 & 0 & 1 \end{vmatrix} = +50$

$D_2 = \begin{vmatrix} 1 & 1 & -1 & 4 \\ 3 & 0 & 1 & 11 \\ 2 & 1 & -1 & 4 \\ 1 & 0 & 1 & 1 \end{vmatrix} = -10 \quad D_4 = \begin{vmatrix} 1 & 2 & -1 & 1 \\ 3 & 1 & 1 & 0 \\ 2 & -3 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{vmatrix} = 10$

则 $x_1 = \dfrac{D_1}{D} = \dfrac{5}{8} \quad x_2 = \dfrac{D_2}{D} = \dfrac{1}{8} \quad x_3 = \dfrac{D_3}{D} = -\dfrac{5}{8} \quad x_4 = \dfrac{D_4}{D} = -\dfrac{1}{8}$

(3) $D=\begin{vmatrix}2&1&-5&1\\1&-3&0&-6\\0&2&-1&2\\1&4&-7&0\end{vmatrix}=27\quad D_1=\begin{vmatrix}1&1&-5&1\\2&-3&0&-6\\-1&2&-1&2\\0&4&-7&0\end{vmatrix}=27$

$$D_2=\begin{vmatrix}2&1&-5&1\\1&0&0&-6\\0&0&-1&2\\1&-1&-7&0\end{vmatrix}=15\quad D_3=\begin{vmatrix}2&1&1&1\\1&-3&2&-6\\0&2&0&2\\1&4&-1&0\end{vmatrix}=6\quad D_4=\begin{vmatrix}2&1&-5&2\\1&-3&0&0\\0&2&-1&0\\1&4&-7&-1\end{vmatrix}=-12$$

故 $x_1=\dfrac{D_1}{D}=1\quad x_2=\dfrac{D_2}{D}=\dfrac{5}{9}\quad x_3=\dfrac{D_3}{D}=\dfrac{2}{9}\quad x_4=\dfrac{D_4}{D}=-\dfrac{4}{9}$

2.

依题意, $\begin{vmatrix}a&1&1\\1&a&1\\1&2a&1\end{vmatrix}=0$ 即 $a(1-a)=0$ 时有非零解.

$\therefore$ 当 $a=0$ 或 $a=1$ 时有非零解.

## (B)

1.

证明: 设 $P(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$

则由 $P(x_i)=y_i$, 可知 $a_0,a_1,\ldots,a_{n-1}$ 为该方程组的解.

由于 $D=\begin{vmatrix}1&x_1&\cdots&x_1^{n-1}\\1&x_2&\cdots&x_2^{n-1}\\\vdots&\vdots&\ddots&\vdots\\1&x_n&\cdots&x_n^{n-1}\end{vmatrix}=\prod\limits_{1\le i<j\le n}(x_i-x_j)\neq 0$

故该方程组有唯一解.

即存在唯一的次数小于 $n$ 的多项式 $P(x)$, 使 $P(x_i)=y_i$.

2.

由 $\left(\begin{array}{ccc:c}1&1&b&4\\-1&b&1&b^2\\1&-1&2&-4\end{array}\right)\to\left(\begin{array}{ccc:c}1&1&b&4\\0&b+1&b+1&b^2+4\\0&-2&2-b&-8\end{array}\right)$

则当 $b=-1$ 时, 方程组无解.

$b \neq -1$ 时 $\rightarrow \left(\begin{array}{ccc:c} 1 & 1 & b & 4 \\ 0 & 1 & 1 & \frac{b^2+4}{b+1} \\ 0 & 0 & 4-b & \frac{2b(b-4)}{b+1} \end{array}\right)$

故当 $b \neq -1, 4$ 时, 方程组有唯一解.

$b = 4$ 时, 方程组有无穷解, 此时 $\rightarrow \left(\begin{array}{ccc:c} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$

取自由未知量 $x_3$ 为1, 得基础解系 $\eta_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$

取 $x_3 = 0$, 得特解 $\eta^* = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$

故 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$, $k$ 为任意常数.

## (C)

1.

设圆上的点为 $(x, y)$, 则

设圆方程为 $A(x^2+y^2) + Dx + Ey + F = 0$

则 $\begin{cases} A(x^2+y^2) + Dx + Ey + F = 0 \\ A(x_1^2+y_1^2) + Dx_1 + Ey_1 + F = 0 \\ A(x_2^2+y_2^2) + Dx_2 + Ey_2 + F = 0 \\ A(x_3^2+y_3^2) + Dx_3 + Ey_3 + F = 0 \end{cases}$

由该方程组为非零解, 则

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

此即过这三点的圆方程.

2.

设平面方程为 $Ax+By+Cz+D=0$

则 $\begin{cases} Ax+By+Cz+D=0 \\ Ax_0+By_0+Cz_0+D=0 \end{cases}$

由于该平面与 $\pi_1,\pi_2$ 垂直, 则

$$(A,B,C)\cdot(a_1,b_1,c_1)=0 \quad (A,B,C)\cdot(a_2,b_2,c_2)=0$$

即 $\begin{cases} Aa_1+Bb_1+Cc_1=0 \\ Aa_2+Bb_2+Cc_2=0 \\ Ax_0+By_0+Cz_0+D=0 \end{cases}$

该方程组有非零解, 则 $\begin{vmatrix} x & y & z & 1 \\ a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ x_0 & y_0 & z_0 & 1 \end{vmatrix}=0$

# 习题3.1

## (A)

1.

(1) $\alpha_1+2\alpha_2-\alpha_3=(3,8,-8)^{\mathrm{T}}$

(2) $(\alpha_1+\alpha_2)+2(\alpha_2+\alpha_3)-3(\alpha_3+\alpha_1)=(3,12,-19)^{\mathrm{T}}$

(3) $(\alpha_1-\alpha_2)+(\alpha_2-\alpha_3)+(\alpha_3-\alpha_1)=(0,0,0)^{\mathrm{T}}$

2.

依题意,有$\gamma=\frac{5}{3}\beta-\frac{2}{3}\alpha=\frac{1}{3}(4,10,16)^{\mathrm{T}}$

3.

$\gamma=\frac{3}{2}\beta-\frac{1}{2}\alpha=(1,2,3,4)^{\mathrm{T}}$

4.

(1) 依题意,

$$\begin{aligned}V_1 &= \{x_1\boldsymbol{\varepsilon}_1 + x_2\boldsymbol{\varepsilon}_2 \mid x_1, x_2 \in \mathbf{R}\} \\ &= \left\{(x_1, x_2, 0, 0)^{\mathrm{T}} \mid x_1, x_2 \in \mathbf{R}\right\}\end{aligned}$$

(2) 对任意向量 $\alpha = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$, 总有 $\alpha = a\boldsymbol{\varepsilon}_1 + b\boldsymbol{\varepsilon}_2 + c\boldsymbol{\varepsilon}_3 + d\boldsymbol{\varepsilon}_4$,

故 $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\varepsilon}_3, \boldsymbol{\varepsilon}_4$ 生成的子空间为 $\mathbf{R}^4$

5.

$V = \left\{k_1\alpha + k_2\beta = (k_1, k_1, k_2)^{\mathrm{T}} \mid k_1, k_2 \in \mathbf{R}\right\}$, 几何意义为过 $z$ 轴的平面.

6.

(1) 不是. 设 $f(x) \in V$, 则 $f(x) - f(x) = 0 \notin V$

$\therefore V$ 不能构成线性空间

(2) 是. 首先, $V$ 中的矩阵有加法运算, 且满足交换律, 结合律.

$V$ 中有零矩阵 $O$, 使得对任意 $A \in V$, 有 $A + O = A$.

对 $\forall A \in V, \exists -A \in V$, 使 $A + (-A) = 0$, 从而性质 (1) $\sim$ (4) 满足.

由矩阵乘法定义易验证性质 (5) $\sim$ (8) 也满足.

故 $V$ 是线性空间.

## (B)

1.

证明: $\because V_1 \cap V_2 \subset V_1,\ V_1 \cap V_2 \subset V_2$, 则:

(i) $\alpha + \beta = \beta + \alpha$

(ii) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(iii) 又由于 $0 \in V_1,\ 0 \in V_2$, 故 $0 \in V_1 \cap V_2$

对任何 $\alpha \in V_1 \cap V_2$, 有 $\alpha + 0 = \alpha$

(iv) 对 $\alpha \in V_1 \cap V_2$, 一定有 $\beta \in V_1 \cap V_2$, 使 $\alpha + \beta = 0$

(v) $1\alpha = \alpha$

(vi) $\lambda(\mu\alpha) = (\lambda\mu)\alpha$

(vii) $(\lambda+\mu)\alpha = \lambda\alpha + \mu\alpha$

(viii) $\lambda(\alpha+\beta) = \lambda\alpha + \lambda\beta$

故 $V_1 \cap V_2$ 也是 $\mathbf{R}$ 上的线性空间

2.

证明同1, 易验证满足八条性质.

3.

可以.

(1) $a \oplus b = ab = ba = b \oplus a$

(2) $(a \oplus b) \oplus c = ab \oplus c = abc = a \oplus (b \oplus c)$

(3) $a \oplus 1 = a \cdot 1 = a$ （存在零元为1）

(4) $a \oplus \dfrac{1}{a} = a \cdot \dfrac{1}{a} = 1$ （存在负元素为$\dfrac{1}{a}$）

(5) $1 \odot a = a^1 = a$

(6) $k \odot (l \odot a) = k \odot a^l = a^{lk} = a^{kl} = (kl) \odot a$

(7) $(k+l) \odot a = a^{k+l} = a^k \cdot a^l = a^k \oplus a^l = k \odot a \oplus l \odot a$

(8) $k \odot (a \oplus b) = k \odot ab = k \odot a \oplus k \odot b$

故可以构成线性空间

4.

(1) 证明：设 $0_1$，$0_2$ 为两个零元素, 则

$$\alpha + 0_1 = \alpha, \quad \alpha + 0_2 = \alpha$$

于是

$$\begin{cases} 0_1 + 0_2 = 0_1 \\ 0_2 + 0_1 = 0_2 \end{cases} \Rightarrow 0_1 = 0_1 + 0_2 = 0_2 + 0_1 = 0_2$$

故零元素唯一

(2) 证明：设 $\alpha$ 有两个负元素 $\beta$，$\gamma$, 则 $\alpha + \beta = 0$，$\alpha + \gamma = 0$

则 $\beta = \beta + 0 = \beta + (\alpha + \gamma) = (\alpha + \beta) + \gamma = 0 + \gamma = \gamma$

(3) 证明：$\alpha + 0\alpha = 1\alpha + 0\alpha = (1+0)\alpha = 1\alpha = \alpha$，所以 $0\alpha = 0$，

$\alpha + (-1)\alpha = [1 + (-1)]\alpha = 0\alpha = 0$

故 $(-1)\alpha = -\alpha$. $k0 = k[\alpha + (-1)\alpha] = k\alpha + (-k)\alpha = [k + (-k)]\alpha = 0\alpha = 0$

(4) 证明：若 $k \neq 0$，则 $\frac{1}{k}(k\alpha) = \frac{1}{k} \cdot 0 = 0$ 而 $\frac{1}{k}(k\alpha) = (\frac{1}{k} \cdot k)\alpha = 1\alpha = \alpha$ 故 $\alpha = 0$

# 习题3.2

## (A)

1.

(1) $\left(\begin{array}{cccc:c} 1 & -2 & 3 & -1 & 1 \\ 3 & -1 & 5 & -3 & 2 \\ 2 & 1 & 2 & -2 & 3 \end{array}\right) \to \left(\begin{array}{cccc:c} 1 & 0 & \frac{7}{5} & -1 & 0 \\ & 1 & -\frac{9}{5} & 0 & 0 \\ & & & & -1 \end{array}\right)$

故不能表示.

(2) $\left(\begin{array}{cccc:c} 1 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 2 & 3 \\ 1 & 2 & 2 & 3 & 2 \end{array}\right) \to \left(\begin{array}{cccc:c} 1 & 2 & 0 & 0 & 1 \\ & & 1 & 0 & 2 \\ & & & 1 & -1 \end{array}\right)$ 故可以表示.

$\beta = \alpha_1 + 2\alpha_3 - \alpha_4$

2.

(1) 有零向量 $\alpha_3^{\mathrm{T}}$,故可取 $k_1 = k_2 = 0, k_3 = 1$ 使 $k_1\alpha_1^{\mathrm{T}} + k_2\alpha_2^{\mathrm{T}} + k_3\alpha_3^{\mathrm{T}} = 0$. 故线性相关.

(2) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 5 & 3 \\ 3 & 6 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, 故 $\alpha_3 + \alpha_1 - \alpha_2 = 0$ 线性相关

(3) $\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 6 \\ 3 & 6 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, 故线性无关.

3.

(1) 错误. 如 $\alpha_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(2) 错误. 如 $\alpha_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(3) 正确. 若 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 线性相关, 则存在不全为 0 的 $k_1, k_2, \cdots, k_s$, 使

$$k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$$

不妨设 $k_i \neq 0$, 则 $\alpha_i = -\frac{1}{k_i}(k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s)$.

即 $\alpha_i$ 可以由其余的向量线性表示,矛盾!

故向量组线性无关.

(4) 正确. 为(3)的逆否命题

(5) 错误. 如 $\alpha_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

(6) 正确. 依题意, 存在不全为 0 的 $k_1, k_2, \cdots, k_s$ 使 $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s = 0$

则 $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_s\alpha_s + 0\beta_1 + 0\beta_2 + \cdots + 0\beta_t = 0$

故整体组相关

(7) 错误. 如 $\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \beta_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \beta_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

4.

$A\alpha = \begin{pmatrix} a \\ 2a+3 \\ 3a+4 \end{pmatrix}$ 与 $\alpha = \begin{pmatrix} a \\ 1 \\ 1 \end{pmatrix}$ 线相关, 则对应元素成比例, 有

$$\frac{a}{a} = \frac{2a+3}{1} = \frac{3a+4}{1}$$

得 $a = -1$

5.

设 $k_1(\alpha_1 + \alpha_2) + k_2(\alpha_2 + \alpha_3) + k_3(\alpha_3 + \alpha_1) = 0$

则整理得 $(k_1 + k_3)\alpha_1 + (k_1 + k_2)\alpha_2 + (k_2 + k_3)\alpha_3 = 0$

由于 $\alpha_1,\alpha_2,\alpha_3$ 线性无关, 则

$$\begin{cases} k_1+k_3=0 \\ k_1+k_2=0 \\ k_2+k_3=0 \end{cases} \Rightarrow \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

故 $\alpha_1+\alpha_2,\alpha_2+\alpha_3,\alpha_3+\alpha_1$ 线性无关.

又 $(\alpha_1-\alpha_2)+(\alpha_2-\alpha_3)+(\alpha_3-\alpha_1)=\alpha_1-\alpha_2+\alpha_2-\alpha_3+\alpha_3-\alpha_1=0$,

故取 $k_1',k_2',k_3'$ 为 $1,1,1$,

可知 $\alpha_1-\alpha_2,\alpha_2-\alpha_3,\alpha_3-\alpha_1$ 线性相关.

## (B)

1.

(1) 相关. 因为4个3维向量必然线性相关.

(2) 无关. 考虑 $\beta_1^{\mathrm{T}}=(1,0,0)$, $\beta_2^{\mathrm{T}}=(0,2,3)$, $\beta_3^{\mathrm{T}}=(0,0,4)$ 则

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{vmatrix} = 8 \neq 0$$

故 $\beta_1^{\mathrm{T}},\beta_2^{\mathrm{T}},\beta_3^{\mathrm{T}}$ 线性无关. 则其延伸组 $\alpha_1^{\mathrm{T}},\alpha_2^{\mathrm{T}},\alpha_3^{\mathrm{T}}$ 必线性无关.

2.

**充分性**: 若 $\alpha_i$ 可以由 $\alpha_1,\alpha_2,\cdots,\alpha_{i-1}$ 线性表示, 则

$$\alpha_i=k_1\alpha_1+k_2\alpha_2+\cdots+k_{i-1}\alpha_{i-1}$$

即

$$k_1\alpha_1+k_2\alpha_2+\cdots+k_{i-1}\alpha_{i-1}-\alpha_i=0$$

取 $k_i=-1$, $k_{j+1}=0\ (j\le i-1)$, 可知 $\alpha_1,\alpha_2,\cdots,\alpha_m$ 线性相关.

**必要性**: 由 $\alpha_1,\alpha_2,\cdots,\alpha_m\ (\alpha_i\neq 0)$ 线性相关, 则存在不全为0的 $k_1,k_2,\cdots,k_m$, 使

$$k_1\alpha_1+k_2\alpha_2+\cdots+k_m\alpha_m=0$$

则从 $k_m$ 开始往前, 必有一个 $k_i\ (i\neq 1)$ 使 $k_i\neq 0$ (否则 $k_1\alpha_1=0$, 又 $\alpha_1\neq 0$, 矛盾). 故有

$$k_1\alpha_1+k_2\alpha_2+\cdots+k_i\alpha_i=0$$

$$\Rightarrow \alpha_i=-\frac{1}{k_i}(k_1\alpha_1+k_2\alpha_2+\cdots+k_{i-1}\alpha_{i-1})$$

3.

证明: 据已知有

$$r(\alpha_1,\alpha_2,\cdots,\alpha_r)=r(\alpha_1,\alpha_2,\cdots,\alpha_r,\beta)$$
$$r(\alpha_1,\alpha_2,\cdots,\alpha_{r-1})+1=r(\alpha_1,\alpha_2,\cdots,\alpha_{r-1},\beta)$$

考察

$$\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{r-1}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_{r-1}\end{pmatrix}=\alpha_r$$

$$\begin{aligned}r(\alpha_1,\alpha_2,\ldots,\alpha_{r-1},\alpha_r)&\le r(\alpha_1,\alpha_2,\ldots,\alpha_{r-1})+1\\&=r(\alpha_1,\alpha_2,\ldots,\alpha_{r-1},\beta)\\&\le r(\alpha_1,\alpha_2,\ldots,\alpha_{r-1},\alpha_r,\beta)\\&=r(\alpha_1,\alpha_2,\ldots,\alpha_r).\end{aligned}$$

故 $r(\alpha_1,\alpha_2,\cdots,\alpha_{r-1})+1=r(\alpha_1,\alpha_2,\cdots,\alpha_r)$, 即 $\alpha_r$ 不能由 $\alpha_1,\cdots,\alpha_{r-1}$ 表出.

由于 $r(\alpha_1,\alpha_2,\cdots,\alpha_{r-1},\beta)=r(\alpha_1,\alpha_2,\cdots,\alpha_r,\beta)=r(\alpha_1,\alpha_2,\cdots,\alpha_{r-1},\beta,\alpha_r)$

故 $\alpha_r$ 可以由 $\alpha_1,\alpha_2,\cdots,\alpha_{r-1},\beta$ 线性表出.

## (C)

1.

证明: 设 $k_1\alpha + k_2 A\alpha + \cdots + k_m A^{m-1}\alpha = 0$ 则

$$A^{m-1}(k_1\alpha + k_2 A\alpha + \cdots + k_m A^{m-1}\alpha) = 0$$

$$\Rightarrow k_1 A^{m-1}\alpha = 0$$

又 $A^{m-1}\alpha \neq 0$

$\therefore k_1 = 0$ 同理依次对假设式乘 $A^{m-2}, A^{m-3}, \cdots, A, 1$, 有

$$k_2 = 0,\ k_3 = 0,\ \cdots,\ k_m = 0$$

故 $\alpha, A\alpha, \cdots, A^{m-1}\alpha$ 线性无关.

2.

证明: 设

$$k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0 \tag{1}$$

则

$$k_1 A\alpha_1 + k_2 A\alpha_2 + k_3 A\alpha_3 = 0$$

即

$$(k_1 + k_2)\alpha_1 + (k_1 + k_3)\alpha_2 + k_3\alpha_3 = 0 \tag{2}$$

(2) - (1) 得

$$k_2\alpha_1 + k_3\alpha_2 = 0 \tag{3}$$

则

$$k_2 A\alpha_1 + k_3 A\alpha_2 = 0$$

即

$$k_2\alpha_1 + k_3\alpha_1 + k_3\alpha_2 = 0 \tag{4}$$

(4) - (3) $\Rightarrow k_3\alpha_1 = 0$, 又 $\alpha_1 \neq 0$

$\therefore k_3 = 0$

将 $k_3$ 往(3)、(1)回代可得

$$k_2 = 0,\ k_1 = 0$$

故 $\alpha_1, \alpha_2, \alpha_3$ 线性无关.

3.

证明: 设 $k_1\beta_1+k_2\beta_2+\cdots+k_s\beta_s=0$

可得 $(k_1+k_2)\alpha_1+(k_1+k_2)\alpha_2+(k_2+k_3)\alpha_3+\cdots+(k_{s-1}+k_s)\alpha_s=0$

由于 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 线性无关, 故

$$\begin{cases} k_1+0+\cdots+k_s=0 \\ k_1+k_2+\cdots+0=0 \\ 0+k_2+k_3+\cdots+0=0 \\ \cdots \\ 0+\cdots+k_{s-1}+k_s=0 \end{cases}$$

其系数行列式

$$|A|=\begin{vmatrix} 1 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 1 \end{vmatrix}=\begin{vmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1-(-1)^{s-1} \end{vmatrix}=1-(-1)^{s-1}$$

则当 $s$ 为奇数时 $|A|=2$, 向量组线性无关;

当 $s$ 为偶数时 $|A|=0$, 向量组线性相关.

# 习题3.3

## (A)

1.

(1) 由 $\begin{pmatrix} 1 & 0 & 2 & 0 \\ & 1 & -3 & 0 \\ & & & 0 \end{pmatrix}$ 知 $\alpha_1,\alpha_2$ 为一个极大线性无关组. 秩为2.

(2) 由 $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 4 & 5 \end{pmatrix}\to\begin{pmatrix} 1 & & & -\frac{5}{2} \\ & 1 & & \frac{15}{4} \\ & & 1 & \frac{5}{4} \end{pmatrix}$ 知 $\alpha_1,\alpha_2,\alpha_4$ 为一个极大线性无关组. 秩为3.

2.

(1) 不正确. 不一定是极大线性无关组

(2) 正确. 若秩大于等于r, 则存在r个线性无关的向量, 与已知矛盾. 故命题正确.

(3) 错误. 如 $(1,0)^{\mathrm{T}}$, $(0,1)^{\mathrm{T}}$, $(2,0)^{\mathrm{T}}$.

(4) 正确. 由秩的定义, 存在 $r$ 个线性无关的向量, 故其中的 $r-1$ 个向量必线性无关

3.

依题意, $r(\alpha_1,\alpha_2,\cdots,\alpha_r)=r(\alpha_1,\alpha_2,\cdots,\alpha_{r+1})=r$
则向量组A的秩为r.
又 $\alpha_1,\alpha_2,\cdots,\alpha_r$ 是r个线性无关的向量,
设 $\alpha_j\ (j=1,2,\cdots,n)$ 为向量组 $\alpha_1,\alpha_2,\cdots,\alpha_n$ 中的任一向量
则 $\alpha_1,\alpha_2,\cdots,\alpha_r,\alpha_j$ 线性相关
故 A中任一向量都可以由 $\alpha_1,\alpha_2,\cdots,\alpha_r$ 线性表示
从而 $\alpha_1,\alpha_2,\cdots,\alpha_r$ 是极大线性无关组.

4.

过渡矩阵为

$$\begin{aligned}
Q&=(\alpha_1,\alpha_2,\alpha_3)^{-1}(\beta_1,\beta_2,\beta_3)\\
&=\begin{pmatrix}\frac{1}{2}&0&\frac{1}{2}\\ \frac{1}{2}&0&-\frac{1}{2}\\ -\frac{1}{2}&1&\frac{1}{2}\end{pmatrix}\begin{pmatrix}1&1&0\\ -2&2&1\\ 1&-1&-2\end{pmatrix}\\
&=\begin{pmatrix}1&0&-1\\ 0&1&1\\ -2&1&0\end{pmatrix}
\end{aligned}$$

在基 $(\alpha_1,\alpha_2,\alpha_3)$ 下: $\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=(\alpha_1,\alpha_2,\alpha_3)^{-1}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}$

在基 $(\beta_1,\beta_2,\beta_3)$ 下: $\begin{pmatrix}y_1\\ y_2\\ y_3\end{pmatrix}=Q^{-1}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=\begin{pmatrix}0\\ 1\\ -1\end{pmatrix}$

5.

(1) $(\alpha,\beta)=1\times(-2)+2\times1+1\times\dfrac{1}{2}=\dfrac{1}{2}$

$$(\alpha,\gamma)=1\times 2+2\times(-2)+1\times 2=0$$
$$(\beta,\gamma)=2\times(-2)-2\times 1+2\times\frac{1}{2}=-5$$

(2) $\dfrac{1}{|\alpha|}\alpha=\dfrac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}$

(3) $\theta=\arccos\dfrac{|(\alpha,\beta)|}{|\alpha||\beta|}=\arccos\dfrac{1}{3\sqrt{14}}$

6.

(1) $\begin{pmatrix}1&\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1&0\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}\frac{3}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{pmatrix}\neq E$

$$\begin{pmatrix}2&2&1\\2&1&-2\\1&2&2\end{pmatrix}\begin{pmatrix}2&2&1\\2&1&2\\1&-2&2\end{pmatrix}=\begin{pmatrix}9&8&0\\8&9&4\\0&4&9\end{pmatrix}\neq E$$

$$\begin{pmatrix}2&2&1\\2&1&-2\\1&2&2\end{pmatrix}\begin{pmatrix}2&2&1\\2&1&2\\1&-2&2\end{pmatrix}=\begin{pmatrix}9&&\\&9&\\&&9\end{pmatrix}\neq E$$

$$\frac{1}{3}\begin{pmatrix}2&-2&1\\2&1&-2\\1&2&2\end{pmatrix}\cdot\frac{1}{3}\begin{pmatrix}2&2&1\\-2&1&2\\1&-2&2\end{pmatrix}=\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}=E$$

故 $\dfrac{1}{3}\begin{pmatrix}2&-2&1\\2&1&-2\\1&2&2\end{pmatrix}$ 是正交矩阵.

7.

$\beta_1=\alpha_1=\begin{pmatrix}1\\0\\1\end{pmatrix}$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{2}{3}\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

## (B)

1.

设 $\alpha_3 = (x, y, z)^{\mathrm{T}}$, 则 $\begin{cases} x + 2y + 2z = 0 \\ -2x + y = 0 \end{cases}$

由于 $\begin{pmatrix} 1 & 2 & 2 \\ -2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & & \frac{2}{5} \\ & 1 & \frac{4}{5} \end{pmatrix}$,

故可取 $x = -\frac{2}{5}z$, $y = -\frac{4}{5}z$

又由 $x^2 + y^2 + z^2 = 1$

$\Rightarrow \alpha_3 = \pm\frac{1}{3\sqrt{5}}\begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$

2.

因 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 中有 $r_1$ 个向量线性无关,

$\beta_1, \beta_2, \cdots, \beta_t$ 中有 $r_2$ 个向量线性无关,

则整体组中的线性无关的向量个数一定大于 $r_1$ 与 $r_2$ 中的较大者.

又因 $\forall \alpha_1, \alpha_2, \cdots, \alpha_{r_1}, \alpha_{r_1+1}$ 线性相关,

则 $\alpha_1, \alpha_2, \cdots, \alpha_{r_1}, \alpha_{r_1+1}, \beta_1, \beta_2, \cdots, \beta_{r_2}$ 也必然线性相关,

故秩 $r < r_1 + r_2 + 1$,

综上, $\max\{r_1, r_2\} \le r_3 \le r_1 + r_2$.

3.

设 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\beta = 0$

则 $k_4\beta^2 = 0$, 从而有 $k_4 = 0$.

又 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, $\therefore k_1 = k_2 = k_3 = 0$

故 $\alpha_1, \alpha_2, \alpha_3, \beta$ 线性无关.

4.

依题意, 有 $(\alpha_1, \alpha_2, \cdots, \alpha_m) \cdot \beta = 0$

设 $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_m\alpha_m + k_{m+1}\beta = 0$

$\Rightarrow k_1\alpha_1\beta + k_2\alpha_2\beta + \cdots + k_m\alpha_m\beta + k_{m+1}\beta^2 = 0$

$\Rightarrow k_{m+1}\beta^2 = 0$

$\Rightarrow k_{m+1} = 0$

又 $\alpha_1, \alpha_2, \cdots, \alpha_m$ 线性无关, 则 $k_1 = k_2 = \cdots = k_m = 0$

从而 $\alpha_1, \alpha_2, \cdots, \alpha_m, \beta$ 线性无关.

## (C)

1.

证明: **必要性**: 设 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性无关, $\alpha$ 是任意一个 $n$ 维向量.

$\because n+1$ 个 $n$ 维向量必线性相关,

故 $\alpha$ 可以由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示.

**充分性**: 设任何一个 $n$ 维向量可由 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性表示.

故 $e_1 = (1, 0, \cdots, 0)$, $e_2 = (0, 1, \cdots, 0)$, $\cdots$, $e_n = (0, 0, \cdots, 1)$ 能被 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 表示.

又 $e_1, e_2, \cdots, e_n$ 这组向量可以表示任何 $n$ 维向量, 故 $e_1, e_2, \cdots, e_n$ 与 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 等价,

所以 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性无关.

2.

证明: 设 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 是某一向量组中的线性无关部分组.

在向量组考虑向量 $\beta$, 若 $\beta$ 可由 $\alpha_1, \alpha_2, \cdots, \alpha_s$ 表示, 则放弃此向量,

否则将 $\beta$ 添加至 $\alpha_1, \alpha_2, \cdots, \alpha_s$, 即 $\alpha_{s+1} = \beta$.

如此遍历下去, 遍历整个原向量组, 使得扩充的部分组 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 满足:

(1) 线性无关,

(2) 原向量组中任一向量都可以由此部分组表示,

则该部分组即扩充为了一个极大线性无关组.

3.

证明: 当 $s \le m - r$ 时,

$\because r(\alpha_{k_1}, \alpha_{k_2}, \cdots, \alpha_{k_s}) > 0 \ge r + s - m$

此时成立.

当 $s > m - r$ 时, 则设 $\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_r}$ 为一个极大线性无关组.

则 $s$ 个向量中必有 $s+r-m$ 个向量 $\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_{(s+r-m)}}$ 属于极大线性无关组, 故

$$
\begin{aligned}
r(\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_s}) &= r(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{(s+r-m)}} + \alpha_{k_i} + \cdots) \\
&> r(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{(s+r-m)}}) = s+r-m
\end{aligned}
$$

综上, 向量组B的秩大于等于 $r-(m-s)$.

# 习题3.4

1.

基可取 $\alpha_1 = 1, \alpha_2 = i$

若 $k_1\alpha_1 + k_2\alpha_2 = 0$ 则 $k_1 = 0, k_2 = 0$, 故 $\alpha_1, \alpha_2$ 是该空间的一组基.

维数为2.

2.

设 $\alpha_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

则若 $k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 = 0$, 即 $\begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & k_3 \\ -k_2 & -k_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

则 $k_1 = k_2 = k_3 = 0$, 从而 $\alpha_1, \alpha_2, \alpha_3$ 线性无关.

所以 $\alpha_1, \alpha_2, \alpha_3$ 是该空间的一组基.

维数为3.

3.

定义A的一个部分组 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 满足:

(1) 向量组 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 线性无关

(2) 向量组A的任意向量可以由 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 线性表示,

则称 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 为A的一个极大线性无关组.

极大线性无关组所含个数称为A的秩.

4.

假设对于一向量有

$$A = x_1A_1 + x_2A_2 + \cdots + x_nA_n \tag{1}$$

$$A = y_1A_1 + y_2A_2 + \cdots + y_nA_n \tag{2}$$

则 (1) − (2) 有

$$(x_1-y_1)A_1+(x_2-y_2)A_2+\cdots+(x_n-y_n)A_n=0$$

由于 $A_1,A_2,\cdots,A_n$ 线性无关, 故 $\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}y_1\\y_2\\\vdots\\y_n\end{pmatrix}$, 故坐标唯一.

5.

若有 $m$ 阶矩阵 $\boldsymbol{P}=(p_{ij})$, 使

$$(\beta_1,\beta_2,\cdots,\beta_m)=(\alpha_1,\alpha_2,\cdots,\alpha_m)\begin{pmatrix}p_{11}&p_{12}&\cdots&p_{1m}\\p_{21}&p_{22}&\cdots&p_{2m}\\\vdots&\vdots&\ddots&\vdots\\p_{m1}&p_{m2}&\cdots&p_{mm}\end{pmatrix}$$

则称 $\boldsymbol{P}$ 为基 $\alpha_1,\alpha_2,\cdots,\alpha_m$ 到基 $\beta_1,\beta_2,\cdots,\beta_m$ 的过渡矩阵.

若 $\alpha$ 在基 $\alpha_1,\alpha_2,\cdots,\alpha_m$ 下坐标为 $x_1,x_2,\cdots,x_m$, 在基 $\beta_1,\beta_2,\cdots,\beta_m$ 下坐标为 $y_1,\cdots,y_m$.
则

$$\begin{pmatrix}x_1\\x_2\\\vdots\\x_m\end{pmatrix}=\begin{pmatrix}p_{11}&p_{12}&\cdots&p_{1m}\\p_{21}&p_{22}&\cdots&p_{2m}\\\vdots&\vdots&\ddots&\vdots\\p_{m1}&p_{m2}&\cdots&p_{mm}\end{pmatrix}\begin{pmatrix}y_1\\y_2\\\vdots\\y_m\end{pmatrix}$$

# 习题3.5

## (A)

1.

(1) $\begin{pmatrix}1&2&-3&4\\2&4&-6&8\end{pmatrix}\to\begin{pmatrix}1&2&-3&4\\0&0&0&0\end{pmatrix}$

故秩 $r=1$

(2) $\begin{pmatrix}2&-1&3&-2&4\\4&-2&5&1&7\\2&-1&1&8&2\end{pmatrix}\to\begin{pmatrix}1&-\frac{1}{2}&0&\frac{13}{2}&\frac{1}{2}\\&&1&-5&1\\&&&&\end{pmatrix}$,

故秩 $r=2$

(3) $$\begin{pmatrix}3&2&-1&-3&-1\\2&-1&3&1&-3\\2&0&5&1&8\\5&1&2&-2&-4\end{pmatrix}\to\begin{pmatrix}1&0&0&-\frac{2}{5}&-\frac{3}{5}\\&1&0&-\frac{18}{25}&\frac{27}{5}\\&&1&\frac{9}{25}&\frac{14}{5}\\&&&&\end{pmatrix}$$

故秩 $r=3$

2.

(1) $$\begin{pmatrix}1&3&0&2&2\\-1&0&3&1&-2\\2&7&1&5&4\\4&4&-8&6&8\end{pmatrix}\to\begin{pmatrix}1&0&-3&0&2\\&1&1&0&0\\&&&1&0\\&&&&\end{pmatrix}\text{秩为 }3$$

则极大线性无关组可取 $\alpha_1,\alpha_2,\alpha_4$, 且有

$$\alpha_3=-3\alpha_1+\alpha_2,\quad \alpha_5=2\alpha_1$$

(2) $$\begin{pmatrix}1&2&3&1&3\\1&3&4&2&2\\2&7&9&3&7\\3&7&10&2&12\end{pmatrix}\to\begin{pmatrix}1&0&1&0&3\\&1&1&0&1\\&&&1&-2\\&&&&\end{pmatrix}$$

则秩为 3. $\alpha_1,\alpha_2,\alpha_4$ 是一个极大无关组, 且

$$\alpha_3=\alpha_1+\alpha_2,\quad \alpha_5=3\alpha_1+\alpha_2-2\alpha_4$$

3.

(1) 由 $\begin{vmatrix}1&-1&-2&3\\1&-3&-5&2\\1&1&a&4\\1&7&10&7\end{vmatrix}=0\Rightarrow a=1$, 又由 $\begin{vmatrix}1&-1&3&0\\1&-3&2&-1\\1&1&4&1\\1&7&7&b\end{vmatrix}=0\Rightarrow b=4$

(2) $$\begin{pmatrix}1&-1&-2&3&0\\1&-3&-5&2&-1\\1&1&1&4&1\\1&7&10&7&4\end{pmatrix}\to\begin{pmatrix}1&0&-\frac{1}{2}&\frac{7}{2}&\frac{1}{2}\\&1&\frac{3}{2}&\frac{1}{2}&\frac{1}{2}\\&&&&\end{pmatrix}$$

故 $\alpha_1,\alpha_2$ 是一个极大无关组,

$$\alpha_3=-\frac{1}{2}\alpha_1+\frac{3}{2}\alpha_2,\quad \alpha_4=\frac{7}{2}\alpha_1+\frac{1}{2}\alpha_2,\quad \alpha_5=\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2.$$

4.

(1) 由 $\begin{pmatrix}1&3&a&7\\-1&2&3&8\\2&-1&0&b\\0&1&2&3\end{pmatrix}\to\begin{pmatrix}1&3&a&7\\&1&\frac{3+a}{5}&3\\&&14-2a&7+b\\&&\frac{7-a}{5}&0\end{pmatrix}$

以及秩为 2 知

$$\begin{cases}14-2a=0\\7+b=0\\\dfrac{7-a}{5}=0\end{cases}\Rightarrow\begin{cases}a=7\\b=-7\end{cases}$$

(2) 由 $\begin{pmatrix}1&3&7&7\\-1&2&3&8\\2&-1&0&-7\\0&1&2&3\end{pmatrix}\to\begin{pmatrix}1&0&1&-2\\&1&2&3\\&&&\\&&&\end{pmatrix}$

则 $\alpha_1,\alpha_2$ 是一个极大无关组, $\alpha_3=\alpha_1+2\alpha_2$, $\alpha_4=-2\alpha_1+3\alpha_2$.

(3) 由于任何两列都线性无关, 故有 $C_4^2=6$ 个极大无关组.

5.

(1) 错误. 如 $A=\begin{pmatrix}1&&&\\&1&&\\&&0&\\&&&0\end{pmatrix}$

(2) 正确. 若A的所有 $r-1$ 阶子式均为0, 则秩 $r^*<r-1$, 矛盾.

(3) 正确. 若存在 $r+1$ 阶子式不为零, 则秩 $r^*\geq r+1$, 矛盾.

(4) 错误. 如 $A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

(5) 错误. 如 $A = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

(6) 正确. 若存在 $r$ 阶子式不为零, 则秩 $r^* \geq r$ 的逆否命题即该命题.

或者用反证法: 若秩 $r^* \geq r$, 则存在一个 $r$ 阶子式不为零, 矛盾!

6.

由 $r(AB) = m \leq \min\{r(A), r(B)\} \leq r(B)$

知 $B$ 的列向量组的秩 $r(\beta_1, \beta_2, \cdots, \beta_m) \geq m$

故 $B$ 的列向量组线性无关.

7.

依题意, $(\alpha_1, \alpha_2, \cdots, \alpha_n)x = 0$ 只有零解, 则 $\alpha_1, \alpha_2, \cdots, \alpha_n$ 线性无关.

故 $r(A) = r(\alpha_1, \alpha_2, \cdots, \alpha_n) = n$

## (B)

1.

$$A=\begin{pmatrix}1&\lambda&-1&2\\2&-1&\lambda&5\\1&10&-6&1\end{pmatrix}\xrightarrow{\text{初等行变换}}\begin{pmatrix}1&\lambda&-1&2\\&-1-2\lambda&\lambda+2&1\\&9-3\lambda&\lambda-3&0\end{pmatrix}$$

故当 $\lambda=3$ 时, $r(A)=2$

当 $\lambda\neq3$ 时, $r(A)=3$.

2.

由于 $A$ 是秩为1的 $m\times n$ 矩阵, 则存在可逆矩阵 $P,Q$, 使

$$A=P\cdot\begin{pmatrix}1&&&\\&0&&\\&&\ddots&\\&&&0\end{pmatrix}_{m\times n}\cdot Q$$

则 $A=P\cdot\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}_{m\times1}\cdot\begin{pmatrix}1&0&\cdots&0\end{pmatrix}_{1\times n}\cdot Q$

故取 $\alpha=P\cdot\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}_{m\times1}$ $\beta^T=\begin{pmatrix}1&0&\cdots&0\end{pmatrix}_{1\times n}Q$ 即证.

3.

**必要性**: 若 $r(A)=r(B)$, 则

$$r(\alpha_1,\alpha_2,\dots,\alpha_n)=r(\alpha_1,\alpha_2,\dots,\alpha_{s-1},\alpha_{s+1},\dots,\alpha_n)\leq n-1$$

故 $\alpha_1,\alpha_2,\dots,\alpha_n$ 线性相关,

即所划去的行可用其余的行线性表示.

**充分性**: 若所划去的行可由其余的行线性表示, 则 $r(B)\leq r(A)$

又

$$r(\alpha_1,\alpha_2,\dots,\alpha_n)\geq r(\alpha_1,\alpha_2,\dots,\alpha_{s-1},\alpha_{s+1},\dots,\alpha_n)$$

则 $B$ 中的极大无关组也是 $A$ 中的极大无关组,
故 $r(B) = r(A)$.

4.

记 $r(A) = r$. 若 $s < m - r$, 则 $r(B) \geq 0 > s + r - m$, 成立.
若 $s \geq m - r$, 设 $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}$ 是 $A$ 中行向量组的一个极大无关组,
则 $B$ 中必可取到 $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_r}$ 中的 $s + r - m$ 个元素 $\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_{(s+r-m)}}$
故 $r(B) \geq r(\alpha_{k_1}, \alpha_{k_2}, \ldots, \alpha_{k_{(s+r-m)}}) = s + r - m$
综上, $r(B) \geq r(A) + s - m$.

## (C)

1.

证明：由 $r\begin{pmatrix} A & O \\ E & B \end{pmatrix} \geq r\begin{pmatrix} A & O \\ O & B \end{pmatrix} = r(A) + r(B)$

又 $\begin{pmatrix} A & O \\ E & B \end{pmatrix} \xrightarrow{r_1 - Ar_2} \begin{pmatrix} O & -AB \\ E & B \end{pmatrix} \xrightarrow{c_2 + Bc_1} \begin{pmatrix} O & -AB \\ E & O \end{pmatrix} \to \begin{pmatrix} AB & O \\ O & E \end{pmatrix}$

故 $r(AB) + r(E) = r\begin{pmatrix} AB & O \\ O & E \end{pmatrix} = r\begin{pmatrix} A & O \\ E & B \end{pmatrix} \geq r(A) + r(B)$

即 $r(AB) \geq r(A) + r(B) - n$.

2.

**必要性**: 设 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 线性无关.
$(\beta_1, \beta_2, \cdots, \beta_r) = (\alpha_1, \alpha_2, \cdots, \alpha_r)C$
且 $\beta_1, \beta_2, \cdots, \beta_r$ 线性无关, 则

$$(\beta_1, \beta_2, \cdots, \beta_r)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = (\alpha_1, \alpha_2, \cdots, \alpha_r)C\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = 0, \text{当且仅当 } C\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = 0 \text{ 且 } \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = 0$$

即齐次线性方程组 $CX = 0$ 只有零解, 于是 $|C| \neq 0$

**充分性**: 若 $|C| \neq 0$, 则

$$(\alpha_1, \alpha_2, \cdots, \alpha_r) = (\beta_1, \beta_2, \cdots, \beta_r)C^{-1}$$

即 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 可以由 $\beta_1, \beta_2, \cdots, \beta_r$ 线性表示.

故 $\alpha_1, \alpha_2, \cdots, \alpha_r$ 与 $\beta_1, \beta_2, \cdots, \beta_r$ 等价, 于是

$$r(\alpha_1, \alpha_2, \cdots, \alpha_r) = r(\beta_1, \beta_2, \cdots, \beta_r) = r$$

故 $\beta_1, \beta_2, \cdots, \beta_r$ 线性无关.

3.

令 $K = \begin{pmatrix} a_{11} & \cdots & a_{1r} & \cdots & a_{1s} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nr} & \cdots & a_{ns} \end{pmatrix}$, 设 $r(K) = r \le \min(n, s)$

不失一般性, 不妨设 $K$ 前 $r$ 列是极大线性无关组, 则

$$\begin{cases} \beta_1 = a_{11}\alpha_1 + \cdots + a_{n1}\alpha_n \\ \cdots \\ \beta_r = a_{1r}\alpha_1 + \cdots + a_{nr}\alpha_n \\ \cdots \\ \beta_s = a_{1s}\alpha_1 + \cdots + a_{ns}\alpha_n \end{cases}$$

下面证明 $\beta_1, \beta_2, \cdots, \beta_r$ 是极大线性无关组.

设 $k_1\beta_1 + k_2\beta_2 + \cdots + k_r\beta_r = 0 \Rightarrow (k_1a_{11} + k_2a_{12} + \cdots + k_ra_{1r})\alpha_1 + \cdots + (k_1a_{n1} + \cdots + k_ra_{nr})\alpha_n = 0$

则 $\begin{cases} a_{11}k_1 + a_{12}k_2 + \cdots + a_{1r}k_r = 0 \\ \cdots \\ a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nr}k_r = 0 \end{cases}$

该方程组系数矩阵秩为 $r$, 故只有零点 $k_1 = k_2 = \cdots = k_r = 0$,

故 $\beta_1, \beta_2, \cdots, \beta_r$ 线性无关.

其次, 任意添加一个向量 $\beta_j$ 后, 方程组的秩 $r < r + 1$, 则有非零解, 即线性相关.

故向量组 $\beta_1, \beta_2, \cdots, \beta_s$ 的秩等于 $K$ 的秩,

故 $\beta_1, \cdots, \beta_s$ 线性无关 $\Leftrightarrow r(K) = s$.

# 习题4.1

## (A)

1.

(1) 由于 $\begin{pmatrix} 3 & 4 & -7 & 1 \\ 2 & 1 & -6 & 0 \\ -1 & 2 & 5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & & -\frac{17}{5} & -\frac{1}{5} \\ & 1 & \frac{4}{5} & \frac{2}{5} \\ & & & \end{pmatrix}$

故取 $x_3, x_4$ 为自由未知量. 令 $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 得基础解系

$$\eta_1 = \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 1 \\ 0 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{pmatrix}$$

故 $x = c_1\eta_1 + c_2\eta_2 = c_1 \begin{pmatrix} \frac{17}{5} \\ -\frac{4}{5} \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{pmatrix}$

(2) 由于 $\begin{pmatrix} 2 & 3 & 1 & 0 \\ -5 & 7 & 0 & 1 \end{pmatrix}$ 则取 $x_3, x_4$ 为自由未知量.

令 $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, 得 $\eta_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix} \quad \eta_2 = \begin{pmatrix} 0 \\ 1 \\ -3 \\ -7 \end{pmatrix}$

故 $x = c_1 \begin{pmatrix} 1 \\ 0 \\ -2 \\ 5 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -3 \\ -7 \end{pmatrix}$

(3) 由于 $\begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & 2 & 0 & -1 \\ 5 & 6 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & & -1 & -2 \\ & 1 & 1 & \frac{3}{2} \\ & & & \end{pmatrix}$ 则取 $x_3, x_4$ 为自由未知量

令 $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, 得 $\eta_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$ $\eta_2 = \begin{pmatrix} 4 \\ -3 \\ 0 \\ 2 \end{pmatrix}$

故 $x = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 4 \\ -3 \\ 0 \\ 2 \end{pmatrix}$

(4) 由于 $\begin{pmatrix} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 2 & -1 & -1 \\ -1 & 2 & -1 & -2 & -3 \\ 2 & -4 & 3 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & & & -3 \\ & & 1 & & 2 \\ & & & 1 & 2 \\ & & & & \end{pmatrix}$ 故取 $x_2, x_5$ 为自由未知量

令 $\begin{pmatrix} x_2 \\ x_5 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ 得 $\eta_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\eta_2 = \begin{pmatrix} 3 \\ 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}$

故 $x = c_1 \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 0 \\ -2 \\ -2 \\ 1 \end{pmatrix}$

(5) 由于系数矩阵为 $\begin{pmatrix} 1 & 1 & 1 & 1 \\ & & & \\ & & & \end{pmatrix}$, 则取 $x_2, x_3, x_4$ 为自由未知量.

则令 $\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ 得 $\eta_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

故 $x = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

(6) 同(5), 有 $x = c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_{n-1} \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

2.

(1) 对 I, 有 $\begin{pmatrix} 1 & 2 & 3 & -1 \\ 3 & 2 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & -\frac{1}{2} \end{pmatrix} \Rightarrow x_{\mathrm{I}} = a_1 \begin{pmatrix} -1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}$

对 II, 有 $\begin{pmatrix} 2 & 3 & 1 & 1 \\ 2 & 2 & 2 & -1 \\ 5 & 5 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & & & -\frac{5}{6} \\ & 1 & & \frac{7}{6} \\ & & 1 & -\frac{5}{6} \end{pmatrix} \Rightarrow x_{\mathrm{II}} = b \begin{pmatrix} -5 \\ -7 \\ 5 \\ 6 \end{pmatrix}$

故取 $a_1 = 5a_0$, $a_2 = 3a_0$ 时 $x_{\mathrm{I}} = a_0 \begin{pmatrix} 5 \\ -7 \\ 5 \\ 6 \end{pmatrix}$ 为公共解.

故全部非空公共解为 $x = c\begin{pmatrix}5\\-7\\5\\6\end{pmatrix}$

3.

依题意，系数行列式为0. 即

$$\begin{vmatrix}1&1&a\\-1&a&1\\1&-1&2\end{vmatrix}=\begin{vmatrix}1&1&a\\0&a+1&a+1\\0&-2&2-a\end{vmatrix}=(a+1)\begin{vmatrix}1&1&a\\&1&1\\&&4-a\end{vmatrix}=(a+1)(4-a)=0$$

当 $a=-1$ 时 $\begin{pmatrix}1&1&-1\\-1&-1&1\\1&-1&2\end{pmatrix}\to\begin{pmatrix}1&&\frac{1}{2}\\&1&-\frac{3}{2}\\&&\end{pmatrix}$

则通解为 $c\begin{pmatrix}-1\\3\\2\end{pmatrix}$

当 $a=4$ 时 $\begin{pmatrix}1&1&4\\-1&4&1\\1&-1&2\end{pmatrix}\to\begin{pmatrix}1&&3\\&1&1\\&&\end{pmatrix}$

则通解为 $c\begin{pmatrix}-3\\-1\\1\end{pmatrix}$

4.

依题意，系数行列式为 0. 即

$$\begin{vmatrix}a&-2&3\\1&a+2&3\\2&1&-1\end{vmatrix}=\begin{vmatrix}a+6&0&0\\7&a+5&0\\2&1&-1\end{vmatrix}=-(a+5)(a+6)=0$$

当 $a=-5$ 时，由 $\begin{pmatrix}-5&-3&3\\1&-3&3\\2&1&-1\end{pmatrix}\to\begin{pmatrix}1&0&0\\0&1&-1\\0&0&0\end{pmatrix}$ 知，通解为 $c\begin{pmatrix}0\\1\\1\end{pmatrix}$

当 $a=-6$ 时, 由 $\begin{pmatrix}-6 & -3 & 3\\ 1 & -4 & 3\\ 2 & 1 & -1\end{pmatrix} \to \begin{pmatrix}1 & 0 & -\frac{1}{4}\\ 0 & 1 & -\frac{7}{4}\\ 0 & 0 & 0\end{pmatrix}$ 知, 通解为 $c\begin{pmatrix}1\\7\\9\end{pmatrix}$

5.

$\begin{pmatrix}\xi_1\\ \xi_1+2\xi_2\\ \xi_1+2\xi_2+3\xi_3\end{pmatrix} = \begin{pmatrix}1 & & \\ 1 & 2 & \\ 1 & 2 & 3\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}$

由于 $\begin{vmatrix}1 & & \\ 1 & 2 & \\ 1 & 2 & 3\end{vmatrix} = 6 \neq 0$. 故 $r\begin{pmatrix}\xi_1\\ \xi_1+2\xi_2\\ \xi_1+2\xi_2+3\xi_3\end{pmatrix} = r\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}$

故 $\xi_1, \xi_1+2\xi_2, \xi_1+2\xi_2+3\xi_3$ 也是 $Ax=0$ 的基础解系.

6.

显然 $\beta, \alpha_1, \alpha_2, \ldots, \alpha_s$ 与 $\beta, \beta+\alpha_1, \ldots, \beta+\alpha_s$ 可以互相线性表示, 故

$$r(\beta, \alpha_1, \alpha_2, \ldots, \alpha_s) = r(\beta, \beta+\alpha_1, \ldots, \beta+\alpha_s)$$

又 $\beta$ 不是 $Ax=0$ 的解, 故 $\beta, \alpha_1, \ldots, \alpha_s$ 线性无关,

$\therefore r(\beta, \alpha_1, \ldots, \alpha_s) = s+1$

$\therefore r(\beta, \beta+\alpha_1, \ldots, \beta+\alpha_s) = s+1$

故 $r(\beta+\alpha_1, \ldots, \beta+\alpha_s) = s$

即 $\beta+\alpha_1, \beta+\alpha_2, \ldots, \beta+\alpha_s$ 线性无关.

7.

由于 $r(\begin{pmatrix}A & O\\ O & B\end{pmatrix} = r(A)+r(B))$

又 $r(\begin{pmatrix}A & O\\ O & B\end{pmatrix}) \le r(\begin{pmatrix}A & O\\ E & B\end{pmatrix})$ 且 $\begin{pmatrix}A & O\\ E & B\end{pmatrix} \xrightarrow{c_2-c_1B} \begin{pmatrix}A & -AB\\ E & O\end{pmatrix} \xrightarrow{r_1-Ar_2} \begin{pmatrix}O & -AB\\ E & O\end{pmatrix}$

故 $r(AB)+r(E) = r(\begin{pmatrix}O & -AB\\ E & O\end{pmatrix}) = r(\begin{pmatrix}A & O\\ E & B\end{pmatrix}) \ge r(\begin{pmatrix}A & O\\ O & B\end{pmatrix}) = r(A)+r(B)$

$\because AB=O$, 则 $r(AB)=0$, 故 $r(A)+r(B) \le n$.

8.

证明: 当 $r(A)=n$ 时, $|A| \neq 0$, 则 $|A^*| = |A|^{n-1} \neq 0$, 故 $r(A^*)=n$.

当 $r(A) = n-1$ 时, 则 $A$ 中至少有一个 $n-1$ 阶子式不为0, 故 $A^*$ 中至少有1个不为0的元素, 故 $r(A^*) \geq 1$

又 $AA^* = |A|E = O$, 故 $r(A) + r(A^*) \leq n$ 即 $r(A^*) \leq 1$

故 $r(A^*) = 1$

当 $r(A) < n-1$ 时, $A$ 的每个 $n-1$ 阶子式均为零, 故 $A^* = O$

故 $r(A^*) = 0$

9.

证明: 假设 $A$ 不可逆

则由 $AB = AC \Rightarrow A(B-C) = O$

则 $(B-C)^{\mathrm{T}}A^{\mathrm{T}} = O$, 则 $A^{\mathrm{T}}$ 是 $(B-C)^{\mathrm{T}}x = O$ 的解.

由于 $A$ 不可逆, 则 $r(A^{\mathrm{T}}) < n$

又 $r(A^{\mathrm{T}}) = n - r(B-C)$

故 $r(B-C) > 0$ 则 $B \neq C$, 矛盾!

故 $A$ 可逆

## (B)

1.

设所求齐次方程组为 $Ax = O$

由 $n - r(A) = 2$, 且 $n = 4$, 知 $r(A) = 2$

又 $A(\alpha_1, \alpha_2) = 0$, 则 $\begin{pmatrix} \alpha_1^{\mathrm{T}} \\ \alpha_2^{\mathrm{T}} \end{pmatrix} A^{\mathrm{T}} = O$

考虑 $Bx = \begin{pmatrix} \alpha_1^{\mathrm{T}} \\ \alpha_2^{\mathrm{T}} \end{pmatrix} x = O$, 由 $n - r(B) = 4 - 2 = 2$, 知 $Bx = O$ 的基础解系是 $A^{\mathrm{T}}$ 的列向量

$$B = \begin{pmatrix} \alpha_1^{\mathrm{T}} \\ \alpha_2^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 2 & 1 & -5 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -6 & -1 \\ 0 & 1 & 7 & 2 \end{pmatrix}$$

则基础解系为 $\begin{pmatrix} 6 \\ -7 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$

故 $A$ 可取 $\begin{pmatrix} 6 & -7 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{pmatrix}$

则所求齐次方程组为 $\begin{cases} 6x_1 - 7x_2 + x_3 = 0 \\ x_1 - 2x_2 + x_4 = 0 \end{cases}$

2.

由 $\alpha_4 = \alpha_1 + 2\alpha_2 - \alpha_3$ 知 $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)\begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \end{pmatrix} = 0$

即 $\begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \end{pmatrix}$ 是一组解.

又由于 $\alpha_2, \alpha_3, \alpha_4$ 线性无关, 则 $r(A) = 3$

故基础解系的解的个数为 $n - r(A) = 1$

从而 $Ax = O$ 的通解为 $c\begin{pmatrix} 1 \\ 2 \\ -1 \\ -1 \end{pmatrix}$

3.

基础解系中向量个数为 $n - r(A) = 1$

又由于 $A$ 的各行元素之和均为0, 则

$$A \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \alpha_1 + \alpha_2 + \cdots + \alpha_n = 0,$$

故 $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ 是特解

$Ax = 0$ 的通解为 $c\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$

4.

(1) 考虑

$$A = \begin{vmatrix} a_1+b & a_2 & \cdots & a_n \\ a_1 & a_2+b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n+b \end{vmatrix}$$

当 $b = 0$ 时, 行列式所有列对应成比例, $A = 0$

当 $b \neq 0$ 时

$$A = \begin{vmatrix} a_1+b & a_2 & \cdots & a_n \\ a_1 & a_2+b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n+b \end{vmatrix} = \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ 0 & a_1+b & a_2 & \cdots & a_n \\ 0 & a_1 & a_2+b & \cdots & a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_1 & a_2 & \cdots & a_n+b \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a_1 & a_2 & \cdots & a_n \\ -1 & b & 0 & \cdots & 0 \\ -1 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & b \end{vmatrix} = \begin{vmatrix} 1 & \frac{a_1}{b} & \frac{a_2}{b} & \cdots & \frac{a_n}{b} \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{vmatrix} = 1 + \frac{\sum\limits_{i=1}^{n} a_i}{b}$$

则当 $\sum\limits_{i=1}^{n} a_i = -b$ 时, $A = 0$,

综上, 当 $\sum\limits_{i=1}^{n} a_i \neq -b$ 且 $b \neq 0$ 时, 方程组仅有零解.



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(2) 当 $b=0$ 时，则 $\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \to \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

则不妨 $a_1 \neq 0$，通解为

$$c_1 \begin{pmatrix} -a_2 \\ a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -a_3 \\ 0 \\ a_1 \\ \vdots \\ 0 \end{pmatrix} + \cdots + c_{n-1} \begin{pmatrix} -a_n \\ 0 \\ 0 \\ \vdots \\ a_1 \end{pmatrix}$$

当 $\sum\limits_{i=1}^{n} a_i = -b$ 时，则

$$\begin{pmatrix} a_1+b & a_2 & \cdots & a_n \\ a_1 & a_2+b & \cdots & a_n \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_n+b \end{pmatrix} \to \begin{pmatrix} a & a_2 & \cdots & a_n & \\ -b & b & & & \\ -b & & b & & \\ \vdots & & & \ddots & \\ -b & & & & b \end{pmatrix} \to \begin{pmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \\ & & & & 0 \end{pmatrix}$$

则通解为 $c \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

## (C)

1.

(1) 由 $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$

则 $x_3, x_4$ 为自由未知量，取 $\begin{pmatrix} x_3 \\ x_4 \end{pmatrix}$ 为 $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



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则 $\eta_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\eta_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ 则 $x = c_1 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

(2) 依题意, 由 $\begin{pmatrix} -c_2 \\ c_2 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -k_2 \\ k_1 + 2k_2 \\ k_1 + 2k_2 \\ k_2 \end{pmatrix} \Rightarrow \begin{cases} c_1 = c_2 \\ c_2 = k_2 \\ k_1 = -k_2 \end{cases}$

则 I 与 II 有公共解 $c \begin{pmatrix} -1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

2.

记 $A = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ $B = (\beta_1, \beta_2, \ldots, \beta_n)$

若 $(\alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n)x = 0$ 有解,

此即非零公共解.

由于 $(A\ B)$ 的列向量组由 $A$, $B$ 的列向量扩充而成

故 $(A\ B)$ 的列向量组可以由 $A$ 的列极大线性无关组, $B$ 的列极大无关组线性表示, 则

$$r(A\ B) \leq r(A) + r(B) < n$$

从而 $(A\ B)x = 0$ 有非零解,

即 $Ax = 0$ 与 $Bx = 0$ 有非零公共解.

# 习题4.2

## (A)

1.

(1) 由 $\left(\begin{array}{cccc:c} 2 & 1 & 3 & 3 & 1 \\ 1 & 1 & 1 & 2 & 0 \\ 1 & -2 & 4 & 1 & 4 \end{array}\right) \to \left(\begin{array}{cccc:c} 1 & 0 & 2 & 0 & \frac{1}{2} \\ 0 & 1 & -1 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right)$

则取自由未知量为 0, 得特解 $\eta^* = \frac{1}{2}\begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}$

又基础解系为 $\eta = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, 故解为 $x = \frac{1}{2}\begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(2) 由 $\left(\begin{array}{ccc:c} 1 & 1 & -1 & -1 \\ 2 & -5 & 3 & 2 \\ 7 & -7 & 2 & 1 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 0 & 0 & -\frac{3}{7} \\ 0 & 1 & 0 & -\frac{4}{7} \\ 0 & 0 & 1 & 0 \end{array}\right)$

故解为 $x = \frac{1}{7}\begin{pmatrix} -3 \\ -4 \\ 0 \end{pmatrix}$

(3) 由 $\left(\begin{array}{ccc:c} 1 & 0 & 3 & 2 \\ -1 & 3 & 0 & 1 \\ 2 & 1 & 7 & 5 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 0 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

自由未知量为 $x_3$

取 $x_3 = 0$, 得 $\eta^* = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ 取 $x_3 = 1$, 得 $\eta = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$



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则 $x=\begin{pmatrix}2\\1\\0\end{pmatrix}+c\begin{pmatrix}-3\\-1\\1\end{pmatrix}$

(4) 由 $\left(\begin{array}{cccc:c}1&0&3&1&2\\1&-3&0&1&-1\\2&1&7&2&5\\4&2&14&0&6\end{array}\right)\to\left(\begin{array}{cccc:c}1&0&3&0&1\\0&1&1&0&1\\0&0&0&1&1\\0&0&0&0&0\end{array}\right)$

自由未知量为 $x_3$

取 $x_3=0$, 得 $\eta^*=\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$ 取 $x_3=1$, 得 $\eta=\begin{pmatrix}-3\\-1\\1\\0\end{pmatrix}$

故 $x=\begin{pmatrix}1\\1\\0\\1\end{pmatrix}+c\begin{pmatrix}-3\\-1\\1\\0\end{pmatrix}$

2.

由 $\left(\begin{array}{cccc:c}1&3&2&1&1\\0&1&a&-a&-1\\1&2&0&3&1\end{array}\right)\to\left(\begin{array}{cccc:c}1&3&2&1&1\\0&1&a&-a&-1\\0&0&a-2&2-a&0\end{array}\right)$

则当 $a=2$ 时, 无解.

当 $a\neq 2$ 时, 通解为 $\dfrac{1}{a-2}\begin{pmatrix}7a-10\\2-2a\\1\\0\end{pmatrix}+c\begin{pmatrix}-3\\0\\1\\1\end{pmatrix}$

3.

系数行列式

$$A = \begin{vmatrix} 1 & 1 & a \\ -1 & a & 1 \\ 1 & -1 & 2 \end{vmatrix} = (a+1)\begin{vmatrix} 1 & 1 & a \\ & 1 & 1 \\ & & 4-a \end{vmatrix} = (a+1)(4-a)$$

则当 $a \neq 4$ 且 $a \neq -1$ 时, 方程组为唯一解.

当 $a = 4$ 时, 方程组有无穷多解, 由于

$$\left(\begin{array}{ccc:c} 1 & 1 & 4 & 4 \\ -1 & 4 & 1 & 16 \\ 1 & -1 & 2 & -4 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

故通解为 $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} + c\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$

当 $a = -1$ 时, 由于 $\left(\begin{array}{ccc:c} 1 & 1 & -1 & 4 \\ -1 & -1 & 1 & -1 \\ 1 & -1 & 2 & -4 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

故为方程组无解.

4.

由 $\left(\begin{array}{ccccc:c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 3 & 2 & 1 & 1 & -3 & a \\ 5 & 4 & 3 & 3 & -1 & b \end{array}\right) \to \left(\begin{array}{ccccc:c} 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 2 & 6 & 3 \\ & & & & & a \\ & & & & & a-2 \end{array}\right)$

则由方程组有解得 $\begin{cases} a = 0 \\ b = 2 \end{cases}$

则 $\to \left(\begin{array}{ccccc:c} 1 & 0 & -1 & -1 & -5 & -2 \\ 0 & 1 & 2 & 2 & 6 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ 自由未知量为 $x_3, x_4, x_5$,



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取 $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$，得 $\eta^* = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

取 $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

得基础解系 $\eta_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \eta_3 = \begin{pmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

则 $x = \begin{pmatrix} -2 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

5.

考虑 $A = (\alpha_1, \alpha_2, \alpha_3)$，研究 $Ax = \beta$ 的解.

(1) 即 $Ax = \beta$ 有唯一解，则 $|A| \neq 0$

即 $\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ a & -2 & 1 \\ 10 & 5 & 4 \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -2-\frac{a}{2} & 1-\frac{a}{2} \\ 0 & 0 & -1 \end{vmatrix} = (2 + \dfrac{a}{2}) \neq 0$

则 $a \neq -4$. 此时 $\beta$ 由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示，且表示法唯一.

(2) 即 $Ax = \beta$ 无解，则 $|A| = 0 \Rightarrow a = -4$

由于 $\left(\begin{array}{ccc:c} -4 & -2 & 1 & 1 \\ 2 & 1 & 1 & b \\ 10 & 5 & 4 & c \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{b}{2} \\ 0 & 0 & 3 & 1+2b \\ 0 & 0 & -1 & c-5b \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{b}{2} \\ 0 & 0 & -1 & c-5b \\ 0 & 0 & 0 & 3c-13b+1 \end{array}\right)$

故当 $3c - 13b + 1 \neq 0$，$\beta$ 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示.（此时 $a = -4$）

(3) 即 $Ax=\beta$ 有无穷多解, 则 $a=-4$ 且 $3c-13b+1=0$

则 $\left(\begin{array}{ccc:c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{b}{2} \\ 0 & 0 & -1 & c-5b \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & \frac{1}{2} & 0 & \frac{b+1}{6} \\ 0 & 0 & 1 & \frac{1+2b}{3} \\ 0 & 0 & 0 & 0 \end{array}\right)$

自由未知量为 $x_2$.

令 $x_2=0 \Rightarrow \eta^*=\begin{pmatrix} \frac{b-1}{6} \\ 0 \\ \frac{1+2b}{3} \end{pmatrix}$

令 $x_2=1 \Rightarrow \eta=\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

则 $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\eta^*+k\eta=\begin{pmatrix} \frac{b-1-3k}{6} \\ k \\ \frac{2b+1}{3} \end{pmatrix}$

故 $\beta=\dfrac{b-1-3k}{6}\alpha_1+k\alpha_2+\dfrac{2b+1}{3}\alpha_3$

6. C

A 为特解 + 通解,

B 中 $\dfrac{1}{2}\eta_1+k_1\xi_1$ 是 $Ax=\dfrac{1}{2}\beta$ 的解, $\dfrac{1}{2}\eta_2+k_2(\xi_1+\xi_2)$ 是 $Ax+\dfrac{1}{2}\beta$ 的解.

故 $k_1\xi_1+k_2(\xi_1+\eta_2)+k_2(\eta_1+\eta_2)$ 是 $Ax=\beta$ 的解.

C 应是 $Ax=2\beta$ 的解, D 同 B 理可证.

7.

由 $\left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 2 & 3 & a+2 & 3 \\ 1 & a & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -1 & a & 1 \\ 0 & a-2 & -3 & -1 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 2 & 1 & 1 \\ 0 & -1 & a & 1 \\ 0 & 0 & (a-3)(a+1) & a-3 \end{array}\right)$

则 $a=-1$ 时无解.

8.

证明: $\lambda_1\eta_1+\lambda_2\eta_2+\cdots+\lambda_s\eta_s=(1-\lambda_2-\lambda_3-\cdots-\lambda_s)\eta_1+\lambda_2\eta_2+\cdots+\lambda_s\eta_s=\eta_1+\sum\limits_{i=2}^{s}\lambda_i(\eta_i-\eta_1)$

因为 $\eta_i-\eta_1$ 是导出组的解, 故 $\sum\limits_{i=2}^{s}\lambda_i(\eta_i-\eta_1)$ 也是导出组的解.

又 $\eta_1$ 是原方程一个特解, 故 $\lambda_1\eta_1+\lambda_2\eta_2+\cdots+\lambda_s\eta_s$ 也是 $Ax=\beta$ 的解.

9.

(1) 设 $k_1\xi_1+k_2\xi_2+\cdots+k_{n-r}\xi_{n-r}+k_{n-r+1}\eta=0$

则 $k_1A\xi_1+k_2A\xi_2+\cdots+k_{n-r}A\xi_{n-r}+k_{n-r+1}A\eta=0$

即 $k_{n-r+1}\beta=0$

故 $k_{n-r+1}=0$

则 $\xi_1,\xi_2,\ldots,\xi_{n-r},\eta$ 线性无关.

(2) 设 $k_1(\xi_1+\eta)+k_2(\xi_2+\eta)+\cdots+k_{n-r}(\xi_{n-r}+\eta)+k_{n-r+1}\eta=0$

即 $k_1\xi_1+k_2\xi_2+\cdots+k_{n-r}\xi_{n-r}+(k_1+k_2+\cdots+k_{n-r+1})\eta=0$

则由(1)知 $k_1+k_2+\cdots+k_{n-r+1}=0$,

则 $k_1\xi_1+k_2\xi_2+\cdots+k_{n-r}\xi_{n-r}=0$ 又 $\because\xi_1,\xi_2,\ldots,\xi_{n-r}$ 线性无关

故 $k_1=k_2=\cdots=k_{n-r}=0$

$\Rightarrow k_{n-r+1}=0$

故 $\xi_1+\eta,\xi_2+\eta,\ldots,\xi_{n-r}+\eta,\eta$ 线性无关.

## (B)

1.

由于 $$\left(\begin{array}{cccc:c}1&1&-2&3&0\\2&1&-6&4&-1\\3&2&a&7&-1\\1&-1&-b&-2&b\end{array}\right)\to\left(\begin{array}{cccc:c}1&1&-2&3&0\\0&-1&-2&-2&-1\\0&-1&a+6&-2&-1\\0&-2&-4&-4&b\end{array}\right)\to\left(\begin{array}{cccc:c}1&0&-4&1&-1\\0&1&2&2&1\\0&0&a+8&0&0\\0&0&0&0&b+2\end{array}\right)$$

故当 $b=-2$ 时方程组有解.

(1) 当 $b=-2$ 且 $a=-8$ 时, 自由未知量为 $x_1,x_4$,

取 $\begin{pmatrix}x_1\\x_4\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, 得特解 $\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$

取 $\begin{pmatrix} x_1 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, 得基础解系

$$\eta_1 = \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$$

故此时 $x = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$

2.

(1) $X$ 应为 3 行 2 列矩阵, 才满足矩阵乘法.

(2) 由 $\left(\begin{array}{ccc:c} 1 & 1 & 3 & 1 \\ 2 & 1 & 4 & s \\ 1 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 0 & 1 & s-1 \\ 0 & 1 & 2 & 2-s \\ 0 & 0 & 0 & 1-s \end{array}\right)$ 知当 $s = 1$ 时, $\begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ s \\ 0 \end{pmatrix}$ 有解.

此时 $x_3$ 为自由未知量,

取 $x_3 = 0$, 得 $\eta^* = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, 又基础解系 $\eta = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

故通解为 $x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

由 $\left(\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 2 & 1 & 4 & 1 \\ 1 & 0 & 1 & t \end{array}\right) \to \left(\begin{array}{ccc:c} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & t+1 \end{array}\right)$ 知当 $t = -1$ 时, $\begin{pmatrix} 2 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ t \end{pmatrix}$ 有解.

此时 $x_3$ 为自由未知量, 取 $x_3 = 0$, 可得 $\eta^* = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$, 又基础解系 $\eta = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

故通解为 $x = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

(3) $\begin{pmatrix} 0 & -1 \\ 1 & 3 \\ 0 & 0 \end{pmatrix}$

3.

(1) 由 $\left(\begin{array}{cccc:c} 1 & 1 & 0 & -2 & -6 \\ 4 & -1 & -1 & -1 & 1 \\ 3 & -1 & -1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{cccc:c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right)$

取 $x_4$ 为自由未知量, 令 $x_4 = 0$, 得特解 $\eta^* = \begin{pmatrix} -2 \\ -4 \\ -5 \\ 0 \end{pmatrix}$

令 $x_4 = 1$, 得基础解系 $\eta = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$

则通解为 $\begin{pmatrix} -2 \\ -4 \\ -5 \\ 0 \end{pmatrix} + c \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$

(2) 由

$$\left(\begin{array}{cccc:c} 1 & m & -1 & -1 & -5 \\ 0 & n & -1 & -2 & -11 \\ 0 & 0 & 1 & -2 & -t+1 \end{array}\right) \to \left(\begin{array}{cccc:c} 1 & 0 & 0 & -3+\frac{4m}{n} & -7+\frac{24m}{n} \\ 0 & 1 & 0 & -\frac{4}{n} & -\frac{12}{n}+t \\ 0 & 0 & 1 & -2 & -t+1 \end{array}\right)$$



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则

$$\left(\begin{array}{cccc:c} 1 & 0 & 0 & -3+\frac{4m}{n} & -7+\frac{24m}{n} \\ 0 & 1 & 0 & -\frac{4}{n} & -\frac{12}{n}+t \\ 0 & 0 & 1 & -2 & -t+1 \end{array}\right) = \left(\begin{array}{cccc:c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -1 & -4 \\ 0 & 0 & 1 & -2 & -5 \end{array}\right)$$

$\Rightarrow m=2,\ n=4,\ t=6.$

4.

$A=\alpha\beta^{\mathrm{T}}=\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}1 & \frac{1}{2} & 0\end{pmatrix}=\begin{pmatrix}1 & \frac{1}{2} & 0\\ 2 & 1 & 0\\ 1 & \frac{1}{2} & 0\end{pmatrix}$

$B=\beta^{\mathrm{T}}\alpha=\begin{pmatrix}1 & \frac{1}{2} & 0\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix}=2$

则 $2B^2A^2X=A^4X+B^4X+Y$

$\Leftrightarrow 8A^2X-A^4X-16X=Y$

又 $A^2=\alpha\beta^{\mathrm{T}}\alpha\beta^{\mathrm{T}}=2\alpha\beta^{\mathrm{T}}=2A$, 故 $A^4=2^3A=8A$

则 $(8A-16E)X=Y$, 故 X 是该方程组的解.

又 $8A-16E=\begin{pmatrix}-8 & 4 & 0\\ 16 & -8 & 0\\ 8 & 4 & -16\end{pmatrix}$

由 $\left(\begin{array}{ccc:c} -8 & 4 & 0 & 0\\ 16 & -8 & 0 & 0\\ 8 & 4 & -16 & 8\end{array}\right)\to\left(\begin{array}{ccc:c} 1 & 0 & -1 & \frac{1}{2}\\ 0 & 1 & -2 & 1\\ 0 & 0 & 0 & 0\end{array}\right)$, 自由未知量为 $x_3$.

取 $x_3=0$, 则 $\eta^*=\begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix}$, 取 $x_3=1$, 得基础解系 $\eta=\begin{pmatrix}1\\2\\1\end{pmatrix}$

则 $X=\begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix}+c\begin{pmatrix}1\\2\\1\end{pmatrix}$

5.

$$A = (\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = \begin{pmatrix} 1 & 1 & 1 & 1 & 2 & 2 \\ 0 & 1 & -1 & 2 & 1 & 1 \\ 2 & 3 & a+2 & a+3 & a+6 & a+4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & 2 & 1 & 1 \\ 0 & 0 & a+1 & a-1 & a+1 & a-1 \end{pmatrix}$$

若 $a = -1$, 则 $\beta_1, \beta_3$ 不能由 $\alpha_1, \alpha_2, \alpha_3$ 线性表示, 此时向量组 I 与向量组 II 不等价;

若 $a \neq -1$, 则 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, $\beta_1, \beta_2, \beta_3$ 线性无关, 此时 I 与 II 等价.

6.

依题意, $\eta_{n-r+1} - \eta_1, \eta_{n-r} - \eta_1, \ldots, \eta_s - \eta_1$ 是 $Ax = 0$ 的 $r$ 个线性无关解

则 $Ax = \beta$ 的通解为

$$\begin{aligned} x &= \eta_1 + k_1(\eta_{n-r+1} - \eta_1) + \cdots + k_r(\eta_s - \eta_1) \\ &= \left(1 - \sum_{i=1}^{r} k_i\right) \eta_1 + k_1\eta_2 + k_2\eta_3 + \cdots + k_r\eta_{n-r+1} \end{aligned}$$

记 $\lambda_1 = 1 - \sum\limits_{i=1}^{r} k_i$, $\lambda_i = k_{r+2-i}$, 则 $\sum\limits_{i=1}^{n-r+1} \lambda_i = 1$

故 $Ax = \beta$ 的通解为 $\lambda_1\eta_1 + \lambda_2\eta_2 + \cdots + \lambda_{n-r+1}\eta_{n-r+1}$, 其中 $\lambda_i$ 满足 $\sum\limits_{i=1}^{n-r+1} \lambda_i = 1$.

# 习题5.1

## (A)

1.

(1) $A$ 的特征多项式为

$$f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$$

$\therefore A$ 有两个特征值 $\lambda_1 = 1$, $\lambda_2 = -1$

对于特征值 $\lambda_1 = 1$, 解 $(\lambda_1 E - A)x = 0$, 即 $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$

可得一个特征向量为 $(1,1)^{\mathrm{T}}$. (全部为 $k\begin{pmatrix}1\\1\end{pmatrix}$)

对于特征值 $\lambda_2=-1$, 解 $(\lambda_2 E-A)x=0$, 即 $\begin{pmatrix}-1&-1\\-1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0$

得一个特征向量为 $(1,-1)^{\mathrm{T}}$. (全部为 $k\begin{pmatrix}1\\-1\end{pmatrix}$)

(2) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-11&-25\\4&\lambda+9\end{vmatrix}=(\lambda-1)^2$,

特征值为 $1,1$

特征值 1 的特征向量由 $\begin{pmatrix}-10&-25\\4&10\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=0$

得 $k(-5,2)^{\mathrm{T}}$.

(3) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda+2&-1\\-5&\lambda-2\end{vmatrix}=(\lambda-3)(\lambda+3)$,

特征值为 $3,-3$

特征值 $-3$ 的特征向量为 $k_1(1,-1)^{\mathrm{T}}$

特征值 3 的特征向量为 $k_2(1,5)^{\mathrm{T}}$

(4) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda+1&-2\\-8&\lambda+1\end{vmatrix}=(\lambda+5)(\lambda-3)$,

特征值为 $-5,3$

特征值为 $-5$ 的特征向量为 $k_1(1,-2)^{\mathrm{T}}$

特征值为 3 的特征向量为 $k_2(1,2)^{\mathrm{T}}$

2.

(1) $A$ 的特征多项式为 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-3&-6&-6\\0&\lambda-2&0\\3&12&\lambda+6\end{vmatrix}=(\lambda+3)(\lambda-2)\lambda$,

特征值 $-3,0,2$.

特征值 $-3$ 的特征向量为 $k_1(1,0,-1)^{\mathrm{T}}$

特征值 0 的特征向量为 $k_2(-2,0,1)^{\mathrm{T}}$

特征值 2 的特征向量为 $k_3(2,-5,3)^{\mathrm{T}}$

(2) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-3 & 2 & 0\\ 1 & \lambda-3 & 1\\ 5 & -7 & \lambda+1\end{vmatrix}=(\lambda-1)(\lambda-2)^2,$

特征值 $1,2,2$

特征值 1 的特征向量为 $k_1(1,1,1)^{\mathrm{T}}$

特征值 2 的特征向量为 $k_2(-2,-1,1)^{\mathrm{T}}$

(3) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & -1 & -1\\ -1 & \lambda-1 & -1\\ -1 & -1 & \lambda-1\end{vmatrix}=\lambda^2(\lambda-3),$

特征值 $0,0,3$

特征值 0 的特征向量为 $k_1(-1,1,0)^{\mathrm{T}}+k_2(-1,0,1)^{\mathrm{T}}$

特征值 3 的特征向量为 $k_3(1,1,1)^{\mathrm{T}}$

(4) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-4 & 2 & 1\\ -5 & \lambda+2 & 1\\ 2 & -1 & \lambda-1\end{vmatrix}=(\lambda-1)^3,$

特征值 $1,1,1$

特征值 1 的向量为 $k(-1,-2,1)^{\mathrm{T}}$

(5) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-2 & 8 & 4\\ 2 & \lambda-2 & -2\\ -7 & 14 & \lambda+9\end{vmatrix}=(\lambda+1)(\lambda+2)^2,$

特征值 $-1,-2,-2$

特征值 $-1$ 的特征向量为 $k_1(-4,2,7)^{\mathrm{T}}$

特征值 $-2$ 的特征向量为 $k_2(2,1,0)^{\mathrm{T}}+k_3(1,0,1)^{\mathrm{T}}$

(6) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-23 & -10 & -16\\ 8 & \lambda+2 & 6\\ 27 & 11 & \lambda+19\end{vmatrix}=(\lambda-1)^2\lambda,$

特征值 $0,1,1$

特征值 $0$ 的特征向量为 $k_1(-14,5,17)^{\mathrm{T}}$

特征值 $1$ 的特征向量为 $k_2(-6,2,7)^{\mathrm{T}}$.

3.

(1) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & -1 & 0 & 0\\ 0 & \lambda-1 & -1 & 0\\ 0 & 0 & \lambda-1 & 0\\ 0 & 0 & 0 & \lambda-1\end{vmatrix}=(\lambda-1)^4,$

特征值 $1,1,1,1$

特征值 $1$ 的特征向量为 $k_1(0,0,0,1)^{\mathrm{T}}+k_2(1,0,0,0)^{\mathrm{T}}$

(2) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & -1 & -1 & -1\\ -1 & \lambda-1 & 1 & 1\\ -1 & 1 & \lambda-1 & 1\\ -1 & 1 & 1 & \lambda-1\end{vmatrix}=(\lambda+2)(\lambda-2)^3,$

特征值 $-2,2,2,2$

特征值为 $-2$ 的特征向量为 $k_1(-1,1,1,1)^{\mathrm{T}}$

特征值为 $2$ 的特征向量为 $k_2(1,1,0,0)^{\mathrm{T}}+k_3(1,0,1,0)^{\mathrm{T}}+k_4(1,0,0,1)^{\mathrm{T}}$

4.

证明: 由 $A\alpha=\lambda\alpha$, 左乘 $A^{-1}$, 有

$$A^{-1}(A\alpha)=A^{-1}(\lambda\alpha)$$

$$\Longrightarrow\ \lambda A^{-1}\alpha=\alpha\Rightarrow A^{-1}\alpha=\frac{1}{\lambda}\alpha$$

则 $\dfrac{1}{\lambda}$ 为 $A^{-1}$ 的特征值.

5.

(1) 证明: $\because A^2\alpha=A(A\alpha)=A(\lambda\alpha)=\lambda(A\alpha)=\lambda^2\alpha$

设当 $n=k$ 时, $A^k\alpha=\lambda^k\alpha$

当 $n=k+1$ 时, $A^{k+1}\alpha=A(A^k\alpha)=A(\lambda^k\alpha)=\lambda^k A\alpha=\lambda^{k+1}\alpha$

故 $A^n\alpha=\lambda^n\alpha$. $\because A^n=O$, 则 $\lambda^n\alpha=0$

又 $\alpha\neq 0$, $\therefore \lambda^n=0$ 从而 $\lambda=0$

(2) 证明: 设 $A\alpha=\lambda\alpha$, 同时左乘 $A$, 有

$$A^2\alpha=\lambda A\alpha=\lambda^2\alpha$$

又 $A^2=A$, 故 $\lambda\alpha=A\alpha=A^2\alpha=\lambda^2\alpha$

$\Rightarrow \lambda(\lambda-1)\alpha=0$ 又 $\alpha\neq 0$

$\Rightarrow \lambda=0$ 或 $\lambda=1$

(3) 证明: 设 $A\alpha=\lambda\alpha$, 同时左乘 $A$, 有

$$A^2\alpha=\lambda A\alpha=\lambda^2\alpha$$

又 $\alpha=E\alpha=A^2\alpha$

$\Rightarrow (\lambda^2-1)\alpha=0$ 又 $\alpha\neq 0$

$\Rightarrow \lambda=1$ 或 $-1$

6.

证明: 设 $A\alpha=\lambda\alpha$

则 $(kE+A)\alpha=k\alpha+A\alpha=k\alpha+\lambda\alpha=(k+\lambda)\alpha$

$\therefore k+\lambda$ 是 $kE+A$ 的特征值

7.

$$f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & -1 & -1\\ -1 & \lambda-1 & -1\\ -1 & -1 & \lambda-1\end{vmatrix}=\lambda^2(\lambda-3),$$

特征值为 $0,0,3$

特征值 0 的特征向量为 $k_1(-1,1,0)^{\mathrm{T}}+l(-1,0,1)^{\mathrm{T}}$

特征值 3 的特征向量为 $k_2(1,1,1)^{\mathrm{T}}$

故特征值 0 的特征子空间为 $\left\{k\alpha+l\beta \mid \alpha=(-1,1,0)^{\mathrm{T}},\beta=(-1,0,1)^{\mathrm{T}},k,l\in\mathbf{R}\right\}$

特征值 3 的特征子空间为 $\left\{k\alpha \mid \alpha=(1,1,1)^{\mathrm{T}},k\in\mathbf{R}\right\}$.

8.

依题意，$A\alpha=\lambda\alpha$ 有一个特征向量为 $(1,-2,1)^{\mathrm{T}}$.

则

$$\begin{pmatrix} k & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & k \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}=\lambda\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\Rightarrow\begin{pmatrix} k-2 \\ -2 \\ k-2 \end{pmatrix}=\begin{pmatrix} \lambda_0 \\ -2 \\ \lambda_0 \end{pmatrix}$$

则 $\lambda_0=k-2$, 则

$$|\lambda_0E-A|=\begin{vmatrix} -2 & -1 & 0 \\ -1 & k-4 & -1 \\ 0 & -1 & -2 \end{vmatrix}=0 \quad \text{得 } k=3$$

$$\text{则}f_A(\lambda)=|\lambda E-A|=\begin{vmatrix} \lambda-3 & -1 & 0 \\ -1 & \lambda-2 & -1 \\ 0 & -1 & \lambda-3 \end{vmatrix}=(\lambda-3)\begin{vmatrix} \lambda-2 & 1 \\ -1 & \lambda-3 \end{vmatrix}+\begin{vmatrix} -1 & -1 \\ 0 & \lambda-3 \end{vmatrix}$$

$$=(\lambda-3)(\lambda-1)(\lambda-4)$$

故 $A$ 的特征值为 $3,1,4$.

9.

$\because A$ 可逆，则 $|A|\neq 0$, $\therefore \lambda\neq 0$

则由 $A^{-1}\alpha=\lambda\alpha\Rightarrow A\alpha=\dfrac{1}{\lambda}\alpha$

故 $\alpha$ 也是 $A$ 的特征向量，则

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}=\lambda\begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}\Leftrightarrow\begin{pmatrix} 3+k \\ 2k+2 \\ 3+k \end{pmatrix}=\begin{pmatrix} \lambda \\ \lambda k \\ \lambda \end{pmatrix}\Rightarrow\begin{cases} 3+k=\lambda \\ 2k+2=\lambda k \\ 3+k=\lambda \end{cases}$$

得 $k=1$ 或 $-2$.

10.

(1) 由第5题(1)的证明知由 $A\alpha=\lambda\alpha$ 可得 $A^m\alpha=\lambda^m\alpha$

$$\begin{aligned}\text{则} f(A)\alpha &= (A^3-2A^2-A+2E)\alpha \\ &= A^3\alpha-2A^2\alpha-A\alpha+2\alpha \\ &= (\lambda^3-2\lambda^2-\lambda+2)\alpha \\ &= f(\lambda)\alpha\end{aligned}$$

故 $f(\lambda)$ 是 $f(A)$ 的特征值.

(2) 由10.(1)的公式可得

1.(1) $0,0$ (2) $0,0$ (3) $-40,8$ (4) $-168,8$

11.

证明: 同10.(1)有 $\varphi(A)\alpha=(a_mA^m+a_{m-1}A^{m-1}+\cdots+a_0E)\alpha=(a_m\lambda^m+\cdots+a_0)\alpha=\varphi(\lambda)\alpha$
即 $\varphi(\lambda)$ 是 $\varphi(A)$ 的特征值.

## (B)

1.

证明: 设 $A=\begin{pmatrix}a & b\\ b & a\end{pmatrix}$

则 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-a & -b\\ -b & \lambda-a\end{vmatrix}=(\lambda-a-b)\begin{vmatrix}1 & -b\\ 1 & \lambda-a\end{vmatrix}=(\lambda-a-b)(\lambda-a+b)$

故 $A$ 一定有两个实特征值 $a+b$, $a-b$.

2.

证明: $\because |\lambda E-A^{\mathrm{T}}|=|(\lambda E-A)^{\mathrm{T}}|=|\lambda E-A|$,
$\therefore A$ 与 $A^{\mathrm{T}}$ 有相同的特征多项式

3. 4.

只考虑 $n$ 维向量. 首先, 由于

$$r(\alpha\beta^{\mathrm{T}})\le\min\left\{r(\alpha),r(\beta^{\mathrm{T}})\right\}=1$$

故矩阵 $\alpha\beta^{\mathrm{T}}$ 必有 $\lambda=0$ 的特征值. 记 $A=\alpha\beta^{\mathrm{T}}$
则 $\lambda=0$ 的 $Ax=0$ 所含的向量个数

$$n-r(A)\ge n-1$$

则 $\lambda$ 的为0的代数重数至少有 $n-1$ 个,
故 $A$ 最多只有1个不为0的特征值. 记 $k=\beta^{\mathrm{T}}\alpha$

又

$$(\alpha\beta^{\mathrm{T}})\alpha = \alpha \cdot k = k\alpha$$

则 $k = \beta^{\mathrm{T}}\alpha$ 是一个特征值.

则 $A$ 的特征多项式为 $\lambda^{n-1}(\lambda - \beta^{\mathrm{T}}\alpha)$

特征值为 $\beta^{\mathrm{T}}\alpha$, 0 ($n-1$ 重)

5.

(1) $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 4 & 1 & 1 \\ 12 & \lambda - 1 & -5 \\ -4 & 2 & \lambda \end{vmatrix} = \lambda(\lambda-2)(\lambda-3)$

(2) 特征值为 $0, 2, 3$.

特征值 0 的特征向量为 $k_1(1,2,2)^{\mathrm{T}}$

特征值 2 的特征向量为 $k_2(3,-1,7)^{\mathrm{T}}$

特征值 3 的特征向量为 $k_3(-1,1,-2)^{\mathrm{T}}$

(3) 由A组5.(1)的结论, 有 $A^5$ 的特征值为 $\lambda^5$, $A+2E$ 的特征值为 $\lambda+2$

故 $A^5$ 的特征值为 $0, 32, 243$

$A+2E$ 的特征值为 $2, 4, 5$

6.

证明: 由定理5.1知 $m=2$ 时成立.

假设当 $m=k$ 时成立, 则立 $m=k+1$ 时, 对于

$$k_1x_1 + k_2x_2 + \cdots + k_kx_k + k_{k+1}x_{k+1} = 0 \tag{1}$$

对(1)用 $A$ 左乘两端有

$$k_1\lambda_1x_1 + k_2\lambda_2x_2 + \cdots + k_k\lambda_kx_k + k_{k+1}\lambda_{k+1}x_{k+1} = 0 \tag{2}$$

(1) 与 (2) 消去 $x_{k+1}$, 有

$$k_1(\lambda_{k+1}-\lambda_1)x_1 + k_2(\lambda_{k+1}-\lambda_2)x_2 + \cdots + k_{k+1}(\lambda_{k+1}-\lambda_k)x_k = 0$$

由归纳假设知 $x_1, x_2, \ldots, x_k$ 线性无关. 故

$$k_1 = k_2 = \cdots = k_k = 0$$

则 $k_{k+1}x_{k+1} = 0 \Rightarrow k_{k+1} = 0$

从而 $x_1,x_2,\dots,x_k,x_{k+1}$ 线性无关.
综上, 由数学归纳法知定理5.2成立.

7.
$m=2$ 时, 设 $\lambda_1,\lambda_2$ 是两个不同的特征值, 设

$$k_1x_{11}+k_2x_{12}+\cdots+k_{k_1}x_{1k_1}+l_1x_{21}+\cdots+l_{k_2}x_{2k_2}=0 \tag{1}$$

对 (1) 两边左乘 $A$, 有

$$k_1Ax_{11}+k_2Ax_{12}+\cdots+k_{k_1}Ax_{1k_1}+l_1Ax_{21}+\cdots+l_{k_2}Ax_{2k_2}=0$$

由于 $x_{1i}(i=1,2,\dots,k_1)$ 和 $x_{2j}(j=1,2,\dots,k_2)$ 分别是 $A$ 的属于 $\lambda_1,\lambda_2$ 的特征向量, 则

$$k_1\lambda_1x_{11}+k_2\lambda_1x_{12}+\cdots+k_{k_1}\lambda_1x_{1k_1}+l_1\lambda_2x_{21}+\cdots+l_{k_2}\lambda_2x_{2k_2}=0 \tag{2}$$

(2) $-\lambda_2$ (1), 有

$$k_1(\lambda_1-\lambda_2)x_{11}+k_2(\lambda_1-\lambda_2)x_{12}+\cdots+k_{k_1}(\lambda_1-\lambda_2)x_{1k_1}=0$$

由于 $x_{11},x_{12},\dots,x_{1k_1}$ 线性无关, 且 $\lambda_1-\lambda_2\neq 0$, 从而

$$k_1=k_2=\cdots=k_{k_1}=0$$

再代入 (1) 有

$$l_1x_{21}+l_2x_{22}+\cdots+l_{k_2}x_{2k_2}=0$$

由于 $x_{21},x_{22},\dots,x_{2k_2}$ 线性无关, 则 $l_1=l_2=\cdots=l_{k_2}=0$
因此向量组 $x_{11},x_{12},\dots,x_{1k_1},x_{21},\dots,x_{2k_2}$ 线性无关. $m=2$ 时成立.
假设 $m=n$ 时结论成立, 则立 $m=n+1$ 时, 设

$$k_{11}x_{11}+k_{12}x_{12}+\cdots+k_{1k_1}x_{1k_1}+k_{21}x_{21}+\cdots+k_{2k_2}x_{2k_2}+\cdots+k_{n1}x_{n1}+\cdots+k_{nk_n}x_{nk_n}+\cdots+k_{(n+1)k_{n+1}}x_{(n+1)k_{n+1}}=0$$

同左乘 $A$ 后有

$$k_{11}\lambda_1x_{11}+k_{12}\lambda_1x_{12}+\cdots+k_{1k_1}\lambda_1x_{1k_1}+\cdots+k_{(n+1)1}\lambda_{n+1}x_{(n+1)1}+\cdots+k_{(n+1)k_{n+1}}\lambda_{n+1}x_{(n+1)k_{n+1}}=0$$

消去 $x_{(n+1)1},x_{(n+1)2},\dots,x_{(n+1)k_{n+1}}$, 有

$$k_{11}(\lambda_{n+1}-\lambda_1)x_{11}+k_{12}(\lambda_{n+1}-\lambda_1)x_{12}+\cdots+k_{nn}(\lambda_{n+1}-\lambda_n)x_{nk_n}=0$$

由归纳假设知 $k_{11}=k_{12}=\cdots=k_{nk_n}=0$, 往回代入, 有

$$k_{(n+1)1}x_{(n+1)1}+k_{(n+1)2}x_{(n+1)2}+\cdots+k_{(n+1)k_{n+1}}x_{(n+1)k_{n+1}}=0$$

从而 $k_{(n+1)1}=k_{(n+1)2}=\cdots=k_{(n+1)k_{n+1}}=0$
综上, 结论成立.

8.

依题意, 设 $f(t)=t^2+at+b$

则有 $f(\lambda)=\lambda^2+a\lambda+b=0$

考虑 $f(A)\alpha=A^2\alpha+aA\alpha+b\alpha$

$$\begin{aligned}&=\lambda^2\alpha+a\lambda\alpha+b\alpha\\&=(\lambda^2+a\lambda+b)\alpha\\&=0\end{aligned}$$

又 $\alpha\neq0$,

则 $f(A)=0$

9.

证明: $f_C(\lambda)=|\lambda E_{m+n}-C|=\begin{vmatrix}\lambda E_m-A & D\\ 0 & \lambda E_n-B\end{vmatrix}=|\lambda E_m-A|\cdot|\lambda E_n-B|=f_A(\lambda)f_B(\lambda)$

10.

(1) $f_A(\lambda)=\begin{vmatrix}\lambda-1 & -2 & 1 & -2\\ -2 & \lambda-4 & -2 & 1\\ & & \lambda-1 & -2\\ & & 1 & \lambda+2\end{vmatrix}=\begin{vmatrix}\lambda-1 & -2\\ -2 & \lambda-4\end{vmatrix}\cdot\begin{vmatrix}\lambda-1 & -2\\ 1 & \lambda+2\end{vmatrix}=(\lambda-5)(\lambda+1)\lambda^2$

故特征值为 $-1,5,0,0$.

特征值为 $-1$ 的特征向量是 $k_1(-7,4,-2,2)^{\mathrm{T}}$

特征值为 $5$ 的特征向量是 $k_2(1,2,0,0)^{\mathrm{T}}$

特征值为 $0$ 的特征向量是 $k_3(-2,1,0,0)^{\mathrm{T}}$

(2) $f_A(\lambda)=\begin{vmatrix}\lambda-3 & 1 & & \\ -5 & \lambda-3 & & \\ & & \lambda-1 & 1\\ & & -1 & \lambda-3\end{vmatrix}=\begin{vmatrix}\lambda-3 & 1\\ -5 & \lambda-3\end{vmatrix}\cdot\begin{vmatrix}\lambda-1 & 1\\ -1 & \lambda-3\end{vmatrix}=(\lambda-2)^2(\lambda+2)$

故特征值为 $2,2,-2$.

特征值为 $-2$ 的特征向量是 $k_1(1,5,0)^{\mathrm{T}}$

特征值为 $2$ 的特征向量是 $k_2(0,0,-1,1)^{\mathrm{T}}+k_3(1,1,0,0)^{\mathrm{T}}$

11.

(1)

$$f_A(\lambda)=\begin{vmatrix}\lambda-1 & 1 & \cdots & 1\\ 1 & \lambda-1 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & \lambda-1\end{vmatrix}=\begin{vmatrix}1 & 1 & 1 & \cdots & 1\\ 0 & \lambda-1 & 1 & \cdots & 1\\ 0 & 1 & \lambda-1 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 1 & 1 & \cdots & \lambda-1\end{vmatrix}=\begin{vmatrix}1 & 1 & 1 & \cdots & 1\\ -1 & \lambda-2 & & & \\ -1 & & \lambda-2 & & \\ \vdots & & & \ddots & \\ -1 & & & & \lambda-2\end{vmatrix}$$

$$=(\lambda-2)\begin{vmatrix}1 & 1 & 1 & \cdots & 1 & 1\\ -1 & \lambda-2 & & & & \\ -1 & & \lambda-2 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 1 & -1\\ -1 & & & & & \lambda-2\end{vmatrix}=\begin{vmatrix}\lambda-2 & 1 & 1 & \cdots & 1 & 1\\ -\lambda+2 & \lambda-2 & & & & \\ -\lambda+2 & & \lambda-2 & & & \\ \vdots & & & \ddots & & \\ 0 & & & & 1 & -1\\ -\lambda & & & & & \lambda-2\end{vmatrix}$$

$$=\begin{vmatrix}\lambda+n-2 & 1 & \cdots & 1 & 1\\ & \lambda-2 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \lambda-2\end{vmatrix}=(\lambda+n-2)(\lambda-2)^{n-1}$$

则 $A$ 的特征值为 2 和 $2-n$. 又 $n-r(2E-A)=n-1$, $n-r((2-n)E-A)=1$.

(2) 当 $\lambda$ 为 2 时, 特征向量是 $k_1\begin{pmatrix}-1\\1\\0\\0\\\vdots\\0\end{pmatrix}+k_2\begin{pmatrix}-1\\0\\1\\0\\\vdots\\0\end{pmatrix}+\cdots+k_{n-1}\begin{pmatrix}-1\\0\\0\\0\\\vdots\\1\end{pmatrix}$;

当 $\lambda$ 为 $2-n$ 时, 特征向量是 $k_n\begin{pmatrix}-1\\0\\0\\\vdots\\1\end{pmatrix}$.

## 习题5.2

### (A)

1.

(1) 不能. 因 $r(E-A)=r\begin{pmatrix}0&-1\\0&0\end{pmatrix}=1$, 说明 $(E-A)x=0$ 的基础解系中只有一个解向量, 即 $\lambda=1$ (二重根) 只有一个特征向量, 所以 $\begin{pmatrix}1&-1\\0&1\end{pmatrix}$ 不能对角化.

(2) 可以. 如 $\begin{pmatrix}\frac{-11+\sqrt{97}}{2}&\\&\frac{-11-\sqrt{97}}{2}\end{pmatrix}$

2.

(1) 由 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-2&-1&1\\-1&\lambda-2&-1\\1&-1&\lambda\end{vmatrix}=(\lambda-1)^2(\lambda-2)$, 则特征值为 $1,1,2$

又 $\lambda=1$ 时, 由 $\begin{pmatrix}-1&-1&1\\-1&-1&1\\1&-1&1\end{pmatrix}\to\begin{pmatrix}1&1&-1\\0&0&0\\0&0&0\end{pmatrix}$ 故基础解系 $\alpha_1=(1,-1,0)^{\mathrm{T}}$, $\alpha_2=(1,0,1)^{\mathrm{T}}$

$\lambda=2$ 时, 由 $\begin{pmatrix}0&-1&1\\-1&0&1\\-1&-1&0\end{pmatrix}\to\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$ 故基础解系 $\alpha_3=(1,1,1)^{\mathrm{T}}$

故 $P=(\alpha_1,\alpha_2,\alpha_3)=\begin{pmatrix}1&1&1\\-1&0&1\\0&1&1\end{pmatrix}$, $P^{-1}AP=\mathrm{diag}(1,1,2)$

(2) 由 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & -2 & -3\\ -2 & \lambda-1 & -3\\ -3 & -3 & \lambda-6\end{vmatrix}=\lambda(\lambda+1)(\lambda-9)$, 则特征值为 $0,-1,9$

$\lambda=0$ 时, 由 $\begin{pmatrix}-1 & -2 & -3\\ -2 & -1 & -3\\ -3 & -3 & -6\end{pmatrix}\to\begin{pmatrix}1 & & 1\\ & 1 & 1\\ & & \end{pmatrix}$, 故基础解系 $\alpha_1=(1,1,-1)^{\mathrm{T}}$

$\lambda=-1$ 时, 由 $\begin{pmatrix}-2 & -2 & -3\\ -2 & -2 & -3\\ -3 & -3 & -7\end{pmatrix}\to\begin{pmatrix}1 & 1 & \\ & & 1\\ & & \end{pmatrix}$, 故基础解系 $\alpha_2=(-1,1,0)^{\mathrm{T}}$

$\lambda=9$ 时, 由 $\begin{pmatrix}8 & -2 & -3\\ -2 & 8 & -3\\ -3 & -3 & 3\end{pmatrix}\to\begin{pmatrix}1 & & -\frac{1}{2}\\ & 1 & -\frac{1}{2}\\ & & \end{pmatrix}$, 故基础解系 $\alpha_3=(1,1,2)^{\mathrm{T}}$

故 $P=\begin{pmatrix}1 & -1 & 1\\ 1 & 1 & 1\\ -1 & 0 & 2\end{pmatrix}$, $P^{-1}AP=\mathrm{diag}(0,-1,9)$

(3) 由 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda & 2 & 2\\ -2 & \lambda+4 & 2\\ 2 & -2 & \lambda\end{vmatrix}=\lambda(\lambda+2)^2$, 特征值为 $0,-2,-2$

$\lambda=0$ 时, 由 $\begin{pmatrix}0 & 2 & 2\\ -2 & 4 & 2\\ 2 & -2 & 0\end{pmatrix}\to\begin{pmatrix}1 & & 1\\ & 1 & 1\\ & & \end{pmatrix}$ 得基础解系 $\alpha_1=(1,1,-1)^{\mathrm{T}}$

$\lambda=2$ 时, 由 $\begin{pmatrix}-2 & 2 & 2\\ -2 & 2 & 2\\ 2 & -2 & -2\end{pmatrix}\to\begin{pmatrix}1 & -1 & -1\\ & & \\ & & \end{pmatrix}$ 得 $\alpha_2=(1,1,0)^{\mathrm{T}}$, $\alpha_3=(1,0,1)^{\mathrm{T}}$

故 $P=\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ -1 & 0 & 1\end{pmatrix}$, $P^{-1}AP=\mathrm{diag}(0,-2,-2)$

(4) $f_A(\lambda)=\begin{vmatrix}\lambda-8 & 2 & 1\\ 2 & \lambda-5 & 2\\ 3 & 6 & \lambda-6\end{vmatrix}=(\lambda-1)(\lambda-9)^2$, 特征值为 1, 9, 9

$\lambda=1$ 时, 由 $\begin{pmatrix}-7 & 2 & 1\\ 2 & -4 & 2\\ 3 & 6 & -5\end{pmatrix}\to\begin{pmatrix}1 & & -\frac{1}{3}\\ & 1 & -\frac{2}{3}\\ & & \end{pmatrix}$, 得基础解系 $\alpha=(1,2,3)^{\mathrm{T}}$

$\lambda=9$ 时, 由 $\begin{pmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 3 & 6 & 3\end{pmatrix}\to\begin{pmatrix}1 & 2 & 1\\ & & \\ & & \end{pmatrix}$, 得基础解系 $\alpha_2=(-2,1,0)^{\mathrm{T}}$, $\alpha_3=(-1,0,1)^{\mathrm{T}}$

则 $P=\begin{pmatrix}1 & -2 & -1\\ 2 & 1 & 0\\ 3 & 0 & 1\end{pmatrix}$, $P^{-1}AP=\mathrm{diag}(1,9,9)$

3.

(1) $f_A(\lambda)=\begin{vmatrix}\lambda-7 & 12 & -6\\ -10 & \lambda+19 & -10\\ -12 & 24 & \lambda-13\end{vmatrix}=(\lambda+1)(\lambda-1)^2$, 特征值为 $-1,1,1$

$\lambda=-1$ 时, 由 $\begin{pmatrix}-8 & 12 & -6\\ -10 & 18 & -10\\ -12 & 24 & -14\end{pmatrix}\to\begin{pmatrix}1 & & -\frac{1}{2}\\ & 1 & -\frac{5}{6}\\ & & \end{pmatrix}$, 得 $\alpha_1=(3,5,6)^{\mathrm{T}}$

$\lambda=1$ 时, 由 $\begin{pmatrix}-6 & 12 & -6\\ -10 & 20 & -10\\ -12 & 24 & -12\end{pmatrix}\to\begin{pmatrix}1 & -2 & 1\\ & & \\ & & \end{pmatrix}$, 得 $\alpha_2=(2,1,0)^{\mathrm{T}}$, $\alpha_3=(-1,0,1)^{\mathrm{T}}$

则 $P=\begin{pmatrix}3 & 2 & -1\\ 5 & 1 & 0\\ 6 & 0 & 1\end{pmatrix}$, $P^{-1}AP=\begin{pmatrix}-1 & & \\ & 1 & \\ & & 1\end{pmatrix}\Rightarrow A=P\begin{pmatrix}-1 & & \\ & 1 & \\ & & 1\end{pmatrix}P^{-1}$

故 $A^n=P\cdot\begin{pmatrix}-1 & & \\ & 1 & \\ & & 1\end{pmatrix}^n P^{-1}$

故 $n$ 为奇数时 $A^n = P \cdot \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} P^{-1} = A$

$n$ 为偶数时 $A^n = P \cdot \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} P^{-1} = PP^{-1} = E$

(2) $f_A(\lambda) = \begin{vmatrix} \lambda-3 & 0 & 0 \\ -9 & \lambda-6 & -5 \\ 12 & 6 & \lambda+5 \end{vmatrix} = \lambda(\lambda-1)(\lambda-3)$, 特征值为 $0, 1, 3$

$\lambda = 0$ 时, 由 $\begin{pmatrix} -3 & 0 & 0 \\ -9 & -6 & -5 \\ 12 & 6 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & & \\ & 1 & \frac{5}{6} \\ & & \end{pmatrix}$, 基础解系 $\alpha_1 = (0, -5, 6)^{\mathrm{T}}$

$\lambda = 1$ 时, 由 $\begin{pmatrix} -2 & 0 & 0 \\ -9 & -5 & -5 \\ 12 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & & \end{pmatrix}$, 基础解系 $\alpha_2 = (0, -1, 1)^{\mathrm{T}}$

$\lambda = 3$ 时, 由 $\begin{pmatrix} 0 & 0 & 0 \\ -9 & -3 & -5 \\ 12 & 6 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & & \frac{1}{3} \\ & 1 & \frac{2}{3} \\ & & \end{pmatrix}$, $\alpha_3 = (-1, -2, 3)^{\mathrm{T}}$,

则 $P = \begin{pmatrix} 0 & 0 & -1 \\ -5 & -1 & -2 \\ 6 & 1 & 3 \end{pmatrix}$

$P^{-1}AP = \mathrm{diag}(0, 1, 3)$,

则 $A = P\,\mathrm{diag}(0, 1, 3)P^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ -5 & -1 & -2 \\ 6 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -3 & -6 & -5 \\ -1 & 0 & 0 \end{pmatrix}$

故 $A^n = P\begin{pmatrix}0 & & \\ & 1 & \\ & & 3^n\end{pmatrix}P^{-1} = \begin{pmatrix}0 & 0 & -1 \\ -5 & -1 & -2 \\ 6 & 1 & 3\end{pmatrix}\begin{pmatrix}0 & & \\ & 1 & \\ & & 3^n\end{pmatrix}\begin{pmatrix}1 & 1 & 1 \\ -3 & -6 & -5 \\ -1 & 0 & 0\end{pmatrix}$

$= \begin{pmatrix}3^n & 0 & 0 \\ 2\times 3^n - 27 & 6 & 5 \\ 2\times 3^n + 27 & -6 & -5\end{pmatrix}$

(3) $f_A(\lambda) = \begin{vmatrix}\lambda - 122 & 100 \\ -150 & \lambda + 123\end{vmatrix} = (\lambda-2)(\lambda+3)$, 特征值为 $2, -3$

$\lambda = 2$ 时, 由 $\begin{pmatrix}-120 & 100 \\ -150 & 125\end{pmatrix} \to \begin{pmatrix}1 & -\frac{5}{6} \\ 0 & 0\end{pmatrix}$, 得 $\alpha_1 = (5,6)^{\mathrm{T}}$

$\lambda = -3$ 时, 由 $\begin{pmatrix}-125 & 100 \\ -150 & 120\end{pmatrix} \to \begin{pmatrix}1 & -\frac{4}{5} \\ 0 & 0\end{pmatrix}$ 得 $\alpha_2 = (4,5)^{\mathrm{T}}$

故 $P = \begin{pmatrix}5 & 4 \\ 6 & 5\end{pmatrix}$, $P^{-1}AP = \begin{pmatrix}2 & \\ & -3\end{pmatrix}$

故 $A = P\begin{pmatrix}2 & \\ & -3\end{pmatrix}P^{-1}$

则 $A^n = P\begin{pmatrix}2^n & \\ & (-3)^n\end{pmatrix}P^{-1} = \begin{pmatrix}5 & 4 \\ 6 & 5\end{pmatrix}\begin{pmatrix}2^n & \\ & (-3)^n\end{pmatrix}\begin{pmatrix}5 & -4 \\ -6 & 5\end{pmatrix}$

$= \begin{pmatrix}-24\times(-3)^n + 25\times 2^n & 20\times(-3)^n - 20\times 2^n \\ -30\times(-3)^n + 30\times 2^n & 25\times(-3)^n - 24\times 2^n\end{pmatrix}$

(4) $f_A(\lambda) = \begin{vmatrix}\lambda - 10 & -4 \\ 24 & \lambda + 10\end{vmatrix} = (\lambda-2)(\lambda+2)$, 特征值为 $2, -2$

则 $A = P\begin{pmatrix}2 & \\ & -2\end{pmatrix}P^{-1} = 2P\begin{pmatrix}1 & \\ & -1\end{pmatrix}P^{-1}$

则当 $n$ 为奇数时, $A^n = 2^nP\begin{pmatrix}1 & \\ & -1\end{pmatrix}P^{-1} = 2^nA$

$n$ 为偶数时, $A^n = 2^n P \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} P^{-1} = 2^n P P^{-1} = 2^n E$

4.

证明: 依题意, $P^{-1}AP = B$

则 $(P^{-1}AP)^k = B^k$

即 $B^k = P^{-1}APP^{-1}AP \cdots P^{-1}AP = P^{-1}A^kP$, 则 $B^k$ 与 $A^k$ 也相似.

5.

证明: $\because A \sim B, C \sim D$, 则存在可逆矩阵 $P_1, P_2$, 使

$$B = P_1^{-1}AP_1 \quad D = P_2^{-1}CP_2$$

则 $\begin{pmatrix} B & \\ & D \end{pmatrix} = \begin{pmatrix} P_1^{-1}AP_1 & \\ & P_2^{-1}CP_2 \end{pmatrix} = \begin{pmatrix} P_1^{-1} & \\ & P_2^{-1} \end{pmatrix} \begin{pmatrix} A & \\ & C \end{pmatrix} \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}$

$$= \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}^{-1} \begin{pmatrix} A & C \\ & \end{pmatrix} \begin{pmatrix} P_1 & \\ & P_2 \end{pmatrix}$$

故 $\begin{pmatrix} A & \\ & C \end{pmatrix} \sim \begin{pmatrix} B & \\ & D \end{pmatrix}$

6.

证明:

(1) 设 $A$ 可对角化为 $B$, 则 $B = P^{-1}AP$

故 $(P^{-1}AP)^{\mathrm{T}} = B^{\mathrm{T}} = B = P^{-1}AP$

即 $P^{\mathrm{T}}A^{\mathrm{T}}(P^{\mathrm{T}})^{\mathrm{T}} = P^{-1}AP$

$\Rightarrow A^{\mathrm{T}} = (PP^{\mathrm{T}})^{-1}APP^{\mathrm{T}}$, 从而 $A^{\mathrm{T}} \sim A$

(2) 取 $Y = PP^{\mathrm{T}}$, 由(1)知 $A^{\mathrm{T}} = Y^{-1}AY$

故 $AY - YA^{\mathrm{T}} = 0$

7.

$$f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda - 1 & & & \\ -a & \lambda - 1 & & \\ -2 & -b & \lambda - 2 & \\ -2 & -3 & -c & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)^2, \text{特征值 } 1, 1, 2, 2$$

由于 $A$ 可对角化, 故 $(E-A)x=0$ 与 $(2E-A)x=0$ 的基础解系均有2个解向量,

则 $r(E-A)=2$, $r(2E-A)=2$.

对 $E-A=\begin{pmatrix}0&&&\\-a&0&&\\-2&-b&-1&\\-2&-3&-c&-1\end{pmatrix}$ 与 $2E-A=\begin{pmatrix}1&&&\\-a&1&&\\-2&-b&0&\\-2&-3&-c&0\end{pmatrix}$,

由于它们秩为 2, 故 3 阶子式的行列式 $\begin{vmatrix}-a&&\\-2&-b&-1\\-2&-3&-c\end{vmatrix}$ 与 $\begin{vmatrix}-a&1&\\-2&-b&\\-2&-3&-c\end{vmatrix}$ 均为 0

$\Rightarrow a(bc-3)=0$ 且 $c(ab-2)=0$

当 $a=0$ 时, 则 $-2c=0\Rightarrow c=0$, 此时 $r(E-A)=2$, $r(2E-A)=2$, 可对角化

当 $a\neq 0$ 时, 则由 $a(bc-3)=0\Rightarrow bc=3$ 又 $c(ab-2)=0$, 则 $ab=2$

但由 $\begin{pmatrix}1&&&\\-a&1&&\\-2&-b&0&\\-2&-3&-c&0\end{pmatrix}\to\begin{pmatrix}1&&&\\&1&&\\&&-c&\\&&&0\end{pmatrix}\to\begin{pmatrix}1&0&1-c\\0&1&-c\\0&0&0\end{pmatrix}$ 则 $c=0$, 矛盾, 故 $a=0$.

综上, $a=c=0$, $b\in\mathbf{R}$ 时, $A$ 可对角化.

8.

由于 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1&&&&\\1&\lambda-2&&&\\1&2&\lambda-3&&\\\vdots&\vdots&\vdots&\ddots&\\1&2&3&\cdots&\lambda-n\end{vmatrix}=(\lambda-1)(\lambda-2)\cdots(\lambda-n)$,

故 $A$ 有 $n$ 个特征根, 则共有 $n$ 个无关特征向量, 可以对角化.

则其相似标准形为 $\begin{pmatrix}1&&&&\\&2&&&\\&&3&&\\&&&\ddots&\\&&&&n\end{pmatrix}$

## (B)

1.

设一组二维向量 $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$, 则

$$\begin{aligned}\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} &= \begin{pmatrix} 3x_{n-1}-2x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^2\begin{pmatrix} x_{n-2} \\ x_{n-3} \end{pmatrix} \\ &\cdots \\ &= \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^{n-2}\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}, \quad \text{记 } A = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}\end{aligned}$$

又由 $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda-3 & 2 \\ -1 & \lambda \end{vmatrix} = (\lambda-2)(\lambda-1)$, 知有特征值 2,1. 则

$\lambda = 2$ 时, 由 $\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ & \end{pmatrix}$ 得 $\alpha_1 = (2,1)^{\mathrm{T}}$

$\lambda = 1$ 时, 由 $\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ & \end{pmatrix}$ 得 $\alpha_2 = (1,1)^{\mathrm{T}}$

故 $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & 2 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ -1 & 1 \end{pmatrix}$

则 $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}^{n-2} = \begin{pmatrix} 1 & -2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & 2 \end{pmatrix}^{n-2}\begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2^{n-1}-1 & 2-2^{n-1} \\ -1+2^{n-2} & -2^{n-2}+2 \end{pmatrix}$

于是 $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 2^{n-1}-1 & 2-2^{n-1} \\ -1+2^{n-2} & -2^{n-2}+2 \end{pmatrix}\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3\times 2^{n-1}-2 \\ 3\times 2^{n-2}-2 \end{pmatrix}$

故 $x_n = 3\times 2^{n-1} - 2$

2.

由 $\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 7x_{n-1}-12x_{n-2} \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 7 & -12 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}$, 记 $A = \begin{pmatrix} 7 & -12 \\ 1 & 0 \end{pmatrix}$

由 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-7 & 12\\ -1 & \lambda\end{vmatrix}=(\lambda-5)(\lambda-2)$, 得 $\lambda$ 为2,5. 则

$2E-A=\begin{pmatrix}-5 & 12\\ -1 & 2\end{pmatrix}\to\begin{pmatrix}-5 & 2\\ & \end{pmatrix}$ 则 $\alpha_1=(2,5)^{\mathrm{T}}$

$5E-A=\begin{pmatrix}-2 & 12\\ -1 & 5\end{pmatrix}\to\begin{pmatrix}-2 & 2\\ & \end{pmatrix}$ 则 $\alpha_2=(1,1)^{\mathrm{T}}$, 故 $P=\begin{pmatrix}2 & 1\\ 5 & 1\end{pmatrix}$, $P^{-1}=\frac{1}{3}\begin{pmatrix}1 & -1\\ -5 & 2\end{pmatrix}$

则 $A=P\begin{pmatrix}2 & \\ & 5\end{pmatrix}P^{-1}$, 则 $A^{n-2}=P\begin{pmatrix}2 & \\ & 5\end{pmatrix}^{n-2}P^{-1}\begin{pmatrix}-3\\ 2\end{pmatrix}$

故 $\begin{pmatrix}x_n\\ x_{n-1}\end{pmatrix}=\begin{pmatrix}2 & 1\\ 5 & 1\end{pmatrix}\begin{pmatrix}2 & \\ & 5\end{pmatrix}^{n-2}\begin{pmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{5}{3} & -\frac{2}{3}\end{pmatrix}\begin{pmatrix}-3\\ 2\end{pmatrix}=\begin{pmatrix}11\times 3^{n-1}-9\times 4^{n-1}\\ 11\times 3^{n-2}-9\times 4^{n-2}\end{pmatrix}$

因此 $x_n=11\times 3^{n-1}-9\times 4^{n-1}$

3.

由

$$\begin{aligned}\begin{pmatrix}x_n\\ y_n\\ z_n\end{pmatrix}=\begin{pmatrix}5x_{n-1}-3y_{n-1}+2z_{n-1}\\ 6x_{n-1}-4y_{n-1}+4z_{n-1}\\ 4x_{n-1}-4y_{n-1}+5z_{n-1}\end{pmatrix}&=\begin{pmatrix}5 & -3 & 2\\ 6 & -4 & 4\\ 4 & -4 & 5\end{pmatrix}\begin{pmatrix}x_{n-1}\\ y_{n-1}\\ z_{n-1}\end{pmatrix}\\ &=\begin{pmatrix}5 & -3 & 2\\ 6 & -4 & 4\\ 4 & -4 & 5\end{pmatrix}^2\begin{pmatrix}x_{n-2}\\ y_{n-2}\\ z_{n-2}\end{pmatrix}\\ &=\begin{pmatrix}5 & -3 & 2\\ 6 & -4 & 4\\ 4 & -4 & 5\end{pmatrix}^{n-1}\begin{pmatrix}x_1\\ y_1\\ z_1\end{pmatrix}\end{aligned}$$

设 $A=\begin{pmatrix}5 & -3 & 2\\ 6 & -4 & 4\\ 4 & -4 & 5\end{pmatrix}$

则 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-5 & 3 & -2\\ -6 & \lambda+4 & -4\\ -4 & 4 & \lambda-5\end{vmatrix}=(\lambda-1)(\lambda-2)(\lambda-3)$

又 $E-A=\begin{pmatrix}-4&3&-2\\-6&5&-4\\-4&4&-4\end{pmatrix}\to\begin{pmatrix}1&&-1\\&1&-2\\&&\end{pmatrix}$ 得 $\alpha_1=\begin{pmatrix}1\\2\\1\end{pmatrix}$

$2E-A=\begin{pmatrix}-3&3&-2\\-6&6&-4\\-4&4&-3\end{pmatrix}\to\begin{pmatrix}1&-1&\\&&1\\&&\end{pmatrix}$ 得 $\alpha_2=\begin{pmatrix}1\\1\\0\end{pmatrix}$

$3E-A=\begin{pmatrix}-2&3&-2\\-6&7&-4\\-4&4&-2\end{pmatrix}\to\begin{pmatrix}1&&-\frac{1}{2}\\&1&-1\\&&\end{pmatrix}$ 得 $\alpha_3=\begin{pmatrix}1\\2\\2\end{pmatrix}$, 故 $P=\begin{pmatrix}1&1&1\\2&1&2\\1&0&2\end{pmatrix}$

则 $A=P\begin{pmatrix}1&&\\&2&\\&&3\end{pmatrix}P^{-1}$,

故 $A^{n-1}=P\begin{pmatrix}1&&\\&2^{n-1}&\\&&3^{n-1}\end{pmatrix}P^{-1}=\begin{pmatrix}1&1&1\\2&1&2\\1&0&2\end{pmatrix}P\begin{pmatrix}1&&\\&2^{n-1}&\\&&3^{n-1}\end{pmatrix}\begin{pmatrix}-2&2&-1\\2&-1&0\\1&-1&1\end{pmatrix}$

则 $\begin{pmatrix}x_n\\y_n\\z_n\end{pmatrix}=A^{n-1}\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}=\begin{pmatrix}3-2\times3^{n-1}\\6-4\times3^{n-1}\\3-4\times3^{n-1}\end{pmatrix}$

4.

证明: 由 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1&-1&-1&\cdots&-1\\&\lambda-1&-1&\cdots&-1\\&&\lambda-1&\cdots&-1\\&&&\ddots&\vdots\\&&&&\lambda-1\end{vmatrix}=(\lambda-1)^n$, 则特征值为1 (n重根)

又 $r(E-A)=n-1$, 故 $A$ 只有1个线性无关的特征向量, 则 $A$ 不能与对角阵相似.

5.

(1) 依题意, $tr(A)=tr(B)$ 且 $|A|=|B|$

则 $\begin{cases} 1+4+a=2+2+b \\ \begin{vmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & a \end{vmatrix} = \begin{vmatrix} 2 & & \\ & 2 & \\ & & b \end{vmatrix} \end{cases}$

得 $\begin{cases} a=5 \\ b=6 \end{cases}$

(2) 由(1)知 $A$ 的特征值为2,2,6, 则

$2E-A=\begin{pmatrix} 1 & 1 & -1 \\ -2 & -2 & 2 \\ 3 & 3 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ & & \\ & & \end{pmatrix}$ 得 $\alpha_1=(-1,1,0)^{\mathrm{T}}$, $\alpha_2=(1,0,1)^{\mathrm{T}}$

$6E-A=\begin{pmatrix} 5 & 1 & -1 \\ -2 & 2 & 2 \\ 3 & 3 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & & -\frac{1}{3} \\ & 1 & \frac{1}{3} \\ & & 0 \end{pmatrix}$ 得 $\alpha_3=(1,-2,3)^{\mathrm{T}}$

故 $P=\begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix}$

6.

证明: 设 $\lambda_1,\lambda_2,\ldots,\lambda_n$ 为 $A$ 的 $n$ 个不同的特征值, 则存在可逆矩阵 $P$ 使

$$P^{-1}AP=\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}=\Lambda_1$$

记 $P=(\beta_1,\beta_2,\ldots,\beta_n)$, 则 $p_i\ (i=1,2,\ldots,n)$ 也是 $B$ 的特征向量, 则记对应特征值 $\mu_i$

$$Bp_i=\mu_i p_i$$

则有 $P^{-1}BP = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_n \end{pmatrix} = \Lambda_2$

于是 $P^{-1}ABP = (P^{-1}AP)(P^{-1}BP) = \Lambda_1\Lambda_2 = \Lambda_2\Lambda_1 = (P^{-1}BP)(P^{-1}AP) = P^{-1}BAP$

于是 $AB = BA$

7.

由于 $A \sim B$, 若 $r(A) = n$, 则由于 $A \sim B$, 则 $|A| = |B|$, 且 $A^{-1} \sim B^{-1}$.

于是 $P^{-1}((nA^{-1})P = |B|B^{\mathrm{T}}$, 即 $A^*$ 与 $B^*$ 相似.

若 $r(A) < n$, 则 $A, B$ 不可逆, 必 $\exists \delta > 0$, 当 $t \in (0, \delta)$, 使 $|tE + A| \neq 0$, $|tE + B| \neq 0$

记 $A_t = tE + A$, $B_t = tE + B$, 则 $B_t = tE + B = tE + P^{-1}AP = P^{-1}(tE + A)P = P^{-1}A_tP$

则由 $r(A) = n$ 的证明知 $B_t^* \sim A_t^*$ 即 $(tE + B)^* = P^{-1}(tE + A)^*P$

上式两端矩阵均为 $t$ 的多项式, 由于当 $t \in (0, \delta)$ 时, 对应元素相似, 则取 $t \to 0$, 有 $A^*$ 与 $B^*$ 相似.

8.

证明: 设 $\lambda$ 是 $A$ 的任一特征值, $\alpha$ 是 $\lambda$ 的特征向量

则 $A\alpha = \lambda\alpha$, 左乘 $A$, 为

$$A^2\alpha = \lambda A\alpha = \lambda^2\alpha, \text{ 又 } A^2 = A$$

则 $(\lambda^2 - \lambda)\alpha = 0$ 又 $\alpha \neq 0$, $\therefore \lambda^2 - \lambda = 0$

于是 $A$ 的特征值只能是0或1

又秩 $r(A) = r$, 故特征值为0的特征向量有 $n - r$ 个

则 $A$ 的相似标准形为 $\begin{pmatrix} \overbrace{\quad\quad\quad}^{r\text{ 个}} & & & & & \\ 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}_n = \begin{pmatrix} E_r & \\ & O \end{pmatrix}$

9.

设 $A\alpha = \lambda\alpha$, 则 $A^2\alpha = \lambda^2\alpha \Rightarrow \lambda = \pm 1$

故仿8知 $A \sim \begin{pmatrix} E_r & \\ & -E_{n-r} \end{pmatrix}$

10.

反证法. 假设 $A$ 可对角化, 设

$$A \sim \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} = \Lambda,\ \text{即设}\ A = P\Lambda P^{-1}$$

由 $A^k = O$ 得

$$A^k = (P\Lambda P^{-1})(P\Lambda P^{-1})\cdots(P\Lambda P^{-1}) = O$$

$$\Rightarrow P\Lambda^k P^{-1} = O \Rightarrow \Lambda^k = O$$

则 $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, 则 $A = O$, 与题设矛盾.

11.

依题意, 设 $\Lambda = \begin{pmatrix} a & \\ & b \end{pmatrix}$

则由 $\begin{cases} |A| = |\Lambda| \\ tr(A) = tr(\Lambda) \end{cases}$

得 $\begin{cases} a+b=6 \\ ab=8 \end{cases} \Rightarrow \begin{cases} a=2 \\ b=4 \end{cases}$ 或 $\begin{cases} a=4 \\ b=2 \end{cases}$

从而 $\Lambda$ 为 $\begin{pmatrix} 2 & \\ & 4 \end{pmatrix}$ 或 $\begin{pmatrix} 4 & \\ & 2 \end{pmatrix}$

## (C)

1.

(1) 设 $X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$, 则 $X' = AX$

又由 $f_A(\lambda) = \begin{vmatrix} \lambda-1 & -2 \\ -4 & \lambda-8 \end{vmatrix} = \lambda^2 - 9\lambda = \lambda(\lambda-9)$ 得特征值为 0,9.

又 $\lambda = 0$ 时, 由 $\begin{pmatrix} -1 & -2 \\ -4 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_1 = (-2, 1)^{\mathrm{T}}$

又 $9E-A=\begin{pmatrix}8&-2\\-4&1\end{pmatrix}\to\begin{pmatrix}-4&1\\0&0\end{pmatrix}$ 得 $\alpha_2=(1,4)^{\mathrm{T}}$, 则 $P=\begin{pmatrix}-2&1\\1&4\end{pmatrix}$, $\Lambda=\begin{pmatrix}0&\\&9\end{pmatrix}$

故由 $X=PY$ 可得 $Y'=\begin{pmatrix}0&\\&9\end{pmatrix}Y\Rightarrow\begin{cases}\dfrac{\mathrm{d}my_1}{\mathrm{d}t}=0\\\dfrac{\mathrm{d}y_2}{\mathrm{d}t}=9y_2\end{cases}\Rightarrow Y=\begin{pmatrix}c_1\\c_2e^{9t}\end{pmatrix}$

则 $X=\begin{pmatrix}-2&1\\1&4\end{pmatrix}\begin{pmatrix}c_1\\c_2e^{9t}\end{pmatrix}=\begin{pmatrix}-2c_1+c_2e^{9t}\\c_1+4c_2e^{9t}\end{pmatrix}$

(2) 设 $X=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$, $A=\begin{pmatrix}2&-1\\-1&2\end{pmatrix}$, 则 $X'=AX$

由 $f_A(\lambda)=\begin{vmatrix}\lambda-2&1\\1&\lambda-2\end{vmatrix}=(\lambda-1)(\lambda-3)$ 得特征值 1,3

则 $E-A=\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\to\begin{pmatrix}1&-1\\0&0\end{pmatrix}$ 得 $\alpha_1=(1,1)^{\mathrm{T}}$

$3E-A=\begin{pmatrix}1&1\\1&1\end{pmatrix}\to\begin{pmatrix}1&1\\0&0\end{pmatrix}$ 得 $\alpha_2=(1,-1)^{\mathrm{T}}$, 则 $P=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$, $\Lambda=\begin{pmatrix}1&\\&3\end{pmatrix}$

由 $X=PY$ 可得 $Y'=\begin{pmatrix}1&\\&3\end{pmatrix}Y\Rightarrow\begin{cases}\dfrac{\mathrm{d}y_1}{\mathrm{d}t}=y_1\\\dfrac{\mathrm{d}y_2}{\mathrm{d}t}=3y_2\end{cases}\Rightarrow Y=\begin{pmatrix}c_1e^{\mathrm{T}}\\c_2e^{3t}\end{pmatrix}$

故 $X=\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}c_1e^t\\c_2e^{3t}\end{pmatrix}=\begin{pmatrix}c_1e^t+c_2e^{3t}\\c_1e^t-c_2e^{3t}\end{pmatrix}$

(3) 设 $X=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}$, $A=\begin{pmatrix}1&-2\\-1&4\end{pmatrix}$, 则 $X'=AX$

由 $f_A(\lambda)=\begin{vmatrix}\lambda-1&2\\-1&\lambda-4\end{vmatrix}=(\lambda-2)(\lambda-3)$ 得特征值2,3.

则 $2E-A=\begin{pmatrix}1&2\\-1&-2\end{pmatrix}\to\begin{pmatrix}1&2\\0&0\end{pmatrix}$ 得 $\alpha_1=(-2,1)^{\mathrm{T}}$

$3E-A=\begin{pmatrix}2&2\\-1&-1\end{pmatrix}\to\begin{pmatrix}1&1\\0&0\end{pmatrix}$ 得 $\alpha_2=(1,-1)^{\mathrm{T}}$, 则 $P=\begin{pmatrix}-2&1\\1&-1\end{pmatrix}$, $\Lambda=\begin{pmatrix}2&\\&3\end{pmatrix}$

则由 $X=PY \Rightarrow Y'=\begin{pmatrix}2 & \\ & 3\end{pmatrix}Y \Rightarrow \begin{cases}\dfrac{\mathrm{d}y_1}{\mathrm{d}t}=y_1\\ \dfrac{\mathrm{d}y_2}{\mathrm{d}t}=y_2\end{cases} \Rightarrow Y=\begin{pmatrix}c_1e^{2t}\\ c_2e^{3t}\end{pmatrix}$

$\Rightarrow X=\begin{pmatrix}-2 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}c_1e^{2t}\\ c_2e^{3t}\end{pmatrix}=\begin{pmatrix}-2c_1e^{2t}+c_2e^{3t}\\ c_1e^{2t}-c_2e^{3t}\end{pmatrix}$

又 $x_1(0)=x_2(0)=1$ 得 $\begin{cases}c_1=-2\\ c_2=-3\end{cases}$

故 $X=\begin{pmatrix}4e^{2t}-3e^{3t}\\ -2e^{2t}+3e^{3t}\end{pmatrix}$

(4) 记 $X=\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}$, $A=\begin{pmatrix}3 & 1 & 1\\ -12 & 0 & 5\\ 4 & -2 & -1\end{pmatrix}$, 则 $X'=AX$

由 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-3 & -1 & -1\\ 12 & \lambda & -5\\ -4 & 2 & \lambda+1\end{vmatrix}=\lambda^3-2\lambda^2+5\lambda-62$, 考察函数图像, 此方程有且只有一个实根. 故该微分方程组解的形式过于复杂, 但计算方法与(3)相同. 此处怀疑是题目未设置好. 但教材也没给出答案, 故不作过多讨论.

(5) 记 $X=\begin{pmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{pmatrix}$, $A=\begin{pmatrix}1 & 1 & 1\\ 0 & 3 & 3\\ -2 & 1 & 1\end{pmatrix}$, 则 $X'=AX$

由 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & -1 & -1\\ 0 & \lambda-3 & -3\\ -2 & -1 & \lambda-1\end{vmatrix}=\lambda(\lambda-2)(\lambda-3)$ 故特征值为0,2,3.

则 $0E-A=\begin{pmatrix}-1 & -1 & -1\\ 0 & -3 & -3\\ 2 & 1 & -1\end{pmatrix}\to\begin{pmatrix}1 & & \\ & 1 & 1\\ & & \end{pmatrix}$ 则 $\alpha_1=(0,-1,1)^{\mathrm{T}}$

$2E-A=\begin{pmatrix}1 & -1 & -1\\ 0 & -1 & -3\\ -2 & -1 & 1\end{pmatrix}\to\begin{pmatrix}1 & & 2\\ & 1 & 3\\ & & \end{pmatrix}$ 则 $\alpha_2=(2,3,-1)^{\mathrm{T}}$

$$3E-A=\begin{pmatrix}2&-1&-1\\0&0&-3\\2&-1&2\end{pmatrix}\to\begin{pmatrix}1&-\frac{1}{2}&\\&&1\\&&\end{pmatrix} \text{ 则 } \alpha_3=(1,2,0)^{\mathrm{T}}$$

$$\text{故 } P=\begin{pmatrix}0&2&1\\-1&3&2\\1&-1&0\end{pmatrix},\Lambda=\begin{pmatrix}0&&\\&2&\\&&3\end{pmatrix}, \text{ 则由 } X=PY$$

$$\text{有 } Y'=\begin{pmatrix}0&&\\&2&\\&&3\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\Rightarrow\begin{cases}\dfrac{\mathrm{d}y_1}{\mathrm{d}t}=0\\\dfrac{\mathrm{d}y_2}{\mathrm{d}t}=2y_2\\\dfrac{\mathrm{d}y_3}{\mathrm{d}t}=3y_3\end{cases}\Rightarrow Y=\begin{pmatrix}c_1\\c_2e^{2t}\\c_3e^{3t}\end{pmatrix}$$

$$\text{故 } X=\begin{pmatrix}0&2&1\\-1&3&2\\1&-1&0\end{pmatrix}\begin{pmatrix}c_1\\c_2e^{2t}\\c_3e^{3t}\end{pmatrix}=\begin{pmatrix}2c_2e^{2t}+c_3e^{3t}\\-c_1+3c_2e^{2t}+2c_3e^{3t}\\c_1-c_2e^{2t}\end{pmatrix} \text{ 又 } \begin{cases}x_1(0)=3\\x_2(0)=3\\x_3(0)=1\end{cases}$$

$$\text{得 }\begin{cases}2c_2+c_3=3\\-c_1+3c_2+2c_3=3\\c_1-c_2=1\end{cases}\Rightarrow\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}2\\1\\1\end{pmatrix} \text{ 故}$$

$$X=\begin{pmatrix}2e^{2t}+e^{3t}\\-2+3e^{2t}+2e^{3t}\\2-e^{2t}\end{pmatrix}$$

2.

(1) 学习过高等数学(微积分/工科数学分析)可直接求 $\lambda$ 用公式.

这里先使用线性代数方法示范.

令 $x_1(t)=x(t)$, $x_2(t)=x'(t)$

$$\text{则 }\begin{cases}x_1'(t)=x_2(t)\\x_2'(t)=-3x_1(t)-4x_2(t)\end{cases} \quad \text{记 } X=\begin{pmatrix}x_1(t)\\x_2(t)\end{pmatrix}, A=\begin{pmatrix}0&1\\-3&-4\end{pmatrix}$$

则 $X'=AX$, 则由 $f_A(\lambda)=|\lambda E-A|=(\lambda+1)(\lambda+3)\Rightarrow\lambda$ 为 -1, -3

$$\text{则由 } X=PY\Rightarrow Y'=\begin{pmatrix}-1&\\&-3\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}\Rightarrow Y=\begin{pmatrix}c_1e^{-t}\\c_2e^{-3t}\end{pmatrix}$$

又由 $-3\lambda - A = \begin{pmatrix} -3 & -1 \\ 3 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_1 = (-1,3)^{\mathrm{T}}$

由 $-\lambda - A = \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_2 = (-1,1)^{\mathrm{T}}$, 则 $P = \begin{pmatrix} -1 & -1 \\ 1 & 3 \end{pmatrix}$

故 $X = \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-3t} \end{pmatrix} = \begin{pmatrix} -c_1 e^{-t} - c_2 e^{-3t} \\ 3c_1 e^{-t} + c_2 e^{-3t} \end{pmatrix}$

故 $x = c_1 e^{-t} + c_2 e^{-3t}$

(2) 由 $\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2, 3$

则 $x = c_1 e^{2t} + c_2 e^{3t}$

(3) 由 $\lambda^2 - 7\lambda + 6 = 0 \Rightarrow \lambda = 6, 1$

则 $x = c_1 e^{6t} + c_2 e^{t}$ 又 $\begin{cases} x(0) = 1 \\ x'(0) = 1 \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 1 \\ 6c_1 + c_2 = 1 \end{cases}$

得 $x = e^t$

(4) 由 $\lambda^2 - 6\lambda + 8 = 0 \Rightarrow \lambda = 2, 4$ 又 $D^2x - 6Dx + 8x = 8t + 10$

得 $x = \dfrac{8t+10}{(D-2)(D-4)}$ 由 $L(D) = D^2 - 6D + 8$ 且 $L(0) \neq 0$

得 $x = \dfrac{8t+10}{D^2 - 6D + 8} = \left(\dfrac{1}{8} + \dfrac{3}{32}D\right)(8t+10) = t + 2$

故特解为 $x^* = t + 2$

故 $x = c_1 e^{2t} + c_2 e^{4t} + t + 2$

3.

(1) 同2. 先用线性代数思想示范, 然后回归高数公式解题.

令 $x_1(t) = x(t)$, $x_2(t) = x'(t)$

则记 $X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $X' = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} & x_2(t) \\ -4x_1(t) & \end{pmatrix} = AX$

$A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$ 则 $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda & -1 \\ 4 & \lambda \end{vmatrix} = \lambda^2 + 4$ 得入为 $\pm 2i$

$\lambda = 2i$, $2iE - A = \begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix} \to \begin{pmatrix} 2i & -1 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_1 = (1, 2i)^{\mathrm{T}}$

$\lambda = -2i$, $-2iE - A = \begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \to \begin{pmatrix} 2i & 1 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_2 = (-1, 2i)^{\mathrm{T}}$ 则 $P = \begin{pmatrix} 1 & -1 \\ 2i & 2i \end{pmatrix}$

$\Lambda = \begin{pmatrix} 2i & \\ & -2i \end{pmatrix}$, 则由 $X = PY$ 得 $Y' = \begin{pmatrix} 2i & \\ & -2i \end{pmatrix} Y \Rightarrow Y = \begin{pmatrix} c_1 e^{2it} \\ c_2 e^{-2it} \end{pmatrix}$

故 $X = \begin{pmatrix} 1 & -1 \\ 2i & 2i \end{pmatrix} \begin{pmatrix} c_1 e^{2it} \\ c_2 e^{-2it} \end{pmatrix} = \begin{pmatrix} c_1 e^{2it} - c_2 e^{-2it} \\ 2ic_1 e^{2it} + 2ic_2 e^{-2it} \end{pmatrix}$

又由欧拉公式, 则 $x = c_1 e^{2it} - c_2 e^{-2it} = (c_1 + c_2)\cos 2t + (c_1 - c_2) i \sin 2t$

故 $x = C_1 \cos 2t + C_2 \sin 2t$

(2) 由 $\lambda^2 - 6\lambda + 10 = 0 \Rightarrow \lambda = 3 \pm i$

故 $x = e^{3t}(C_1 \cos t + C_2 \sin t)$

(3) $\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} & x_2 \\ -x_1 & \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, 记 $X' = AX$

则 $|\lambda E - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1$ 得 $\lambda$ 为 $\pm i$

$iE - A = \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \to \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_1 = (-1, i)^{\mathrm{T}}$

$-iE - A = \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix} \to \begin{pmatrix} i & -1 \\ 0 & 0 \end{pmatrix}$ 得 $\alpha_2 = (1, i)^{\mathrm{T}}$, 则 $P = \begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}$, $\Lambda = \begin{pmatrix} i & \\ & -i \end{pmatrix}$

则 $Y' = \Lambda Y = \begin{pmatrix} i & \\ & -i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \Rightarrow Y = \begin{pmatrix} c_1 e^{it} \\ c_2 e^{-it} \end{pmatrix}$

则 $X = \begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix} \begin{pmatrix} c_1 e^{it} \\ c_2 e^{-it} \end{pmatrix} = \begin{pmatrix} -c_1 e^{it} + c_2 e^{-it} \\ ic_1 e^{it} + ic_2 e^{-it} \end{pmatrix} = \begin{pmatrix} (c_2 - c_1)\cos t - (c_1 + c_2) i \sin t \\ (c_1 + c_2)\cos t - (c_1 - c_2) i \sin t \end{pmatrix}$

故 $x_1 = C_1 \cos t + C_2 \sin t$, $x_2 = -C_2 \cos t + C_1 \sin t$

(4) 记 $X = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ 则 $X' = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} x_2 \\ -8x_1 - 4x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

则由 $f_A(\lambda)=\begin{vmatrix}\lambda & -1\\ 8 & \lambda+4\end{vmatrix}=\lambda^2+4\lambda+8$ 得 $\lambda$ 为 $-2\pm 2i$

$(-2+2i)E-A=\begin{pmatrix}-2+2i & -1\\ 8 & 2+2i\end{pmatrix}\to\begin{pmatrix}-2+2i & -1\\ 0 & 0\end{pmatrix}$ 则 $\alpha_1=(1,-2+2i)^{\mathrm{T}}$

$(-2-2i)E-A=\begin{pmatrix}-2-2i & -1\\ 8 & 2-2i\end{pmatrix}\to\begin{pmatrix}2+2i & 1\\ 0 & 0\end{pmatrix}$ 则 $\alpha_2=(1,-2-2i)^{\mathrm{T}}$

故 $P=\begin{pmatrix}1 & 1\\ -2+2i & -2-2i\end{pmatrix}$, $\Lambda=\begin{pmatrix}-2+2i & \\ & -2-2i\end{pmatrix}$

由 $X=PY$ 得 $Y'=\begin{pmatrix}-2+2i & \\ & -2-2i\end{pmatrix}Y\Rightarrow Y=\begin{pmatrix}c_1e^{-2+2it}\\ c_2e^{-2-2it}\end{pmatrix}$

则 $X=\begin{pmatrix}1 & 1\\ -2+2i & -2-2i\end{pmatrix}\begin{pmatrix}c_1e^{-2+2it}\\ c_2e^{-2-2it}\end{pmatrix}=\begin{pmatrix}c_1e^{-2+2it}+c_2e^{-2-2it}\\ (-2+2ic_1)e^{-2+2it}+(-2-2ic_2)e^{-2-2it}\end{pmatrix}$

即 $\begin{cases}x_1(t)=e^{-2t}\left(C_1\cos 2x+C_2\sin 2x\right)\\ x_2(t)=e^{-2t}\left[C_2(-2C_2-2C_1)\cos 2x-2(C_1+C_2)\sin 2x\right]\end{cases}$

## 习题5.3

### (A)

1.

(1) $f_A(\lambda)=\begin{vmatrix}\lambda-1 & 2\\ 2 & \lambda-1\end{vmatrix}=(\lambda+1)(\lambda-3)$, 得特征值为 $-1,3$

$3E-A=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\to\begin{pmatrix}1 & 1\\ & \end{pmatrix}$ 得基础解系 $\alpha_1=(1,-1)^{\mathrm{T}}$

$-E-A=\begin{pmatrix}-2 & 2\\ 2 & -2\end{pmatrix}\to\begin{pmatrix}1 & -1\\ & \end{pmatrix}$ 得 $\alpha_2=(1,1)^{\mathrm{T}}$ 单位化 $\begin{cases}p_1=\left(\frac{1}{\sqrt2},-\frac{1}{\sqrt2}\right)^{\mathrm{T}}\\ p_2=\left(\frac{1}{\sqrt2},\frac{1}{\sqrt2}\right)^{\mathrm{T}}\end{cases}$

故 $Q=\begin{pmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ -\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}$, $Q^{-1}AQ=\mathrm{diag}(3,-1)$.

(2) $f_A(\lambda)=\begin{vmatrix}\lambda & 6 & -6\\ 6 & \lambda+3 & 0\\ -6 & 0 & \lambda-3\end{vmatrix}=\lambda(\lambda-9)(\lambda+9)$, 特征值为 $-9,0,9$

$$0E-A=\begin{pmatrix}0 & 6 & -6\\ 6 & 3 & \\ -6 & & -3\end{pmatrix}\to\begin{pmatrix}1 & & -\frac{1}{2}\\ & 1 & -1\\ & & \end{pmatrix}\ 得\ \alpha_1=(-1,1,2)^{\mathrm{T}}$$

$$9E-A=\begin{pmatrix}9 & 6 & -6\\ 6 & 12 & \\ -6 & & 6\end{pmatrix}\to\begin{pmatrix}1 & & -1\\ & 1 & \frac{1}{2}\\ & & \end{pmatrix}\ 得\ \alpha_2=(2,-1,2)^{\mathrm{T}}$$

$$-9E-A=\begin{pmatrix}-9 & 6 & -6\\ 6 & -6 & \\ -6 & & -12\end{pmatrix}\to\begin{pmatrix}1 & & \frac{1}{2}\\ & 1 & -1\\ & & \end{pmatrix}\ 得\ \alpha_3=(-2,-2,1)^{\mathrm{T}}$$

则 $p_1=\left(-\frac{1}{3},\frac{1}{3},\frac{2}{3}\right)^{\mathrm{T}}$, $p_2=\left(\frac{2}{3},-\frac{1}{3},\frac{2}{3}\right)^{\mathrm{T}}$, $p_3=\left(-\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right)^{\mathrm{T}}$

故 $Q=\begin{pmatrix}-\frac{1}{3} & \frac{2}{3} & -\frac{2}{3}\\ \frac{1}{3} & -\frac{1}{3} & -\frac{2}{3}\\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3}\end{pmatrix}$, $Q^{\mathrm{T}}AQ=\mathrm{diag}(0,9,-9)$.

(3) $f_A(\lambda)=\begin{vmatrix}\lambda-4 & -2 & \\ -2 & \lambda-3 & 2\\ & 2 & \lambda-2\end{vmatrix}=\lambda(\lambda-3)(\lambda-6)$, 特征值为 $0,3,6$

$$6E-A=\begin{pmatrix}2 & -2 & \\ -2 & 3 & 2\\ & 2 & 4\end{pmatrix}\to\begin{pmatrix}1 & & 2\\ & 1 & 2\\ & & \end{pmatrix}\ 得\ \alpha_1=(2,2,-1)^{\mathrm{T}}$$

$$3E-A=\begin{pmatrix}-1 & -2 & \\ -2 & 0 & 2\\ & 2 & 1\end{pmatrix}\to\begin{pmatrix}1 & & -1\\ & 1 & \frac{1}{2}\\ & & \end{pmatrix}\ 得\ \alpha_2=(2,-1,2)^{\mathrm{T}}$$

$$0E-A=\begin{pmatrix}-4 & -2 & \\ -2 & -3 & 2\\ & 2 & -2\end{pmatrix}\to\begin{pmatrix}1 & & \frac{1}{2}\\ & 1 & -1\\ & & \end{pmatrix}\ 得\ \alpha_3=(-1,2,2)^{\mathrm{T}}$$

则 $p_1=\left(-\frac{2}{3},-\frac{1}{3},\frac{1}{3}\right)^{\mathrm{T}}$, $p_2=\left(-\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)^{\mathrm{T}}$, $p_3=\left(\frac{1}{3},-\frac{2}{3},-\frac{2}{3}\right)^{\mathrm{T}}$

则 $Q=\begin{pmatrix}-\frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3}\\ \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3}\end{pmatrix}$, $Q^{\mathrm{T}}AQ=\mathrm{diag}(6,3,0)$

(4) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-2 & 2 & 2\\ 2 & \lambda-5 & -4\\ 2 & -4 & \lambda-5\end{vmatrix}=(\lambda-1)^2(\lambda-10)$, 特征值为 $1,1,10$

由 $E-A=\begin{pmatrix}-1 & 2 & 2\\ 2 & -4 & -4\\ 2 & -4 & -4\end{pmatrix}\to\begin{pmatrix}1 & -2 & -2\\ & & \\ & & \end{pmatrix}$ 则 $\alpha_1=(2,1,0)^{\mathrm{T}}$, $\alpha_2=(2,0,1)^{\mathrm{T}}$

$10E-A=\begin{pmatrix}8 & 2 & 2\\ 2 & 5 & -4\\ 2 & -4 & 5\end{pmatrix}\to\begin{pmatrix}1 & & \frac{1}{2}\\ & 1 & -1\\ & & \end{pmatrix}$ 则 $\alpha_3=(1,-2,-2)^{\mathrm{T}}$

由 $\alpha_1,\alpha_2,\alpha_3$ 正交单位化, 有 $p_1=\left(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0\right)^{\mathrm{T}}$, $p_2=\left(\frac{2}{3\sqrt{5}},-\frac{4}{3\sqrt{5}},\frac{\sqrt{5}}{3}\right)^{\mathrm{T}}$, $p_3=\left(-\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)^{\mathrm{T}}$

则 $Q=\begin{pmatrix}\frac{2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & -\frac{1}{3}\\ \frac{1}{\sqrt{5}} & -\frac{4}{3\sqrt{5}} & \frac{2}{3}\\ 0 & \frac{\sqrt{5}}{3} & \frac{2}{3}\end{pmatrix}$, $Q^{-1}AQ=\mathrm{diag}(1,1,10)$

(5) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & -2 & -2\\ -2 & \lambda-1 & -2\\ -2 & -2 & \lambda-1\end{vmatrix}=(\lambda+1)^2(\lambda-5)$, 特征值为 $5,-1,-1$

$5E-A=\begin{pmatrix}4 & -2 & -2\\ -2 & 4 & -2\\ -2 & -2 & 4\end{pmatrix}\to\begin{pmatrix}1 & & -1\\ & 1 & -1\\ & & \end{pmatrix}$ 则 $\alpha_1=(1,1,1)^{\mathrm{T}}$

$-E-A=\begin{pmatrix}-2 & -2 & -2\\ -2 & -2 & -2\\ -2 & -2 & -2\end{pmatrix}\to\begin{pmatrix}1 & 1 & 1\\ & & \\ & & \end{pmatrix}$ 则 $\alpha_2=(1,-1,0)^{\mathrm{T}}$, $\alpha_3=(1,0,-1)^{\mathrm{T}}$

正交单位化，有 $p_1=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)^{\mathrm{T}}$, $p_2=\left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)^{\mathrm{T}}$, $p_3=\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$

则 $Q=\begin{pmatrix}\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}}\end{pmatrix}$, $Q^{-1}AQ=\mathrm{diag}(5,-1,-1)$

(6) $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1 & & 1\\ & \lambda-1 & \\ 1 & & \lambda-1\end{vmatrix}=\lambda(\lambda-1)(\lambda-2)$, 特征值为 $0,1,2$

又由 $0E-A=\begin{pmatrix}-1 & & 1\\ & -1 & \\ 1 & & -1\end{pmatrix}\to\begin{pmatrix}1 & & -1\\ & 1 & \\ & & \end{pmatrix}$ 得 $\alpha_1=(1,0,1)^{\mathrm{T}}$

$E-A=\begin{pmatrix}0 & & 1\\ & 0 & \\ 1 & & 0\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0\\ & & 1\\ & & \end{pmatrix}$ 得 $\alpha_2=(0,1,0)^{\mathrm{T}}$

$2E-A=\begin{pmatrix}1 & & 1\\ & 1 & \\ 1 & & 1\end{pmatrix}\to\begin{pmatrix}1 & & 1\\ & 1 & \\ & & \end{pmatrix}$ 得 $\alpha_3=(1,0,-1)^{\mathrm{T}}$

单位化，得 $p_1=\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$, $p_2=(0,1,0)^{\mathrm{T}}$, $p_3=\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$

则 $Q=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{pmatrix}$, $Q^{-1}AQ=\mathrm{diag}(0,1,2)$

(7) $f_A(\lambda)=\begin{vmatrix}\lambda & -1 & -1 & 1\\ -1 & \lambda & 1 & -1\\ -1 & 1 & \lambda & -1\\ 1 & -1 & -1 & \lambda\end{vmatrix}=(\lambda+3)(\lambda-1)^3$, 特征值为 $1,1,1,-3$

$$E-A=\begin{pmatrix}1&-1&-1&1\\-1&1&1&-1\\-1&1&1&-1\\1&-1&-1&1\end{pmatrix}\to\begin{pmatrix}1&-1&-1&1\\&&&\\&&&\\&&&\end{pmatrix}$$ 得 $\alpha_1=(1,1,0,0)^{\mathrm{T}}$，$\alpha_2=(1,0,1,0)^{\mathrm{T}}$，

$\alpha_3=(1,0,0,-1)^{\mathrm{T}}$

$$-3E-A=\begin{pmatrix}-3&-1&-1&1\\-1&-3&1&-1\\-1&1&-3&-1\\1&-1&-1&-3\end{pmatrix}\to\begin{pmatrix}1&&&-1\\&1&&1\\&&1&1\\&&&\end{pmatrix}$$ 得 $\alpha_4=(1,-1,-1,1)^{\mathrm{T}}$

正交单位化为 $p_1=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)^{\mathrm{T}}$, $p_2=\left(\frac{1}{\sqrt{6}},-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},0\right)^{\mathrm{T}}$,

$p_3=\left(-\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},\frac{1}{2\sqrt{3}},\frac{\sqrt{3}}{2}\right)^{\mathrm{T}}$, $p_4=\left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right)^{\mathrm{T}}$

则 $Q=\begin{pmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{6}}&-\frac{1}{2\sqrt{3}}&\frac{1}{2}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{6}}&\frac{1}{2\sqrt{3}}&-\frac{1}{2}\\0&\frac{1}{\sqrt{6}}&\frac{1}{2\sqrt{3}}&-\frac{1}{2}\\0&0&\frac{\sqrt{3}}{2}&\frac{1}{2}\end{pmatrix}$, $Q^{-1}AQ=\mathrm{diag}(1,1,1,-3)$

(8) $f_A(\lambda)=\begin{vmatrix}\lambda&-1&&\\-1&\lambda&&\\&&\lambda&-1\\&&-1&\lambda\end{vmatrix}=(\lambda-1)^2(\lambda+1)^2$, 则特征值为 $-1,-1,1,1$

$$-E-A=\begin{pmatrix}-1&-1&&\\-1&-1&&\\&&-1&-1\\&&-1&-1\end{pmatrix}\to\begin{pmatrix}1&-1&&\\&&1&-1\\&&&\\&&&\end{pmatrix}$$ 则 $\begin{cases}\alpha_1=(-1,1,0,0)^{\mathrm{T}},\\\alpha_2=(0,0,-1,1)^{\mathrm{T}},\end{cases}$

$$E-A=\begin{pmatrix}1&-1&&\\-1&1&&\\&&1&-1\\&&-1&1\end{pmatrix}\to\begin{pmatrix}1&-1&&\\&&1&-1\\&&&\\&&&\end{pmatrix}$$ 则 $\begin{cases}\alpha_3=(1,1,0,0)^{\mathrm{T}},\\\alpha_4=(0,0,1,1)^{\mathrm{T}},\end{cases}$

故 $p_1=\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)^{\mathrm{T}}$, $p_2=\left(0,0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$,

$p_3=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)^{\mathrm{T}}$, $p_4=\left(0,0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$

则 $Q=\begin{pmatrix}-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\ \frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}&0\\ 0&-\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\ 0&\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\end{pmatrix}$, $Q^{-1}AQ=\mathrm{diag}(-1,-1,1,1)$

2.

证明: 依题意, 存在 $Q$, 使 $Q^{-1}AQ=\mathrm{diag}(a_1,a_2,\dots,a_n)$

由于 $r(A)=r$, 则 $a_1,a_2,\dots,a_n$ 中有 $r$ 个非零, 不妨设为前 $r$ 个, 则

$$A=Q\cdot\mathrm{diag}(a_1,a_2,\dots,a_r,0,\dots,0)\cdot Q^{-1}$$

令 $B_1=Q^{-1}\mathrm{diag}(a_1,0,\dots,0,\dots,0)Q$, $B_2=Q^{-1}\mathrm{diag}(0,a_2,0,\dots,0)Q$, $\dots$, $B_r=Q^{-1}\mathrm{diag}(0,\dots,a_r,0)Q$

则 $Q(B_1+B_2+\cdots+B_r)Q=\mathrm{diag}(a_1,a_2,\dots,a_r,0,\dots,0)=Q^{-1}AQ$

故 $A=B_1+B_2+\cdots+B_r$ 又每个 $B_i$ $(i=1,2,\dots,r)$ 都为对称阵且秩为1

证毕.

3.

$A=\alpha^{\mathrm{T}}\alpha=\begin{pmatrix}1\\-2\\3\end{pmatrix}(1\ -2\ 3)=\begin{pmatrix}1&-2&3\\-2&4&-6\\3&-6&9\end{pmatrix}$

则由 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-1&2&-3\\2&\lambda-4&6\\-3&6&\lambda-9\end{vmatrix}=\lambda^2(\lambda-14)$ 故特征值为 $0,14,0$

又 $0E-A=\begin{pmatrix}-1&2&-3\\2&-4&6\\-3&6&-9\end{pmatrix}\to\begin{pmatrix}1&-2&3\\&&\\&&\end{pmatrix}$ 得 $\alpha_1=(2,1,0)^{\mathrm{T}}$, $\alpha_2=(-3,0,1)^{\mathrm{T}}$

$14E-A=\begin{pmatrix}13&2&-3\\2&10&6\\-3&6&5\end{pmatrix}\to\begin{pmatrix}1&&-\frac{1}{3}\\&1&\frac{2}{3}\\&&\end{pmatrix}$ 得 $\alpha_3=(1,-2,3)^{\mathrm{T}}$

单位正交化 $\rightarrow p_1 = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)^{\mathrm{T}}, p_2 = \left(-\frac{3\sqrt{70}}{70}, \frac{3\sqrt{70}}{35}, \frac{\sqrt{70}}{14}\right)^{\mathrm{T}}, p_3 = \left(\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)^{\mathrm{T}}$

故 $Q = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{3\sqrt{70}}{70} & \frac{1}{\sqrt{14}} \\ \frac{1}{\sqrt{5}} & \frac{3\sqrt{70}}{35} & -\frac{2}{\sqrt{14}} \\ 0 & \frac{\sqrt{70}}{14} & \frac{3}{\sqrt{14}} \end{pmatrix}$

4.

证明: 由于 $Ax = \lambda x$, 则

$$(A\bar{x})^{\mathrm{T}} x = \bar{x}^{\mathrm{T}} A^{\mathrm{T}} x = -\bar{x}^{\mathrm{T}}(Ax) = -\bar{x}^{\mathrm{T}}(\lambda x) = -\lambda(\bar{x}^{\mathrm{T}} x)$$

$$(A\bar{x})^{\mathrm{T}} x = (\overline{A}\bar{x})^{\mathrm{T}} x = (\overline{Ax})^{\mathrm{T}} x = (\overline{\lambda x})^{\mathrm{T}} x = \bar{\lambda}(\bar{x}^{\mathrm{T}} x)$$

则 $(\lambda + \bar{\lambda})\bar{x}^{\mathrm{T}} x = 0$, 又 $x \neq 0$, 于是 $\bar{x}^{\mathrm{T}} x \neq 0$

故 $\lambda + \bar{\lambda} = 0$ 则 $\lambda$ 是纯虚数.

# 习题6.1

## (A)

1.

(1) $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$ 由 $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ 知秩为2

(2) $\begin{pmatrix} 1 & -2 & -4 \\ -2 & -2 & 3 \\ -4 & 3 & 3 \end{pmatrix}$ 由 $\begin{pmatrix} 1 & -2 & -4 \\ -2 & -2 & 3 \\ -4 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ 知秩为3.

(3) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$ 由 $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ & & \\ & & \end{pmatrix}$ 知秩为1

(4) $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ 由 $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ & & & \\ & & & \end{pmatrix}$ 知秩为2

(5) $\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$ 由 $\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ 知秩为4.

2.

(1) $3x_2^2 - 4x_1x_2$

(2) $7x_1^2 + 5x_2^2 + 8x_1x_2$

(3) $-x_1^2 + 2x_2^2 - 3x_3^2 + 8x_1x_2 + 12x_1x_3 - 10x_2x_3$

(4) $-6x_1^2 + 7x_2^2 - 2x_3^2 + 6x_1x_2$

(5) $x_1^2 + x_2^2 - 2x_3^2 - 2x_4^2 + 4x_1x_2 - 2x_3x_4$

(6) $x_1^2 + x_2^2 + x_3^2 + x_4^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_4 + 2x_2x_3 + 2x_2x_4 + 2x_3x_4$

## (B)

1.

证明: 必要性: 由 $A = O$, 则必有 $\alpha^{\mathrm{T}} A \alpha = 0$

充分性: 设 $A = (a_{ij})_{n \times n}$, 则 $X = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$,

由 $X^{\mathrm{T}} A X = 0$, 取 $X$ 为 $\varepsilon_1 = (1, 0, 0, \ldots, 0)^{\mathrm{T}}$, $\varepsilon_2 = (0, 1, \ldots, 0)^{\mathrm{T}}$, ..., $\varepsilon_n = (0, 0, \ldots, 1)^{\mathrm{T}}$.

代入 $X^{\mathrm{T}} A X = 0 \Rightarrow a_{11} = 0, a_{22} = 0, \ldots, a_{nn} = 0$.

又取 $X$ 为 $\varepsilon_i + \varepsilon_j = (0, 0, \ldots, 1, 0, \ldots, 1, \ldots, 0)^{\mathrm{T}}$,

代入 $X^{\mathrm{T}} A X = 0$, 可以得到 $a_{ij} = 0$ $(i \neq j,\ i, j = 1, 2, \ldots, n)$.

则 $A = O$.

2.

证明: 必要性: 由 $A^{\mathrm{T}} = -A$, 则 $(X^{\mathrm{T}} A X)^{\mathrm{T}} = X^{\mathrm{T}} A^{\mathrm{T}} X = -X^{\mathrm{T}} A X$,

又 $X^{\mathrm{T}} A X$ 是一个数, 于是 $(X^{\mathrm{T}} A X)^{\mathrm{T}} = X^{\mathrm{T}} A X$,

则 $X^{\mathrm{T}} A X = 0$.

充分性: 设 $X^{\mathrm{T}} A X = 0$, 取 $X$ 为 $\varepsilon_1 = (1, 0, \ldots, 0)^{\mathrm{T}}$, $\varepsilon_2 = (0, 1, \ldots, 0)^{\mathrm{T}}$, ..., $\varepsilon_n = (0, 0, \ldots, 1)^{\mathrm{T}}$,

代入 $X^{\mathrm{T}} A X = 0 \Rightarrow a_{11} = 0, a_{22} = 0, \ldots, a_{nn} = 0$. 又取 $X = \varepsilon_i + \varepsilon_j$ $(i \neq j)$,

代入 $X^{\mathrm{T}} A X = 0$ 有 $X^{\mathrm{T}} A X = (\varepsilon_i + \varepsilon_j) A (\varepsilon_i + \varepsilon_j)^{\mathrm{T}} = a_{ij} + a_{ji} = 0$ 故 $a_{ij} = -a_{ji}$, 从而 $A$ 为反对称矩阵.

# 习题6.2

## (A)

1.

(1) $A=\begin{pmatrix}2&-2&0\\-2&1&-2\\0&-2&0\end{pmatrix}$,

则 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda-2&2&0\\2&\lambda-1&2\\0&2&\lambda\end{vmatrix}=(\lambda+2)(\lambda-1)(\lambda-4)$,

$\lambda=-2$ 时, $\begin{pmatrix}-4&2&0\\2&-3&2\\0&2&-2\end{pmatrix}\to\begin{pmatrix}1&0&-\frac{1}{2}\\0&1&-1\\0&0&0\end{pmatrix}$, $\alpha_1=(1,2,2)^{\mathrm{T}}$;

$\lambda=1$ 时, $\begin{pmatrix}-1&2&0\\2&0&2\\0&2&1\end{pmatrix}\to\begin{pmatrix}1&0&1\\0&1&\frac{1}{2}\\0&0&0\end{pmatrix}$, $\alpha_2=(2,1,-2)^{\mathrm{T}}$;

$\lambda=4$ 时, $\begin{pmatrix}2&2&0\\2&3&2\\0&2&4\end{pmatrix}\to\begin{pmatrix}1&0&-2\\0&1&2\\0&0&0\end{pmatrix}$, $\alpha_3=(2,-2,1)^{\mathrm{T}}$;

正交单位化, 有 $p_1=\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)^{\mathrm{T}}$, $p_2=\left(\frac{2}{3},\frac{1}{3},-\frac{2}{3}\right)^{\mathrm{T}}$, $p_3=\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right)^{\mathrm{T}}$,

标准形 $y_1^2-2y_2^2+4y_3^2$, $X=\begin{pmatrix}\frac{2}{3}&\frac{1}{3}&\frac{2}{3}\\\frac{1}{3}&\frac{2}{3}&-\frac{2}{3}\\-\frac{2}{3}&\frac{2}{3}&\frac{1}{3}\end{pmatrix}Y$.

(2) $A=\begin{pmatrix}1&-2&-4\\-2&4&-2\\-4&-2&1\end{pmatrix}$,

则 $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda-1 & 2 & 4 \\ 2 & \lambda-4 & 2 \\ 4 & 2 & \lambda-1 \end{vmatrix} = (\lambda+4)(\lambda-5)^2$,

$5E - A = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{1}{2} & 1 \\ & & \\ & & \end{pmatrix}$,

得 $\alpha_1 = (1,-2,0)^{\mathrm{T}}$, $\alpha_2 = (-1,0,1)^{\mathrm{T}}$;

$-4E - A = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & & -1 \\ & 1 & -\frac{1}{2} \\ & & \end{pmatrix}$,

得 $\alpha_3 = (2,1,2)^{\mathrm{T}}$;

正交单位化, 有

$$p_1 = \left(\frac{1}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right)^{\mathrm{T}}, p_2 = \left(\frac{\sqrt{5}}{6}, -\frac{2\sqrt{5}}{3}, -\frac{\sqrt{5}}{6}\right)^{\mathrm{T}}, p_3 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right),$$

标准形 $5y_1^2 + 5y_2^2 - 4y_3^2$, $X = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{\sqrt{5}}{6} & \frac{2}{3} \\ 0 & -\frac{2\sqrt{5}}{3} & \frac{1}{3} \\ -\frac{1}{\sqrt{5}} & -\frac{\sqrt{5}}{6} & \frac{2}{3} \end{pmatrix} Y$.

(3) $A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,

则 $f_A(\lambda) = |\lambda E - A| = \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = (\lambda + \frac{1}{2})^2(\lambda - 1)$,

$-\frac{1}{2}E - A = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ & & \\ & & \end{pmatrix}$,

得 $\alpha_1 = (1,-1,0)^{\mathrm{T}}$, $\alpha_2 = (1,0,-1)^{\mathrm{T}}$;

$$E-A=\begin{pmatrix}1 & -\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & 1 & -\frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2} & 1\end{pmatrix}\to\begin{pmatrix}1 & & -1\\ & 1 & -1\\ & & \end{pmatrix},$$

得 $\alpha_3=(1,1,1)^{\mathrm{T}}$

则 $p_1=\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)^{\mathrm{T}}$, $p_2=\left(-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$,

$p_3=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$,

标准形 $-\frac{1}{2}y_1^2-\frac{1}{2}y_2^2+y_3^2$, $X=\begin{pmatrix}-\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}Y.$

(4) $A=\begin{pmatrix}0 & 0 & 0 & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & 0\\ 0 & \frac{1}{2} & 0 & 0\\ \frac{1}{2} & 0 & 0 & 0\end{pmatrix}$,

则 $f_A(\lambda)=|\lambda E-A|=\begin{vmatrix}\lambda & 0 & 0 & -\frac{1}{2}\\ 0 & \lambda & -\frac{1}{2} & 0\\ 0 & -\frac{1}{2} & \lambda & 0\\ -\frac{1}{2} & 0 & 0 & \lambda\end{vmatrix}=(\lambda-\frac{1}{2})^2(\lambda+\frac{1}{2})^2$,

$$\frac{1}{2}E-A=\begin{pmatrix}\frac{1}{2} & 0 & 0 & -\frac{1}{2}\\ 0 & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & -\frac{1}{2} & \frac{1}{2} & 0\\ -\frac{1}{2} & 0 & 0 & \frac{1}{2}\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

得 $\alpha_1=(1,0,0,1)^{\mathrm{T}}$, $\alpha_2=(0,1,1,0)^{\mathrm{T}}$;

$$-\frac{1}{2}E-A=\begin{pmatrix}-\frac{1}{2} & 0 & 0 & -\frac{1}{2}\\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0\\ 0 & -\frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & 0 & 0 & -\frac{1}{2}\end{pmatrix}\to\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix},$$

得 $\alpha_3=(1,0,0,-1)^{\mathrm{T}}$, $\alpha_4=(0,1,-1,0)^{\mathrm{T}}$;

标准形 $\frac{1}{2}y_1^2+\frac{1}{2}y_2^2-\frac{1}{2}y_3^2-\frac{1}{2}y_4^2$, $X=\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0\\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0\end{pmatrix}Y$

2.

(1)

$$\begin{aligned}&x_1^2+2x_2^2-x_3^2+2x_1x_2-2x_3x_1\\ &=[x_1^2+2x_1(x_2-x_3)+(x_2-x_3)^2]-(x_2-x_3)^2+2x_2^2-x_3^2\\ &=(x_1+x_2-x_3)^2+x_2^2+2x_2x_3-2x_3^2\\ &=(x_1+x_2-x_3)^2+(x_2+x_3)^2-3x_3^2\end{aligned}$$

则 $\begin{cases}y_1=x_1+x_2-x_3\\ y_2=x_2+x_3\\ y_3=x_3\end{cases}$,

得 $X=\begin{pmatrix}1 & -1 & 2\\ & 1 & -1\\ & & 1\end{pmatrix}Y$, 化为 $y_1^2+y_2^2-3y_3^2$.

(2) 令 $\begin{cases}x_1=y_1+y_2\\ x_2=y_1-y_2\\ x_3=y_3\end{cases}$

则 $f=2y_1^2-2y_2^2+4y_1y_3=2(y_1^2+2y_1y_3+y_3^2)-2y_2^2-2y_3^2=2(y_1+y_3)^2-2y_2^2-2y_3^2$,

则 $\begin{cases}z_1=y_1+y_3\\ z_2=y_2\\ z_3=y_3\end{cases}\Rightarrow\begin{cases}y_1=z_1-z_3\\ y_2=z_2\\ y_3=z_3\end{cases}$

故 $X=\begin{pmatrix}1 & 1 & -1\\ 1 & -1 & -1\\ 0 & 0 & 1\end{pmatrix}Z$, 化为 $2z_1^2-2z_2^2-2z_3^2$.

(3)

$$\begin{aligned}
&x_1^2 - x_2^2 + 2x_1x_2 + 4x_3x_1 \\
&= [x_1^2 + 2(x_2 + 2x_3)x_1 + (x_2 + 2x_3)^2] - (x_2 + 2x_3)^2 - x_2^2 \\
&= (x_1 + x_2 + 2x_3)^2 - x_2^2 - (x_2 + 2x_3)^2
\end{aligned}$$

令 $\begin{cases} y_1 = x_1 + x_2 + 2x_3 \\ y_2 = x_2 \\ y_3 = x_2 + 2x_3 \end{cases}$

得 $X = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} Y$, 化为 $y_1^2 - y_2^2 - y_3^2$.

(4) 令 $X = \begin{pmatrix} 1 & -1 \\ 1 & -1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} Y$, 则 $x_1x_4 + x_2x_3 = y_1^2 - y_4^2 + y_2^2 - y_3^2$

3.

| | 秩 | 正惯性指数 | 负惯性指数 | 符号差 |
|---|---|---|---|---|
| 1. | | | | |
| (1) | 3 | 2 | 1 | 1 |
| (2) | 3 | 2 | 1 | 1 |
| (3) | 3 | 1 | 2 | -1 |
| (4) | 4 | 2 | 2 | 0 |
| 2. | | | | |
| (1) | 3 | 2 | 1 | 1 |
| (2) | 3 | 1 | 2 | -1 |
| (3) | 3 | 1 | 2 | -1 |
| (4) | 4 | 2 | 2 | 0 |

4.

依题意 $A \sim \begin{pmatrix} 9 & & \\ & 9 & \\ & & 0 \end{pmatrix}$ 且 $A\beta_3 = 0$

考虑 $PP^{\mathrm{T}} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} (1\ 2\ 2) = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 4 \\ 2 & 4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 2 \\ & & \\ & & \end{pmatrix}$

由于不同的特征值的特征向量正交. 故 $PP^{\mathrm{T}}\alpha = 0$ 的解即其他特征向量

则可取 $\alpha_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, $\alpha_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 0 & & \\ & 9 & \\ & & 9 \end{pmatrix}$

故 $A = Q\Lambda Q^{-1} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & & \\ & 9 & \\ & & 9 \end{pmatrix} \begin{pmatrix} \frac{1}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & -\frac{5}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{4}{9} & -\frac{5}{9} \end{pmatrix} = \begin{pmatrix} 8 & -2 & -2 \\ -2 & 5 & -4 \\ -2 & -4 & 5 \end{pmatrix}$

# 习题6.3

## (A)

1.

(1) 由 $A = \begin{pmatrix} 1 & -4 & 1 \\ -4 & 1 & -2 \\ 1 & -2 & 2 \end{pmatrix}$, $f_A(\lambda) = \begin{vmatrix} \lambda - 1 & 4 & -1 \\ 4 & \lambda - 1 & 2 \\ -1 & 2 & \lambda - 2 \end{vmatrix}$,

由于 $\begin{vmatrix} 1 & -4 \\ -4 & 1 \end{vmatrix} = -7 < 0$, 故 $A$ 不是正定的.

(2) $A = \begin{pmatrix} 7 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 6 \end{pmatrix}$, 又 $|7| > 0$, $\begin{vmatrix} 7 & -2 \\ -2 & 8 \end{vmatrix} = 52 > 0$, $\begin{vmatrix} 7 & -2 & 0 \\ -2 & 8 & -2 \\ 0 & -2 & 6 \end{vmatrix} = 284 > 0$, 故 $A$ 是正定的.

(3) $A = \begin{pmatrix} 99 & -6 & 24 \\ -6 & 130 & -30 \\ 24 & -30 & 70 \end{pmatrix}$, 又 $|99| > 0$, $\begin{vmatrix} 99 & -6 \\ -6 & 130 \end{vmatrix} > 0$, $\begin{vmatrix} 99 & -6 & 24 \\ -6 & 130 & -30 \\ 24 & -30 & 70 \end{vmatrix} = 743040 > 0$, 故 A 是正

定的.

(4) $A=\begin{pmatrix}10&4&12\\4&2&-14\\12&-14&1\end{pmatrix}$, 又 $|A|=-3588<0$, 不是正定的

2.

证明: 依题意, $X^{\mathrm{T}}AX$, $X^{\mathrm{T}}BX$ 是正定二次型, 则

$$X^{\mathrm{T}}(A+B)X=X^{\mathrm{T}}AX+X^{\mathrm{T}}BX,$$

由于 $X^{\mathrm{T}}AX>0$, $X^{\mathrm{T}}BX>0$, 故 $X^{\mathrm{T}}(A+B)X>0$,

故 $X^{\mathrm{T}}(A+B)X$ 也是正定的,

从而 $A+B$ 也是正定矩阵.

3.

(1) $A=\begin{pmatrix}4&-1&2\\-1&1&-1\\2&-1&t\end{pmatrix}$, 则 $|A|>1$, $\begin{vmatrix}4&-1\\-1&1\end{vmatrix}=3>0$, $\begin{vmatrix}4&-1&2\\-1&1&-1\\2&-1&t\end{vmatrix}>0$,

得 $t>\dfrac{4}{3}$.

(2) $A=\begin{pmatrix}2&t&-3\\t&3&1\\-3&1&4\end{pmatrix}$, 由 $\begin{vmatrix}2&t\\t&3\end{vmatrix}>0$ 得 $t^2<6$, 又 $\begin{vmatrix}2&t&-3\\t&3&1\\-3&1&4\end{vmatrix}>0$,

得 $-4t^2-72t-49>0$,

但由 $t>-\sqrt{6}$,

则 $4t^2+72t+49>193-72\sqrt{6}>16>0$, 故它不可能正定

4.

证明: 由于 $A^{\mathrm{T}}=A$, 则 $A=A^{\mathrm{T}}A^{-1}A$, 即 $A^{-1}$ 合同于 $A$,

由于 $A$ 是正定的, 故 $A^{-1}$ 也正定.

5.

证明: 由于 $f(x_1,x_2,\ldots,x_n)=X^{\mathrm{T}}AX=\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}a_{ij}x_ix_j$ 正定,

取 $X_i=\varepsilon_i^{\mathrm{T}}=(0,\ldots,0,1,0,\ldots,0)^{\mathrm{T}}$ (其中第 $i$ 个分量 $x_i=1$),

则 $X_i^{\mathrm{T}}AX_i = a_{ii}x_i^2 = a_{ii} > 0$.