Determining the Radius of Orbit of GPS Satellites

Research Question:

What is the relationship between the radius and time period of an object undergoing circular motion?

Subject: Physics

Word Count: 3994

Contents

Introduction - 3

Background Understanding - 5

- Kepler’s Law of Planetary Motion - 7
- Centripetal Force - 10

Experiments - 11

- Experiment 1 - 12
  - Data Analysis - 16
  - Conclusion - 21
- Experiment 2 - 23
  - Data Analysis - 27
  - Conclusion - 32
- Evaluation of Methodology - 33

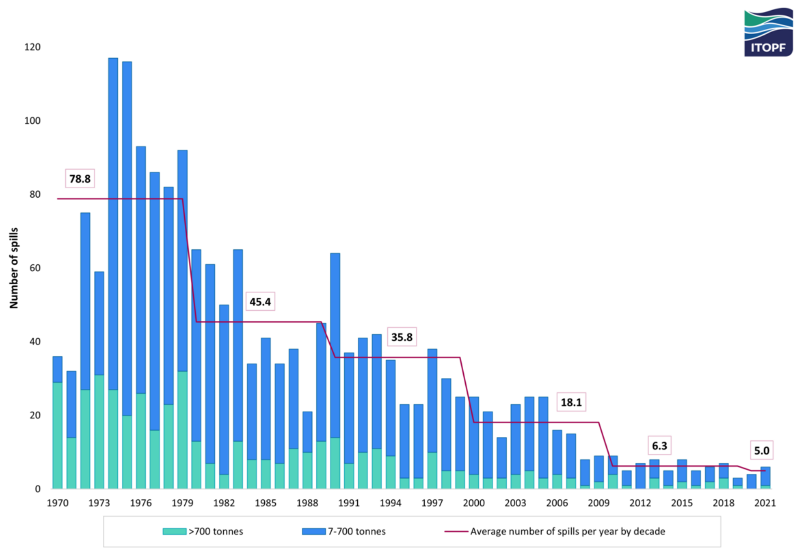
Implications of the experimental results - 34

Bibliography - 38

Introduction

Global Positioning System (GPS) is the most crucial technology of our time. Individuals, aeroplanes, cargo ships and the military rely on it everyday. It is also used for accurate timekeeping on which huge industries rely ranging from telecommunication to financial markets.

GPS wasn't available to the public and was only in military use for a long time, until 1983 when Korean Air Lines' Flight 007 entered Soviet airspace after a navigation error and was shot down, killing all 269 passengers[1]. This incident resulted in President Ronald Reagan ordering the United States military to make the Global Positioning System available for civilian use so that similar incidents could be avoided in the future. And today, navigation is unheard of without the reliance on GPS and has led to far fewer plane crashes and a dramatic decrease[2] in the global oil spillage to name a few.



[1] Korean Airlines flight shot down by Soviet Union. History.com. https://www.history.com/this-day-in-history/korean-airlines-flight-shot-down-by-soviet-union. Published November 13, 2009. Accessed October 21, 2021.
[2] Statistics. ITOPF. https://www.itopf.org/knowledge-resources/data-statistics/statistics/. Accessed October 21, 2021.

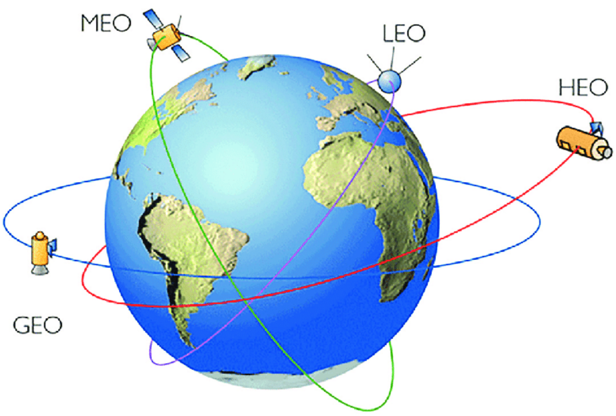
The use of GPS has revolutionised the economy in a number of ways. Perhaps the most significant is in the area of transportation. It has made it possible for vehicles to be routed more efficiently, resulting in time and fuel savings which has a ripple effect throughout the economy, as businesses are able to get their products to market faster and at a lower cost. It has also made it possible for people to find their way around more easily, which has boosted tourism and has also helped emergency services[3] to respond more quickly to incidents.

For the US alone, it's estimated that GPS has generated roughly $1.4 trillion in economic benefits and loss of GPS would result in a $1 billion per-day impact on the nation. The impacts would be even higher nowadays because of the widespread adoption of GPS as it has become a fundamental necessity to civilization.[4]

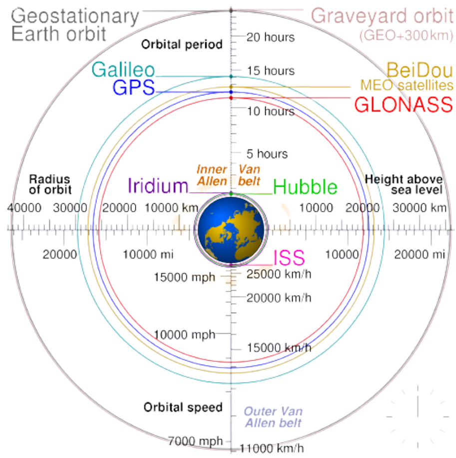
[3] Dziuk B. How GPS tracking can help in emergency service response. How GPS Tracking Can Help in Emergency Service Response. https://info.rastrac.com/blog/emergency-response-use-gps-tracking. Accessed October 20, 2021.

[4] Economic benefits of the Global Positioning System to the U.S. Private Sector Study. NIST. https://www.nist.gov/news-events/news/2019/10/economic-benefits-global-positioning-system-us-private-sector-study. Published February 22, 2021. Accessed October 21, 2021.

Background Understanding



Figure[5]: Types of orbits



Figure[6]: Orbits Comparison

There are different orbits that satellites can follow, depending on its applications. Weather satellites are typically on Polar Orbits as they can observe the same spot twice a day (at night & daytime). GPS satellites fly at an altitude of approximately 20,200 km[7]. These satellites are in geosynchronous orbit around the earth as they have a period of 12 hours[8]; their orbits are in sync with the rotation of earth by a factor of half. This time period is chosen along with other factors to ensure that at any given time, there are at least 4 satellites present (out of 24) at any given location on the earth, which is required to measure the location accurately using trilateration[9].

---

[5] Satellite orbits. https://cosmospnw.com/wp-content/uploads/2020/08/Satelite-Orbits.png. Published December 5, 2018. Accessed October 23, 2021.
[6] File:Comparison Satellite Navigation Orbits.svg. Wikipedia. https://en.wikipedia.org/wiki/File:Comparison_satellite_navigation_orbits.svg. Published August 5, 2020. Accessed October 25, 2021.
[7] Space segment. GPS.gov: Space Segment. https://www.gps.gov/systems/gps/space/. Accessed October 29, 2021.
[8] The Global Positioning System. https://lweb.cfa.harvard.edu/space_geodesy/ATLAS/gps.html. Accessed November 3, 2021.
[9] GISGeography. How GPS receivers work - trilateration. GIS Geography. https://gisgeography.com/trilateration-triangulation-gps/. Published May 31, 2021. Accessed November 8, 2021.

Trilateration is the process of determining the location of a point by measurement of distances of spheres looking at where all the points intersect to determine the location of the receiver as illustrated by the diagram. It requires that we know the satellite's position/distance from the receiver at any given moment with high accuracy.

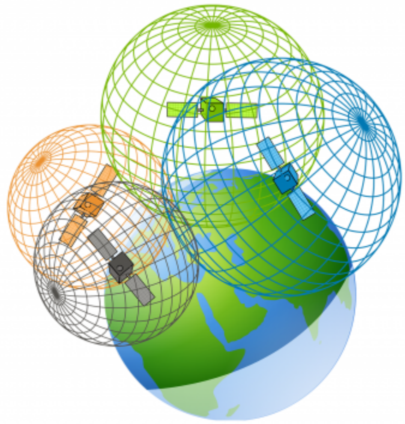


Figure:[9] trilateration

To calculate its position, a GPS device measures its distance from multiple GPS satellites. Firstly, the receiver calculates the time taken to receive the signal from the satellite. As time taken in calculated and the speed of light (signals from GPS) is constant, we can use the distance formula $distance = speed \times time$ to calculate the distance. However, even an inaccuracy of "1/100th of a second would translate into a measurement error of 1,860 miles to the GPS receiver."[10] For high accuracy, GPS satellites have atomic clocks on board. Therefore, it is incredibly important that the orbital period is precisely known for these satellites to calculate the distance, which as this investigation will show requires a very specific orbital radius and speed. Atmospheric drag causes orbital decay, however it's not a significant effect for GPS satellites orbiting at a really high altitude.

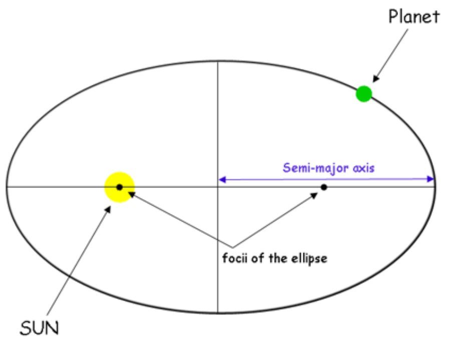
[10] Satellite Navigation - GPS - space segment. Satellite Navigation - GPS - Space Segment | Federal Aviation Administration. https://www.faa.gov/about/office_org/headquarters_offices/ato/service_units/techops/navservices/gnss/gps/spacesegments. Accessed November 16, 2021.

Kepler’s Law of Planetary Motion

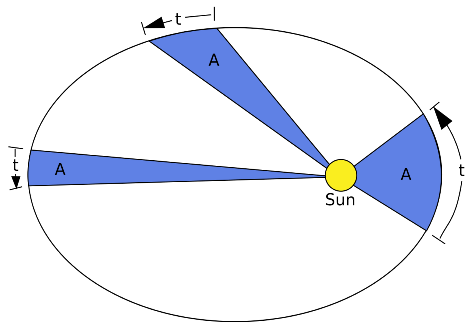
In 1609, German astronomer Johannes Kepler formulated what is now known as Kepler’s laws of planetary motion. These laws describe the motion of planets around the sun nonetheless they can also be used to describe the motion of satellites around the Earth. Kepler's laws[11] are a set of three laws of planetary motion as follows:

- The law of orbits: All planets move in elliptical orbits, with the sun at one focus. Meaning that the planet to Sun distance is constantly changing as the planet goes around its orbit.
- The Law of Areas: A line that connects a planet to the sun sweeps out equal areas in equal times. Hence, a planet is moving fastest when it is closest to the sun and slowest when it’s farthest.
- The Law of Periods: The square of the period of any planet is proportional to the cube of the semimajor axis of its orbit $\left(T^2 \propto R^3\right)$[12]. Mercury, the innermost planet, takes only 88 days to orbit the Sun while Saturn requires 10,759 days to do the same.[13]

[11] Kepler's laws. http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html. Accessed November 16, 2021.
[12] Physics. http://physics.bu.edu/~redner/211-sp06/class16/kepler3.html. Accessed November 22, 2021.
[13] NASA. Planetary fact sheet. NASA. https://nssdc.gsfc.nasa.gov/planetary/factsheet/. Accessed November 22, 2021.



Figure[14]: Kepler's First Law



Figure[15]: Kepler's Second Law

From Kepler's second law we know that a satellite covers equal area in an equal interval of time. So if the orbit were not to be circular it would speed up and down significantly in its orbit as described above. "The relative movement between a GPS receiver and a GPS satellite causes the received signal frequency to differ from its nominal frequency due to Doppler effects. This frequency difference, observable in GPS measurements, is referred to as the Doppler shift"[16] If the velocity of the satellite were to change in the orbit, that is if the orbit was not circular, this effect would vary depending on satellite's position in orbit making it harder to calculate this shift and this was in the 80s where we didn't have high computation chips in the palm of our hands as we have today. Also as discussed above, small changes can hugely affect the accuracy of the calculation, therefore, it can be inferred that circular orbits are the best type of orbit for GPS satellites it would have the least variance no matter where at its orbit it is.

[14] Admin P. Biography of Johannes Kepler German astronomer and observer. Astronomía Astrónomos Universo. https://www.astrojem.net/biography-of-johannes-kepler-german-astronomer-and-observer/. Published July 1, 2021. Accessed November 22, 2021.
[15] File:Kepler's law 2 en.svg. Wikimedia Commons. https://commons.wikimedia.org/wiki/File:Kepler%27s_law_2_en.svg. Accessed November 22, 2021.
[16] On the relativistic Doppler effects and high accuracy velocity ... https://www.researchgate.net/publication/238619351_On_the_relativistic_Doppler_Effects_and_high_accuracy_velocity_determination_using_GPS. Accessed December 2, 2021.

Similarly, Kepler's Third Law is extremely important for satellites in orbits as it exhibits the relationship between time period and radius. NASA has calculated the required period of a GPS satellite to make a formation that covers "virtually any point on the planet"[17] at any given moment as discussed above would require a period of 12 hours. Since we want an orbital period of 12 hours, using the proportionality $T^2 \propto R^3$ from Kepler's Third Law allows us to find its orbital radius as shown later on.

Now that Kepler's Laws have allowed us to deduce the type of orbit the GPS satellites should have and the altitude required to attain its desired time period, engineers also need to determine the speed it should have as the satellite will need to be at a specific speed in order to maintain that orbit;[18] any higher it would spiral out, any lower spiral inwards to Earth. As we know that the satellite is following a circular orbit,[19] we can leverage the centripetal force[20] equation which describes the motion of objects in circular orbits to compute the velocity required.

---

[17] Space segment. GPS.gov: Space Segment. https://www.gps.gov/systems/gps/space/. Accessed December 2, 2021.
[18] Orbits. https://physics.highpoint.edu/~jregester/potl/Mechanics/Orbits/orbitsA.htm. Accessed December 2, 2021.
[19] GPS space segment. GPS Space Segment - Navipedia. https://gssc.esa.int/navipedia/index.php/GPS_Space_Segment. Accessed December 2, 2021.
[20] What is a centripetal force? (article). Khan Academy. https://www.khanacademy.org/science/physics/centripetal-force-and-gravitation/centripetal-forces/a/what-is-centripetal-force. Accessed December 2, 2021.

Centripetal Force

A force that acts on a body moving in a circular path and is directed towards the centre around which the body is moving. The GPS satellite is set to be in a circular orbit around the earth due to reasons mentioned above, therefore we will investigate the relationships that exist between the Force, Radius and Time period on an object undergoing circular motion.

Equation[21] for the centripetal force:

$$F = \frac{m_b v^2}{R}$$

$F = force$
$m_b = mass\ of\ the\ bung$
$R = radius$
$T = time\ period$

Since we know[22] the following:

$$v = \omega R$$
$$w = \frac{2\pi}{T}$$
$$v = \frac{2\pi}{T} \cdot R$$

$\omega = angular\ velocity$
$T = time\ period$

We can substitute for v in the equation of the centripetal force to get it in terms of quantities of our interest: Force, Radius and Time period:

$$F = \frac{m_b\left(\frac{2\pi R}{T}\right)^2}{R}$$

$$F = \frac{4\pi^2 m_b R}{T^2}$$

[21] Derivation of centripetal force. https://www.gb.nrao.edu/~rmaddale/Education/Wvsta'98/Centripetal.html. Accessed December 3, 2021.
[22] Tsokos KA. Circular Motion. In: *Physics for the IB Diploma*. Cambridge, United Kingdom: Cambridge University Press; 2015.

Experiments

As it's not feasible and practical to launch satellites in different orbits to work out the relationship practically, we will be conducting experiments that will closely simulate the environment to determine the relationships radius, time period and force have from the equation above and proving Kepler's Third Law.

	Independent	Control	Measuring	Relationship
Experiment 1	Force (F)	Radius (R)	Period (T)	$\frac{1}{T^2} \propto R$
Experiment 2	Radius (R)	Force (F)	Period (T)	$T^2 \propto F$

Experiment 1

Aim of this experiment will be to investigate how changing the mass *(M)*, which will cause the tension force on the string to increase, affects the time period of the bung.

Hypothesis: As we increase the mass, the time period should decrease as they are inversely proportional to each other: $\frac{1}{T^2} \propto R$. Therefore, the bung should take shorter time for a full rotation.

Variables:

Independent variable - Radius *(R)* of the string

Dependant variable - Time period *(T)*

Control Variable - Mass *(M)*

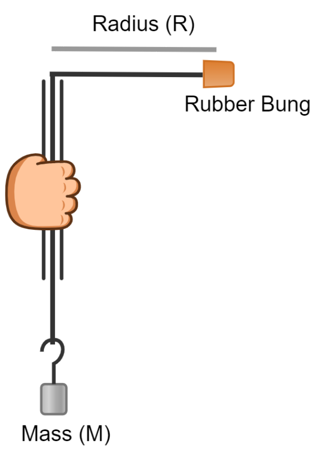
Apparatus

- Tube
- String
- Rubber bung
- Mass
- Timer
- Safety glasses
- Metre ruler

Method

- Cut a string *(Choosing a length that is comfortable to spin the bung in, not too short as the time period will be shorter, therefore harder to time with the increase in mass; not too long as it might get tilted horizontally)*
- Tie a bung to the piece of string and ensure they are securely attached
- Measure a distance *(R) from the centre of mass of the bung and* using a dark coloured marker, mark the point *(Make sure to mark least 1cm both ways from the point so you can clearly see the spot when it is spinning so that it stays at that level)*
- Use a tube as a holder and put one end of the string through it
- Hold the tube, making sure there is no one too near, whirl the bung around your head, keeping the indicator just at the border of the tube.
- Experiment with rotating the bung around the tube to familiarise yourself until you are capable of rotating it horizontally with as less of a vertical tilt as possible.
- While you continue rotating it your partner should measure the period of rotation. *(Measure the time period of 20 rotations as measuring one time period by eye can get really difficult as at higher mass, the bung needs to be spun faster for the centripetal force to counteract the tension force due to the mass, therefore it will spin very fast according to the proportionality)*
- Make sure you spin the bung at least 10 times before you start the timer, this will help in making sure the bung is stable and spinning horizontally.

- Change the mass with increments of $100g$ making sure the radius is kept the same and repeat the steps above *(Using at least 7 different masses)*
- Repeat the above steps at least 5 more times to find the average and eliminate anomalies if any.



$R\ -\ Radius\ of\ the\ string\ from\ the\ top\ of\ tube\ to\ the\ centre\ of\ mass\ of\ the\ rubber\ bung\ in\ metres$

$M\ -\ mass\ of\ the\ mass\ in\ kg$

Experiment 1 Results

Mass (Kg)	Force (N)	Time for twenty periods (s)							$\frac{1}{T^2}$ (s^{-2})	
		Trail 1	Trial 2	Trail 3	Trail 4	Trail 5	Average	Uncertainty	Value	Uncertainty
0.300	2.94	8.76	8.75	8.70	8.28	8.54	8.61	0.24	5.40	0.30
0.400	3.92	7.58	7.57	7.86	7.21	8.44	7.73	0.62	6.69	1.06
0.500	4.91	6.81	7.08	6.90	7.16	7.11	7.01	0.18	8.14	0.41
0.600	5.89	6.78	6.78	6.88	6.09	6.31	6.57	0.40	9.27	1.12
0.700	6.87	6.38	6.60	6.63	6.36	6.06	6.41	0.29	9.75	0.87
0.800	7.85	6.03	6.10	6.33	6.57	6.22	6.25	0.27	10.24	0.88
0.900	8.83	5.89	5.77	5.54	5.91	5.99	5.82	0.23	11.81	0.91

Data processing

The following shows the steps by which the values in the table were calculated:

To get the Force (N) from Mass (kg), we multiply the mass by 9.81 - acceleration due to gravity:

$$0.300\ kg\ \times 9.81ms^{-2} = 2.94\ N$$

To calculate the average time taken for 20 rotations as set out in the experiment, we get the sum of the 5 trials we conducted and divide by the number of trials:

$$\frac{8.76+8.75+8.70+8.28+8.54}{5} = 8.61\ s$$

To calculate the uncertainty in time, we do half the range

$$\frac{1}{2}(8.76 - 8.28) = 0.24\ s$$

To calculate $\frac{1}{T^2}$, we divide 1 by the square of the average time multiplied by $\frac{1}{20}$ as while doing the experiment, a time period of 20 rotations was recorded

$$\frac{1}{T^2} = \frac{1}{\left(\frac{8.61}{20}\right)^2} = 5.40\ s^{-2}$$

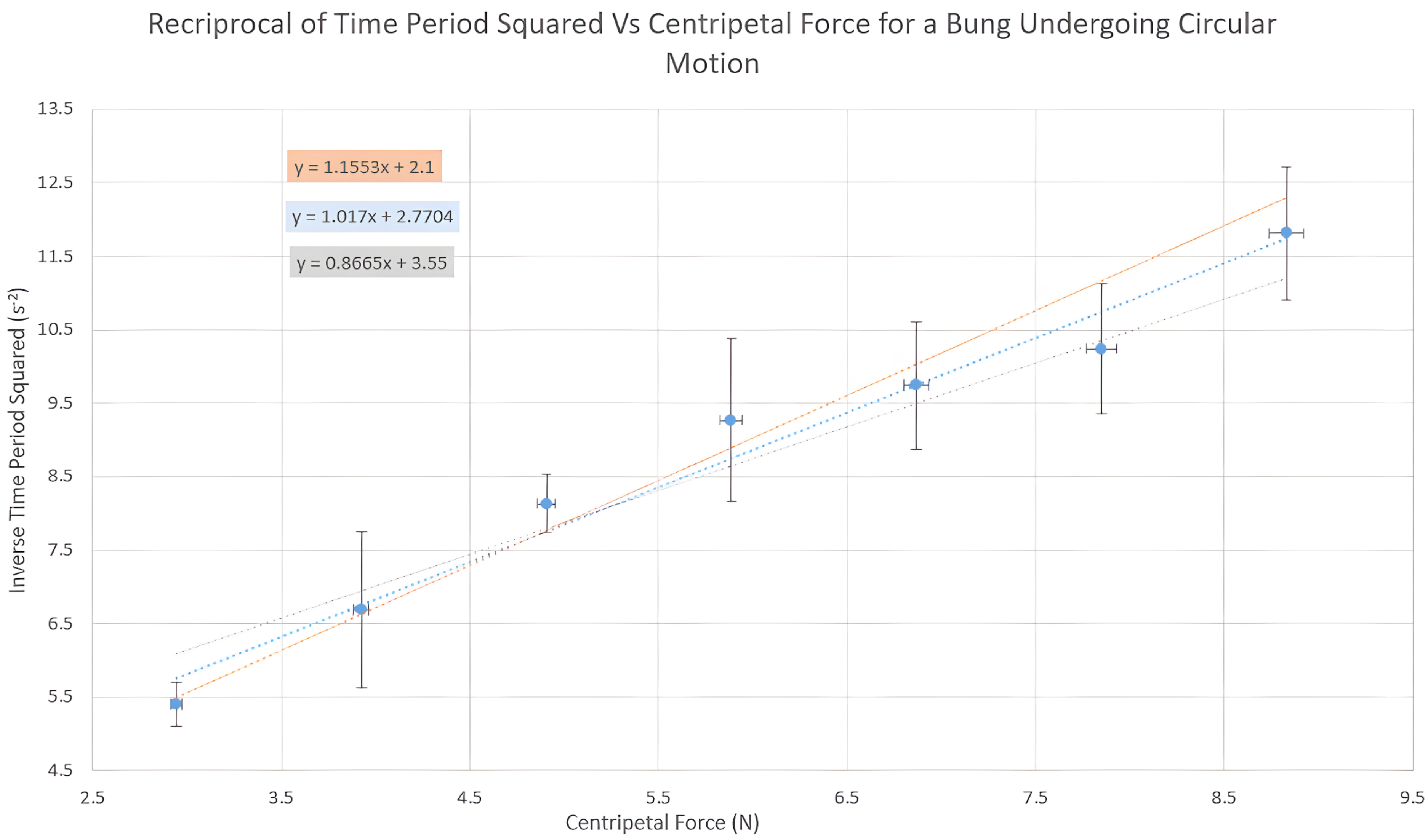
To calculate the fractional uncertainty in time, we do uncertainty divided by average time and multiply that by 2 as T is raised to the power 2

$$2\left(\frac{0.24}{8.61}\right) = 0.056\ s^{-2}$$

Finally to calculate the absolute uncertainty in $\frac{1}{T^2}$, we multiply the the fractional uncertainty by $\frac{1}{T^2}$

$$5.40 \times 0.056 = 0.30\ s^{-2}$$

We can then repeat these steps for all sets of repeats and plot the graph as shown below.


Reciprocal of Time Period Squared Vs Centripetal Force for a Bung Undergoing Circular Motion
y = 1.1553x + 2.1
y = 1.017x + 2.7704
y = 0.8665x + 3.55
Inverse Time Period Squared (s^{-2})
13.5
12.5
11.5
10.5
9.5
8.5
7.5
6.5
5.5
4.5
2.5
3.5
4.5
5.5
6.5
7.5
8.5
9.5
Centripetal Force (N)

Equation of centripetal force as we derived above:

$$F = \frac{4\pi^2 m_b R}{T^2}, \text{ let } m_b \text{ be the mass of the bung}$$

Writing in the from $y = mx + c$ with the intention of making the gradient linear, therefore we will be plotting $\frac{1}{T^2}$

Writing the equation in the form $y = mx + c$, with y being $\frac{1}{T^2}$, the value we measured, and x being F, the value we changed:

$$\frac{1}{T^2} = \left(\frac{1}{4\pi^2 m_b R}\right) F + 0$$

$$\therefore Gradient = \frac{1}{4\pi^2 m_b R}$$

From this we know that if the y-intercept doesn't pass through the origin, we can postulate a systematic error.

As we have 3 lines of best fit from the graph above, we can find out the uncertainties in the gradient and the y-intercept.

The percentage uncertainty in the gradient is half the range divided by the best gradient:

Half the range: $\frac{1}{2}(1.1553 - 0.8665) = 0.1444$

Percentage uncertainty = $\left(\frac{0.1444}{1.017}\right) \times 100$ = 14%

Absolute uncertainty in gradient = $1.017 \times 0.14 = \pm 0.142$

Similarly doing the same for the y-intercept:

Half the range: $\frac{1}{2}(3.55 - 2.1) = 0.725$

Percentage uncertainty = $\left(\frac{0.725}{2.7704}\right) \times 100$ = 26%

Absolute uncertainty = $2.7704 \times 0.26 = \pm 0.725$

Substituting the variables from the graph above:

$$\frac{1}{T^2} = (m \pm \Delta m)\, F + (c \pm \Delta c)$$

$$\frac{1}{T^2} = (1.017 \pm 0.142)\, F + (2.7704 \pm 0.725)$$

We showed earlier that the $Gradient\ (G) = \frac{1}{4\pi^2 m_b R}$

Now that we know the gradient from the graph above, we can substitute it into the following equation and find out the mass of the bung.

$$m_b = \frac{1}{4\pi^2 GR}$$

$$m_b = \frac{1}{4\pi^2 (1.017)(0.4)}$$

$$m_b = 0.06226\ kg$$

There are uncertainties in the value of m_b which we can find as shown below.

Percentage uncertainty in $Gradient = 12.93\%$ from above.

Percentage uncertainty in radius is found by dividing the absolute uncertainty in the radius (0.02, the uncertainty, when marking the radius with a marker, 1 cm of thickness was marked both ways to better see the marking when the bung was spinning) divided by the radius of the orbit:

$$\text{Percentage uncertainty} = \left(\frac{0.02}{0.40}\right) \times 100 = 5\%$$

By adding these percentage uncertainties together the uncertainty in the mass of the bung can be determined:

$$m_b = 0.06226\,kg \pm 19\%$$

Multiplying the fractional uncertainty with the mass of the bung gives us the absolute uncertainty.

$$m_b = 62 \pm 11.8\,g$$

The mass of the bung as measured by a pan balance is: $0.041 \pm 0.001\ kg$

$0.041kg$ is not in the range $(0.062 \pm 0.0118\,kg)$

$$\text{Percentage Difference} = \frac{0.062-0.041}{0.041} \times 100$$

$$= 51.2\%$$

Systematic Error: Force of friction

When spinning the bung, the string was rubbed against the cylinder which meant that some of the force was lost due to friction. We expected the equation of the line to pass through the origin, which it did not. If we find out where the x-intercept of the line is, we can decide that the x-intercept was the force of friction acting on the string.

$\frac{1}{T^2} = 1.017F + 2.7704$

Finding the x-intercept of the following line can help us find the force of friction

$0 = 1.017F + 2.7704$

$F = \frac{-2.7704}{1.017}$

$F =- 2.724\ N$

Conclusion

The initial theory before conducting this experiment was that as the force *(the mass)* increased the time period would get shorter. Based on the equation of centripetal force $F = \frac{4\pi^2 m_b R}{T^2}$ as derived above, the expected relation would be $T^2 \propto \frac{1}{F}$ if the radius of the orbit was kept constant which is in line with my proposed theory. The following is the equation of the line as found from the experiment.

$$\frac{1}{T^2} = (1.017 \pm 0.142)\,F + (2.7704 \pm 0.725)$$

As can be observed on the graph above, the line was linear as hypothesised. The line didn't pass through the origin including with the uncertainty range postulating that there must be a systematic error. This is likely caused by the friction between the string and the tube, which was calculated using the X intercept of the graph above as 2.724N

The experiment was not precise as shown by the substantial error bars in the graph above. The time period data was spread out with the average range across repeats being 0.63 with the average time period of 20 rotations being 6.91, which accounts for approximately 10% spread between the data on average.

The experiment data was mostly reliable as the data points were close to the line of best fit. By comparing the value of the mass of the bung (m_b) as determined by the graph with the actual measured value of the bung, we can find the accuracy of the experiment. The experiment was not as accurate as hoped since the value of the bung as measured by the pan balance (41g) was not within the tolerance calculated from the gradient ($m_b = 62 \pm 11.8\ g$**)** and had a percentage difference of 51% from the measured mass.

Experiment 2

Aim: Investigate how changing the radius *(R)* affects the time period of the bung. This will help us determine the time periods of satellites placed on a higher or lower orbit which is crucial information as described above.

Hypothesis: As we reduce the radius of the orbit *(altitude)*, the time period should get smaller and smaller as $T^2 \propto R$

Apparatus

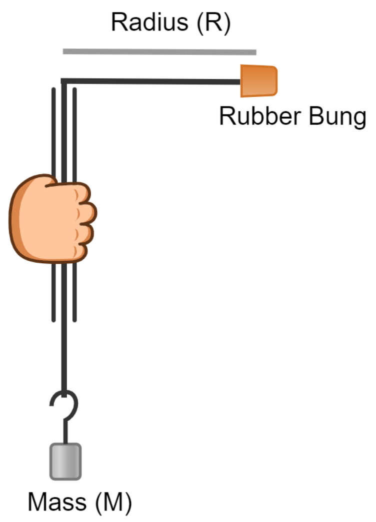
- Tube
- String
- Rubber bung
- Mass
- Timer
- Safety glasses
- Metre ruler

Method

- Cut a string *(Choose a long length to start with which is easier to spin and you can get smaller from that length)*
- Tie a bung to the piece of string and ensure they are securely attached
- Measure a distance *(R) from the centre of mass of the bung and* using a dark coloured marker, mark the point *(Make sure to mark least 1cm both ways from the point so you can clearly see the spot when it is spinning so that it stays at that level)*
- Use a tube as a holder and put one end of the string through it
- Hold the tube, making sure there is no one too near, whirl the bung around your head, keeping the indicator just at the border of the tube.
- Experiment with rotating the bung around the tube to familiarise yourself until you are capable of rotating it horizontally with as less of a vertical tilt as possible.
- While you continue rotating it your partner should measure the period of rotation. *(Measure the time period of 20 rotations as measuring one time period at a time can get really difficult as at a smaller radius, it will take a shorter amount of time to complete a revolution, therefore it will spin very fast)*
- Make sure you spin the bung at least 10 times before you start the timer, this will help in making sure the bung is stable and spinning horizontally.
- Change the radius *(R),* keeping mass the same and repeat the steps above with decrements of $5\ cm$ each time using at least 7 different radius lengths *(you can mark the radius with a different colour marker or use a completely different string, however, using the same string is recommended as it will provide more precise results as the knot used to tie the string to the bung might change slightly each time.)*
- Repeat the above steps at least 5 more times to find the average and eliminate anomalies if any.

Modifications to the method

- The previous experiment had a high uncertainty apparent through huge error bars and the measured accuracy of the mass of the bung was low. As a human had to press the timer, reaction time (visual stimuli is approximately 180–200 ms[23]) played a role. On average the time taken for According to the hypothesis the rubber bung will be spinning faster and faster as the rotations was $6.91s$, assuming $200ms$ of delay on both the start and the end, that's about $2 \cdot (200\,ms)/6.91s = 6\%$ of uncertainty introduced that could be removed by the use of technology. This would be of an even bigger concern as we are reducing the radius in this experiment which is proportional to the time period squared $(T^2 \propto R)$. Therefore, we will be using a camera to record the spinning bung in a high frame rate (60 fps) and later, process the individual frames to get accurate timings of the time period.



R $-$ $Radius\ of\ the\ string\ from\ the\ top\ of\ glass\ tube\ to\ the\ centre\ of\ mass\ of\ the\ rubber\ bung\ in\ metres$

M $-$ $mass\ of\ the\ mass\ in\ kg$

[23] Jain A, Bansal R, Kumar A, Singh KD. A comparative study of visual and auditory reaction times on the basis of gender and physical activity levels of medical first year students. International journal of applied & basic medical research. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4456887/. Published 2015. Accessed December 26, 2021.

Experiment 2 Results

	Time Period for 20 Rotations (s)							Units (s^2)	
Length (m)	Test 1	Test 2	Test 3	Test 4	Test 5	Average	Uncertainty in time	Time Period Squared	Uncertainty in T^2
0.10	4.08	4.03	4.07	4.06	4.03	4.05	0.03	0.041	0.001
0.15	4.23	4.21	4.26	4.24	4.25	4.24	0.03	0.045	0.001
0.20	5.14	5.16	5.11	5.04	5.09	5.11	0.06	0.065	0.002
0.25	5.11	5.2	5.07	5.16	5.19	5.15	0.07	0.066	0.002
0.30	5.25	5.28	5.29	5.28	5.28	5.28	0.02	0.070	0.001
0.35	6.12	6.13	6.09	6.04	6.16	6.11	0.06	0.093	0.002
0.40	6.21	6.26	6.27	6.25	6.26	6.25	0.03	0.098	0.001

Data processing

The following shows the steps by which the values in the table were calculated:

To calculate the average time taken for 20 rotations as set out in the experiment, we get the sum of the 5 trials we conducted and divide by the number of trials:

$$\frac{4.08+4.03+4.07+4.06+4.03}{5} = 4.05\ s$$

To calculate the uncertainty in time, we do half the range, range being the difference between the maximum and minimum values.

$$\frac{1}{2}(4.08 - 4.03) = 0.025\ s$$

While doing the experiment, time period of 20 rotations were recorded, therefore to calculate the T^2, we divide the average period by 20.

$$T^2 = \left(\frac{4.05}{20}\right)^2 = 0.041\ s^2$$

To calculate the fractional uncertainty, we do uncertainty divide by average and multiply that by 2 as T is raised to the power 2

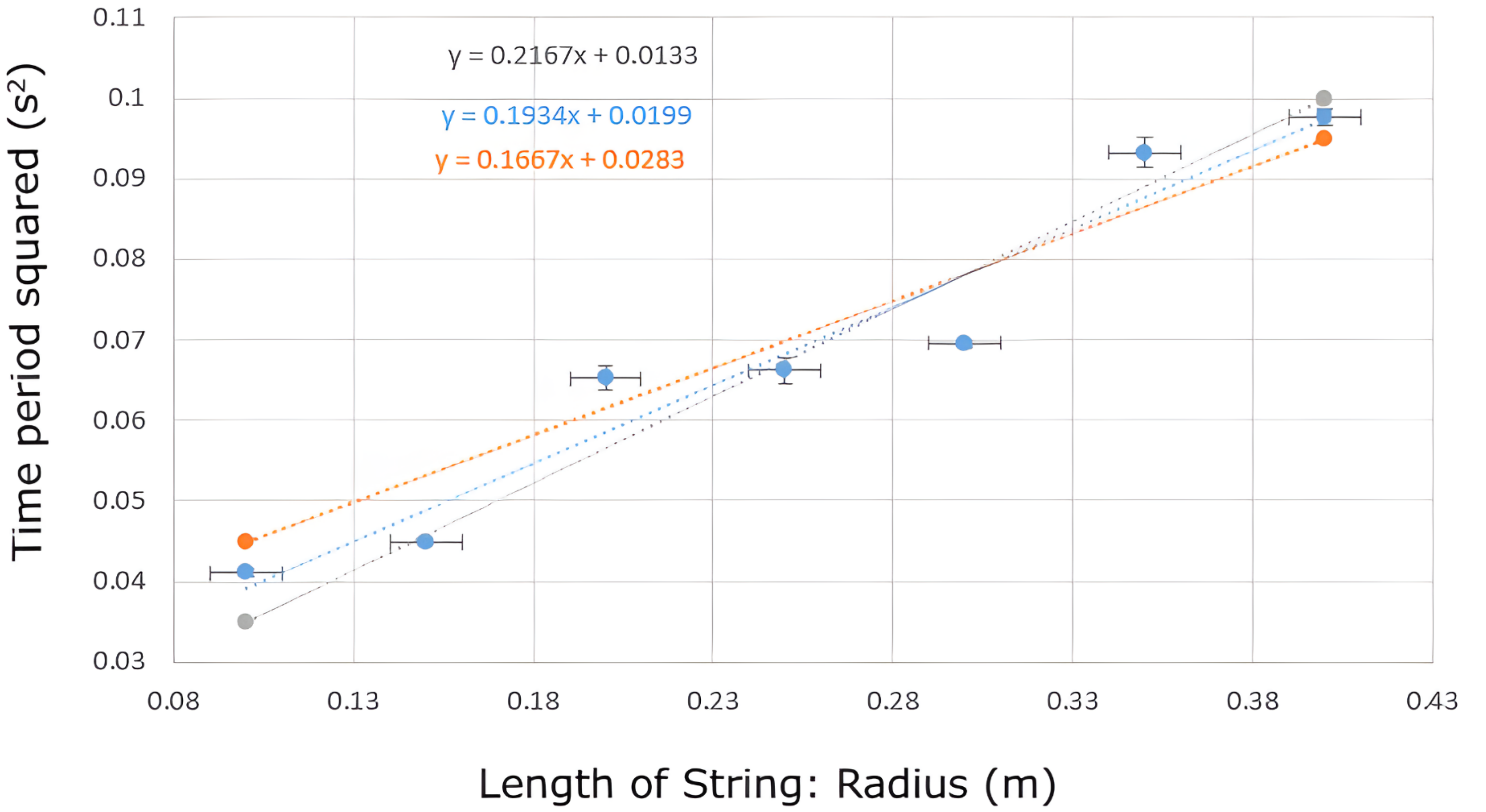
$$2 \times \left(\frac{0.025}{4.05}\right) = 0.0123\ s^2$$

Finally to calculate the absolute uncertainty in the T^2, we multiply the the fractional uncertainty by T^2

$$0.041 \times 0.0123 = 0.001\ s^2$$

We can then repeat these steps for all sets of repeats and plot the graph as shown below.

Time period squared vs Radius for a Bung Undergoing Circular Motion



Equation of centripetal force as we derived above:

$$F = \frac{4\pi^2 m_b R}{T^2}, \ let\ m_b\ be\ the\ mass\ of\ the\ bung$$

Writing the equation in the form $y = mx + c$, with y being T^2, the value we measured, and x being R, the value we changed:

$$T^2 = \left(\frac{4\pi^2 m_b}{F}\right) R + 0$$

$$\therefore Gradient = \frac{4\pi^2 m_b}{F}$$

As we have 3 lines of best fit from the graph above, we can find out the uncertainties in the gradient and the y-intercept.

The percentage uncertainty in the gradient is half the range divided by the best gradient:

Half the range: $\frac{1}{2}(0.2167 - 0.1667) = 0.025$

Percentage uncertainty in gradient = $\left(\frac{0.025}{0.1934}\right) \times 100$ = 12.93%

Absolute uncertainty in gradient = $0.1934 \times 0.1293 = \pm 0.025$

Similarly doing the same for the y-intercept:

Half the range: $\frac{1}{2}(0.0283 - 0.0133) = 0.0075$

Percentage uncertainty = $\left(\frac{0.0075}{0.0199}\right) \times 100$ = 37.68%

Absolute uncertainty = $0.0199 \times 0.3768 = \pm 0.0075$

Substituting the variables from the graph above and the uncertainties:

$$T^2 = (0.1934 \pm 0.025)R + (0.0199 \pm 0.0075)$$

We showed earlier that the $Gradient\ (G) = \frac{4\pi^2 m_b}{F}$

Now, as we know the gradient, by comparing the value of the mass of the bung (m_b) as determined by the graph with the actual measured value of the bung, we can find the accuracy of the experiment.

$$m_b = \frac{FG}{4\pi^2}$$

Substituting the values into the equation:

We used a mass of $0.6\ kg$, multiplying that with the acceleration due to gravity

$$F = 9.81 \times 0.600$$

$$F = 5.89\ N$$

From the graph, the gradient $G = 0.1934$

$$m_b = \frac{(5.89) \times (0.1934)}{4\pi^2}$$

$$m_b = 0.0289\ kg$$

Similarly as the previous experiment, we can find the uncertainties in m_b as shown.

Percentage uncertainty in $Gradient = 12.93\%$ from above.

Percentage uncertainty in Force is found by dividing the absolute uncertainty in the radius (to the nearest grams (0.001kg) as per the specification) divided mass used:

$$\text{Percentage uncertainty} = \left(\frac{0.001}{0.6}\right) \text{x } 100 = 0.17\%$$

By adding the percentage uncertainties in gradient and force together the percentage uncertainty in the mass of the bung used can be determined:

$$m_b = 0.0289\, kg \pm 13.1\%$$

Multiplying the fractional uncertainty with the mass of the bung gives us the absolute uncertainty.

$$m_b = 28.9 \pm 3.8\, g$$

The mass of the bung as measured by a pan balance is: $0.041 \pm 0.001\ kg$

$0.041kg$ is not in the range $(0.0289 \pm 0.0038\, kg)$

Percentage Difference = $\frac{0.0289-0.041}{0.041} \times 100$

$= 29.5\%$

Conclusion

The initial theory before conducting this experiment was that as the radius of the orbit decreased, the time period would get shorter. Based on the equation of centripetal force $F = \frac{4\pi^2 m_b R}{T^2}$ as derived above by substitution, the expected relation would be that $T^2 \propto R$ if the force *(mass of the object being orbited)* was kept constant which is in line with my proposed theory. The following is the equation of the line as found from the experiment.

$$T^2 = (0.1934 \pm 0.025)R + (0.0199 \pm 0.0075)$$

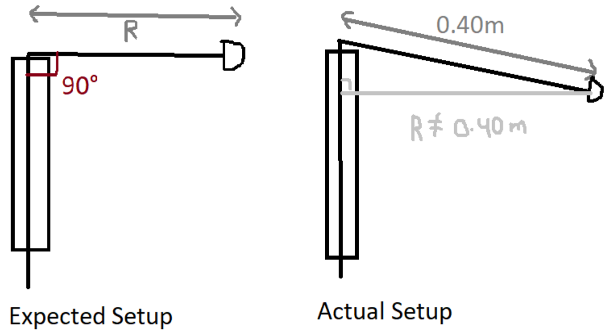
As can be observed on the graph above, the line was linear as expected. Although the origin does not fall within (0.0199 ± 0.0075)**,** the line passed closely through the origin as seen from the equation above, as explained in the previous experiment, this is likely due to friction between the string and the tube.

The experiment was precise as it can be observed in the graph that the error bars were small. But the experiment data was not as reliable as the previous experiment as the data points were not close to the line of best fit. The experiment was more accurate than the previous experiment as the determined mass of the bung by the experiment was only 30% off the measured mass.

Evaluation of Methodology

As the length R got shorter and/or the mass increased, the velocity of the bung would get higher therefore it was increasingly harder to balance the bung perfectly horizontally.

Validity of the experiment



Due to physical constraints, the actual setup of the experiment was as shown on the diagram, so according to the Pythagorean theorem, the actual radius was less than the anticipated 0.40m which meant that the time period was lower than expected as the time period squared is proportional to radius $(T^2 \propto R)$ as shown experimentally in Experiment 2.

Systematic Error

There was friction between the string and the end of the tube which affected the Force. This was apparent while doing the experiment as the string actually snapped and flew off in the tangential direction. This means the tension force on the bug was lower than what was being provided by the mass, therefore the velocity was lower than expected.

Impact of modifications on Precision

Precision was low in Experiment 1, this might be due to human error as described above, when a camera was used in Experiment 2, the precision in the results increased.

Implications of the experimental results on orbital satellites

From experiments one and two, we showed that $T^2 \propto \frac{1}{F}$ and $T^2 \propto R$, combining the proportionalities from the two experiments together we get, that

$$T^2 \propto \frac{R}{F}$$

From Newton's law of universal gravitation: $F = \frac{GMm}{R^2}$, where (G) being the gravitational constant (6.6743×10^{-11}), (M) being the mass of the Earth, (m) being the mass of the satellite and (R) being the radius of the orbit, we can deduce that $F \propto \frac{1}{R^2}$

Substituting F for the proportionality $F \propto \frac{1}{R^2}$ from Newton's law of gravitation, into the combined proportionality from experiments combined $T^2 \propto \frac{R}{F}$ we get that:

$$T^2 \propto \frac{R}{\left(\frac{1}{R^2}\right)}$$

$$T^2 \propto R^3$$

Using the two experiments from above and Newton's law of universal gravitation, we have successfully proved Kepler's Third Law, which states that proportionality $T^2 \propto R^3$ It implies that the time period for the satellite to orbit the Earth decreases rapidly with the decrease in the radius of its orbit which is what we observed in experiment 2 where we changed the radius of the orbit to observe its time period. This is important to know as described above as we want the orbital period to be 12 hours, we can use this to calculate the radius it should orbit to maintain that time period.

Furthermore, as described previously, we need to know at what speed the satellite needs to be to maintain its orbit as it spiralling inwards or outwards would pose risks of not being able to provide GPS signals to certain areas of the earth as the constellation as shown in the figure[24] would break which would have significant economic implication of estimated $\$1\ billion$ dollars per day and military implications as discussed above and essentially disrupt how we go about with our daily lives.

As GPS satellites fly in a circular orbit, we can equate the circular motion equation with Newton's Law of gravitation which will allow us to find the velocity required to maintain the specific altitude found using the previous proportionality $T^2 \propto R^3$

[24] How does GPS work? - Constellation. NASA. https://spaceplace.nasa.gov/gps/en/. Published June 27, 2019. Accessed January 6, 2022.

From above we know that:

$$F = \frac{mv^2}{R} \text{ and } F = \frac{GMm}{R^2}$$

$$\frac{mv^2}{R} = \frac{GMm}{R^2}$$

Dividing both sides by m and R gives us the following:

$$v^2 = \frac{GM}{R}$$

The equation $v = \sqrt{\frac{GM}{R}}$ now permits us to calculate the velocity required to maintain that orbital radius (R) as gravitational constant (G) and mass of the earth (M) which is $5.97\times10^{24}\ kg$ are known constants.

As discussed above, we can define $v = \frac{2\pi R}{T}$ for uniform circular motion, substituting this value of v into the equation gives us the following as we want the equation in terms of time period as that is what we measured in the experiment above.

$$\left(\frac{2\pi R}{T}\right)^2 = \frac{GM}{R}$$

$$\frac{4\pi^2 R^2}{T^2} = \frac{GM}{R}$$

Rearranging to make T^2 the subject, we get the following and successfully prove Kepler's Third Law.

$$T^2 = \frac{4\pi^2 R^3}{GM},\ T^2 \propto R^3$$

With the proof of Kepler's third law, we know the required orbital radius to launch the satellite at to maintain the orbital time period of 12 hours, using the equation $T^2 = \frac{4\pi^2 R^3}{GM}$. And as the satellites require to have a specific velocity to maintain that orbit, we substitute the desired orbital radius to be maintained into the equation $v^2 = \frac{GM}{R}$ as found above and find out exactly the required velocity to maintain the required orbit.

Bibliography

Kepler's laws. http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html. Accessed November 16, 2021.

Physics. http://physics.bu.edu/~redner/211-sp06/class16/kepler3.html. Accessed November 22, 2021.

The Global Positioning System. https://lweb.cfa.harvard.edu/space_geodesy/ATLAS/gps.html. Accessed November 3, 2021.

Economic benefits of the Global Positioning System to the U.S. Private Sector Study. NIST. https://www.nist.gov/news-events/news/2019/10/economic-benefits-global-positioning-system-us-private-sector-study. Published February 22, 2021. Accessed October 21, 2021.

Dziuk B. How GPS tracking can help in emergency service response. How GPS Tracking Can Help in Emergency Service Response. https://info.rastrac.com/blog/emergency-response-use-gps-tracking. Accessed October 20, 2021.

File:Comparison Satellite Navigation Orbits.svg. Wikipedia. https://en.wikipedia.org/wiki/File:Comparison_satellite_navigation_orbits.svg. Published August 5, 2020. Accessed October 25, 2021.

GISGeography. How GPS receivers work - trilateration. GIS Geography. https://gisgeography.com/trilateration-triangulation-gps/. Published May 31, 2021. Accessed November 8, 2021.

Korean Airlines flight shot down by Soviet Union. History.com. https://www.history.com/this-day-in-history/korean-airlines-flight-shot-down-by-soviet-union. Published November 13, 2009. Accessed October 21, 2021.

NASA. Planetary fact sheet. NASA. https://nssdc.gsfc.nasa.gov/planetary/factsheet/. Accessed November 22, 2021.

Orbits. https://physics.highpoint.edu/~jregester/potl/Mechanics/Orbits/orbitsA.htm. Accessed December 2, 2021.

Satellite Navigation - GPS - space segment. Satellite Navigation - GPS - Space Segment | Federal Aviation Administration. https://www.faa.gov/about/office_org/headquarters_offices/ato/service_units/techops/navservices/gnss/gps/spacesegments. Accessed November 16, 2021.

Satellite Navigation - GPS - space segment. Satellite Navigation - GPS - Space Segment | Federal Aviation Administration. https://www.faa.gov/about/office_org/headquarters_offices/ato/service_units/techops/navservices/gnss/gps/spacesegments. Accessed November 3, 2021.

Satellite orbits.
https://cosmospnw.com/wp-content/uploads/2020/08/Satelitte-Orbits.png.
Published December 5, 2018. Accessed October 23, 2021.

Space segment. GPS.gov: Space Segment.
https://www.gps.gov/systems/gps/space/. Accessed October 29, 2021.

Statistics. ITOPF.
https://www.itopf.org/knowledge-resources/data-statistics/statistics/. Accessed October 21, 2021.

Trilateration . GPS.gov: Trilateration .
https://www.gps.gov/multimedia/tutorials/trilateration/. Accessed November 5, 2021.

Derivation of centripetal force.
https://www.gb.nrao.edu/~rmaddale/Education/Wvsta'98/Centripetal.html.
Accessed December 3, 2021.

Admin P. Biography of Johannes Kepler, German astronomer and observer.
Astronomía Astrónomos Universo.
https://www.astrojem.net/biography-of-johannes-kepler-german-astronomer-and-observer/. Published July 1, 2021. Accessed November 22, 2021.

File:Kepler's law 2 en.svg. Wikimedia Commons.
https://commons.wikimedia.org/wiki/File:Kepler%27s_law_2_en.svg. Accessed November 22, 2021.

GPS space segment. GPS Space Segment - Navipedia. https://gssc.esa.int/navipedia/index.php/GPS_Space_Segment. Accessed December 2, 2021.

How does GPS work? - Constellation. NASA. https://spaceplace.nasa.gov/gps/en/. Published June 27, 2019. Accessed January 6, 2022.

Jain A, Bansal R, Kumar A, Singh KD. A comparative study of visual and auditory reaction times on the basis of gender and physical activity levels of medical first year students. International journal of applied & basic medical research. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4456887/. Published 2015. Accessed December 26, 2021.

On the relativistic Doppler effects and high accuracy velocity ... https://www.researchgate.net/publication/238619351_On_the_relativistic_Doppler_Effects_and_high_accuracy_velocity_determination_using_GPS. Accessed December 2, 2021.

Orbits and Kepler's laws. NASA. https://solarsystem.nasa.gov/resources/310/orbits-and-keplers-laws/. Accessed January 6, 2022.

Satellite Navigation - GPS - space segment. Satellite Navigation - GPS - Space Segment | Federal Aviation Administration.

https://www.faa.gov/about/office_org/headquarters_offices/ato/service_units/techops/navservices/gnss/gps/spacesegments. Accessed January 12, 2022.

Space segment. GPS.gov: Space Segment. https://www.gps.gov/systems/gps/space/. Accessed December 2, 2021.

Teacher guidance the physics of GPS – Trilateration - STEM. https://www.stem.org.uk/system/files/elibrary-resources/2018/08/Quantum%20Tech_Teacher%20guide_GPS%20and%20Trilateration.pdf. Accessed December 14, 2021.

Tsokos KA. Circular Motion. In: *Physics for the IB Diploma*. Cambridge, United Kingdom: Cambridge University Press; 2015.

What is a centripetal force? (article). Khan Academy. https://www.khanacademy.org/science/physics/centripetal-force-and-gravitation/centripetal-forces/a/what-is-centripetal-force. Accessed December 2, 2021.