MATH 166: Introductory Probability and Statistics

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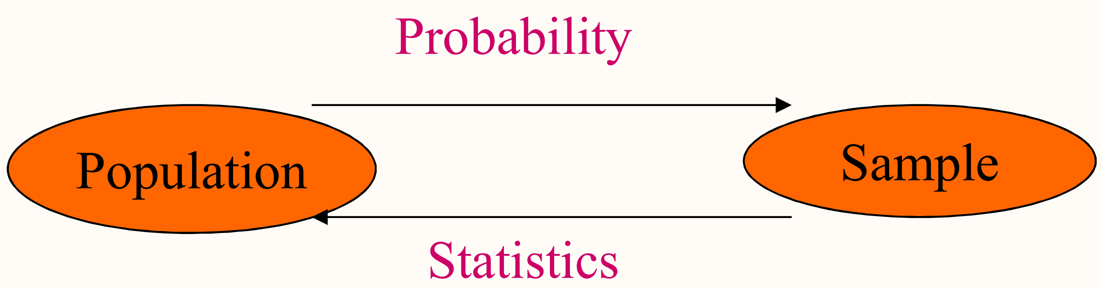
The Concept of Probability

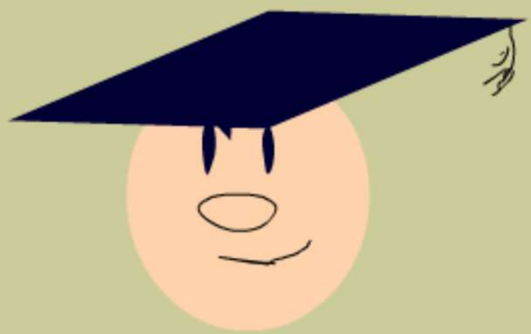
Why Learn Probability?

- Nothing in life is certain. In everything we do, we gauge the chances of successful outcomes, from business to medicine to the weather
- A probability provides a quantitative description of the chances or likelihoods associated with various outcomes
- It provides a bridge between descriptive and inferential statistics

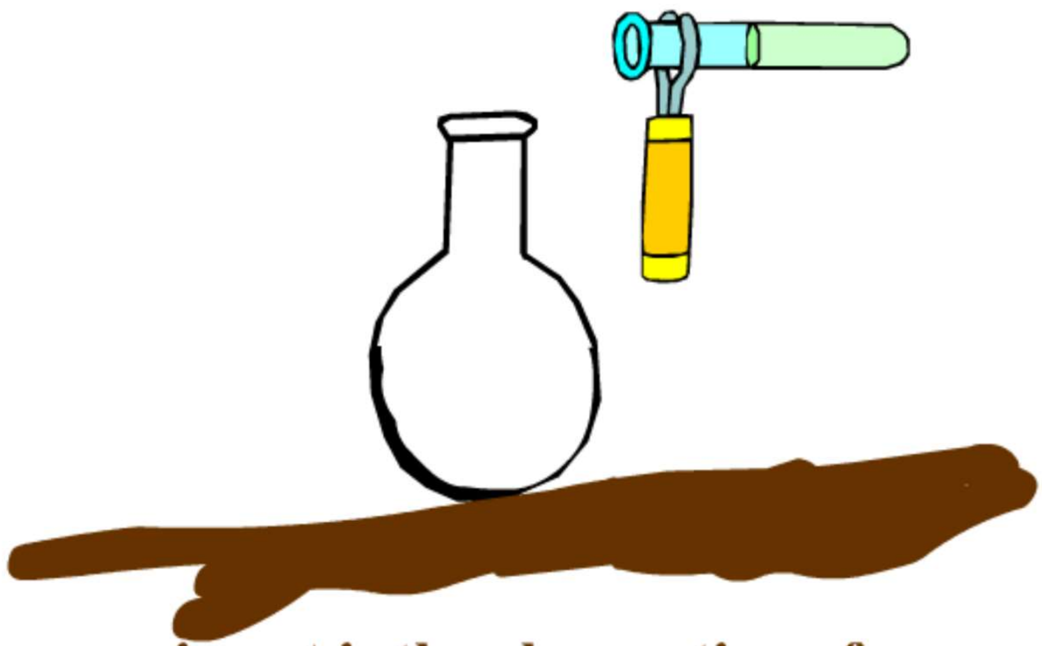


Definition

A **probability** is a measure of the likelihood that an event in the future will happen.



Definition



An experiment is the observation of some activity or the act of taking some measurement.

Definition

Experiment: A fair die is cast.

An Outcome is the particular result of an experiment.

Possible outcomes: The numbers 1, 2, 3, 4, 5, 6

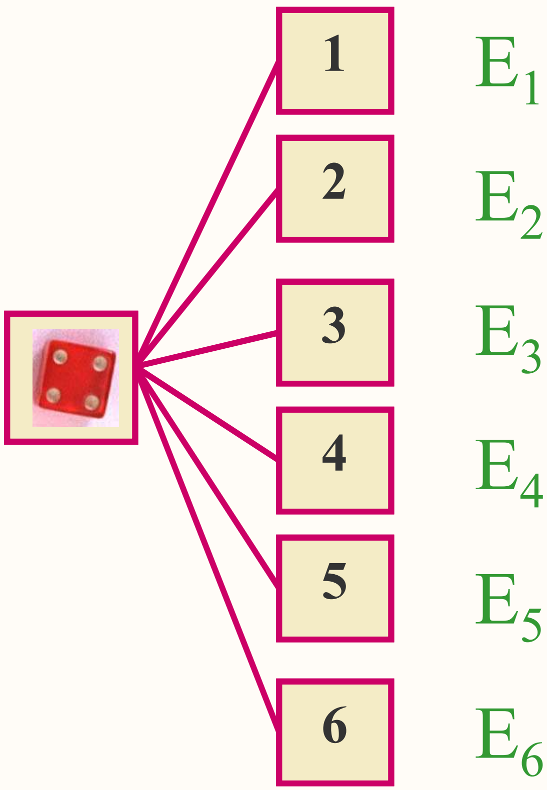
An Event is the collection of one or more outcomes of an experiment.

One possible event: The occurrence of an even number. That is, we collect the outcomes 2, 4, and 6.

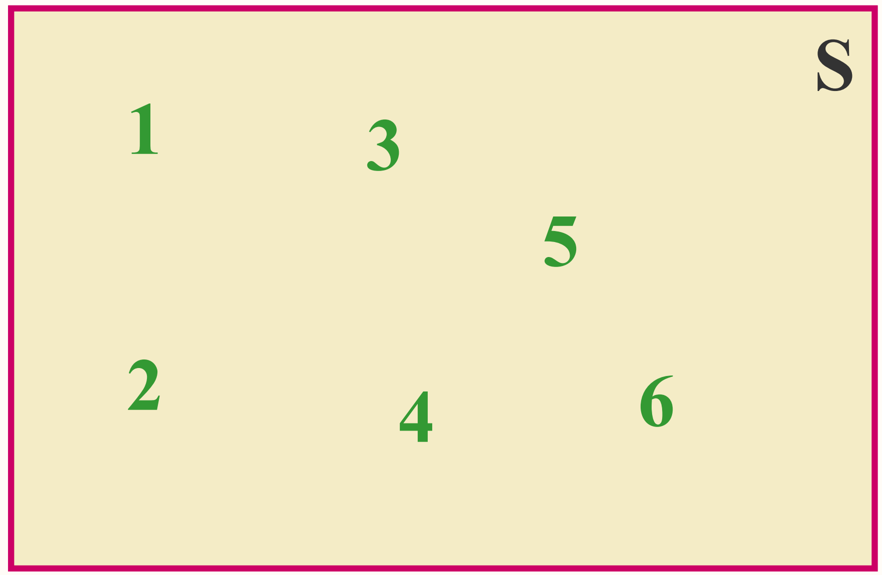
Sample space: is the set of all outcomes of an experiments

Example

- **The die toss:**
- Simple events: Sample space:



S ={1, 2, 3, 4, 5, 6}



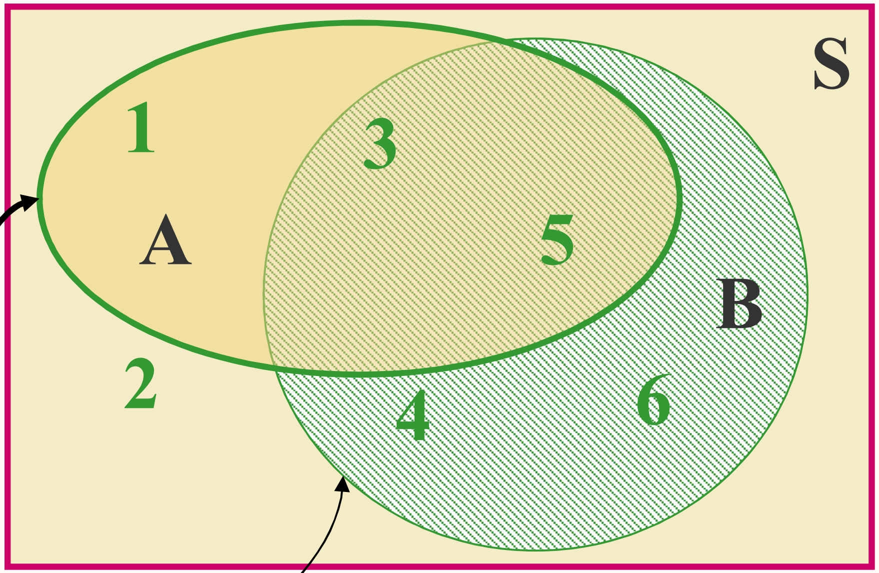
Definition

- An **event** is a collection of one or more **simple events.**

- **The die toss:**
  - –A: an odd number
  - –B: a number > 2

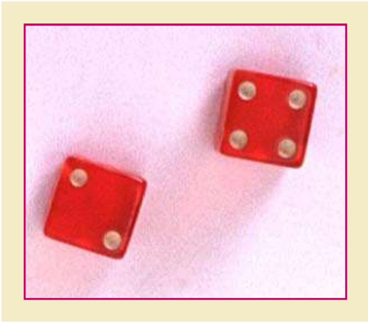
A ={1, 3, 5}

B ={3, 4, 5, 6}



Definition

Experiments and Events



- **Experiment: Record an age**
  - A: person is 30 years old
  - B: person is older than 65
- **Experiment: Toss a die**
  - A: observe an odd number
  - B: observe a number greater than 2

Definition

Experiments and Sample space

Experiment	Sample space
Toss one coin	Head, Tail
Roll a die	1, 2, 3, 4, 5, 6
Answer a true/false question	True, False
Toss two coins	Head-head, tail-tail, head-tail, tail-head

Definition

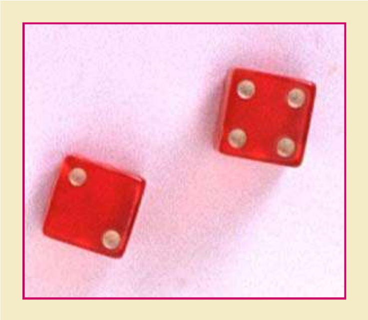
Events are **Mutually Exclusive** if the occurrence of any one event means that none of the others can occur at the same time.

Mutually exclusive: Rolling a 2 precludes rolling a 1, 3, 4, 5, 6 on the same roll.

Events are **Independent** if the occurrence of one event does not affect the occurrence of another.

Independence: Rolling a 2 on the first throw does not influence the probability of a 3 on the next throw. It is still a one in 6 chance.

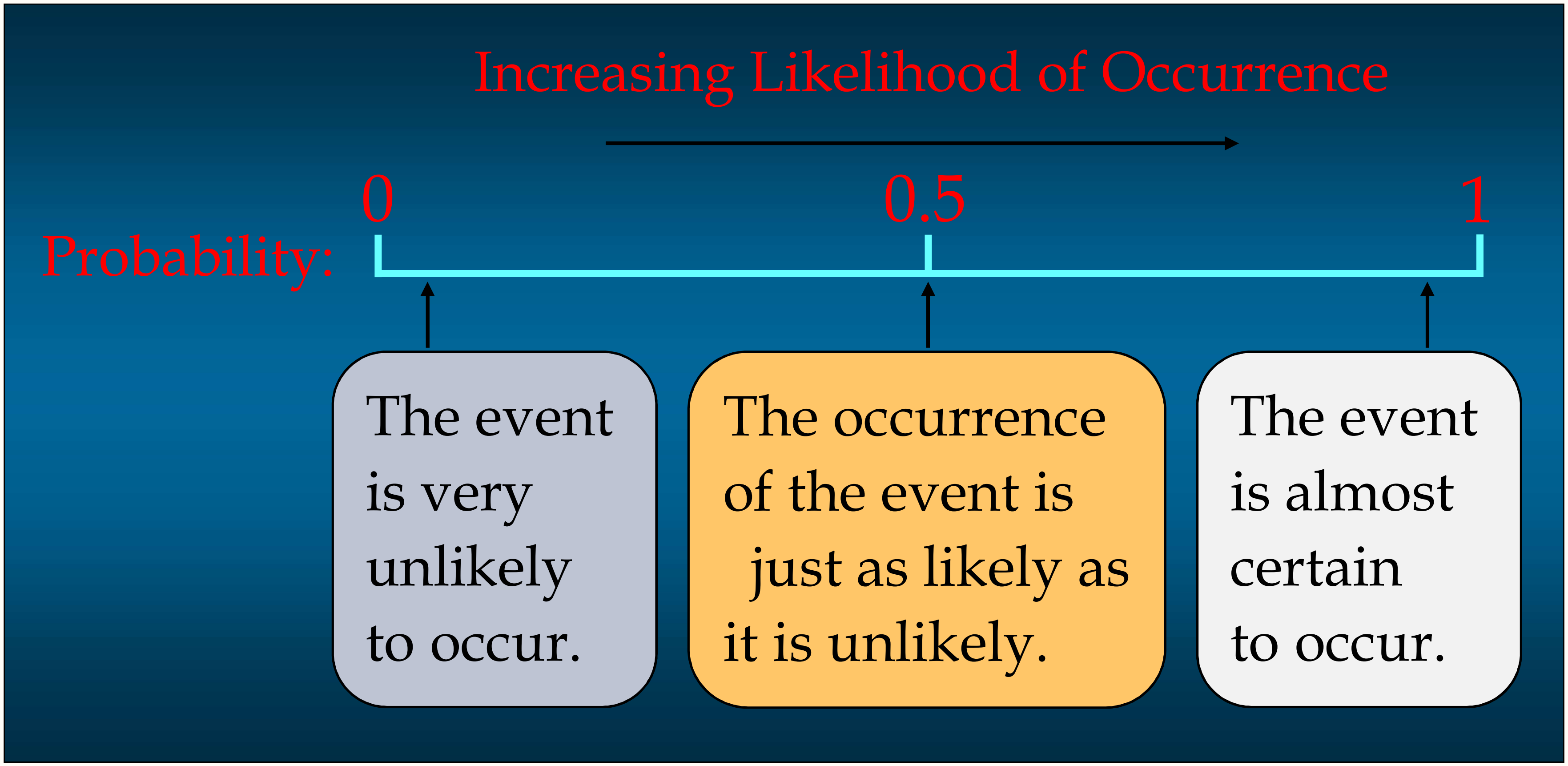
Probability of an Event



- P(A) must be between 0 and 1.
  - If event A can never occur, P(A) = 0.
  - If event A always occurs when the experiment is performed, P(A) =1.
- The sum of the probabilities for all simple events in S equals 1.

• The **probability of an event A** is found by adding the probabilities of all the simple events contained in A.

Probability of an Event



Finding Probability

- The probability of an event A is equal to the sum of the probabilities of the simple events contained in A

- If the simple events in an experiment are **equally likely,** you can calculate

$$P(A) = \frac{n_A}{N} = \frac{\text{number of simple events in A}}{\text{total number of simple events}}$$

Example

Toss a fair coin once. What is the probability of observing a head?

E_i	$P(E_i)$
H	1/2
T	1/2

$P(H) = 1/2$

Example

Toss a fair coin twice. What is the probability of observing at least one head?

1st Coin	2nd Coin	E_i	$P(E_i)$
H	H	HH	¼
	T	HT	¼
T	H	TH	¼
	T	TT	¼

P(at least 1 head)

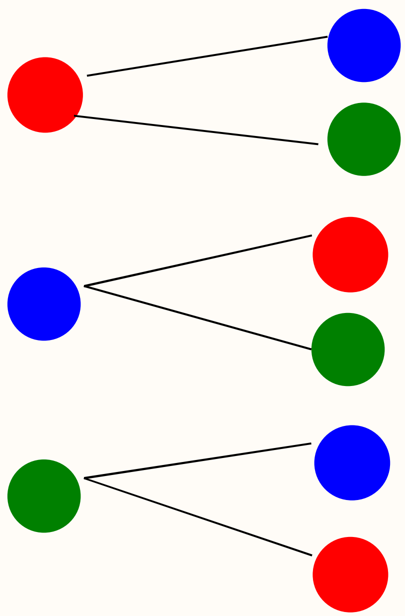
$= P(E_1) + P(E_2) + P(E_3)$

$= 1/4 + 1/4 + 1/4 = 3/4$

Example

A bowl contains three balls, one red, one blue and one green. A child selects two balls at random. What is the probability that at least one is red?

1st ball 2nd ball



E_i	$P(E_i)$
RB	1/6
RG	1/6
BR	1/6
BG	1/6
GB	1/6
GR	1/6

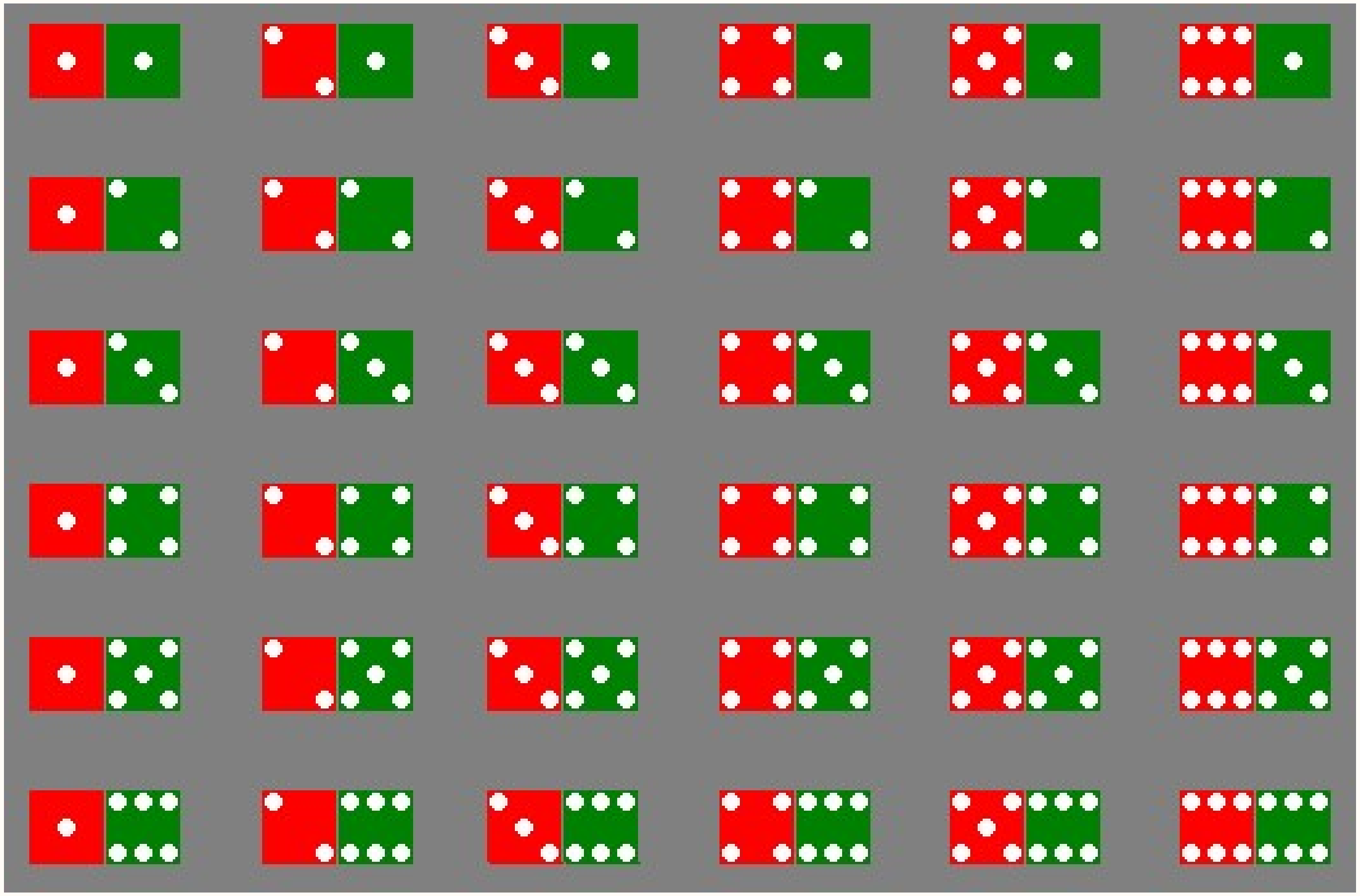
P(at least 1 red)

= P(RB) + P(BR)+ P(RG) + P(GR)

= 4/6 = 2/3

Example

The sample space of throwing a pair of dice is



Example

What is the probability that the

- Dice add to 3
- Dice add to 6
- Red die show 1
- Green die show 1

Simple events	Probability
(1,2),(2,1)	2/36
(1,5),(2,4),(3,3),(4,2),(5,1)	5/36
(1,1),(1,2),(1,3),(1,4),(1,5),(1,6)	6/36
(1,1),(2,1),(3,1),(4,1),(5,1),(6,1)	6/36

Basic Rules of Probability

Special Rule of Addition

If two events A and B are mutually exclusive, the **Special Rule of Addition** states that the Probability of A or B occurring equals the sum of their respective probabilities.

$$\mathbf{P(\mathit{A} \text{ or } \mathit{B}) = P(\mathit{A}) + P(\mathit{B})}$$

Example

New England Commuter Airways recently supplied the following information on their commuter flights from Boston to New York:

Arrival	Frequency
Early	100
On Time	800
Late	75
Canceled	25
Total	1000

Example

If A is the event that a flight arrives early, then $P(A) = 100/1000 = .10$.

If B is the event that a flight arrives late, then $P(B) = 75/1000 = .075$.

The probability that a flight is either early or late is:

$P(A \text{ or } B) = \mathrm{P}(A) + \mathrm{P}(B) = .10 + .075 = .175.$

The Complement Rule

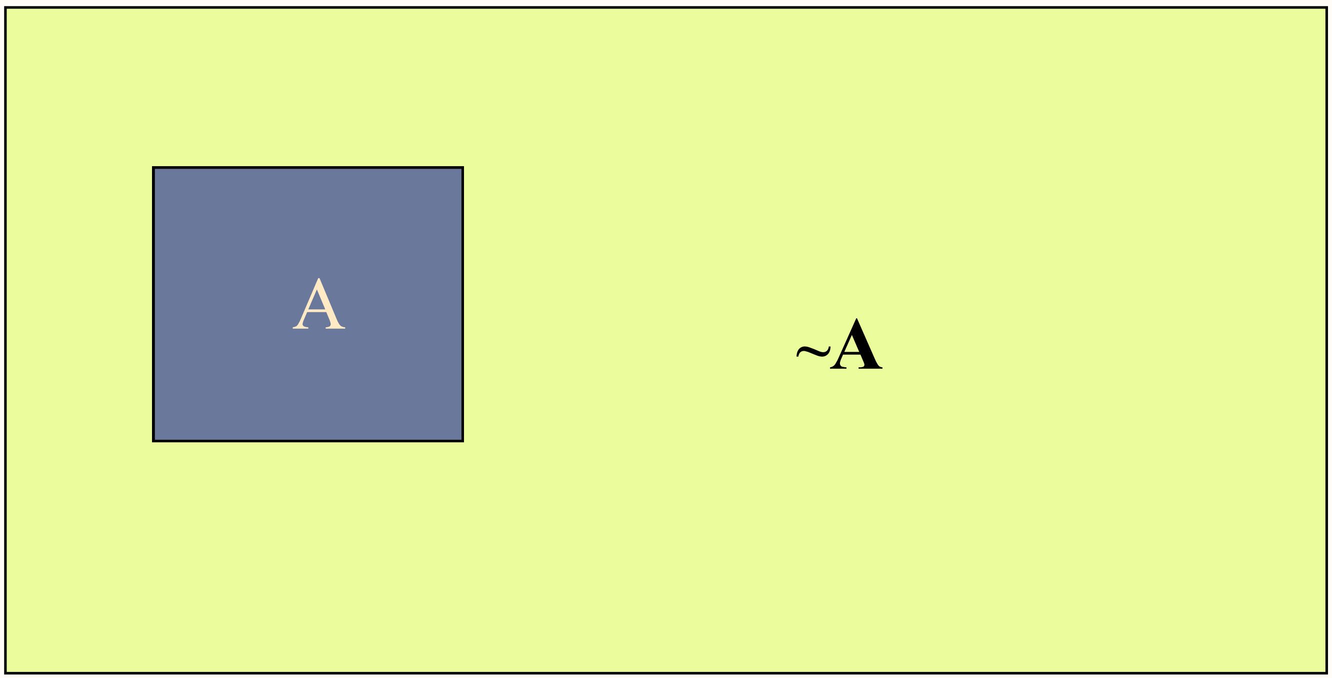
The Complement Rule is used to determine the probability of an event occurring by subtracting the probability of the event *not* occurring from 1.

If P(A) is the probability of event A and $P(\sim A)$ is the complement of A,

$$P(A) + P(\sim A) = 1 \text{ or } \quad P(A) = 1 - P(\sim A).$$

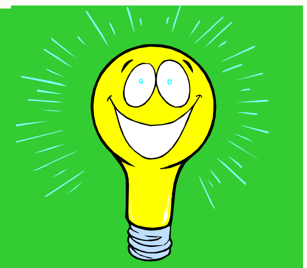
The Complement Rule

A Venn Diagram illustrating the complement rule would appear as:



Example

Recall example 3. Use the complement rule to find the probability of an early (A) or a late (B) flight

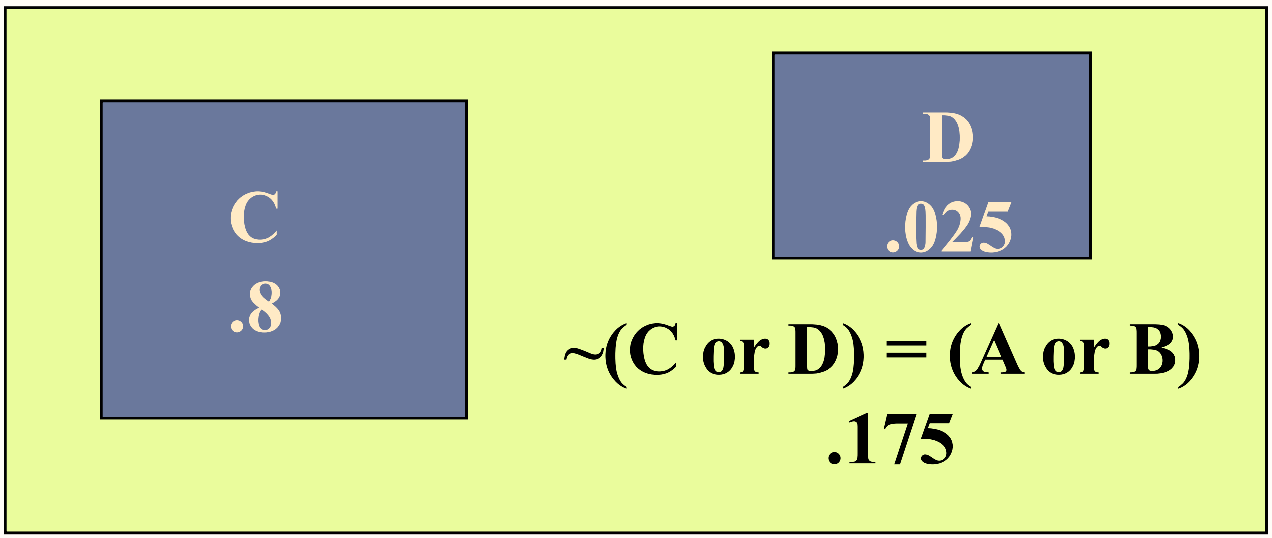


If C is the event that a flight arrives on time, then P(C) = 800/1000 = .8.

If D is the event that a flight is canceled, then P(D) = 25/1000 = .025.

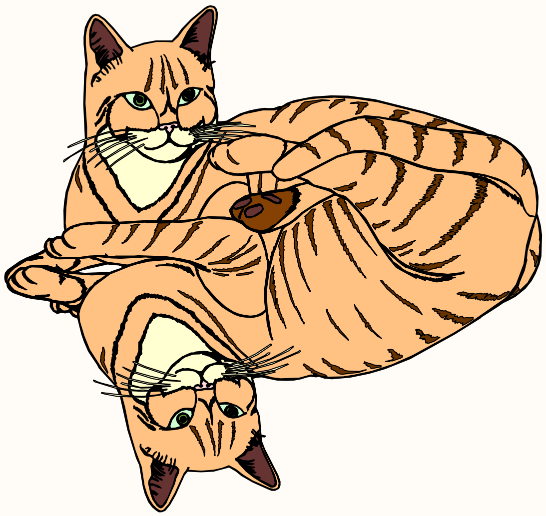
Example

$$\begin{aligned} P(A \text{ or } B) &= 1 - P(C \text{ or } D) \\ &= 1 - [.8 + .025] \\ &= .175 \end{aligned}$$



General Rule of Addition

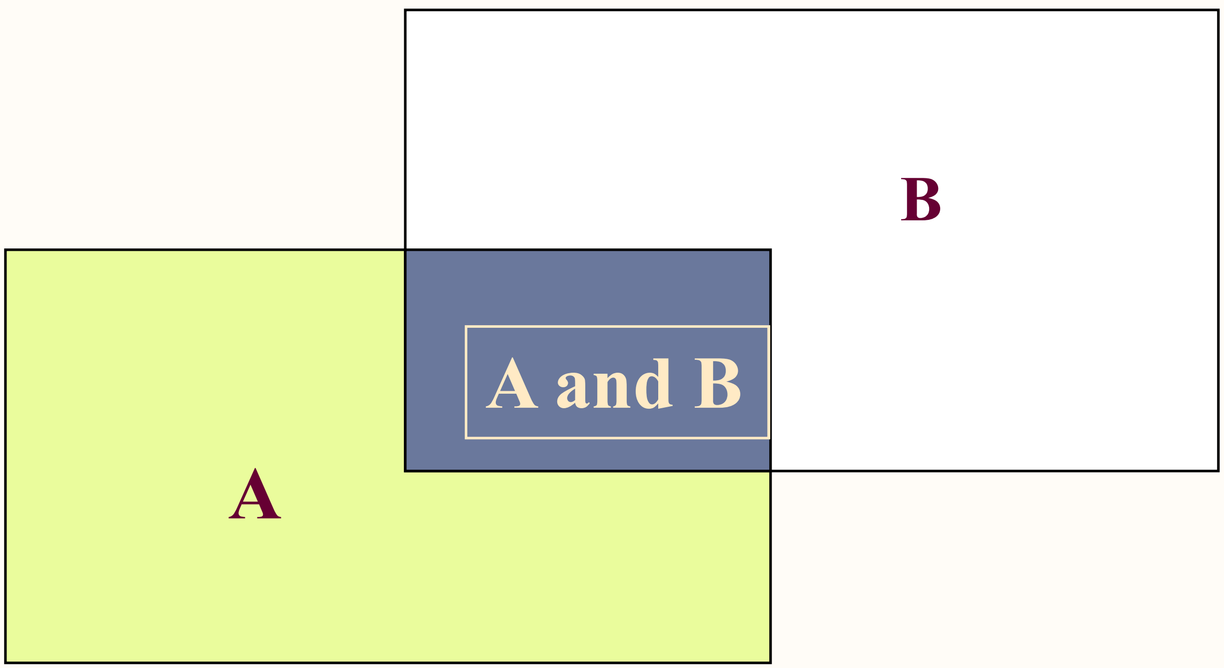
If A and B are two events that are not mutually exclusive, then $P(A \text{ or } B)$ is given by the following formula:



$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

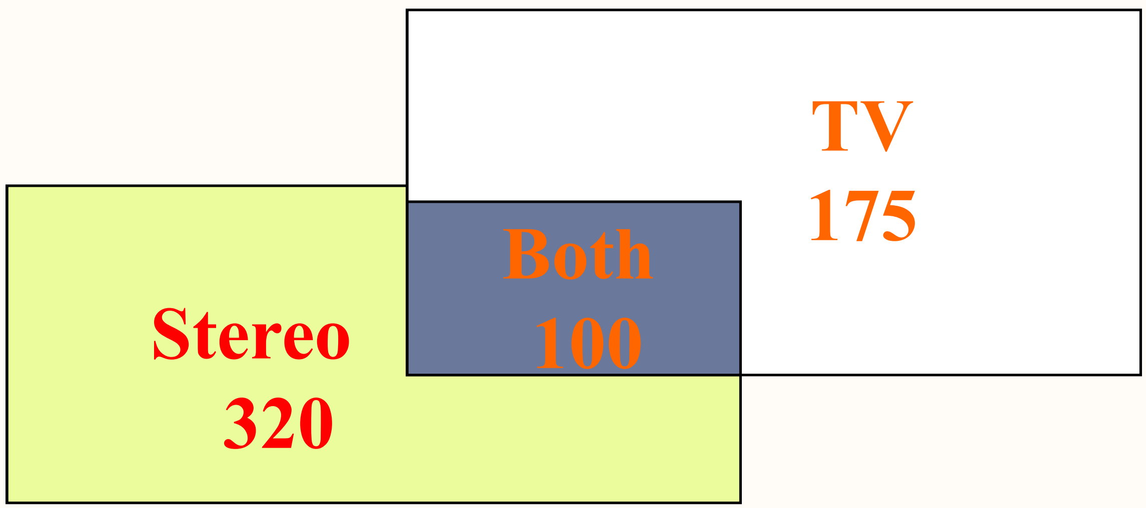
General Rule of Addition

The Venn Diagram illustrates this rule:



Example

In a sample of 500 students, 320 said they had a stereo, 175 said they had a TV, and 100 said they had both. 5 said they had neither.



Example

If a student is selected at random, what is the probability that the student has only a stereo or TV? What is the probability that the student has both a stereo and TV?

$$P(\text{S or TV}) = P(\text{S}) + P(\text{TV}) - P(\text{S and TV})$$

$$= 320/500 + 175/500 - 100/500$$

$$= .79.$$

$$\text{P(S and TV)} = 100/500$$

$$= .20$$

Joint Probability

A **Joint Probability** measures the likelihood that two or more events will happen concurrently.

An example would be the event that a student has both a stereo and TV in his or her dorm room.

Special Rule of Multiplication

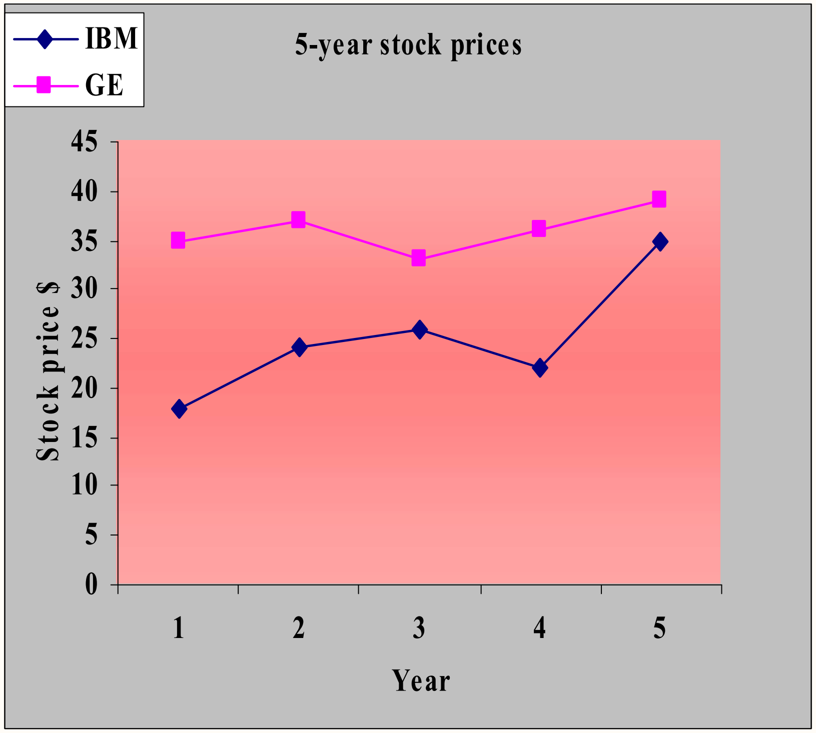
The **Special Rule of Multiplication** requires that two events A and B are *independent*.

- **Two events A and B are independent if the occurrence of one has no effect on the probability of the occurrence of the other.**

- **This rule is written:** $P(A \text{ and } B) = P(A)P(B)$

Example

Chris owns two stocks, IBM and General Electric (GE). The probability that IBM stock will increase in value next year is .5 and the probability that GE stock will increase in value next year is .7. Assume the two stocks are independent. What is the probability that both stocks will increase in value next year?



$P(IBM \text{ and } GE) = (.5)(.7) = .35$

Example

What is the probability that at least one of these stocks increases in value in the next year? This means that either one can increase or both.

P(at least one)
= P(IBM but not GE)
+ P(GE but not IBM)
+ P(IBM and GE)

$$
\begin{aligned}
&(.5)(1-.7)\\
&+ (.7)(1-.5)\\
&\underline{+ (.7)(.5)}\\
&= .85
\end{aligned}
$$

Conditional Probability

A Conditional Probability is the probability of a particular event occurring, given that another event has occurred.

The probability of event *B* occurring given that the event *A* has occurred is written:

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A\ and\ B)}{P(A)}$$

Example

A number from the sample space $S = \{2, 3, 4, 5, 6, 7, 8, 9\}$ is randomly selected. Given the defined events A and B,

A: selected number is odd, and

B: selected number is a multiple of 3

find the following probabilities.

a) $P(B)$ b) $P(A \text{ and } B)$ c) $P(B/A)$

a) $B = \{3, 6, 9\}$ $P(B) = 3/8$

b) $P(A \text{ and } B) = P(\{3, 5, 7, 9\} \cap \{3, 6, 9\})$

$= P(\{3, 9\}) = 2/8 = 1/4$

c) Probability of B given A has occurred:

$$P(B/A) = \frac{P(A \text{ and } B)}{P(A)} = \frac{1/4}{4/8} = 1/2$$

Example

Given a family with two children, find the probability that both are boys, given that at least one is a boy.

Conditional Probability $\quad P(B/A) = \dfrac{P(A \text{ and } B)}{P(A)}$

S= {gg, gb, bg, bb}

A = at least one boy $\quad$ A = {gb, bg, bb}

B = both are boys $\quad$ B = {bb}

$P(A \text{ and } B) = P(\{gb, bg, bb\} \cap \{bb\}) = P(\{bb\}) = 1/4$

$P(A) = P(\{gb, bg, bb\}) = 3/4$

$$\frac{P(A \text{ and } B)}{P(A)} = \frac{1/4}{3/4} = 1/3$$

The General Rule of Multiplication

The **General Rule of Multiplication** is used to find the joint probability that two events will occur.

It states that for two events A and B, the joint probability that both events will happen is found by multiplying the probability that event A will happen by the conditional probability of B given that A has occurred.

The General Rule of Multiplication

The joint probability, $P(A \text{ and } B)$, is given by the following formula:

$$P(A \text{ and } B) = P(A)P(B/A)$$

or

$$P(A \text{ and } B) = P(B)P(A/B)$$

Example

The Dean of the School of Business at Owens University collected the following information about undergraduate students in her college:

Major	Male	Female	Total
Accounting	170	110	280
Finance	120	100	220
Marketing	160	70	230
Management	150	120	270
Total	600	400	1000

Example

If a student is selected at random, what is the probability that the student is a female (*F*) accounting major (*A*)?

$P(A \text{ and } F) = 110/1000.$

Given that the student is a female, what is the probability that she is an accounting major?

$$P(A|F) = P(A \text{ and } F)/P(F)$$

$$= [110/1000]/[400/1000] = .275$$

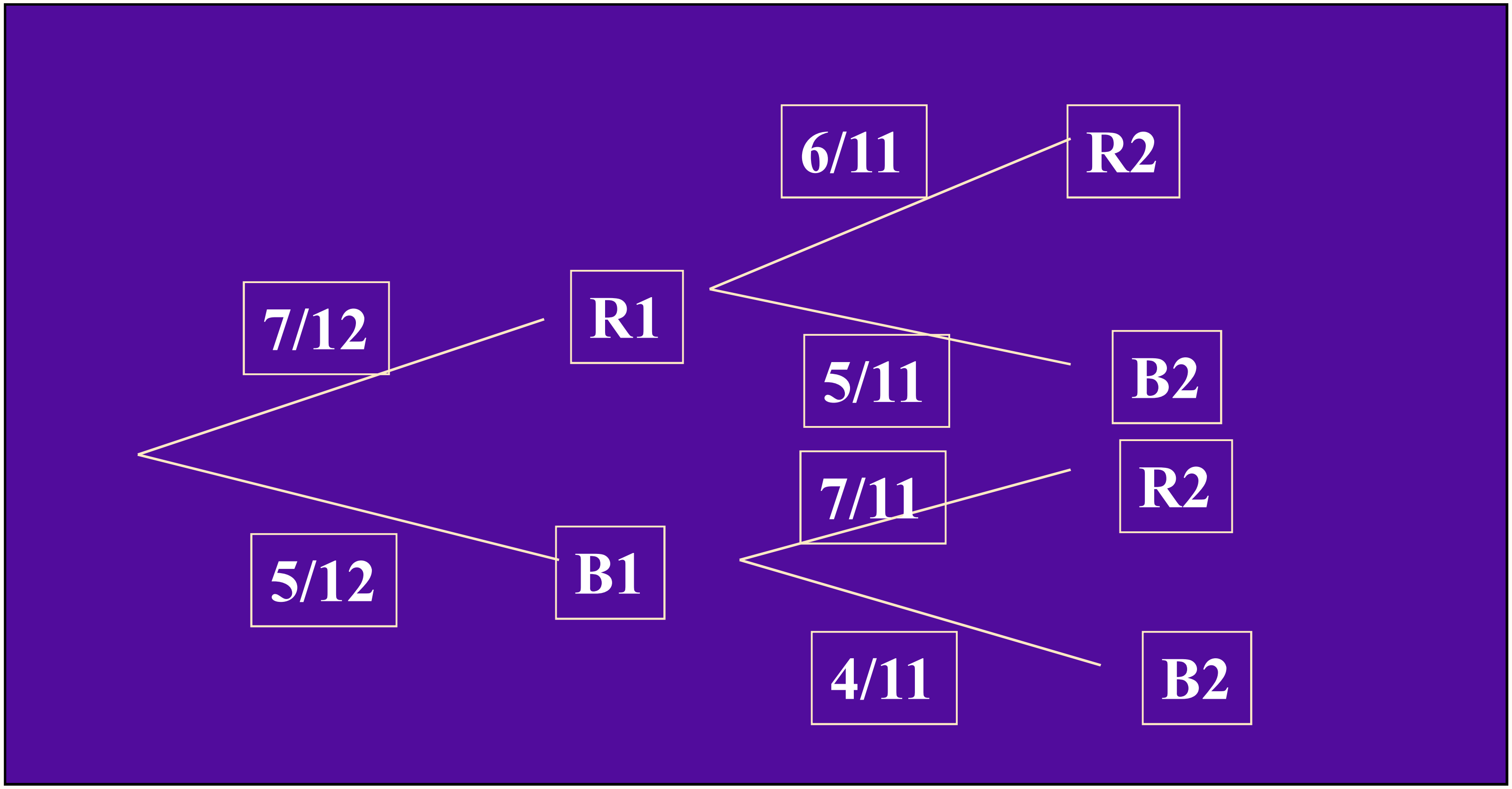
Tree Diagram

A Tree Diagram is useful for portraying conditional and joint probabilities.

It is particularly useful for analyzing business decisions involving several stages.

Example

In a bag containing 7 red chips and 5 blue chips you select 2 chips one after the other without replacement. Construct a tree diagram showing this information.



Some Principles of Counting

The *mn* Rule

- If an experiment is performed in two stages, with ***m*** ways to accomplish the first stage and ***n*** ways to accomplish the second stage, then there are ***mn*** ways to accomplish the experiment.

- This rule is easily extended to ***k*** stages, with the number of ways equal to

$$n_1 n_2 n_3 \ldots n_k$$

Example: Toss two coins. The total number of simple events is:

$$2 \times 2 = 4$$

Example

Example: Toss three coins. The total number of simple events is:

$$2 \times 2 \times 2 = 8$$

Example: Toss two dice. The total number of simple events is:

$$6 \times 6 = 36$$

Example: Toss three dice. The total number of simple events is:

$$6 \times 6 \times 6 = 216$$

Example

Example 10: Dr. Delong has 10 shirts and 8 ties. How many shirt and tie outfits does he have?

$$\boxed{(10)(8) = 80}$$

Permutation

- The number of ways you can arrange ***n*** distinct objects, taking them ***r*** at a time is

$$P_r^n = \frac{n!}{(n-r)!}$$

where $n! = n(n-1)(n-2)...(2)(1)$ and $0! \equiv 1$.

Example: How many 3-digit lock combinations can we make from the numbers 1, 2, 3, and 4?

The order of the choice is important!

$$P_3^4 = \frac{4!}{1!} = 4(3)(2) = 24$$

Combination

- The number of distinct combinations of ***n*** distinct objects that can be formed, taking them ***r*** at a time is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

Example: Three members of a 5-person committee must be chosen to form a subcommittee. How many different subcommittees could be formed?

The order of the choice is not important!

$$C_3^5 = \frac{5!}{3!(5-3)!} = \frac{5(4)(3)(2)1}{3(2)(1)(2)1} = \frac{5(4)}{(2)1} = 10$$

Example

There are 12 players on the KNUST basketball team. Coach Thompson must pick five players among the twelve on the team to comprise the starting lineup. How many different groups are possible? (Order does not matter.)

$$ {}_{12}C_5 = \frac{12!}{5!(12-5)!} = 792 $$

Example

Suppose that in addition to selecting the group, he must also rank each of the players in that starting lineup according to their ability (order matters).

$$_{12}P_5 = \frac{12!}{(12-5)!} = 95{,}040$$