CSM 166: Discrete Mathematics for Computer Science

Multinomial Coefficients

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Content

Multinomial Coefficients

Binomial Coefficients

- The number of r-combinations from a set with n elements is often denoted by $\binom{n}{k}$
- This is also called a **binomial coefficient** as it occurs as coefficient in the expansion of powers of binomial expressions such as $(a+b)^n$.
- A binomial expression is simply the sum of two terms, such as $x+y$.

Theorem 1 (Binomial Theorem)

Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

where $\binom{n}{j}$ is called the ***binomial coefficient****.*

Proof left as exercise

Binomial Coefficients

Example 1

What is the expansion of $(x+y)^4$

Solution:

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4$$

Binomial Coefficients

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Binomial Coefficients

Example 2

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

Solution:

It follows from the binomial theorem that this coefficient is

$$\binom{25}{13} = \frac{25!}{13!12!} = 5,200,300$$

Binomial Coefficients

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Example 3

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

Solution:

$(2x-3y)^{25} = (2x+(-3y))^{25}$

By the binomial theorem

$$(2x+(-3y))^{25} = \sum_{J=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^{j}$$

The coefficient of $x^{12}y^{13}$ is obtained when $j = 13$:

$$\binom{25}{13} 2^{12}(-3)^{13} = -\frac{25!}{13!12!} 2^{12}3^{13}$$

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Binomial Coefficients

Note:

Let n be a nonnegative integer. Then

1. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

2. $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

3. $$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Binomial Coefficients

Theorem 2 (Pascal's Identity)

Let n and k be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Pascal's Identity and Triangle

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$
$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$
$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$
$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$
$$\cdots$$

(a)

By Pascal's identity:

$$\binom{6}{4} + \binom{6}{5} = \binom{7}{5}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

. . .

(b)

Multinomial Coefficients

Definition 1

Let $k_1, k_1, \dots k_r$ be integers satisfying the relation $k_1 + k_2 + \cdots + k_r = n$. Then

$$\binom{n}{k_1, k_2 \dots k_r} = \frac{n!}{k_1!k_2!\dots k_r!}$$
$$= \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\dots\binom{n-\sum_{i=1}^{r-i} k_i}{k_r}$$

is called a ***multinomial coefficient****.*

NB: *Binomial coefficients are special case of $r = 2$*

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Multinomial Coefficients

The multinomial coefficient $\binom{n}{k_1,k_2,\ldots,k_r}$ is:

- the number of ways to put interchangeable objects into r boxes, so that box i has k_i objects in it, for $1 \le i \le r$.

- the number of ways to choose k_1 interchangeable objects from objects, then to choose k_2 from what remains, then to choose k_3 from what remains, ..., then to choose k_{r-1} from what remains.

Multinomial Coefficients

The multinomial coefficient $\binom{n}{k_1, k_2, \ldots, k_r}$ is:

- the number of unique permutations of a word with n letters and r distinct letters, such that the ith letter occurs k_i times.

Multinomial Coefficients

Example 4

A police department of 10 officers wants to have 5 patrol the streets of KNUST, 2 doing paperwork, and 3 at the Icy-cup shops on campus. How many ways can this be done?

Solution

$$\binom{10}{5,2,3} = \frac{10!}{5!2!3!} = 2520$$

Multinomial Coefficients

Theorem 3

For any $x_1, \ldots, x_r$ and $n > 1$

$$(x_1 + \cdots + x_r)^n = \sum_{\substack{(k_1,\ldots,k_r) \\ k_1+\cdots+k_r=n}} \binom{n}{k_1, k_2, \ldots, k_r} x_1^{k_1} x_2^{k_2} \ldots x_r^{k_r} \tag{1}$$

Multinomial Coefficients I

Example 5

Expand $(x+y+z)^3$

Solution

$$
\begin{aligned}
(x+y+z)^3 = {}& {}^mC_{3,0,0}x^3y^0z^0 + {}^mC_{0,3,0}x^0y^3z^0 + {}^mC_{0,0,3}x^0y^0z^3 \\
&+ {}^mC_{2,1,0}x^2y^1z^0 + {}^mC_{2,0,1}x^2y^0z^1 + {}^mC_{1,2,0}x^1y^2z^0 \\
&+ {}^mC_{0,2,1}x^2y^2z^1 + {}^mC_{1,0,2}x^1y^0z^2 + {}^mC_{0,1,2}x^0y^1z^2 \\
&+ {}^mC_{1,1,1}x^1y^1z^1
\end{aligned}
$$

Multinomial Coefficients II

$$= \frac{3!}{3!0!0!}x^3 + \frac{3!}{0!3!0!}y^3 + \frac{3!}{3!0!0!}z^3 + \frac{3!}{2!1!0!}x^2y$$
$$+ \frac{3!}{2!0!1!}x^2z + \frac{3!}{1!2!0!}xy^2 + \frac{3!}{0!2!1!}y^2z + \frac{3!}{1!0!2!}xz^2$$
$$+ \frac{3!}{0!1!2!}yz^2 + \frac{3!}{1!1!1!}xyz$$

$$= x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3xy^2 + 3y^2z$$
$$+ 3xz^2 + 3yz^2 + 6xyz$$

Multinomial Coefficients

Example 6

1. Evaluate the following
   a) $\binom{6}{4,2,0}$

   b) $\binom{5}{3,2}$

   c) $\binom{10}{5,3,0,2}$

2. Find the number m of ways that 9 toys can be divided between 4 children if the youngest is to receive 3toys and each of the others 2 toys.

Multinomial Coefficients

Exercise A: Determine the coefficient of the following terms in the indicated multinomial expressions.

i) xyz^2 in $(2x-y-z)^4$

ii) xyz^{-2} in $x-2y+3z^{-1}$

iii) $w^3x^2yz^2$ in $(2w-x+3y-2z)^8$

iv) $x^{11}y^4z^2$ in $(2x^3-3xy^2+z^2)^6$

v) $x^3y^4z^5$ in $(x-2y+3z)^{12}$

Exercise B:

1. The letters $B, C, E, E, N, R, S, S, Y, Z, Z, Z, Z$ are arranged at random. Determine the probability that these letters will spell the word SZCZEBRZESZYN

2. Expand $(x+y+z)^6$. Hence with $x=y=z=0.3$ evaluate 0.96 to six decimal places.

3. Find the number of distinct permutations of the letters in the word MISSISSIPPI.

End of Lecture

Questions...???

Thanks

Reference Books

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2. H. Levy, F. Lessman Finite Difference Equations.Dover books on mathematices
3. Gary Chartrand. Ping Zhang. Discrete Mathematics 1^{th}
4. Oscar Leven. Discrete Mathematics: An open introduction. 2nd Edition. 2013