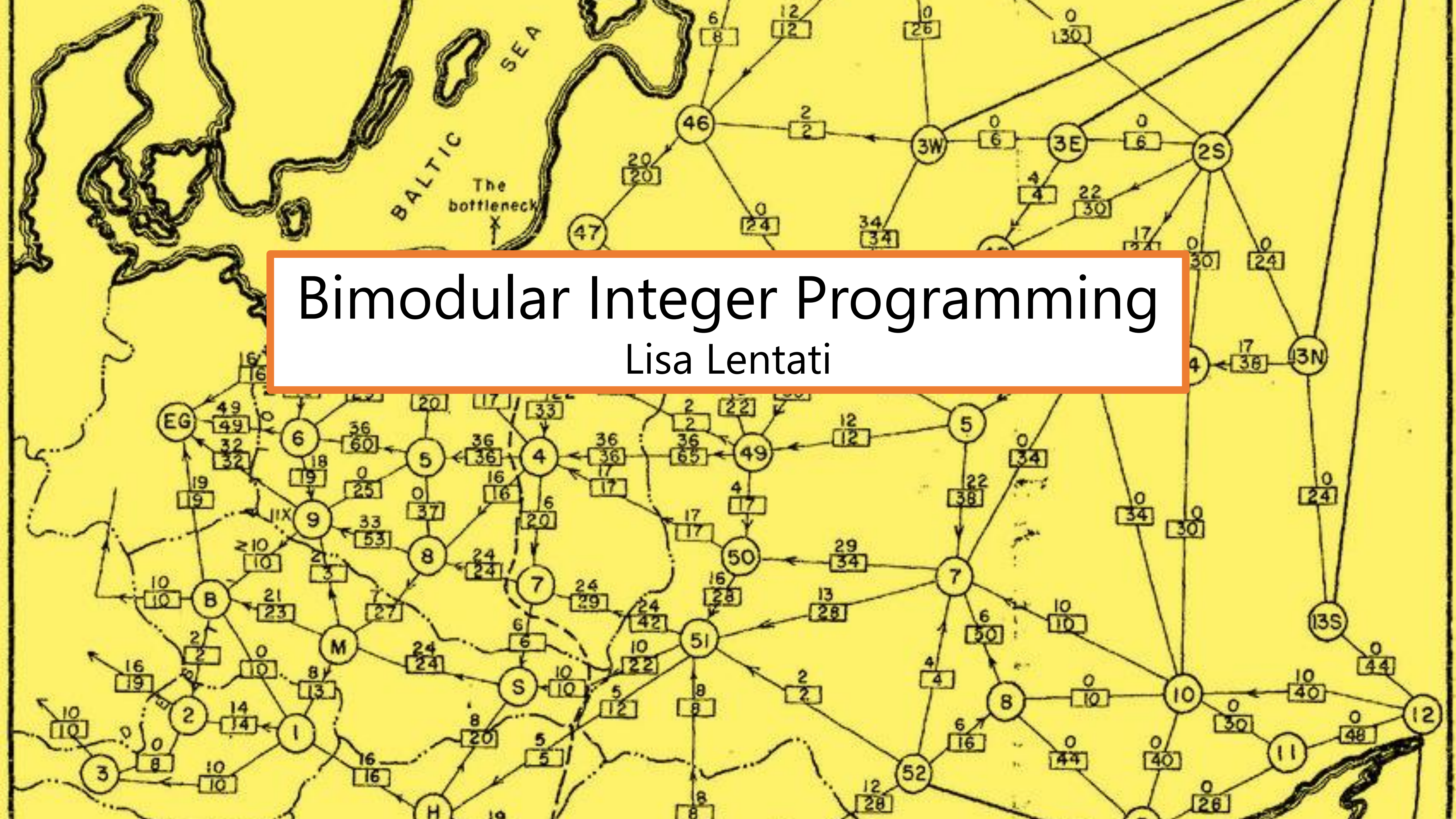


Bimodular Integer Programming

Lisa Lentati

Topic

- *Dissertation*: Solving bimodular integer programs in strongly polynomial time

  Supervisor: Dr. Giacomo Zambelli

- *Paper*: A strongly polynomial algorithm for bimodular integer linear programming (2017)

  Authors: S. Artmann, R. Weismantel, and R. Zenklusen

Structure

- Introduction to integer programming
- Complexity of integer programming and other associated problems
- Bimodular integer programming
- Outline of an algorithm to solve bimodular integer programming

Integer linear programming

- An integer linear program (IP) is an optimization problem of the form,

$$\max\{c^{\top}x : Ax \leq b, x \in \mathbb{Z}^n\}$$

  where $c \in \mathbb{Z}^n, b \in \mathbb{Z}^m$ and $A \in \mathbb{Z}^{m \times n}$, rank($A$)= n.

Fruit	Cost	Weight	Vitamin C	Potassium
1 Banana	0.30€	120g	10mg	420mg
1 Orange	0.50€	130g	70mg	240mg

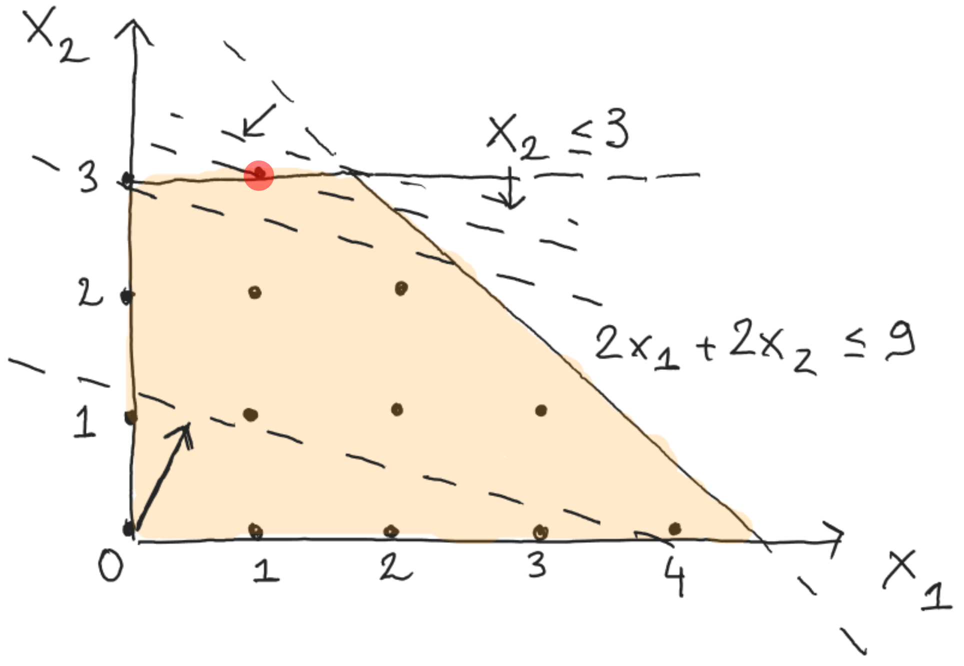
$$
\begin{array}{lrrrrl}
\min & 0.3x_B & +\,0.5x_O & & & \text{Cost} \\
 & 10x_B & +70x_O & \geq & 65 & \text{Vitamin C} \\
 & 420x_B & +240x_O & \geq & 3000 & \text{Potassium} \\
 & 120x_B & +130x_O & \leq & 2000 & \text{Weight} \\
 & & x_O & \geq & 0 & \\
 & x_B & & \geq & 0 & \\
 & & & & & x_B, x_O \in \mathbb{Z}
\end{array}
$$

- $A = \begin{pmatrix} -10 & -70 \\ -420 & -240 \\ 120 & 130 \\ 0 & -1 \\ -1 & 0 \end{pmatrix}, b = \begin{pmatrix} -65 \\ -300 \\ 2000 \\ 0 \\ 0 \end{pmatrix}, x = \begin{pmatrix} x_B \\ x_o \end{pmatrix}, c = \begin{pmatrix} -0.3 \\ -0.5 \end{pmatrix}$

Graphical representation of an IP

- $$\begin{array}{lllll} \max & x_1 & +3x_2 & & \\ & 2x_1 & +2x_2 & \leq & 9 \\ & & x_2 & \leq & 3 \\ & & -x_2 & \leq & 0 \\ & -x_1 & & \leq & 0 \end{array}$$

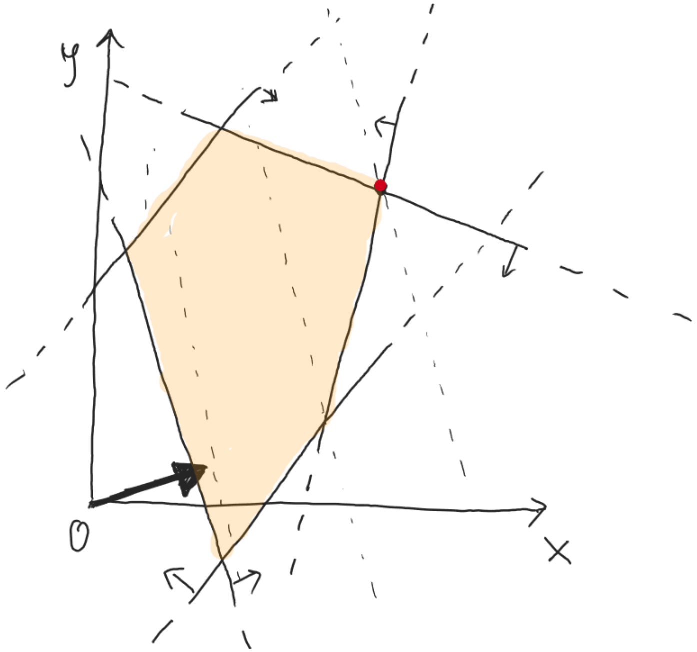
$$x_1, x_2 \in \mathbb{Z}$$



Complexity of

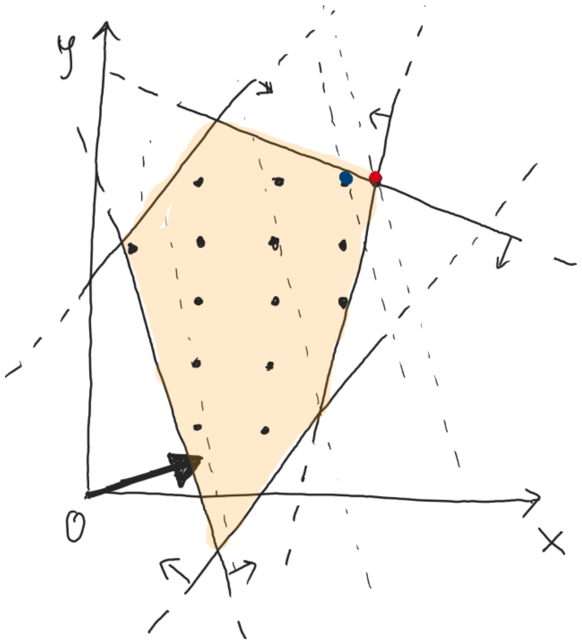
Linear programming

- $\max\{c^\top x : Ax \leq b, x \in \mathbb{R}^n\}$



Integer programming

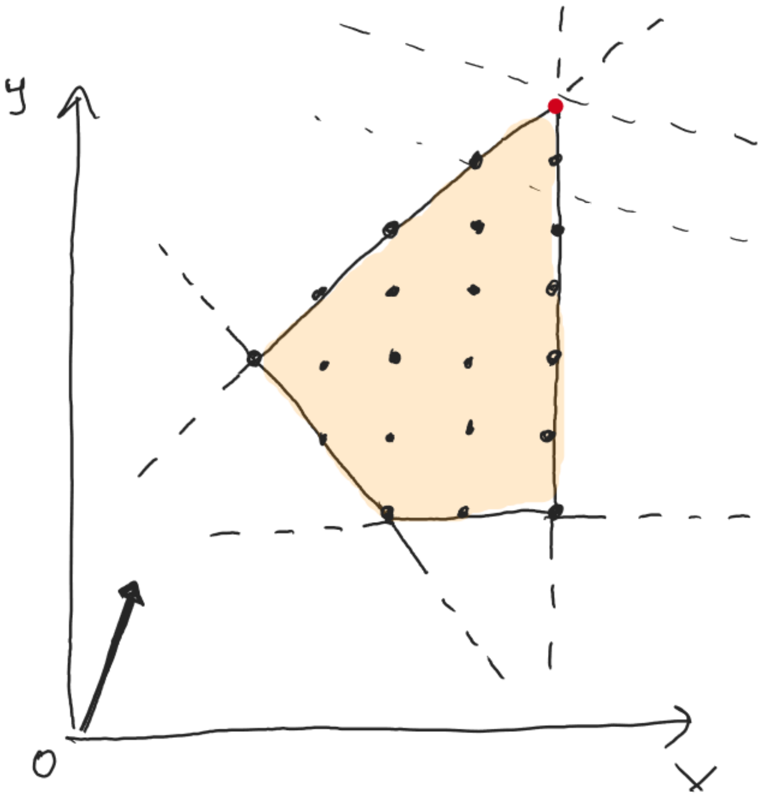
- $\max\{c^\top x : Ax \leq b, x \in \mathbb{Z}^n\}$



Totally unimodular integer programs

- <u>Totally unimodular</u> (TU) matrix: all sub-determinants have values 0, ±1

- <u>Unimodular</u> matrix: the $n \times n$ sub-determinants have values 0, ±1



Can we extend the result further?

- Yes!

- $n \times n$ sub-determinants have values 0, ±1, ±2 .

- A bimodular integer program (BIP) is an optimization problem of the form,
$$\max\{c^\top x : Ax \leq b, x \in \mathbb{Z}^n\}$$
where $c \in \mathbb{Z}^n, b \in \mathbb{Z}^m$ and $A \in \mathbb{Z}^{m\times n}$, rank($A$)= n and the $n \times n$ sub-determinants of A are at most 2 in absolute value.

Outline of the algorithm

1. Transform BIP to a TU problem with an additional parity constraint

$$\max\{c^\top x : Ax \le 0, x \in \mathbb{Z}^n_{\ge 0}, \textstyle\sum_{i\in S} x_i \text{ odd}\} \qquad \text{(CPTU)}$$

2. Iteratively decompose CPTU into smaller CPTU problems
   - The number of subproblems doesn’t grow exponentially

3. Solving the base problems

Thank you!

- Slides and dissertation: https://github.com/LisaLentati/bip

Appendix: from unimodular to TU

$$A = \left.\overbrace{\begin{pmatrix} \begin{bmatrix} \quad Q \quad \end{bmatrix} \\ \\ \\ \end{pmatrix}}^{n}\right\}m$$

$\mathrm{rank}(A) = n$

$\det(Q) = \pm 1$

$$\bar{A} = AQ^{-1} = \left.\overbrace{\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \\ & & & & \\ & & & & \boxed{B} \end{pmatrix}}^{n}\right\}m$$

- $\max\{c^\top x : Ax \leq b,\ x \in \mathbb{Z}^n\}$, with A unimodular
- $x = Q^{-1}\mathrm{y}$
- $\max\{c^\top Q^{-1}\mathrm{y} : \bar{A}\mathrm{y} \leq bQ^{-1},\ \mathrm{y} \in \mathbb{Z}^n\}$, with $\bar{A}$ totally unimodular