DDA/STA 4001 | Stochastic Processes | Fall 2018

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Homework 5

Due: October 9, 2018

1. Let $X = \{X_n : n = 0, 1, 2, \ldots\}$ be a DTMC on state space $\{1, 2, 3\}$ with transition probability matrix

$$P = \begin{pmatrix} .25 & .5 & .25 \\ .5 & 0 & .5 \\ .25 & .25 & .5 \end{pmatrix}.$$

Assume that

$$C(1) = \$ - 5, \quad C(2) = \$1, \quad \text{and } C(3) = \$10.$$

   (a) Compute $\mathbb{E}[\sum_{n=0}^{10} C(X_n) | X_0 = 1]$. For this part and other parts, you may use **Matlab**.

   (b) Use Monte Carlo method to estimate $\mathbb{E}[\sum_{n=0}^{10} C(X_n) | X_0 = 1]$. How many episodes are needed in order for the half width of the CI is within \$.1? Give the estimate of your CI.

   (c) Assume $\beta = .9$. Compute $\mathbb{E}[\sum_{n=0}^{10} \beta^n C(X_n) | X_0 = 1]$. ($\beta$ is known as the discount factor.)

   (d) Assume $\beta = .9$. Compute $v(1) = \mathbb{E}[\sum_{n=0}^{\infty} \beta^n C(X_n) | X_0 = 1]$.

   (e) Do value iteration 10 times starting with $v^0 = C$. List $v^1, \ldots v^{10}$. How large k should be in order for $\left| v^k(1) - v(1) \right| \leq .01$?

2. For Problem 4 in Assignment 4,

   (a) Assume $X_0 = 2$. Find the probability that the DTMC first reaches 0 before reaching 4.

   (b) Find $T_{1,3}$, the expected number of timeslots needed for the DTMC to first reach state 3 when initially starting from state 1.

   (c) Find $T_{2,2}$, the expected number of timeslots needed for the DTMC to first return to state 2 when initially starting from state 2.

3. Let $X = \{X_n : n = 0, 1, \ldots, \}$ be a DTMC on state space S. Define $Y_n = (X_n, X_{n+1})$. Prove that $Y = \{Y_n : n = 0, 1, 2, \ldots\}$ is a DTMC. Specify its state space and the transition matrix.

4. Let $X = \{X_n : n = 0, 1, \ldots, \}$ be a DTMC on state space S. Define $Y_n = X_{2n}$. Prove that $Y = \{Y_n : n = 0, 1, 2, ...\}$ is a DTMC. Specify its state space and the transition matrix.

5. Consider a jewelry store that only sells diamond rings and operates as follows. Each week, the store is open from Monday-Friday. The weekly (5-day) demand is random, and has distribution

$$D = \begin{cases} 0, & \text{w.p. } 1/6 \\ 1, & \text{w.p. } 1/6 \\ 2, & \text{w.p. } 1/6 \\ 3, & \text{w.p. } 1/6 \\ 4, & \text{w.p. } 1/6 \\ 5, & \text{w.p. } 1/6. \end{cases}$$

Assume that each ring sells for \$100, and any rings unsold by the end of Friday require cleaning on Saturday, which costs \$10 per ring. After a week's worth of sales, the store owner reviews inventory on Saturday morning, and decides how many rings to order. The ring supplier offers two shipping options: standard or express shipping. Standard shipping costs \$15 per ring, and the order arrives on the following Friday evening (a week after it is placed) after the store closes. Express shipping costs \$35 per ring, but the order arrives on the evening of the next day (Sunday).

Consider the following ordering policy: each Saturday morning, the store owner looks at the inventory and sees x rings. She then orders $(3-x)^+$ rings via standard shipping, and then places an express order to ensure she starts out on Monday with 5 rings (if she has more than 5 rings on Saturday morning, no order is placed). Let X_n be the number of rings in inventory on the morning of the nth Saturday, $n = 0, 1, 2, 3 \ldots$

   a. Prove that $\{X_n\}$ is a DTMC.
   b. Write down the state space and transition matrix of this DTMC.
   c. Calculate (Matlab ok) the expected profit in 10 weeks of the store starting with 0 ring.

6. Consider the same setup as in the previous problem, except that the express shipping option is now replaced by a "rustic" shipping option. Bearded millenials now transport the rings in person for a fraction of the cost (\$5 per ring), but also take 3 weeks to deliver the ring, i.e. an order placed on the nth Saturday morning will arrive on the Friday before the $(n+3)$rd Saturday morning. Let $R_n^{(1)}$ be the number of rings to be delivered via rustic shipping one week from the nth Saturday morning. Similarly, let $R_n^{(2)}$ be the number of rings to be delivered two weeks from the nth Saturday morning.

Consider the following ordering policy: on the morning of the nth Saturday, the store owner orders $(5 - X_n)^+$ rings via rustic shipping. She also orders $(5 - X_n - R_n^{(1)})^+$ rings via standard shipping.

a. Model the system as a DTMC (it is no longer enough to keep track of just X_n). What is the state space? Write down the transition probability matrix **or** the transition diagram (whichever you prefer).

c. Calculate (Matlab ok) the expected profit in 10 weeks of the store starting with 0 ring.

7. Let $X = \{X_n : n = 0, 1, \ldots\}$ be a DTMC with state space S and transition matrix P. Recall that by definition, for any nonnegative integer n, any $i_0, \ldots, i_{n-1}, i, j \in S$, the following Markov property holds:

$$\mathbb{P}\{X_{n+1} = j | X_0 = i_0, X_1 = i_1, X_{n-1} = i_{n-1}, X_n = i\} = P_{ij}. \tag{1}$$

(a) Using (1) to prove that

$$\begin{aligned} &\mathbb{P}\{X_{n+2} = k, X_{n+1} = j | X_0 = i_0, X_1 = i_1, X_{n-1} = i_{n-1}, X_n = i\} \\ &\quad = \mathbb{P}\{X_{n+2} = k, X_{n+1} = j | X_n = i\} \end{aligned} \tag{2}$$

for any nonnegative integer n, any $i_0, \ldots, i_{n-1}, i, j, k \in S$.

(b) Give an expression of the right side of (2) in terms of the transition matrix P.