Simulation of Vibrating Strings with Nonlinearity

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Motivation

When a string is plucked, various vibrational modes are produced, in a distribution that depends on the location and strength of the pluck, as well as intrinsic string properties. As the string exchanges energy with its surroundings, the amplitude of the various modes can mix and evolve. There are two extreme views on how the sound will evolve: periodicity, meaning the sound will eventually return to the original sound; and chaotic, meaning that knowledge of the initial pluck will be lost, so that it will not be possible by analyzing a sound sometime after the pluck to determine information about the original pluck. We explored this topic computationally. Our hypothesis is that the system starts to exhibit chaotic behavior when the nonlinearity crosses a certain boundary.

Wave Equation

An ideal string vibrates according to the equation $\partial_t^2 y = c^2 \partial_x^2 y$, where y is the displacement of a point on the string, and c is the speed of the wave. However, a real string has some non-linear properties such as stiffness and viscoelasticity. Then wave equation becomes

$$\partial_t^2 y = c^2 \partial_x^2 y - e\partial_x^4 y + b\partial_x^2 v \tag{1}$$

Where e and b are the coefficients of stiffness and viscosity. We assumed that the length of the string L is 1.5 m, and the fundamental frequency is 67 Hz, so $c = 200$ m/s is a reasonable number. e can be calculated with the parameters of a real string according to the equation

$$e = \frac{EI}{\rho A} = \frac{Er^2}{4\rho},$$

where E is the Young's Modulus, and I is the moment of inertia, ρ is the density, A is the cross-section area and r is the radius of the string. We estimated that the e of a silk string is $10\ \mathrm{m}^4/\mathrm{s}^2$. However, b hasn't been experimentally measured for silk strings.

Method

We used C language to construct our string model. In our model, a string is decomposed into N points attached with x and y positions. Each point moves under the forces exerted by its neighboring points due to their relative positions. The acceleration can be calculated with the wave equations. We first simulated the behavior of an ideal plucked string using Euler's method. Then the nonlinear terms were added to the string model and Runge-Kutta 4th Order (RK4) method was used to evolve the string with time to allow a larger time step but the same accuracy.

We used GNU Science Library Fast Fourier Transform (FFT) package to generate the corresponding Fourier frequency spectrum from the positions of the string components. We checked if the simulation program was functioning by looking at if the frequency spectrum was constant for any given initial pluck condition when nonlinear terms were zero. Since we set the initial shape of a plucked string as a triangle, the acceleration of the two end points and the top point would be huge and this was not physical, so we rounded off the corners by letting the string evolve for some time with the top point fixed. We also used python to animate the movement of string and see whether it is behaving physically. Since energy is conserved when there is only stiffness, while viscoelasticity, do give rise to loss of energy, we had another way of checking our program.

In order to assess string's chaotic behavior, we plucked the string at two positions very close to each other and let them evolve to calculate the Lyapunov Exponent

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \left|\frac{\delta_t}{\delta_0}\right|$$

where δ_0 and δ_t are the separation vector at time 0 and time t.

$$\delta = \sqrt{\sum (F_1(n) - F_2(n))^2}$$

is calculated in the frequency space after Fourier transform. Since different frequencies were out of phase, the maximum magnitude of each frequency in an oscillation period was selected and stored in $F(n)$. As δ_t kept decreasing exponentially after 60 seconds, we used python curve fit module to find the exponential fitting with the form of $\lambda = \alpha e^{-\beta t} + \lambda_\infty$ and λ_∞ is the value that λ approaches when time goes to infinity. Finally, we were able to change the value of e and see how λ changes accordingly.

The computer model of a plucked string accomplishes the following steps:

1. Pluck the two strings at two adjacent positions and let the strings relax.
2. Calculate the acceleration and then the velocity at each point.
3. Use RK4 method to take a time step dt.
4. Use FFT to convert position space to frequency space after each dt.
5. Pick out the maximum of each frequency component in one vibration period T.
6. Calculate the separation vector and Lyapunov Exponent after T.
7. Repeat step 2 to step 6.

Result

When the number of points N increases from 64 to 5096, the string exhibits less wriggling and moves more smoothly. We used $N = 512$ for most of the simulation since it is a fairly good tradeoff between the running time of the program on a PC and smoothness of the string motion. For this value of N and L, $dt = 10^{-6}$ ensures that the result is accurate enough. In each run of simulation, we change the plucking position (one half, one third, etc...) and e and let the string evolve for over 60 s. We found that when e increased from 0 to 4.5 with a step of 0.5, the Lyapunov exponent first increased fast and then became steady, and it was true for any plucking position.

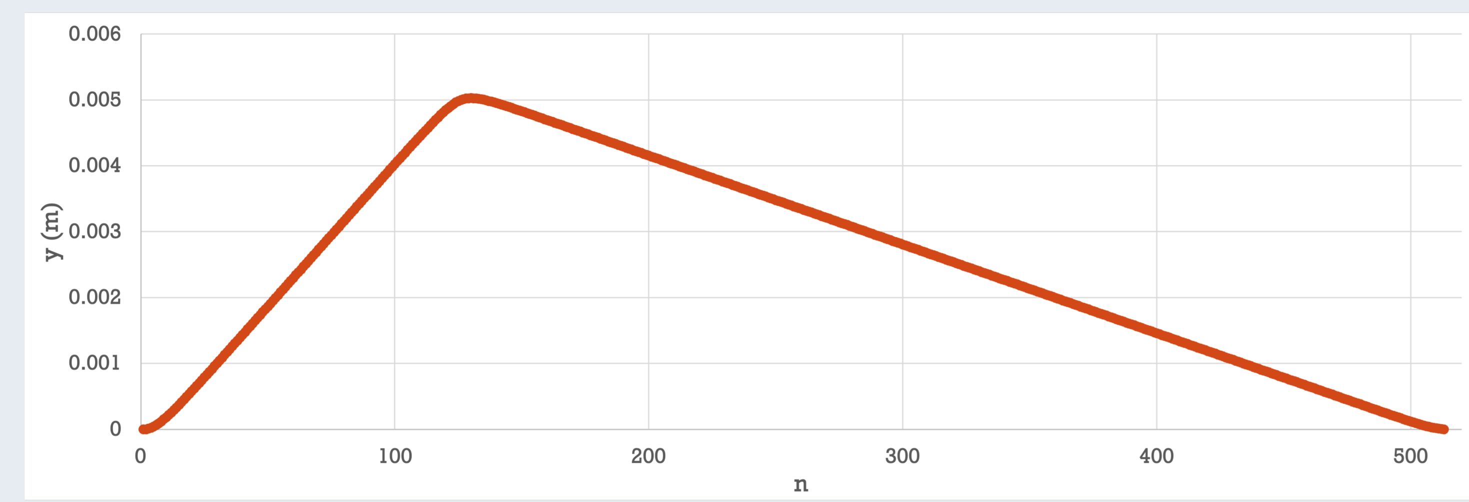

Figure 1: The shape of a plucked string after relaxation

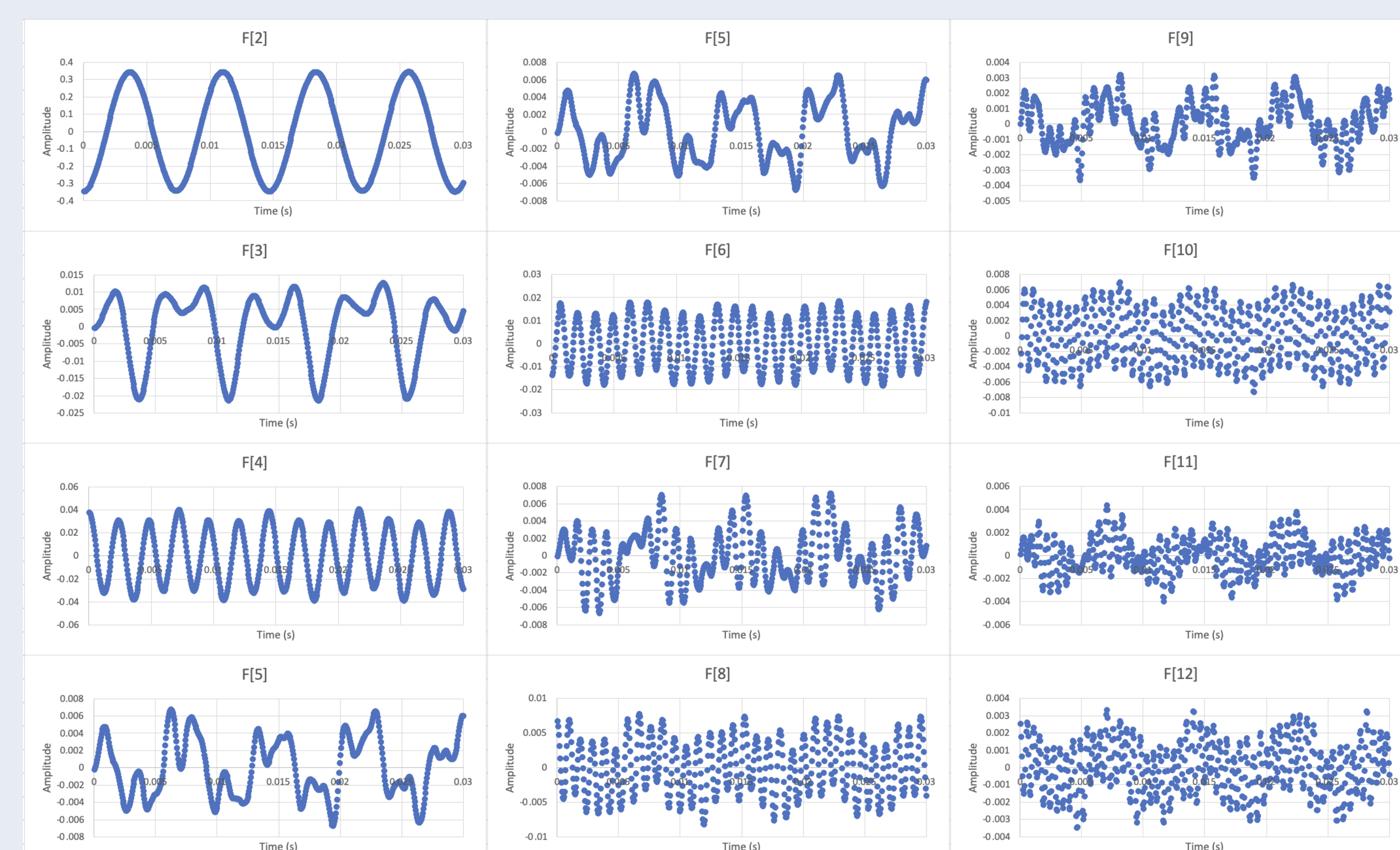

Figure 2: Oscillation of different frequency components in two string oscillation periods

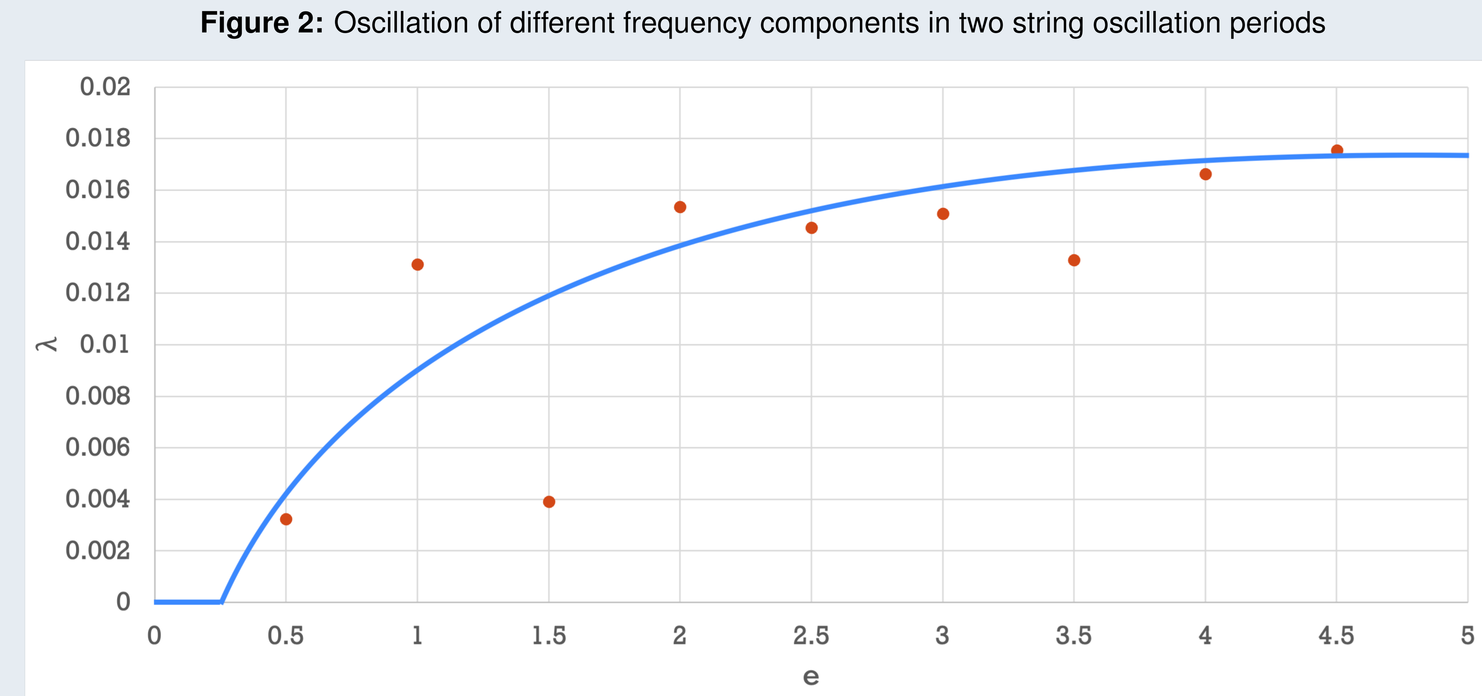

Figure 3: Lyapunov exponent changing with stiffness coefficient.
The blue curve is of form anticipated by theory.

Conclusions

Theory predicts that when b is zero and e is smaller than a critical number e_{critical}, λ should be zero, but when e is larger than e_{critical}, λ should increase with a logarithmic style envelope. In this summer research, we haven't found the e_{critical} yet and we still need to do more simulation in the region of $0 < e < 0.5$ with a larger N and smaller dt, to acquire more accurate result. We have only been exploring stiffness and not gone to the effect of viscoelasticity. We will continue to study it in the future.

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