Учреждение образования
“БЕЛОРУССКИЙ ГОСУДАРСТВЕННЫЙ УНИВЕРСИТЕТ
ИНФОРМАТИКИ И РАДИОЭЛЕКТРОНИКИ”
Кафедра информатики

Ответ по лабораторной работе №9
Методы Эйлера и Рунге-Кутта

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Содержание:

Цель выполнения задания: Изучить решение задачи Коши для обыкновенных дифференциальных уравнений методом Эйлера и методом Рунге-Кутта.

Краткие теоретические сведения.

Рассмотрим дифференциальное уравнение $y' = f(x, y)$ с начальным условием $y(x_0) = y_0$. Будем предполагать, что *f(x,y)* непрерывная и непрерывно дифференцируемая по y функция в окрестности замкнутой области

$$D = \{(x, y) \mid a \le x \le b, c \le y \le d\},$$

содержащей внутри себя точку (x_0, y_0).

Требуется решить задачу Коши: найти непрерывно дифференцируемую функцию *y=y(x)*, такую что $y'(x) = f(x, y(x))$ при всех $x \in [a,b]$ и $y(x_0) = y_0$.

Разобьем отрезок *[a, b]* с помощью точек разбиения $a = x_0, x_1, ..., x_n = b$ с шагом $h = (b-a)/n$. Тогда узлы разбиения имеют вид $x_k = x_0 + kh,\ k = \overline{0,n}$.

Пусть $y(x_0), y(x_1),...,y(x_n)$ - значения функции в точках разбиения.

1) Метод Эйлера

Построим рекуррентную последовательность:

$$y_{k+1} = y_k + hf(x_k, y_k), \qquad k = 0,1... \tag{9.1}$$

$$y_0 = y(x_0),$$

которую называют последовательностью Эйлера. Соединяя ломаными все точки (x_k, y_k), полученные из рекуррентной последовательности Эйлера, получим ломаную линию, приближающую график решения $y = y(x)$. Функция, график которой совпадает с ломаной Эйлера, принимается за приближенное решение задачи Коши.

Точность метода Эйлера на всем отрезке [*a, b*] будет *O(h)*.

Для повышения точности вычислений иногда используется модифицированный метод Эйлера, в котором рекуррентная последовательность Эйлера вычисляется по формулам

$$y_{k+1} = y_k + hf(x_k + \frac{h}{2}, y_k + \frac{h}{2} f(x_k, y_k)), \qquad k = 0,1...,n-1. \qquad (9.2)$$

Модифицированный метод Эйлера обычно дает более точное приближение решения.

Пример. Пусть требуется решить задачу Коши:

$$\begin{cases} y' = -y, \ x \in [0,1] \\ y(0) = 1. \end{cases}$$

Полагая $h = 0{,}2$ и используя метод Эйлера, получим, как легко убедиться, из формулы Эйлера (9.1)

$$y_{k+1} = y_k + 0.2 \cdot (-y_k) = 0.8 \cdot y_k .$$

С другой стороны, используя модифицированный метод Эйлера, получим в силу формулы (2) рекуррентную последовательность

$$y_{k+1} = y_k + 0.2 \cdot (-y_k) = 0.82 \cdot y_k \ .$$

Поскольку точным решением задачи Коши, как легко проверить, является функция $y = e^{-x}$, можно сравнить точность обоих методов.

	0	1	2	3	4	5
x_k	0	0.2	0.4	0.6	0.8	1
y_k	1	0.8	0.64	0.572	0.4086	0.3277
$y_k^{модиф}$	1	0.82	0.6724	0.5514	0.4521	0.3708

e^{-x}	1	0.8187	0.6703	0.5488	0.4493	0.3679

Общепризнанным недостатком метода Эйлера является его не достаточно высокая точность. Несомненным достоинством метода Эйлера является его простота.

2) Метод Рунге-Кутта четвертого порядка.

На каждом шаге производится вычисление коэффициентов K_1, K_2, K_3, K_4:

$K_1 = hf(x_k, y_k)$;

$K_2 = hf(x_k + \frac{h}{2}, y_k + \frac{K_1}{2})$;

$K_3 = hf(x_k + \frac{h}{2}, y_k + \frac{K_2}{2})$;

$K_4 = hf(x_k + h, y_k + K_3)$.

Затем вычисляем

$y_{k+1} = y_k + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$.

Данный метод имеет точность $O(h^4)$ на $[a,b]$.

Рассмотрим пример, который мы использовали для иллюстрации точности метода Эйлера.

Пример. Требуется решить задачу Коши:

$\begin{cases} y' = -y \\ y(0) = 1 \end{cases}$ на отрезке [0, 1].

Выберем шаг $h = 0{,}2$. Результат вычислений поместим в таблицу.

	0	1	2	3	4	5
x_k	0	0.2	0.4	0.6	0.8	1
y_k	1	0.8187	0.6703	0.5487	0.4493	0.3678
e^{-x}	1	0.8187	0.6703	0.5488	0.4493	0.3679

Таким образом, метод Рунге-Кутта 4-го порядка отличается очень высокой точностью. К определенным его недостаткам относится большая сложность и трудоемкость (на каждом шаге необходимо четырежды вычислять значения функции f вместо одного раза в методе Эйлера).

Отметим, что на практике выбирают начальную длину шага h таким образом, чтобы $h^4 < \varepsilon$, где ε - заданная точность вычисления решения. Затем шаг выбирают вдвое меньшим и останавливают вычисления, если разность полученных значений y_k со значениями, полученными при начальном выборе шага меньше ε. В противном случае шаг еще раз уменьшают вдвое и т.д.

ЗАДАНИЕ. С помощью метода Эйлера, а затем метода Рунге-Кутта найти с точностью до 0.001 решения следующих уравнений на отрезке [0; 1].

$$y' = \frac{a\left(1 - y^2\right)}{(1+m)x^2 + y^2 + 1}, \; y(0) = 0,$$

где значения параметров a и m принимают следующие значения для вариантов k.

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14
m	1.0	1.5	2.0	1.0	1.5	2.0	1.0	1.5	2.0	1.0	1.5	2.0	1.0	2.0
a	0.5	0.7	0.9	1.1	1.3	0.5	0.7	0.9	1.1	1.3	0.5	0.7	0.9	1.0

Шаг интегрирования h, обеспечивающий требуемую точность, выбирать в процессе вычисления из сравнения результатов, полученных с h и $h/2$. В случае необходимости шаг h должен быть уменьшен.

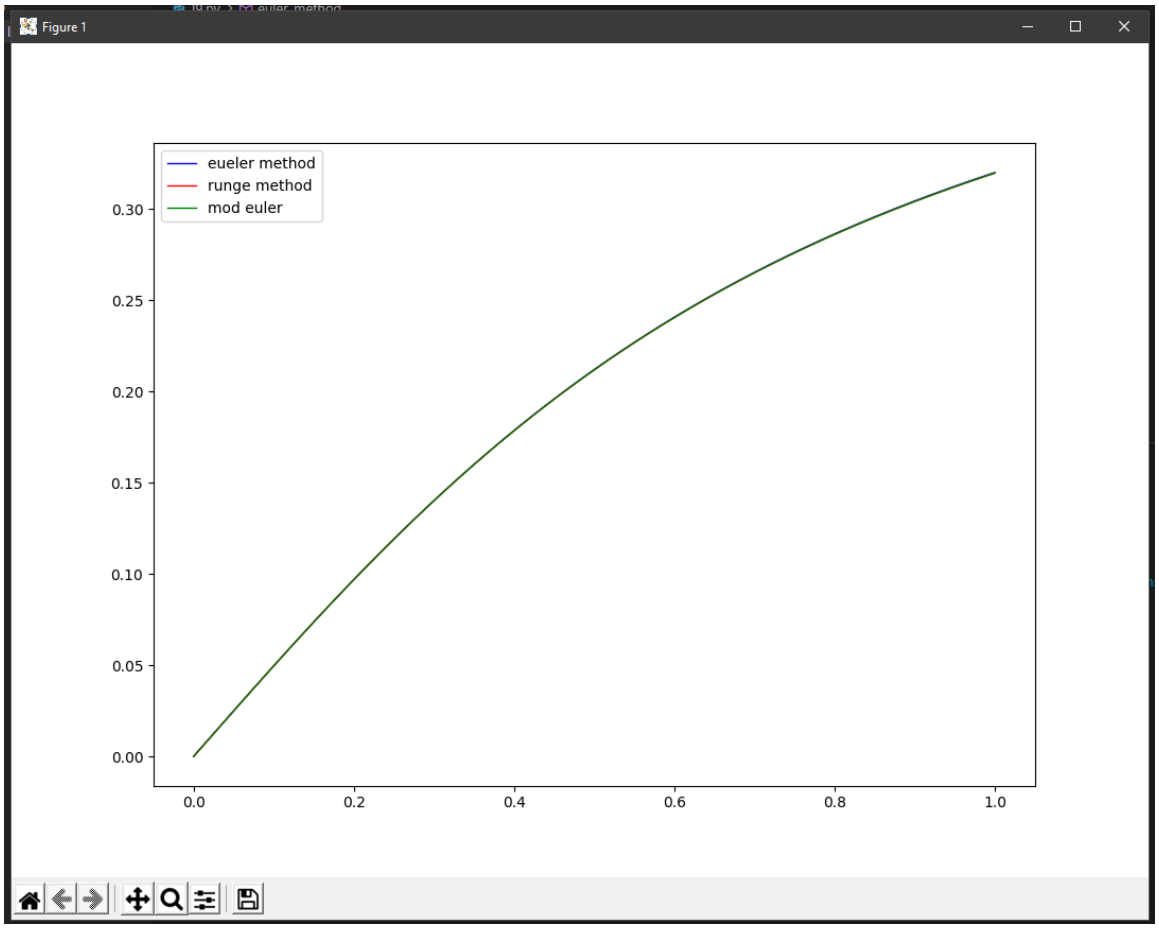
Сравнить результаты.

Программная реализация

Задаются значения параметров, вычисляется результат и выводится:

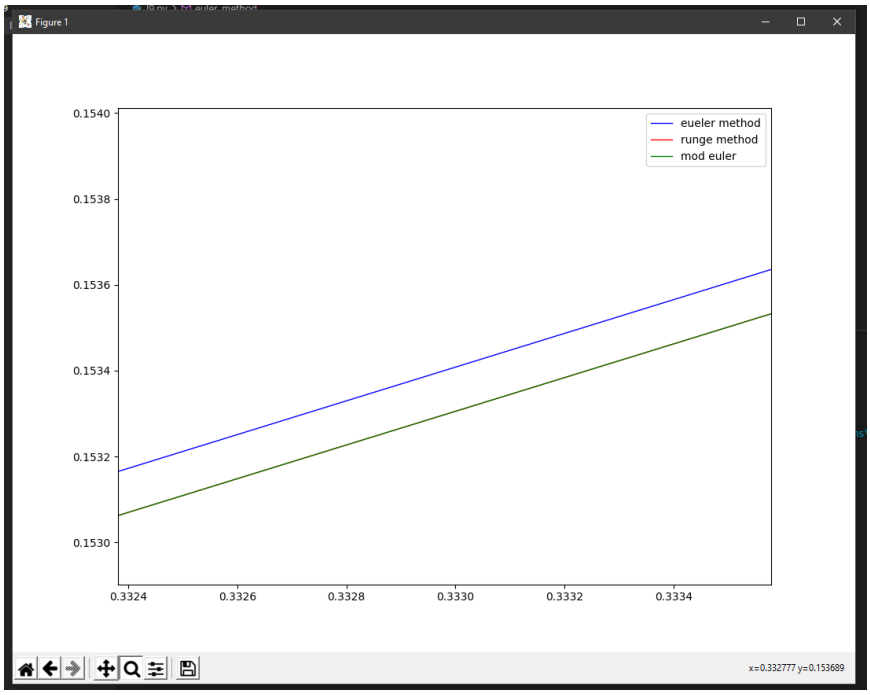
```
a = 0.5
m = 1.
acc = 1e-3
fr = int(abs(np.log10(acc)))

def main():
    tasks = [euler_method, runge_method, modified_euler]
    for task in tasks:
        res = task(f, solution_range, initial_value, acc)
        print("step: ", res[2])
        print("values (x, y):\n", res[1])
        print("\n")
    plt.legend()
    plt.show()
```

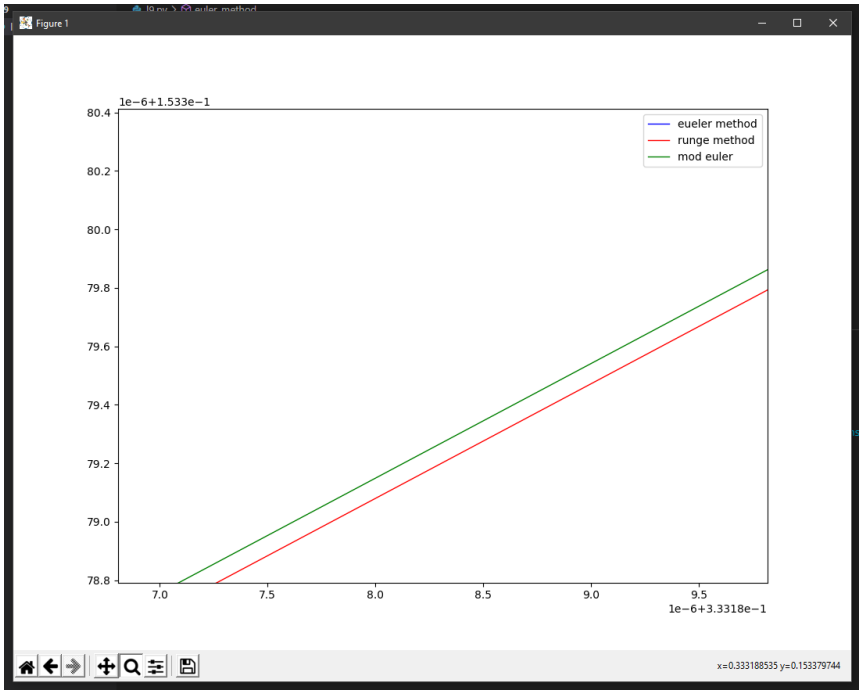


Кривые почти легли в одну.

Если приблизить:



Еще приблизить (к зеленой кривой):



Так же выводятся результаты вычислений в виде списка значений формата (x, y(x))

Для метода Эйлера:

```
step:  0.002
values (x, y):
 [(0, 0), (0.002, 0.001), (0.004, 0.002), (0.006, 0.003), (0.008, 0.004), (0.01, 0.005), (0.012, 0.006), (0.014, 0.007), (0.016, 0.008), (0.018,
 0.009), (0.02, 0.01), (0.021, 0.011), (0.023, 0.012), (0.025, 0.013), (0.027, 0.014), (0.029, 0.015), (0.031, 0.016), (0.033, 0.017), (0.035, 0
.018), (0.037, 0.019), (0.039, 0.02), (0.041, 0.02), (0.043, 0.021), (0.045, 0.022), (0.047, 0.023), (0.049, 0.024), (0.051, 0.025), (0.053, 0.0
26), (0.055, 0.027), (0.057, 0.028), (0.059, 0.029), (0.061, 0.03), (0.062, 0.031), (0.064, 0.032), (0.066, 0.033), (0.068, 0.034), (0.07, 0.035
), (0.072, 0.036), (0.074, 0.037), (0.076, 0.038), (0.078, 0.039), (0.08, 0.04), (0.082, 0.041), (0.084, 0.042), (0.086, 0.043), (0.088, 0.044),
 (0.09, 0.045), (0.092, 0.046), (0.094, 0.047), (0.096, 0.048), (0.098, 0.048), (0.1, 0.049), (0.102, 0.05), (0.104, 0.051), (0.105, 0.052), (0.
107, 0.053), (0.109, 0.054), (0.111, 0.055), (0.113, 0.056), (0.115, 0.057), (0.117, 0.058), (0.119, 0.059), (0.121, 0.06), (0.123, 0.061), (0.1
25, 0.062), (0.127, 0.063), (0.129, 0.064), (0.131, 0.065), (0.133, 0.065), (0.135, 0.066), (0.137, 0.067), (0.139, 0.068), (0.141, 0.069), (0.1
43, 0.07), (0.145, 0.071), (0.146, 0.072), (0.148, 0.073), (0.15, 0.074), (0.152, 0.075), (0.154, 0.076), (0.156, 0.077), (0.158, 0.078), (0.16,
 0.078), (0.162, 0.079), (0.164, 0.08), (0.166, 0.081), (0.168, 0.082), (0.17, 0.083), (0.172, 0.084), (0.174, 0.085), (0.176, 0.086), (0.178, 0
.087), (0.18, 0.088), (0.182, 0.088), (0.184, 0.089), (0.186, 0.09), (0.188, 0.091), (0.189, 0.092), (0.191, 0.093), (0.193, 0.094), (0.195, 0.0
95), (0.197, 0.096), (0.199, 0.097), (0.201, 0.097), (0.203, 0.098), (0.205, 0.099), (0.207, 0.1), (0.209, 0.101), (0.211, 0.102), (0.213, 0.103
), (0.215, 0.104), (0.217, 0.104), (0.219, 0.105), (0.221, 0.106), (0.223, 0.107), (0.225, 0.108), (0.227, 0.109), (0.229, 0.11), (0.23, 0.111),
 (0.232, 0.111), (0.234, 0.112), (0.236, 0.113), (0.238, 0.114), (0.24, 0.115), (0.242, 0.116), (0.244, 0.117), (0.246, 0.117), (0.248, 0.118),
(0.25, 0.119), (0.252, 0.12), (0.254, 0.121), (0.256, 0.122), (0.258, 0.122), (0.26, 0.123), (0.262, 0.124), (0.264, 0.125), (0.266, 0.126), (0.
268, 0.127), (0.27, 0.127), (0.271, 0.128), (0.273, 0.129), (0.275, 0.13), (0.277, 0.131), (0.279, 0.132), (0.281, 0.132), (0.283, 0.133), (0.28
5, 0.134), (0.287, 0.135), (0.289, 0.136), (0.291, 0.136), (0.293, 0.137), (0.295, 0.138), (0.297, 0.139), (0.299, 0.14), (0.301, 0.141), (0.303
, 0.141), (0.305, 0.142), (0.307, 0.143), (0.309, 0.144), (0.311, 0.144), (0.312, 0.145), (0.314, 0.146), (0.316, 0.147), (0.318, 0.148), (0.32,
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2), (0.357, 0.163), (0.359, 0.164), (0.361, 0.164), (0.363, 0.165), (0.365, 0.166), (0.367, 0.167), (0.369, 0.167), (0.371, 0.168), (0.373, 0.16
9), (0.375, 0.169), (0.377, 0.17), (0.379, 0.171), (0.381, 0.172), (0.383, 0.172), (0.385, 0.173), (0.387, 0.174), (0.389, 0.174), (0.391, 0.175
), (0.393, 0.176), (0.395, 0.177), (0.396, 0.177), (0.398, 0.178), (0.4, 0.179), (0.402, 0.179), (0.404, 0.18), (0.406, 0.181), (0.408, 0.181),
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5, 0.194), (0.447, 0.195), (0.449, 0.196), (0.451, 0.196), (0.453, 0.197), (0.455, 0.198), (0.457, 0.198), (0.459, 0.199), (0.461, 0.199), (0.46
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9), (0.82, 0.29), (0.822, 0.291), (0.824, 0.291), (0.826, 0.291), (0.828, 0.292), (0.83, 0.292), (0.832, 0.292), (0.834, 0.293), (0.836, 0.293),
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0.873, 0.3), (0.875, 0.3), (0.877, 0.3), (0.879, 0.301), (0.881, 0.301), (0.883, 0.301), (0.885, 0.302), (0.887, 0.302), (0.889, 0.302), (0.891,
 0.303), (0.893, 0.303), (0.895, 0.303), (0.896, 0.304), (0.898, 0.304), (0.9, 0.304), (0.902, 0.305), (0.904, 0.305), (0.906, 0.305), (0.908, 0
.306), (0.91, 0.306), (0.912, 0.306), (0.914, 0.307), (0.916, 0.307), (0.918, 0.307), (0.92, 0.308), (0.922, 0.308), (0.924, 0.308), (0.926, 0.3
09), (0.928, 0.309), (0.93, 0.309), (0.932, 0.309), (0.934, 0.31), (0.936, 0.31), (0.938, 0.31), (0.939, 0.311), (0.941, 0.311), (0.943, 0.311),
 (0.945, 0.312), (0.947, 0.312), (0.949, 0.312), (0.951, 0.313), (0.953, 0.313), (0.955, 0.313), (0.957, 0.313), (0.959, 0.314), (0.961, 0.314),
 (0.963, 0.314), (0.965, 0.315), (0.967, 0.315), (0.969, 0.315), (0.971, 0.316), (0.973, 0.316), (0.975, 0.316), (0.977, 0.316), (0.979, 0.317),
 (0.98, 0.317), (0.982, 0.317), (0.984, 0.318), (0.986, 0.318), (0.988, 0.318), (0.99, 0.318), (0.992, 0.319), (0.994, 0.319), (0.996, 0.319), (
0.998, 0.32), (1.0, 0.32)]
```

Для метода Рунге-Кутта

```
step:  0.002
values (x, y):
 [(0, 0), (0.002, 0.001), (0.004, 0.002), (0.006, 0.003), (0.008, 0.004), (0.01, 0.005), (0.012, 0.006), (0.014, 0.007), (0.016, 0.008),
 (0.018, 0.009), (0.02, 0.01), (0.021, 0.011), (0.023, 0.012), (0.025, 0.013), (0.027, 0.014), (0.029, 0.015), (0.031, 0.016), (0.033, 0
.017), (0.035, 0.018), (0.037, 0.019), (0.039, 0.02), (0.041, 0.02), (0.043, 0.021), (0.045, 0.022), (0.047, 0.023), (0.049, 0.024), (0.
051, 0.025), (0.053, 0.026), (0.055, 0.027), (0.057, 0.028), (0.059, 0.029), (0.061, 0.03), (0.062, 0.031), (0.064, 0.032), (0.066, 0.03
3), (0.068, 0.034), (0.07, 0.035), (0.072, 0.036), (0.074, 0.037), (0.076, 0.038), (0.078, 0.039), (0.08, 0.04), (0.082, 0.041), (0.084,
 0.042), (0.086, 0.043), (0.088, 0.044), (0.09, 0.045), (0.092, 0.046), (0.094, 0.047), (0.096, 0.047), (0.098, 0.048), (0.1, 0.049), (0
.102, 0.05), (0.104, 0.051), (0.105, 0.052), (0.107, 0.053), (0.109, 0.054), (0.111, 0.055), (0.113, 0.056), (0.115, 0.057), (0.117, 0.0
58), (0.119, 0.059), (0.121, 0.06), (0.123, 0.061), (0.125, 0.062), (0.127, 0.063), (0.129, 0.064), (0.131, 0.065), (0.133, 0.065), (0.1
35, 0.066), (0.137, 0.067), (0.139, 0.068), (0.141, 0.069), (0.143, 0.07), (0.145, 0.071), (0.146, 0.072), (0.148, 0.073), (0.15, 0.074)
, (0.152, 0.075), (0.154, 0.076), (0.156, 0.077), (0.158, 0.078), (0.16, 0.078), (0.162, 0.079), (0.164, 0.08), (0.166, 0.081), (0.168,
0.082), (0.17, 0.083), (0.172, 0.084), (0.174, 0.085), (0.176, 0.086), (0.178, 0.087), (0.18, 0.088), (0.182, 0.088), (0.184, 0.089), (0
.186, 0.09), (0.188, 0.091), (0.189, 0.092), (0.191, 0.093), (0.193, 0.094), (0.195, 0.095), (0.197, 0.096), (0.199, 0.097), (0.201, 0.0
97), (0.203, 0.098), (0.205, 0.099), (0.207, 0.1), (0.209, 0.101), (0.211, 0.102), (0.213, 0.103), (0.215, 0.104), (0.217, 0.104), (0.21
9, 0.105), (0.221, 0.106), (0.223, 0.107), (0.225, 0.108), (0.227, 0.109), (0.229, 0.11), (0.23, 0.111), (0.232, 0.111), (0.234, 0.112),
 (0.236, 0.113), (0.238, 0.114), (0.24, 0.115), (0.242, 0.116), (0.244, 0.117), (0.246, 0.117), (0.248, 0.118), (0.25, 0.119), (0.252, 0
.12), (0.254, 0.121), (0.256, 0.122), (0.258, 0.122), (0.26, 0.123), (0.262, 0.124), (0.264, 0.125), (0.266, 0.126), (0.268, 0.127), (0.
27, 0.127), (0.271, 0.128), (0.273, 0.129), (0.275, 0.13), (0.277, 0.131), (0.279, 0.132), (0.281, 0.132), (0.283, 0.133), (0.285, 0.134
), (0.287, 0.135), (0.289, 0.136), (0.291, 0.136), (0.293, 0.137), (0.295, 0.138), (0.297, 0.139), (0.299, 0.14), (0.301, 0.14), (0.303,
 0.141), (0.305, 0.142), (0.307, 0.143), (0.309, 0.144), (0.311, 0.144), (0.312, 0.145), (0.314, 0.146), (0.316, 0.147), (0.318, 0.148),
 (0.32, 0.148), (0.322, 0.149), (0.324, 0.15), (0.326, 0.151), (0.328, 0.151), (0.33, 0.152), (0.332, 0.153), (0.334, 0.154), (0.336, 0.
154), (0.338, 0.155), (0.34, 0.156), (0.342, 0.157), (0.344, 0.157), (0.346, 0.158), (0.348, 0.159), (0.35, 0.16), (0.352, 0.16), (0.354
, 0.161), (0.355, 0.162), (0.357, 0.163), (0.359, 0.163), (0.361, 0.164), (0.363, 0.165), (0.365, 0.166), (0.367, 0.166), (0.369, 0.167)
, (0.371, 0.168), (0.373, 0.169), (0.375, 0.169), (0.377, 0.17), (0.379, 0.171), (0.381, 0.171), (0.383, 0.172), (0.385, 0.173), (0.387,
 0.174), (0.389, 0.174), (0.391, 0.175), (0.393, 0.176), (0.395, 0.176), (0.396, 0.177), (0.398, 0.178), (0.4, 0.179), (0.402, 0.179), (
0.404, 0.18), (0.406, 0.181), (0.408, 0.181), (0.41, 0.182), (0.412, 0.183), (0.414, 0.183), (0.416, 0.184), (0.418, 0.185), (0.42, 0.18
5), (0.422, 0.186), (0.424, 0.187), (0.426, 0.188), (0.428, 0.188), (0.43, 0.189), (0.432, 0.19), (0.434, 0.19), (0.436, 0.191), (0.438,
 0.192), (0.439, 0.192), (0.441, 0.193), (0.443, 0.193), (0.445, 0.194), (0.447, 0.195), (0.449, 0.195), (0.451, 0.196), (0.453, 0.197),
 (0.455, 0.197), (0.457, 0.198), (0.459, 0.199), (0.461, 0.199), (0.463, 0.2), (0.465, 0.201), (0.467, 0.201), (0.469, 0.202), (0.471, 0
.203), (0.473, 0.203), (0.475, 0.204), (0.477, 0.204), (0.479, 0.205), (0.48, 0.206), (0.482, 0.206), (0.484, 0.207), (0.486, 0.207), (0
.488, 0.208), (0.49, 0.209), (0.492, 0.209), (0.494, 0.21), (0.496, 0.211), (0.498, 0.211), (0.5, 0.212), (0.502, 0.212), (0.504, 0.213)
, (0.506, 0.214), (0.508, 0.214), (0.51, 0.215), (0.512, 0.215), (0.514, 0.216), (0.516, 0.217), (0.518, 0.217), (0.52, 0.218), (0.521,
0.218), (0.523, 0.219), (0.525, 0.219), (0.527, 0.22), (0.529, 0.221), (0.531, 0.221), (0.533, 0.222), (0.535, 0.222), (0.537, 0.223), (
0.539, 0.223), (0.541, 0.224), (0.543, 0.225), (0.545, 0.225), (0.547, 0.226), (0.549, 0.226), (0.551, 0.227), (0.553, 0.227), (0.555, 0
.228), (0.557, 0.229), (0.559, 0.229), (0.561, 0.23), (0.562, 0.23), (0.564, 0.231), (0.566, 0.231), (0.568, 0.232), (0.57, 0.232), (0.5
72, 0.233), (0.574, 0.233), (0.576, 0.234), (0.578, 0.235), (0.58, 0.235), (0.582, 0.236), (0.584, 0.236), (0.586, 0.237), (0.588, 0.237
), (0.59, 0.238), (0.592, 0.238), (0.594, 0.239), (0.596, 0.239), (0.598, 0.24), (0.6, 0.24), (0.602, 0.241), (0.604, 0.241), (0.605, 0.
242), (0.607, 0.242), (0.609, 0.243), (0.611, 0.243), (0.613, 0.244), (0.615, 0.244), (0.617, 0.245), (0.619, 0.245), (0.621, 0.246), (0
.623, 0.246), (0.625, 0.247), (0.627, 0.247), (0.629, 0.248), (0.631, 0.248), (0.633, 0.249), (0.635, 0.249), (0.637, 0.25), (0.639, 0.2
5), (0.641, 0.251), (0.643, 0.251), (0.645, 0.252), (0.646, 0.252), (0.648, 0.253), (0.65, 0.253), (0.652, 0.254), (0.654, 0.254), (0.65
6, 0.255), (0.658, 0.255), (0.66, 0.256), (0.662, 0.256), (0.664, 0.257), (0.666, 0.257), (0.668, 0.258), (0.67, 0.258), (0.672, 0.258),
 (0.674, 0.259), (0.676, 0.259), (0.678, 0.26), (0.68, 0.26), (0.682, 0.261), (0.684, 0.261), (0.686, 0.262), (0.688, 0.262), (0.689, 0.
263), (0.691, 0.263), (0.693, 0.263), (0.695, 0.264), (0.697, 0.264), (0.699, 0.265), (0.701, 0.265), (0.703, 0.266), (0.705, 0.266), (0
.707, 0.267), (0.709, 0.267), (0.711, 0.267), (0.713, 0.268), (0.715, 0.268), (0.717, 0.269), (0.719, 0.269), (0.721, 0.27), (0.723, 0.2
7), (0.725, 0.27), (0.727, 0.271), (0.729, 0.271), (0.73, 0.272), (0.732, 0.272), (0.734, 0.273), (0.736, 0.273), (0.738, 0.273), (0.74,
 0.274), (0.742, 0.274), (0.744, 0.275), (0.746, 0.275), (0.748, 0.275), (0.75, 0.276), (0.752, 0.276), (0.754, 0.277), (0.756, 0.277),
(0.758, 0.278), (0.76, 0.278), (0.762, 0.278), (0.764, 0.279), (0.766, 0.279), (0.768, 0.28), (0.77, 0.28), (0.771, 0.28), (0.773, 0.281
), (0.775, 0.281), (0.777, 0.281), (0.779, 0.282), (0.781, 0.282), (0.783, 0.283), (0.785, 0.283), (0.787, 0.283), (0.789, 0.284), (0.79
1, 0.284), (0.793, 0.285), (0.795, 0.285), (0.797, 0.285), (0.799, 0.286), (0.801, 0.286), (0.803, 0.287), (0.805, 0.287), (0.807, 0.287
), (0.809, 0.288), (0.811, 0.288), (0.812, 0.288), (0.814, 0.289), (0.816, 0.289), (0.818, 0.289), (0.82, 0.29), (0.822, 0.29), (0.824,
0.291), (0.826, 0.291), (0.828, 0.291), (0.83, 0.292), (0.832, 0.292), (0.834, 0.292), (0.836, 0.293), (0.838, 0.293), (0.84, 0.293), (0
.842, 0.294), (0.844, 0.294), (0.846, 0.295), (0.848, 0.295), (0.85, 0.295), (0.852, 0.296), (0.854, 0.296), (0.855, 0.296), (0.857, 0.2
97), (0.859, 0.297), (0.861, 0.297), (0.863, 0.298), (0.865, 0.298), (0.867, 0.298), (0.869, 0.299), (0.871, 0.299), (0.873, 0.299), (0.
875, 0.3), (0.877, 0.3), (0.879, 0.3), (0.881, 0.301), (0.883, 0.301), (0.885, 0.301), (0.887, 0.302), (0.889, 0.302), (0.891, 0.302), (
0.893, 0.303), (0.895, 0.303), (0.896, 0.303), (0.898, 0.304), (0.9, 0.304), (0.902, 0.304), (0.904, 0.305), (0.906, 0.305), (0.908, 0.3
05), (0.91, 0.306), (0.912, 0.306), (0.914, 0.306), (0.916, 0.307), (0.918, 0.307), (0.92, 0.307), (0.922, 0.308), (0.924, 0.308), (0.92
6, 0.308), (0.928, 0.309), (0.93, 0.309), (0.932, 0.309), (0.934, 0.31), (0.936, 0.31), (0.938, 0.31), (0.939, 0.31), (0.941, 0.311), (0
.943, 0.311), (0.945, 0.311), (0.947, 0.312), (0.949, 0.312), (0.951, 0.312), (0.953, 0.313), (0.955, 0.313), (0.957, 0.313), (0.959, 0.
313), (0.961, 0.314), (0.963, 0.314), (0.965, 0.314), (0.967, 0.315), (0.969, 0.315), (0.971, 0.315), (0.973, 0.316), (0.975, 0.316), (0
.977, 0.316), (0.979, 0.316), (0.98, 0.317), (0.982, 0.317), (0.984, 0.317), (0.986, 0.318), (0.988, 0.318), (0.99, 0.318), (0.992, 0.31
8), (0.994, 0.319), (0.996, 0.319), (0.998, 0.319), (1.0, 0.32)]
```

Для модифицированного метода Эйлера:

```
step:  0.002
values (x, y):
 [(0, 0), (0.002, 0.001), (0.004, 0.002), (0.006, 0.003), (0.008, 0.004), (0.01, 0.005), (0.012, 0.006), (0.014, 0.007), (0.016, 0.008),
 (0.018, 0.009), (0.02, 0.01), (0.021, 0.011), (0.023, 0.012), (0.025, 0.013), (0.027, 0.014), (0.029, 0.015), (0.031, 0.016), (0.033, 0
.017), (0.035, 0.018), (0.037, 0.019), (0.039, 0.02), (0.041, 0.02), (0.043, 0.021), (0.045, 0.022), (0.047, 0.023), (0.049, 0.024), (0.
051, 0.025), (0.053, 0.026), (0.055, 0.027), (0.057, 0.028), (0.059, 0.029), (0.061, 0.03), (0.062, 0.031), (0.064, 0.032), (0.066, 0.03
3), (0.068, 0.034), (0.07, 0.035), (0.072, 0.036), (0.074, 0.037), (0.076, 0.038), (0.078, 0.039), (0.08, 0.04), (0.082, 0.041), (0.084,
 0.042), (0.086, 0.043), (0.088, 0.044), (0.09, 0.045), (0.092, 0.046), (0.094, 0.047), (0.096, 0.047), (0.098, 0.048), (0.1, 0.049), (0
.102, 0.05), (0.104, 0.051), (0.105, 0.052), (0.107, 0.053), (0.109, 0.054), (0.111, 0.055), (0.113, 0.056), (0.115, 0.057), (0.117, 0.0
58), (0.119, 0.059), (0.121, 0.06), (0.123, 0.061), (0.125, 0.062), (0.127, 0.063), (0.129, 0.064), (0.131, 0.065), (0.133, 0.065), (0.1
35, 0.066), (0.137, 0.067), (0.139, 0.068), (0.141, 0.069), (0.143, 0.07), (0.145, 0.071), (0.146, 0.072), (0.148, 0.073), (0.15, 0.074)
, (0.152, 0.075), (0.154, 0.076), (0.156, 0.077), (0.158, 0.078), (0.16, 0.078), (0.162, 0.079), (0.164, 0.08), (0.166, 0.081), (0.168,
0.082), (0.17, 0.083), (0.172, 0.084), (0.174, 0.085), (0.176, 0.086), (0.178, 0.087), (0.18, 0.088), (0.182, 0.088), (0.184, 0.089), (0
.186, 0.09), (0.188, 0.091), (0.189, 0.092), (0.191, 0.093), (0.193, 0.094), (0.195, 0.095), (0.197, 0.096), (0.199, 0.097), (0.201, 0.0
97), (0.203, 0.098), (0.205, 0.099), (0.207, 0.1), (0.209, 0.101), (0.211, 0.102), (0.213, 0.103), (0.215, 0.104), (0.217, 0.104), (0.21
9, 0.105), (0.221, 0.106), (0.223, 0.107), (0.225, 0.108), (0.227, 0.109), (0.229, 0.11), (0.23, 0.111), (0.232, 0.111), (0.234, 0.112),
 (0.236, 0.113), (0.238, 0.114), (0.24, 0.115), (0.242, 0.116), (0.244, 0.117), (0.246, 0.117), (0.248, 0.118), (0.25, 0.119), (0.252, 0
.12), (0.254, 0.121), (0.256, 0.122), (0.258, 0.122), (0.26, 0.123), (0.262, 0.124), (0.264, 0.125), (0.266, 0.126), (0.268, 0.127), (0.
27, 0.127), (0.271, 0.128), (0.273, 0.129), (0.275, 0.13), (0.277, 0.131), (0.279, 0.132), (0.281, 0.132), (0.283, 0.133), (0.285, 0.134
), (0.287, 0.135), (0.289, 0.136), (0.291, 0.136), (0.293, 0.137), (0.295, 0.138), (0.297, 0.139), (0.299, 0.14), (0.301, 0.14), (0.303,
 0.141), (0.305, 0.142), (0.307, 0.143), (0.309, 0.144), (0.311, 0.144), (0.312, 0.145), (0.314, 0.146), (0.316, 0.147), (0.318, 0.148),
 (0.32, 0.148), (0.322, 0.149), (0.324, 0.15), (0.326, 0.151), (0.328, 0.151), (0.33, 0.152), (0.332, 0.153), (0.334, 0.154), (0.336, 0.
154), (0.338, 0.155), (0.34, 0.156), (0.342, 0.157), (0.344, 0.157), (0.346, 0.158), (0.348, 0.159), (0.35, 0.16), (0.352, 0.16), (0.354
, 0.161), (0.355, 0.162), (0.357, 0.163), (0.359, 0.163), (0.361, 0.164), (0.363, 0.165), (0.365, 0.166), (0.367, 0.166), (0.369, 0.167)
, (0.371, 0.168), (0.373, 0.169), (0.375, 0.169), (0.377, 0.17), (0.379, 0.171), (0.381, 0.171), (0.383, 0.172), (0.385, 0.173), (0.387,
 0.174), (0.389, 0.174), (0.391, 0.175), (0.393, 0.176), (0.395, 0.176), (0.396, 0.177), (0.398, 0.178), (0.4, 0.179), (0.402, 0.179), (
0.404, 0.18), (0.406, 0.181), (0.408, 0.181), (0.41, 0.182), (0.412, 0.183), (0.414, 0.183), (0.416, 0.184), (0.418, 0.185), (0.42, 0.18
5), (0.422, 0.186), (0.424, 0.187), (0.426, 0.188), (0.428, 0.188), (0.43, 0.189), (0.432, 0.19), (0.434, 0.19), (0.436, 0.191), (0.438,
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 (0.455, 0.197), (0.457, 0.198), (0.459, 0.199), (0.461, 0.199), (0.463, 0.2), (0.465, 0.201), (0.467, 0.201), (0.469, 0.202), (0.471, 0
.203), (0.473, 0.203), (0.475, 0.204), (0.477, 0.204), (0.479, 0.205), (0.48, 0.206), (0.482, 0.206), (0.484, 0.207), (0.486, 0.207), (0
.488, 0.208), (0.49, 0.209), (0.492, 0.209), (0.494, 0.21), (0.496, 0.211), (0.498, 0.211), (0.5, 0.212), (0.502, 0.212), (0.504, 0.213)
, (0.506, 0.214), (0.508, 0.214), (0.51, 0.215), (0.512, 0.215), (0.514, 0.216), (0.516, 0.217), (0.518, 0.217), (0.52, 0.218), (0.521,
0.218), (0.523, 0.219), (0.525, 0.219), (0.527, 0.22), (0.529, 0.221), (0.531, 0.221), (0.533, 0.222), (0.535, 0.222), (0.537, 0.223), (
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8), (0.994, 0.319), (0.996, 0.319), (0.998, 0.319), (1.0, 0.32)]
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Заключение.

Изучить решение задачи Коши для обыкновенных дифференциальных уравнений методом Эйлера и методом Рунге-Кутта.