#1

Лабораторная работа №5

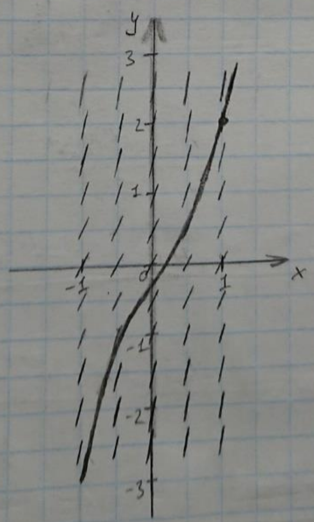
№1

$$y' = 2 + y^2$$

$$2 + y^2 = k$$

$$y = \pm\sqrt{k-2}$$

$k = tg(\alpha)$	α°	Ур-е изоклин
2	~63,4	$y = 0$
2,25	~66	$y = \pm 0,5$
3	~71,5	$y = \pm 1$
4,25	~76,8	$y = \pm 1,5$
6	~80,5	$y = \pm 2$
8,25	~83	$y = \pm 2,5$



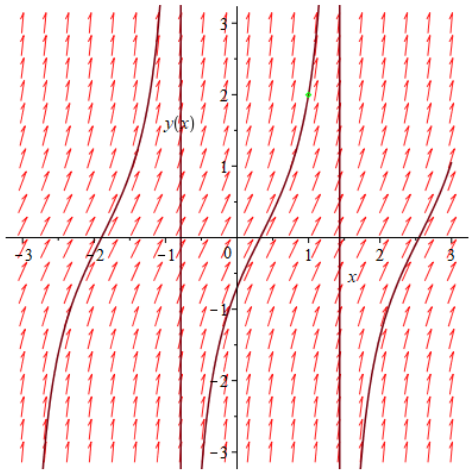
```
with(DEtools) :
expr := diff(y(x), x) = 2 + (y(x))^2;
```

$$expr := \frac{d}{dx}\, y(x) = 2 + y(x)^2$$

```
solution := rhs(combine(dsolve({expr, y(1) = 2}), trig));
```

$$solution := \sqrt{2}\,\tan\left(\arctan\left(\sqrt{2}\right) + \sqrt{2}\,x - \sqrt{2}\right)$$

```
xr := -3 ..3 :
yr := -3 ..3 :
p1 := dfieldplot(expr, y(x), x = xr, y = yr) :
p2 := plot(solution, x = xr, y = yr) :
p3 := plot([[1, 2]], style = point, color = green) :
plots[display](p1, p2, p3);
```



#2

1)

№2

$M_0(12, 2)$, $a = 20$

Ур-е нормали в т. $M(x_0, y(x_0))$

$$g(z) = y(x_0) - \frac{1}{y'(x_0)}(z - x_0)$$

$$g(0) = y(x_0) + \frac{x_0}{y'(x_0)}$$

$g(z)$ пересек O_y в т. $N\left(0, y(x_0) + \frac{x_0}{y'(x_0)}\right)$

$$|\overrightarrow{MN}| = \sqrt{(x-0)^2 + \left(y(x_0) + \frac{x_0}{y'(x_0)} - y(x_0)\right)^2} =$$

$$= \sqrt{x_0^2 + \left(\frac{x_0}{y'(x_0)}\right)^2} = 20 = a.$$

$$\frac{1}{(y'(x_0))^2} = \frac{400 - x_0^2}{x_0^2}$$

$$y'(x) = \frac{|x|}{\sqrt{400 - x^2}}$$

$$y(x_0) = \pm\int \frac{x\,dx}{\sqrt{400 - x^2}} = \pm\frac{1}{2}\int \frac{d(400 - x^2)}{\sqrt{400 - x^2}} =$$

$$= \pm\sqrt{400 - x_0^2} + C$$

$x > 0$:

$2 = 16 + C$, $C = -14$

$x < 0$:

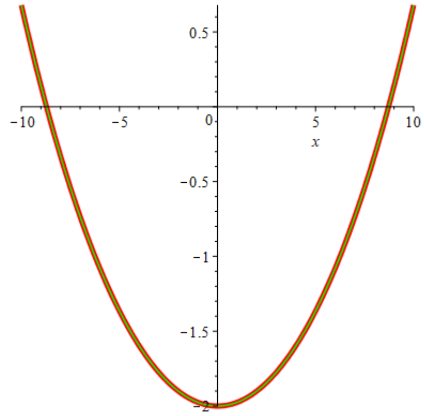
$C = 18$

$$y(x_0) = \pm\sqrt{400 - x^2} \mp 16 - 2$$

```
with(DEtools) :
a := 20;
ans := dsolve({sqrt((x^2 + (x/diff(y(x), x))^2)) = a, y(12) = 2});

p1 := plot(rhs(ans), color = red, thickness = 5) :
p2 := plot(-sqrt(400 - x^2) + 18, color = green) :
plots[display](p1, p2);
```

$$a := 20$$

$$ans := y(x) = (x-20)(x+20)\,RootOf\left((x^2-400)\_Z^2+1\right) + 2 + 256\,RootOf\left(256\_Z^2-1\right)$$



#2)

2) $M_0(-1, \sqrt{e})$, $a = -1$

Касательная в т. $M(x_0, y(x_0))$

$$f(z) = y(x_0) + y'(x_0)(z - x_0)$$

$$0 = y(x_0) + y'(x_0)(z - x_0)$$

$$-\frac{y(x_0)}{y'(x_0)} = z - x_0$$

$$z = x_0 - \frac{y(x_0)}{y'(x_0)}$$

Кас. пересек Ox в т. $N\left(x_0 - \frac{y(x_0)}{y'(x_0)}, 0\right)$

$$|\vec{MN}| = \sqrt{\left(x_0 - x_0 + \frac{y(x_0)}{y'(x_0)}\right)^2 + (y(x_0) - 0)^2} = \sqrt{\left(\frac{y(x_0)}{y'(x_0)}\right)^2 + (y(x_0))^2}$$

Проекция $\vec{MN}$ на Ox:

$$\sqrt{\left(\frac{y(x_0)}{y'(x_0)}\right)^2 + \left(y^2(x_0) - y^2(x_0)\right)} = \left|\frac{y(x_0)}{y'(x_0)}\right|$$

$$\left|x_0 - \frac{y(x_0)}{y'(x_0)} - x_0\right| = \left|\frac{y(x_0)}{y'(x_0)}\right|$$

$$x_0\left|\frac{y(x_0)}{y'(x_0)}\right| = a = -1$$

$$\pm x_0\frac{y(x_0)}{y'(x_0)} = 1$$

$$\pm\frac{y(x_0)}{y'(x_0)} = \frac{1}{x}$$

$$\frac{y'(x_0)}{y(x_0)} = \pm x_0 \Rightarrow \frac{dy}{y} = \pm x\,dx$$

$$\ln|y| = \pm\frac{x^2}{2}$$

$$y = Ce^{\pm\frac{x^2}{2}}$$

$$C = e^{\frac{1}{2}\mp\frac{1}{2}}$$

$$y = e^{\pm\frac{x^2}{2}+\frac{1}{2}\mp\frac{1}{2}}$$

$with(DEtools):$

$dsolve\left(\left\{diff(y(x),x) = x\cdot y(x),\ y(-1) = \exp\left(\frac{1}{2}\right)\right\}\right);$

$$y(x) = e^{\frac{x^2}{2}}$$

$xin := -5..5:$

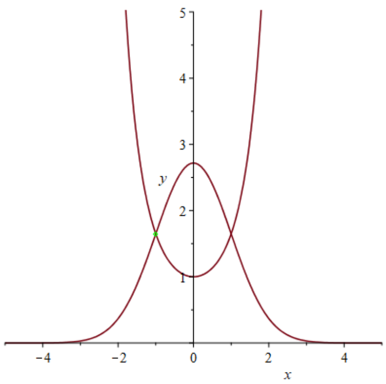
$yin := 0..5:$

$p1 := plot\left(\exp\left(\frac{x^2}{2}\right), x = xin, y = yin\right):$

$p2 := plot\left(\exp\left(-\frac{x^2}{2}+1\right), x = xin, y = yin\right):$

$p3 := plot\left(\left[\left[-1, \exp\left(\frac{1}{2}\right)\right]\right], style = point, color = green\right):$

$plots[display](p1, p2, p3);$



#3

№3

$$y' = \frac{-10x+26y-16}{37x+y-38}$$

$$\begin{vmatrix}-10 & 26\\ 37 & 1\end{vmatrix} = -10-37\cdot 26 \neq 0 \Rightarrow \text{ед. реш.}$$

$$\begin{cases}-10\alpha+26\beta=16 & |\cdot 3{,}7\\ 37\alpha+\beta=38\end{cases}$$

$$97{,}2\beta = 97{,}2$$

$$\begin{cases}\beta=1\\ \alpha=1\end{cases} \Rightarrow (1;1) \text{ — особая т.}$$

with(DEtools) :

$expr := diff(y(x), x) = \dfrac{-10\cdot x + 26\cdot y(x) - 16}{37\cdot x + y(x) - 38};$

$$expr := \frac{d}{dx}\, y(x) = \frac{-10\,x + 26\,y(x) - 16}{37\,x + y(x) - 38}$$

$solve(\{-10\cdot\alpha + 26\cdot\beta - 16 = 0, 37\cdot\alpha + \beta - 38 = 0\});$

$$\{\alpha = 1, \beta = 1\}$$

$$\begin{cases} x = u+1 \\ y = \vartheta+1 \end{cases} \qquad \begin{cases} dx = du \\ dy = d\vartheta \end{cases}$$

$$\frac{d\vartheta}{du} = \frac{-10u - 10 + 26\vartheta + 26 - 16}{37u + 37 + \vartheta + 1 - 38} = \frac{-10u + 26\vartheta}{37u + \vartheta} \quad \Big| : u \neq 0$$

$$\frac{d\vartheta}{du} = \frac{-10 + 26\frac{\vartheta}{u}}{37 + \frac{\vartheta}{u}}$$

$$z = \frac{\vartheta}{u}$$

$$\vartheta = zu, \quad \vartheta' = z'u + z$$

$$uz' = \frac{-10 + 26z - z(37+z)}{37+z}$$

$$uz' = \frac{-10 + 26z - 37z - z^2}{37+z} = \frac{-10 - 11z - z^2}{37+z}$$

$$uz' = \frac{-(z+1)(z+10)}{37+z}$$

$$\frac{z+37}{(z+1)(z+10)}\,dz = -\frac{du}{u}$$

$$-\ln|Cu| = \int\frac{4}{z+1}\,dz - \int\frac{3}{z+10}\,dz$$

$$-\ln|Cu| = 4\ln|z+1| - 3\ln|z+10|$$

$$\ln\left|\frac{1}{Cu}\right| = \ln\left|\frac{(z+1)^4}{(z+10)^3}\right|$$

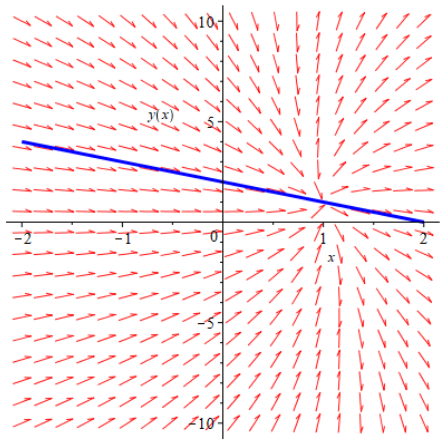
$$\frac{1}{Cu} = \frac{\left(\frac{\vartheta}{u}+1\right)^4}{\left(\frac{\vartheta}{u}+10\right)^3}, \quad \frac{1}{C(x-1)} = \frac{(y+x-2)^4}{(x-1)(y+10x-11)^3}$$

$(y+10x-11)^3 = C(y+x-2)^4$ – общее решение

dsolve(expr);

$$-4\ln\left(-\frac{y(x) - 2 + x}{x-1}\right) + 3\ln\left(-\frac{y(x) - 11 + 10\,x}{x-1}\right) - \ln(x-1) - \_C1 = 0$$

```
xin := -2 ..2 :
yin := -10 ..10 :
DEplot(expr, y(x), x = xin, y = yin, [y(0) = 2], linecolor = blue);
```



$$\begin{vmatrix} 37-\lambda & 1 \\ -10 & 26-\lambda \end{vmatrix} = 0$$

$$(\lambda-37)(\lambda-26)+10 = \lambda^2-63\lambda+972=0$$

$$D = 63^2-4\cdot 972 = 81$$

$$\lambda_1 = 36$$

$$\lambda_2 = 27$$

$\lambda_1 \neq \lambda_2$, $\lambda_1>0$, $\lambda_2>0$ $\Rightarrow$ неустойчивый узел.

```
with(Student[LinearAlgebra]) :
A := ⟨⟨37, 1⟩|⟨−10, 26⟩⟩ :
Eigenvectors(A);
```

$$\begin{bmatrix} 27 \\ 36 \end{bmatrix}, \begin{bmatrix} 1 & 10 \\ 1 & 1 \end{bmatrix}$$

#4

$2(xy'+y)=xy^2$, $y(1)=2$

$$y' = \frac{y^2}{2} - \frac{y}{x}$$

$y' + \frac{y}{x} = \frac{y^2}{2}$ — ур-е Бернулли

$$\frac{y'}{y^2} + \frac{1}{xy} = \frac{1}{2}$$

$z = \frac{1}{y}$, $z' = -\frac{y'}{y^2} \Rightarrow \frac{y'}{y^2} = -z'$

$$-z' + \frac{z}{x} = \frac{1}{2}$$

$z' - \frac{z}{x} = -\frac{1}{2}$ — ЛДУ

$$z = u\vartheta$$

$$u'\vartheta + u\left(\vartheta' - \frac{\vartheta}{x}\right) = -\frac{1}{2}$$

$$\begin{cases} \frac{\vartheta'}{\vartheta} = \frac{1}{x} \\ u'\vartheta = -\frac{1}{2} \end{cases} \qquad \begin{cases} \frac{d\vartheta}{\vartheta} = \frac{dx}{x} \\ du = -\frac{1}{2x}\,dx \end{cases}$$

$$u = -\frac{1}{2}\ln|Cx|$$

$$z = -\frac{x}{2}\ln|Cx|$$

$y = \frac{1}{z} = -\frac{2}{x\ln|Cx|}$ — общее решение

$$2 = -\frac{2}{\ln|C|}$$

$$\ln|C| = -1$$

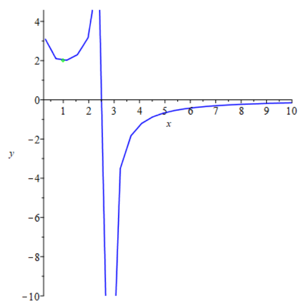
$C = \frac{1}{e} \Rightarrow y = -\frac{2}{x(\ln|x|-1)}$ — решение задачи Коши

```
expr := 2(x·y'+y) = x·y^2;

dsolve(expr);
sol := dsolve({expr, y(1) = 2});

xin := −10 ..10 :
yin := −10 ..10 :
p1 := plots[implicitplot](sol, x=xin, y=yin, color=blue) :
p2 := plots[pointplot]([[1, 2]], color=green) :
plots[display](p1, p2);
```

$$expr := 2x\left(\frac{d}{dx}y(x)\right) + 2y(x) = xy(x)^2$$

$$y(x) = -\frac{2}{(\ln(x) - 2\_C1)\,x}$$

$$sol := y(x) = -\frac{2}{(\ln(x)-1)\,x}$$



#5, 1)

№ 5

1) $x = y' \arccos(y') + \sqrt{1-y'^2}$

$$\begin{cases} y' = t \\ x = t\arccos(t) + \sqrt{1-t^2} \end{cases}$$

$$dx = \arccos(t)\,dt - \frac{t\,dt}{\sqrt{1-t^2}} - \frac{t\,dt}{\sqrt{1-t^2}} =$$

$$= \arccos(t)\,dt - \frac{2t\,dt}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = t, \quad dx = \frac{dy}{t}$$

$$dy = \left(t\arccos(t) - \frac{2t^2}{\sqrt{1-t^2}}\right)dt$$

$$\int t\arccos(t)\,dt = t^2\arccos(t) - \int t\left(\arccos(t) - \frac{t}{\sqrt{1-t^2}}\right)dt = t^2\arccos(t) - \int t\arccos(t) + \int\frac{t^2}{\sqrt{1-t^2}}\,dt$$

$$\int t\arccos(t)\,dt = \frac{1}{2}t^2\arccos(t) + \frac{1}{2}\int\frac{t^2}{\sqrt{1-t^2}}\,dt$$

$$\int\frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\int\frac{t}{\sqrt{1-t^2}}\,dt^2 = -\frac{1}{2}\int\frac{t}{\sqrt{1-t^2}}\,d(1-t^2) =$$

$$= -\int t\,d\sqrt{1-t^2} = -t\sqrt{1-t^2} + \int\sqrt{1-t^2}\,dt =$$

$$= -t\sqrt{1-t^2} + \int\frac{1}{\sqrt{1-t^2}}\,dt - \int\frac{t^2}{\sqrt{1-t^2}}\,dt$$

$$\int\frac{t^2}{\sqrt{1-t^2}}\,dt = \frac{1}{2}\left(\arcsin(t) - t\sqrt{1-t^2}\right)$$

$$\begin{cases} y = \frac{1}{2}t^2\arccos(t) + \frac{3}{4}\left(t\sqrt{1-t^2} - \arcsin(t)\right) + C \\ x = t\arccos(t) + \sqrt{1-t^2} \end{cases}$$

– общее реш. ДУ в парам. виде

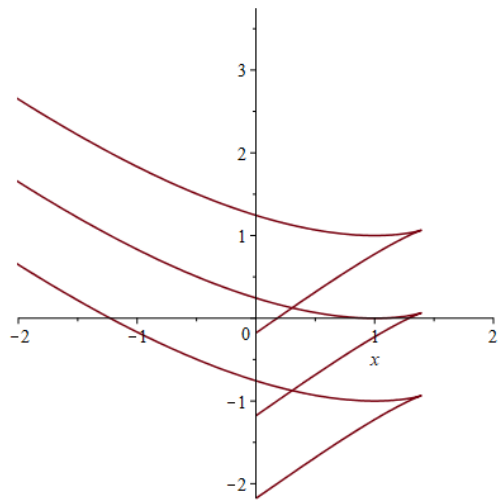
```
with(DEtools) :
xparam := t·arccos(t) + sqrt(1 − t^2) :
expr := xparam;
diffed := t·(diff(expr, t) + diff(expr, x));
res := int(diffed, t);
```

$$expr := t\arccos(t) + \sqrt{-t^2+1}$$

$$diffed := t\left(\arccos(t) - \frac{2t}{\sqrt{-t^2+1}}\right)$$

$$res := \frac{t^2\arccos(t)}{2} + \frac{3t\sqrt{-t^2+1}}{4} - \frac{3\arcsin(t)}{4}$$

```
xin := −2 ..2 :
tin := −1 ..1 :
a = array(1 ..3) :
for i from −1 by 1 to 1 do
a[i + 2] := plot([xparam, res + i, t = tin], x = xin) :
end do:
plots[display](a[1], a[2], a[3]);
```



2)

$$2)\quad y = (y'^2+2)\,sh(y') - 2y'\,ch(y')$$

$$\begin{cases} y' = t \\ y = (t^2+2)\,sh(t) - 2t\,ch(t) \end{cases}$$

$$dy = (2t\cdot sh(t) + (t^2+2)\cdot ch(t) - 2\cdot ch(t) - 2\cdot t\cdot sh(t))\,dt$$

$$dy = t\cdot dx$$

$$dx = 2\,sh(t)\,dt + \frac{(t^2+2)\,ch(t)}{t}\,dt - \frac{2\,ch(t)}{t}\,dt - 2\,sh(t)\,dt = t\,ch(t)\,dt + \frac{2\,ch(t)\,dt}{t} - \frac{2\,ch(t)\,dt}{t} = t\,ch(t)\,dt$$

$$x = \int t\,ch(t)\,dt = \int t\,d\,sh(t) = t\,sh(t) - \int sh(t)\,dt = t\cdot sh(t) - ch(t) + C$$

$$\begin{cases} x = t\cdot sh(t) - ch(t) + C \\ y = (t^2+2)\,sh(t) - 2t\,ch(t) \end{cases}$$

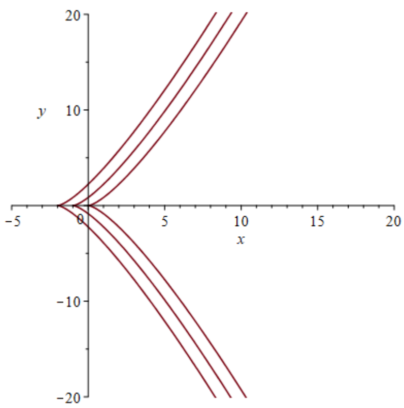
— общее решение в парам. виде.

```
expr := y(x) = (t^2 + 2)·sinh(t) − 2·t·cosh(t);
yparam := rhs(expr) :

diffed := diff(yparam, t) / t;

res := int(diffed, t);

tin := −π .. π :
xin := −5 ..20 :
yin := −20 ..20 :
a = array(1 ..3) :
for i from −1 by 1 to 1 do
 a[i + 2] := plot([res + i, yparam, t = tin], x = xin, y = yin) :
 end do:
plots[display](a[1], a[2], a[3]);
```

$$expr := y(x) = (t^2+2)\sinh(t) - 2t\cosh(t)$$

$$diffed := \frac{(t^2+2)\cosh(t) - 2\cosh(t)}{t}$$

$$res := t\sinh(t) - \cosh(t)$$



#6

№6

$$y = xy' + \underbrace{y'^2 + 1}_{g(y')} \quad \text{— ур. с Клеро}$$

$$\begin{cases} y' = t \\ y = xt + t^2 + 1 \end{cases}$$

$$dy = t\,dx + x\,dt + 2t\,dt$$

$$dy = t\,dx$$

$$t\,dx = t\,dx + (x + 2t)\,dt$$

$$(x + 2t)\,dt = 0$$

$$\begin{cases} x + 2t = 0 \\ dt = 0 \end{cases} \qquad \begin{cases} x = -2t \\ t = c \end{cases}$$

$x = -2t$ — особое решение

$y = Cx + C^2 + 1$ — общее решение.

$$x = -2y'$$

$$-\frac{x}{2} = \frac{dy}{dx}$$

$$dy = -\frac{x\,dx}{2}$$

$$y = -\frac{x^2}{4} + C$$

```
t := diff(y(x), x) :
expr := y(x) = x·t + t^2 + 1 :
res := dsolve(expr);
```

$res := y(x) = -\frac{x^2}{4} + 1, y(x) = \_C1^2 + \_C1\,x + 1$

```
a := array(1..7) :
for i from -3 by 1 to 3 do
a[i + 4] := plot(subs(_C1 = i, rhs(res[2]))) :
end do:
plots[display](a[1], a[2], a[3], a[4], a[5], a[6], a[7]);
```