DECOMPOSITION METHOD

Dr. Gabriel Obed Fosu
Department of Mathematics
Kwame Nkrumah University of Science and Technology
Google Scholar: https://scholar.google.com/citations?user=ZJfCMyQAAAAJ&hl=en&oi=ao
ResearchGate ID: https://www.researchgate.net/profile/Gabriel_Fosu2

Outline

Outline of Presentation

Definition (Cholesky Decomposition)

The decomposition is such that

$$A = R^T R \tag{1}$$

That is every symmetric positive definite matrix A can be decomposed into a product of a unique upper triangular matrix R and its transpose.

The derivation is done for 3×3 matrix but could be generalized.

$$A = R^T R \tag{2}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} r_{11} & 0 & 0 \\ r_{21} & r_{22} & 0 \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} \tag{3}$$

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Find the second column of R. We already know r_{11}, r_{12} and r_{13}, so we only need to equate the second and third entries of the second column of both sides.

$$a_{22} = r_{12}^2 + r_{22}^2 \implies r_{22} = \sqrt{a_{22} - r_{12}^2} \tag{9}$$

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$$a_{23} = r_{12}r_{13} + r_{23}r_{22} \quad \Longrightarrow \quad r_{23} = \frac{a_{23} - r_{12}r_{13}}{r_{22}} \tag{10}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} r_{11}^2 & r_{11}r_{12} & r_{11}r_{13} \\ r_{11}r_{12} & r_{12}^2 + r_{22}^2 & r_{12}r_{13} + r_{22}r_{23} \\ r_{11}r_{13} & r_{12}r_{13} + r_{23}r_{22} & r_{13}^2 + r_{23}^2 + r_{33}^2 \end{bmatrix} \tag{11}$$

Finding the third column of R.

$$a_{33} = r_{13}^2 + r_{23}^2 + r_{33}^2 \implies r_{33} = \sqrt{a_{33} - r_{13}^2 - r_{23}^2} \tag{12}$$

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Generally

$$r_{ki} = \frac{a_{ki} \cdot \sum_{j=1}^{i-1} r_{ij} \cdot r_{kj}}{r_{ii}}; \qquad r_{kk} = \sqrt{a_{kk} \cdot \sum_{j=1}^{k-1} r_{kj}^2} \tag{13}$$

Using Cholesky Decomposition to Solve Equations

1. Factor A into the product of R^T and R: that is

$$Ax = b \tag{14}$$
$$R^T Rx = b \tag{15}$$
$$R^T (Rx) = b \tag{16}$$

$$\text{Let } Rx = y \quad \text{where } y = n \times 1 \text{ vector} \tag{17}$$

Now equation (16) becomes

$$R^T y = b \tag{18}$$

2. First solve $R^T y = b$ to find y by forward substitution.
3. Then solve $Rx = y$ to find x by back substitution.

Example

Solve

$$\begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix}$$

with the Cholesky decomposition method

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$$A = \begin{bmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{bmatrix}$$

then

$$r_{11} = \sqrt{a_{11}} = \sqrt{25} = 5; \quad r_{12} = \frac{a_{12}}{r_{11}} = \frac{15}{5} = 3; \quad r_{13} = \frac{a_{13}}{r_{11}} = \frac{-5}{5} = -1 \tag{19}$$

$$r_{22} = \sqrt{a_{22} - r_{12}^2} = \sqrt{18 - 3^2} = 3 \tag{20}$$

$$r_{23} = \frac{a_{23} - r_{12}r_{13}}{r_{22}} = \frac{0 - 3(-1)}{3} = 1 \tag{21}$$

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Therefore

$$R = \begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \qquad R^T = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \tag{23}$$

1. First solve $R^T y = b$ to find y by forward substitution.

$$\begin{bmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 35 \\ 33 \\ 6 \end{bmatrix} \tag{24}$$

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$$5y_1 = 35 \implies y_1 = 7 \tag{25}$$

$$3(7) + 3y_2 = 33 \implies y_2 = \frac{12}{3} = 4 \tag{26}$$

$$-(7) + 4 + 3y_3 = 6 \implies y_3 = \frac{9}{3} = 3 \tag{27}$$

1. Then solve $Rx = y$ to find x by back substitution.

$$\begin{bmatrix} 5 & 3 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix} \tag{28}$$

$$3x_3 = 3 \implies x_3 = 1 \tag{29}$$

$$3x_2 + 1 = 4 \implies x_2 = 1 \tag{30}$$

$$5x_1 + 3(1) - 1 = 7 \implies x_1 = 1 \tag{31}$$

Solution by Cholesky Decomposition method is

$$x_1 = x_2 = x_3 = 1$$

Outline of Presentation

1. The Cholesky Decomposition
2. Gram-Schmidt Orthonormalization
3. QR Decomposition

Gram-Schmidt Orthonormalization

The Gram-Schmidt process takes a set of linearly independent vectors

$$S = \{v_1,\ v_2, \cdots, v_n\} \in R^m \tag{32}$$

and transforms them into a set of orthonormal vectors

$$S' = \{e_1,\ e_2, \cdots, e_n\} \tag{33}$$

1. The orthonormal set $S' = \{e_1,\ e_2, \cdots, e_n\}$ spans the same n-dimensional subspace as S.
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2. It follows that $m \geq n$ or the vectors would be linearly dependent.

An upper triangular matrix R is also obtained during the derivation process.

1. Assume $A = [v_1 \;\; v_2 \cdots v_{n-1} \;\; v_n]$ is an $m \times n$ matrix with columns $v_1 \; v_2 \cdots v_{n-1} \; v_n$.
2. The Gram-Schmidt process can be used to factor A into a product

$$A = QR \tag{34}$$

where $Q^{m \times n}$ has orthonormal columns, and $R^{n \times n}$ is an upper-triangular matrix.

3. The decomposition comes directly from the Gram-Schmidt process by using the r_{ij} values we defined in the description of Gram-Schmidt.

Step for the orthonormalization

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2. Define

$$r_{12} = \langle v_2,\ e_1 \rangle, \qquad u_2 = v_2 - r_{12} e_1 \tag{36}$$

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$$e_2 = \frac{u_2}{||u_2||} = \frac{u_2}{r_{22}}; \quad r_{22} = ||u_2|| \tag{37}$$

1. Similarly, we define

$$r_{13} = \langle v_3,\ e_1\rangle, \quad r_{23} = \langle v_3,\ e_2\rangle \quad u_3 = v_3 - r_{13}e_1 - r_{23}e_2 \tag{38}$$

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1. Similarly, we define

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then

$$e_3 = \frac{u_3}{||u_3||} = \frac{u_3}{r_{33}}; \quad r_{33} = ||u_3|| \tag{39}$$

Generally, the formula for computing a general vector, e_i, is:

$$u_i = v_i - \sum_{j=1}^{i-1} r_{ij} e_j; \quad 1 \leq i \leq n \tag{40}$$

where

$$r_{ij} = \langle v_i, \ e_j \rangle \tag{41}$$

then

$$e_i = \frac{u_i}{||u_i||_2} = \frac{u_i}{r_{ii}}; \qquad r_{ii} = ||u_i|| \tag{42}$$

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The sequence $e_1, \cdots, e_k$ is the required set of orthonormal vectors, and the process is known as **Gram-Schmidt orthonormalization**

Example

Given

$$S = \begin{bmatrix} 1 & 3 & 3 \\ -1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \tag{43}$$

Decompose S into Q a set orthonormal vectors and R an upper triangular matrix.

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Decompose S into Q a set orthonormal vectors and R an upper triangular matrix.

We first observe that the column vectors are linearly independent in $\mathbb{R}^3$. That is

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

Step for the orthonormalization

Iteration 1

$$r_{11} = ||v_1|| = \sqrt{11} = 3.3166 \quad (44)$$

$$e_1 = \frac{v_1}{||v_1||} = \frac{v_1}{r_{11}} = \frac{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}{\sqrt{11}} = \begin{bmatrix} 0.30151 \\ -0.30151 \\ 0.90453 \end{bmatrix} \quad (45)$$

Iteration 2

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$$u_2 = v_2 - r_{12}e_1 = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} - 4.2212 \begin{bmatrix} 0.30151 \\ -0.30151 \\ 0.90453 \end{bmatrix} = \begin{bmatrix} 1.7273 \\ 2.2727 \\ 0.18182 \end{bmatrix} \tag{47}$$

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$$r_{22} = ||u_2|| = \sqrt{1.72^2 + 2.27^2 + 0.18^2} = 2.8604 \tag{48}$$

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$$e_2 = \frac{u_2}{||u_2||} = \frac{u_2}{r_{22}} = \frac{\begin{bmatrix} 1.7273 \\ 2.2727 \\ 0.18182 \end{bmatrix}}{2.8604} = \begin{bmatrix} 0.60386 \\ 0.79455 \\ 0.063564 \end{bmatrix} \tag{49}$$

Iteration 3

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$$u_3 = v_3 - r_{13}e_1 - r_{23}e_2 \tag{51}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} - 4.8242 \begin{bmatrix} 0.30151 \\ -0.30151 \\ 0.90453 \end{bmatrix} - 3.7185 \begin{bmatrix} 0.60386 \\ 0.79455 \\ 0.063564 \end{bmatrix} = \begin{bmatrix} -0.7 \\ 0.5 \\ 0.4 \end{bmatrix} \tag{52}$$

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$$e_3 = \frac{u_3}{||u_3||} = \frac{u_3}{r_{33}} = \frac{\begin{bmatrix} -0.7 \\ 0.5 \\ 0.4 \end{bmatrix}}{0.94868} = \begin{bmatrix} -0.737861 \\ 0.52705 \\ 0.42164 \end{bmatrix} \tag{54}$$

In summary

$$[e_1 \ e_2 \ e_3] = \begin{bmatrix} 0.30151 & 0.60386 & -0.737861 \\ -0.30151 & 0.79455 & 0.52705 \\ 0.90453 & 0.063564 & 0.42164 \end{bmatrix} \tag{55}$$

In summary

$$[e_1\ e_2\ e_3] = \begin{bmatrix} 0.30151 & 0.60386 & -0.737861 \\ -0.30151 & 0.79455 & 0.52705 \\ 0.90453 & 0.063564 & 0.42164 \end{bmatrix} \quad (55)$$

and

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 3.3166 & 4.2212 & 4.8242 \\ 0 & 2.8604 & 3.7185 \\ 0 & 0 & 0.94868 \end{bmatrix} \quad (56)$$

Outline of Presentation

QR Decomposition

Definition (QR decomposition)

If A is a full rank $m \times n$ matrix, $m \geq n$, then there exists an $m \times n$ matrix Q with orthonormal columns and an $n \times n$ upper-triangular matrix R such that

$$A = QR \tag{57}$$

Using QR decomposition to Solve Equations

1. Factor A into the product of Q and R: that is

$$Ax = b \tag{58}$$

$$QRx = b \tag{59}$$

$$Q^{-1}QRx = Q^{-1}b \tag{60}$$

$$Rx = Q^{-1}b \tag{61}$$

$$Rx = Q^{T}b \tag{62}$$

Then solve for x by back substitution.

Note

A matrix P is orthogonal if

$$P^T P = I \implies P^T = P^{-1} \tag{63}$$

Example

Solve

$$\begin{bmatrix} 1 & 3 & 3 \\ -1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

using the QR decomposition method

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \tag{64}$$

From the earlier computation, A is decomposed as

$$Q = [e_1 \ e_2 \ e_3] = \begin{bmatrix} 0.30151 & 0.60386 & -0.737861 \\ -0.30151 & 0.79455 & 0.52705 \\ 0.90453 & 0.063564 & 0.42164 \end{bmatrix} \tag{65}$$

and

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 3.3166 & 4.2212 & 4.8242 \\ 0 & 2.8604 & 3.7185 \\ 0 & 0 & 0.94868 \end{bmatrix} \tag{66}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} 3.3166 & 4.2212 & 4.8242 \\ 0 & 2.8604 & 3.7185 \\ 0 & 0 & 0.94868 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.30151 & -0.30151 & 0.90453 \\ 0.60386 & 0.79455 & 0.063564 \\ -0.737861 & 0.52705 & 0.42164 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} 3.3166 & 4.2212 & 4.8242 \\ 0 & 2.8604 & 3.7185 \\ 0 & 0 & 0.94868 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.30151 & -0.30151 & 0.90453 \\ 0.60386 & 0.79455 & 0.063564 \\ -0.737861 & 0.52705 & 0.42164 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3.3166 & 4.2212 & 4.8242 \\ 0 & 2.8604 & 3.7185 \\ 0 & 0 & 0.94868 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.9196 \\ 0.8581 \\ 0.9487 \end{bmatrix} \tag{67}$$

$$Rx = Q^T b$$

$$\begin{bmatrix} 3.3166 & 4.2212 & 4.8242 \\ 0 & 2.8604 & 3.7185 \\ 0 & 0 & 0.94868 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.30151 & -0.30151 & 0.90453 \\ 0.60386 & 0.79455 & 0.063564 \\ -0.737861 & 0.52705 & 0.42164 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3.3166 & 4.2212 & 4.8242 \\ 0 & 2.8604 & 3.7185 \\ 0 & 0 & 0.94868 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.9196 \\ 0.8581 \\ 0.9487 \end{bmatrix} \tag{67}$$

$$0.94x_3 = 0.94 \implies x_3 = 1 \tag{68}$$

$$2.86x_2 + 3.72(1) = 0.86 \implies x_2 = \frac{-2.86}{2.86} - 1 \tag{69}$$

$$3.32x_1 + 4.22(-1) + 4.82(1) = 3.92 \implies x_1 = 1 \tag{70}$$

Determinant

It follows that

$$
\begin{aligned}
|\det(A)| &= |\det(QR)| && (71)\\
&= |\det(Q)||\det(R)| && (72)\\
&= |\det(R)| && (73)\\
&= |r_{11}r_{22}r_{33}\cdots r_{nn}|, && (74)
\end{aligned}
$$

since the determinant of an upper-triangular matrix is the product of its diagonal elements.

Note

$$Q^TQ = I, \implies \det Q^TQ = \det Q^T \det(Q) = (\det(Q))^2 = I \qquad (75)$$

so

$$|\det(Q)| = 1 \qquad (76)$$

Given that

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} 3.3166 & 4.2212 & 4.8242 \\ 0 & 2.8604 & 3.7185 \\ 0 & 0 & 0.94868 \end{bmatrix} \tag{77}$$

then the determinant of the matrix

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} \tag{78}$$

is

$$\det A = r_{11} r_{22} r_{33} = 3.3166(2.8604)(0.94868) = 9 \tag{79}$$

Exercise

Solving the following system of equations using the (1) Cholesky and (2) QR decomposition and hence find the determinants.

$$\begin{aligned} 10x + 4y - 2z &= 20 \\ 3x + 12y - z &= 28 \\ x + 4y + 7z &= 2 \end{aligned}$$

$$\begin{aligned} 2a + b + c + d &= 2 \\ 4a + 2c + d &= 3 \\ 3a + 2b + 2c &= -1 \\ a + 3b + 2c + 6d &= 2 \end{aligned}$$

END OF LECTURE

THANK YOU