Ex 1.1

$$\det \begin{pmatrix} -8-\lambda & 11 \\ -6 & 9-\lambda \end{pmatrix}$$

$$= (-8-\lambda)(9-\lambda) - (-6)(11) = -72 + 8\lambda - 9\lambda + \lambda^2$$
$$+66$$

$$= \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) = 0$$

$$\Rightarrow \lambda_2 = 3, \lambda_1 = -2$$

$$\text{Kern} \begin{pmatrix} -11 & 11 \\ -6 & 6 \end{pmatrix}$$

$$\sim \begin{pmatrix} -11 & 11 \\ 0 & 0 \end{pmatrix}$$

$$x_1 = \mu$$
$$x_2 = \mu \in \mathbb{R}$$

$\Rightarrow$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is Basis for λ_2

$$\ker \begin{pmatrix} -6 & 11 \\ -6 & 11 \end{pmatrix}$$

$$\sim \begin{vmatrix} -6 & 11 \\ 0 & 0 \end{vmatrix}$$

$x_1 = \frac{11}{6}\mu$

$x_2 = \mu \in \mathbb{R}$

$\Rightarrow \begin{pmatrix} \frac{11}{6} \\ 1 \end{pmatrix}$, equivalent

$\begin{pmatrix} 11 \\ 6 \end{pmatrix}$ is a Basis for

$\lambda_1 = -2$

Ex 1.2

$$y_1' = -8y_1 + 11y_2$$
$$y_2' = -6y_1 + 9y_2 \qquad : (\mathrm{I})$$

We need two functions $y = (y_1, y_2)$
$z = (z_1, z_2)$
which both fullfill both equations (I)

let $y = (11 \cdot e^{-2x}, 6e^{-2x})$
$z = (e^{3x}, e^{3x})$

$$y_1' = 11e^{-2x}(-2) = -22e^{-2x}$$
$$= -88e^{-2x} + 66e^{-2x}$$
$$= -8y_1 + 11y_2.$$

and

$$y_2' = -12e^{-2x}$$
$$= -66e^{-2x} + 54e^{-2x}$$
$$= -6y_1 + 9y_2$$

$\Longrightarrow$ y is one solution

$$z_1' = 3e^{3x}$$

$$= -8e^{3x} + 11e^{3x}$$

$$= -8z_1 + 11z_2$$

and

$$z_2' = 3e^{3x}$$

$$= -6e^{3x} + 9e^{3x}$$

$$= -6z_1 + 9z_1$$

z is a solution

it follows from the hint

$\Rightarrow$ $c_1 \cdot y + c_2 z$ are solutions $c_1, c_2 \in \mathbb{R}$

$\square$

Ex 1.3

Notice $A\bar{x} = y$

$$A\tilde{x} = 1$$

$$\Rightarrow \quad A\bar{x} + A\tilde{x} = y + 1$$

$$A(\bar{x} + \tilde{x}) = y + 1$$

$$A(\bar{x} + \tilde{x}) - 1 = y$$

We already have $A\bar{x} = y$ from 1.2

Now we solve $Ax = 1$

$$\left|\begin{array}{cc|c} -8 & 11 & 1 \\ -6 & 9 & 1 \end{array}\right|$$

$$\Rightarrow \quad x_2 = x_1 = \frac{1}{3}$$

So the solution is

$$c_1 e^{-2x}\begin{pmatrix} 11 \\ 6 \end{pmatrix} + c_2 e^{3x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

test

$$y_1' + 1 = -8\left(c_1 e^{-2x}\begin{pmatrix} 11 \\ 6 \end{pmatrix} + c_2 e^{3x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}\right)$$
$$+ 11\left(c_1 e^{-2x}\begin{pmatrix} 11 \\ 6 \end{pmatrix} + c_2 e^{3x}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}\right)$$

$$= \underbrace{\begin{pmatrix} -8\left(c_1 e^{-2x}\begin{pmatrix}11\\6\end{pmatrix} + c_2 e^{3x}\begin{pmatrix}1\\1\end{pmatrix}\right) \\ -11\left(c_1 e^{-2x}\begin{pmatrix}11\\6\end{pmatrix} + c_2 e^{3x}\begin{pmatrix}1\\1\end{pmatrix}\right)\end{pmatrix}}_{= y'} \underbrace{\begin{pmatrix} + -8\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\end{pmatrix} \\ + 11\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\end{pmatrix}\end{pmatrix}}_{= \begin{pmatrix}1\\1\end{pmatrix}}$$

analogously for the 2nd equation

$\square$

Ex 2

Compute characteristic Polyn.

$$\det\begin{pmatrix} -3-\lambda & -4 \\ 10 & 9-\lambda \end{pmatrix}$$

$$= (-3-\lambda)(9-\lambda) - (-4)10$$

$$= \lambda^2 - 6\lambda - 27 + 40$$

$$= \lambda^2 - 6\lambda + 13$$

find roots

$$\lambda^2 - 6\lambda + 13 = 0$$

$$= (\lambda - (3+2i))(\lambda - (3-2i))$$

$$= \lambda^2 - 3\lambda + 2\lambda i - 3\lambda - 2\lambda i + (3-2i)(3+2i)$$

$$= \lambda^2 - 6\lambda + 9 + 4$$

$$= \lambda^2 - 6\lambda + 13$$

compute Eig. vectors

$$\left(\begin{array}{cc|c} -3-(3+2i) & -4 & 0 \\ 10 & 9-(3+2i) & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -6-2i & -4 & 0 \\ 10 & 6-2i & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} -6-2i & -4 & 0 \\ 0 & \frac{40}{-6-2i} + 6-2i & 0 \end{array}\right)$$

$$\frac{40}{-6-2i} + 6-2i = \frac{40 + -(6+2i)(6-2i)}{-6-2i}$$

$$= \frac{40-40}{-6-2i}$$

$$= 0$$

$$= \frac{-24i - 8}{40}$$

$$x_2 = \mu \in \mathbb{C}$$

$$x_1 = \frac{4\mu}{-6-2i}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mu \begin{pmatrix} \frac{4}{-6-2i} \\ 1 \end{pmatrix}$$

now for $\lambda_2 = 3-2i$

$$\left(\begin{array}{cc|c} -3-3+2i & -4 & 0 \\ 10 & 9-3+2i & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -6+2i & -4 & 0 \\ 10 & 6+2i & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} -6+2i & -4 & 0 \\ 0 & \dfrac{40+(6+2i)\cdot(-(6-2i))}{-6+2i} & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} -6+2i & -4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mu \begin{pmatrix} \frac{4}{-6+2i} \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad v_2 = \bar{v}_1 \quad , \text{for} \quad v_1 = \begin{pmatrix} \frac{4}{-6-2i} \\ 1 \end{pmatrix}$$

$\square$

Ex 2.2

from the script we have

$$y_j(x) = \left(\operatorname{Re} v_j \cos(\operatorname{Im} \lambda_j) - \operatorname{Im} v_j \sin(\operatorname{Im} \lambda_j)\right) \cdot e^{\operatorname{Re} \lambda_j}$$

$$y_i(x) = \operatorname{Re} v_j \sin(\operatorname{Im} \lambda_j) - \operatorname{Im} v_j \cos(\operatorname{Im} \lambda_j) \cdot e^{\operatorname{Re} \lambda_j}$$

where $y_j(x), y_i(x)$ are Basis for all solutions, therefore the linear combination is

$$c_2 y_j(x) + c_1 y_i(x) \qquad , c_1, c_2 \in \mathbb{C}$$

$$= \left(c_2 \cdot \operatorname{Re}(v_j) \cos(\operatorname{Im}(\lambda_j)) + c_1 \operatorname{Re}(v_j) \sin(\operatorname{Im}(\lambda_j))\right) e^{\operatorname{Re} \lambda_j}$$
$$+ \left(c_1 \operatorname{Im}(v_j) \cos(\operatorname{Im}(\lambda_j)) - c_2 \operatorname{Im}(v_j) \sin(\operatorname{Im}(\lambda_j)\right) e^{\operatorname{Re} \lambda_j}$$

for $v_j = \begin{pmatrix} -2 \\ 3+i \end{pmatrix}, \lambda_j = 3+2i$

we get

$$\begin{pmatrix} -2 \\ 3 \end{pmatrix} \left(c_2 \cos(2x) + \sin(2x)\, c_1\right) e^{3x}$$
$$+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \left(\cos(2x)\, c_1 - \sin(2x)\, c_2\right) e^{3x}$$

assuming there is a little typo in the exercise $\square$

Ex 3.1

compute Eigenvalues

$$\det\begin{pmatrix} -5-\lambda & 1 \\ -1 & -3-\lambda \end{pmatrix}$$

$$= (-5-\lambda)(-3-\lambda) - (-1)(1)$$

$$= \lambda^2 + 5\lambda + 3\lambda + 15 + 1$$

$$= \lambda^2 + 8\lambda + 16$$

$$= (\lambda+4)(\lambda+4)$$

$$\Rightarrow \lambda = -4 \,, \quad \mu_A(-4) = 2$$

$$\ker\begin{pmatrix} -5+4 & 1 \\ -1 & -3+4 \end{pmatrix}$$

$$\sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mu \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

since $\dim(\ker(A+4I)) < \mu(-4) = 2$

$A = \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix}$ is not

diagonazible

Ex 3.2

first sub task

Geometric multiplicity $= 1$

implies there is only one

Jordan block, therefore

$$JNF_A = \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix}$$

2nd sub task

$$\begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix} = P^{-1} A P$$

$$\Longleftrightarrow$$

$$P \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix} P^{-1} = A$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -4 & 1 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -3 \\ -4 & -7 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 1 \\ -1 & -3 \end{pmatrix} = A$$

Ex 3.3

from the script we know

$$e^{tA} = e^{P^{-1} tJ P} = P^{-1} e^{tJ} P$$

for $t = (x - x_0)$
and also that e^{Ax}

is a general solution
to linear ODE's

instead of a diagonalizable
Matrix we use JNF, also
from script

$$e^{tJ} = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} \cdot e^{t(-4)}$$

$$= \begin{vmatrix} e^{-t4} & te^{-t4} \\ 0 & e^{-t4} \end{vmatrix}$$

which means, that the span

of this matrix is the solution space and the generalized solution

$$e^{At} y_0 = P e^{Jt} P^{-1} y_0$$

$$= (P e^{Jt})(P^{-1} y_0)$$

$$y_0 = \begin{pmatrix} y(x_0) \\ y(x_1) \end{pmatrix}$$

$$\Rightarrow P^{-1} y_0 = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Where yo is the vector of initial conditions

$$\Rightarrow (P e^{Jt})(P^{-1} y_0)$$

$$= P \; e^{Jt} \underbrace{(P^{-1} y_0)}_{= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t4} & t e^{-t4} \\ 0 & e^{-t4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} .$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 e^{-t4} + c_2 e^{-t4} \\ c_2 e^{-t4} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{-t4} + c_2 e^{-t4} + c_2 e^{-t4} \\ c_1 e^{-t4} + c_2 e^{-t4} + 2 c_2 e^{-t4} \end{bmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} c_1 e^{-4t} + c_2 e^{-4t} \left(t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(e^{-4t} c_1 + t e^{-4t} c_2 \right) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-4t} c_2$$

$\square$

E4.1

find roots of polynomial

$p(s) = s^3 + s^2 - s - 1 = 0$

$p(1) = 0$

$$
\begin{array}{l}
s^3 + s^2 - s - 1 : (s-1) = s^2 + 2\cdot s + 1 \\
-(s^3 - s^2) \\
\quad 2\cdot s^2 - s - 1 \\
\quad -(2\cdot s^2 - 2\cdot s) \\
\qquad s - 1 \\
\qquad -(s-1) \\
\hline
\qquad 0
\end{array}
$$

$p(s) = (s+1)^2 (s-1)$

from script we know

$\{ x^{\ell} e^{\lambda_m x} :$
$m \in \{1, \dots, k\}, \ell \in \{0, 1, \dots, n_m - 1\}$

forms a basis for

solution space, where α_m are the roots and n_m is the algebraic multiplicity of that root.

so $\{xe^{-x}, e^{-x}, e^{x}\}$ a basis for

$$\{y \in \mathbb{R}^{\mathbb{R}} : y^{(3)} + y^{(2)} - y^{(1)} - y^{(0)} = 0\}$$

Ex 8.2

$$y^{(4)} + 2y^{(2)} + y^{(0)} = 0$$

form Polynomial

$$p(s) = s^4 + 2s^2 + s^0 = 0$$

$$= s^4 + 2s^2 + 1 = 0$$

i is root since $P(i) = 0$

$$s^4 + 2 \cdot s^2 + 1 : (s - i) = s^3 + is^2 + s + i$$
$$-(s^4 - is^3)$$

$$is^3 + 2 \cdot s^2 + 1$$
$$-(-is^3 - i^2 s^2)$$

$$s^2 + 1$$
$$-(s^2 - is)$$

$$si + 1$$
$$-(si - i^2)$$

$$\overline{\quad 0}$$

more roods

$$s^3 + is^2 + s + i = 0$$

$$s = i$$

$$i^3 - i + i + i = -i - i + i + i = 0$$

$$s^3 + is^2 + s + i : (s - i) = s^2 + 2is - 1$$
$$-(s^3 - is^2)$$

$$2is^2 + s + i$$
$$-(2is^2 + 2s)$$

$$-s + i$$
$$-(-s + i)$$

$$\overline{\quad\quad 0 \quad\quad}$$ more rook

$$s^2 + 2is - 1 = 0$$

$$(s + i)^2$$

$$\Rightarrow p(s) = (s - i)^2 (s + i)^2$$

so the roots are $i, -i$

so the basis is

$$\{ e^{ix}, e^{-ix}, xe^{ix}, xe^{-ix} \}$$

now we could somehow find the coefficients with the initial conditions

...