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Numerik 1 – Homework 4

Deadline: 12.04.2024, 10:00 PM

Upload any relevant code and a PDF with everything other than the code (comments, proofs, etc...). The content of the PDF file must be clearly readable and, for the codes, it must be clear which file (or part of file) solves which exercise.

Hint. *If you use Python you may find useful the packages* `numpy` *and* `matplotlib.pyplot`.

Note: You can not use preimplemented functions that automatically let you interpolate and evaluate polynomials (such as `polyfit`, `polyval`) or differentiate functions (such as `gradient`, `diff`) to solve the exercises.

Exercise 1 (Analytical task: Simpson's quadrature, 13 points)

(a) We consider a continuous function f defined on an interval $[a, b]$. Provide the expression of the second-order polynomial p_2 defined such that $p_2(a) = f(a)$, $p_2(b) = f(b)$ and $p_2((a + b)/2) = f((a + b)/2)$.

(b) By integrating p_2, show that the integral $\int_a^b f(x)\,\mathrm{d}x$ can be approximated by

$$Q_s(f, a, b) := \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

This approximation is referred to as the Simpson's quadrature.

(c) Show that the Simpson's quadrature $Q_s(f, a, b)$ has a third order of accuracy, *i.e.* it is exact for polynomials up to order 3.
HINT: Use the linearity of the integral and the canonical basis $(1, x, x^2, ..)$ for the polynomials.

(d) Assuming that $f \in \mathcal{C}^4([0, h])$ and using a Taylor expansion, show that

$$\int_0^h f(x)\,\mathrm{d}x - Q_s(f, 0, h) = \frac{1}{24}\left(\int_0^h f^{(4)}(\xi_x)x^4\,\mathrm{d}x - Q_s(f^{(4)}(\xi_x)x^4, 0, h)\right),$$

where ξ_x is a point in $]0, x[$.

(e) Show that the Simpson's quadrature rule verifies the following estimate for $f \in \mathcal{C}^4([0, h])$:

$$\left|\int_0^h f(x)dx - Q_s(f, 0, h)\right| \le C \max_{x\in[0,h]} \left|f^{(4)}(x)\right| h^5,$$

where C is a constant not dependent on f or h.

Exercise 2 (Simple integration, 11 points)

Consider the following quadrature formulae to integrate the general function f over the general interval $[a, b]$ ($Q(f, a, b) \approx \int_a^b f(x)\mathrm{d}x$):

i) **Midpoint rule**

$$Q_m(f, a, b) = (b-a)f\left(\frac{a+b}{2}\right);$$

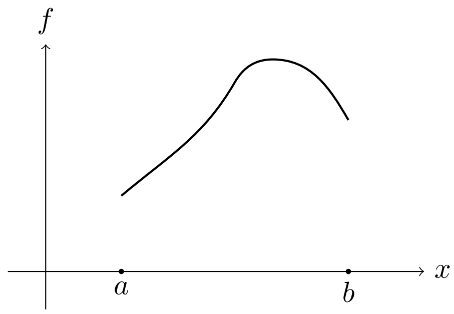
ii) **Trapezoidal method**

$$Q_t(f,a,b) = \frac{b-a}{2}(f(a)+f(b));$$

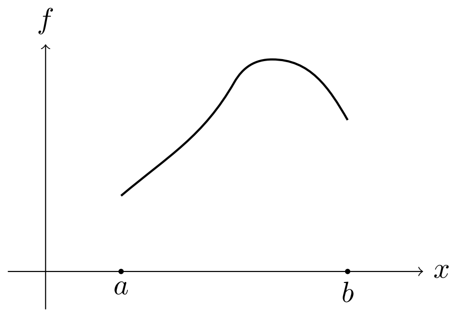
iii) **Simpson's method**

$$Q_s(f,a,b) = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

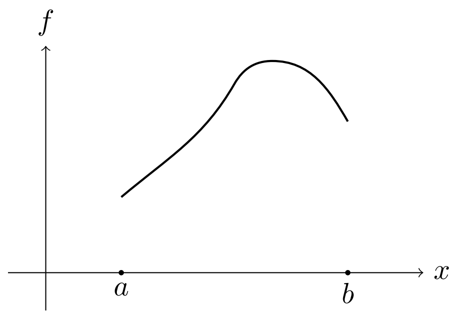
(a) Using a sketch like the ones below, draw a graphical interpretation of (i) the midpoint rude, (ii) the trapezoidal method, (iii) the Simpson's method.
HINT: For the Simpson's method, Exercise 1.b) can be helpful.



(i) Midpoint rule



(ii) Trapezoidal method



(iii) Simpson's method

(b) Implement three functions called `midpoint_rule`, `trapezoidal_method` and `simpsons_method`, performing the simple integration through the mentioned quadrature formulae in the general interval $[a,b]$. The functions must take as inputs:

- the function f,
- the extremum a,
- the extremum b,

and return as ouput the approximated value of the integral computed through the quadrature formula.

(c) Test the functions to evaluate the integral of $f(x) = e^x$ in the interval $[0,1]$ and compute the error[1] with respect to the exact value of the integral

$$\left|\int_0^1 f(x)dx - Q(f,0,1)\right|$$

Report and comment the results. Which quadrature formula gives the best approximation? Why?

[1]The error is always meant in absolute value also in the following unless explicitly specified.

(d) Using the functions previously defined, evaluate the errors committed through the quadrature formulas with respect to the exact integration for $a = 0$ and different values of $b = h = 2^{-n}$ for $n = 1, 2, \ldots 7$. Plot all the errors as function of h in a single plot in log-log coordinates. Add references curves of the form $h^p,\ p \in \mathbb{N}$. What can you conclude regarding the order of convergence of the three quadrature formulae? Regarding Simpson's rule, is it in agreement with the conclusions of Exercise 1?

Exercise 3 (Composite integration, 8 points)
Quadrature formulae are more often used in a composite way. To integrate a general function f over a general interval $[L, R]$, we first define $n + 1$ equidistant nodes

$$L = x_0 < x_1 < \cdots < x_n = R,$$

where $x_{i+1} = x_i + h$ and $h = (R - L)/n$. On each subinterval $\{x_i, x_{i+1}\}$, we apply the simple quadrature formula.

(a) Implement three functions performing the composite integration through the mentioned quadrature formulas in the general interval $[L, R]$: `midpoint_composite`, `trapezoidal_composite` and `simpsons_composite`. The functions must take as inputs:

- the function f,
- the extremum L,
- the extremum R,
- the number of desired subintervals n,

and return as output the approximated value of the integral computed through the quadrature formula.
HINT: In each function make a loop over the subintervals and call the functions defined in the previous exercise.

(b) Test the functions to evaluate the integral of $f(x) = e^x$ in the interval $[0, 1]$ with $n = 100$ subintervals and compute the error with respect to the exact value of the integral. Report and comment the results.

(c) Make a convergence analysis by computing the error in the composite integration of $f(x) = e^x$ in the interval $[0, 1]$ using m subintervals for the Midpoint, trapezoidal and Simpson's methods, for $m = 2^n$ and $n = 1, 2, \ldots, 9$. Plot the errors with respect to $h = 1/m$ in log-log scale in a single plot with suitable references of the type h^p with $p \in \mathbb{N}$. What can you conclude about the orders of convergence?