Digit Recognition

with Support Vector Machines

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Outline

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2. Our Approach
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4. Multi-Class Classification
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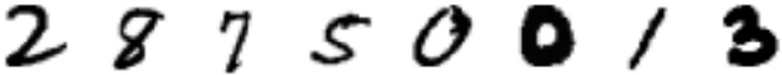
Introduction to Our Data Set

Main Goal: train algorithm to recognize handwritten digits

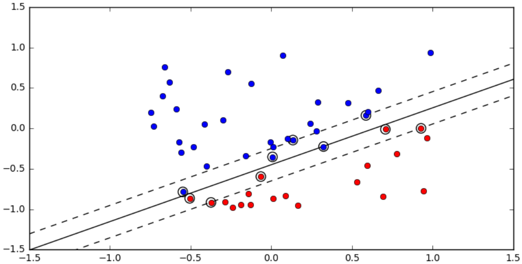


Data:

- 42,000 greyscale images
- 28 by 28 pixels each
- partitioned into ten classes

Our Approach

We want to use the concept of SVMs.



- **Problem I:** SVMs are binary classifiers
- **Problem II:** Need to solve optimization problem

Our Approach

1. Implement solver for our QP
   - 3 versions
2. Implement basic SVM algorithm
   - linear kernel / Gaussian kernel
   - Parameter optimization
3. Combine individual SVMs in different ways
   - 3 versions
4. Validate and compare results

Sequential Minimal Optimization (SMO)

- The primal Soft Margin SVM QP is equivalent to solving the **dual problem:**

$$\begin{aligned} \text{minimize} \quad & d(\alpha) := \frac{1}{2}\alpha^T Q\alpha - 1^T\alpha \\ \text{s.t.} \quad & 0 \leq \alpha \leq C \quad \text{and} \quad y^T\alpha = 0, \end{aligned} \tag{1}$$

where $q_{ij} = y_i y_j k(x_i, x_j)$, x_i the data, y_i the labels, k the kernel function and C the penalty term

- Since Q is spsd, satisfying the KKT conditions guarantees a solution to (1).

Sequential Minimal Optimization (SMO)

- Lagrangian of dual objective d:
  $\mathcal{L}(\alpha, \delta, \mu, \beta) = d(\alpha) - \delta^T \alpha + \mu^T(\alpha - C) - \beta \alpha^T y$
- **KKT conditions** for dual Lagrangian:

$$\left.\begin{aligned} \nabla_\alpha \mathcal{L}(\alpha^*, \delta^*, \mu^*, \beta^*) &= 0 \\ \delta_i^* &\geq 0 \\ \delta_i^* \alpha_i^* &= 0 \\ \mu_i^* &\geq 0 \\ \mu_i^*(\alpha_i^* - C) &= 0 \\ \alpha_i^* &\text{ feasible} \end{aligned}\right\} \text{for all } i \in \{1, \ldots, l\}$$

- Define $F_i(\alpha) := y_i(\partial_i d)(\alpha) = \sum_{j=1}^{l} \alpha_j y_j k(x_i, x_j) - y_i$.

Sequential Minimal Optimization (SMO)

- The KKT conditions are equivalent to:

$$b_{up}(\alpha) := \min_{i \in I_{up}(\alpha)} F_i(\alpha) \geq \max_{j \in I_{low}(\alpha)} F_j(\alpha) =: b_{low}(\alpha),$$

  where $I_{up}(\alpha), I_{low}(\alpha) \subset \{1, \dots, l\}$:
  - $I_{up}(\alpha) := \{i \mid \alpha_i < C, y_i = 1 \text{ or } \alpha_i > 0, y_i = -1\}$
  - $I_{low}(\alpha) := \{j \mid \alpha_j < C, y_j = -1 \text{ or } \alpha_j > 0, y_j = 1\}$.
- Relax to $b_{up}(\alpha) \geq b_{low}(\alpha) - \tau$ for some tolerance $\tau > 0$.
- A pair $(i, j) \in I_{up}(\alpha) \times I_{low}(\alpha)$ with $F_i(\alpha) < F_j(\alpha) - \tau$ is called **τ–violating.**

Sequential Minimal Optimization (SMO)

- Any algorithm of the following form converges after finitely many steps:

Algorithm (General SMO type algorithm)

Let $\tau > 0$. *Initialize* $k = 0$ *and* $\alpha^0 = 0$ *and generate iterates* α^k, $k \in \mathbb{N}$, *as follows:*

1. *If* α^k *satisfies* $b_{up}(\alpha^k) \geq b_{low}(\alpha^k) - \tau$, *stop. Else* ***pick*** *a* τ*-violating pair* $(i, j) \in I_{up}(\alpha^k) \times I_{low}(\alpha^k)$.
2. ***Minimize*** d *only in* α_i^k *and* α_j^k, *leaving* α_n^k *fixed for* $n \notin \{i, j\}$ *and respecting constraints.* $\rightarrow$ *Obtain* α^{new}.
3. *Set* $k := k + 1$, $\alpha^k := \alpha^{new}$ *and go to Step 1.*

Sequential Minimal Optimization (SMO)

- Each step of GSMO is only a (clipped) **one-dimensional QP** $\rightarrow$ analytic solution known $\rightarrow$ **cheap**.
- Two heuristics for choosing violating pair:
  - WSS1: steepest possible **gradient**

$$(i_{up}, j_{low}) \in \operatorname{argmin}_{i \in I_{up}(\alpha)} F_i(\alpha) \times \operatorname{argmax}_{j \in I_{low}(\alpha)} F_j(\alpha)$$

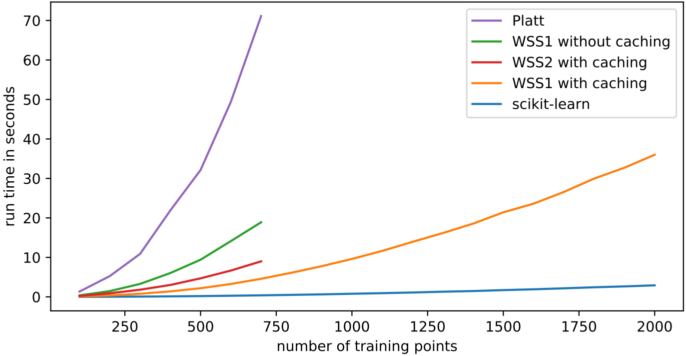
  - WSS2: maximal possible **decrease** in d

$$(i, j) \in \operatorname{argmin}_{i \in I_{up}(\alpha)} F_i(\alpha) \times I_{low}(\alpha) : d(\alpha^{new}) - d(\alpha) \rightarrow \min$$

- WSS2 seems promising, but is too expensive.

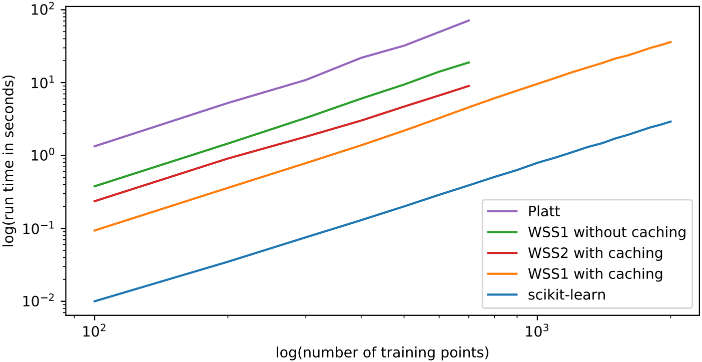
Sequential Minimal Optimization (SMO)

- **Run time comparison** of algorithms with Gaussian kernel and labels by first ECOC classifier on our digits:



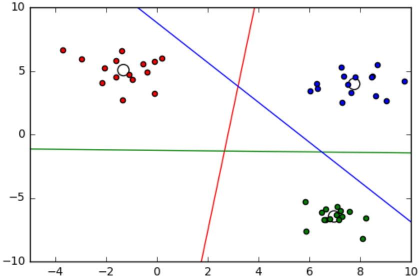
Sequential Minimal Optimization (SMO)

- All algorithms seem to have **polynomial order 2.**
- Our WSS1 with caching runs about 5.8 times as long as scikit-learn SVC.



Multi-Class Classification

- Choose k groups of the classes
- Train k SVMs that separate each group from the rest
- Compare outcome to what would arise for each digit.
- **Problem:** Points may not be classified uniquely.
- Handle overlappings by minimizing distance to barycenters



Multi-Class Classification

1. One-vs-All

Idea: For each $i \in \{0, 1, \ldots, 9\}$, train an SVM that separates class i from the rest

Class	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9
0	–1	1	1	1	1	1	1	1	1	1
1	1	–1	1	1	1	1	1	1	1	1
2	1	1	–1	1	1	1	1	1	1	1
3	1	1	1	–1	1	1	1	1	1	1
4	1	1	1	1	–1	1	1	1	1	1
5	1	1	1	1	1	–1	1	1	1	1
6	1	1	1	1	1	1	–1	1	1	1
7	1	1	1	1	1	1	1	–1	1	1
8	1	1	1	1	1	1	1	1	–1	1
9	1	1	1	1	1	1	1	1	1	–1

Multi-Class Classification

2. Error Correcting Output Codes

Idea: Relabeling with large Hamming distance according to:

Class	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}
0	1	1	–1	–1	–1	–1	1	–1	1	–1	–1	1	1	–1	1
1	–1	–1	1	1	1	1	–1	1	–1	1	1	–1	–1	1	–1
2	1	–1	–1	1	–1	–1	–1	1	1	1	1	–1	1	–1	1
3	–1	–1	1	1	–1	1	1	1	–1	–1	–1	–1	1	–1	1
4	1	1	1	–1	1	–1	1	1	–1	–1	1	1	–1	–1	1
5	–1	1	–1	–1	1	1	–1	–1	1	1	–1	–1	–1	–1	1
6	1	–1	1	1	1	–1	–1	–1	–1	1	–1	1	–1	–1	1
7	–1	–1	–1	1	1	1	1	–1	1	–1	1	1	–1	–1	1
8	1	1	–1	1	–1	1	1	–1	–1	1	–1	–1	–1	1	1
9	–1	1	1	1	–1	–1	–1	–1	1	–1	1	–1	–1	1	1

Results & Conclusions

# training points	One-vs-All uniquely classfied, linear	One-vs-All with bary-centers, linear	One-vs-All uniquely classfied, Gaussian	One-vs-All with bary-centers, Gaussian	ECOC, linear	ECOC, Gaussian
500						
1000						
2000						
5000						
10000						

Table: Correctly Classified Digits

Results & Conclusions

# training points	One-vs-All uniquely classfied, linear	One-vs-All with bary-centers, linear	One-vs-All uniquely classfied, Gaussian	One-vs-All with bary-centers, Gaussian	ECOC, linear	ECOC, Gaussian
500	65.9%	74.1%	75.4%	83.3%	74.2%	87.4%
1000	68.2%	75.0%	84.3%	89.0%	78.0%	92.7%
2000	70.2%	76.4%	89.8%	91.9%	77.8%	94.3%
5000	70.0%	73.8%	88.9%	91.6%	82.0%	95.2%
10000	64.6%	67.5%	88.0%	90.6%	82.5%	95.4%

Table: Correctly Classified Digits

Results & Conclusions

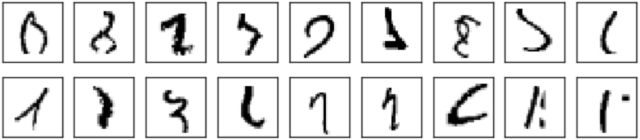


Figure: Visualizing very illegible digits