# Automated Market Making with Synchronized Liquidity Pools

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**Abstract**

We propose a new approach to automated market making (AMM) by synchronizing two constant function market makers (CFMM). Our methodology combines the advantages of each individual CFMM and reduces the downside risk for liquidity providers (LP). The approach can be extended to any combination of AMMs. The application of our technique described in this paper is suitable for assets with high volatility that can deviate from the entry price for LPs. The deviation risk, compared to a buy-and-hold strategy and often referred to as impermanent loss can counterbalance the benefit of liquidity mining, that is the earning of associated trading fees. We show in this paper how our approach provides a way to largely limit the impact of such risk and compare the behavior of the new AMM to the well known exchanges, such as Uniswap and Balancer.

## 1 Introduction

With the recent surge of Decentralized Finance (DeFi), a variety of blockchain-powered applications aimed at creating decentralized alternatives to traditional financial services have emerged. With very large accessibility, DeFi brings benefits such as greater transparency, decentralization, enhanced security and peer-to-peer global transactions among many other advantages as discussed e.g. in [10]. The underlying mechanics rely on smart-contract technologies and equivalents. More specifically, the emergence of the Ethereum blockchain [9], which allows the encoding of arbitrary smart-contract functionalities and execution on a blockchain has been a catalyst for DeFi applications. DeFi reached impressive amounts of capital, with 100 billion USD in September 2021 up from 20 billion USD within a year, see [1]. The industry proves to be in constant expansion and provides continuous innovation.

DeFi has allowed its users to lend or borrow with services on the blockchain, see for example [24, 3], among others. A crucial DeFi application is the decentralization of asset exchanges. Until recently, exchanging digital and traditional assets was only feasible on classic systems that share an accepted and common design known as a continuous-limit order-book, see e.g. [18] for details. Such an order-book consists of a list of all bids and offers from buyers and sellers in the system, i.e. prospective buyers place a limit buy order, which specifies a maximum price at which they are willing to buy an asset. Other types of orders such as market order or stop loss are usually available depending on the Centralized Exchange (CEX). While offering a range of advantages, CEXs have also experienced problems ranging from high-profile thefts, some of which are reviewed in [8], to offenses such as price manipulation [16].

Decentralized exchanges (DEXs), improve several aspects of CEXs, for example, security vulnerabilities, centralized control of assets, custodian challenges and more as reported in [10]. There have been interesting

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designs to decentralize, at least partially, the continuous limit order-book. For example, a counter-party can select an order in the order book and present it to the smart contract with a signed counter-order. The smart contract executes the order and counter-order, clearing the order from the order book. In this model traders themselves perform order matching, an approach that has been used for example by Etherdelta. Similar off-chain trade matching with on-chain settlement enforced by smart contracts has been proposed by dYdX [23] and IDex [22] among others, where the exchange itself performs the matching of orders. dYdX [23] has, for example, seen large volume increases in recent market setups. Fully decentralized exchanges have been recently built upon protocols acting as Automated Market Makers (AMM). AMMs are algorithmic agents that provide liquidity in electronic markets, a topic that has been well studied in algorithmic game theory, see [27], and for which an early work is the logarithmic market scoring rule introduced in [19].

The first fully decentralized exchanges for digital assets have been built around Constant Function Market Maker models (CFMM), see e.g. [20, 4, 26, 13]. The mechanism links two or more reserves of the different participating assets dynamically, relying on a driving constant function with specific properties. The liquidity available on the reserves and the CFMM function then jointly determine the market price of any two assets.

Liquidity providers (LPs) in DEXs can generate revenues by providing their funds as liquidity to the DEX of their choice, which will allow traders to exchange the assets of interest. In other words the liquidity provided will allow trading of the digital assets to be fully handled, in a decentralized manner, on the blockchain. Funds provided by liquidity providers are protected because the custody and exchange logic is processed and guaranteed by the smart contract directly.

Traders will generally pay a trading fee for each exchange, which is then shared among the liquidity providers and represents a direct remuneration for the funds provided. Over time, the trading fees get accumulated and LPs can see substantial return on their capital. However, as a counterpart for that reward, LPs also face a risk associated to the change of spot price value, commonly referred to as impermanent loss (IL). The term "impermanent" is employed in the field since, without withdrawal, the LP has a non-zero probability to recover this loss if the spot reverts back to its initial value. An intuitive view of the phenomenon is when the market moves heavily, a LP can recover through the DEX dynamics, more of the cheaper assets and less of the valuable assets than when he/she entered the AMM. In practice however, trading fees accumulated during that time can counterbalance the impermanent loss if the LP holds his/her position long enough in the DEX and if the market does not deviate too drastically. Because of the last point, this risk therefore remains a main concern for LPs and an area of innovation among various participants in the DEX industry. Uniswap and associated constant product two-assets CFMM [4], represent one of the most widely used DEX by volume as of today. However, it also embeds an impermanent loss profile which can strongly negatively impact the returns of the liquidity provider. In Uniswap V3 [5], the authors provide a way to optimize liquidity and therefore mitigate indirectly the risk of impermanent loss by increasing returns from trading fees in a designated range of spot chosen by the LP. However, the intrinsic properties of the underlying Uniswap CFMM being unchanged, the impermanent loss impact can worsen if the spot exits the fixed range where the liquidity was concentrated.

A detailed analysis of the Uniswap V2 market maker is provided in [8]. In [26], Balancer has generalized the Uniswap formula by introducing the geometric mean market maker CFMM. A precise analysis of geometric mean markets can be found for example in [15, 6, 7]. Geometric mean markets with asymmetric weights can improve the impermanent loss profile and therefore increase the returns of LPs on either the rise or the fall of the spot, but not both. Indeed, the IL on one side of the spot deviation stays uncovered as with Uniswap and slippage properties for traders are significantly worse on one trade direction as well. As in [5], there are also different approaches which do not directly amend the impermanent loss profile of the AMM itself. Bancor V2 [21] provides a system of insurance pools as well as a protection of impermanent loss, paid-out with their own protocol's token. It is an efficient solution in most market regimes, however, as discussed in detail in [25], this methodology presents a systemic risk under a stressed market scenario; more precisely there is an exposure to a downward spiral risk that can highly affect both liquidity providers and the protocol's token holders. Dodo [12] relies fully on external price oracles, such as ChainLink [14], to create a market maker algorithm. One of the main advantages of Dodo, similarly to Bancor V2, is to allow single sided liquidity provisioning by construction.

The impermanent loss is also improved, however, liquidity providers bear inventory risks that have similar disadvantages to impermanent loss, particularly in highly volatile market conditions. Additionally, market making with external oracles may raise issues for non-liquid assets.

Some analyses have discussed the returns and impermanent loss of liquidity providers with some possibilities to statically or dynamically hedge the IL risk. For example [15, 7] analyse LPs' returns under geometric mean markets and CFMM more generally and introduce some hedging possibilities. In this paper we propose to work directly on the intrinsic mechanism of the automated market maker with a new approach in order to reduce the impermanent loss. This new methodology has advantages in that it can be used in combination with other non-AMM specific improvements of the impermanent loss, such as the ones proposed in [21, 5] to only cite a few. Our approach, which we denominate as sync-AMM standing for Synchronized Automated Market Maker, proposes to combine the properties of two, or possibly more, CFMM and therefore obtain improved joint-properties. The synchronization process allows to align the spot prices of the CFMMs at play for each new trade. While we discuss a duo of geometric mean markets in the current work, it is possible to work with a combination of other types of CFMM such as the one proposed in [13]. Numerical results show that even under highly volatile scenarios, the LPs returns are significantly improved compared with other CFMMs. In the market setup tested and over the paths analyzed, the liquidity providers simulated in our test case, were able to obtain positive returns from trading fees compared to a buy-and-hold strategy with spot deviations rising and falling by a factor of 150.

We note that, this work provided the theoretical and computational background for the decentralized exchange at the AllianceBlock Foundation named AllianceDEX.

The remainder of this paper is organized as follows. In Section 2, we define the automated market market models of interest as well as discuss the definition of standard nomenclature of the field. Additionally, we derive and provide a detailed analysis of the impermanent loss and slippage in geometric mean markets. Secondly, in Section 3, we discuss the main contribution of the paper and a new approach to AMM which provides a considerably improved impermanent loss profile. Additionally, Section 4 provides details on the simulation framework used for the testing of the sync-AMM as well as showcase its impermanent loss profile under different market scenarios. Finally, Section 5 concludes with a brief summary of the contributions.

### 1.1 Disclaimer

The results discussed by the authors in this article do not constitute, in any form, an investment advice in the associated AllianceBlock decentralized exchange or any other mentioned decentralized exchange. The authors and AllianceBlock are not responsible for any loss incurred as a result of the use of the AllianceBlock decentralized exchange or any information discussed and reported in this paper. This article is meant to provide informational research and does not aim to detail the risks involved in trading or liquidity providing associated with the automated market maker models of interest.

## 2 Definitions

### 2.1 Framework definition

We consider a set of $n$-assets denoted $(\alpha_i)_{i\leq n}$ and we additionally define $\beta = \alpha_n$ for writing convenience. The spot $\left(S^i_{t\geq 0}\right)_{i\leq n}$ associated with the asset pair $\alpha_i, \beta$ denotes the amount of units of $\beta$ needed to buy one unit of $\alpha_i$ at time $t$. Naturally, this would imply that $S^n_t \equiv 1$ by construction at any time. In the remainder of the paper, when no additional information is provided, we assume that the numéraire of choice will be the asset $\beta$, that is all quantities will be denominated in amounts of $\beta$. Moreover, when referring to the pair $\alpha$, $\beta$ without specific indices, it signifies that we are working under the two-assets case where $\alpha \equiv \alpha_1$ and $\beta \equiv \alpha_2$.We assume the existence of a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ with a real-world measure associated $\mathbb{P}$. The spot $S$, as well as all subsequent stochastic processes defined, are assumed adapted to the filtration $\mathcal{F}$.

In the AMM framework, and the now well established constant market maker function, it is common practice to have reserves amounts appear explicitly in the formulation and definition of the CFMM function, which as discussed in extensive details in [6] allows to link a valid trade to the reserves time evolution. While the scope of CFMM covers a large set of possible automated market maker, in this paper we will focus primarily on Uniswap V2 [4] and Balancer V1 [26], that are examples of constant product market makers. Let us denote pool reserve sizes as $\left(R^i_{t\geq 0}\right)_{i\leq n}$, a positive quantity that represents the reserve amounts of asset $\alpha_i$. Where no super-script is specified, the process $\boldsymbol{R}_t$ is a $n$-dimensional representation of each reserve with $R^i_t$ as element.

A trade will give rise to a constant proportion of fee denoted $(1-\gamma)$, where $0 < \gamma \leq 1$ and with $\gamma = 1$ for the case where no trading fees are considered. This means that any amount a trader is willing to sell will be scaled by $\gamma$ to compute the actual input amount of the trade, which will naturally provide a lower output amount. The reserves are however updated with the total input amounts such that liquidity providers are rewarded for providing liquidity to the DEX.

Following the definition in [6], we let a trade be a tuple of vector values, $(\boldsymbol{\Lambda}, \boldsymbol{\Delta})$ where $\boldsymbol{\Lambda}$, a $n$-dimensional real valued vector is the output amounts resulting from the AMM following a valid trade and $\boldsymbol{\Delta}$, also a $n$-dimensional real valued vector is the input amounts for a given trade. Each element of the vector, that is $\left(\Lambda^i, \Delta^i\right)$ are the output and input amounts respectively of asset $\alpha_i$ for a specific trade. Let us write the following definition,

**Definition 1.** An automated market maker is a constant function market maker if and only if there exists a continuous, once differentiable with continuous derivatives function with respect to all variables, $\phi : \left(\mathbb{R}^+\right)^n \times \left(\mathbb{R}^+\right)^n \times \left(\mathbb{R}^+\right)^n \to \mathbb{R}$ , such that for any given valid trade $(\boldsymbol{\Lambda}, \boldsymbol{\Delta})$ at a positive time $t$,

$$\phi(\boldsymbol{R}_t, \boldsymbol{\Lambda}, \boldsymbol{\Delta}) = \phi(\boldsymbol{R}_t, \boldsymbol{0}, \boldsymbol{0})\,, \tag{2.1}$$

and where each underlying asset is active, namely that, for all $i \in [\![1, n]\!]$,

$$\partial_{R^i}\psi(\boldsymbol{R}_t) \neq 0\,, \tag{2.2}$$

where $\psi(\boldsymbol{R}_t) = \phi(\boldsymbol{R}_t, \boldsymbol{0}, \boldsymbol{0})$ and $\partial_{R^i}\psi$ denotes the partial derivative of $\psi$ with respect to the $i-$th element of $\boldsymbol{R}_t$.

If the trade is executed, the above formula should be understood with reserves immediately prior to the jump associated to the trade, that is $\boldsymbol{R}_{t-}$ since the value of the reserves is updated at trade time.

As described in [6], this function is not necessarily unique for a given AMM, for example a constant scaling of the constant product of Uniswap would still behave in a similar fashion resulting in an equivalent CFMM. While continuity and differentiability is not a necessary assumption, we will assume this to be verified in the remainder of the paper. A classic example is the Balancer [26] $n$-assets expression which accounts for Uniswap as a limit case with,

$$\phi(\boldsymbol{R}_t, \boldsymbol{\Lambda}, \boldsymbol{\Delta}) = \prod_{i=1}^{n}\left(R^i_t + \gamma\Delta^i - \Lambda^i\right)^{w_i}, \tag{2.3}$$

where $w_i \in \,]0, 1[$ and $\sum_{i=1}^n w_i = 1$ and where Uniswap is the duo of assets case with weights of 0.5.

*Remark.* In the remainder of the article, when not specified, 'Uniswap' will refer to the Uniswap V2 DEX [4], while 'Balancer' will refer to the Balancer V1 DEX [26].

## 2.2 Dynamical properties

Quantities defined in the previous section such as pool reserves, are jump processes and their value will change at trade or liquidity addition/withdrawal times. Therefore, we define two random time sets; the set of liquidity addition and withdrawal times $\Theta = \{\theta_1, \theta_2, ...\}$ with $(\theta_i)_{i\geq 1} \in \mathbb{R}^+ \cup \{\infty\}$, and the set of trading times

$\mathcal{T} = \{\tau_1, \tau_2, ...\}$, with $(\tau_i)_{i\geq 1} \in \mathbb{R}^+ \cup \{\infty\}$. This also allows to define the processes $(\boldsymbol{\Lambda}_t)_{t\geq 0}$ and $(\boldsymbol{\Delta}_t)_{t\geq 0}$ that are zero except on trade times $t \in \mathcal{T}$ where they are linked together by (2.1). We note that a market participant performing a trade at time $t$ can choose to either provide the input amounts $\boldsymbol{\Delta}_t$ which are asset quantities to be sold, or output amounts $\boldsymbol{\Lambda}_t$ that are asset quantities to be received. Any combination is theoretically possible for each element, namely $\Lambda_t^i$ and $\Delta_t^i$. However, a rational trader is unlikely to have both $\Lambda_t^i$ and $\Delta_t^i$ to be jointly non-zero [6].

The evolution of the pool reserves can be summarized by jump processes, namely, for any $t \in \mathcal{T}$,

$$\boldsymbol{R}_t = \boldsymbol{R}_{t-} + \boldsymbol{\Delta}_t - \boldsymbol{\Lambda}_t \,. \tag{2.4}$$

And for any $t \in \Theta$,

$$\boldsymbol{R}_t = \boldsymbol{R}_{t-} + \boldsymbol{L}_t \,,$$

where $\boldsymbol{L}_t$ is the liquidity addition or withdrawal at time $t \in \Theta$ which is precisely zero for any $t \notin \Theta$.

We note that the paths discussed in this paper are all right-continuous with left limit. More details about path properties of jump processes can be found in [11].

*Remark* 2. In the remainder of the paper we will indistinguishably use $\Delta_t^i$ and $\Lambda_t^i$ as processes, and $\Delta^i$, $\Lambda^i$ without specific time index as arguments of a function of the trade sizes. Therefore, a quantity $y_t$ which holds a direct relationship to trade sizes can be understood as the value $y_t$ for the trade at time $t$, or as the function $y_t(\boldsymbol{\Delta}, \boldsymbol{\Lambda})$ which still depends on reserve sizes but can have varying trade sizes. Additionally, we note that $\boldsymbol{\Delta}_t$ and $\boldsymbol{\Lambda}_t$ can effectively be defined as trades sizes on the timeline since all events are assumed to happen sequentially; where two traders cannot trade at the same time.

## 2.3 Reported, marginal and effective spot price

The spot value bears a primary importance in the definition of a market. In the foreign exchange (FX) market, the spot is defined as the ratio of notional values involved in a FX cash exchange operation; that is an order to trade an amount of $N^{\text{ccy1}}$ for an amount of $N^{\text{ccy2}}$ specifies the spot value as $N^{\text{ccy2}}/N^{\text{ccy1}}$ . The spot value of the Uniswap V2 decentralized exchange [8] follows a similar definition. Indeed, the spot is defined as the ratio of the reserves at any given time. However, for other CFMM the spot is not necessarily intuitive and is linked to the trading function $\phi$. Let us provide hereafter a few definitions that will be useful for the remainder of this paper.

The reported spot $\left(S_t^i\right)_{i\in[\![1,n]\!]}$ is defined as the price of the AMM at time $t \geq 0$. Following [6, 7] and with Definition 1 in Section 2.1 $\psi(\boldsymbol{R}_t) = \phi(\boldsymbol{R}_t, \boldsymbol{0}, \boldsymbol{0})$, the reported price $S_t^i$ is defined as,

$$S_t^i = \frac{\partial_{R^i}\psi(\boldsymbol{R}_t)}{\partial_{R^n}\psi(\boldsymbol{R}_t)}, \tag{2.5}$$

since the chosen numéraire is $\beta$, that is the asset associated to the reserve $R^n$.

Additionally, we can also write the reported spot from asset $i$ to asset $j$ as,

$$S_t^{i,j} = \frac{S_t^i}{S_t^j} = \frac{\partial_{R^i}\psi(\boldsymbol{R}_t)}{\partial_{R^j}\psi(\boldsymbol{R}_t)} \,.$$

Because $\phi$ has been assumed differentiable in all variables, the above spots are unique [6]. As an example, the reported spot for Balancer using

$$\psi_{\text{Bal}}(\boldsymbol{R}_t) = \prod_{i=1}^{n} \left(R_t^i\right)^{w_i} , \tag{2.6}$$

leads to the ratio of the weights and pool sizes,

$$S_t^i = \frac{w_i R_t^n}{w_n R_t^i}\,. \tag{2.7}$$

For Uniswap, as $\psi_{\text{Uni}}(\boldsymbol{R}_t) = \sqrt{R_t^1}\sqrt{R_t^2}$ , the reported spot is simply the ratio of the pool sizes which is similar to a FX cash trade as expected.

The marginal price $\left(\hat{S}_t^i\right)_{i\in[\![1,n]\!]}$ is defined as the spot price for an infinitesimal trade size where we have $\hat{S}_t^i = \gamma S_t^i$ from [7]. Hence, the reported and marginal prices differ only by the scaling attributed to trading fees.

Finally, the effective spot $\left(\bar{S}_t^i\right)_{i\in[\![1,n]\!]}$ is the price of a trade of arbitrary size, which generally accounts for slippage, see sub-section 2.6 . The latter would depend on the trade size and theoretically should be written as $\bar{S}_t^i(\boldsymbol{\Lambda},\boldsymbol{\Delta})$ as per Remark 2, which we will omit for ease of notation. For a trade where all input amounts are zero except for asset $\alpha_i$, we can define for any $i \in [\![1,n]\!]$,

$$\bar{S}_t^i(\boldsymbol{\Lambda},\boldsymbol{\Delta}) = \frac{\sum_{j=1}^n \Lambda^j S_t^j}{\Delta^i}\,. \tag{2.8}$$

## 2.4 Liquidity providers

A CFMM, and more generally, an automated market maker relies on liquidity providers to give traders the ability to exchange assets. Each liquidity provider owns a share of the total reserve upon entry in the DEX. This share will fluctuate generally when other liquidity providers join or exit the liquidity pools. Liquidity events follow the below assumption,

**Assumption 3.** *Adding or withdrawing liquidity does not change the reported spot of the AMM or impact the liquidity holdings of other liquidity providers.*

For a particular LP $m$ we denote his/her share by $\left(\boldsymbol{X}_{t\geq 0}^m\right)_{m\leq U_t}$ where $U_t \in \mathbb{N}^*$ is the total number of liquidity providers in the liquidity pool at time $t$. We note that $U_t$ is increasing, such that a liquidity provider who exits the AMM, will get a share of zero but still be accounted for in the numbering. Additionally, we can define the holdings of the liquidity provider $\boldsymbol{r}_t^m$ for which the $i-$th element is denoted $r_t^{i,m}$ and represents the amount of asset $\alpha_i$ which belongs to the $m$-th LP, and where, with $\odot$ the element-wise product,

$$\boldsymbol{r}_t^m = \boldsymbol{X}_t^m \odot \boldsymbol{R}_t^m\,.$$

In the case where the share $\boldsymbol{X}_t^m$ does not depend on a specific asset, we will simply denote it as $X_t^m$, which is the case for geometric mean markets such as Uniswap or Balancer. Indeed, when entering the DEX at time $t_0$ with a quantity $r_{t_0}^j$ of asset $\alpha_j$, a liquidity provider gets a share for the corresponding pool,

$$X_{t_0}^{j,m} = \frac{r_{t_0}^{j,m}}{R_{t_0}^j}\,. \tag{2.9}$$

For Uniswap and Balancer, given Assumption 3, to keep every spot constant around liquidity events, one needs the above share $X^{j,m}$ to be the same for any $j \in [\![1,n]\!]$. However we note that this does not represent a generic property of AMMs.

Additionally, to ease notations and when the reserves of only one liquidity provider are involved, we will drop the superscript such that for a liquidity provider $m$ we will directly write $\boldsymbol{r}_t \equiv \boldsymbol{r}_t^m$ where $\boldsymbol{r}$ is implicitly assumed to be associated to the LP $m$ of interest. We also define $(\boldsymbol{L}_t)_{t\geq 0}$, the process of liquidity addition or removal where each component, $L_t^i$, represents the liquidity change in the corresponding pool of asset $\alpha_i$.

### 2.4.1 Liquidity addition in geometric mean CFMM

In this section the aim is to discuss the fact that, with geometric mean CFMM, one can work with each liquidity provider's holdings rather than the DEX reserves and ignore liquidity events. That simplifies the calculation of the IL in the following sections. For geometric mean CFMM and Balancer, the Assumption 3 implies for any $t \in \Theta$ and $i \in [\![1, n]\!]$,

$$S_t^i = S_{t-}^i \,, \tag{2.10}$$

which combined with (2.7) provides,

$$\frac{\left(R_{t-}^n + L_t^n\right)}{R_{t-}^n} = \frac{\left(R_{t-}^i + L_t^i\right)}{R_{t-}^i}.$$

From the above equation and with an addition/withdraw ratio $\lambda = \frac{L_{t-}^n}{R_{t-}^n}$ chosen on one of the assets, for instance $\alpha_n$, one can get

$$\boldsymbol{R}_t = \boldsymbol{R}_{t-}(1+\lambda)\,. \tag{2.11}$$

Given liquidity movements are always proportional to reserve sizes for Uniswap and Balancer, the LP share is independent of the assets for any $t \geq 0$. Therefore, with $\boldsymbol{r}_t$ being the holdings of the considered LP $m$, we can write,

$$\boldsymbol{r}_t = X_t^m \boldsymbol{R}_t\,. \tag{2.12}$$

The share of each liquidity provider remains constant between two consecutive liquidity events. On each new event $\theta \in \Theta$, when another LP $j$ adds or withdraws liquidity in proportion $\lambda$, reserves are updated following (2.11). Therefore, the share of the considered LP $m \neq j$ jumps according to,

$$X_\theta^m = \frac{X_{\theta-}^m}{1+\lambda}\,. \tag{2.13}$$

Using the above equations and the geometric mean markets condition $\sum_{i=1}^n w_i = 1$, we get the following useful property, for any time $t > 0$,

$$\psi_{\mathrm{Bal}}(\boldsymbol{r}_t) = X_t^m \psi_{\mathrm{Bal}}(\boldsymbol{R}_t)\,, \tag{2.14}$$

By combining (2.11), (2.12) and (2.13), we find that around liquidity events, the holdings of LPs remain constant which is in line with Assumption 3. Therefore,

$$\psi_{\mathrm{Bal}}(\boldsymbol{r}_\theta) = \psi_{\mathrm{Bal}}(\boldsymbol{r}_{\theta-})\,. \tag{2.15}$$

Additionally, when no transaction fee is involved, for any time $t$ between any two consecutive liquidity events, $(\theta_k, \theta_{k+1})$,

$$\psi_{\mathrm{Bal}}(\boldsymbol{R}_t) = \psi_{\mathrm{Bal}}(\boldsymbol{R}_{\theta_k})\,,$$

from which one gets,

$$\psi_{\mathrm{Bal}}(\boldsymbol{r}_t) = \psi_{\mathrm{Bal}}(\boldsymbol{r}_{\theta_k}) = \psi_{\mathrm{Bal}}(\boldsymbol{r}_{t_0})\,, \tag{2.16}$$

where $t_0$ is the entry-time of the $m$-th LP. Equation (2.16) shows that each LP's holdings follow the CFMM equation.

## 2.5 Impermanent loss for geometric mean markets

As discussed, liquidity providers will earn trading fees by providing their assets to the DEX. These will then be used as reserves for trading. This means that LPs make a choice to switch from a static buy-and-hold portfolio, where the amount of assets stays constant over time, to a dynamic portfolio where their assets will fluctuate according to the AMM dynamic and market movements. It is natural to compare the evolution of

LPs holdings when they enter a DEX versus a buy-and-hold portfolio. In the crypto-currency field, this is commonly known as the impermanent loss. In this section, we show that as the name suggests, without any trading fees, providing liquidity to a CFMM DEX such as Uniswap or Balancer will always result in holdings being worth less than under a buy-and-hold strategy. We will look at the details of the expression of the IL for geometric mean markets with and without trading fees. Since Balancer is a generalization of the two-asset constant product CFMM, this will allow to conclude about Uniswap as well. We remind that the reported spot $\left(S^i_{t\geq 0}\right)_{i\leq n}$ associated with the asset pair $(\alpha_i, \beta)$ denotes the amount of units of $\beta$ needed to buy one unit of $\alpha_i$ at time $t$ and where the numéraire of choice will be the asset $\beta$. Let us suppose that a liquidity provider entered the DEX at time $t_0$. We drop all liquidity provider's specific superscript in that section for ease of notation. The holdings of the LP at time $t \geq t_0$ are worth under numéraire $\beta$,

$$P_t = \sum_{i=1}^{n} r^i_t S^i_t \,. \tag{2.17}$$

The same assets, if not invested in a DEX but kept in a static portfolio would be worth,

$$P^{\mathrm{S}}_t = \sum_{i=1}^{n} r^i_{t_0} S^i_t \,. \tag{2.18}$$

The impermanent loss at time $t$, $(\mathrm{IL}_t)_{t\geq 0}$ is the relative difference of the portfolio of the liquidity provider with respect to his/her holdings under a buy-and-hold strategy,

$$\mathrm{IL}_t = \frac{P_t - P^{\mathrm{S}}_t}{P^{\mathrm{S}}_t} \,. \tag{2.19}$$

Balancer's constant product function $\psi_{\mathrm{Bal}}(\boldsymbol{R}_t)$ is defined in (2.6). To ignore the jump processes of the reserves or the LP shares $X$, we use the individual liquidity provider holdings $\boldsymbol{r}_t$, see (2.16). By doing so, the expression of the reported spots depends only on the holdings of individual liquidity providers,

$$S^i_t = \frac{w_i r^n_t}{w_n r^i_t} \,. \tag{2.20}$$

In the next sections we calculate the impermanent loss for each LP without and with trading fees.

### 2.5.1 Impermanent loss without trading fees

**Proposition 4.** *The impermanent loss for a liquidity provider entering at time $t_0$, in a geometric mean CFMM without trading fees, is given by, for any $t \geq t_0$,*

$$IL_t = \frac{\prod_{i=1}^{n} \left(z^i_t\right)^{w_i}}{\sum_{i=1}^{n} w_i z^i_t} - 1 \,, \tag{2.21}$$

*with for any $i \in [\![1, n]\!]$,*

$$z^i_t = S^i_t / S^i_{t_0} \,, \tag{2.22}$$

*and $IL_t \leq 0$.*

*Proof.* Without loss of generality, we set $t_0 = 0$. Our goal is to compute (2.19) with (2.17) and (2.18). We remind here that by definition we have $S^n_t = 1$. Using the expression of the spot $S^i_t$ from (2.20), we can write

$$P_t = \sum_{i=1}^{n-1} r^i_t \frac{w_i r^n_t}{w_n r^i_t} + r^n_t = \frac{r^n_t}{w_n} \,. \tag{2.23}$$

Similarly, $P_t^S$ can be rewritten as,

$$P_t^{\mathrm{S}} = \sum_{i=1}^{n} r_0^i S_t^i = \sum_{i=1}^{n} r_0^i S_0^i \left( \frac{S_t^i}{S_0^i} \right).$$

By using $r_0^i S_0^i = \frac{w_i r_0^n}{w_n}$ from (2.20) and (2.22) we can rewrite $P_t^S$ as,

$$P_t^{\mathrm{S}} = \sum_{i=1}^{n} r_0^i S_0^i z_t^i = \sum_{i=1}^{n} \frac{w_i r_0^n}{w_n} z_t^i = P_0 \sum_{i=1}^{n} w_i z_t^i \,. \tag{2.24}$$

That gives,

$$\mathrm{IL}_t = \frac{r_t^n / r_0^n}{\sum_{i=1}^{n} w_i z_t^i} - 1 \,. \tag{2.25}$$

By using (2.20), we can rewrite (2.6),

$$\psi_{\mathrm{Bal}}(\boldsymbol{r}_t) = \prod_{i=1}^{n} \left( r_t^i \right)^{w_i} = \prod_{i=1}^{n} \left( \frac{w_i r_t^n}{w_n S_t^i} \right)^{w_i} = r_t^n \prod_{i=1}^{n} \left( \frac{w_i}{w_n S_t^i} \right)^{w_i} . \tag{2.26}$$

Without taking into account trading fees, equation (2.16) holds. Hence, combining (2.16) with the above expression we have,

$$\frac{r_t^n}{r_0^n} = \prod_{i=1}^{n} \left( \frac{S_t^i}{S_0^i} \right)^{w_i} = \prod_{i=1}^{n} \left( z_t^i \right)^{w_i} . \tag{2.27}$$

We conclude the proof by replacing (2.27) into (2.25). Furthermore, according to the Jensen's inequality and since the logarithm is a concave function, we have,

$$\ln \prod_{i=1}^{n} \left( z_t^i \right)^{w_i} = \sum_{i=1}^{n} w_i \ln \left( z_t^i \right) \leq \ln \left( \sum_{i=1}^{n} w_i z_t^i \right),$$

which shows that the impermanent loss in (2.21) is always negative. □

For Uniswap, $n = 2$, we can define $z_t \equiv z_t^1$ and by definition, the second asset being $\beta$, we have $z_t^2 \equiv 1$. Therefore, the expression of the IL becomes,

$$\mathrm{IL}^{\mathrm{Uni}}(z_t) = \frac{2\sqrt{z_t}}{z_t + 1} - 1 \,, \tag{2.28}$$

It is interesting to notice that,

$$\mathrm{IL}^{\mathrm{Uni}}(z) = \mathrm{IL}^{\mathrm{Uni}}(1/z) \,,$$

which means that under Uniswap by ignoring the trading fee, for a liquidity provider that enters the pool, the risk of loss against a buy-and-hold portfolio will be similar if the spot increases or decreases by a factor $\eta$ or $1/\eta$, respectively. This symmetry suggests to plot the impermanent loss curves on a logarithmic scale.

For Balancer with two assets, we additionally introduce $w \equiv w_1$ and therefore $w_2 = 1 - w$. In this case the expression of the IL becomes,

$$\mathrm{IL}^{\text{Bal-2}}(z_t) = \frac{z_t^w}{w z_t + 1 - w} - 1 \,. \tag{2.29}$$

Contrary to Uniswap where we have perfect symmetry for high and low spot returns, Balancer shows asymmetry. For $w < 0.5$ one has smaller IL for any return $z < 1$ compared to high spot regimes $z > 1$. Symmetric conclusions hold for $w > 0.5$. Therefore Balancer provides an effective solution for improving the impermanent

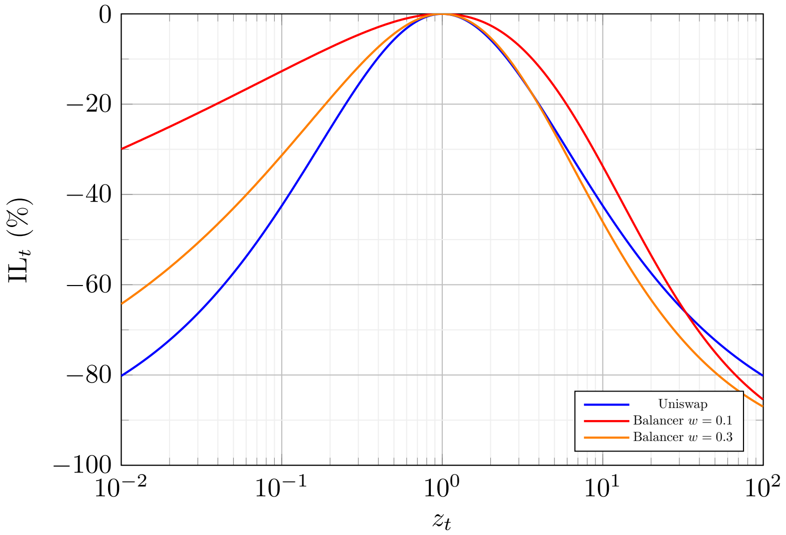


**Figure 2.1:** Impermanent loss under Uniswap, Balancer $w = 0.1$ and Balancer $w = 0.3$ as a function of $z_t$ on a log-scale. The Balancer IL is improved on the downside spot movements when the weight is decreased.

loss on one direction of spot movement. However, if the spot deviates to the opposite side, Balancer is close to Uniswap in terms of impermanent loss. We will also see in sub-section 2.6 that Balancer with weights diverging from 0.5 does not offer good slippage properties on one trading direction.

In Figure 2.2 and 2.1 one can see the asymmetric improvement of Balancer's impermanent loss depending on the weights. For instance with $w = 0.1$, on Figure 2.2, if the spot drops by 90% compared to the entry level of the LP, Balancer offers an improvement of IL, namely from −42.5% under Uniswap to −12.7%. However, on the flip side when spot increases by a factor of 10, the improvement is smaller; from −42.5% under Uniswap to −33.7% only. For the same weight, one can also notice that the Balancer IL becomes worse than Uniswap when the spot increases by a factor greater than 30.

### 2.5.2 Impermanent loss with trading fees

As seen in sub-section 2.5, providing liquidity without earning trading fees will lead to a loss due to the unfavorable balance of assets as soon as the spot deviates from the entry level. Trading fees are a very efficient way of cancelling this potential loss and more importantly allowing to earn returns when sufficient trading volumes accumulate over time.

In the following we assume that, if trades on different assets are executed at the same time, we decompose these as sequential trades each involving one asset for another. That is for any trade, if $\Delta_t^i > 0$ and $\Lambda_t^j > 0$, then for any $k \neq i$, $\Delta_t^k = 0$ and for any $m \neq j$, $\Lambda_t^m = 0$ . We introduce for every $\tau \in \mathcal{T}$, the input asset index $(k_\tau)_{\tau \in \mathcal{T}} \in [\![1, n]\!]$ which is defined by the trader's decision to exchange asset $\alpha_{k_\tau}$ for another asset $(\alpha_i)_{i \leq n, i \neq k_\tau}$.

**Proposition 5.** *The impermanent loss for a liquidity provider entering at time $t_0$, in a geometric mean CFMM, with trading fees for any $t \geq t_0$, is given by,*

$$IL_t = \rho_t \frac{\prod_{i=1}^n \left(z_t^i\right)^{w_i}}{\sum_{i=1}^n w_i z_t^i} - 1\,, \tag{2.30}$$

*with for any $i \in [\![1, n]\!]$,*

$$z_t^i = S_t^i / S_{t_0}^i\,, \tag{2.31}$$

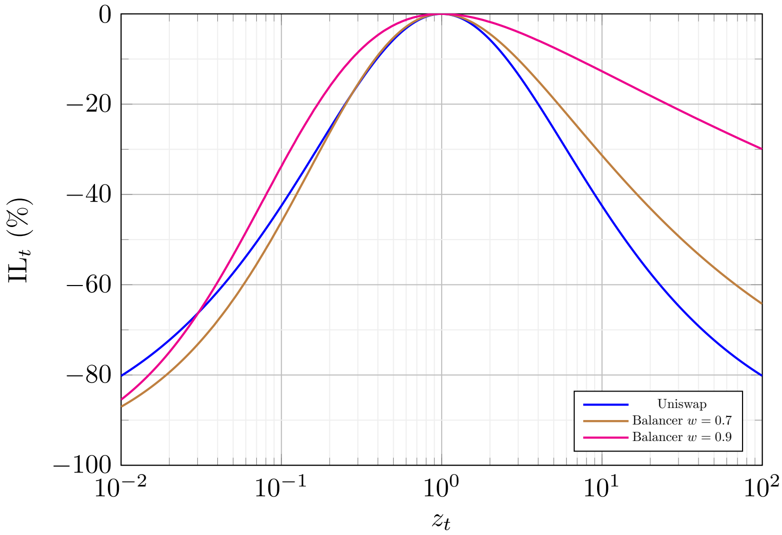


**Figure 2.2:** Impermanent loss under Uniswap, Balancer $w = 0.7$ and Balancer $w = 0.9$ as a function of $z_t$ on a log-scale. The Balancer IL is improved on the upside spot movements when the weight is increased.

*and where,*

$$\rho_t = \prod_{\tau \in \mathcal{T}, t_0 \leq \tau \leq t} \left( \frac{1 + V_\tau}{1 + \gamma V_\tau} \right)^{w_{k_\tau}} \geq 1, \qquad \forall \tau \in \mathcal{T},\, V_\tau = \frac{\Delta_\tau^{k_\tau}}{R_{\tau-}^{k_\tau}} \,. \tag{2.32}$$

*$V_\tau$ is the trading volume in percentage of the pool size at trade time $\tau$. The parameter $\rho_t$ takes into account all the trading volumes between $t_0$ when the liquidity provider joins the pools, and time $t$.*

*Proof.* We first recall that the geometric mean CFMMs follows at any time,

$$\psi_{\mathrm{Bal}}(\boldsymbol{R}_t) = \prod_{i=1}^{n} \left(R_t^i\right)^{w_i} = C_t \,,$$

where the newly defined quantity $C_t$ jumps due to liquidity movements, but also with trading fees according to the way the protocols are built. A fact that was ignored in the previous result in Proposition 4. We recall that a rational trader will not buy and sell the same asset for a given trade as per [6]. Therefore, following our assumption where each trade involves only one input and output asset at a time, for any $\tau \in \mathcal{T}$, there exists $j \in [\![1, n]\!]$ such that we can consider trades from asset $\alpha_{k_\tau}$ to asset $\alpha_j$ and as per Section 3 write,

$$\left(R_{\tau-}^{k_\tau} + \gamma \Delta_\tau^{k_\tau}\right)^{w_{k_\tau}} \left(R_{\tau-}^{j} - \Lambda_\tau^{j}\right)^{w_j} \prod_{i \neq k_\tau, j}^{n} \left(R_{\tau-}^i\right)^{w_i} = \prod_{i=1}^{n} \left(R_{\tau-}^i\right)^{w_i} = C_{\tau-} \,.$$

After the trade occurs, the remaining fraction of the input amount $(1 - \gamma)\Delta_\tau^{k_\tau}$, accounting for the fees is injected in the reserve associated to the asset $\alpha_{k_\tau}$. This impacts the value $C_\tau$ which gets updated according to,

$$C_\tau = \left(R_{\tau-}^{k_\tau} + \Delta_\tau^{k_\tau}\right)^{w_{k_\tau}} \left(R_{\tau-}^{j} - \Lambda_\tau^{j}\right)^{w_j} \prod_{i \neq k_\tau, j}^{n} \left(R_{\tau-}^i\right)^{w_i} = C_{\tau-} \left( \frac{1 + V_\tau}{1 + \gamma V_\tau} \right) \geq C_{\tau-} \,. \tag{2.33}$$

The constant value is increased after each trade due to the commission. By using the liquidity provider holdings rather than the reserve sizes we can ignore the jumps of $C_t$ due to liquidity movements. Let us denote,

$$c_t = \psi_{\mathrm{Bal}}(\boldsymbol{r}_t) = \prod_{i=1}^{n} \left(r_t^i\right)^{w_i} .$$

Since $\boldsymbol{r}$ is not impacted by a liquidity event as seen in Assumption 3, the newly defined value $c_t$ only jumps after each trade. Using the fact that the share $X^m$ of each LP is constant between two liquidity events with equations (2.14) and (2.33), one obtains for $\tau \in \mathcal{T}$,

$$c_\tau = c_{\tau-} \left( \frac{1 + V_{\tau-}}{1 + \gamma V_{\tau-}} \right)^{w_{k_\tau}} . \tag{2.34}$$

The above equation implies

$$\frac{\psi_{\mathrm{Bal}}(\boldsymbol{r}_t)}{\psi_{\mathrm{Bal}}(\boldsymbol{r}_0)} = \frac{c_t}{c_{t_0}} = \rho_t, \tag{2.35}$$

with $\rho_t$ as defined in (2.32). Combining (2.26) and the above equation (2.35), one finds

$$\frac{r_t^n}{r_0^n} = \frac{c_t}{c_{t_0}} \prod_{i=1}^{n} \left( \frac{S_t^i}{S_0^i} \right)^{w_i} = \rho_t \prod_{i=1}^{n} \left(z_t^i\right)^{w_i} . \tag{2.36}$$

Finally, replacing (2.36) within (2.25) gives the result of the proposition. □

The parameter $\rho_t$ increases with each new trade starting from 1 at inception when the LP joins the pools. The longer the LP holds his/her position in the DEX, the higher will be the value of $\rho_t$. It is also worth noting that if trading volume increases, this parameter grows faster, improving the IL even more efficiently. In Figure 2.3 several profiles of IL are represented with different fixed values of $\rho_t$. The gradual increase of $\rho_t$ shifts the impermanent loss profile higher and reduces the area where the IL is negative. For example, when trading fees bring $\rho_t$ from 1 to 1.7, compared to a buy-and-hold strategy, liquidity providing remains profitable even when the spot changes by a factor of 10 (either increase or decrease). The profile of IL obtained with $\rho_t \equiv 1$ corresponds to the case with no trading commission described in the previous section.

## 2.6 Slippage

Slippage is the difference between the price at which a trade is expected to get executed and the actual price at which it occurs. With exchanges relying on order books [18], slippage can be positive or negative, depending on whether the difference is favorable or not. For DEXs, such as Uniswap or Balancer, the prices depend on the liquidity pools sizes which have slower dynamic properties than order books. The resulting slippage is often unfavorable for the trader.

We are defining the slippage as the difference of the effective spot of asset $i$, $\bar{S}_t^i$, which is the price of a trade of arbitrary size and its associated marginal price $\hat{S}_t^i$ as discussed in sub-section 2.3. The rationale behind this definition is that a trader submitting an order with very small size will experience a trade flow in line with $\hat{S}_t^i$. However, if the trade size is larger, the effective spot of the trade will differ from $\hat{S}_t^i$ and the difference is generally seen as a slippage. Other definitions or views are possible as well. Additionally, and to simplify reasoning, we ignore the impact of trading fees such that, following [6], $\hat{S}_t^i = S_t^i$ that is, the marginal price is equal to the reported price and we can write the slippage $\Upsilon_t$ without trading fees as,

$$\Upsilon_t = \frac{\bar{S}_t^i - S_t^i}{S_t^i} .$$

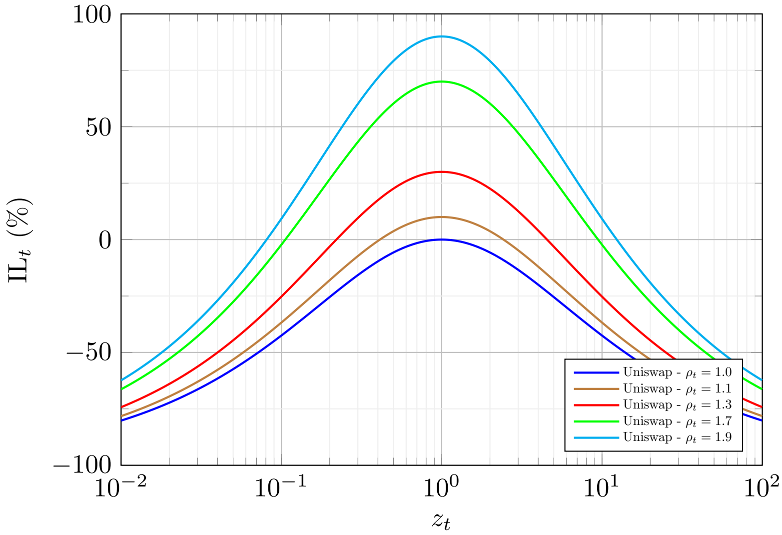


**Figure 2.3:** Impact of trading fees on the profile of impermanent loss for a LP joining a DEX similar to Uniswap. The accrual of commissions increases $\rho_t$ gradually. Different IL profiles are represented for $\rho_t = \{1, 1.1, 1.3, 1.7, 1.9\}$. $\rho_t = 1$ corresponds to the case without trading fees. The $x$-axis is $z_t$ on a log-scale.

We focus on the case of two assets for the geometric mean market CFMM. Let us consider a trader who exchanges $\Delta_t^1 \geq 0$ to obtain $\Lambda_t^2$ at time $t$. By writing the two-assets equation of (2.3) with $\gamma = 1$ we have,

$$\left(R_{t-}^1 + \Delta_t^1\right)^w \left(R_{t-}^2 - \Lambda_t^2\right)^{1-w} = \left(R_{t-}^1\right)^w \left(R_{t-}^2\right)^{1-w}.$$

This gives,

$$\Lambda_t^2 = R_{t-}^2 \left(1 - \left(1 + \frac{\Delta_t^1}{R_{t-}^1}\right)^{-\frac{w}{1-w}}\right).$$

Let us remark that expression (2.8) simplifies to $\bar{S}_t^1 = \Lambda_t^2 / \Delta_t^1$ in the case of two assets with $\alpha \equiv \alpha_1$ and $\beta \equiv \alpha_2$.

By using the expression 2.7; one finds,

$$\Upsilon_t^{\alpha \to \beta} = \frac{1-w}{w} \frac{R_{t-}^1}{\Delta_t^1} \left(1 - \left(1 + \frac{\Delta_t^1}{R_{t-}^1}\right)^{-\frac{w}{1-w}}\right) - 1, \tag{2.37}$$

This slippage, when defined as a function of the trade size $\Delta^1$ converges to zero when $\Delta^1 \to 0$. For Uniswap, with $w = 0.5$, expression (2.37) becomes $\Upsilon_t^{\alpha \to \beta} = -\Delta_t^1 / \left(R_{t-}^1 + \Delta_t^1\right)$. The negative slippage here, means when selling a non-zero quantity $\Delta_t^1$ at time $t$, the obtained spot is less favorable than if the trade size was marginal. Symmetrically for a trader selling an amount $\Delta_t^2 \geq 0$ to obtain $\Lambda_t^1$ one finds,

$$\Upsilon_t^{\beta \to \alpha} = \frac{w}{1-w} \frac{R_{t-}^2}{\Delta_t^2} \left(1 - \left(1 + \frac{\Delta_t^2}{R_{t-}^2}\right)^{-\frac{1-w}{w}}\right) - 1, \tag{2.38}$$

In the case of Uniswap expression (2.38) simplifies to $\Upsilon_t^{\beta \to \alpha} = -\Delta_t^2 / \left(R_{t-}^2 + \Delta_t^2\right)$ which can be similarly interpreted as above.

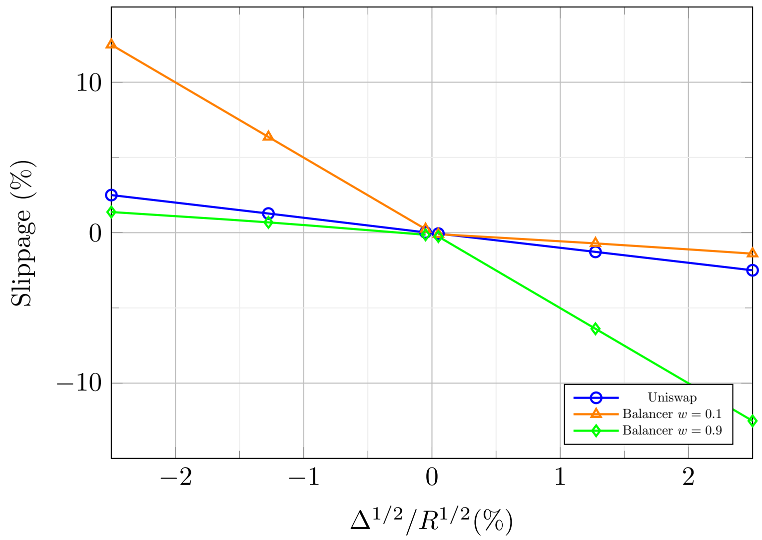


**Figure 2.4:** Slippage for geometric mean markets, with Uniswap and Balancer $w = \{0.1, 0.9\}$ as a function of $\Delta^1/R^1$ for positive values on the abscissa and $-\Delta^2/R^2$ for negative values on the abscissa. The ordinate is the slippage for a trade to sell $\alpha$ for $\beta$ on positive abscissa values. Similarly, the ordinate is - slippage for a trade to sell $\beta$ for $\alpha$ on negative abscissa values. One can see symmetrical slippage for Uniswap for both trade directions but asymmetric slippage for each Balancer. For example, Balancer with weight 0.1 is favorable for traders who sell $\alpha$ for $\beta$.

On Figure 2.4, both functions $\Upsilon^{\beta\to\alpha} : \mathbb{R}^+ \to \mathbb{R}^-$ and $\Upsilon^{\alpha\to\beta} : \mathbb{R}^+ \to \mathbb{R}^-$ are represented on the same graphic,

$$\Upsilon^{\alpha\to\beta}(x) = \frac{1-w}{xw}\left(1-(1+x)^{-\frac{w}{1-w}}\right) - 1, \quad \Upsilon^{\beta\to\alpha}(x) = \frac{w}{x(1-w)}\left(1-(1+x)^{-\frac{1-w}{w}}\right) - 1\,,$$

where negative abscissas are mapped to $x \to$ -$\Upsilon^{\beta\to\alpha}(-x)$ and positive abscissas, mapped to $x \to \Upsilon^{\alpha\to\beta}(x)$.

One can see that Balancer with weights different from 0.5, offers very asymmetric behavior in terms of slippage depending on the trade direction. For example low weights offers low slippage when selling $\alpha$ for $\beta$. However from $\beta$ to $\alpha$, the slippage is much higher than Uniswap. The situation is symmetric for high weights. We remind here that Balancer with asymmetric weights helps improve the IL for one direction of spot movement but not both. In conclusion, using Balancer with low or high weights partially improves the IL either on the increase or decrease of the spot but also offers very asymmetric slippage depending on the trade direction.

# 3 Synchronized AMM

In this article, we introduce a new type of AMM which combines the properties of several CFMMs together. The goal is two-fold, on the one hand this approach allows to gain flexibility on the features of the resulting AMM, on the other hand, as a direct application of this technique and driving result of the paper, we propose an AMM with strongly reduced downside risk for liquidity providers. The latter comes from the interesting properties of geometric mean markets CFMM with asymmetric weights which improve the impermanent loss either on the rise or on the fall of the spot, depending on the value of the weights of the geometric mean as seen in sub-section 2.5.1.

To avoid complex notations, we focus on the two-CFFM synchronized AMM. This bears similarities with the two-pools Balancer with aligned spot prices discussed in [7]. In sub-section 2.1, we have denoted $(\boldsymbol{R}_t)_{t\geq 0}$

the reserves for a given CFMM, which we will use here to account for the reserves of a first CFMM ($\mathcal{G}_1$). Therefore, we also denote $\left(\bar{\boldsymbol{R}}_t\right)_{t\geq 0}$ the reserve process for a second CFMM ($\mathcal{G}_2$). Both $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined by their associated constant function $\phi_1(\boldsymbol{R}_t,\cdot,\cdot)$ and $\phi_2\left(\bar{\boldsymbol{R}}_t,\cdot,\cdot\right)$ respectively. We propose the following definition,

**Definition 6.** Let $\boldsymbol{S}_{t-}$ be the reported spot of $\mathcal{G}_1$ and $\boldsymbol{Y}_{t-}$ be the reported spot of $\mathcal{G}_2$ at a positive time $t-$ prior to a trade or a liquidity addition or withdrawal. An automated market maker is a synchronized CFFM if and only if, for any valid trade $(\boldsymbol{\Lambda},\boldsymbol{\Delta})$, there exist a vector $\boldsymbol{q}\in[0,1]^n$ such that,

$$\phi_1(\boldsymbol{R}_{t-},\boldsymbol{\eta},\boldsymbol{q}\odot\boldsymbol{\Delta})=\phi_1(\boldsymbol{R}_{t-},\boldsymbol{0},\boldsymbol{0})\,, \tag{3.1}$$

as well as,

$$\phi_2\left(\bar{\boldsymbol{R}}_{t-},\bar{\boldsymbol{\eta}},(\boldsymbol{1}-\boldsymbol{q})\odot\boldsymbol{\Delta}\right)=\phi_2\left(\bar{\boldsymbol{R}}_{t-},\boldsymbol{0},\boldsymbol{0}\right),$$

with $\boldsymbol{\Lambda}=\bar{\boldsymbol{\eta}}+\boldsymbol{\eta}$, $\odot$ the element-wise (Hadamart) product and $\boldsymbol{1}$ the unit vector of $n$-entries. Additionally, the reported spots, $\boldsymbol{S}_t$ and $\boldsymbol{Y}_t$ after the trade and at liquidity events must agree, that is the following relationship must hold for any $i\in[\![1,n]\!]$ and at any time $t\geq 0$,

$$\frac{\partial_{R^i}\psi_1(\boldsymbol{R}_t)}{\partial_{R^n}\psi_1(\boldsymbol{R}_t)}=\frac{\partial_{R^i}\psi_2\left(\bar{\boldsymbol{R}}_t\right)}{\partial_{R^n}\psi_2\left(\bar{\boldsymbol{R}}_t\right)}\,, \tag{3.2}$$

with,

$$\psi_{1/2}(\boldsymbol{.})=\phi_{1/2}(\boldsymbol{.},\boldsymbol{0},\boldsymbol{0}),$$

and where, as per (2.4),

$$\begin{aligned}\boldsymbol{R}_t&=\boldsymbol{R}_{t-}+\boldsymbol{q}\odot\boldsymbol{\Delta}-\boldsymbol{\eta}\\ \bar{\boldsymbol{R}}_t&=\bar{\boldsymbol{R}}_{t-}+(1-\boldsymbol{q})\odot\boldsymbol{\Delta}-\bar{\boldsymbol{\eta}}\,.\end{aligned}$$

Less formally, this entails that the synchronization allows to split any valid trade between $\mathcal{G}_1$ and $\mathcal{G}_2$, making it a valid trade for both $\mathcal{G}_1$ and $\mathcal{G}_2$ individually. Additionally, the reported spots before and after the trade should match perfectly. Without loss of generality, since the reported spots of both $\mathcal{G}_1$ and $\mathcal{G}_2$ are matching, we will denote the reported spot of the synchronized AMM by $(\boldsymbol{S}_t)_{t\geq 0}$ , following the reported spot notation of $\mathcal{G}_1$.

## 3.1 Two-assets synchronized geometric mean AMM

In this section, as a concrete example of the above general definition, we consider a two assets AMM built with the synchronization of two geometric mean CFMM. In the remainder of the article we refer to this automated market maker as sync-AMM. There are two reasons behind this choice; the first is that two assets within two synchronized CFMM is a natural configuration for such a system and therefore also the easiest both numerically and theoretically. The second and most important reason is that by combining two geometric CFMM, with well chosen weights, we can take advantage of their best properties and combine them to improve the impermanent loss for LPs. It is important here to pin-point the fact that traders will not be able to interact with the two internals CFMM, as this will be done following the procedure below, and will result in a standard AMM from an end-user perspective. The trade splitting and rerouting into the two internal CFMM as well as the synchronization of the reported spots is handled by the new type of DEX.

This section derives the main result of this article and the two-assets synchronized geometric mean CFMM will be discussed in extensive details in the remainder of the paper. We write,

$$\psi_1(\boldsymbol{R}_t)=\left(R_t^1\right)^w\left(R_t^2\right)^{1-w}\,,\qquad \psi_2\left(\bar{\boldsymbol{R}}_t\right)=\left(\bar{R}_t^1\right)^{\bar{w}}\left(\bar{R}_t^2\right)^{1-\bar{w}}\,, \tag{3.3}$$

where the driving weights of the geometric mean $\mathcal{G}_1$ and $\mathcal{G}_2$ are $w$ and $\bar{w}$ respectively. The Definition 6 can be written naturally under that setup, more specifically, the spot equality condition (3.2) becomes for any $t$,

$$\frac{w}{1-w}\frac{R_t^2}{R_t^1}=\frac{\bar{w}}{1-\bar{w}}\frac{\bar{R}_t^2}{\bar{R}_t^1}. \tag{3.4}$$

This combined with (3.1) provides insights on the necessary and sufficient condition for what is required at trade times $t \in \mathcal{T}$. We also assume that reserves of the geometric mean CFMM are strictly positive for any time $t > 0$,

$$R_t^{1/2} > 0, \qquad \bar{R}_t^{1/2} > 0. \tag{3.5}$$

Finally, while most of the derivations and discussions below can be made under the assumptions that $w$ and $\bar{w}$ are not linked together. In our case, we will work with

$$\bar{w} = 1 - w\,. \tag{3.6}$$

#### 3.1.1 Trade events and trade routing

Let us denote a valid trade $(\boldsymbol{\Lambda}, \boldsymbol{\Delta})$ at time $t \in \mathcal{T}$ and $q \in [0, 1]$ a trade split. We also assume that at $t-$ the reported spot of $\mathcal{G}_1$ and $\mathcal{G}_2$ are synchronized following (3.4), and we therefore write,

$$S_{t-} = Y_{t-}\,.$$

Without loss of generality, we work under the case where the trader wishes to sell a predefined amount $\Delta_1 > 0$ of asset $\alpha_1$ and is expecting to receive an amount $\Lambda_2$ of asset $\alpha_2 \equiv \beta$. The rational trader is not expecting to input any amount of asset $\beta$ for that operation, such that $\Delta_2 = 0$ and will not receive any amount of asset $\alpha_1$ such that $\Lambda_1 = \eta_1 = \bar{\eta}_1 = 0$. For $\mathcal{G}_1$ with (3.1) we therefore have,

$$\left(R^1_{t-} + \gamma q \Delta_1\right)^w \left(R^2_{t-} - \eta_2\right)^{1-w} = \left(R^1_{t-}\right)^w \left(R^2_{t-}\right)^{1-w}, \tag{3.7}$$

and similarly for $\mathcal{G}_2$,

$$\left(\bar{R}^1_{t-} + \gamma(1-q)\Delta_1\right)^{\bar{w}} \left(\bar{R}^2_{t-} - \bar{\eta}_2\right)^{1-\bar{w}} = \left(\bar{R}^1_{t-}\right)^{\bar{w}} \left(\bar{R}^2_{t-}\right)^{1-\bar{w}}. \tag{3.8}$$

Since at $t-$ we have (3.4), before the trade we can write the condition on the output amounts $\eta_1$ and $\eta_2$ which are unique solutions of (3.7) and (3.8), namely,

$$\eta_2 = R^2_{t-}\left(1 - \left(\frac{R^1_{t-}}{R^1_{t-} + \gamma q \Delta_1}\right)^{\frac{w}{1-w}}\right), \qquad \bar{\eta}_2 = \bar{R}^2_{t-}\left(1 - \left(\frac{\bar{R}^1_{t-}}{\bar{R}^1_{t-} + \gamma(1-q)\Delta_1}\right)^{\frac{\bar{w}}{1-\bar{w}}}\right),$$

which provides the unique solution for the output amount of the synchronized AMM

$$\Lambda_2 = \eta_2 + \bar{\eta}_2. \tag{3.9}$$

To find the term $q$, we use the fact that after the trade, reserves get updated and the spots are required to remain synchronized according to (3.4) with the new reserves. This gives

$$\frac{w}{1-w}\frac{R^2_{t-}\left(\frac{R^1_{t-}}{R^1_{t-}+\gamma q\Delta_1}\right)^{\frac{w}{1-w}}}{R^1_{t-} + q\Delta_1} = \frac{\bar{w}}{1-\bar{w}}\frac{\bar{R}^2_{t-}\left(\frac{\bar{R}^1_{t-}}{\bar{R}^1_{t-}+\gamma(1-q)\Delta_1}\right)^{\frac{\bar{w}}{1-\bar{w}}}}{\bar{R}^1_{t-} + (1-q)\Delta_1}. \tag{3.10}$$

By using $S_{t-} = \frac{w}{1-w}\frac{R^2_{t-}}{R^1_{t-}}$ and $Y_{t-} = \frac{\bar{w}}{1-\bar{w}}\frac{\bar{R}^2_{t-}}{\bar{R}^1_{t-}}$, the above expression is simplified into

$$S_{t-}\frac{\left(1 + \gamma\frac{q\Delta_1}{R^1_{t-}}\right)^{\frac{-w}{1-w}}}{1 + \frac{q\Delta_1}{R^1_{t-}}} = Y_{t-}\frac{\left(1 + \gamma(1-q)\frac{\Delta_1}{\bar{R}^1_{t-}}\right)^{\frac{-\bar{w}}{1-\bar{w}}}}{1 + (1-q)\frac{\Delta_1}{\bar{R}^1_{t-}}}\,. \tag{3.11}$$

Here we use the fact that $S_{t-} = Y_{t-}$ and $S_{t-} > 0$, such that we remove the spots from (3.11). This condition translates in defining a continuous function $f : [0, 1] \to \mathbb{R}$, such that $q$ is solution of $f(q) = 0$, where

$$f(q) = \left(1 + (1-q)\frac{\Delta_1}{\bar{R}^1_{t-}}\right)\left(1 + \gamma(1-q)\frac{\Delta_1}{\bar{R}^1_{t-}}\right)^{\frac{\bar{w}}{1-\bar{w}}} - \left(1 + \frac{q\Delta_1}{R^1_{t-}}\right)\left(1 + \gamma\frac{q\Delta_1}{R^1_{t-}}\right)^{\frac{w}{1-w}}.$$

Additionally we note that,

$$f(0) = \left(1 + \frac{\Delta_1}{\bar{R}^1_{t-}}\right)\left(1 + \gamma\frac{\Delta_1}{\bar{R}^1_{t-}}\right)^{\frac{\bar{w}}{1-\bar{w}}} - 1 > 0, \qquad f(1) = 1 - \left(1 + \frac{\Delta_1}{R^1_{t-}}\right)\left(1 + \gamma\frac{\Delta_1}{R^1_{t-}}\right)^{\frac{w}{1-w}} < 0,$$

and since $f$ is a continuous and strictly decreasing function we conclude that there exists a unique $q_0$ such that $f(q_0) = 0$.

*Remark.* This is of paramount importance as it signifies that any trade can be performed with a unique split and routing into $\mathcal{G}_1$ and $\mathcal{G}_2$ that also allows to keep both $\mathcal{G}_1$ and $\mathcal{G}_2$ with a synchronized reported spot at any time $t$ around a trade event. We see in the next section that synchronization holds also during liquidity addition and withdrawal. Therefore the reported spots for both $\mathcal{G}_1$ and $\mathcal{G}_2$ will remain synchronized at any given time $t$.

### 3.1.2 Liquidity addition and withdrawal

Liquidity addition in the synchronized AMM follows two requirements. On the one hand, we require that the spot should not move when liquidity is changed, which is a generally standard desired property as part of Assumption 3. On the other hand, thanks to extra degrees of freedom within the sync-AMM, we can also require that the ratio of the assets added follows the current reported spot, which is a property shared with Uniswap. Namely, assuming the liquidity provider desires to add a liquidity amount $r^1_{t_0}$ of asset $\alpha \equiv \alpha_1$ at joining time $t_0$, the amount of asset $\beta \equiv \alpha_2$ to put alongside will follow,

$$r^2_{t_0} = S_{t_0} r^1_{t_0}.$$

We also make use of the link between the weights in $\mathcal{G}_1$ and $\mathcal{G}_2$, from equation (3.6). Additionally, we have seen in (3.4) that liquidity addition in geometric mean market makers should be proportional to the current reserve sizes, such that, for any $\theta \in \Theta$ and for a given $\lambda > 0$ the reserve increase is,

$$\boldsymbol{R}_\theta = \boldsymbol{R}_{\theta-}(1+\lambda).$$

This naturally leads to matching spots at $t_0-$ and $t_0$ for both $\mathcal{G}_1$ and $\mathcal{G}_2$, where we define two addition factors, $\lambda$ and $\bar{\lambda}$ respectively and where we get,

$$\boldsymbol{R}_{t_0} = \boldsymbol{R}_{t_0-}(1+\lambda), \quad \bar{\boldsymbol{R}}_{t_0} = \bar{\boldsymbol{R}}_{t_0-}\left(1+\bar{\lambda}\right).$$

Spot equality is therefore guaranteed by construction, the only extra condition would translate in finding the tuple $\left(\lambda, \bar{\lambda}\right)$ such that,

$$\lambda R^1_{t_0-} + \bar{\lambda}\bar{R}^1_{t_0-} = r^1_{t_0}, \quad \lambda R^2_{t_0-} + \bar{\lambda}\bar{R}^2_{t_0-} = S_{t_0-} r^1_{t_0}, \tag{3.12}$$

where we used the fact that the reported spots of $\mathcal{G}_1$ and $\mathcal{G}_2$ match precisely at $t_0-$. Since we have assumed (3.6) as well as (3.4) which ensures equality of reported spots, the unique solution to the linear system (3.12) is,

$$\lambda = w\frac{r^1_{t_0}}{R^1_{t_0-}}, \quad \bar{\lambda} = (1-w)\frac{r^1_{t_0}}{\bar{R}^1_{t_0-}}.$$

This means that the liquidity amount is split between $\mathcal{G}_1$ and $\mathcal{G}_2$ with proportions $w$ and $1-w$ respectively. The liquidity provider shares are then updated naturally for each of the geometric mean market makers $\mathcal{G}_1$ and $\mathcal{G}_2$ independently following (2.13). It is worth noting that $\mathcal{G}_1$ and $\mathcal{G}_2$ having a spot as defined in (2.7), the AMM dispatches a share $w$ of asset $\alpha$ and a share $(1-w)$ of asset $\beta$ into the first internal pool $\mathcal{G}_1$. Symmetrically, the AMM dispatches a share $(1-w)$ of $\alpha$ assets and a share $w$ of tokens $\beta$ into the second internal pool $\mathcal{G}_2$.

Liquidity removal is similar to liquidity withdrawal in the individual geometric mean market makers $\mathcal{G}_1$ and $\mathcal{G}_2$. For both $\mathcal{G}_1$ and $\mathcal{G}_2$, the spots naturally do not move since the share of the liquidity providers in the reserves for asset $\alpha$ and reserve for asset $\beta$ are under the same proportion following (2.13) and therefore the withdrawal is done by a down-scaling of the reserve sizes.

### 3.1.3 Impermanent loss of a synchronized AMM without trading fees

We consider here the case where the AMM accounts for two assets $\alpha \equiv \alpha_1$ and $\beta \equiv \alpha_2$ and provide the impermanent loss for the sync-AMM discussed in Section 3.1 in the case where there is no trading fees. We recall that for each individual internal geometric CFMM $\mathcal{G}_1$ and $\mathcal{G}_2$, the total value of the assets of a liquidity provider under numéraire $\beta$ and at time $t \geq 0$, is $p_t = \frac{r_t^2}{1-w}$ for $\mathcal{G}_1$ and $\bar{p}_t = \frac{\bar{r}_t^2}{1-\bar{w}}$ for $\mathcal{G}_2$ according to (2.23). Here, $r_t^2$ and $\bar{r}_t^2$ are the holdings of the LP for asset $\beta$ in the corresponding CFMM.

At any time $t \geq 0$, the total holding $P_t$ of the LP is the sum of his holdings in the two internal pools,

$$P_t = p_t + \bar{p}_t \,.$$

Let us denote $Q_{t_0}^{\beta}$ the total amount of liquidity added at time $t_0$ by the LP into the sync-AMM denominated in amounts of $\beta$. Following the details in sub-section 3.1.2, the sync-AMM will distribute this initial liquidity by dispatching a fraction $\nu Q_{t_0}^{\beta}$ into $\mathcal{G}_1$ and $(1-\nu)Q_{t_0}^{\beta}$ into $\mathcal{G}_2$. By combining the generic equations (2.23), (2.24) and (2.27) we can write the difference between the holdings in the DEX versus a buy-and-hold strategy,

$$P_t - P_t^S = \nu Q_{t_0}^{\beta}(z_t^w - w z_t - (1-w)) + (1-\nu) Q_{t_0}^{\beta}\left(z_t^{\bar{w}} - \bar{w} z_t - (1-\bar{w})\right).$$

That gives the expression of the impermanent loss,

$$\mathrm{IL}(z_t) = \frac{\nu(z_t^w - w z_t - (1-w)) + (1-\nu)(z_t^{\bar{w}} - \bar{w} z_t - (1-\bar{w}))}{\nu[w z_t + (1-w)] + (1-\nu)[\bar{w} z_t + (1-\bar{w})]}.$$

As a sanity check if both internal pools are following Uniswap dynamics $w = \bar{w} = \frac{1}{2}$, one recovers $\mathrm{IL}(z)^{\mathrm{Uni}} = \frac{2\sqrt{z}}{z+1} - 1$. Following (3.6), we set $\bar{w} = 1-w$ and the choice of $\nu$ can be restricted by imposing the buy-and-hold portfolio to behave similarly to Uniswap; that means that at inception the liquidity provider will have to provide assets with equal quantity as discussed in sub-section 3.1.2. For this purpose, the following equation must be satisfied,

$$(\nu w + (1-\nu)(1-w)) = \frac{1}{2},$$

which leads to

$$\nu = \frac{1}{2}\,. \tag{3.13}$$

The above equation means the internal CFMM $\mathcal{G}_1$ and $\mathcal{G}_2$ receive equal quantities of asset $\alpha$ and $\beta$. This choice provides a rather elegant form for the impermanent loss of the sync-AMM,

$$\mathrm{IL}(z) = \frac{z^w + z^{1-w}}{z+1} - 1\,. \tag{3.14}$$

It is interesting to notice that (3.14) is always smaller than zero, as expected, but greater than $\mathrm{IL}^{\mathrm{Uni}}(z)$ for any $z \geq 0$. Indeed,

$$\mathrm{IL}(z) - \mathrm{IL}^{\mathrm{Uni}}(z) = \frac{z^w + z^{1-w} - 2\sqrt{z}}{z+1} = \frac{(z^w - \sqrt{z})^2}{z^w(z+1)} \geq 0\,.$$

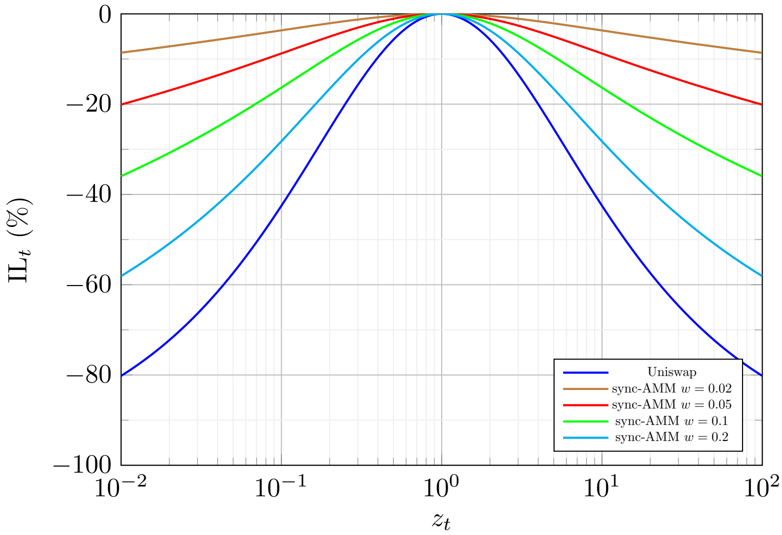


**Figure 3.1:** Impermanent loss without trading fees for LP under Uniswap and Sync-AMM with different values of $w$, namely $w = \{0.02, 0.05, 0.1, 0.2\}$. The plot is as a function of $z_t$ on a log-scale.

The sync-AMM therefore improves the impermanent loss compared to Uniswap. Moreover the smaller the weight $w$, the higher the improvement. However, small $w$ have lower quality for the slippage, which brings a trade-off between the improvement of the IL and the spot slippage to take into consideration.

As seen on Figure 3.1, the new AMM takes advantages of the best features of each CFMM and helps improve the IL on both increase and decrease of the spot performance. For example, the synchronized AMM with $w = 0.1$, leads to an IL of $-16.3\%$ against $-42.5\%$ for Uniswap if the spot either increases or decreases by a factor of 10 with respect to the LP entry value. The IL for the new AMM is perfectly symmetric as chosen in (3.6).

The gradual accumulation of trading fees combined with this new profile of impermanent loss is the building-block of improved returns for liquidity providers. More details and associated simulations are provided in Section 4.

### 3.1.4 Expression of slippage

A trade on the sync-AMM is decomposed in two portions which are dispatched to the internal geometric mean CFMMs. A root-finding algorithm solves for trade splitting proportion $q_t$ that keeps the internal CFMM spot synchronized as per sub-section 3.1 for a given trade. We assume that (3.6) holds, and using the result of Section 2.6, we can obtain the effective spot price after a trade. For example, selling a strictly positive amount $\Delta_t^1$ of asset $\alpha$ to obtain $\Lambda_t^2$ of asset $\beta$,

$$\bar{S}_t^1 = \frac{1}{\Delta_t^1}\left[R_{t-}^2\left[1-\left(1+\frac{q_t\Delta_t^1}{R_{t-}^1}\right)^{-\frac{w}{1-w}}\right]+\bar{R}_{t-}^2\left[1-\left(1+\frac{(1-q_t)\Delta_t^1}{\bar{R}_{t-}^1}\right)^{-\frac{1-w}{w}}\right]\right].$$

And we also have the marginal spot defined as,

$$\lim_{\Delta^1\to 0}\bar{S}_t^1(\Delta^1) = q_t\frac{w}{1-w}\frac{R_{t-}^2}{R_{t-}^1}+(1-q_t)\frac{1-w}{w}\frac{\bar{R}_{t-}^2}{\bar{R}_{t-}^1}.$$

And from the spot equality in (3.4), we can write the expression of the slippage without trading fees as,

$$\Upsilon_t^{\alpha\to\beta}(\Delta^1) = \frac{1-w}{w}\frac{R_{t-}^1}{\Delta^1}\left[1-\left(1+\frac{q_t\Delta^1}{R_{t-}^1}\right)^{-\frac{w}{1-w}}\right] + \frac{w}{1-w}\frac{\bar{R}_{t-}^1}{\Delta^1}\left[1-\left(1+\frac{(1-q_t)\Delta^1}{\bar{R}_{t-}^1}\right)^{-\frac{1-w}{w}}\right].$$

The above expression is the sum of the slippage of each internal CFMM taking respectively the input trades $q_t\Delta_t^1$ and $(1-q_t)\Delta_t^1$. Given that the trade split $q_t$ is a function of the input trade size and the states of the internal reserves just before a trade, it makes the slippage calculation more complex than a more standard CFMM. And because of this, the slippage is not a function of $\Delta_t^1/\big(R_{t-}^1+\bar{R}_{t-}^1\big)$ anymore and cannot be obtained in closed form. If only the total value of $R_t^{1/2}+\bar{R}_t^{1/2}$ were to be reported as reserves publicly, but not the individual reserves of $\mathcal{G}_1$ and $\mathcal{G}_2$, it would be more difficult to perform front-running due to the missing information required to calculate the slippage.

# 4 Simulations and numerical results

In this section, we discuss the simulation of the synchronized two-assets AMM introduced in sub-section 3.1. First and foremost we will use the following simplified market model; we suppose the market spot$(Z_t)_{t\geq 0}$ representing one unit of $\alpha$ in terms of $\beta$, follows a geometric Brownian motion stochastic differential equation under $\mathbb{P}$ with,

$$\begin{cases} dZ_t &= \mu Z_t dt + \sigma Z_t dW_t \\ Z_0 &\geq 0, \end{cases} \tag{4.1}$$

where $W$ is a Brownian motion under $\mathbb{P}$, and where $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$. The diffusion parameters $\mu$ and $\sigma$ are not calibrated to the market but arbitrarily chosen values which align with the high growth-rate and volatility sometimes encountered on crypto-currency markets. For example, over the second half of March 2021, Bitcoin realized volatility ranged from 70% to 85%, and similarly Ethereum realized volatility ranged from 60% to 95% (source [2]). More precisely, we present the set of results obtained with the below driving numbers,

$$Z_0 = 137, \quad \mu = 25\%, \quad \sigma = 120\%.$$

The trading fees are assumed fixed at 0.3% per trade and the simulations are run over periods of 3 years with an average amount of 50 000 trades per year, excluding arbitrage trades.

The state variables of the AMMs simulated are calculated following details from sub-section 2.2. Extensive details about Monte Carlo simulations as well as sampling procedure for both (4.1) and random times as described in sub-section 4.1 can be found in [17].

## 4.1 Trade generation and arbitrage

Our assumption for the different simulations is that trades are simulated such that inter-arrival times follow an exponential distribution of mean $\frac{1}{\lambda}$, which provides an average of $\lambda$ jumps per unit of time. We chose an average of 50 000 trades per year, randomly happening on the timeline. Let $t \in \mathcal{T}$ be a trade time, the trade size associated to the trade is defined as a Gaussian random variable with mean 0 and standard deviation $N_t^1/400$, where resulting sizes are truncated to stay within the bounds,

$$\big[-N_t^1/100, N_t^1/100\big],$$

where we denoted that for any $t \geq 0$,

$$N_t^1 = R_t^1 + \bar{R}_t^1, \quad N_t^2 = R_t^2 + \bar{R}_t^2.$$

A positive trade size is assumed to be a trade to buy or sell asset $\alpha$ for asset $\beta$, whereas a negative trade size, is a trade to buy or sell asset $\beta$ for asset $\alpha$. The buy or sell order choice is determined randomly with equal probability.

The trade generation discussed above accounts for any spontaneous trades done by traders but does not account for a quoted market spot, e.g. the spot value available on a centralized exchange. Where a market spot is available outside of the sync-AMM, it is possible for arbitrage opportunities to arise, and we assume that in between any two trades, an arbitrageur will try to take advantage of the AMM reported spot divergence with the market spot. More precisely, at time $t = \frac{t_{k+1}+t_k}{2}$ in between non-arbitrage related trades $(t_k, t_{k+1})_{k\in\mathbb{N}} \in \mathcal{T}^2$, an arbitrageur will perform the following operations. If the market spot $Z_t$ is higher than the sync-AMM reported spot $S_t$, that means one can sell $\alpha$ for $\beta$ more expensively in the market than on the DEX. Therefore the arbitrageur will optimize a sell order of asset $\beta$ for asset $\alpha$ in the DEX,

$$A_t = \max_{(\mathbf{\Lambda},\mathbf{\Delta})\in\mathcal{H}_t} \left(Z_t\Lambda^1 - \Delta^2\right),$$

symmetrically if the market spot $Z_t$ is smaller than the sync-AMM reported spot $S_t$, the arbitrageur will optimize a sell order of asset $\alpha$ for asset $\beta$ in the DEX,

$$A_t = \max_{(\mathbf{\Lambda},\mathbf{\Delta})\in\mathcal{H}_t} \left(\frac{\Lambda^2}{Z_t} - \Delta^1\right),$$

where $\mathcal{H}_t$ is the set of all valid trade of the sync-AMM at time $t$. If $A_t > 0$, then the arbitrageur will execute the trade to sell $\Delta^1$ or $\Delta^2$ of asset $\alpha$ or $\beta$ respectively, recover the output amount $\Lambda^2$ or $\Lambda^1$ of asset $\beta$ or $\alpha$ and sell this amount back on the centralized exchange at spot price $(Z_t)^{-1}$ or $Z_t$ respectively, therefore realizing a profit. We note that $A_t$ is not necessarily positive even if the market and reported spot differ, this is due to the impact of trading fees. For example, in [8], the authors show that for Uniswap, it is not efficient to perform an arbitrage trade if the reported spot is smaller that $Z_t/\gamma$ and higher than $\gamma Z_t$.

## 4.2 Liquidity addition and withdrawal

The liquidity addition follows very closely the discussion in sub-section 3.1.2, where each liquidity provider, can either join as a new provider in the AMM or withdraw his/her liquidity. Our assumptions are simplistic but provide a good model for the analysis of the liquidity provider returns, e.g. impermanent loss.

We will work with an initial liquidity amount, brought by a first liquidity provider at time $t = 0$,

$$N_0^1 = 10\,000\,000, \quad N_0^2 = Z_0 N_0^1 = 1\,370\,000\,000.$$

Each subsequent liquidity provider will bring precisely 5% of the initial reserve size $N_0^1$ when he/she joins the sync-AMM. The liquidity provider will also deposit $Z_0 N_0^1$ into the reserve of $\beta$ as part of the liquidity addition. Withdrawal can be done in full or partially by the LP in both of the underlying synchronized CFMM. For ease of testing and showcase, we assume that the user exits fully from both synchronized CFMM, at the same time. The associated return is discussed hereafter and displays both the impact of impermanent loss and the rewards thanks to trading fees.

Over a $3Y$ time frame, 20 new liquidity providers will enter the AMM with entry times uniformly distributed over the timeline. While this assumption is simplistic, it allows to validate the dynamics of the DEX.

## 4.3 Liquidity provider returns

When entering the AMM, the liquidity provider returns, which we will refer to as impermanent loss, can be calculated at every point in time. More specifically, since the liquidity provider can decide to exit the DEX at any given time, it is possible to evaluate the profit-and-loss of the holdings compared to a simple buy-and-hold

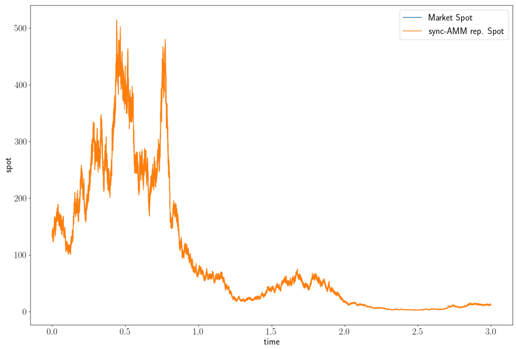


**Figure 4.1:** Paths 1: Market spot $Z_t$, sync-AMM reported spot $S_t$. $X$ axis is the time in years.

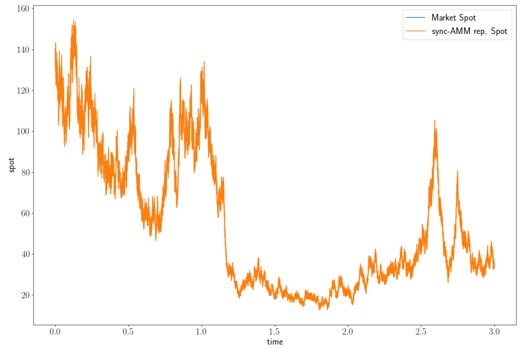


**Figure 4.2:** Paths 2: Market spot $Z_t$, sync-AMM reported spot $S_t$. $X$ axis is the time in years.

strategy started at the LP entry time in the DEX. This without the need for the LP to effectively exit the liquidity pools, which allows to assess the robustness of impermanent loss over different exit points, therefore provides a large scope of possible outcomes.

## 4.4 Numerical results

The aggregated results for the liquidity provider returns and impermanent loss are computed with 30 paths. While this is low to achieve any reasonable convergence properties on the market spot distribution, this amount of paths combined with the number of possible exit points allows the gathering of a considerable amount of information and scenario outcomes on the liquidity provider PnL and IL. Exit points can happen and profit-and-loss can be calculated at every time step, which will range between 50 000 and 100 000 times per year with the lower bound being the amount of trades independent of arbitrage and the upper bound which accounts for the extra amount of trades a year attributed to arbitrage.

Out of the total number of paths, we hereafter extract 4 of them to display dynamic behaviors. In Figures 4.1 4.2 4.3 4.4 we display the time series of the market spot as well as the sync-AMM reported spot. We can see that the spot moves in a large range of values which provides useful stress test cases.

We also display in Figure 4.5 4.6 4.7 4.8 the impermanent loss of all LPs who entered the given AMM at different time and therefore different spot values. Consequently, they are exposed to different effects of market moves and their profit-and-loss will naturally look different. We note that an important feature of the sync-AMM, which is shared by most CFMM such as Uniswap or Balancer and follows Assumption 3, is that liquidity addition or withdrawal does not impact the PnL of other liquidity providers within the DEX.

*Remark.* The various colors in Figures 4.5 4.6 4.7 4.8 correspond to the IL of different liquidity providers, entering at different times as well as on possibly different paths, which provides an aggregated view of a large amount of possible outcomes and scenarios.

### 4.4.1 Time series under stressed markets

In addition to the above, in Table 2, we display the values of the IL with trading fees, i.e. the liquidity providers returns with trading fees compared with a buy-and-hold strategy started time $t = 0$. One can notice that in both cases (when the initial spot increases or decreases by a factor of 100), the IL of the sync-AMM is considerably better than on Uniswap and either a Balancer with low weight or high weight depending on the

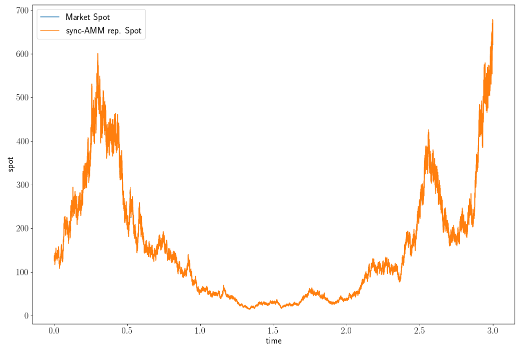


**Figure 4.3:** Paths 3: Market spot $Z_t$, sync-AMM reported spot $S_t$. $X$ axis is the time in years.

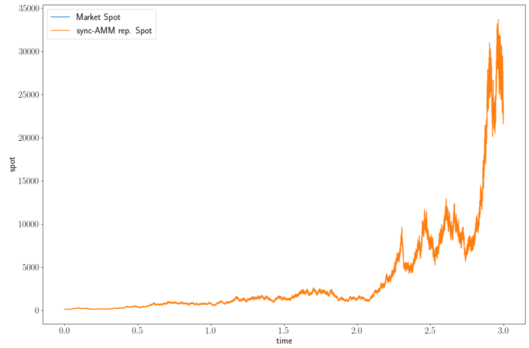


**Figure 4.4:** Paths 4: Market spot $Z_t$, sync-AMM reported spot $S_t$. $X$ axis is the time in years.

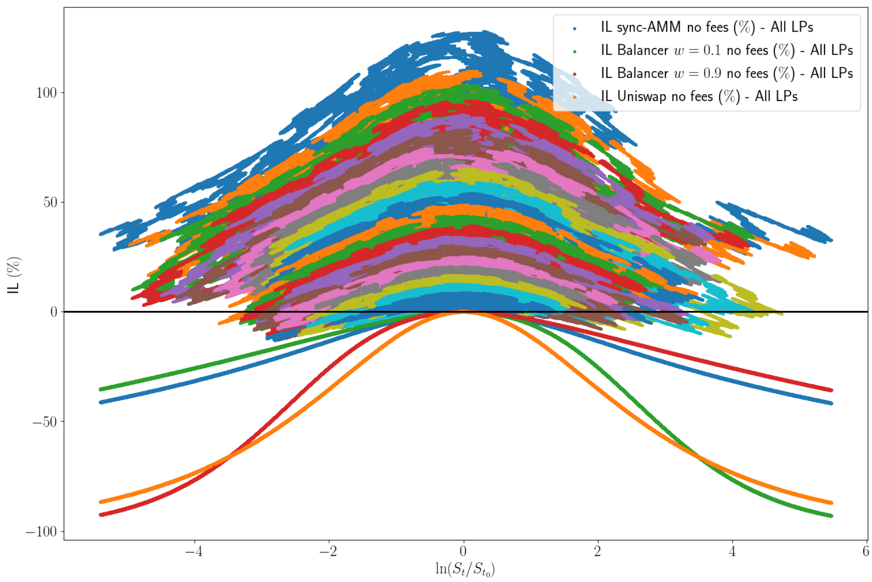


**Figure 4.5:** Sync-AMM $w = 0.1$: Impermanent loss with trading fees for all liquidity providers who entered the AMM. Each color corresponds to a given liquidity provider and path.

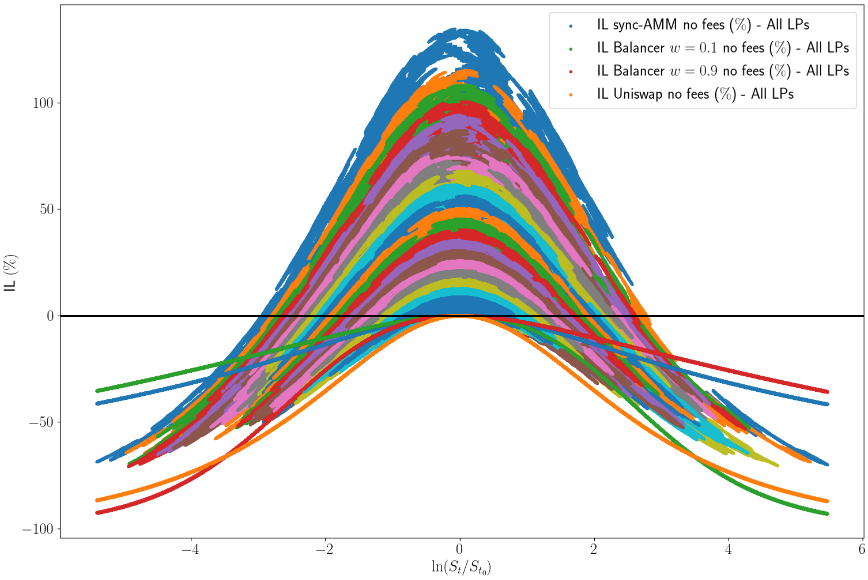


**Figure 4.6:** Uniswap: Impermanent loss with trading fees for all liquidity providers who entered the AMM. Each color corresponds to a given liquidity provider and path

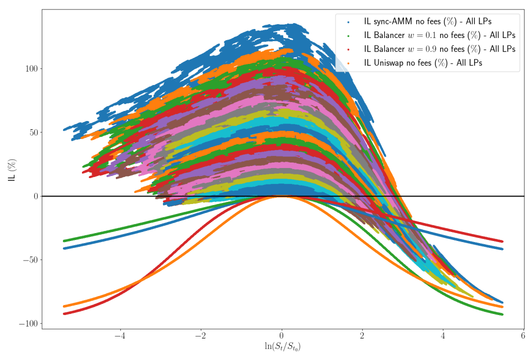

**Figure 4.7:** Balancer $w = 0.1$: Impermanent loss with trading fees for all liquidity providers who entered the AMM. Each color corresponds to a given liquidity provider and path

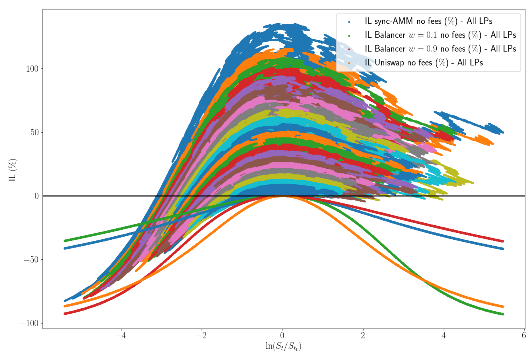

**Figure 4.8:** Balancer $w = 0.9$: Impermanent loss with trading fees for all liquidity providers who entered the AMM. Each color corresponds to a given liquidity provider and path

| | LP returns: IL with trading fees | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Downward trend | | | | | Upward trend | | | |
| **Time** | **3Y** | **2Y** | **1Y** | **6M** | **Start** | **6M** | **1Y** | **2Y** | **3Y** |
| **Spot** (ratio) | 1.35 (/101.5) | 3.79 (/36.1) | 31.14 (/4.4) | 31.05 (/4.4) | 137 ($\times$1) | 1 119.59 ($\times$8.2) | 1 693.07 ($\times$12.4) | 5 210.34 ($\times$38.0) | 14 260.38 ($\times$104.1) |
| IL sync-AMM $w = 0.1$ | 48.24% | 25.04% | 20.81% | 5.21% | 0% | −1.84% | 7.41% | 23.61% | 46.24% |
| IL Uniswap | −53.56% | −42.78% | 3.10% | −10.46% | 0% | −28.26% | −29.72% | −44.34% | −54.24% |
| IL Balancer $w = 0.1$ | 65.18% | 36.91% | 24.42% | 7.79% | 0% | −17.21% | −19.39% | −46.11% | −66.71% |
| IL Balancer $w = 0.9$ | −66.05% | −44.01% | 14.60% | −0.34% | 0% | 2.25% | 13.61% | 35.61% | 63.84% |
| IL Balancer $w = 0.33$ | −24.01% | −20.55% | 9.20% | −5.17% | 0% | −31.67% | −35.48% | −55.88% | −68.92% |
| IL Balancer $w = 0.67$ | −68.38% | −54.31% | 1.71% | −11.53% | 0% | −19.07% | −16.75% | −22.13% | −24.89% |

**Table 1:** Liquidity provider returns *with trading fees* compared to a buy-and-hold strategy, i.e. "impermanent loss with trading fees". The initial spot value is $S_0 = 137$; the right-hand side of the table displays a path that increases significantly whereas the left-hand side displays a path that decreases significantly.

| | LP returns: IL without trading fees | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | Downward trend | | | | | Upward trend | | | |
| **Time** | **3Y** | **2Y** | **1Y** | **6M** | **Start** | **6M** | **1Y** | **2Y** | **3Y** |
| **Spot** (ratio) | 1.35 (/101.5) | 3.79 (/36.1) | 31.14 (/4.4) | 31.05 (/4.4) | 137 ($\times$1) | 1 119.59 ($\times$8.2) | 1 693.07 ($\times$12.4) | 5 210.34 ($\times$38.0) | 14 260.38 ($\times$104.1) |
| IL sync-AMM $w = 0.1$ | −36.02% | −28.15% | −8.28% | −8.26% | 0% | −14.36% | −18.32% | −28.65% | −36.22% |
| IL Uniswap | −80.27% | −67.59% | −22.38% | −22.30% | 0% | −37.74% | −47.12% | −68.50% | −80.55% |
| IL Balancer $w = 0.1$ | −30.02% | −22.60% | −6.57% | −6.54% | 0% | −28.23% | −39.48% | −69.55% | −85.90% |
| IL Balancer $w = 0.9$ | −85.55% | −68.24% | −13.51% | −13.45% | 0% | −11.17% | −14.27% | −23.05% | −30.23% |
| IL Balancer $w = 0.33$ | −67.59% | −54.90% | −17.73% | −17.67% | 0% | −40.68% | −51.44% | −74.98% | −86.75% |
| IL Balancer $w = 0.67$ | −86.49% | −74.03% | −23.23% | −23.15% | 0% | −29.67% | −37.20% | −55.75% | −67.90% |

**Table 2:** Liquidity provider returns *without trading fees* compared to a buy-and-hold strategy, i.e. "impermanent loss without trading fees". The initial spot value is $S_0 = 137$; the right-hand side of the table displays a path that increases significantly whereas the left-hand side displays a path that decreases significantly.

spot deviation direction. Because of the latest remark, the sync-AMM is then consistent both on the rise and fall of the spot which provides a strong improvement of impermanent loss in the two scenarios. As a benchmark, we also provide the impermanent loss without trading fees in Table 1 in which we notice a similar behavior where the sync-AMM provides consistent improvement compared to Uniswap in both up- and down-trend scenarios.

In Figure 4.9, we display the same sample paths as per Table 2 but with the entire time-series of the liquidity providers returns where it is possible to display more precisely the different changes over time of the impermanent loss with the addition of trading fees. The sync-AMM improves the liquidity provider returns, even when large swings of spot prices occur.

### 4.4.2 Slippage

In Figures 4.10 , 4.11 , 4.12 and 4.13 we display the slippage of the sync-AMM DEX which can be compared to the ones of Uniswap as well as Balancer. The impact of trading fees can be observed as the gap at the origin which shifts all curves above or below the $x$-axis by the fee amount. The average slippage becomes symmetrical in the sync-AMM compared to Balancer which displays a highly asymmetric behavior when weights are either small or close to 1. This feature favors trading in both directions as opposed to Balancer in such cases. Additionally, an interesting feature of the sync-AMM is a slippage that depends on the current market

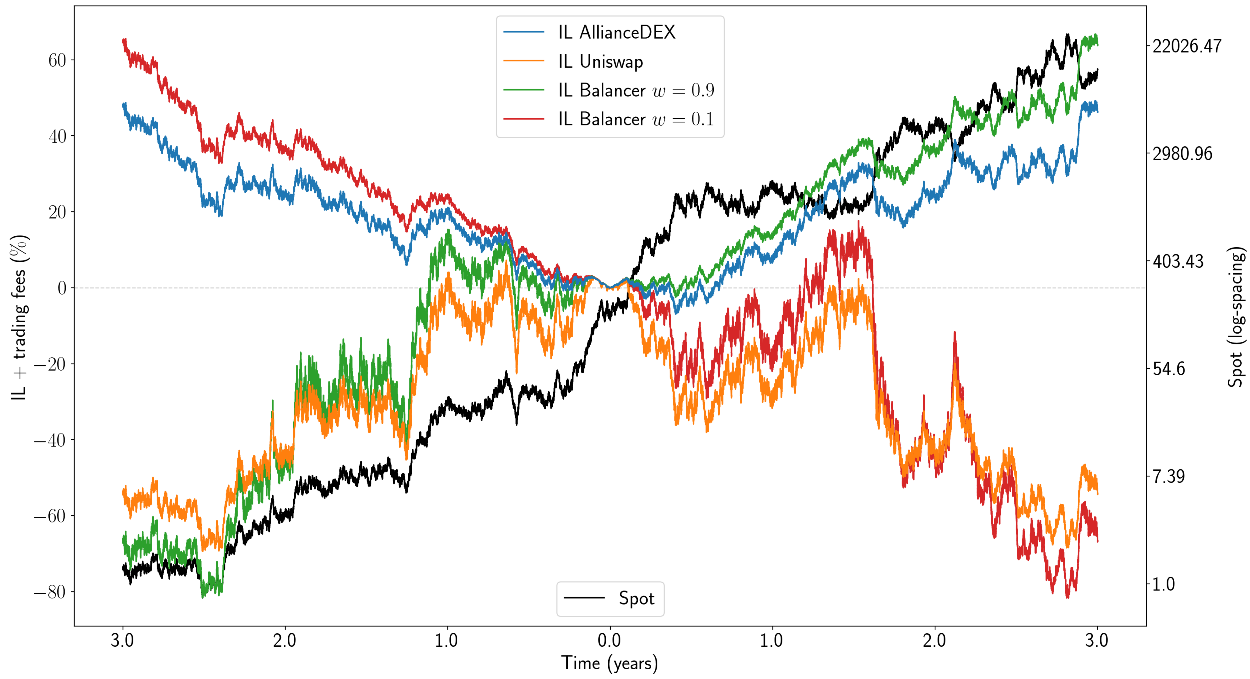


**Figure 4.9:** Liquidity provider returns *with trading fees* compared to a buy-and-hold strategy, i.e. "impermanent loss with trading fees". The initial spot value is $S_0 = 137$; the right-hand side of the graphic displays a path that increases significantly whereas the left-hand side displays a path that decreases significantly.

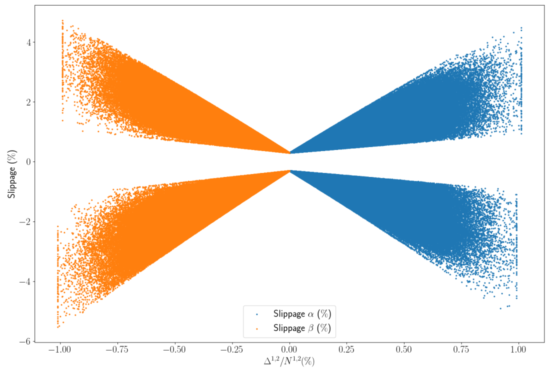


**Figure 4.10:** Slippage of the sync-AMM with weight $w = 0.1$.

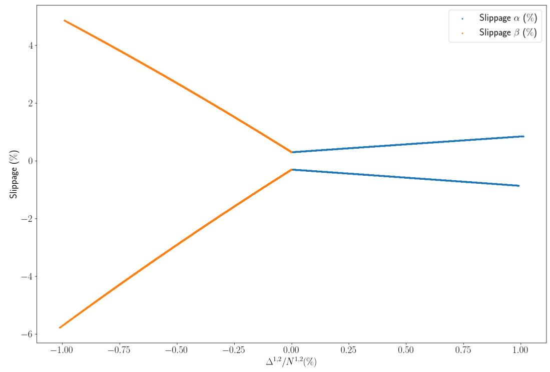


**Figure 4.11:** Slippage of Balancer with weight $w = 0.1$.

and reserve states and not only on the ratio of the trade size with respect to the associated reserve size like geometric mean markets CFMM. This feature displays the slippage as an apparently random value between bounded ranges when plotted with respect to the trade and reserve size ratio. That contrasts with Balancer or Uniswap, where the slippage has a functional dependence on the same ratio. Additionally, if the individual reserves of pools $\mathcal{G}_1$ and $\mathcal{G}_2$ were not visible publicly, slippage would appear stochastic and potentially could help making front-running more challenging. For the same volume of liquidity, Balancer and sync-AMM have higher slippage than Uniswap, however, if the reserve sizes were increased by the appropriate factor compared to the reserves in a Uniswap-type AMM, then the average slippage would be comparable between the DEXs, while preserving the improved impermanent loss as discussed in previous sections.

# 5 Conclusion

With the recent innovations in the DEX industry and decentralized finance more generally, new aspects of risk exposures have emerged for investors providing liquidity to decentralized exchanges. In particular, the impermanent loss, or the risk of loss compared to a buy-and-hold strategy when the spot deviates largely from the LP's entry-point, has been a topic of strong research and development in the field. Up until now, tackling the impermanent loss problem has relied mainly on extra-features which do not modify the intricacies of the AMM itself. In this article, we focused on the underlying mechanisms of the market maker by proposing a new approach combining properties of different CFMMs. Our driving example combines two dual-asset geometric mean markets with symmetric weights and provides an improvement of the impermanent loss in every spot market scenario resulting in an average increase of the returns for liquidity providers. This setup brought us to discuss impermanent loss in geometric mean markets used by well known DEXs, such as Uniswap V2 and Balancer V1 where we provided analytical formulas of the IL for the generic case of $n-$assets with and without trading fees. Our numerical results show that the sync-AMM combined with the usual 0.3% trading fees, lead to positive returns in a large amount of market scenarios, even when the spot deviates drastically from the liquidity providers entry points. Striking examples, are sample paths which lead to an increase or decrease of the asset price by a factor of 150 and where, in that instance and on the paths analyzed, the sync-AMM still provided positive returns, compared to a buy-and-hold portfolio. The symmetrical average slippage makes the proposed DEX suitable for trading in both directions as opposed to Balancer with uneven weights. Finally

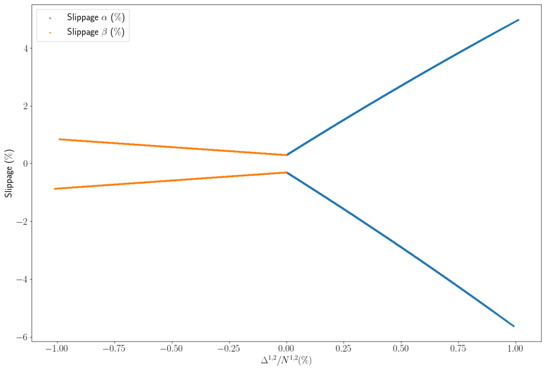


**Figure 4.12:** Slippage of Balancer with weight $w = 0.9$.

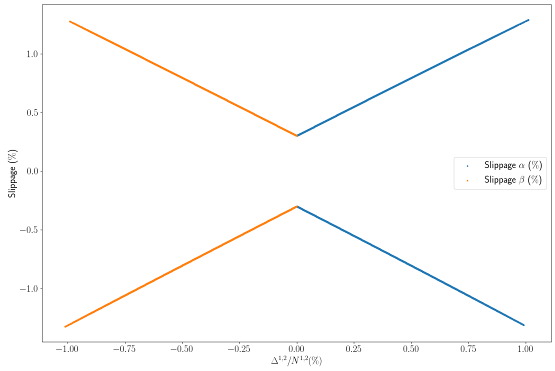


**Figure 4.13:** Slippage of Uniswap.

this new approach to automated market making could potentially open other research topics on additional improvements of the impermanent loss, reduction of slippage costs as well as new features such as single-sided liquidity provisioning.

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