Definitions and Proofs in latex

Def. Topology

Let X be some set and $\mathcal{T} \subseteq \mathcal{P}(X)$ be a collection

Then $\mathcal{T}$ is a topology iff

1. $X \in \mathcal{T}$
2. $\forall A \subseteq \mathcal{T} : |A| < \infty \Rightarrow \bigcap_{a \in A} a \in \mathcal{T}$
3. $\forall A \subseteq \mathcal{T} : \bigcup_{a \in A} a \in \mathcal{T}$

We call X a topological space iff it has a topology

For a topology $\mathcal{T}$ we call $A \in \mathcal{T}$ open

Def. Closed

$Y \subseteq X$ is said to be closed iff $Y^c := X \setminus Y$ is open

lemma

Let X be a topological space then for any collection $\{A_i\}_{i \in I}$ of closed sets we have

1. if $|I| < \infty$ then $\bigcup_{i \in I} A_i$ is closed
2. $\bigcap_{i \in I} A_i$ is closed

Proof of 1.

$(\bigcup_{i \in I} A_i) = ((\bigcup_{i \in I} A_i)^c)^c = (\bigcap_{i \in I} A_{ci})^c$

Since each A_i is closed, A_{ci} is open and I is finite so the finite intersection is open as well, taking its complement gives a closed set.

Proof of 2.

$(\bigcap_{i \in I} A_i) = ((\bigcap_{i \in I} A_i)^c)^c = (\bigcup_{i \in I} A_{ci})^c$

again, A_i is closed so A_{ci} is open so the union of opens is open, thus its complement is closed

Def. closure

Let $A \subseteq X$ be a subset of a topological space X, then

the closure of A is given by $\quad A := \bigcap_{A}$

By the preceeding lemma, the closure is obviously closed since its an intersection of closed sets

Theorem 1, characterization of closed

Let $A \subseteq X$ be a subset of a topological space X, then

A is closed iff $A = A$

Proof

We first prove the left right implication (=>) so suppose A is closed, then A is a closed set containing A so

$A \ A$, A is closed, by definition of the closure this impies $\quad A \subseteq A$

But any of the sets forming the intersecton contain A so we get

$A = \bigcap_{A} A$, therefore we have

$A \subseteq A \subseteq A$ which gives us $A = A$

Now lets proceed with the right left implication (<=)

Suppose we have $A = A$ then

$A = \bigcap_{A}$, by the preceeding lemma, an arbitrary intersection of closed sets is closed

so we are immediately done.

Def neigbourhood

Let $\in X$ be a point in some topological space X then the neighbourhood of is given by

$() := \{\ |\ \in \}$

Theorem 2, consistency with alternative definition

Let A be a subset of a topological space X then

A is closed iff $\forall \in X \setminus A \in () : \ \subseteq X \setminus A$

Proof

(=>)

assume A is closed, then

$$A = \ A = \bigcap_{A}$$

which implies

$$A^c = \bigcup_{A}{}^c$$

also notice that $\ A$ implies ${}^c \subseteq A^c = X \setminus A$

Now let $\in X \setminus A$ be some arbitrary point

we now get from the preceeding lines

$$\in \bigcup_{A}{}^c$$

so there exists at least one $\ A$ that is closed and $\in {}^c$

Since ^{c} is open and $\in {}^c$ this means that ${}^c \in ()$

And given that ${}^c \subseteq A^c = X \setminus A$, we are done with this direction.

(<=)

So suppose now that

$\forall \in X \setminus A \in () : \ \subseteq X \setminus A$

This defines a collection $\{\}_{\in X \setminus A}$ where we pick one neigbourhood per , satisfying the preceeding statement

Its hard to escape the noticing that

$$A^c = X \setminus A = \bigcup_{\in X \setminus A} \{\} \subseteq \bigcup_{\in X \setminus A} \ \subseteq X \setminus A = A^c$$

where the first inclusion is given by the fact that there exists such a neighbourhood for each point in X

we conclude by recalling that the intersection of arbitrary closed sets, is closed which happens to be $(A^c)^c = A$

lemma 3

Let A be some subset of some topological space X

$\in \ A$ iff $\forall \in () : \ A$

Proof

(=>)

Suppose $\in \ A$, so by definiton of the closure for any closed set with $\ A$ we have

$\in$

Now suppose towards absurdity that

$\in () : \ A = $

This implies $A \subseteq {}^c$,since is open, ^{c} is closed and it contains A so we have

$\in {}^c$ but this is absurd since $\in$ which would imply that

^{c}

So there cannot exist such which proofs that

$\forall \in () : \ A$

(<=)

Now we consider any point $\in X$ that satisfies

$\forall \in () : \quad A$

Now assume towards absurdity that there exists a closed set with $A \subseteq$ and

This implies that $\in {}^c$ which is open and thus ${}^c \in ()$

So we have ${}^c \quad A$, but this cannot be true because

$A \subseteq$ implies $A^c \quad {}^c$ so ${}^c \quad A =$

this is absurd, so such a cannot exist, which means is a member of every such so

$\forall \quad A : \quad \in$ which implies

$\in \quad A$ and we are done

Def. Filter

A collection of subsets of X is called a filter iff

1. $X \in$
2. if $A \in$ and $A \subseteq \quad \subseteq X$ then $\in$
3. if $A \in$ then $A \quad \in$

A filter is said to be proper iff

Theorem 4, bradley

Let A be a subset of some topological space X then

$\in \quad A$ iff $: () \subseteq A \in$

Proof

(=>)

suppose $\in \quad A$

Define the filter base $:= \{ \quad A | \quad \subseteq X \quad \in () \}$

and the filter $= \{ \subseteq X | \in \quad \subseteq \}$

Because $A \subseteq X$ and $A \in$ we have $X \in$ satisfying the first property of a filter

Now suppose that $\in$ then any superset that is also a subset of X , is in the filter because

$(\quad A) \in \quad : \quad A \subseteq$ and therefore we have

$A \subseteq \quad \subseteq$ which implies $\in$

which proves the second property of a filter

Lastly, consider $\in$ we aim to show $\in$

Its clear that $\in ()$ with

$A \subseteq$ and

$A \subseteq$ which gives

$A \quad A$
$= \quad A \subseteq$

where $\in ()$ because finite intersections of opens is open and both contain by def.

so therefore $A \in$

which means that $\in$

which proves tha is indeed a filter

Now recall from lemma 3 that because $\in \quad A$ we have

$\forall \in () : \quad A$

So for any superset A we conclude

which shows that the filter is proper

Lastly, because of $A \subseteq A$ we have $A \in$ but also

$\forall \in () : \quad A \subseteq$ which implies

$() \subseteq$ which shows that the filter 'converges' to

this concludes this direction of the proof

(<=)

Let $\in X$ be some point and let be a proper filter satisfying the assumptions, that is

$() \subseteq$ and $A \in$

by the third property of any filter we have for any $\in ()$

$A \in$

since the filter is proper A

by lemma 3 we get $\in \quad A$ which concludes this proof