Universität Zürich
Institut für Mathematik
MAT221
Fall Semester 2023

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Sheet 3 (Outer measures and Lebesgue's measure)

Deadline: October 13th, 10:00 a.m.
We write λ (λ^*) for Lebesgue's (outer) measure and $\mathcal{M}_{\lambda^*}$ for the σ-algebra of Lebesgue-measurable sets.

Exercise 1 (3 points) Let $\mu^* : \mathcal{P}(X) \to [0, \infty]$ an outer measure on X and a sequence of pairwise disjoint μ^*-measurable sets $(E_n)_1^\infty \subseteq \mathcal{M}_{\mu^*}$. Prove that

$$\mu^* \left(E \cap \bigcup_{n \geq 1} E_n \right) = \sum_{n \geq 1} \mu^*(E \cap E_n), \quad \text{for every } E \subseteq X.$$

Exercise 2 (3 points) Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection of $\mathbb{R}^2$ onto $\mathbb{R}$, i.e., $\pi(x, y) = x$. Define a function $\mu^* : \mathcal{P}(\mathbb{R}^2) \to [0, +\infty]$ by $\mu^*(A) = \lambda^*(\pi(A))$.

1. Show that μ^* is an outer measure on $\mathbb{R}^2$.
2. Show that $B \in \mathcal{M}_{\mu^*}$ if and only if there are $B_0, B_1 \in \mathcal{M}_{\lambda^*}$ such that $B_0 \subseteq B_1$, $\lambda^*(B_1 \setminus B_0) = 0$ and $B_0 \times \mathbb{R} \subseteq B \subseteq B_1 \times \mathbb{R}$.

Exercise 3 (Sufficient and necessary conditions for Lebesgue's measurability, 6 points) Prove the following assertions about a set $E \subseteq \mathbb{R}$ are equivalent:

1. $E \in \mathcal{M}_{\lambda^*}$.
2. For every $\varepsilon > 0$ there is an open set $G = G_\varepsilon$ such that $E \subseteq G$ and $\lambda^*(G \setminus E) < \varepsilon$.
3. There is a set $K \in \mathcal{G}_\delta$ such that $E \subseteq K$ and $\lambda^*(K \setminus E) = 0$.
4. For every $\varepsilon > 0$ there is a closed set $F = F_\varepsilon$ such that $F \subseteq E$ and $\lambda^*(E \setminus F) < \varepsilon$.
5. There is a set $H \in \mathcal{F}_\sigma$ such that $H \subseteq E$ and $\lambda^*(E \setminus H) = 0$.

Note: $\mathcal{G}_\delta$ ($\mathcal{F}_\sigma$) denotes the collection of intersections (unions) of countable families of open (closed) real sets.

Exercise 4 (4 points) Let $E \in \mathcal{M}_{\lambda^*}$ such that $\lambda(E) > 0$. Show that for every $\alpha \in (0, 1)$ there is an open interval $I \subset \mathbb{R}$ such that $\lambda(E \cap I) > \alpha \lambda(I)$.

Exercise 5 (4 points) Denote by λ_0 the restriction of Lebesgue's measure to $\mathcal{B}([0, 1])$, the Borel sets of $[0, 1]$. Suppose there exist $E \in \mathcal{B}([0, 1])$ and $\alpha \in (0, 1)$ such that

$$\lambda_0(E \cap J) > \alpha \lambda_0(J),$$

for every interval $J \subseteq [0, 1]$. Show that $\lambda_0(E) = 1$.