Quadratic isoparametric element programming for a two-dimensional elastic problem

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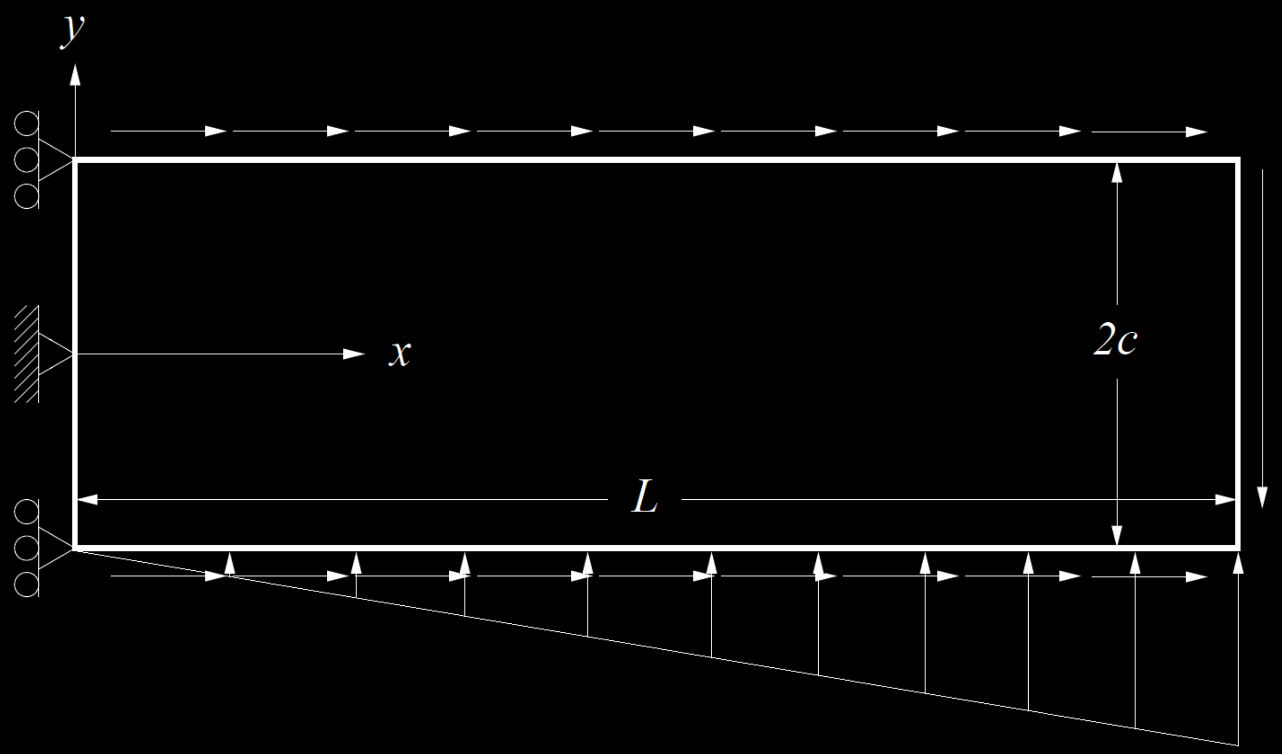
Problem Statement

- to analyze the displacement and stress of cantilever beam structure under the action of multiple loads

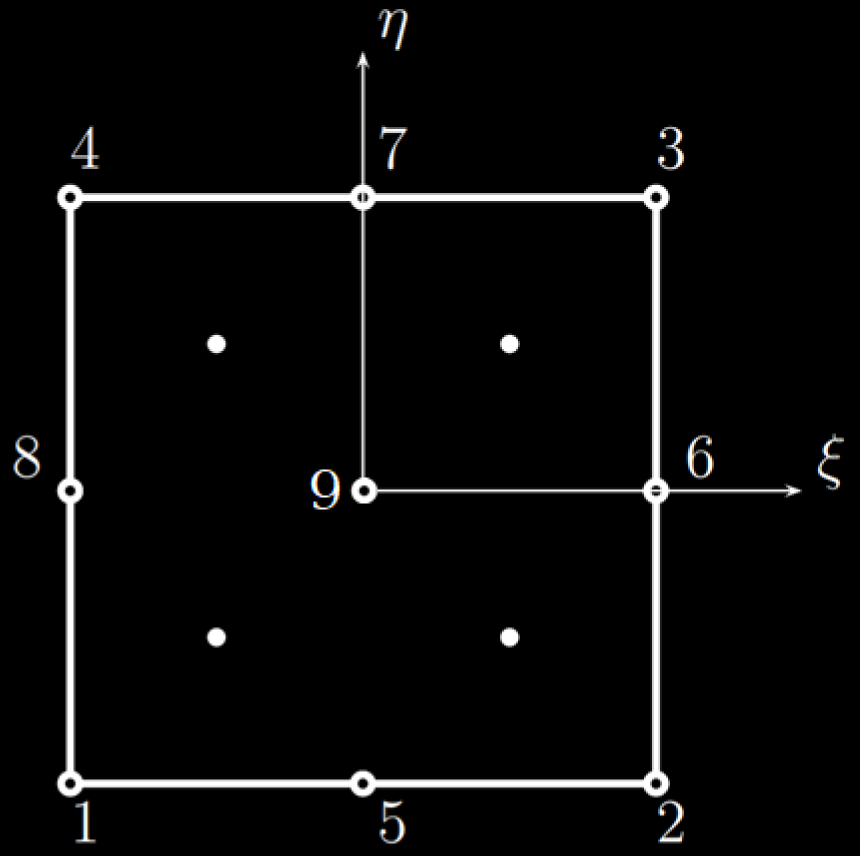


Element type

- Quadrilateral nine-node isoparametric element
- Shape functions

$$N_1(\xi,\eta) = 0.25\xi\eta(\xi-1)(\eta-1)$$
$$N_2(\xi,\eta) = 0.25\xi\eta(\xi+1)(\eta-1)$$
$$N_3(\xi,\eta) = 0.25\xi\eta(\xi+1)(\eta+1)$$
$$N_4(\xi,\eta) = 0.25\xi\eta(\xi-1)(\eta+1)$$
$$N_5(\xi,\eta) = 0.5\eta(1-\xi^2)(1-\eta)$$

$$N_6(\xi,\eta) = 0.5\xi(1+\xi)(1-\eta^2)$$
$$N_7(\xi,\eta) = 0.5\eta(1-\xi^2)(1+\eta)$$
$$N_8(\xi,\eta) = -0.5\xi(1-\xi)(1-\eta^2)$$
$$N_9(\xi,\eta) = (1-\xi^2)(1-\eta^2)$$



Calculation

- For each element, we have following controlling equations
  - $\boldsymbol{\sigma} = D\boldsymbol{\varepsilon} = DB\boldsymbol{u^e}$
  - $\boldsymbol{\varepsilon} = B\boldsymbol{u^e}$ (when using Gauss interpolation)
  - $\boldsymbol{k} = \int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{B}t|\boldsymbol{J}|\mathrm{d}\xi\mathrm{d}\eta$, or $\boldsymbol{K}^e = \sum_{i=1}^{3}\sum_{j=1}^{3} \boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{D}\boldsymbol{B}t|\boldsymbol{J}|w_i w_j$
  - $\boldsymbol{R}^e = \sum_e \boldsymbol{N}^{\mathrm{T}}\boldsymbol{P}^e + \int_{\Omega^e}\mathbf{N}^{\mathrm{T}}\boldsymbol{f}t|\boldsymbol{J}|\mathrm{d}\xi\mathrm{d}\eta + \int_{\boldsymbol{S}^e}\mathbf{N}^{\mathrm{T}}\bar{\boldsymbol{f}}t\,\mathrm{d}s$
- For the solution of whole structure, we need to solve

$$\boldsymbol{K}\boldsymbol{u^e} = \boldsymbol{R}$$

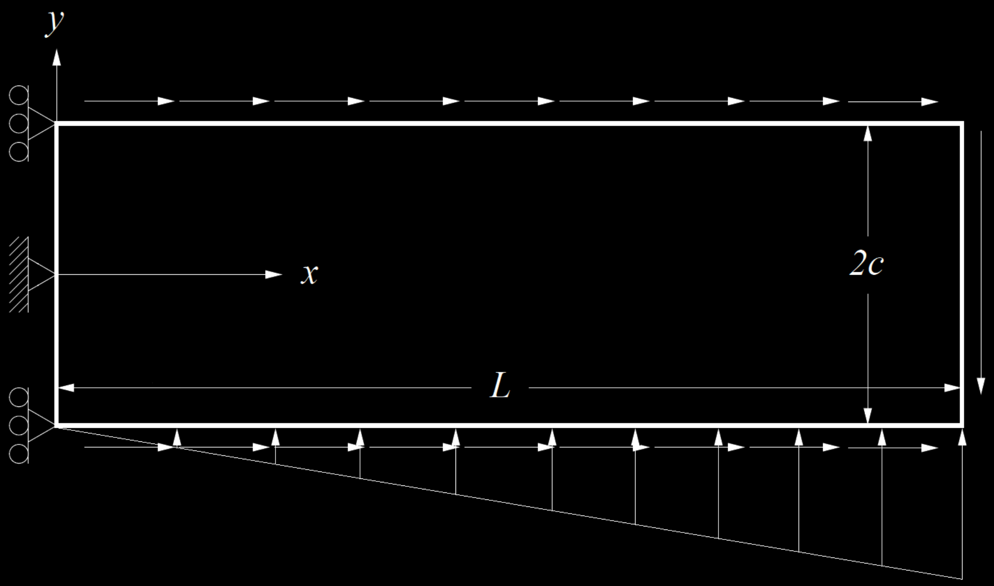
where

$$\boldsymbol{K} = \sum_e \boldsymbol{C}_e^{\mathrm{T}}\boldsymbol{k}\boldsymbol{C}_e, \quad \boldsymbol{R} = \sum_e \boldsymbol{C}_e^{\mathrm{T}}\boldsymbol{R}^e$$

Stress boundaries

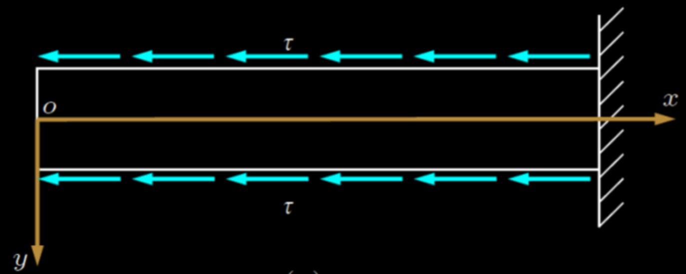
- According to the basic theory of elasticity, complex stress conditions of primary problem can be obtained by the superposition of the following simple stress conditions

(a) The upper and lower boundaries are subjected to uniformly distributed symmetric shear forces
(b) The upper boundary is subjected to gradient distribution pressure
(c) The right boundary is subjected to concentrated force

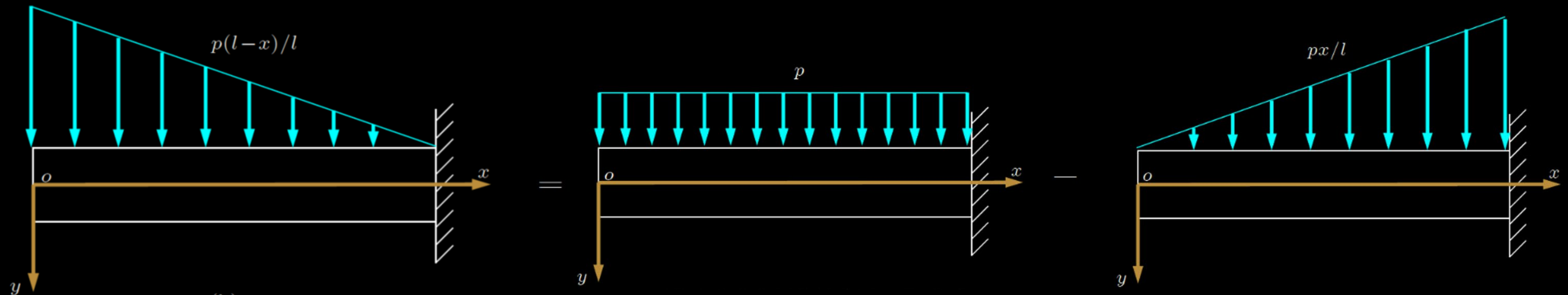


Superposition principle

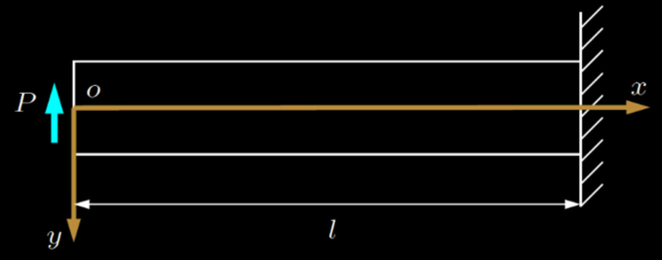
The coordinate frame of the above calculation results is different from that original problem. Therefore, it is necessary to do the transformation “ $x \rightarrow l - x$; $\tau(x, y) \rightarrow -\tau(x, y)$ ” when calculating the theoretical solution of the original problem.



(a)



(b)



(c)

About stress boundaries

- Denote $f_i(x, y)$、$g_i(x, y)$、$h_i(x, y)$ stand for σ_{11}、σ_{22}、σ_{12}（σ_{21}）respectively, the analytical solution can be obtained by Maple as follow

（a）$f3(x, y) := \frac{s}{c}x$; $g3(x, y) := 0$; $h3(x, y) := -\frac{s}{c}y$

（b）$f7(x, y) := -\frac{3px^2y}{4c^3} + \frac{3p\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right)}{4c^3} + \frac{x^3py}{4c^3l} - x\left(\frac{py^3}{2c^3l} - \frac{3py}{10cl}\right)$;

$g7(x, y) := -\frac{p\left(1 - \frac{3y}{2c} + \frac{y^3}{2c^3}\right)}{2} - x\left(-\frac{py^3}{4c^3l} + \frac{3py}{4cl} - \frac{p}{2l}\right)$;

$h7(x, y) := \frac{3p\left(-c^2 + y^2\right)x}{4c^3} + \frac{x^2\left(-\frac{3py^2}{4c^3l} + \frac{3p}{4cl}\right)}{2} + \frac{py^4}{8c^3l} - \frac{3py^2}{20cl} + \frac{cp}{40l}$

（c）$f4(x, y) := \frac{Pxy}{\frac{2}{3}c^3}$; $g4(x, y) := 0$; $h4(x, y) := \frac{P\left(c^2 - y^2\right)}{\frac{4}{3}c^3}$

Parameters	Meaning
L	Length of beam
c	The distance from the outside edge of the beam to the neutral axis
P	The amount of concentrated force on the right edge of the beam (direction negative along the Y-axis)
p	The peak size of the triangular load on the lower edge of the beam (the direction is positive along the Y-axis)
s	The magnitude of the shear force on the upper/lower edge of the beam (direction is positive along the X-axis)

Programming

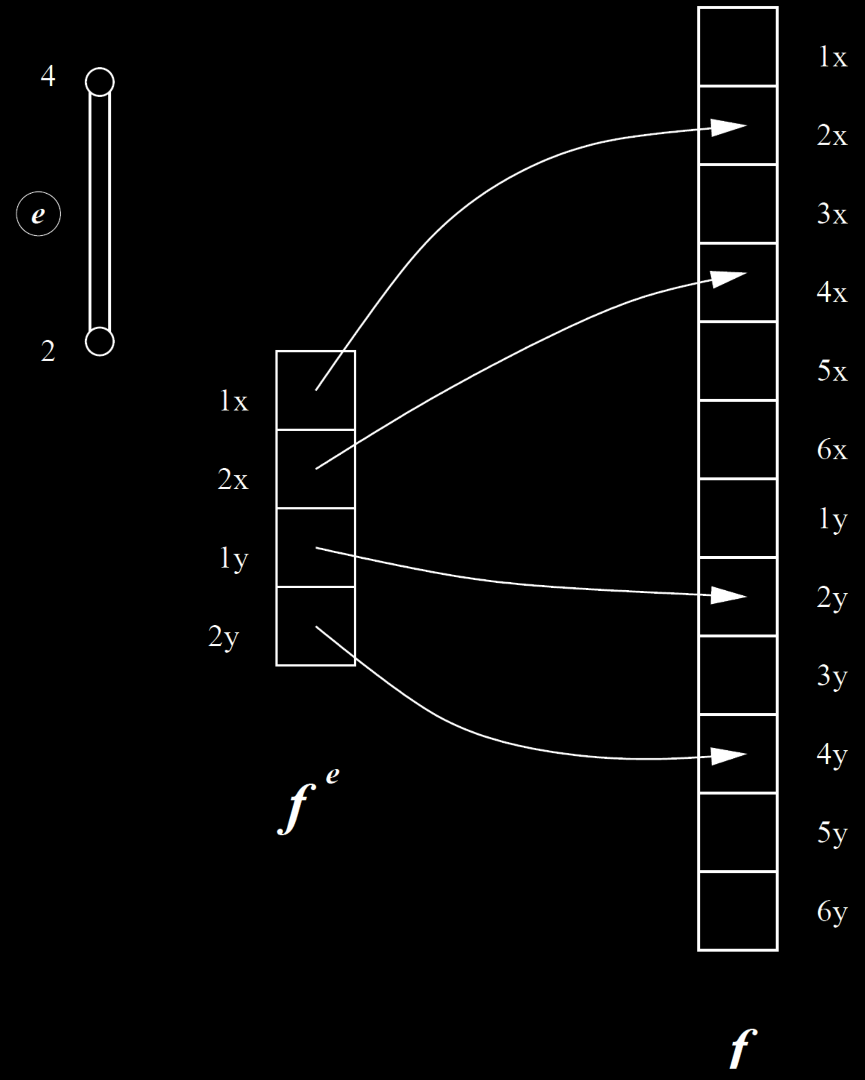
- The code was written by Jack Chessa (back to then he studied at Purdue University). My work is as follow.

  ---fix certain bugs
  ---add the boundary of the top edge and the bottom edge
  ---stress boundaries are given based on elasticity solution of beam
  ---consider the influence of the beam weight, however may lead to distortion of the left boundary mesh because the stress boundaries are determined with no consideration of gravity (improvement still needed)
  ---based on the above, when applying gravity, you need to first reset the displacement boundary from few points to full points on the left edge given the structure is shallow cantilever beam (line 189-190 in 'beam.m’)

- Modified code can be found at https://github.com/Liskelleo.
- Repository: FEM-Programing-of-Shallow-Cantilever-Beam.

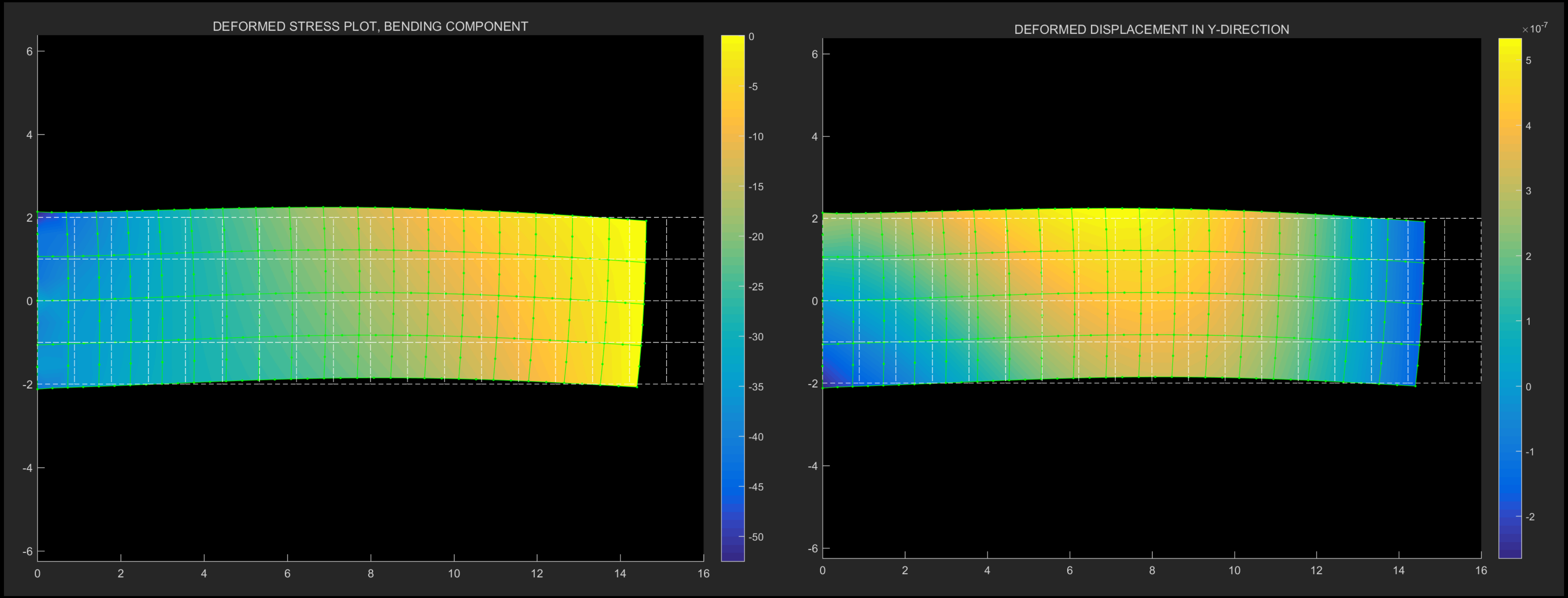
Code efficiency

- It is worth mentioning that Jack Chessa uses Dof index mode to store node displacement, and pays great attention to code efficiency.
- Because MATLAB is an interpreted programming language, Jack Chessa avoided the use of “for” loop nesting in the process of coding, which greatly improves the execution time of the code.
- See reference [5] for the relevant content.



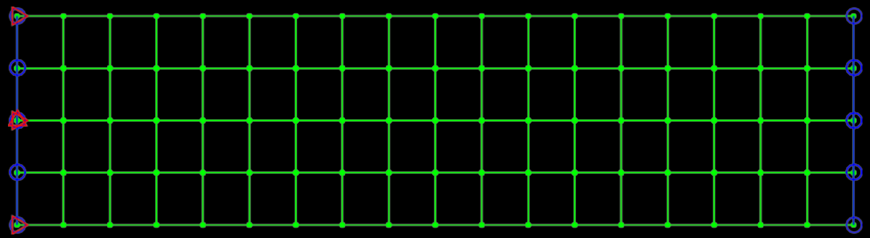
Example 1

Plane stress problem, cantilever shallow beam, length 16, height 2c=4, apply uniform shear force s=-5 on the upper and lower boundary, apply positive gradient triangle load (peak size p = 3) on the lower boundary, apply concentrated force P = 4.5 on the right boundary, regardless of the beam's own weight.

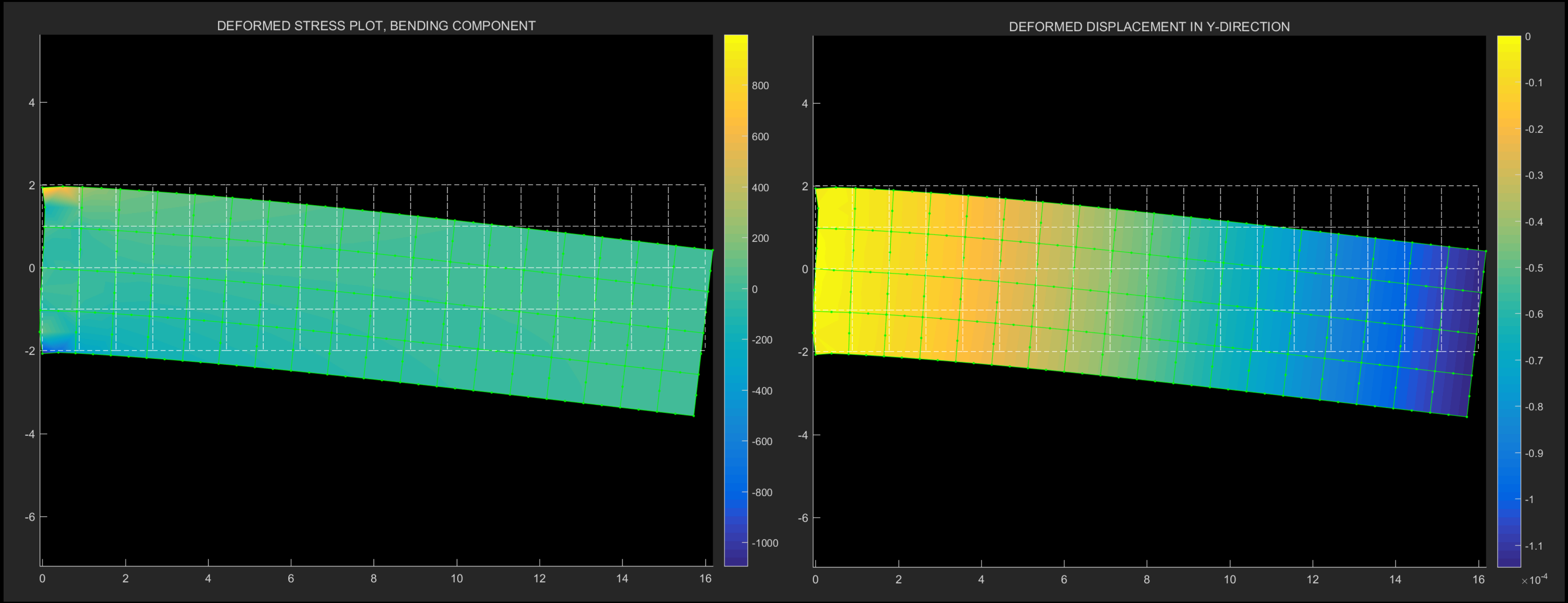


Example 2

The geometric model and the stress situation are the same as example 1, but the beam's self-weight G=1 is taken into account.



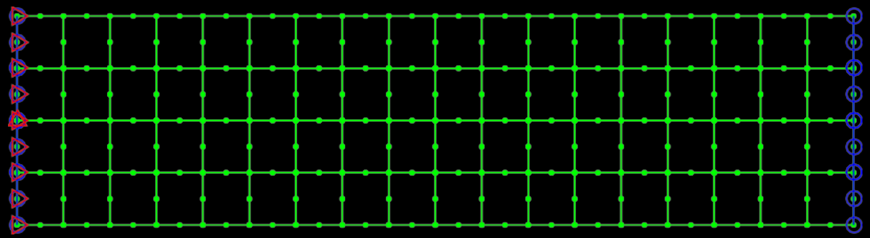
Case 1: relax the constrain of the fix end

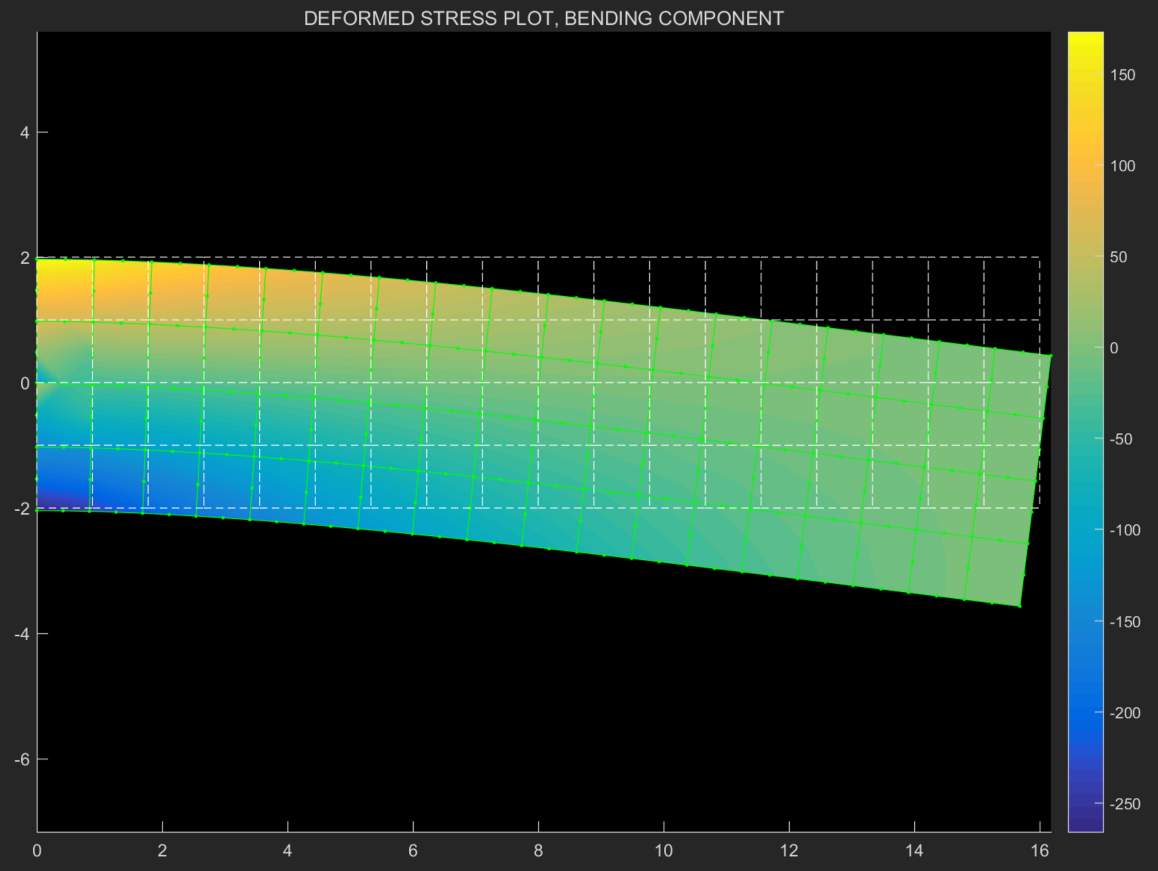


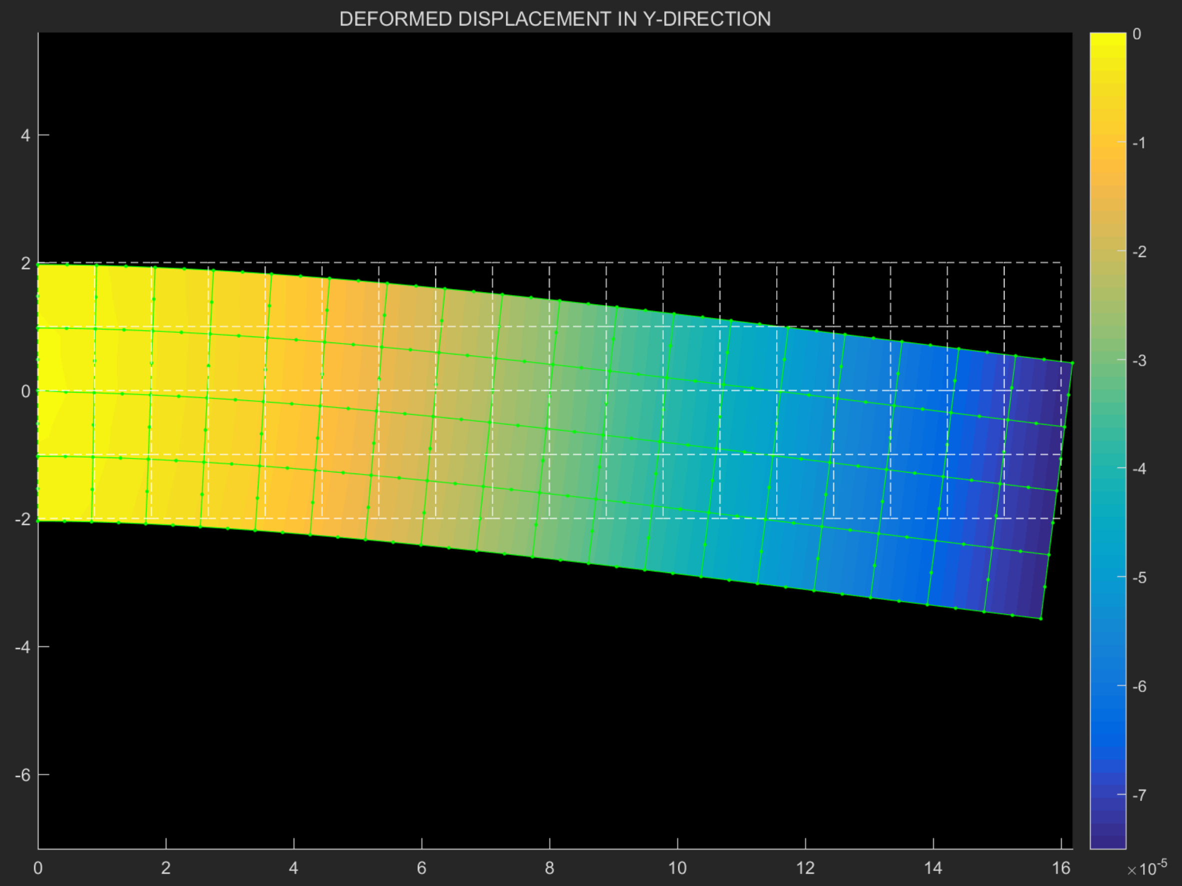
Example 2

The geometric model and the stress situation are the same as example 1, but the beam's self-weight G=1 is taken into account.

Case 2: do not relax the constrain of the fix end







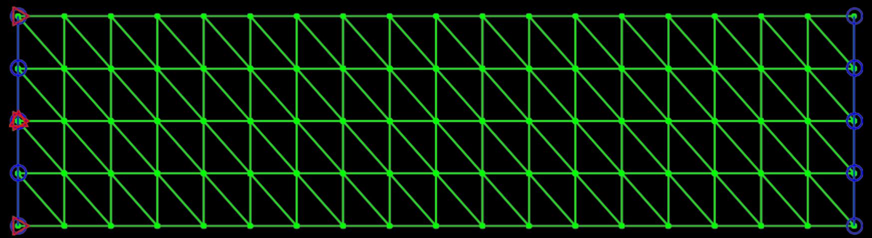
Conclusion 1

- Under the premise of considering the beam body weight, if the displacement boundary of the fixed support is relaxed, the grid distortion at the fixed support end will be caused.
- Compared with the calculation results of the two cases, it can be found that the numerical solution of stress and transverse displacement is nearly 4 times larger and nearly 1 order of magnitude larger when the constraint is relaxed than when the constraint is not relaxed, which is inconsistent with the actual results.

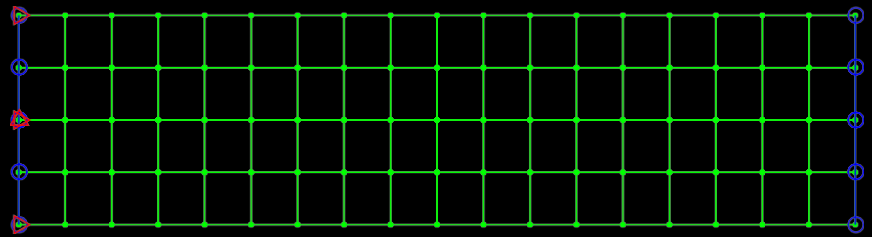
Example 3

The geometric model and the force situation are the same as example 1, but the isoparametric element (triangular three nodes, quadrilateral four nodes) is used to calculate respectively.

The calculation results are compared with example 1 as follows:



(Triangular three nodes, T3-type)

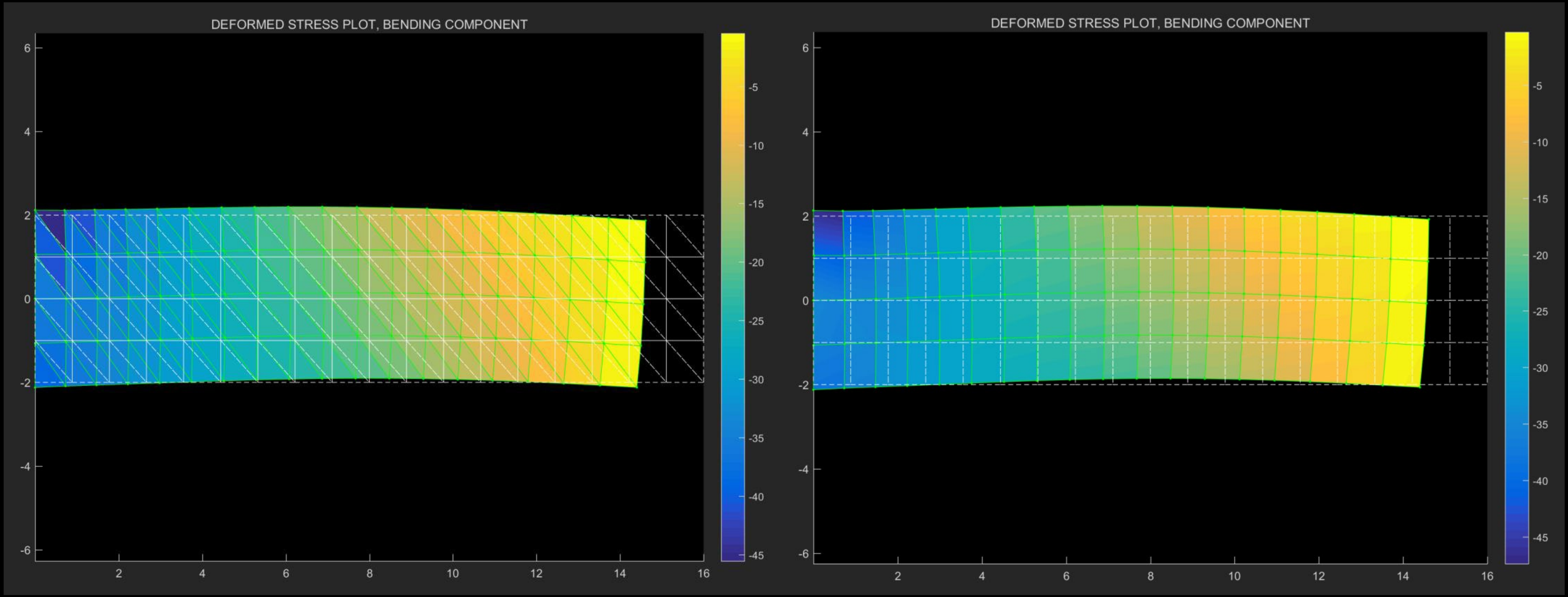


(Quadrilateral four nodes, Q4/9-type)

Absolute Value	T3-type	Q4-type	Q9-type
Maximum stress	≈45	≈47	≈52
Maximum displacement in y-direction ($\times 10^{-7}$)	≈4.1	≈5.1	≈5.2

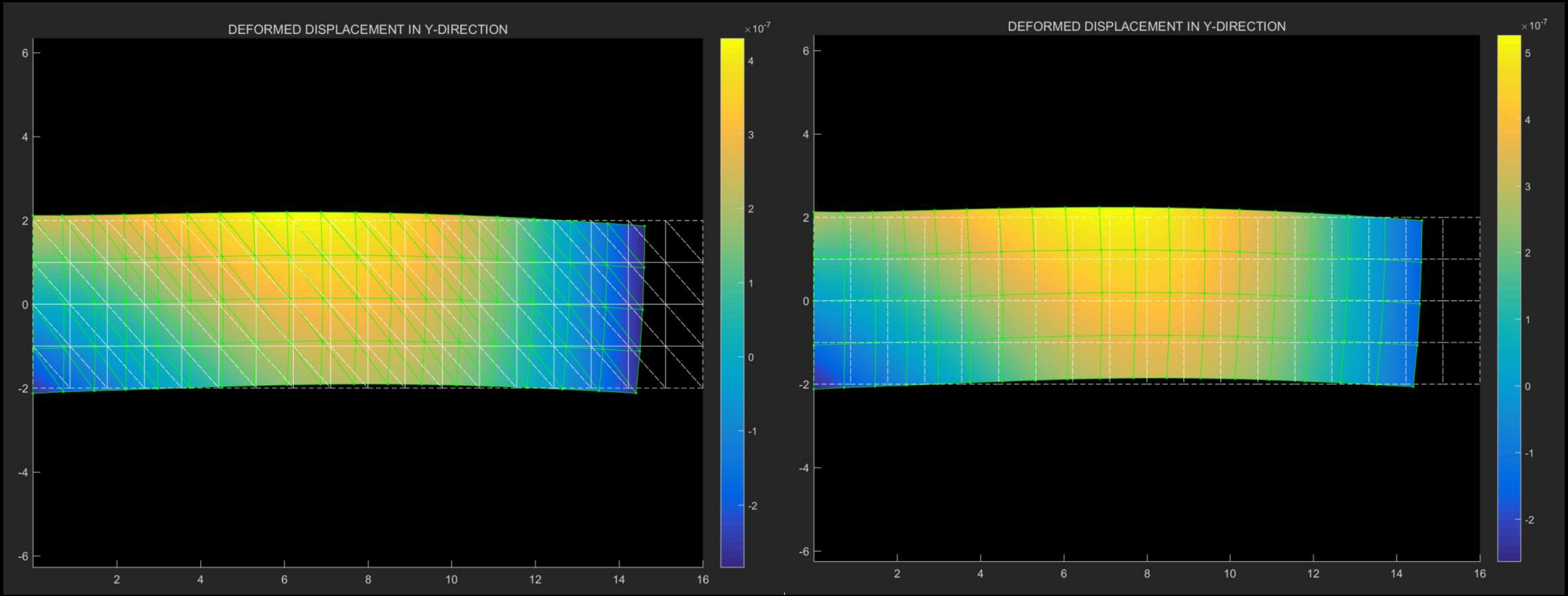
Example 3

Comparation of stress



Example 3

Comparation of displacement in y-direction



Conclusion 2

- Compared with example 1, the second isoparametric element is changed to the first isoparametric element. It is not difficult to find that with the reduction of junction number, the element is rigid, the numerical result gradually decreases, and the smoothness of stress concentration area gradually becomes worse.
- This shows that the boundary of the primary element can not well represent the ground bending of the beam, and there is a shear self-locking phenomenon.
- The appearance of this phenomenon means that the strain energy is producing shear deformation rather than the desired bending deformation, so the total deflection becomes smaller, that is, the element is too rigid.

Reference

- [1] Zhenyu Yang, Wenyi Gong, Zixing Lu. Methods for solving elastic mechanics of beams under Several typical Loads [J]. Mechanics and Practice, 2023, 45(1): 206-212. (in Chinese) DOI: 10.6052/1000-0879-22-112.
- [2] Zhilun Xu. Elastic Mechanics (5th Ed.)[M]. Higher Education Press, 2016.
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- [4] Gui-Fang Wang. Cantilever gravity stress of deep beam [J]. Journal of Sichuan University (engineering science), 2000(02): 8-12. (in Chinese) DOI: 10.15961/j.jsuese.2000.02.003.
- [5] Jack Chessa. Programing the Finite Element Method with Matlab. 2002.10.