Exercises for the lecture
“Mathematische Modellierung in der Klimaforschung W16/17”

Due to Tuesday, 2016-10-08, Room A011 or via e-mail to
`nadolski@math.fu-berlin.de`

In the tutorial we have seen that conservative methods are given by schemes of the form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n\right) \tag{1}$$

where Q_i^n denotes the average value in the i-th cell to time $t = t_n$

$$Q_i^n \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} q(x, t_n)\,\mathrm{d}x \tag{2}$$

and $F_{i-\frac{1}{2}}^n$ is some approximation to the average flux along $x = x_{i-\frac{1}{2}}$

$$F_{i-\frac{1}{2}}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t^{n+1}} f\left(q\left(x_{i-\frac{1}{2}}, t\right)\right)\mathrm{d}t. \tag{3}$$

We assume, for now, that $F_{i-\frac{1}{2}}^n$ depends only on the states Q_{i-1}^n and Q_i^n, thus

$$F_{i-\frac{1}{2}}^n = \mathcal{F}\left(Q_{i-1}^n, Q_i^n\right) \tag{4}$$

for some *numerical flux* function $\mathcal{F}$.

Exercise 1 (Central-Difference (10 points))**.** Consider the linear transport equation

$$\begin{cases} q_t(x,t) + (aq(x,t))_x & = 0 \qquad \forall (x,t) \in [0,1] \times \mathbb{R}_0^+ \\ q(x,0) & = q_0(x), \end{cases} \tag{5}$$

with periodic boundary conditions. The correct solution to (5) is

$$q(x,t) = q_0(x - at). \qquad \forall (x,t) \in [0,1] \times \mathbb{R}_0^+ \tag{6}$$

We chose the discretisation $\Delta x = \frac{1}{k}$ and $\Delta t = \lambda \Delta x$. Furthermore let the grid points be defined as $x_{i-\frac{1}{2}} = i\Delta x$ and $t_n = n\Delta t$. As initial values we define (2) by

$$Q_i^n := q_0\left(x_i\right) \quad 0 \le i \le k \tag{7}$$

with $x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2}$. From the boundary conditions we have

$$Q^n_{-1} = Q^n_k \quad \text{and} \tag{8}$$

$$Q^n_{k+1} = Q^n_0 \tag{9}$$

Implement scheme (1) with this discretisation and the numerical flux $\mathcal{F}$ given by

(i) $F^n_{i-\frac{1}{2}} = \mathcal{F}\left(Q^n_{i-1}, Q^n_i\right) = \frac{a}{2}\left(Q^n_{i-1} + Q^n_i\right)$

(ii) $F^n_{i-\frac{1}{2}} = \mathcal{F}\left(Q^n_{i-1}, Q^n_i\right) = aQ^n_{i-1}$

for $0 \leq i \leq k$. Test the scheme for initial values $q_0(x) = \sin(x)$ and $q_0(x) = \chi_{[\frac{1}{3},\frac{2}{3}]}(x)$ with $a \in \{-1, 0, +1\}$, $k \in \{10, 100\}$ and $\lambda \in \{\frac{1}{2}, 1, \frac{3}{2}\}$. Use a programming language of your choice and plot some time steps against the true solution given in (6).

What do you observe?