DEPARTMENT OF INFORMATICS

TECHNISCHE UNIVERSITÄT MÜNCHEN

Bachelor's Thesis in Informatics: Games Engineering

Tsunami simulation in the ExaHyPE framework

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Tsunamisimulation innerhalb des ExaHyPE-Frameworks

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I confirm that this bachelor's thesis in informatics: games engineering is my own work and I have documented all sources and material used.

Munich, 18.12.2017 Lisa Scheller

Acknowledgments

Abstract

List of Abbreviations

ODE ordinary differential equation

PDE partial differential equation

Nomenclature

$f'(q)$ Jacobi matrix of $f(q)$

q vector of unknowns

q_x partial derivative $\frac{\delta q}{\delta x}$

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2.1.1 Definition

Partial differential equations (PDEs) are used to describe a multitude of problems. In contrast to ordinary differential equations (ODEs) PDEs contain partial derivatives, whereas ODEs only contain derivatives of one variable. There are a multitude of phenomenons that can be described by PDEs, ranging from fluid mechanics and traffic simulation over thermodynamic problems to quantum mechanics. In this bachelor's thesis we restrict ourselves to a certain subclass of PDEs, the so-called hyperbolic PDEs. The core equations of this thesis, the shallow water equations, which will be introduced in detail in the next section,as well as the standard advection equation, are hyperbolic PDEs.

Initially let

$$q_t + Aq_x = 0 \tag{2.1}$$

be a one-dimensional, homogenous, first-order system of PDEs. q is a vector from $\mathbb{R}^m$ that contains the unknowns of the equations and A a $m \times m$ matrix. If this equation shall describe a system of hyperbolic PDEs, the matrix A has to meet certain conditions. Firstly, the matrix has to be diagonizable [Lev04], thus it must be possible to describe the matrix in the following way:

$$A = S * D * S^{-1} \tag{2.2}$$

, where D is a matrix that is zero for all entries not on the diagonal, and S is a transformation matrix. Secondly, the eigenvalues of A have to be real. This leads to the following definition:

Definition 2.1.1. "A linear system of the form

$$q_t + Aq_x = 0$$

is called hyperbolic if the $m \times m$ matrix A is diagonizable with real eigenvalues."[Lev04]

Other types of PDEs are elliptic or parabolic PDEs.

2.1.2 Classification

There are several sub-groups of hyperbolic PDEs, in regard to the eigenvalues of A as well as in regard to the structure of the equations.

Eigenvalues

Since a symmetric matrix is always diagonizable with real eigenvalues, all symmetric matrices can be used to describe a hyperbolic problem [Lev04]. These equations are called symmetric hyperbolic.
If the matrix has distinct eigenvalues and is diagonizable, the system of PDEs is called strictly hyperbolic. [Lev04]
In contrast, a matrix that is not diagonizable but has real eigenvalues is called weakly hyperbolic.[Lev04]

Structure of equations

In the simplest case of a hyperbolic PDE A is not a matrix, but a constant scalar. Equation (2.1) simplifies to

$$q_t + aq_x = 0 \tag{2.3}$$

and a has to be real for the system to be hyperbolic.
If the scalar a or the matrix A are time-dependent or depend on another value (e.g. $a(x,t)$, $A(x,p)$), they have to fulfill the criteria for a system of hyperbolic PDEs for each value of t (or in the examples also of x and p).

Conservation laws A special group of hyperbolic PDEs are called conservation laws. In the simplest case (1 dimension) they have the following structure:

$$q(x,t)_t + f(q(x,t))_x = 0 \tag{2.4}$$

or in the quasilinear form

$$q_t + f'(q)q_x = 0 \tag{2.5}$$

[Lev04] $f(q)$ can be linear as well as non-linear in regard to q.
However, it is often necessary to use the integral formulation instead of (2.4). This is caused by the formation of discontinuities, which are often seen in the solutions of hyperbolic PDEs.

The integral formulation The main benefit of using the integral formulation is that its validity is not impacted by discontinuities. Since handling discontinuities is an important part of solving hyperbolic PDEs, it is vital to always be certain if the differential formulation can be used or not.

The integral formulation of a hyperbolic PDE looks as follows:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)\,\mathrm{d}x = F(x_2,t) - F(x_1,t) \tag{2.6}$$

$f(x,t)$ represent fluxes going either into or out from the element. The differential formlation ((2.4)), which is used mostly, can be derived from the integral formulation. [Lev04]

2.1.3 Examples

The complexity of hyperbolic PDEs ranges from relatively simple, with known exact solutions, to very complex with no known analytic solutions. One example of a very basic hyperbolic PDE is the one-dimensional advection equation with constant speed:

$$q_t + uq_x = 0 \tag{2.7}$$

, where u is the constant velocity. This equation is solved by

$$q(x,t) = q_0(x - ut) \tag{2.8}$$

, where $q_0(x) = q(x,0)$ are the known initial conditions.

Even though this equation is very simple, many problems can be put into a formulation resembling it. So it plays a role even at solving more difficult problems.[Lev04]
Another, more complex example are the Maxwell equations

$$\begin{aligned} E_t &= \nabla \times B \\ B_t &= -\nabla \times E \end{aligned} \tag{2.9}$$

([Stu13]), which are used to describe electromagnetic fields. Most problems which are solved by waves can be described by hyperbolic PDEs. Of course this also includes the description of water waves. Because of this the shallow water equations, which will be covered in detail in the next chapter, are also hyperbolic.

2.1.4 Numerical difficulties

The main problem with solving hyperbolic PDEs is that the solutions (or even the initial conditions) often contain discontinuities, which are difficult to solve. Especially methods that assume the solution is sufficiently smooth often fail at providing the correct solution. Another problem is that the solutions often can be described as several waves of different types. The composition of types depends on the problem and has to be computed first. If the waves interact which each other, it complicates the solution further.

2.2 The shallow water equations

This thesis focuses on solving the shallow water equations in one dimension, which describe water height and momentum of a fluid. Their main characteristic is that the vertical scale is much smaller than the horizontal scale. This is true for ocean waves and even tsunami waves, which are also described by the shallow water equations.

2.2.1 Equations in one dimension

The one-dimensional shallow water equations consist of two equations. The first one is:

$$h_t + (uh)_x = 0 \tag{2.10}$$

and the second one, which is a little bit more complicated, is:

$$(hu)_t + (hu^2 + \frac{1}{2}gh^2)_x = 0 \tag{2.11}$$

If $q(\vec{x}, t) = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} h \\ hu \end{pmatrix}$ is introduced and $f(q) = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{bmatrix} = \begin{bmatrix} q_2 \\ (q_2)^2/q_1 + \frac{1}{2}gq_1 \end{bmatrix}$ is defined, then the equations can be written as

$$q_t + f(q)_x = 0 \tag{2.12}$$

For smoot solutions, the formulation

$$q_t + f'(q)q_x = 0 \tag{2.13}$$

is also valid.[Lev04]

2.2.2 Properties

The eigenvalues and eigenvectors of $f'(q)$ are important properties of the equation. They have to be known to be able to compute a numeric or exact solution to any problem. Fortunately, they are quite simple to compute. They are:

$$\lambda_1 = u - \sqrt{gh} \tag{2.14}$$

and

$$\lambda_2 = u + \sqrt{gh} \tag{2.15}$$

This leads to the eigenvectors

$$\vec{v_1} = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix} \tag{2.16}$$

and

$$\vec{v_2} = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix} \tag{2.17}$$

[Lev04].
Because the eigenvalues (and thus the eigenvectors) depend on h and u, which are the unknowns of the equation system and can be different for every point in the domain, every of these points has different eigenvalues. It is not sufficient to compute one set of eigenvalues at the beginning and then never change it again, like with the constant velocity advection equation. The eigenvalues have to be computed for each point or cell in the domain and for each timestep.
If a comparison is made between the advection equation and the shallow equations (both for a hump of water in gaussian form), the main difference is that using the advection equation the wave doesn't change shape and just travels around, whereas using the shallow water equations a change of shape can be observed.

2.2.3 Example problems

There are to types of problems that will be approximated by applying the shallow water equations. The first one has continuous starting conditions and is the already descriped gaussian water hump. The second one is the dam break problem, a so called Riemann problem with initial velocities zero and a difference in inital height.

Riemann problems

Definition 2.2.1. A problem with inital conditions of the form

$$q(x,0) = \begin{cases} q_l & x \leq 0 \\ q_r & 0 \leq x \end{cases}$$

is called a Riemann problem[Lev04].

The main characteristic of a Riemann problem is the discontinuity. To solve such a problem analytically, it has to be fragmented into several states. Two of them are the outer left and right states, where the solution takes the value q_l or q_r respectively. The middle states depend on the exact composition of the problem, the possibilities for the cases viewed here are either 2 shock waves, 2 rarefaction waves or one of each. In every case a middle value q_m has to be computed. The exact way to solve a Riemann problem as well as an explanation of the different types of waves will be given in chapter 3. The analytical solution for a dam-break problem will also be computed.

2.2.4 Two-dimensional equations

In two dimension the shallow water equations can be written as

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + (hu^2 + \frac{1}{2}gh^2)_x + (huv)_y &= 0 \\ (hv)_t + (huv)_x + (hv^2 + \frac{1}{2}gh^2)_y &= 0 \end{aligned} \tag{2.18}$$

[Lev04] This differs from the one-dimensional form by the addition of a additional dimension y. Also the velocity is now described by a vector, $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$. The equations look very similar to and can be solved using similar techniques as for the one-dimensioal case, however for the sake of complexity this thesis will only cover the one-dimensional equations.

3 Methods for solving hyperbolic partial differential equations

There are a multitude of numerical methods to solve PDEs. However, some of them are more suited to the task of solving hyperbolic PDEs than others. As pointed out earlier especially discontinuities break some methods. So it's important to choose a method that handles these discontinuities well if a hyperbolic PDE has to be solved numerically.

3.1 Finite Differences

3.1.1 General description

The finite differences method is one of the better known numerical schemes. It approximates derivatives by replacing them with a difference quotient of the form

$$\frac{u(x+h) - u(x)}{h} \tag{3.1}$$

and solving the resulting equation for $u(x+h)$. The result can then be integrated by using an integration scheme such as explicit Euler etc. The domain is divided into intervals of length h, for which the equation is evaluated. Higher accuracy can be achieved by choosing a smaller h or using a higher order integration scheme such as Runge-Kutta.

3.1.2 Advantages and disadvantages

Advantages The method is easy to understand and since only one value is computed per interval it can also be very efficient[JSH08]. Also it is possible to modify the scheme to get high order accuracy.

Disadvantages However since the finite difference becomes quite nonsensical at a discontinuity it is not well suited for solving hyperbolic PDEs. Also it doesn't allow more complex geometry because it only supports equidistant intervals [JSH08]. This becomes even more of a problem for solving higher dimensional problems.

3.1.3 Example

3.2 Finite Volumes

3.2.1 General description

3.2.2 Advantages and disadvantages

3.2.3 The CFL condition

3.3 Finite Elements

3.3.1 General description

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3.4 Discontinuous Galerkin methods

[JSH08]

3.4.1 General description

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