Alternative $k = -1$ loop quantum cosmology

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An alternative quantization of the gravitational Hamiltonian constraint of the $k = -1$ Friedmann-Robertson-Walker model is proposed by treating the Euclidean term and the Lorentzian term independently, mimicking the treatment of full loop quantum gravity. The resulting Hamiltonian constraint operator for the $k = -1$ model with a massless scalar field is successfully constructed, and is shown to have the corrected classical limit. Compared to the former quantization schemes in the literature where only the Euclidean term is quantized, the new quantum dynamics of the $k = -1$ model with a massless scalar field indicates that the classical big-bang singularity is replaced by an asymmetric quantum bounce.

I. INTRODUCTION

How to quantize general relativity (GR) in a consistent manner is a great challenge to theoretical physics. One of the promising candidates is the so-called loop quantum gravity (LQG) which is a nonperturbative approach to quantum GR [1–4]. In the past three decades, LQG has made remarkable progress, such as making the natural predictions of the discretized geometries and providing the microscopic interpretation of BH entropy [5–12]. The nonperturbative quantization procedure of LQG has been successfully applied to the metric $f(R)$ theories [13, 14], scalar-tensor theories [15, 16], higher-dimensional gravity [17], and so on [18]. Despite these achievements, the dynamics of full LQG is still an unsolved issue. To gain a certain level of understanding of the dynamics, the quantization ideas and technologies developed in LQG have also be applied to its symmetry-reduced models, such as the Friedmann-Robertson-Walker (FRW) models and the spherically symmetric black hole models, leading to loop quantum cosmology (LQC) and loop quantum black hole models [19, 20]. The most successful feature of LQC is that it can resolve the classical big bang singularity by a quantum bounce due to the quantum geometry effects. We refer to [19, 21–23] for more complete reviews on LQC.

In full LQG, the gravitational Hamiltonian constraint is a combination of the so-called Euclidean term and the Lorentzian term. In the spatially flat, $k = 0$ FRW model, the Lorentzian term and the Euclidean term are proportional to each other. Thus one often combines these two terms into one term proportional to the Euclidean term, and then quantizes the Euclidean term to obtain the well-defined gravitational Hamiltonian constraint operator [19, 22]. It turns out that in this quantization scheme the classical big-bang singularity is replaced by a symmetric quantum bounce for the $k = 0$ FRW model with a massless scalar field in the framework of LQC [22]. Note that in full LQG, the Lorentzian term is quantized independently by employing the Thiemann's trick [24]. Thus, to mimic the full LQG quantization procedure in the $k = 0$ model of LQC, the Euclidean term and the Lorentzian term were treated independently [25–27]. This alternative quantization scheme leads to an asymmetric quantum bounce, which relates the spatially flat FRW model with an asymptotic de Sitter universe, and thus an effective cosmological constant and an effective Newtonian constant can be obtained [28–30].

As in the $k = 0$ model, the quantization technologies for the gravitational Hamiltonian constraint developed in LQG have been extended to the $k = -1, +1$ models [31–38]. Compared to the $k = 0$ model where the spin connection vanishes and hence the Ashtekar connection equals to the extrinsic curvature multiplied by the Immirzi parameter, the Lorentzian term is not proportional to the total Euclidean term, but is proportional to the part of the Euclidean term involving the extrinsic curvature due to the nonvanishing spin connection for both the $k = -1$ model and the $k = +1$ model. Hence in the literatures one often absorbs the Lorentzian term into a part of the Euclidean term, and then quantize the two parts of the Euclidean term, respectively. It turns out that, as the $k = 0$ model with the similar treatment, the resulting $k = -1$ LQC model also predicts a vacuum repulsion in the high curvature regime that would lead to a symmetric bounce [31]. Moreover, the $k = -1$ model of LQC also possesses some new features that never appears in the $k = 0$ model, for example, due to a vacuum repulsion in the high curvature regime, the scale factor has the minimum value as $a_{\min} = \gamma\sqrt{\Delta}$ [31]. It is natural to ask whether the treatment of the Lorentzian term independently, mimicking the treatment in the full theory, can be directly carried to the $k = -1$ model, and whether an asymmetric bounce can still be held for the $k = -1$ model. This is the main motivation of the present paper. In this paper, we consider an alternative quantization of the gravitational Hamiltonian constraint in the $k = -1$ model by treating the Lorentzian term independently.

This paper is organized as follows. The canonical formulation of the $k = -1$ model is briefly recalled in Sec. II. Then we propose an alternative gravitational Hamiltonian constraint operator by treating the Lorentzian term independently, and provide a new quantum dynamics for the $k = -1$ model in Sec. III. The effective theory of the new quantum dynamics and its asymptotic behavior are studied in Sec. IV. Summary

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is included in the last section.

II. CANONICAL FORMULATION OF THE $k = -1$ MODEL

According to the cosmological principle, the line elements of the homogenous isotropic cosmological models take as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{1}{1-kr^2}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right], \tag{2.1}$$

where $a(t)$ is the scale factor, and $k = -1, 0, 1$ for the open, flat, and closed FRW models, respectively.

In what follows, we present the canonical formulation of the $k = -1$ model following Ref. [31]. For the spatially noncompact $k = 0, -1$ models with topology homeomorphic to $\mathbb{R}^3$, one introduces an "elemental cell" $\mathcal{V}$ on the homogeneous spatial manifold $\mathbb{R}^3$ and restrict all integrals to this elemental cell. Then one chooses a fiducial metric ${}^o q_{ab} = {}^o\omega_a^i\, {}^o\omega_b^j \delta_{ij}$ on $\mathbb{R}^3$ with ${}^o\omega_a^i$ being the left- and right-invariant fiducial one-forms in the $k = 0$ model, and only the left-invariant fiducial one-forms in the $k = -1$ model. Here $a, b, \cdots$ denote the spatial indices while $i, j, \cdots = 1, 2, 3$. Denote V_o the volume of $\mathcal{V}$ measured by the fiducial metric ${}^o q_{ab}$. The left-invariant one-forms ${}^o\omega_a^i$ satisfy the Maurer-Cartan equation

$$\mathrm{d}\, {}^o\omega^i + \frac{1}{2}C^i{}_{jk}\, {}^o\omega^j\, {}^o\omega^k = 0, \tag{2.2}$$

where for the $k = -1$ model the structure constants read

$$C^i{}_{jk} = \delta^i_j\delta_{k1} - \delta^i_k\delta_{j1}, \tag{2.3}$$

while for the $k = 0$ model they take zero. The corresponding left-invariant vector fields ${}^o e_i^a$ are dual to ${}^o\omega_a^i$, satisfying ${}^o e_i^a\, {}^o\omega_a^j = \delta_i^j$ and ${}^o e_i^a\, {}^o\omega_b^i = \delta_b^a$. The commutators between the left-invariant vector fields read

$$[{}^o e_i, {}^o e_j] = C^k{}_{ij}\, {}^o e_k. \tag{2.4}$$

Classically, the dynamical variables of LQC are obtained by symmetrically reducing those of full LQG. In the full theory, the dynamical variables consist of the $su(2)$-valued connection A_a^i and the densitized triad $\tilde{E}_j^b$ with the nontrivial Poisson bracket

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \kappa\gamma\delta_a^b\delta_j^i\delta(x, y), \tag{2.5}$$

where $\kappa = 8\pi G$ with G being the Newtonian constant, and γ is the Immirzi parameter [39, 40]. The connection A_a^i is related to the spin connection Γ_a^i and the extrinsic curvature K_a^i by $A_a^i = \Gamma_a^i + \gamma K_a^i$. It turns out that the symmetry-reduced extrinsic curvature K_a^i is diagonal in the basis of left-invariant one-forms for the $k = 0, -1$ models. While, unlike the $k = 0$ model where Γ_a^i vanishes, the symmetry-reduced spin connection Γ_a^i in the $k = -1$ model takes the form [31]

$$\Gamma_a^i = -\epsilon^{1ij}\, {}^o\omega_a^j, \tag{2.6}$$

and thus it is nondiagonal. Hence the symmetry-reduced connection and densitized triad for the $k = -1$ model read [31]

$$A_a^i = -\epsilon^{1ij}\, {}^o\omega_a^j + cV_o^{-\frac{1}{3}}\, {}^o\omega_a^i \equiv A_j^i V_o^{-\frac{1}{3}}\, {}^o\omega_a^j, \tag{2.7}$$

$$\tilde{E}_i^a = pV_o^{-\frac{2}{3}}\sqrt{\det({}^o q)}\, {}^o e_i^a, \tag{2.8}$$

where

$$A_j^i = \begin{pmatrix} c & 0 & 0 \\ 0 & c & -V_o^{\frac{1}{3}} \\ 0 & V_o^{\frac{1}{3}} & c \end{pmatrix}, \tag{2.9}$$

the variables c and p are only functions of t, and $\det({}^o q)$ denotes the determinant of ${}^o q_{ab}$. Hence the gravitational phase space of the $k = -1$ model consists of conjugate pairs (c, p). The nontrivial Poisson bracket reads

$$\{c, p\} = \frac{\kappa}{3}\gamma. \tag{2.10}$$

Note that the variables c and p are related to the scale factor a by $|p| = a^2 V_o^{\frac{2}{3}}$ and $c = \gamma\dot{a}V_o^{\frac{1}{3}}$. The physical volume V of the elemental cell $\mathcal{V}$ measured by the spatial (physical) metric $q_{ab} = |p|V_o^{-\frac{2}{3}}\, {}^o q_{ab}$ is related to p via $V = |p|^{3/2}$. In the improved scheme, it is convenient to choose the following variables to simplify the dynamics [41]

$$b := \frac{\bar{\mu}c}{2}, \qquad v := \frac{\mathrm{sgn}(p)|p|^{3/2}}{2\pi\gamma\ell_{\mathrm{p}}^2\sqrt{\Delta}}, \tag{2.11}$$

where $\ell_{\mathrm{p}} \equiv \sqrt{G\hbar}$ denotes the Planck length, $\mathrm{sgn}(p)$ is the signature of p, $\Delta \equiv 4\sqrt{3}\,\pi\gamma\ell_{\mathrm{p}}^2$ is the minimum nonzero eigenvalue of the area operator in full LQG [42], and $\bar{\mu} \equiv \sqrt{\Delta/|p|}$. The Poisson bracket between b and v is given by

$$\{b, v\} = \frac{1}{\hbar}. \tag{2.12}$$

As in the $k = 0$ model, the Gauss and diffeomorphism constraints of the gravitational part are automatically satisfied for the symmetry-reduced variables in Eqs. (2.7) and (2.8) in the $k = -1$ model, and thus the classical dynamics is encoded in the Hamiltonian constraint. The gravitational Hamiltonian constraint of the $k = -1$ model reads

$$\mathcal{H}_{\mathrm{grav}}^{k=-1} := \int_{\mathcal{V}} \mathrm{d}^3x\, \frac{\tilde{E}_i^a\tilde{E}_j^b}{2\kappa\sqrt{\det(q)}}\left[\epsilon^{ij}{}_k\, {}^{(A)}F_{ab}^k - 2(1+\gamma^2)K_{[a}^i K_{b]}^j\right]$$