

Sallen-Key Low-pass Filter

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March 9, 2013, rev. June 10, 2013

In 1955 R.P. Sallen and E.L. Key introduced a set of circuits for implementing second-order low-pass, high-pass, and band-pass filter sections. These popular circuits are known as the Sallen-Key filters. The circuit shown in Figure 1 is known as the unity gain low-pass filter and is capable of implementing a pair of poles, real or complex, anywhere on the s-plane. All four components are generally of non-equal values.

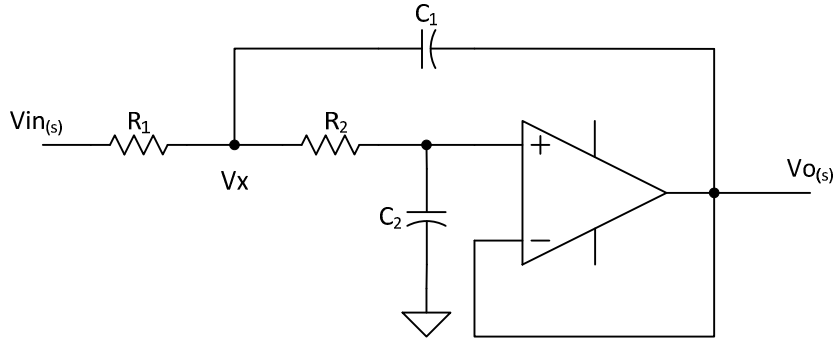


Figure 1: Sallen-Key unity gain low-pass filter circuit

Analysis

The s subscript for the voltages in the following is omitted for convenience. Using Kirchoff's current law we can write that the current through R_1 is equal to the sum of the currents through R_2 and C_1 . The node at the junction of R_1 , R_2 , and C_1 is labeled V_x for temporary convenience. We capitalize on the observation that the voltage at the non-inverting input of the op-amp is equal to V_o .

$$\frac{V_{in} - V_x}{R_1} = \frac{V_x - V_o}{R_2} + \frac{V_x - V_o}{\frac{1}{C_1 s}} \quad (1)$$

Next we multiply through by $R_1 R_2$ to clear the fractions.

$$R_2(V_{in} - V_x) = R_1(V_x - V_o) + R_1 R_2 C_1 s(V_x - V_o) \quad (2)$$

We now use the fact that the non-inverting input to the op-amp is V_x through the voltage divider formed by R_2 and C_2 .

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$$V_{in}^+ = V_o = \frac{V_x}{R_2 C_2 s + 1} \quad (3)$$

Solving for V_x in terms of V_o .

$$V_x = V_o (R_2 C_2 s + 1) \quad (4)$$

Substituting the equation for V_x .

$$R_2 (V_{in} - V_o (R_2 C_2 s + 1)) = R_1 (V_o (R_2 C_2 s + 1) - V_o) + R_1 R_2 C_1 s (V_o (R_2 C_2 s + 1) - V_o) \quad (5)$$

Simplifying.

$$V_o (R_1 R_2 C_1 C_2 s^2 + R_1 C_2 s + R_2 C_2 s + 1) = V_{in} \quad (6)$$

Now we can develop the complete transfer function.

$$\frac{V_o(s)}{V_{in}(s)} = \frac{1}{R_1 R_2 C_1 C_2 s^2 + (R_1 + R_2) C_2 s + 1} \quad (7)$$

Converting to standard form.

$$\frac{V_o(s)}{V_{in}(s)} = \frac{\frac{1}{R_1 R_2 C_1 C_2}}{s^2 + \frac{R_1 + R_2}{R_1 R_2 C_1} s + \frac{1}{R_1 R_2 C_1 C_2}} \quad (8)$$

The standard second-order form is given by Equation 9.

$$\frac{V_o(s)}{V_{in}(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (9)$$

We equate the terms of Equation 8 to Equation 9. The natural frequency is given by Equation 10.

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$$\omega_n = \sqrt{\frac{1}{R_1 R_2 C_1 C_2}} \quad (10)$$

We now can solve for the damping ratio as follows.

$$2\zeta\omega_n = \frac{R_1 + R_2}{R_1 R_2 C_1} \quad (11)$$

Substituting ω_n .

$$\frac{2\zeta}{\sqrt{R_1 R_2 C_1 C_2}} = \frac{R_1 + R_2}{R_1 R_2 C_1} \quad (12)$$

Solving for ζ .

$$\zeta = \frac{R_1 + R_2}{2} \sqrt{\frac{C_2}{R_1 R_2 C_1}} \quad (13)$$

With Equations 10 and 13 we can analyze an existing circuit to determine its natural frequency and damping ratio.

Design

Design is the process of inverting the analytical equations to derive the design equations. In the analytical equations we have four knowns and two unknowns and a unique solution. When those equations are inverted we will have two knowns and four unknowns. The solution set will be infinite. We need to determine a practical solution as not all possible solutions work well.

One popular approach to reducing the number of unknowns is to let $R_1 = R_2 = R$, some arbitrary chosen resistor. Then the solution for the required capacitors, C_1 and C_2 , is determined. The logic of this approach is sound but it is very likely that the computed C values will be non-standard. Capacitors are commonly available in fairly coarse steps and two or more standard capacitors will have to be placed in some series or parallel combination to achieve the odd value. Resistors are available in fine steps so a close standard value is probably available.

Given the preceding issue the obvious thing to do might be to let $C_1 = C_2 = C$ and then solve for the two resistor values. Then we can choose a standard capacitor value with confidence that there will be a close standard resistor value available for whatever we calculate. Again, the logic of this approach is sound but there is a hidden flaw that will cause this concept to fail in all cases

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where ζ is less than 1.0 – which is where most filters are designed. This type of flawed concept can occur in many aspects of engineering. It is important to always develop the complete math and not trust intuition. The complete math might confirm intuition or might reveal a major flaw. In the following development the flaw remains hidden in the clutter of equation terms until the last step where it literally jumps off the page.

In engineering math there is an old saying, “Ratios are your friend.” Solving for ratios rather than the values comprising the ratios reduces the number of unknowns thus making the process easier. The following development reduces the four unknown components to a set of two ratios, (C_2/C_1) and (R_2/R_1) . The math seems to explode and that is typical but the end result is simple and very revealing.

Since we prefer not to work with square roots we will square the analytical equation for ζ .

$$\zeta^2 = \frac{(R_1 + R_2)^2}{4R_1R_2} \left(\frac{C_2}{C_1} \right) \quad (14)$$

Expanding and working towards ratio form.

$$\zeta^2 = \frac{R_1^2 + 2R_1R_2 + R_2^2}{4R_1R_2} \left(\frac{C_2}{C_1} \right) = \left(\frac{R_1}{4R_2} + \frac{1}{2} + \frac{R_2}{4R_1} \right) \left(\frac{C_2}{C_1} \right) = \left(\frac{1}{4(R_2/R_1)} + \frac{1}{2} + \frac{(R_2/R_1)}{4} \right) \left(\frac{C_2}{C_1} \right) \quad (15)$$

$$\left(\frac{R_2}{R_1} \right) \zeta^2 = \left(\frac{1}{4} + \frac{(R_2/R_1)}{2} + \frac{(R_2/R_1)^2}{4} \right) \left(\frac{C_2}{C_1} \right) \quad (16)$$

This can be re-arranged into a quadratic equation in (R_2/R_1) .

$$\frac{(C_2/C_1)}{4} \left(\frac{R_2}{R_1} \right)^2 + \left(\frac{(C_2/C_1)}{2} - \zeta^2 \right) \left(\frac{R_2}{R_1} \right) + \frac{(C_2/C_1)}{4} = 0 \quad (17)$$

The solution is.

$$\left(\frac{R_2}{R_1} \right) = \frac{\zeta^2 - \frac{(C_2/C_1)}{2} + \sqrt{\left(\frac{(C_2/C_1)}{2} - \zeta^2 \right)^2 - \frac{(C_2/C_1)^2}{4}}}{\frac{(C_2/C_1)}{4}} \quad (18)$$

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Simplifying.

$$\left(\frac{R_2}{R_1}\right) = \frac{2\zeta^2 - \left(\frac{C_2}{C_1}\right) + 2\zeta\sqrt{\zeta^2 - \left(\frac{C_2}{C_1}\right)}}{\left(\frac{C_2}{C_1}\right)} \quad (19)$$

The flaw of the concept of arbitrarily letting $C_1 = C_2 = C$ is now revealed. If ζ is less than 1.0 (and for the majority of filter designs it is) then (R_2/R_1) will have both real and imaginary parts – not practical. However, Equation 19 provides us with what we need to properly design the filter.

Step 1. The first step is to determine practical standard values for the capacitors. The concept is based on the design note, *Choosing Resistors and Capacitors for Op-Amp Active Filters* written by this author and available on the EE431 class website. Rather than trying to directly determine a C_1 or C_2 it is better to determine a nominal capacitance that is the geometric mean of C_1 and C_2 . From the design note we would compute the following.

$$C = \sqrt{C_1 C_2} = \frac{4 \times 10^{-7}}{\sqrt{F_n}} \quad (20)$$

The capacitances are in farads and F_n is the natural frequency in Hz ($\omega_n/2\pi$). This is the rough geometric center of practical capacitor values to use for the particular natural frequency. In the case for $\zeta < 1$, C_1 is going to be higher than this mean and C_2 is going to be lower. That reverses if $\zeta > 1$.

Step 2. Compute $C_1 = C/\zeta$ and round C_1 up to a convenient standard value. Although you can round down it is often better to round up as that will help in obtaining more practical values for C_2 and the resistors. You can round up to the nearest larger value but feel free to round up perhaps six standard values – i.e. you can round C_1 up to about three times the calculated value. This gives you plenty of flexibility and is not the absolute limit. You can use less or more but understand that that may drive other components towards non-practical values. Round up regardless of the value for ζ . If ζ is much greater than 1.0 it may be necessary to round up a factor of ten or more to obtain practical component values.

Step 3. Using the selected standard value for C_1 , compute $C_2 < \zeta^2 C_1$ and round C_2 down at least one standard value and perhaps as many as six standard values – i.e. to about one third of the computation at most. You can go lower but understand that the resistor values may become too large to be practical.

Step 4. At this point both C_1 and C_2 are now known standard values. Compute (C_2/C_1) and substitute into Equation 19 to determine the required ratio of the resistors, (R_2/R_1) .

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Step 5. From Equation 10 we can write.

$$(R_1 R_2) = \frac{1}{(2\pi F_n)^2 C_1 C_2} \quad (21)$$

At this time we do not yet know the individual values of R_1 and R_2 but we do know the product.

Step 6. We can now write the following equation.

$$\left(\frac{R_2}{R_1}\right) R_1^2 = (R_1 R_2) \quad (22)$$

Note that the parentheses indicate values we have recently calculated (Steps 4 and 5) in the aggregate. The only unknown is the actual value of R_1 . We can write:

$$R_1 = \sqrt{\frac{(R_1 R_2)}{(R_2/R_1)}} \quad (23)$$

We round the computed value of R_1 to the nearest standard resistor value.

Step 7. Using the aggregate value of $(R_1 R_2)$ and the actual selected value of R_1 from the previous step we can now compute R_2 .

$$R_2 = \frac{(R_1 R_2)}{R_1} \quad (24)$$

We round the computed value of R_2 to the nearest standard resistor value.

Step 8. The last step is to check the design. Substitute the four component values into the analytical equations and see if the result is in agreement with the specification. If not then an error has been made. Another point is that depending on your choice for C_1 it is possible that C_2 might be a very small value such as in the low hundreds or less picofarads. In high frequency applications this is normal and we are careful to minimize stray capacitance. Natural stray capacitance in the ten or so picofarad range will add to C_2 thus causing error when we are working a low frequencies and are not so careful. Another possibility is that one or more of the resistors might be well into the megohm range. We try to avoid using multi-megohm resistors if practical. If the natural frequency is less than about ten Hertz then high resistance values are to be expected. If either results is out of a desirable range then you might want to consider redoing the

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design with C_1 chosen to be a higher value as that will cause C_2 to be larger and the resistors to be smaller.

Example: Design a Sallen-Key low-pass filter with a natural frequency of 1 kHz and a Q of 2.

Solution:

We note that $\zeta = 1/2Q$ so $\zeta = 0.25$.

Step 1: $C = 4E-7/\text{sqrt}(1,000) = 12.6 \text{ nF}$.

Step 2: $C_{\text{nominal}} = 12.6 \text{ nF} / 0.25 = 50.6 \text{ nF}$. **Choose $C_1 = 68 \text{ nF}$** .

Step 3: $C_{2\text{max}} = 0.25^2 * 68 \text{ nF} = 4.3 \text{ nF}$. **Choose $C_2 = 3.3 \text{ nF}$** .

Step 4: $(C_2/C_1) = (3.3 \text{ nF}/68 \text{ nF}) = 0.0485$. Using Equation 19, $(R_2/R_1) = 2.79$.

Step 5: Using equation 21, $(R_1 R_2) = 1.13E08 \text{ ohms}^2$.

Step 6: $R_1 = \text{sqrt}(1.13E08 / 2.79) = 6,364$. **Choose $R_1 = 6.2K$** .

Step 7: $R_2 = 1.13E08/6.2K = 18.2K$. **Choose $R_2 = 18K$** .

Step 8: Analyze the results using Equations 10 and 13. We have $F_n = 1006 \text{ Hz}$ and $Q = 1.98$. Both results are in agreement with the specification.

When the amplifier gain is not exactly unity

The unity gain situation is preferred but there may be cases where the gain is either higher or lower than unity. At high frequencies the op-amp may be replaced with a common-collector or common-drain amplifier which naturally has a voltage gain somewhat less than unity. In other cases the filter is asked to also provide a voltage gain for scaling purposes. How does an amplifier gain other than unity affect the analysis and design? The derivation is a good exercise for the student. The starting point is to modify Equation 1 to include the amplifier gain, G. G can be greater or less than unity.

$$\frac{V_{in} - V_x}{R_1} = \frac{V_x - V_o/G}{R_2} + \frac{V_x - V_o}{\frac{1}{C_1 s}} \quad (25)$$

The following transfer function results after performing the same steps as previous.

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$$\frac{V_{o(s)}}{V_{in(s)}} = \frac{\frac{G}{R_1 R_2 C_1 C_2}}{s^2 + \frac{R_1 + R_2}{R_1 R_2 C_1} s + \frac{G}{R_1 R_2 C_1 C_2}} \quad (26)$$

The analytical results are then computed as follows.

$$\omega_n = \sqrt{\frac{G}{R_1 R_2 C_1 C_2}} \quad (27)$$

$$\zeta = \frac{R_1 + R_2}{2} \sqrt{\frac{C_2}{R_1 R_2 C_1 G}} \quad (28)$$

Note that an increase in gain raises the natural frequency and lowers the damping. The analytical results indicate that if C_2 were scaled by G in comparison to the unity gain case then there would be no change in the natural frequency and damping. The process for design is then the same as if the gain were unity – only the actual value of C_2 should be scaled by the gain, G .

As an example suppose the raw maximum value of C_2 computed to be 13.5 nF in a system with $G = 4$. For the unity gain case you would round that down to the standard capacitor values of 12, or maybe 10 nF or perhaps 8.2 nF. However, the capacitance scaled by 4 needs to be a standard value. So, you would round C_2 down to 11.75, 9.75, or 8.25 nF respectively in the above design process so that the actual value for C_2 would be either 47, 39, or 33 nF.

Figure 2 shows a typically case where the circuit performs two functions at the same time – filtering and scaling.

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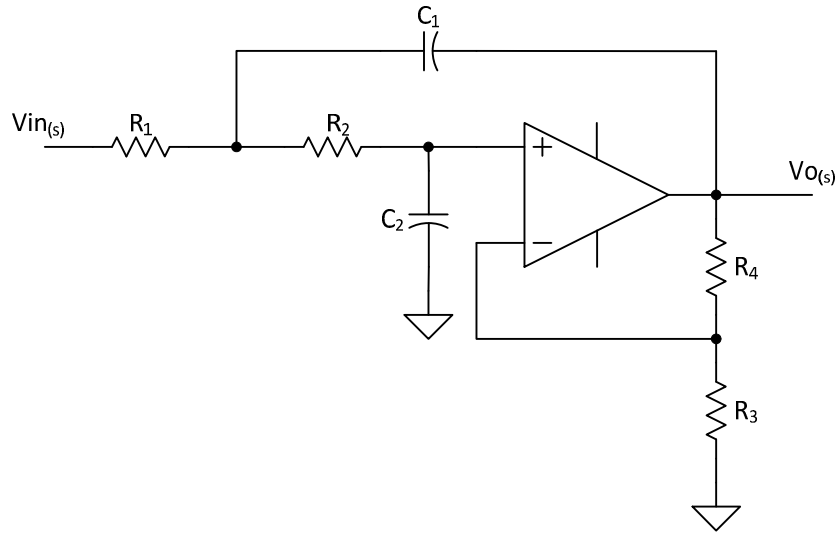


Figure 2: Sallen-Key circuit for simultaneous filtering and scaling

For more information see the following web link.

http://en.wikipedia.org/wiki/Sallen%E2%80%93Key_topology