CSM 165: Discrete Mathematics for Computer Science

Chapter 1: Propositional and first order predicate logic

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Content

Propositional Equivalence

Inference

Definition 1 (Tautology)

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology

Definition 2 (Contradiction)

A compound proposition that is always false is called a contradiction

Definition 3 (Contingency)

A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Example 1

Table 1 : A tautology and a Contradiction

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

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Definition 4 (Logical Equivalence)

*Compound propositions that have the same truth values in all possible cases are called **logically equivalent**.*

The compound propositions p and q are also called **logically equivalent** if $p \leftrightarrow q$ is a **tautology**. The notation $p \equiv q$ denotes that p and q are logically equivalent.

De Morgan's Laws

1. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
2. $\neg(p \vee q) \equiv \neg p \wedge \neg q$

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Logical Equivalence

Example 2

1. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent

Table 2 : Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

2. Show that $p \rightarrow q$ and $\neg p \vee q$ and equivalent.
3. Show that $p \wedge (q \vee r)$ and $(p \vee q) \wedge (p \vee r)$.

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Logical Equivalence

Solution to example 2 question 3

Table 3 : Truth Table for $p \wedge (q \vee r)$ and $(p \vee q) \wedge (p \vee r)$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Precedence of Logical Operators

Table 4 : Precedence of Logical Operators

Operators	Names	Precedence
$\neg$	Negation	1
$\wedge$	Conjunction	2
$\vee$	Disjunction	3
$\rightarrow$	Implication	4
$\leftrightarrow$	Biconditional	5

Table 5 : Logical Equivalences

Equivalence	***Name***
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Table 6 : Logical Equivalences Involving Conditional Statements.

$p \to q \equiv \neg p \vee q$
$p \to q \equiv \neg q \to \neg p$
$p \vee q \equiv \neg p \to q$
$p \wedge q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \wedge \neg q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
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Table 7 : Equivalences Involving Biconditional Statements.

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
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$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Inference

Definition 5

Premise: *It is the proposition on the basis of which we would be able to draw a conclusion.*
It can be thought of as an evidence or assumption.

Conclusion: *It is the a proposition that is reached from a given set of premises.*

Argument: *Sequence of statements that ends with a conclusion.*

Valid Argument: *An argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false. OR*

Let A and B be two statement formulas. We say that "B ***logically follows from*** *A" or "B is a valid conclusion of the premise A" iff* $A \to B$ *is a tautology.*

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Validity Using Truth Table

Example 3

Determine whether the following conclusion C follows logically from the premises H_1 and H_2.

1. H_1 $P \rightarrow Q$ $H_2{:}P$ $C{:}Q$
2. H_1 $P \rightarrow Q$ $H_2{:}\neg P$ $C{:}Q$
3. H_1: $P \rightarrow Q$ $H_2{:}\neg(p \wedge Q)$ C:$\neg P$
4. $H_1{:}\neg P$ $H_2{:}P \leftrightarrow Q$ C:$\neg(P \wedge Q)$

P	Q	$P \rightarrow Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	P$\leftrightarrow Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
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3. $H_1{:}\ P \rightarrow Q \quad H_2{:}\neg(p \wedge Q) \quad \mathrm{C}{:}\neg P$
4. $H_1{:}\neg P \quad H_2{:}P \leftrightarrow Q \quad \mathrm{C}{:}\neg(P \wedge Q)$

P	Q	$P \rightarrow Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	P$\leftrightarrow Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	F
F	F	T	T	T	T	T

Rules of Inference

Example 4

Consider:

"If you have a current password, then you can log onto the network".

"You have a current password".

Therefore, "You can log onto the network."

Let P = you have a current password
q = you can log onto the network

Argument form:

$$\begin{array}{l} p \rightarrow q \\ \underline{p \quad\quad} \\ \therefore q \end{array} \qquad\qquad ((p \rightarrow q) \wedge p) \rightarrow q$$

This form of argument is valid because whenever all its premises are true, the conclusion must also be true

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Rules of Inference

Rule	Tautology	Name
p $\underline{p \rightarrow q}$ $\therefore q$	$(p \wedge (p \rightarrow)) \rightarrow q$	Modus ponens
$\neg q$ $\underline{p \rightarrow q}$ $\therefore \neg p$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens
$p \rightarrow q$ $\underline{q \rightarrow r}$ $\therefore p \rightarrow r$	$(p \rightarrow q) \wedge (\rightarrow r) \rightarrow (p \rightarrow r)$	Hypothetical syllogism

Rules of Inference

$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \rightarrow q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \rightarrow (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \rightarrow p$	Simplification
p p $\therefore p \land q$	$((p) \land (q)) \rightarrow (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \rightarrow (q \lor r)$	Resolution

End of Lecture

Questions...???

Thanks

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