Computational Models – Exercise 1

Due Wednesday, 29 March 2023

Each student must solve the problems on their own. If you encounter difficulties, you may ask a classmate for a hint or the general idea. However, detailed discussion, note-taking, or sharing of written solutions is not allowed. Do not write down your answers while communicating with other people or show the answers for feedback.

Our grading app has severe limitations, such as no zoom tool. To make sure we can grade your work, please follow these technical guidelines:

Submit a **single PDF file** through Moodle.

The file size is limited to **10 MB**. If necessary, google *reduce PDF file size*.

Fill in your answers **on this form*** in the allocated spaces. The space provided gives you an indication of the expected length and level of detail of the answer. You may add a little more space if you need.

Include everything from this form in your submission. In particular, **include the problem statements**. Do not delete any text or omit pages, just add your answers.

Ensure your answers are **legible** (easy to read) at zoom 100% on a standard computer screen. Your text should be **large**, **sharp**, and in **high contrast** with the background.

Do not squeeze scanned solutions to fit in the space, as the text will become small.

Verify that pages are properly **ordered** and **oriented**.

The page size must be **A4**. Before submitting your file, check its page size using Acrobat Reader: go to File > Properties > Description and confirm that Page Size is around 21 × 29 cm. Note that scanning A4 pages does not guarantee the resulting page size will be A4, due to scaling. If necessary, google *resize PDF to A4*.

Do not add your answers as PDF comments. If you can drag them in Acrobat Reader, they are comments. If necessary, google *flatten PDF*.

A **5-point bonus** will be given to solutions typed in a word processor. Hand-sketched illustrations or diagrams will not deny you this bonus.

If there are technical issues with your submission, you may receive a fine. In extreme cases, your submission may not be graded at all.

If you need help or have questions, please use the course forum at Piazza.

*The only exception is in case you use LaTeX or a similar typesetting system. In that case, copy-paste everything from this file, except for illustrations or other hard-to-reproduce graphical elements. No need to fix corrupted formulas.

Worked with Jemma Diamond - 806839

Question 1 (3 pts)

Read the instructions on page 1 carefully.

1. Have you read the instructions on page 1 carefully? ___yes____

2. What should be the page dimensions (**in centimeters**) of your submitted file?
________21 x 29 cm_________

3. What software can be used to verify the page size? __Acrobat Reader___

Question 2 (21 pts)

Write the elements of the following sets:

a) $2^{\{1,2\}} \times (\{a, b\} \cup 2^{\phi})$
{(ϕ, ϕ) , (ϕ, a) , (ϕ, b) , ({1} , ϕ) , ({1} , a) , ({1} , b) , ({2} , ϕ) , ({2} , a) , ({2} , b) , ({1,2} , ϕ) , ({1,2} , a) , ({1,2} , b) }

b) $\{a, b, c, d\} - \{\{a, b, c, d\}\}$
{a,b,c,d}

c) $2^{\{a,b,c\}} - 2^{\{c,b\}}$
{{a} , {a,b} , {a,c} , {a,b,c}}

d) $2^{2^{\phi}}$
{ ϕ, {ϕ}}

e) $\{a, b, \phi\}^2$

{(a,a) , (a,b) , (a, ϕ) , (b,a) , (b,b) , (b, ϕ) , (ϕ,a) , (ϕ,b) , (ϕ, ϕ)}

f) $\{a, b\} \times \phi$

ϕ

g) $\{a, b\} \times \{\phi\}$

{(a, ϕ) , (b, ϕ)}

Question 3 (20 pts)

Prove (by a formal and complete proof) or disprove (showing a counter example) the following claims:

1. If $L_1 \subseteq L_2$ $then$ $L_1^* \subseteq L_2^*$

True.

Assume $L_1 \subseteq L_2$. Let w ∈ L_1^*. If w = ε, then w ∈ L_2^* by definition. If w ≠ ε, then by definition, there exists $w_1, w_2, w_3, \dots, w_n \in L_1$ s.t w = $w_1 w_2 w_3 \dots w_n$. Since $w_1, w_2, w_3, \dots, w_n \in L_1$, then that implies that $w_1, w_2, w_3, \dots, w_n \in L_2$, as $L_1 \subseteq L_2$. Therefore w = $w_1 w_2 w_3 \dots w_n \in L_2^*$ by definition. Therefore $L_1^* \subseteq L_2^*$

2. $(L_1 \circ L_2)^+ \subseteq (L_1^+ \circ L_2^+)$

False. Counterexample:

Let $L_1 = \{a\}$ and $L_2 = \{b\}$.

Therefore $(L_1 \circ L_2)^+ = (\{a\} \circ \{b\})^+ = (\{ab\})^+ =$ {ab, abab, ababab, ...}

And $(L_1^+ \circ L_2^+) = (\{a\}^+ \circ \{b\}^+) = (\{a, aa, aaa, \dots\} \circ \{b, bb, bbb, \dots\})$

= {ab, abb, abbb, ... , aab, aabb, aabbb, ... , aaab, aaabb, ...} (i.e all possible words where 0 or more a's are concatenated with 0 or more b's)

Therefore abab ∈ $(L_1 \circ L_2)^+$, but abab ∉ $(L_1^+ \circ L_2^+)$.

Therefore $(L_1 \circ L_2)^+ \nsubseteq (L_1^+ \circ L_2^+)$.

3. $(L_1 \circ L_2)^+ = (L_1^+ \circ L_2^+)$

False. Counterexample:

Let $L_1 = \{a\}$ and $L_2 = \{b\}$.

Therefore $(L_1 \circ L_2)^+ = (\{a\} \circ \{b\})^+ = (\{ab\})^+ =$ {ab, abab, ababab, ...}

And $(L_1^+ \circ L_2^+) = (\{a\}^+ \circ \{b\}^+) = (\{a, aa, aaa, \dots\} \circ \{b, bb, bbb, \dots\})$

= {ab, abb, abbb, ... , aab, aabb, aabbb, ... , aaab, aaabb, ...} (i.e all possible words where 0 or more a's are concatenated with 0 or more b's)

Therefore abab ∈ $(L_1 \circ L_2)^+$, but abab ∉ $(L_1^+ \circ L_2^+)$.

Therefore $(L_1 \circ L_2)^+ \nsubseteq (L_1^+ \circ L_2^+)$.

4. $(L_1 \circ L_2)^R = {L_2}^R \circ {L_1}^R$

True.

Let $w \in (L_1 \circ L_2)^R$. Therefore $w^R \in (L_1 \circ L_2)$ and $w^R = w_1 w_2$ where $w_1 \in L_1$ and $w_2 \in L_2$. Since $w^R = w_1 w_2$, that implies that w = $w_2^R w_1^R$, and we know that $w_2^R \in {L_2}^R$ and $w_1^R \in {L_1}^R$. Therefore w ∈ ${L_2}^R \circ {L_1}^R$.

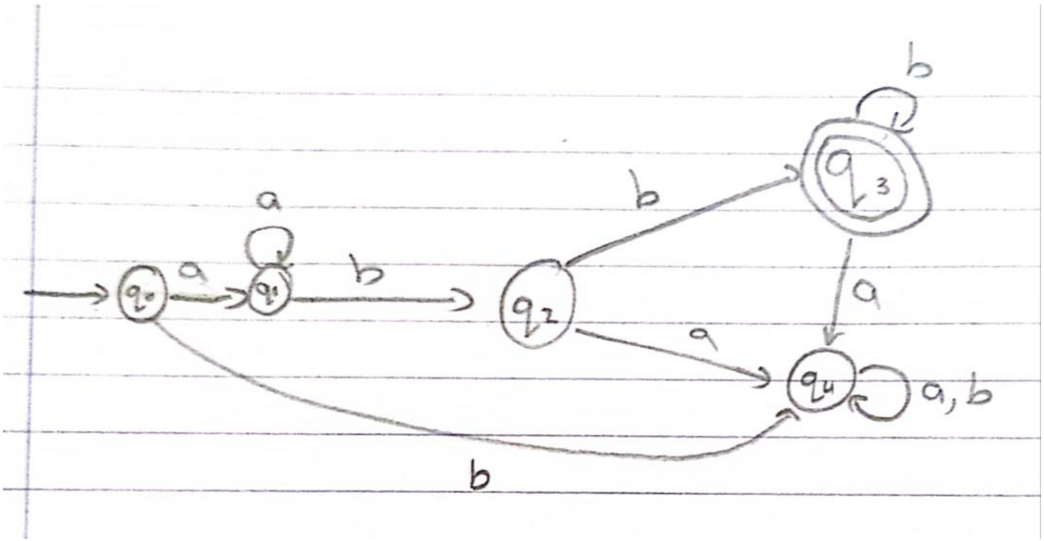
Question 4 (36 pts)

For each one of the following languages define a DFA recognizing it. It might be possible that a language cannot be recognized by a DFA. In such a case write "impossible". In all languages, if not defined otherwise, are defined over alphabet is $\Sigma = \{a, b\}$. You just need to draw the state diagram of your DFA.

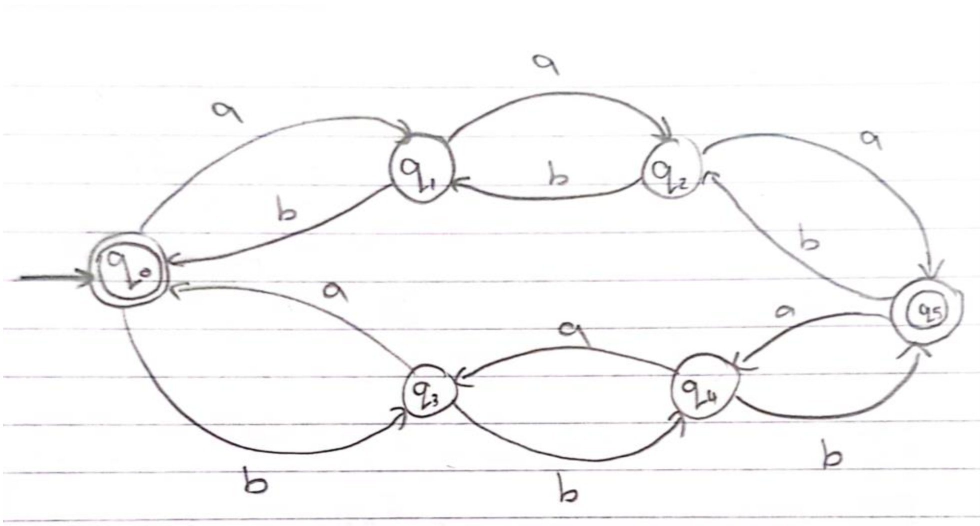
a. $L = \{w \in \Sigma^* \mid \#_a(w) > 2 \ \ and \ \ \#_b(w) = 2\}$

Impossible.

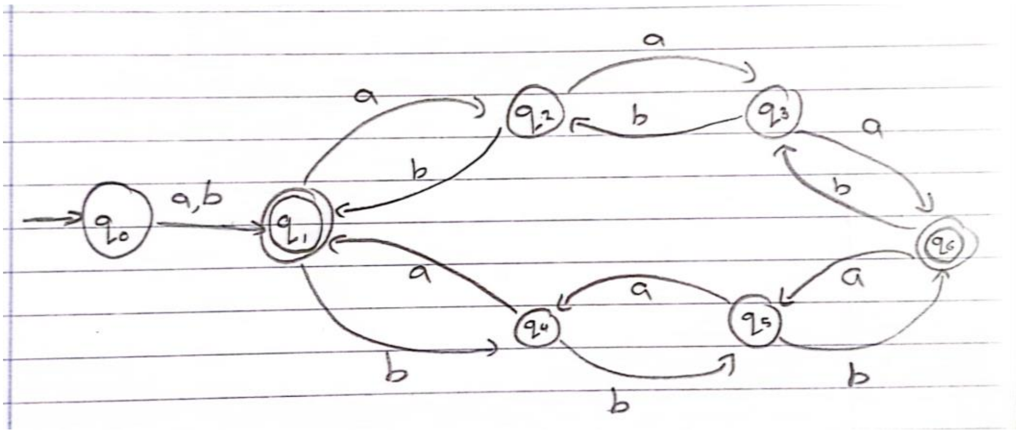
b. $L = \{w \in \Sigma^* \mid \ w \ has \ \boldsymbol{abb} \ but \ does \ not \ have \ \boldsymbol{ba} \ as \ substring\}$



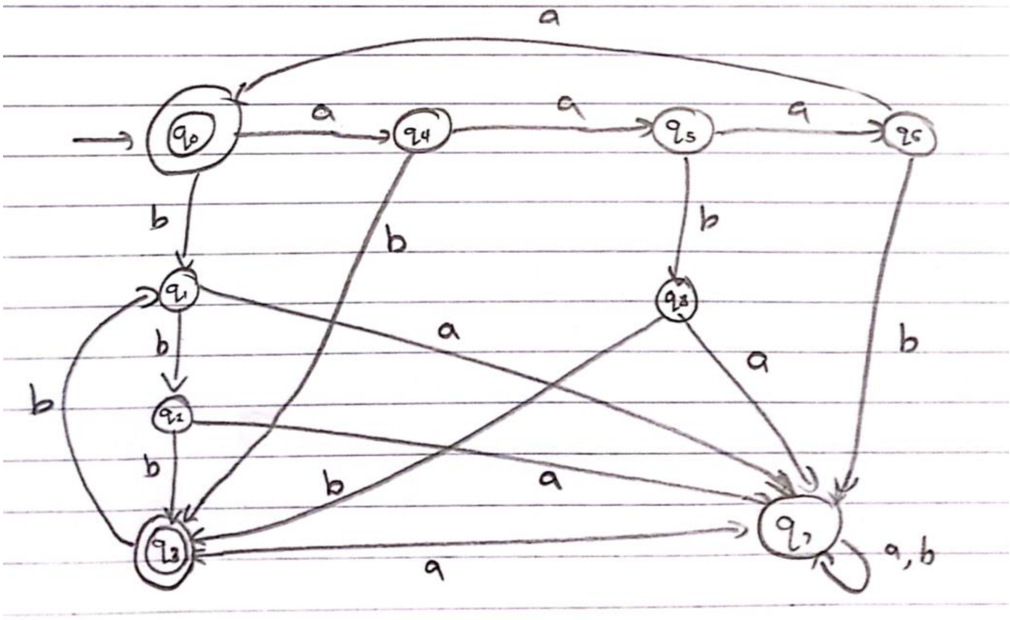
c. $L = \{w \in \Sigma^* \mid \ |\#_a(w) - \#_b(w)| \ mod \ 3 = 0\}$



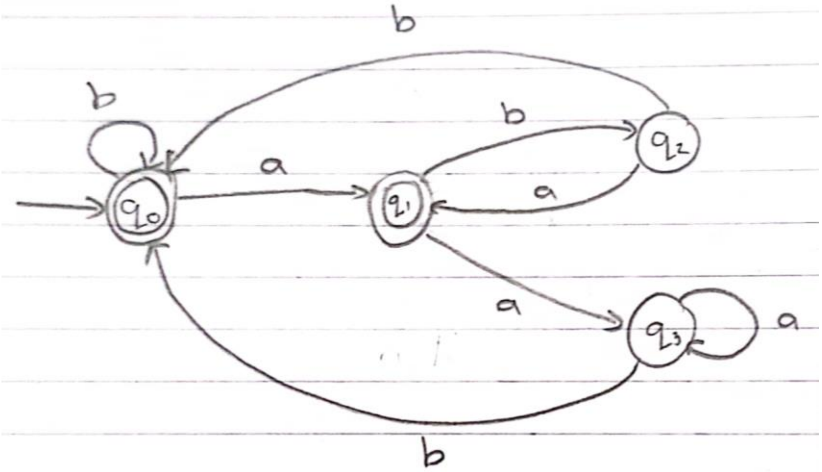
d. $L = \{w \in \Sigma^* \mid \ |\#_a(w) - \#_b(w)|\ mod\ 3 = 1\}$



e. $L = \{a^i b^j \ \mid \ i\ mod\ 4 = j\ mod\ 3\ \}$



f. $L = \Sigma^* - \{w \in \Sigma^* \mid \ \ w = ua\sigma,\ u \in \Sigma^*, \sigma \in \Sigma\}$

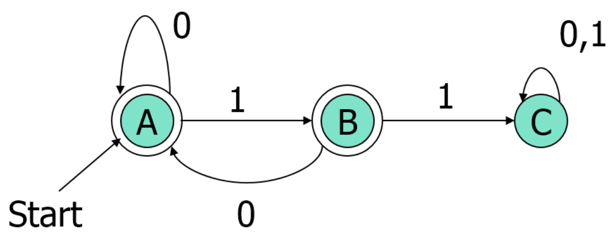


Question 5 (20 pts)

Define a language L: $L = \Sigma^* - \Sigma^*\{11\}\,\Sigma^*$

L is a language of all the words over {0,1} that do not contain 11 as substring.

Given the automaton M:



Prove that $\boldsymbol{L(M) = L}$

The proof is conducted by a bi-directional inclusion:

1st direction: $L(M) \subseteq L$ i.e. $\forall\, w,\ if\ \ w \in L(M)\ then\ \ w \in L$

2nd direction: $L \subseteq L(M)$ i.e. $\forall\, w,\ if\ \ w \in L\ \ then\ \ w \in L(M)$

Prove only the 1st direction. Prove by induction where the claims are:

1. If δ'(A, w) = A, then w has no consecutive 1's and does not end with 1.
2. If δ('A, w) = B, then w has no consecutive 1's and ends with a single 1.

Base Case: for w = ε.

Clearly δ'(A, ε) = A, as when the input is the empty word, the automaton's end state is the starting state. Since w is the empty word, it is trivial that there is no consecutive 1's and that it does not end with a 1. Therefore claim (1) holds (claim (2) is not possible with the empty word as the input, and therefore it holds vacuously). Therefore claim (1) and (2) hold.

Induction Hypothesis: Assume the claim (1) and (2) holds for word u of length n.

Induction step:

Let w = uσ where |u| = n and σ ∈ Σ.

Therefore δ'(A, w) = δ'(A, uσ) = δ (δ'(A, u) , σ) .

1. If δ'(A, u) = A, then by the induction hypothesis, u has no consecutive 1's and it doesn't end with a 1. Since u does not end with a 1, then for any σ ∈ {1,0}, w = uσ will not have any consecutive 1's. If σ = '1' , then δ'(A, w) = δ (δ'(A, u) , σ) = δ (A , 1) = B. In this case w ends with a 1, and as proven before w does not have any consecutive 1's. Therefore if δ'(A, w) = B, then w has no consecutive 1's and ends with a single 1. If σ = '0' , then δ'(A, w) = δ (δ'(A, u) , σ) = δ (A , 0) = A. In this case w ends with a 0, and as proven before w does not have any consecutive 1's. Therefore if δ'(A, w) = A, then w has no consecutive 1's and ends with a single 1.

2. If δ'(A, u) = B, then by the induction hypothesis, u has no consecutive 1's and it ends with a 1. If σ = '0' , then δ'(A, w) = δ (δ'(A, u) , σ) = δ (B , 0) = A. In this case w does not end with a 1, and by the induction hypothesis u does not have any consecutive 1's. Therefore w = uσ has no consecutive 1's and it does not end with a 1. Therefore if δ'(A, w) = A, then w has no consecutive 1's and doesn't end with a 1. If σ = '1' , then δ'(A, w) = δ (δ'(A, u) , σ) = δ (B , 1) = C. Therefore when δ'(A, u) = B, it is not possible to have δ'(A, w) = B, and therefore claim (2) holds vacuously.

Proof of 1st direction:

Let w ∈ $L(M)$. That means that δ'(A, w) = A or δ'(A, w) = B, by the construction of M. If δ'(A, w) = A, then by claim (1), then w has no consecutive 1's and does not end with 1. Therefore w ∈ L as it has no consecutives 1's and therefore it has no '11' substring. If δ'(A, w) = B, then by claim (2), then w has no consecutive 1's and it ends with 1. Therefore w ∈ L as it has no consecutives 1's and therefore it has no '11' substring.