Лабораторная работа №8

№1

$$f(t)=\begin{cases}0, & t<0\\ \frac{t}{a}-1, & 0<t<a\\ 0, & a<t<2a\\ \frac{t}{2a}-1, & 2a<t<4a\end{cases}$$

$$F(p)=\int_0^\infty f(t)e^{-pt}dt=\int_0^a\left(\frac{t}{a}-1\right)e^{-pt}+$$
$$+\int_a^{2a}0\cdot dt+\int_{2a}^\infty\left(\frac{t}{2a}-1\right)e^{-pt}dt$$

$$\int_0^a\left(\frac{t}{a}-1\right)e^{-pt}dt=\int_0^a\frac{te^{-pt}}{a}dt-\int_0^a e^{-pt}dt=$$
$$=-\frac{1}{ap}\int_0^a te^{-pt}d(-pt)+\frac{1}{p}\int_0^a e^{-pt}d(-pt)=-\frac{1}{ap}\int_0^a t\,de^{-pt}+$$
$$+\frac{1}{p}e^{-pt}\Big|_0^a=-\frac{t}{ap}e^{-pt}\Big|_0^a-\frac{1}{ap^2}\int_0^a e^{-pt}d(-pt)+$$
$$+\frac{1}{p}(e^{-ap}-1)=-\frac{e^{-ap}}{p}-\frac{e^{-pt}}{ap^2}\Big|_0^a+\frac{1}{p}(e^{-ap}-1)=$$
$$=-\frac{e^{-ap}}{p}+\frac{e^{-ap}}{p}-\frac{1}{p}-\frac{e^{-ap}}{ap^2}+\frac{1}{ap^2}=$$
$$=\frac{1-e^{-ap}}{ap^2}-\frac{ap}{ap^2}=\frac{1-ap-e^{-ap}}{ap^2}$$

$$\int_{2a}^\infty\left(\frac{t}{2a}-1\right)e^{-pt}dt=\int_{2a}^\infty\frac{te^{-pt}}{2a}dt-\int_{2a}^\infty e^{-pt}dt=$$
$$=-\frac{1}{2ap}\int_{2a}^\infty t\,de^{-pt}+\frac{e^{-pt}}{p}\Big|_{2a}^\infty=-\frac{1}{2ap}t\cdot e^{-pt}\Big|_{2a}^\infty-$$
$$-\frac{1}{2ap^2}\int_{2a}^\infty e^{-pt}d(-pt)+\frac{e^{-pt}}{p}\Big|_{2a}^\infty=-\frac{1}{2ap}t\,e^{-pt}\Big|_{2a}^\infty-$$
$$-\frac{e^{-pt}}{2ap^2}\Big|_{2a}^\infty+\frac{e^{-pt}}{p}\Big|_{2a}^\infty=\frac{e^{-2ap}}{p}+\frac{e^{-2ap}}{2ap^2}-\frac{e^{-2ap}}{p}=$$
$$=\frac{e^{-2ap}}{2ap^2}$$

$$F(p)=\frac{1-ap-e^{-ap}}{ap^2}+\frac{e^{-2ap}}{2ap^2}=\frac{2-2ap-2e^{-ap}+e^{-2ap}}{2ap^2}$$

$assume(a, positive):$

$f(t) := piecewise\left(t < 0, 0, 0 < t < a, \frac{t}{a} - 1, a < t < 2 \cdot a, 0, 2 \cdot a < t < \text{infinity}, \frac{t}{2 \cdot a} - 1\right) : f(t);$

$$\begin{cases} 0 & t < 0 \\ \frac{t}{a\sim} - 1 & 0 < t < a\sim \\ 0 & a\sim < t < 2\,a\sim \\ \frac{t}{2\,a\sim} - 1 & 2\,a\sim < t < \infty \end{cases}$$

$convert(laplace(f(t), t, p), integrate);$

$$-\frac{p\,a\sim + e^{-p\,a\sim} - 1}{p^2\,a\sim} + \left(\lim_{t \to \infty} \frac{2\,e^{-p\,t}\,a\sim p - e^{-p\,t}\,p\,t + e^{-2\,p\,a\sim} - e^{-p\,t}}{2\,p^2\,a\sim}\right)$$

$$f(t) = \eta(t) \cdot \left(\frac{t}{a} - 1\right) - \eta(t-a)\left(\frac{t}{a} - 1\right) + \eta(t-2a)\left(\frac{t}{2a} - 1\right)$$

$$t \xrightarrow{L} \frac{1}{p^2}$$

$$\frac{t}{a} \xrightarrow{L} a \frac{1}{p^2 a^2} = \frac{1}{ap^2}$$

$$1 \xrightarrow{L} \frac{1}{p}$$

$$\frac{t}{a} - 1 \xrightarrow{L} \frac{1}{ap^2} - \frac{1}{p}$$

$$t\,\eta(t) \xrightarrow{L} \frac{1}{p^2}$$

$$(t-a)\eta(t-a) \xrightarrow{L} e^{-ap} \frac{1}{p^2}$$

$$\eta(t-a)\left(\frac{t-a}{a}\right) \xrightarrow{L} e^{-ap} \frac{1}{ap^2}$$

$$\eta(t-2a)(t-2a) \xrightarrow{L} e^{-2ap} \frac{1}{p^2}$$

$$\eta(t-2a)\left(\frac{t-2a}{2a}\right) \xrightarrow{L} \frac{e^{-2ap}}{2ap^2}$$

$$F(t) = \frac{1}{ap^2} - \frac{1}{p} + \frac{e^{-ap}}{ap^2} + \frac{e^{-2ap}}{2ap^2} = \frac{2 - 2ap - 2e^{-ap} + e^{-2ap}}{2ap^2}$$

$convert(laplace(f(t), t, p), integrate);$

$$-\frac{p\,a\sim + e^{-p\,a\sim} - 1}{p^2\,a\sim} + \left(\lim_{t \to \infty} \frac{2\,e^{-p\,t}\,a\sim p - e^{-p\,t}\,p\,t + e^{-2\,p\,a\sim} - e^{-p\,t}}{2\,p^2\,a\sim}\right)$$

№2

$$F(p) = \frac{p}{(p+1)(p^2+4p+5)} = \frac{A}{p+1} + \frac{Bp+C}{p^2+4p+5}$$

$$\begin{cases} A+B=0 \\ 4A+B+C=1 \\ 5A+C=0 \end{cases} \qquad \begin{cases} A=-\frac{1}{2} \\ B=\frac{1}{2} \\ C=\frac{5}{2} \end{cases}$$

$$eq := \frac{p}{(p+1)\cdot(p^2+4\cdot p+5)}:$$

$convert(eq, parfrac)$

$$\frac{p+5}{2\left(p^2+4p+5\right)} - \frac{1}{2\,(p+1)}$$

$$F(p) = -\frac{1}{2}\cdot\frac{1}{p+1} + \frac{p+5}{2(p^2+4p+5)} = -\frac{1}{2}\cdot\frac{1}{p+1} + \frac{p+2}{2((p+2)^2+1)} + \frac{3}{2((p+2)^2+1)}$$

$$-\frac{1}{2}\cdot\frac{1}{p+1} \xrightarrow{L^{-1}} -\frac{1}{2}e^{-t}$$

$$\frac{1}{2}\cdot\frac{p+2}{(p+2)^2+1} \xrightarrow{L^{-1}} \frac{1}{2}\cdot e^{-2t}\cos(t)$$

$$\frac{3}{2}\cdot\frac{1}{(p+2)^2+1} \xrightarrow{L^{-1}} \frac{3}{2}\cdot e^{-2t}\sin(t)$$

$$F(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{-2t}\cos(t) + \frac{3}{2}e^{-2t}\sin(t)$$

$inttrans[invlaplace](eq, p, t);$

$$-\frac{e^{-t}}{2} + \frac{e^{-2t}\left(\cos(t)+3\sin(t)\right)}{2}$$

№3

$$y'' - y = \frac{e^t}{1+e^t} \qquad y(0)=0,\quad y'(0)=0$$

Операторный метод:

$$y(t) \xrightarrow{L} Y(p)$$

$$y'(t) \xrightarrow{L} pY(p) - y(0) = pY(p)$$

$$y'' = p^2 Y(p)$$

$$p^2 Y(p) - pY(p) = \frac{1}{p}$$

$$Y(p) = \frac{1}{p^2(p-1)} = \frac{1}{p-1} - \frac{1}{p^2} - \frac{1}{p}$$

$convert\left(\frac{1}{p^2\cdot(p-1)}, parfrac\right);$

$$\frac{1}{p-1} - \frac{1}{p^2} - \frac{1}{p}$$

$inttrans[invlaplace]\left(\frac{1}{p-1} - \frac{1}{p^2} - \frac{1}{p}, p, t\right);$

$$e^t - t - 1$$

$$y(t) = e^t - t - 1$$

$$y'(t) = e^t - 1$$

$$y(t) = \int_0^t \frac{e^\tau}{1+e^\tau}\left(e^{t-\tau} - 1\right)d\tau = \int_0^t \left(e^{t-\tau}-1\right) d\ln(1+e^\tau) =$$

$$= e^t \int_0^t e^{-\tau} d\ln(1+e^\tau) - \int_0^t d\ln(1+e^\tau) = e^t \int_0^t \frac{1}{1+e^\tau} d\tau -$$

$$- \ln(1+e^\tau)\Big|_0^t$$

$$\int \frac{1}{1+e^\tau} d\tau = \left| \begin{matrix} e^\tau = u \\ du = u\,d\tau \end{matrix} \right| = \int \frac{1}{u(1+u)} du =$$

$$= \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du = \ln(u) - \ln(1+u) = \ln(e^\tau) - \ln(1+e^\tau) =$$

$$= \tau - \ln(1+e^\tau)$$

$$y(t) = e^t\left(\tau - \ln(1+e^\tau)\right)\Big|_0^t - \ln(1+e^\tau)\Big|_0^t =$$

$$= te^t - e^t\ln(1+e^t) + e^t\ln(2) - \ln(1+e^t) + \ln(2) =$$

$$= e^t\left(t + \ln(2) - \ln(1+e^t)\right) + \ln(2) - \ln(1+e^t) =$$

$$= e^t\left(t + \ln\left(\frac{2}{1+e^t}\right)\right) + \ln\left(\frac{2}{1+e^t}\right) =$$

$$= te^t + (e^t+1)\ln\left(\frac{2}{1+e^t}\right).$$

$$int\left(\frac{\exp(\tau)}{1+\exp(\tau)}\cdot(\exp(t-\tau)-1), \tau = 0..t\right);$$

$$e^t\ln(2)+\ln(2)-e^t\ln(1+e^t)+e^t\ln(e^t)-\ln(1+e^t)$$

$$simplify\left(dsolve\left(\left\{y'(0)=0, y(0)=0., y''(t)-y'(t)=\frac{\exp(t)}{1+\exp(t)}\right\}\right)\right);$$

$$y(t)=(-e^t-1)\ln(1+e^t)+e^t\ln(e^t)+\ln(2)\,(1+e^t)$$

Метод Лагранжа:

$$y''-y'=0$$

$$\frac{dy'}{y'}=dt$$

$$y'=C_1e^t$$

$$y=C_1e^t+C_2$$

$$\begin{cases} C_1 = C_1(x) \\ C_2 = C_2(x) \end{cases}$$

$$\begin{cases} C_1'(t)e^t + C_2'(t)=0 \\ C_1'(t)e^t = \frac{e^t}{1+e^t} \end{cases}$$

$$\begin{cases} C_1'(t) = \frac{1}{1+e^t} \\ C_2'(t) = -\frac{e^t}{1+e^t} \end{cases} \qquad \begin{cases} C_1(t) = t-\ln(1+e^t)+C_1 \\ C_2(t) = -\ln(1+e^t)+C_2 \end{cases}$$

$$dsolve\left(\left[C1'(t)\cdot\exp(t)+C2'(t)=0, C1'(t)\cdot\exp(t)=\frac{\exp(t)}{\exp(t)+1}\right], [C1(t), C2(t)]\right);$$

$$\{C1(t)=-\ln(1+e^t)+\ln(e^t)+\_C1, C2(t)=-\ln(1+e^t)+\_C2\}$$

$$y=te^t-e^t\ln(1+e^t)+C_1e^t-\ln(1+e^t)+C_2$$

$$y=te^t+C_1e^t+C_2-(e^t+1)\ln(1+e^t)$$

$$y'=te^t+C_1e^t-e^t\ln(1+e^t)$$

$$\begin{cases} 0 = C_1+C_2-2\ln(2) \\ 0 = C_1-\ln(2) \end{cases}$$

$$\begin{cases} C_1 = \ln(2) \\ C_2 = \ln(2) \end{cases}$$

$$y=te^t+(e^t+1)\ln(2)-(e^t+1)\ln(1+e^t)$$

$$y=te^t+(e^t+1)\cdot\ln\left(\frac{2}{1+e^t}\right)$$

$$simplify\left(dsolve\left(y''(t) - y'(t) = \frac{\exp(t)}{1 + \exp(t)}\right)\right);$$

$$y(t) = \left(-e^{t} - 1\right)\ln\left(1 + e^{t}\right) + e^{t}\_C1 + e^{t}\ln\left(e^{t}\right) + \_C2 + 1$$

$$simplify\left(dsolve\left(\left\{y'(0) = 0, y(0) = 0., y''(t) - y'(t) = \frac{\exp(t)}{1 + \exp(t)}\right\}\right)\right);$$

$$y(t) = \left(-e^{t} - 1\right)\ln\left(1 + e^{t}\right) + e^{t}\ln\left(e^{t}\right) + \ln(2)\left(1 + e^{t}\right)$$

↓

№4

$$\begin{cases} y'' - 3y' + 2y = e^t \\ y(0) = 1, \; y'(0) = 0 \end{cases}$$

$$y(t) \xrightarrow{L} Y(p)$$

$$y'(t) \xrightarrow{L} pY(p) - 1$$

$$y''(t) \xrightarrow{L} p^2 Y(p) - p$$

$$p^2 Y(p) - p - 3pY(p) + 3 + 2Y(p) = \frac{1}{p-1}$$

$$Y(p)(p^2 - 3p + 2) - p + 3 = \frac{1}{p-1}$$

$$Y(p) = \frac{p^2 - 4p + 4}{(p-1)(p^2 - 3p + 2)} = \frac{1}{p-1} - \frac{1}{(p-1)^2}$$

$$y(t) = e^t - t\,e^t = e^t(1-t)$$

$convert\left(\frac{p^2 - 4\cdot p + 4}{(p-1)\cdot(p^2 - 3\cdot p + 2)}, parfrac\right);$

$$\frac{1}{p-1} - \frac{1}{(p-1)^2}$$

$inttrans[invlaplace](\%, p, t);$

$$-e^t(-1 + t)$$

$dsolve(\{y'(0) = 0, y(0) = 1, y''(t) - 3\cdot y'(t) + 2\cdot y(t) = \exp(t)\});$

$$y(t) = (-t + 1)\,e^t$$

№5

$$\begin{cases} x' = x + 4y \\ y' = 2x - y + 9 \end{cases} \qquad \begin{matrix} x(0) = 1 \\ y(0) = 0 \end{matrix}$$

$$x(t) \xrightarrow{L} X(p)$$
$$y(t) \xrightarrow{L} Y(p)$$
$$x'(t) \xrightarrow{L} pX(p) - 1$$
$$y'(t) \xrightarrow{L} pY(p)$$

$$\begin{cases} pX(p) - 1 = X(p) + 4Y(p) \\ pY(p) = 2X(p) - Y(p) + \frac{9}{p} \end{cases}$$

$$\begin{cases} (p-1)X(p) - 4Y(p) = 1 \\ -2X(p) + (p+1)Y(p) = \frac{9}{p} \end{cases}$$

$$\Delta = \begin{vmatrix} p-1 & -4 \\ -2 & p+1 \end{vmatrix} = p^2 - 1 - 8 = p^2 - 9$$

$A := Matrix([[p-1,-4],[-2,p+1]]);$

$$A := \begin{bmatrix} p-1 & -4 \\ -2 & p+1 \end{bmatrix}$$

$LinearAlgebra[Determinant](A);$

$$p^2 - 9$$

$$\Delta X = \begin{vmatrix} 1 & -4 \\ \frac{9}{p} & p+1 \end{vmatrix} = \frac{p^2 + p + 36}{p}$$

$X := Matrix\left(\left[[1,-4], \left[\frac{9}{p}, p+1\right]\right]\right);$

$$X := \begin{bmatrix} 1 & -4 \\ \frac{9}{p} & p+1 \end{bmatrix}$$

$LinearAlgebra[Determinant](A);$

$$\frac{p^2 + p + 36}{p}$$

$$\Delta Y = \begin{vmatrix} p-1 & 1 \\ -2 & \frac{9}{p} \end{vmatrix} = \frac{11p - 9}{p}$$

$Y := Matrix\left(\left[\left[p-1, 1\right], \left[-2, \frac{9}{p}\right]\right]\right);$

$$Y := \begin{bmatrix} p-1 & 1 \\ -2 & \frac{9}{p} \end{bmatrix}$$

$LinearAlgebra[Determinant](A);$

$$\frac{11\,p-9}{p}$$

$$\begin{cases} X(p) = \dfrac{\Delta x}{\Delta} = \dfrac{p^2+p+36}{p(p^2-9)} = \dfrac{7}{3}\cdot\dfrac{1}{p+3} + \dfrac{8}{3}\cdot\dfrac{1}{p-3} - \dfrac{4}{p} \\ Y(p) = \dfrac{\Delta y}{\Delta} = \dfrac{11p-9}{p(p^2-9)} = -\dfrac{7}{3}\cdot\dfrac{1}{p+3} + \dfrac{4}{3}\cdot\dfrac{1}{p-3} + \dfrac{1}{p} \end{cases}$$

$convert\left(\frac{p^2+p+36}{(p^2-9)\cdot p}, parfrac\right);$

$$\frac{7}{3\,(p+3)} + \frac{8}{3\,(p-3)} - \frac{4}{p}$$

$convert\left(\frac{11\cdot p-9}{(p^2-9)\cdot p}, parfrac\right);$

$$-\frac{7}{3\,(p+3)} + \frac{4}{3\,(p-3)} + \frac{1}{p}$$

$$\begin{cases} x(t) = \dfrac{7}{3}e^{-3t} + \dfrac{8}{3}e^{3t} - 4 \\ y(t) = -\dfrac{7}{3}e^{-3t} + \dfrac{4}{3}e^{3t} + 1 \end{cases}$$

$dsolve(\{x(0)=1, y(0)=0, x'(t)=x(t)+4\cdot y(t), y'(t)=2\cdot x(t)-y(t)+9\});$

$$\left\{x(t) = \frac{7\,e^{-3\,t}}{3} + \frac{8\,e^{3\,t}}{3} - 4, y(t) = -\frac{7\,e^{-3\,t}}{3} + \frac{4\,e^{3\,t}}{3} + 1\right\}$$