Homework 1

Deadline: 28 September 2025, 23:30.
All solutions must be in a single PDF file and uploaded to the LMS portal.

1. (0.25 point) Prove that the sum of two functions of bounded first-order variation also has bounded first-order variation.
2. (0.75 point) Prove that the product of two functions of bounded first-order variation also has bounded first-order variation.
3. (1 point) Is it true that the quadratic variation of function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}), & x \in (0,1], \\ 0, & x = 0. \end{cases}$$

   is bounded?
4. (2 points) Verify that the explicit call option price formula

$$C_t = S_t N(d_1) - Ke^{-r(T-t)} N(d_2),$$
$$d_1 = \frac{\ln \frac{S_t}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t}$$

   satisfies the Black–Scholes–Merton equation

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0.$$

Homework 2

Deadline: 5 October 2025, 23:30.
All solutions must be in a single PDF file and uploaded to the LMS portal.

1. (1 point) Find the variance of the sample third-order variation

$$\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^3$$

of Brownian motion on $[0, T]$.

2. (1 point) The Ornstein–Uhlenbeck process is defined by

$$dX_t = -\kappa X_t dt + \sigma dW_t.$$

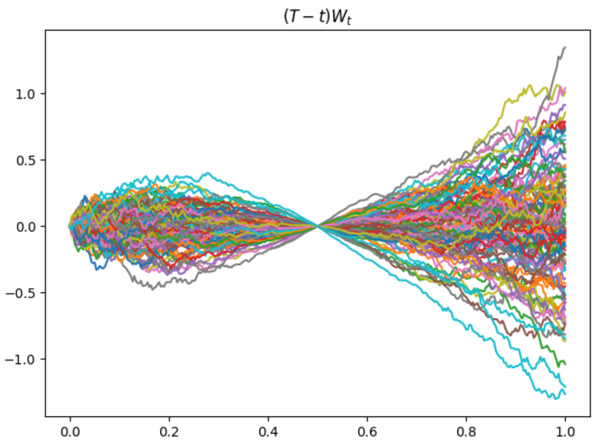
Find the quadratic variation of $\sin(X_t)$.

3. (a) (0.5 point) Show that the process defined as

$$(T - t)W_t$$

is differentiable at point T almost surely.

(b) (0.5 point) Are there other points at which the process is differentiable almost surely?



Homework 3

Deadline: 5 October 2025, 23:45.
All solutions must be in a single PDF file and uploaded to the LMS portal.

1. Consider a structured deposit - a financial product that offers fixed income and additional yield in the form of a call option. The underlying asset of the call option is asset S. The guaranteed income pays an amount S_0 at maturity of the deposit. Assume the funding rate is higher than the model rate.
   (a) (0.5 point) Is it true that the funding component is always positive?
   (b) (0.5 point) Is it true that the funding component is always negative?
2. (1 point) Describe a hedging strategy that replicates the payoff of a call option under a non-zero constant interest rate, for continuous paths crossing the strike price finitely many times.
3. (1 point) Consider a naive hedging strategy for an option with a certain strike price, assuming the underlying asset price follows a Brownian motion. Compute the ratio of the standard deviation of the hedging error to the premium of a zero-strike option. Assume zero interest rates.

Homework 4

Deadline: 19 October 2025, 23:30.
All solutions must be in a single PDF file and uploaded to the LMS portal.

1. (0.25 point) Derive the expression for $\frac{\partial V}{\partial r}$ for a call option within the Black-Scholes-Merton model.
2. (0.25 point) Derive the expression for $\frac{\partial^2 V}{\partial S \partial \sigma}$ for a call option within the Black-Scholes-Merton model.
3. (0.25 point) Derive the expression for $\frac{\partial^2 V}{\partial \sigma^2}$ for a call option within the Black-Scholes-Merton model.
4. (0.25 point) Derive the expression for $\frac{\partial V}{\partial t}$ for a call option within the Black-Scholes-Merton model.
5. (0.5 point) Consider two Brownian motions W_t^1 and W_t^2 such that $[W^1, W^2]_t = \rho t$. Show that $cov(W_t^1, W_t^2) = \rho t$.
6. (0.5 point) Let Y_t follow a Geometric brownian motion, and define $Z_t = \frac{1}{Y_t}$. Find the quadratic covariation $[Y, Z]_t$.

Homework 5

Deadline: 2 November 2025, 23:30.
All solutions must be in a single PDF file and uploaded to the LMS portal.

1. Consider the following estimator of volatility:

$$\sigma_t^\pi = \sqrt{\frac{1}{\tau}\sum_{k=1}^{n}(\Delta \ln S_{t_k})^2}$$

where $\pi = (t_0, \ldots, t_n)$ is a partition of the time segment $[t-\tau, t]$, and the price of the underlying asset follows the process

$$d\ln S_t = b_t dW_t.$$

- (1 point) Derive expressions for $\text{Bias}(\sigma_t^\pi)$ and $\text{Var}(\sigma_t^\pi)$ for the case of piecewise constant volatility, i.e., b_t is constant on each $[t_{k-1}, t_k]$.
- (0.5 point) Analyze how $\text{Bias}(\sigma_t^\pi)$ and $\text{Var}(\sigma_t^\pi)$ change under the following conditions:
  - The partition size $||\pi||$ decreases while τ is held fixed;
  - Time length τ increases while $||\pi||$ is held fixed.

2. Suppose that at the initial time we have estimated $\sigma(t)$ for the model

$$dS_t = rS_t dt + \sigma(t) S_t dW_t.$$

Now consider two option pricing models:

- The modified Black-Scholes-Merton model[1] with the underlying asset process using the estimated $\sigma(t)$;
- The Black-Scholes-Merton model with constant volatility $\sigma = \sqrt{\frac{1}{T}\int_0^T \sigma(u)du}$.

Your tasks are to determine and explain:

- (0.5 point) Will the option's delta computed at the initial time differ between these models?
- (0.5 point) Will it differ at an arbitrary time $t > 0$?
- (0.5 point) Will the option's theta differ between these models?

[1] See Lecture 5, Section 2.

Homework 6

Deadline: 23 November 2025, 23:30.
All solutions must be in a single PDF file and uploaded to the LMS portal.

Consider an Asian call option with payoff at time T and strike K. Assume the underlying asset S_t follows a geometric Brownian motion in risk-neutral measure:

$$dS_t = \sigma(t) S_t d\widetilde{W}_t.$$

where $\sigma(t)$ — determenistic function. Consider the quantity $M_t = \frac{1}{t}\int_0^t S_u du$ and

$$U_t = \frac{tM_t + (T-t)S_t}{T}.$$

1. (0.2 point) Is it true that $U_0 = S_0$ and $U_T = M_T$? Justify your answer.
2. (0.2 point) Can U_t be replicated using a position in asset S over an infinitesimal interval $[t, t+dt]$? If yes, what position is required for replication?
3. (0.3 point) Find the stochastic differential equation for U_t.
4. (0.3 point) Assuming that $S_t/U_t = 1$, find the price of the Asian call option. Compare the obtained price with the price of a vanilla European call option with the same parameters.

Homework 7

Deadline: 30 November 2025, 23:30.
All solutions must be in a single PDF file and uploaded to the LMS portal.

1. (0.25 point) Derive the expression for $\frac{\partial V}{\partial S}$ for a geometric Asian call option within the modified Black-Scholes- Merton model with estimated volatility function from GARCH(1,1).
2. (0.25 point) Derive the expression for $\frac{\partial^2 V}{\partial S^2}$ for a geometric Asian call option within the modified Black-Scholes- Merton model with estimated volatility function from GARCH(1,1).
3. (0.5 point) Consider the instantaneous forward rate $f(t,T) = \lim_{\delta \to 0} F(t,T,T+\delta)$. Find the relationship between the rates $f(t,T_1)$ and $f(t,T_2)$, where $T_1 < T_2$, in the Hull-White model[1] with parameters $\theta(t) = \theta$ and $\sigma(t) = \sigma$.

[1] See Lecture 7 Section 2.

Homework 8

Deadline: 9 December 2025, 23:30.
All solutions must be in a single PDF file and uploaded to the LMS portal.

1. Consider a vanilla call option on an underlying asset S with expiration date T and strike K under Black-Scholes-Merton model.
   (a) (0.5 point) At what price of the underlying asset is the gamma of this option maximized at time t? Justify your answer.
   (b) (0.5 point) Find the value of t at which the gamma reaches its maximum, given that the price of the underlying asset is equal to the option's strike price.

2. (1 point) Let $x \in \mathbb{R}^n$, and consider a function $f(x, z) \in \mathbb{R}$ where $z \in \mathbb{R}^m$. Let $g(x) \in \mathbb{R}^m$ be a vector function of a vector argument and define the composite function $h(x) = f(x, g(x))$, so that $h(x) \in \mathbb{R}$. Using the notation from the lecture, prove that the following holds:

$$\frac{\partial^2 h}{\partial x^2} \neq \frac{\partial^2 f}{\partial x^2}.$$

   What additional terms must be included on the right-hand side to obtain equality? Justify your answer.