P_{155}, 1:

考察级数:

$$\sum_{k=1}^{+\infty}|x_k|P(X=x_k)=\sum_{k=1}^{+\infty}\left|(-1)^{k+1}\frac{3^k}{k}\right|\cdot\frac{2}{3^k}=2\underbrace{\sum_{k=1}^{+\infty}\frac{1}{k}}$$

调和级数发散

柯西准则:

对于级数 $u_1+u_2+\cdots+u_n+\cdots$

发散的充要条件:存在某正数 ε_0,对任何正整数 N,总存在正整数 $m_0\,(>N)$ 和 p_0,有:

$$|u_{m_0+1}+u_{m_0+2}+\cdots+u_{m_0+p_0}|\geqslant\varepsilon_0$$

$\sum_{k=1}^{+\infty}\frac{1}{k}$ 为发散级数,因:

$\forall\varepsilon_0=\frac{1}{2}$,对任何正整数 N,只要 $m>N$ 和 $p_0=m$,有:

$$\left|\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{2m}\right|$$

$$\geqslant\left|\frac{1}{2m}+\frac{1}{2m}+\cdots+\frac{1}{2m}\right|$$

$$=\frac{1}{2}.$$

$\because\sum_{k=1}^{+\infty}|x_k|P(X=x_k)$ 不收敛,

$\therefore X$ 的数学期望不存在.

P_{155}, 2:

考察积分:

$$\int_{-\infty}^{+\infty}|x|f(x)\,dx=\int_{1}^{+\infty}x\cdot\frac{1}{x^2}\,dx=\int_{1}^{+\infty}\frac{1}{x}\,dx=\ln x\Big|_{1}^{+\infty}=+\infty$$,不收敛

$\therefore X$ 的数学期望不存在.

P_{155}, 4:

X	-1	0	1
P	p_1	p_2	p_3

,

X^2	0	1
P	p_2	p_1+p_3

$$EX = (-1)p_1 + 0\cdot p_2 + 1\cdot p_3 = p_3 - p_1 = 0.2$$

$$EX^2 = 0\cdot p_2 + 1\cdot(p_1+p_3) = p_1 + p_3 = 0.6$$

$$\Rightarrow p_1 = 0.2,\quad p_3 = 0.4$$

$$p_2 = 1 - p_1 - p_3 = 0.4$$

P_{156}, 5:

$P(X=k) = \frac{1}{6}(1-\frac{1}{6})^{k-1}$, $k=1,2,\cdots$, 为几何分布.

$$EX = \sum_{k=1}^{+\infty} k\,P(X=k) = \frac{1}{6}\sum_{k=1}^{+\infty} k\left(\frac{5}{6}\right)^{k-1} = 6$$

参考:

4) 几何分布 $G(p)$

$P(X=k) = pq^{k-1}$, $q=1-p$, $k=1,2,3,\cdots$

$$EX = \sum_{k=0}^{+\infty} k\,P(X=k) = p\sum_{k=1}^{+\infty} kq^{k-1}$$

$$= p\sum_{k=1}^{+\infty}(q^k)'_q = p\left(\sum_{k=1}^{+\infty} q^k\right)'_q$$

$$= p\left(\frac{q}{1-q}\right)'_q = p\,\frac{(1-q)+q}{(1-q)^2}$$

$$= \frac{1}{p}$$

P_{156} 7:

$X = 2, 3, 4, \cdots \qquad p+q=1$

$P(X=2) = p^2 + q^2$ 甲胜或乙胜

$P(X=4) = pq\,p^2 + qp\,q^2$

$\vdots$

$$P(X=2m) = \underbrace{pq \cdot pq \cdots pq}_{2(m-1)\text{个}}\, p^2 + \underbrace{qp \cdot qp \cdots qp}_{2(m-1)\text{个}} \cdot q^2$$

$$= p^{m+1} q^{m-1} + p^{m-1} q^{m+1}, \quad m = 1, 2, 3, \cdots$$

$P(X=3) = pq \cdot q + qp \cdot p$

$P(X=5) = pq\,pq \cdot q + qp \cdot qp \cdot p$

$\vdots$

$$P(X=2m+1) = \underbrace{pq \cdot pq \cdots pq}_{2m\text{个}} \cdot q + \underbrace{qp \cdot qp \cdots qp}_{2m\text{个}} \cdot p$$

$$= p^m q^{m+1} + p^{m+1} q^m, \quad m = 1, 2, 3, \cdots$$

$$EX = \sum_{m=1}^{\infty} 2m\left(p^{m+1}q^{m-1} + p^{m-1}q^{m+1}\right) + \sum_{m=1}^{\infty}(2m+1)\left(p^m q^{m+1} + p^{m+1}q^m\right)$$

$$= 2(p^2+q^2)\sum_{m=1}^{\infty} m(pq)^{m-1} + 2(pq^2+p^2q)\sum_{m=1}^{\infty} m(pq)^{m-1} + \sum_{m=1}^{\infty}(pq)^m \underbrace{(p+q)}_{=1}$$

$$= \left(2(p^2+q^2) + 2pq\right)\sum_{m=1}^{\infty} m(pq)^{m-1} + \sum_{m=1}^{\infty}(pq)^m$$

$\because |pq| \le 1 \qquad m a^{m-1}$ 方便写成求导的形式

$$= \left(2(p^2+q^2)+2pq\right)\left(\frac{d}{d(pq)}\sum_{m=1}^{\infty}(pq)^m\right) + \frac{pq}{1-pq}$$

$$= \left(2(p^2+q^2)+2pq\right)\frac{d}{d(pq)}\left(\frac{pq}{1-pq}\right) + \frac{pq}{1-pq}$$

$$= \left(2(p^2+q^2)+2pq\right)\frac{1}{(1-pq)^2} + \frac{pq}{1-pq}$$

$$= \frac{2\underbrace{(p+q)^2}_{=1} - 2pq + pq - (pq)^2}{(1-pq)^2} = \frac{(2+pq)(1-pq)}{(1-pq)^2}$$

$$= \frac{2+pq}{1-pq}$$

P_{156}. 9:

$$f(x)=\frac{1}{2}e^{-|x|},\ -\infty<x<+\infty$$

$$EX=\int_{-\infty}^{+\infty}xf(x)=\int_{-\infty}^{+\infty}\frac{x}{2}e^{-|x|}dx=0$$

↘奇函数.

$$DX=EX^2-(EX)^2=EX^2=\int_{-\infty}^{+\infty}\frac{x^2}{2}e^{-|x|}dx=2\int_0^{+\infty}\frac{x^2}{2}e^{-x}dx$$

$$=-\int_0^{+\infty}x^2de^{-x}=-x^2e^{-x}\Big|_0^{+\infty}+\int_0^{+\infty}e^{-x}2xdx=-2\int_0^{+\infty}xde^{-x}$$

$$=-2xe^{-x}\Big|_0^{+\infty}+2\int_0^{+\infty}e^{-x}dx=-2e^{-x}\Big|_0^{+\infty}=2.$$

P_{156}, 10:

$$EX=\int_{-\infty}^{+\infty}xf(x)dx=\int_0^{+\infty}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}dx,\quad \underline{\text{令}\ln x=y,\ dx=e^ydy}$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}e^{-\frac{(y-\mu)^2}{2\sigma^2}+y}dy=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}e^{-\frac{y^2}{2\sigma^2}+(1+\frac{\mu}{\sigma^2})y-\frac{\mu^2}{2\sigma^2}}dy$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\sqrt{2\pi\sigma^2}\,e^{\frac{\sigma^2(1+\frac{\mu}{\sigma^2})^2}{2}-\frac{\mu^2}{2\sigma^2}}=e^{\mu+\frac{\sigma^2}{2}}$$

$$\boxed{\int_{-\infty}^{+\infty}e^{-ax^2+bx+c}dx=\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}+c}\quad(a>0)}$$

$$EX^2=\int_{-\infty}^{\infty}x^2f(x)dx=\int_0^{+\infty}\frac{x}{\sqrt{2\pi}\sigma}e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}dx,\quad \underline{\text{令}\ln x=y}$$

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}e^{-\frac{(y-\mu)^2}{2\sigma^2}+2y}dy=e^{2\mu+2\sigma^2}$$

$$DX=EX^2-(EX)^2=e^{2\mu+2\sigma^2}-e^{2\mu+\sigma^2}=e^{2\mu+\sigma^2}(e^{\sigma^2}-1).$$

P_{156}, 11:

$$EX = \int_{-\infty}^{+\infty} x f(x)\,dx = \int_0^{+\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\beta x}\,dx, \quad \text{令 } t=\beta x$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{\beta^{\alpha+1}} \underbrace{\int_0^{+\infty} t^\alpha e^{-t}\,dt}_{\Gamma(\alpha+1),\ P_{110}.}$$

$$= \frac{\beta^{-1}}{\Gamma(\alpha)} \Gamma(\alpha+1), \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$$

$$= \frac{\alpha}{\beta}.$$

$$DX = EX^2 - (EX)^2, \quad EX^2 = \int_0^{+\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha+1} e^{-\beta x}\,dx$$

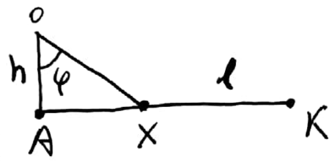
$$= \int_0^{+\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{\beta^{\alpha+2}} t^{\alpha+1} e^{-t}\,dt$$

$$= \frac{\alpha(\alpha+1)}{\beta^2}$$

$$DX = \frac{\alpha^2+\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2}$$

$$= \frac{\alpha}{\beta^2}.$$

P_{157}. 15:



记随机变量X为AX的长度.

$$f(x) = \begin{cases} \frac{1}{l}, & 0 < x < l \\ 0, & \text{其他} \end{cases}, \qquad \tan\varphi = \frac{x}{h}, \quad \varphi = \arctan\frac{x}{h}.$$

$$E\varphi = \int_{-\infty}^{+\infty} \varphi(x) f(x)\,dx = \int_0^l \arctan\frac{x}{h} \cdot \frac{1}{l}\,dx$$

$$= \frac{x}{l}\arctan\frac{x}{h}\Big|_0^l - \frac{1}{l}\int_0^l x\,d\left(\arctan\frac{x}{h}\right)$$

$$= \arctan\frac{l}{h} - \frac{1}{l}\int_0^l \frac{x}{h\left(1+\frac{x^2}{h^2}\right)}\,dx$$

$$= \arctan\frac{l}{h} - \frac{h}{2l}\int_0^l \frac{d\left(\frac{x}{h}\right)^2}{1+\left(\frac{x}{h}\right)^2} = \arctan\frac{l}{h} - \frac{h}{2l}\ln\left(1+\left(\frac{x}{h}\right)^2\right)\Big|_0^l$$

$$= \arctan\frac{l}{h} - \frac{h}{2l}\ln\left(1+\left(\frac{l}{h}\right)^2\right) = \arctan\frac{l}{h} + \frac{h}{l}\ln\frac{h}{\sqrt{h^2+l^2}}.$$

P_{157}, 22:

$\because X, Y$ 独立同分布于 $N(0,\frac{1}{2})$. $\therefore X-Y \sim N(0,\frac{1}{2}+\frac{1}{2}) = N(0,1)$.

令 $T = X-Y \sim N(0.1)$

$$E|X-Y| = E|T| = \int_{-\infty}^{+\infty} |t| \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 2\int_0^{+\infty} \frac{t}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= -2\int_0^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} d(-\frac{t^2}{2}) = -2\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \Big|_0^{+\infty} = \sqrt{\frac{2}{\pi}}.$$

$$E|X-Y|^2 = E|T|^2 = \int_{-\infty}^{+\infty} t^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 2\int_0^{+\infty} \frac{t^2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= -2\int_0^{+\infty} \frac{t}{\sqrt{2\pi}} de^{-\frac{t^2}{2}}. = -2\frac{t}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \Big|_0^{+\infty} + 2\int_0^{\infty} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}} dt$$

$$= 1$$

$$D|X-Y| = E|X-Y|^2 - (E|X-Y|)^2 = 1 - \frac{2}{\pi}.$$

P_{157}. 28:

$$DX = EX^2 - (EX)^2$$

$$E(X-C)^2 = E(X^2 - 2CX + C^2) = EX^2 - 2CEX + C^2.$$

$$DX - E(X-C)^2 = EX^2 - (EX)^2 - (EX^2 - 2CEX + C^2)$$

$$= -((EX)^2 - 2CEX + C^2)$$

$$= -(EX - C)^2 \leq 0$$

$\therefore DX \leq E(X-C)^2$.

P_{157}. 29:

$$\left.\begin{aligned} EX &= \int_a^b x f(x) dx \leq \int_a^b b f(x) dx = b \\ EX &= \int_a^b x f(x) dx \geq \int_a^b a f(x) dx = a \end{aligned}\right\} \Rightarrow a \leq EX \leq b$$

由上题 $DX \leq E(X - \frac{b+a}{2})^2 \leq E(b - \frac{b+a}{2})^2 = \frac{(b-a)^2}{4}$

P_{157}, 24:

令 $X_i=\begin{cases}1, & \text{第i号球放在第i个盒中} \\ 0 & \text{第i号球未放在第i个盒中}\end{cases}$, 则 X_i 的分布律:

X_i	0	1
P	$\frac{n-1}{n}$	$\frac{1}{n}$

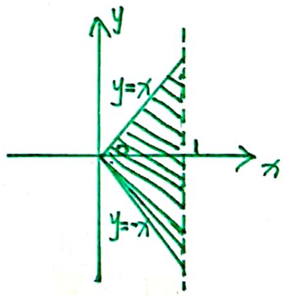
$i=1,2,\cdots,n$

$EX_i = 1\cdot\frac{1}{n}+0\cdot\frac{n-1}{n}=\frac{1}{n}$

$X=X_1+X_2+\cdots+X_n$

$$EX=EX_1+EX_2+\cdots+EX_n = n\cdot\frac{1}{n} = 1$$

P_{158}, 32:



$$EX=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x f(x,y)\,dx\,dy = \int_0^1\int_{-x}^{x} x\,dy\,dx = \int_0^1 2x^2\,dx = \frac{2}{3}$$

$$EY=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} y f(x,y)\,dx\,dy = \int_0^1\int_{-x}^{x} y\,dy\,dx = 0$$

$$EXY=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f(x,y)\,dx\,dy = \int_0^1\int_{-x}^{x} xy\,dy\,dx = 0$$

$$\mathrm{cov}(X,Y)=EXY-EXEY = 0$$

$P_{158}.33$:

$f_X(x)=\int_0^2 \frac{1}{8}(x+y)dy=\frac{x}{4}+\frac{1}{4}$

$f_Y(y)=\int_0^2 \frac{1}{8}(x+y)dx=\frac{y}{4}+\frac{1}{4}$

$EX=\int_0^2 x f_X(x)dx=\frac{7}{6}$

$EY=\int_0^2 y f_Y(y)dy=\frac{7}{6}$

$EX^2=\int_0^2 x^2 f_X(x)dx=\frac{5}{3}$

$EY^2=\int_0^2 y^2 f_Y(y)dy=\frac{5}{3}$

$DX=EX^2-(EX)^2=\frac{11}{36}$

$DY=EY^2-(EY)^2=\frac{11}{36}$

$EXY=\int_0^2\int_0^2 x\cdot y\cdot\frac{1}{8}(x+y)dxdy$

$=\frac{4}{3}$.

$Cov(X,Y)=EXY-EX\cdot EY$

$=-\frac{1}{36}$

$\rho_{XY}=\frac{Cov(X,Y)}{\sqrt{DX\,DY}}=-\frac{1}{11}$.

$P_{158}, 39$:

$$f_X(x)=\begin{cases}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{\pi}dy=\frac{2}{\pi}\sqrt{1-x^2}, & -1\le x\le 1\\ 0, & 其他\end{cases}$$

$$f_Y(y)=\begin{cases}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\frac{1}{\pi}dx=\frac{2}{\pi}\sqrt{1-y^2}, & -1\le y\le 1\\ 0. & 其他\end{cases}$$

$f(x,y)\neq f_X(x)\cdot f_Y(y)$，不相互独立

$EX=\int_{-1}^1 x\,\frac{2}{\pi}\sqrt{1-x^2}\,dx=0$

$EY=\int_{-1}^1 y\cdot\frac{2}{\pi}\sqrt{1-y^2}\,dy=0.$

$EXY=\int_{x^2+y^2\le 1}\frac{xy}{\pi}dxdy=\frac{1}{\pi}\int_0^1 r^3dr\underbrace{\int_0^{2\pi}\cos\theta\sin\theta d\theta}_{=0}$

$x=r\cos\theta$，$y=r\sin\theta$.

$Cov(X,Y)=EXY-EXEY=0-0=0$，不相关.

P_{159}. 41:

设事件A的概率为P_A，事件B的概率为P_B.

$p(X=1)=P_A$. $p(X=-1)=1-P_A$, $p(Y=1)=P_B$, $p(Y=-1)=1-P_B$

① A, B 相互独立 $\Rightarrow$ X, Y 不相关：

$EX = 1\cdot p(X=1) - 1\cdot p(X=-1) = 2P_A - 1$.

$EY = 1\cdot p(Y=1) - 1\cdot p(Y=-1) = 2P_B - 1$

$EXY = 1\cdot 1\cdot p(X=1,Y=1) + 1\cdot(-1)\cdot p(X=1,Y=-1) + (-1)\cdot 1\cdot p(X=-1,Y=1) + (-1)\cdot(-1)p(X=-1,Y=-1)$

(A, B 独立) $= p(X=1)\cdot p(Y=1) - p(X=1)\cdot p(Y=-1) - p(X=-1,Y=1) + p(X=-1)p(Y=-1)$

$= P_AP_B - P_A(1-P_B) - (1-P_A)P_B + (1-P_A)(1-P_B)$

$= (2P_A-1)(2P_B-1)$

$\therefore Cov(X,Y) = EXY - EX\cdot EY = (2P_A-1)(2P_B-1) - (2P_A-1)(2P_B-1) = 0$

$\therefore$ X, Y 不相关.

② X, Y 不相关 $\Rightarrow$ A, B 相互独立.

$\because$ X, Y 不相关 $\therefore cov(X,Y)=0$, $\therefore EXY = EX\cdot EY$

$\therefore p(X=1,Y=1) - p(X=1,Y=-1) - p(X=-1,Y=1) + p(X=-1,Y=-1) = (2P_A-1)(2P_B-1)$ (I)

又$\because p(X=1,Y=1) + p(X=1,Y=-1) + p(X=-1,Y=1) + p(X=-1,Y=-1) = 1$ (II)

(I)+(II) 得: $p(X=1,Y=1) + p(X=-1,Y=-1) = \frac{1}{2}(2P_A-1)(2P_B-1) + \frac{1}{2}$

$= 2P_AP_B - (P_A+P_B) + 1$ (III)

$\because p(X=1,Y=1) = p(AB) = 1 - p(\overline{AB}) = 1 - p(\bar{A}\cup\bar{B}) = 1 - p(\bar{A}) - p(\bar{B}) + p(\bar{A}\bar{B})$

$= 1 - p(X=-1) - p(Y=-1) + p(X=-1,Y=-1)$

$\therefore p(X=1,Y=1) - p(X=-1,Y=-1) = 1-(1-P_A)-(1-P_B) = P_A+P_B-1$ (IV)

(III)+(IV) 得: $p(X=1,Y=1) = P_A\cdot P_B$.

(III)-(I)得: $p(X=1, Y=-1) + p(X=-1, Y=1) = \frac{1}{2} - \frac{1}{2}(2P_A-1)(2P_B-1)$

$$= P_A + P_B - 2P_A P_B \qquad (V)$$

$\because p(X=1, Y=-1) = p(A\bar{B}) = 1 - p(\overline{A\bar{B}}) = 1 - p(\bar{A} \cup B) = 1 - p(\bar{A}) - p(B) + p(\bar{A}B)$

$$= 1 - p(X=-1) - p(Y=1) + p(X=-1, Y=1)$$

$\therefore p(X=1, Y=-1) - p(X=-1, Y=1) = 1 - (1-P_A) - P_B = P_A - P_B \qquad (VI)$

(V)+(VI)得: $p(X=1, Y=-1) = P_A - P_A P_B = P_A(1-P_B)$

类似可证: $p(X=-1, Y=1) = (1-P_A)P_B$, $p(X=-1, Y=-1) = (1-P_A)(1-P_B)$.

((III)+(IV) $\longrightarrow$ (III)-(IV)
(V)+(VI) $\longrightarrow$ (V)-(VI).)

$\therefore p(X=i, Y=j) = p(X=i)p(Y=j)$, $i, j = -1, 1$

$\therefore$ AB 相互独立.