MATRICES

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Outline

Outline of Presentation

Definition

1. A matrix is a rectangular array of numbers, symbols, or anything with m rows and n columns which is used to represent a mathematical object or a property of such an object. The symbol $\mathbb{R}^{m\times n}$ denotes the collection of all $m \times n$ matrices whose entries are real numbers. Matrices represent linear maps.

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2. Matrices will usually be denoted by capital letters, and the notation $A = [a_{ij}]$ specifies that the matrix is composed of entries a_{ij} located in the i_{th} row and j_{th} column of A. Example of 2×3 matrix

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$$A = \begin{bmatrix} 1 & 2 & -9 \\ 3 & -3 & 2 \end{bmatrix}$$
3. A vector is a matrix with either one row or one column.
Column vector (3×1):
$$x = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$
Row vector (1×4):
$$y = \begin{bmatrix} 6 & 1 & 0 & -12 \end{bmatrix}$$

Square and Zero Matrices

Definition (Square matrix)

Matrices of size (n, n) are called **square** matrices or n-**square** matrices of **order** n. Examples of 2×2 and 3×3 matrices are respectively given as

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix}$$

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Definition (The zero matrix)

Each $m \times n$ matrix, all of whose elements are zero, is called the zero matrix (of size $m \times n$) and is denoted by the symbol **0**.

$$S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Identity Matrix

Definition (The identity matrix)

The $n \times n$ matrix $I = [\delta_{ij}]$, defined by $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$, is called the $n \times n$ identity matrix of order n.

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Example of 3 $\times$ 3 identity matrix is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When any $n \times n$ matrix A is multiplied by the identity matrix, either on the left or the right, the result is A. Thus, the identity matrix acts like 1 in the real number system.

Definition (Equality of matrices)

Matrices A and B are said to be equal if they have the same size and their corresponding elements are equal; i.e., A and B have dimension $m \times n$, and $A = [a_{ij}], B = [b_{ij}]$, with $a_{ij} = b_{ij}$ for $1 \leq i \leq m, \quad 1 \leq j \leq n.$

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Definition (Addition of matrices)

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be of the same size. Then $A + B$ is the matrix obtained by adding corresponding elements of A and B; that is,

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

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Definition (Scalar multiple of a matrix)

Let $A = [a_{ij}]$ and t be a number (scalar). Then tA is the matrix obtained by multiplying all elements of A by t; that is,

$$tA = t[a_{ij}] = [ta_{ij}]$$

Definition (Negative of a matrix)

Let $A = [a_{ij}]$. Then $-A$ is the matrix obtained by replacing the elements of A by their negatives.

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Definition (Subtraction of matrices)

Matrix subtraction is defined for two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, in the usual way; that is,

$$A - B = [a_{ij}] - [b_{ij}]$$

Example

- If
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 0 & -1 \end{bmatrix}$$
- $$A + B = \begin{bmatrix} 6 & 8 \\ 3 & 3 \end{bmatrix}$$

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- $$A + B = \begin{bmatrix} 6 & 8 \\ 3 & 3 \end{bmatrix}$$
- $$A - B = \begin{bmatrix} -4 & -4 \\ 3 & 5 \end{bmatrix}$$
- $$2B = \begin{bmatrix} 10 & 12 \\ 0 & -2 \end{bmatrix}$$

Properties

The matrix operations of addition, scalar multiplication, negation and subtraction satisfy the following laws of arithmetic. Let s and t be arbitrary scalars and A, B, C be matrices of the same size

1. $(A + B) + C = A + (B + C)$
2. $A + B = B + A$
3. $0 + A = A$
4. $A + (-A) = 0$
5. $(s + t)A = sA + tA, \quad (s - t)A = sA - tA$
6. $t(A + B) = tA + tB, \quad t(A - B) = tA - tB$
7. $s(tA) = (st)A$
8. $1A = A, \quad 0A = 0, \quad (-1)A = -A$
9. $tA = 0 \implies t = 0$ or $A = 0$

Matrix Product

Let $A = [a_{ij}]$ be a matrix of size $m \times p$ and $B = [b_{jk}]$ be a matrix of size $p \times n$ (i.e., the number of columns of A equals the number of rows of B). Then the product AB is an $m \times n$ matrix. That is, if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

then

$$AB = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

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Example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 11 \end{bmatrix}$$

Matrix multiplication is not commutative $AB \neq BA$

Trace

Definition (Trace)

If A is an $n \times n$ matrix, the trace of A, written **trace(A)**, is the sum of the main diagonal elements; that is,

$$trace(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

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Example

If

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 0 & -1 \end{bmatrix}$$

then

$$trace(A) = 1 + 4 = 5$$

$$trace(B) = 5 + (-1) = 4$$

Properties of Trace

- $trace(A+B) = trace(A) + trace(B)$
- $trace(cA) = c \cdot trace(A)$, where c is a scalar.
- $trace(AB) = trace(BA)$

Power of a Matrix

Definition (k_{th} power of a matrix)

If A is an $n \times n$ matrix, we define A^k as follows:

$$A^0 = I$$

and

$$A^k = A \times A \times A \cdots A \times A; \text{ A occurs k times for } k \geq 1.$$

Example

$$A^4 = A \times A \times A \times A$$

The transpose of a matrix

Definition (The transpose of a matrix)

Let A be an $m \times n$ matrix. Then A^T, the transpose of A, is the matrix obtained by interchanging the rows and columns of A. In other words if $A = [a_{ij}]$, then $(A^T)_{ij} = a_{ji}$.

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 15 \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & 15 \end{bmatrix}$$

Properties of Transpose

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3. $(sA)^T = sA^T$ if s is a scalar
4. $(AB)^T = B^T A^T$ if A is $m \times k$ and B is $k \times n$

Outline of Presentation

Definition (Diagonal Matrix)

The $a_{ii}, 1 \leq i \leq n$, entries of a square matrix are called the **diagonal elements**. If the nondiagonal elements are all zero, then the matrix is called a **diagonal matrix**. It is denoted by $\mathbf{A} = \mathbf{diag}(a_{11}, a_{22}, \ldots, a_{nn})$. Some examples are

$$\begin{bmatrix} 10 & 0 \\ 0 & -13 \end{bmatrix}, \qquad \begin{bmatrix} 14 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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The matrix B_1 is an upper bidiagonal matrix.

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The matrix B_2 is a lower bidiagonal matrix.

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 2 & 10 & 0 & 0 \\ 0 & 9 & 9 & 0 \\ 0 & 0 & -1 & -2 \end{bmatrix} \tag{2}$$

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$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 & 0 \\ 0 & -1 & -3 & 2 & 0 \\ 0 & 0 & 1 & 2 & 10 \\ 0 & 0 & 0 & -6 & 7 \end{bmatrix} \tag{3}$$

Nonsingular matrix

Definition (Nonsingular matrix)

An $n \times n$ matrix A is called nonsingular or invertible if there exists an $n \times n$ matrix B such that

$$AB = BA = I$$

The matrix B is the inverse of A. If A does not have an inverse, A is called **singular**.

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Homogeneous system

A linear system $Ax = 0$ is said to be homogeneous. If A is nonsingular, then $x = A^{-1}(0) = 0$, so the system has only 0 as its solution. It is said to have only the trivial solution.

Symmetric and Skew Symmetric matrix

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Another way of looking at this is that when the rows and columns are interchanged, the resulting matrix is A.

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Another way of looking at this is that when the rows and columns are interchanged, the resulting matrix is A.

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A matrix **A** is skew symmetric if $A^T = -A$.

Example

The matrix $A = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 1 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ is symmetric and $B = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$ is skew symmetric.

Symmetric Definite Matrices

Definition (Symmetric Positive Definite Matrix)

A symmetric matrix A is positive definite if for every nonzero vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$x^T A x > 0 \tag{4}$$

The expression $x^T A x$ is called the **quadratic form** associated with A.

Note

The sum of two positive definite matrices is positive definite.

Definition (Symmetric Positive Semidefinite Matrix)

A is symmetric positive semidefinite if for every nonzero vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$x^T A x \geq 0 \tag{5}$$

Example

The symmetric matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is positive definite because for

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

then

$$x^T A x = [x_1 \ x_2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7}$$

$$= x_1^2 + x_2^2 \tag{8}$$

Since $x_1^2 + x_2^2 > 0$ then A is a symmetric positive definite matrix.

Theorem

1. *If $A = (a_{ij})$ is positive definite, then $a_{ii} > 0$ for all i.*
2. *If $A = (a_{ij})$ is positive definite, then the largest element in magnitude of all matrix entries must lie on the diagonal.*

Example

The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \\ 2 & 5 & 6 \end{bmatrix}$ cannot be positive definite because A has a diagonal element of 0

Example

The matrix $B = \begin{bmatrix} 1 & -1 & 0 & 9 \\ 8 & 45 & 3 & 19 \\ 0 & 15 & 16 & 35 \\ 3 & -55 & 2 & 22 \end{bmatrix}$ cannot be positive definite because the largest element in magnitude (-55) is not on the diagonal of B.

Theorem

Suppose that a real symmetric tridiagonal matrix

$$A = \begin{bmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & b_{n-1} & a_{n-1} \\ & & & a_{n-1} & b_n \end{bmatrix} \tag{9}$$

with diagonal entries all positive is strictly diagonally dominant, that is,

$$b_i > |a_{i-1}| + |a_i|, \quad 1 \leq i \leq n$$

Then A is positive definite.

Definition (Symmetric Negative Definite Matrix)

A is symmetric negative definite if for every nonzero vector $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

$$x^T A x \leq 0 \tag{10}$$

In this case, $-A$ is positive definite.

Definition (Symmetric Indefinite Matrix)

A is symmetric indefinite if $x^T A x$ assumes both positive and negative values.

Alternatively, a matrix is symmetric indefinite if it has both positive and negative eigenvalues.

Definition (Orthogonal Matrix)

A matrix P is orthogonal if

$$P^T P = I \tag{11}$$

The inverse of P is its transpose.

Alternatively, P is orthogonal if and only if the columns of P are orthogonal and have unit length.

$$\langle v_i,\ v_j \rangle = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i \neq j \end{cases} \tag{12}$$

Example

$$D = [e_1 \ e_2 \ e_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

then

$$\langle e_1, \ e_2 \rangle = \langle e_1, \ e_3 \rangle = \langle e_2, \ e_3 \rangle = 0 \tag{14}$$

So the columns are orthogonal, and each has a unit length

$$|e_1| = |e_2| = |e_3| = \sqrt{a_{11}^2 + a_{21}^2 + a_{31}^2} = \sqrt{1^2} = 1 \tag{15}$$

Hence D is an orthogonal matrix.

Definition (Orthonormal)

1. A set of orthogonal vectors, each with unit length, are said to be orthonormal.
2. $D = [e_1 \ e_2 \ e_3]$ is an orthogonal matrix, and each orthogonal vector e_1, e_2 and e_2 has a unit length.
3. Hence $e_1, \ e_2, \cdots, e_n$ are called the **standard orthonormal basis**.

Outline of Presentation

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Definition (Hermitian and Skew Hermitian matrix)

A complex n-square matrix A is said to be **hermitian** if

$$\bar{A}^T = A \text{ or } \bar{z}_{ji} = z_{ij} \tag{16}$$

and **skew hermitian** if

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Example (M is Hermitian and N is skew Hermitian)

$$M = \begin{bmatrix} 2 & 1-i & 0 \\ 1+i & -1 & i \\ 0 & -i & 2 \end{bmatrix} \qquad N = \begin{bmatrix} i & 2+i & 3+2i \\ -2+i & 3i & -3i \\ -3+2i & -3i & 0 \end{bmatrix}$$

Exercises

1. Let A, B, C, D be matrices defined by

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{bmatrix}, C = \begin{bmatrix} -3 & -1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix}, D = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$$

   Which of the following matrices are defined? Compute those matrices which are defined. $A+B, A+C, AB, BA, CD, DC, D^2, (C^T)^T$

2. Rotate the line $y = -x + 3$ 60°counterclockwise about the origin.

3. Let $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right)$ and $\sigma_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the three Pauli matrices. Show that $x\sigma_1 + y\sigma_2 + 2\sigma_3$ is a hermitian matrix for any two real numbers $x, y \in \mathbb{R}$.

END OF LECTURE

THANK YOU