Доманка 1. Долгих Елизавета

1. (0.25 point) Prove that the sum of two functions of bounded first-order variation also has bounded first-order variation.

□ Пусть f и g определены на отрезке $[a,b]$ и имеют на нем огранич. вариацию:

$$V_a^b(f) = \sup_{\Pi_{a,b}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \infty \quad \text{и} \quad V_a^b(g) = \ldots < \infty \quad (1)$$

$\Pi_{a,b}$ – обозначение с лекций

Возьмем $\forall$ разбиение $\{x_i\}_{i=0}^N$ отрезка $[a,b]$ и для него:

$$\sum_{i=1}^{N} |(f+g)(x_i) - (f+g)(x_{i-1})| = \sum_{i=1}^{N} |f(x_i) - f(x_{i-1}) + g(x_i) - g(x_{i-1})| \leq$$

$$\leq \sum_{i=1}^{N} \left(|f(x_i) - f(x_{i-1})| + |g(x_i) - g(x_{i-1})| \right) =$$

$$= \sum_{i=1}^{N} |f(x_i) - f(x_{i-1})| + \sum_{i=1}^{N} |g(x_i) - g(x_{i-1})| < \infty \quad \text{из } (1)$$

Т.к. это было записано для $\forall$ разбиения $[a,b]$, беря супремум по всем разбиениям, получим:

$$V_a^b(f+g) \leq V_a^b(f) + V_a^b(g) < \infty \quad \text{ч.т.д.}$$

$\boxtimes$

2. (0.75 point) Prove that the product of two functions of bounded first-order variation also has bounded first-order variation.

□ Пусть f, g имеют огранич. вариацию на отрезке $[a,b]$

Заметим, что функция огранич. вариации ограничена (на конечном интервале)

Δ Если $V_a^b(f) < \infty$, то $\forall$ фиксир. $x_0 \in [a,b]$, $\forall x \in [a,b]$:

$$|f(x) - f(x_0)| \leq V_{x_0}^x(f) \leq V_a^b(f)$$

$$\hookrightarrow |f(x)| \leq |f(x_0)| + V_a^b(f) < \infty$$

$\blacktriangle$

Т.е. для функций f и g $\exists$ const M_f, M_g: $|f(x)| \leq M_f$, $|g(x)| \leq M_g$ $\forall x \in [a,b]$

Возьмем $\forall$ разбиение $\{x_i\}_{i=0}^N$ отрезка $[a,b]$ и для него:

$$\Delta(fg)_i = f(x_i)g(x_i) - f(x_{i-1})g(x_{i-1}) = f(x_i) \cdot (g(x_i) - g(x_{i-1})) + g(x_{i-1})(f(x_i) - f(x_{i-1})) =$$

$$= f(x_i) \cdot \Delta g_i + g(x_{i-1}) \Delta f_i$$

$$|\Delta(fg)_i| = |f(x_i)\Delta g_i + g(x_{i-1})\Delta f_i| \le |f(x_i)|\cdot|\Delta g_i| + |g(x_{i-1})|\cdot|\Delta f_i| \le$$
$$\le M_f\cdot|\Delta g_i| + M_g\cdot|\Delta f_i|$$

Суммируем по i и возьмём sup по всем разбиениям:

$$V_a^b(fg) \le M_f\cdot V_a^b(g) + M_g\cdot V_a^b(f) < \infty$$

$\Rightarrow$ fg имеет огранич. вариацию ☒

3. (1 point) Is it true that the quadratic variation of function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}), & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

is bounded?

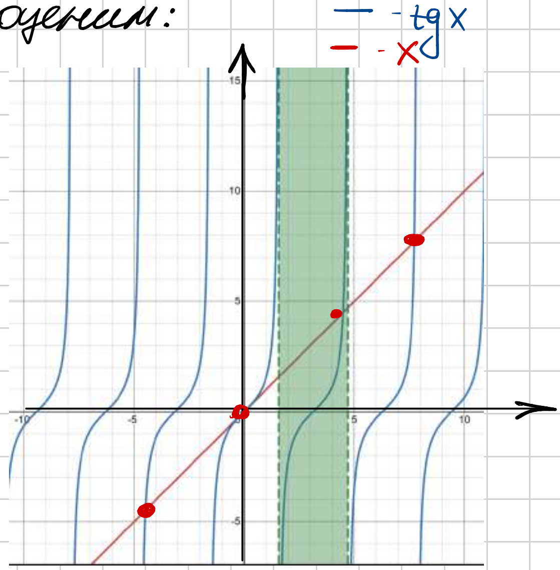
$$V_{0,1}^2(f) = \sup_{\Pi_{a,b}} \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)|^2 \overset{?}{<} \infty$$

$f'(x) = \sin\frac{1}{x} - x\cdot\cos\frac{1}{x}\cdot\frac{1}{x^2} = \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x}$ $\quad (x \in (0;1])$

$f'(x) = 0 \Rightarrow \sin\frac{1}{x} - \frac{1}{x}\cos\frac{1}{x} = 0 \Leftrightarrow \operatorname{tg}\frac{1}{x} = \frac{1}{x}$. Назовем $\{x_k\}_{k\in\mathbb{N}}$ – решения этого ур-я $\operatorname{tg}\frac{1}{x} = \frac{1}{x}$ (т.е. экстремумы нашей функции)

$\sup\limits_{\Pi_{a,b}}$ будет достигаться именно в $\{x_k\}_{k\in\mathbb{N}}$

Т.к. напрямую посчитать не можем, оценим:



Т.к. $\forall k\in\mathbb{N}\ \exists!\ x_k:\ x_k \in \left(\frac{1}{\frac{\pi}{2}+\pi k}; \frac{1}{-\frac{\pi}{2}+\pi k}\right)$,

этот факт следует из π-период. tg и ! реш. из монотонности

пример картинки и 1 такого интервала для $\operatorname{tg} x = x$

то $\sum\limits_{k\in\mathbb{N}} \left|x_{k+1}\sin\left(\frac{1}{x_{k+1}}\right) - x_k\sin\left(\frac{1}{x_k}\right)\right|^2 \le$

$\le \sum\limits_{k\in\mathbb{N}} |x_{k+1} + x_k|^2 \le$

$x_{k+1} + x_k \le \frac{1}{-\frac{\pi}{2}+\pi(k+1)} + \frac{1}{-\frac{\pi}{2}+\pi k} = \frac{1}{\pi}\left(\frac{1}{k+\frac{1}{2}} + \frac{1}{k-\frac{1}{2}}\right) = \frac{1}{\pi}\,\frac{2k}{k^2-\frac{1}{4}} \le \frac{1}{\pi}\cdot\frac{2}{k}$

? $\frac{2k}{k^2-\frac{1}{4}} \le \frac{4}{k}$? проверяем: $\frac{k}{k^2-\frac{1}{4}} \le \frac{2}{k} \overset{k\ge1}{\Leftrightarrow} k^2 \le 2k^2 - \frac{1}{2}$ ✓

$\le \sum\limits_{k\in\mathbb{N}} \frac{1}{\pi^2}\cdot\frac{2^2}{k^2} = \frac{4}{\pi^2}\sum\limits_{k\in\mathbb{N}}\frac{1}{k^2} < \infty$. Т.е. $f(x)$ имеет конечную квадр. вариац. ☒

4. (2 points) Verify that the explicit call option price formula

$$C_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2),$$

$$d_1 = \frac{\ln\frac{S_t}{K} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

satisfies the Black–Scholes–Merton equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0.$$

□ 1) Найдём частные производные по S:

$$\frac{\partial C}{\partial S} = N(d_1) + Sn(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}n(d_2)\cdot\frac{\partial d_2}{\partial S}$$

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}\ ; \quad \frac{\partial d_2}{\partial S} = \frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

$$\Rightarrow \frac{\partial C}{\partial S} = N(d_1) + Sn(d_1)\cdot\frac{1}{S\sigma\sqrt{T-t}} - Ke^{-r(T-t)}n(d_2)\cdot\frac{1}{S\sigma\sqrt{T-t}} =$$

$$\Big\{ n(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}} = \frac{1}{\sqrt{2\pi}}\cdot\exp\{-\tfrac{1}{2}(d_2+\sigma\sqrt{T-t})^2\} =$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\exp\{-\tfrac{1}{2}d_2^2 - \tfrac{1}{2}\sigma^2(T-t) - d_2\sigma\sqrt{T-t}\} = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}}\exp\{-\tfrac{1}{2}\sigma^2(T-t) -$$

$$- d_2\sigma\sqrt{T-t}\} = n(d_2)\cdot\exp\{-\tfrac{1}{2}\sigma^2(T-t) - \frac{\ln(S/K)+(r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\sigma\sqrt{T-t}\} =$$

$$= n(d_2)\cdot\exp\{-\tfrac{1}{2}\sigma^2(T-t) - \ln(S/K) - (r-\tfrac{\sigma^2}{2})(T-t)\} =$$

$$= n(d_2)\cdot\exp\{-\ln(S/K) - r(T-t)\} = n(d_2)\cdot\frac{K}{S}\cdot e^{-r(T-t)} \Big\}$$

То есть $n(d_1) = n(d_2)\frac{K}{S}e^{-r(T-t)}$

$$= N(d_1) + n(d_2)\frac{K}{S}\cdot e^{-r(T-t)}\frac{1}{\sigma\sqrt{T-t}} - Ke^{-r(T-t)}n(d_2)\frac{1}{S\sigma\sqrt{T-t}} = N(d_1)$$

То есть $\frac{\partial C}{\partial S} = N(d_1)$

$$\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}N(d_1) = n(d_1)\frac{\partial d_1}{\partial S} = n(d_1)\cdot\frac{1}{S\sigma\sqrt{T-t}}$$

2) Найдём частную производную по t:

$$\frac{\partial C}{\partial t} = Sn(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}n(d_2)\frac{\partial d_2}{\partial t} =$$

$$\frac{\partial d_1}{\partial t} = \frac{\ln(S/K)+(r+\frac{\sigma^2}{2})(T-t)}{2\sigma(T-t)^{3/2}} - \frac{r+\frac{\sigma^2}{2}}{\sigma\sqrt{T-t}}$$

$$\frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{\sigma}{2\sqrt{T-t}}$$

$$= \cancel{S \cdot n(d_2)\frac{K}{S} e^{-r(T-t)} \frac{\partial d_1}{\partial t}} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)} n(d_2)\left(\cancel{\frac{\partial d_1}{\partial t}} + \frac{\sigma}{2\sqrt{T-t}}\right) =$$

$$= -rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)} n(d_2)\frac{\sigma}{2\sqrt{T-t}}$$

3) подставим в СДУ:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0$$

$$\left[-rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)} n(d_2)\frac{\sigma}{2\sqrt{T-t}}\right] + \frac{1}{2}\sigma^2 S^2 \cdot n(d_1) \cdot \frac{1}{S\sigma\sqrt{T-t}} +$$

$$+ rSN(d_1) - rC =$$

$$= -rKe^{-r(T-t)}N(d_2) - \cancel{Ke^{-r(T-t)} n(d_2)\frac{\sigma}{2\sqrt{T-t}}} + \cancel{\frac{1}{2}\sigma S n(d_2)\frac{K}{S}\frac{e^{-r(T-t)}}{\sqrt{T-t}}} +$$

$$+ rSN(d_1) - rC =$$

$$= -rKe^{-r(T-t)}N(d_2) + rSN(d_1) - rC =$$

$$= r(SN(d_1) - Ke^{-r(T-t)}N(d_2)) - rC = 0$$

Значит СДУ верное!

ч.т.д. ☒

Домашка 2 Долгих Елизавета

1. (1 point) Find the variance of the sample third-order variation

$$\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^3$$

of Brownian motion on $[0, T]$.

□ 1) Рассмотрим сетку $0=t_0<t_1<\ldots<t_n=T$ с $\Delta t_i = t_i - t_{i-1}$

Т.к. $W_{t_i} - W_{t_{i-1}} = \sqrt{\Delta t_i}\, Z_i$, где $\Delta t_i = t_i - t_{i-1}$, $Z_i \sim \mathcal{N}(0,1)$

и такие Z_i независимы

$\Rightarrow |W_{t_i} - W_{t_{i-1}}|^3 = (\Delta t_i)^{3/2} |Z_i|^3$

$\Rightarrow D\left(\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^3\right) = D\left(\sum_{i=1}^{n} (\Delta t_i)^{3/2} |Z_i|^3\right) = \sum_{i=1}^{n} D\left[(\Delta t_i)^{3/2} |Z_i|^3\right] =$

$= \sum_{i=1}^{n} (\Delta t_i)^3 \cdot D\left[|Z_i|^3\right] = \{$для равномерной сетки$\} = n \cdot (\Delta t)^3 \cdot D\,|Z_i|^3$

2) Ищем $D|Z|^3$, где $Z \sim \mathcal{N}(0,1)$

$D|Z|^3 = \mathbb{E}|Z|^6 - (\mathbb{E}|Z|^3)^2$

$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$\mathbb{E}|Z|^3 = \int_{-\infty}^{+\infty} |x|^3 \varphi(x)\,dx = 2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} x^3 e^{-x^2/2}\,dx = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} x^3 e^{-x^2/2}\,dx$$

Замена $u = \frac{x^2}{2}$. Тогда $du = x\,dx$ и $x^2 = 2u$

$\Rightarrow x^3 dx = x^2 \cdot x\,dx = 2u \cdot du$

$$I = \int_0^{+\infty} x^3 e^{-x^2/2}\,dx = \int_0^{+\infty} 2u e^{-u}\,du = 2\int_0^{+\infty} u e^{-u}\,du = 2\Gamma(2) = 2$$

$$\Rightarrow \mathbb{E}|Z|^3 = \frac{2}{\sqrt{2\pi}} \cdot 2 = \frac{2\sqrt{2}}{\sqrt{\pi}}$$

Аналогично, $\mathbb{E}|Z|^6 = \frac{2}{\sqrt{2\pi}} \cdot 2^{5/2} \cdot \frac{15}{8}\sqrt{\pi} = \frac{2}{\sqrt{2\pi}} \cdot 4\sqrt{2}\,\frac{15}{8}\sqrt{\pi} = 15$

$$I = \int_0^{+\infty} x^6 e^{-x^2/2}\,dx = \int_0^{+\infty} (2u)^{5/2} e^{-u}\,du = 2^{5/2} \int_0^{+\infty} u^{5/2} e^{-u}\,du = 2^{5/2}\,\Gamma\left(\frac{7}{2}\right)$$

$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

3) В итоге,

$$\mathbb{D}|Z|^3 = \mathbb{E}|Z|^6 - (\mathbb{E}|Z|^3)^2 = 15 - \left(\frac{2\sqrt{2}}{\sqrt{\pi}}\right)^2 = 15 - \frac{8}{\pi}$$

$$\mathbb{D}\left(\sum_{i=1}^n |W_{t_i} - W_{t_{i-1}}|^3\right) = \sum_{i=1}^n (\Delta t_i)^3 \cdot \mathbb{D}[|Z_i|^3] = \mathbb{D}[|Z|^3] \sum_{i=1}^n (\Delta t_i)^3 =$$

$$= \left(15 - \frac{8}{\pi}\right) \cdot \sum_{i=1}^n (\Delta t_i)^3$$

Для равномерной сетки: $\mathbb{D}\left(\sum_{i=1}^n |W_{t_i} - W_{t_{i-1}}|^3\right) = \left(15 - \frac{8}{\pi}\right) \cdot \frac{T^3}{n^2}$

Ответ: $\mathbb{D}\left(\sum_{i=1}^n |W_{t_i} - W_{t_{i-1}}|^3\right) = \left(15 - \frac{8}{\pi}\right) \sum_{i=1}^n (\Delta t_i)^3$

для равном. сетки: $\mathbb{D}\left(\sum_{i=1}^n |W_{t_i} - W_{t_{i-1}}|^3\right) = \frac{T^3}{n^2}\left(15 - \frac{8}{\pi}\right)$ ⊠

2. (1 point) The Ornstein–Uhlenbeck process is defined by

$$dX_t = -\kappa X_t dt + \sigma dW_t.$$

Find the quadratic variation of $\sin(X_t)$.

□ Применим формулу Ито к $f(x) = \sin(x)$:

$$d\sin(X_t) = \cos(X_t)\,dX_t - \frac{1}{2}\sin(X_t)\underbrace{d\langle X\rangle_t}_{= \sigma^2 dt}$$

$$d\sin(X_t) = \left(-\kappa X_t \cos(X_t) - \frac{\sigma^2}{2}\sin(X_t)\right)dt + \underline{\sigma\cos(X_t)}\,dW_t$$

На лекции обсуждали:

$$d[X]_t = (dX_t)^2 = (\sigma\cos(X_t))^2 dt = \sigma^2\cos^2(X_t)\,dt$$

т.е. буквально из теоремы:

$$[\sin(X)]_t = \int_0^t \sigma^2 \cos^2(X_s)\,ds$$

Ответ: $[\sin(X)]_t = \int_0^t \sigma^2\cos^2(X_s)\,ds$ ⊠

3. (a) (0.5 point) Show that the process defined as

$$(T-t)W_t$$

is differentiable at point T almost surely.

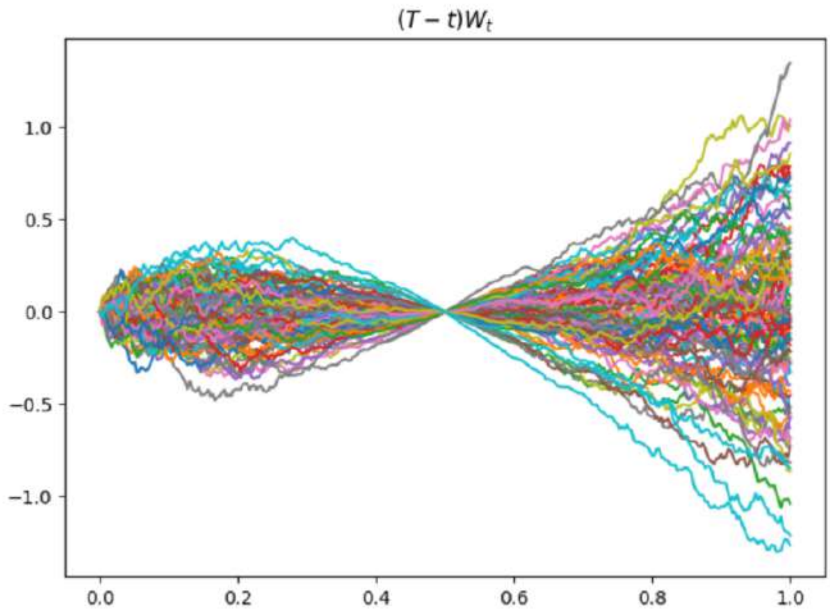
□ a) положим $X_t = (T-t)W_t$

$$((T-t)W_t)'\Big|_{t=T} = \lim_{h\to 0} \frac{X_{T+h} - X_{T-h}}{2h} = \lim_{h\to 0} \frac{(T-(T+h))W_{T+h} - (T-(T-h))W_{T-h}}{2h} =$$

$$= \lim_{h\to 0} \frac{-h\cdot W_{T+h} - hW_{T-h}}{2h} = \frac{1}{2}\lim_{h\to 0}(-W_{T+h} - W_{T-h}) = \left\{ \begin{matrix} \text{непрер.} \\ \text{траекторией п.н.} \end{matrix} \right\} = -W_T$$

$\Rightarrow$ т.е. в точке $t=T$ процесс $X_t = (T-t)W_t$ диффер. п.н.
и $(X_t)'\Big|_{t=T} = -W_T$

(b) (0.5 point) Are there other points at which the process is differentiable almost surely?



Рассмотрим $\forall t_0 \in [0, T)$ и предположим, что X_t диффер-ем в t_0 п.н.

Если X_t диффер-ем п.н. $\Rightarrow$ $\exists$ правые и левые производные п.н. $\Rightarrow$

$$\lim_{h\to 0+} \frac{X_{t_0+h} - X_{t_0}}{h} = \lim_{h\to 0+} \frac{(T-(t_0+h))W_{t_0+h} - (T-t_0)W_{t_0}}{h} =$$

$$= \lim_{h\to 0+} (T-t_0)\frac{W_{t_0+h} - W_{t_0}}{h} - W_{t_0+h}$$

$W_{t_0+h} \to W_{t_0}$ при $h\to 0+$ (из непрер. п.н. траекторий)

НО $\nexists \lim\limits_{h\to 0+} \frac{W_{t_0+h} - W_{t_0}}{h}$ п.н., т.к. траектории БД не дифф. п.н. в $\forall$ точке

$\Rightarrow$ других точек нет

⊠

Домашка 3 Долгих Елизавета

1. Consider a structured deposit - a financial product that offers fixed income and additional yield in the form of a call option. The underlying asset of the call option is asset S. The guaranteed income pays an amount S_0 at maturity of the deposit. Assume the funding rate is higher than the model rate.

   (a) (0.5 point) Is it true that the funding component is always positive?

   (b) (0.5 point) Is it true that the funding component is always negative?

Theorem 3

Under the assumptions above, the hedging error for a European option in the Black-Scholes-Merton model can be decomposed into the next components:

$$Z_t = \underbrace{\frac{Z_0}{D_t}}_{\text{Premium component}} + \underbrace{\int_0^t \frac{D_u}{D_t}\left(r_u^f - r\right)\left(V_u - \frac{\partial V}{\partial s} S_u\right) du}_{\text{Funding component}}$$
$$+ \underbrace{\int_0^t \frac{1}{D_t}\left(\Delta_u^h - \frac{\partial V}{\partial s}\right) dD_u S_u}_{\text{Delta component}}$$
$$+ \underbrace{\frac{1}{2}\int_0^t \frac{D_u}{D_t}\frac{\partial^2 V}{\partial s^2}\left(\sigma^2 S_u^2 du - (dS_u)^2\right)}_{\text{Gamma component}},$$

where the derivatives are taken at (u, S_u).

Вспомним теорему с лекций.

Из нее funding component для опциона-колл:

$$\int_0^t \frac{D_u}{D_t}(r_u^f - r)\underbrace{\left(V_u - \frac{\partial V}{\partial S} S_u\right)}_{=:F(u, S_u)} du$$

На семинаре доказали такой факт для call-опциона:

$$V(t, S_t) - \frac{\partial V}{\partial s}(t, S_t)S_t = V(t, S_t)\left(1 - \frac{S_t}{S_t - x}\right) < 0.$$

Учитывая, что по условию $r_u^f > r$, т.е. $(r_u^f - r) > 0$, и очевидно, $\frac{D_u}{D_t} > 0 \Rightarrow$ для call-опциона

$$\int_0^t \underbrace{\frac{D_u}{D_t}(r_u^f - r)}_{>0} \underbrace{\left(V_u - \frac{\partial V}{\partial S} S_u\right)}_{=:F(u,S_u) < 0} du < 0$$

Но у нас есть ещё и гарантированная выплата $G = S_0$ в момент T.

Для нее: $\tilde{V}_t = S_0 \cdot e^{-r(T-t)}$; $\Delta_t = \frac{\partial \tilde{V}_t}{\partial S} = 0 \Rightarrow \tilde{F}(u, S_u) = \tilde{V}_u > 0$

В итоге, знак зависит от соотношения F и $\tilde{F}$:

- если $S_0 e^{-r(T-t)} = \tilde{F} > |F|$, то fund. comp. > 0
- если $S_0 e^{-r(T-t)} = \tilde{F} = |F|$, то fund. comp. $= 0$
- если $S_0 e^{-r(T-t)} = \tilde{F} < |F|$, то fund. comp. < 0

$\Rightarrow$ Ответ: а) нет ; б) нет

2. (1 point) Describe a hedging strategy that replicates the payoff of a call option under a $\underbrace{\text{non-zero constant interest rate}}_{=:r}$, for continuous paths crossing the strike price finitely many times.

□ Выплата колл-опциона в момент времени T: $X_T = (S_T - K)^+$

Положим $\Delta_t := \mathbb{1}\{S_t > K\}$. ← имеет конечную вариацию!

из-за ⇒ можем применить ф. Ито

При условии, что S_t непрерывна и пересекает конечное число раз $\forall$ кривую вида $C \cdot e^{\mu t}$

Стратегия:

1) В начальный момент времени капитал $V_0 = (S_0 - K)^+ e^{-rT}$

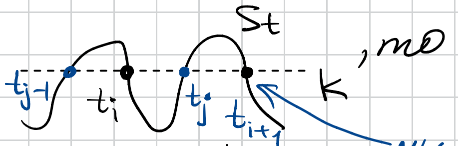
2) В каждый момент времени t поддерживаем позицию из Δ_t единиц рискового актива и $\left[-K \Delta_t e^{-r(T-t)}\right]$ — кол-во безрискового актива

Стоимость портфеля в момент времени t:

$$V_t = S_t \cdot \Delta_t + \left[-K \cdot \Delta_t \cdot e^{-r(T-t)}\right] = S_t \Delta_t - K \Delta_t e^{r(t-T)} = \Delta_t \left(S_t - K e^{r(t-T)}\right)$$

Пусть мы пересекаем уровень K во времена $t_1, t_2, \dots, t_m$ (их конечное число по условию)

– Если



, то у нас минус 1 акция (продаем) плюс $Ke^{r(t-t_j)}$ в момент t_{i+1}

– Если мы в t_j (т.е. $S_t \nearrow$ ------ K), то у нас плюс 1 акция (покупаем) минус $Ke^{-r(T-t)}$

В таком случае стоимость портфеля непрерывна

В конечный момент времени T:

– Если $S_T \leq K$, то рискового актива $\Delta_T = 0$
безрискового 0
$V_T = 0 = (S_T - K)^+$

– Если $S_T > K$, то рискового актива $\Delta_T = 1$
безрискового $-K \cdot \Delta_T e^{-r(T-T)} = -K$
$V_T = S_T - K = (S_T - K)^+$

3. (1 point) Consider a naive hedging strategy for an option with a certain strike price, assuming the underlying asset price follows a Brownian motion. Compute the ratio of the standard deviation of the hedging error to the premium of a zero-strike option. Assume zero interest rates.

□ В наивной стратегии хеджирования $\Delta_t = \mathbb{1}\{S_t \geq K\}$ с $dS_t = dW_t$

С лекции, по формуле Танака для выпуклой функции $f(x) = (x-K)^+$, имеем:

According to Theorem 1 with $X = S$ and $f(x) = (x-K)^+$,

$$(S_t - K)^+ = (S_0 - K)^+ + \int_0^t 1\{S_u \geq K\} dS_u + \frac{1}{2} L_t^K.$$

Thus, the 'naive' hedging strategy gives the next slippage in continuous time when the underlying price is a geometric Brownian motion:

$$Z_T = -\frac{1}{2} L_T^K.$$

где

$$L_t^x = \lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t 1\{x - \varepsilon < X_u < x + \varepsilon\} d[X]_u$$

is the local time of X at point x.

У нас $K=0$, то есть

$$X_T = (S_T - 0)^+ = (S_0 - 0)^+ + \int_0^T \mathbb{1}\{S_u \geq 0\} dS_u + \frac{1}{2} L_T^0 = \{dS_t = dW_t\} =$$

$$= (S_0)^+ + \int_0^T \mathbb{1}\{W_u \geq 0\} dW_u + \frac{1}{2} L_T^0$$

Ошибка хеджирования:

$$Z_T = \left[(S_0)^+ + \int_0^T \mathbb{1}\{W_u \geq 0\} dW_u\right] - (W_T)^+ = -\frac{1}{2} L_T^0 \qquad (1)$$

$$\mathbb{E}[S_T^+] = \mathbb{E}[\max(W_T, 0)] = \frac{1}{\sqrt{2\pi}}$$

С семинара 3 знаем, что $L_T^0 \overset{d}{=} M_T \overset{d}{=} |W_T|$, где $M_t = \sup_{s \leq t} W_s$

$\mathbb{E}[|W_T|] = \sqrt{\frac{2T}{\pi}}$, $\mathbb{E}[|W_T|^2] = T$

$$\Rightarrow \mathbb{D}[L_T^0] = \mathbb{D}[|W_T|] = \mathbb{E}[|W_T|^2] - (\mathbb{E}[|W_T|])^2 = T - \frac{2T}{\pi} = T\left(1 - \frac{2}{\pi}\right)$$

из (1) $\mathbb{D}[Z_T] = \frac{1}{4} \mathbb{D}[L_T^0] = \frac{T}{4} \cdot \left(1 - \frac{2}{\pi}\right) \Rightarrow$

$$\mathrm{std}(Z_T) = \frac{1}{2}\sqrt{T\left(1 - \frac{2}{\pi}\right)}$$

Тогда $k = \frac{std(Z_T)}{\mathbb{E}[S_T^+]} = \frac{\frac{1}{2}\sqrt{T(1-\frac{2}{\pi})}}{\frac{1}{\sqrt{2\pi}}} = \frac{1}{2}\sqrt{T(2\pi-4)} = \sqrt{T(\frac{\pi}{2}-1)}$

Ответ: $k = \sqrt{T(\frac{\pi}{2}-1)}$ $\boxtimes$

Домашка 4 Долгих Елизавета

1. (0.25 point) Derive the expression for $\frac{\partial V}{\partial r}$ for a call option within the Black-Scholes-Merton model. Rho

$$V = C_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2),$$

$$d_1 = \frac{\ln\frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\frac{\partial V}{\partial r} = SN'(d_1)\frac{\partial d_1}{\partial r} - Ke^{-r(T-t)}\left(-(T-t)N(d_2) + N'(d_2)\frac{\partial d_2}{\partial r}\right)$$

$$\frac{\partial d_1}{\partial r} = \frac{T-t}{\sigma\sqrt{T-t}} = \frac{\sqrt{T-t}}{\sigma}; \quad \frac{\partial d_2}{\partial r} = \frac{T-t}{\sigma\sqrt{T-t}} = \frac{\sqrt{T-t}}{\sigma}$$

В первой домашке доказывали равенство

$$N'(d_1) = n(d_1) = \underbrace{n(d_2)}_{=N'(d_2)} \cdot \frac{K}{S}e^{-r(T-t)}$$

$$\Rightarrow \frac{\partial V}{\partial r} = \cancel{S} \cdot N'(d_2)\frac{K}{\cancel{S}}e^{-r(T-t)} \cdot \frac{\sqrt{T-t}}{\sigma} + Ke^{-r(T-t)}(T-t)N(d_2) -$$

$$- Ke^{-r(T-t)}N'(d_2) \cdot \frac{\sqrt{T-t}}{\sigma} =$$

$$= Ke^{-r(T-t)}(T-t)N(d_2)$$

Ответ: $\frac{\partial V}{\partial r} = Ke^{-r(T-t)}(T-t) \cdot N(d_2)$

2. (0.25 point) Derive the expression for $\frac{\partial^2 V}{\partial S \partial \sigma}$ for a call option within the Black-Scholes-Merton model. Vanna

В первой домашке находили $\Delta = \frac{\partial V}{\partial S} = N(d_1)$.

Далее, $\frac{\partial^2 V}{\partial S \partial \sigma} = \frac{\partial \Delta}{\partial \sigma} = N'(d_1)\frac{\partial d_1}{\partial \sigma}$

$$\frac{\partial d_1}{\partial \sigma} = -\frac{\ln\frac{S}{K} + r(T-t)}{\sigma^2\sqrt{T-t}} + \frac{1}{2}\frac{(T-t)}{\sqrt{T-t}} = -\frac{d_1}{\sigma} + \sqrt{T-t}$$

$$\Rightarrow \frac{\partial^2 V}{\partial S \partial \sigma} = N'(d_1) \cdot \left(\sqrt{T-t} - \frac{d_1}{\sigma}\right), \quad \text{где } N'(d_1) = \frac{1}{\sqrt{2\pi}}e^{-\frac{d_1^2}{2}}$$

Ответ: $\frac{\partial^2 V}{\partial S \partial \sigma} = N'(d_1) \cdot \left(\sqrt{T-t} - \frac{d_1}{\sigma}\right) = -N'(d_1)\frac{d_2}{\sigma}$

3. (0.25 point) Derive the expression for $\frac{\partial^2 V}{\partial \sigma^2}$ for a call option within the Black-Scholes-Merton model. Volga

$$\square \quad \vartheta = \frac{\partial V}{\partial \sigma} = S N'(d_1) \cdot \frac{\partial d_1}{\partial \sigma} - K e^{-r(T-t)} N'(d_2) \frac{\partial d_2}{\partial \sigma}$$

$$\frac{\partial d_1}{\partial \sigma} = -\frac{d_1}{\sigma} + \sqrt{T-t}\,; \quad \frac{\partial d_2}{\partial \sigma} = -\frac{d_1}{\sigma} + \sqrt{T-t} - \sqrt{T-t} = -\frac{d_1}{\sigma}$$

$$N'(d_1) = n(d_1) = \underbrace{n(d_2)}_{=N'(d_2)} \cdot \frac{K}{S} e^{-r(T-t)}$$

$$\Rightarrow \vartheta = \frac{\partial V}{\partial \sigma} = S N'(d_1) \cdot \left(\sqrt{T-t} - \frac{d_1}{\sigma}\right) - S N'(d_1) \cdot \left(-\frac{d_1}{\sigma}\right) = S N'(d_1) \sqrt{T-t}$$

$$\begin{cases} \frac{\partial^2 V}{\partial \sigma^2} = \frac{\partial \vartheta}{\partial \sigma} = S\sqrt{T-t} \cdot N''(d_1) \cdot \frac{\partial d_1}{\partial \sigma} \\ N''(d_1) = -d_1 N'(d_1) \end{cases}$$

$$\hookrightarrow \frac{\partial^2 V}{\partial \sigma^2} = -S\sqrt{T-t}\, d_1 \cdot N'(d_1) \cdot \left(\sqrt{T-t} - \frac{d_1}{\sigma}\right)$$

Ответ: $\frac{\partial^2 V}{\partial \sigma^2} = S\sqrt{T-t}\, d_1 \cdot N'(d_1) \left(\frac{d_1}{\sigma} - \sqrt{T-t}\right) = \frac{S\sqrt{T-t}\, N'(d_1)\, d_1 \cdot d_2}{\sigma}$ ⊠

4. (0.25 point) Derive the expression for $\frac{\partial V}{\partial t}$ for a call option within the Black-Scholes-Merton model. Theta

$$\square \quad \begin{cases} \frac{\partial V}{\partial t} = S N'(d_1) \frac{\partial d_1}{\partial t} - K e^{-r(T-t)} \left(r N(d_2) + N'(d_2) \frac{\partial d_2}{\partial t} \right) \\ \frac{\partial d_1}{\partial t} = +\frac{1}{2} \cdot \frac{\ln \frac{S}{K}}{(T-t)^{3/2}} - \left(r + \frac{\sigma^2}{2}\right) \cdot \frac{1}{2} \frac{1}{\sigma \sqrt{T-t}} = \frac{d_1}{2\sqrt{T-t}} - \frac{r + \frac{\sigma^2}{2}}{\sigma\sqrt{T-t}} \\ \frac{\partial d_2}{\partial t} = \frac{\partial d_1}{\partial t} + \frac{1}{2} \frac{\sigma}{\sqrt{T-t}} = \frac{d_1}{2\sqrt{T-t}} - \frac{r}{\sigma\sqrt{T-t}} \\ N'(d_1) = n(d_1) = \underbrace{n(d_2)}_{=N'(d_2)} \cdot \frac{K}{S} e^{-r(T-t)} \end{cases}$$

$$\hookrightarrow \frac{\partial V}{\partial t} = \cancel{S N'(d_1) \left(\frac{d_1}{2\sqrt{T-t}}\right.} - \frac{\cancel{r} + \frac{\sigma^2}{2}}{\sigma\sqrt{T-t}} \Bigg) - r K e^{-r(T-t)} N(d_2) -$$

$$- \cancel{S N'(d_1) \cdot \left(\frac{d_1}{2\sqrt{T-t}}\right.} - \cancel{\frac{r}{\sigma\sqrt{T-t}}} \Bigg) =$$

$$= -S N'(d_1) \frac{\sigma}{2\sqrt{T-t}} - r K e^{-r(T-t)} N(d_2)$$

Ответ: $\frac{\partial V}{\partial t} = -S N'(d_1) \frac{\sigma}{2\sqrt{T-t}} - r K e^{-r(T-t)} N(d_2)$ ⊠

5. (0.5 point) Consider two Brownian motions W_t^1 and W_t^2 such that $[W^1, W^2]_t = \rho t$. Show that $cov(W_t^1, W_t^2) = \rho t$.

□ по формуле Ито для произведения:

$$d(W_t^1 W_t^2) = W_t^1 dW_t^2 + W_t^2 dW_t^1 + d[W^1, W^2]_t$$

$$W_t^1 W_t^2 = \int_0^t W_t^1 dW_t^2 + \int_0^t W_t^2 dW_t^1 + [W^1, W^2]_t$$

$$\mathbb{E}[W_t^1 W_t^2] = 0 + 0 + \mathbb{E}[W^1, W^2]_t$$

$$\Rightarrow \mathbb{E}[W_t^1 W_t^2] = \mathbb{E}[W^1, W^2]_t = \rho t$$

т.к. $\mathbb{E} W_t^1 = \mathbb{E} W_t^2 = 0$

$$\Rightarrow cov(W_t^1, W_t^2) = \mathbb{E}[W_t^1 W_t^2] - \mathbb{E}[W_t^1] \cdot \mathbb{E}[W_t^2] = \rho t$$

т.е. $cov(W_t^1, W_t^2) = \rho t$ ч.т.д. ⊠

6. (0.5 point) Let Y_t follow a Geometric brownian motion, and define $Z_t = \frac{1}{Y_t}$. Find the quadratic covariation $[Y, Z]_t$.

□ пусть $dY_t = \mu Y_t dt + \sigma Y_t dW_t$. применим ф. Ито к $f(y) = \frac{1}{y}$

$$f'(y) = -\frac{1}{y^2}; \quad f''(y) = \frac{2}{y^3}$$

$$\Rightarrow d(Z_t) = df(Y_t) = -\frac{1}{Y_t^2} dY_t + \frac{1}{2} \cdot \frac{2}{Y_t^3} (dY_t)^2 =$$

$$= -\frac{1}{Y_t^2}(\mu Y_t dt + \sigma Y_t dW_t) + \frac{1}{Y_t^3}(\sigma^2 Y_t^2 dt) =$$

$$= (\sigma^2 - \mu) \cdot \frac{1}{Y_t} dt - \sigma \cdot \frac{1}{Y_t} dW_t = (\sigma^2 - \mu) Z_t dt - \sigma Z_t dW_t$$

т.е. $dZ_t = (\sigma^2 - \mu) Z_t dt - \sigma Z_t dW_t$

$$d[Y, Z]_t = dY_t \cdot dZ_t = (\mu Y_t dt + \sigma Y_t dW_t) \cdot ((\sigma^2 - \mu) Z_t dt - \sigma Z_t dW_t) =$$

$$= \sigma Y_t \cdot (-\sigma Z_t) dt = -\sigma^2 dt$$

$$\Rightarrow [Y, Z]_t = -\sigma^2 t$$

Ответ: $[Y, Z]_t = -\sigma^2 t$ ⊠