INTRODUCTION TO NUMERICAL ANALYSIS

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Lecture Outline

Definition

Definition

Numerical analysis and method is the area of mathematics and computer science that creates, analyses, and implements algorithms for solving numerically the problems of continuous mathematics.

1. Such problems originate generally from real-world applications of algebra, geometry, and calculus, and they involve variables which vary continuously.
2. The overall goal of the field is the design and analysis of techniques to give approximate but not exact solutions to problems that do may/may not have analytical solutions.

Definition

1. When an analytical solution does not exist, numerical techniques are employed to solving hard problems.
2. There are times that an analytical solution to a problem may exist but might be cumbersome and time wasting to find such solution. In such instances, it is better to resort to numerical solutions.

Most numerical solutions are iterative.

This implies that a sequence of approximate solution is obtained by repeating a given procedure. These solutions are implemented using computer programs.

Example

Analytically we can find the roots of

$$x^2 - 1 = 0$$

as

$$x = \pm 1$$

Since the highest degree of x is two, then we have two roots.

Example

Find all the five roots of the problem

$$x^5 - 1 = 0$$

It will be very difficult to find analytical (exact) solution to this problem, hence it would require that we find the approximate solution to the polynomial function using numerical methods.

Method vs. Algorithm vs. Implementation

1. Method: a general (mathematical) framework describing the solution process
2. Algorithm: a detailed description of executing the method
3. Implementation: a particular instantiation of the algorithm

We use a pseudocode to describe the algorithms. This pseudocode specifies the form of the input to be supplied and the form of the desired output.

Not all numerical procedures give satisfactory output for arbitrarily chosen input. As a consequence, a stopping technique independent of the numerical technique is incorporated into each algorithm to avoid infinite loops.

Numerical focus

People who employ numerical methods for solving problems have the following concerns:

Accuracy

Very few of our computations will yield the exact answer to a problem, so we will have to understand how much error is made, and how to control (or even diminish) that error. That is the difference between the approximate and exact solutions.

Efficiency

How fast and cheap (memory) can we compute a solution? Does the algorithm take an excessive amount of computer time? This might seem to be an odd question to concern ourselves with. The numerical analyst prefers algorithms that has faster rate of convergence.

Completeness

Is the solution unique or do other solutions exist excluding your obtained approximate solution.

Numerical focus

Stability

1. Does the method produce similar results for similar data?
2. We are interested in choosing methods that will produce dependably accurate results for a wide range of problems.
3. One criterion we will impose on an algorithm whenever possible is that small changes in the initial data produce correspondingly small changes in the final results. An algorithm that satisfies this property is called stable;
4. otherwise the method is unstable, and unstable methods tend to produce unreliable results.
5. Some algorithms are stable only for certain choices of initial data. These are called conditionally stable.

Significant Digits

Definition

There are digits beginning with the leftmost non-zero digit and ending with the rightmost correct digit, including final zeros that are exact.

1. All non-zero digits are considered significant. For example 91 has two significant figures, likewise 123.45 has five significant figures.
2. Zeros appearing anywhere between two non-zero digits are significant. Example 101.1203 has seven significant figures.
3. Leading zeros are not significant. Example 0.00053 has two significant figures.
4. Trailing zeros in a number containing a decimal point are significant. Example 12.2300 has six significant figures.
5. Again 0.0001200 has four significant figures (the zeros before 1 are not significant).
6. In addition 120.00 has five significant figures since it has three trailing zeroes.

Accuracy and Precision

Definition

Accurate to n decimal places means that you can trust n digits to the right of the decimal place.

Accurate to n significant digits means that you can trust a total of n digits as being meaningful beginning with the leftmost nonzero digit.

Example

12.356 has three decimal places but five significant figures.

Rounding and Chopping

Definition

We say that a number x is **chopped** to n digits or figures when all digits that follow the nth digit are discarded and none of the remaining n digits are changed.
Conversely, x is **rounded** to n digits or figures when x is replaced by an n-digit number that approximates x with minimum error.

Example

The results of rounding some three-decimal numbers to two digits are

$$0.217 \approx 0.22, \qquad 0.365 \approx 0.36, \qquad 0.475 \approx 0.48, \qquad 0.592 \approx 0.59, \tag{1}$$

while chopping them gives

$$0.217 \approx 0.21, \qquad 0.365 \approx 0.36, \qquad 0.475 \approx 0.47, \qquad 0.592 \approx 0.59. \tag{2}$$

Absolute and Relative Errors

Suppose that α and β are two numbers, of which one is regarded as an approximation to the other. The error of β as an approximation to α is

$$\alpha - \beta \tag{3}$$

that is, the error equals the exact value minus the approximate value.

Definition

The absolute error of β as an approximation to α is

$$|\alpha - \beta| \tag{4}$$

The relative error of β as an approximation to α is

$$\frac{|\alpha - \beta|}{|\alpha|} \tag{5}$$

Generally

$$\text{absolute error} = |\text{exact value - approximate value}| \tag{6}$$

and

$$\text{relative error} = \frac{|\text{exact value - approximate value}|}{|\text{exact value}|} \tag{7}$$

For practical reasons, the relative error is usually more meaningful than the absolute error.

Example

If $x = 0.00347$ is rounded to $\tilde{x} = 0.0035$, what is its number of significant digits. Again find the absolute error, and relative error. Interpret the results.

Solution

$\tilde{x} = 0.0035$ has two significant digits.

$$
\begin{aligned}
\text{absolute error} &= |\text{exact value - approximate value}| && (8)\\
&= |0.00347 - 0.0035| && (9)\\
&= |-0.00003| && (10)\\
&= 0.00003 && (11)
\end{aligned}
$$

$$
\text{relative error} = \frac{|\text{exact value - approximate value}|}{|\text{exact value}|} = \frac{0.00003}{|0.00347|} = 0.008646 \qquad (12)
$$

Clearly, the relative error is a better indication of the number of significant digits than the absolute error

Differentiable Functions

Definition

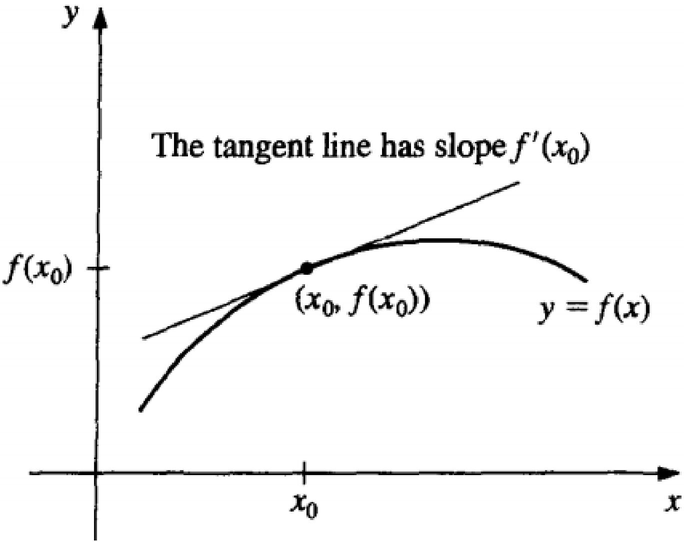
Assume that $f(x)$ is defined on an open interval containing x_0. Then f is said to be differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{13}$$

exists. When this limit exists, it is denoted by $f'(x_0)$ called the derivative of f at x_0.

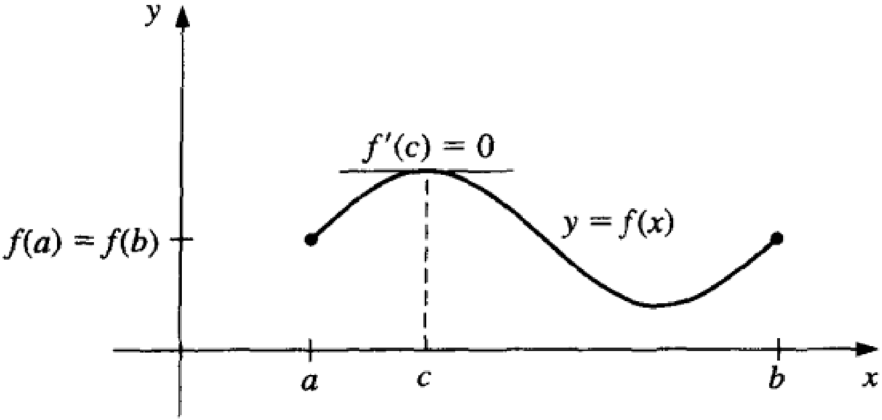
An equivalent way to express this limit is to use the h-increment notation:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = f'(x_0) \tag{14}$$


y
The tangent line has slope $f'(x_0)$
$f(x_0)$
$(x_0, f(x_0))$
$y = f(x)$
x_0
x

Rolle's Theorem

Suppose $f \in [a,\ b]$ and f is differentiable on (a, b). If $f(a) = f(b)$, then a number c in (a, b) exists with $f'(c) = 0$.



Mean-Value Theorem

Definition

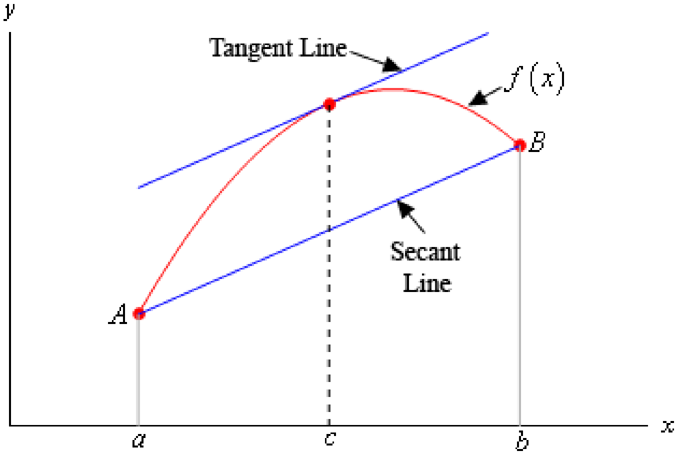
If f is a continuous function on the closed interval $[a, b]$ and possesses a derivative at each point of the open interval (a, b), then

$$f(b) = f(a) + (b - a)f'(c) \quad \Longrightarrow \quad f'(c) = \frac{f(b) - f(a)}{b - a} \tag{15}$$

for some c in (a, b).

The right-hand side could be used as an approximation for $f'(x)$ for any x within the interval (a, b).

Geometrically, the mean value theorem says that there is at least one number $c \in (a, b)$ such that the slope of the tangent line to the graph of $y = f(x)$ at the point $(c, f(c))$ equals the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.



Example

The function $f(x) = sin(x)$ is continuous on the closed interval $[0.1, 2.1]$ and differentiable on the open interval $(0.1, 2.1)$. Thus, by the mean value theorem, there is a number c such that

$$f'(c) = \frac{f(2.1) - f(0.1)}{2.1 - 0.1} = \frac{0.863209 - 0.099833}{2.1 - 0.1} = 0.381688 \tag{16}$$

The solution to

$$f'(c) = cos(c) = 0.381688 \tag{17}$$
$$c = \cos^{-1}(0.381688) = 1.179174 \tag{18}$$

in the interval $(0.1, 2.1)$.

Taylor Series

1. Most students will have encountered infinite series (particularly Taylor series) in their study of calculus.
2. Consequently, this section is particularly important for numerical analysis, and deserves careful study.
3. Once students are well grounded with a basic understanding of Taylor series we can proceed to study the fundamentals of numerical methods with better comprehension.

Definition

Taylor series is a representation of a function as an **infinite sum of terms** that are calculated from the values of the function's derivatives at a single point.

Some examples include

1. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$
2. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
3. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
4. $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$
5. $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$
6. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1$
7. $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \qquad -1 < x <\leq 1$

Example

Use five terms of the Taylor series to approximate $\ln(1.1)$.

Solution

Taking $x = 0.1$, then the first five terms of the series for $ln(1+x)$ gives us

$$\ln(1.1) \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} = 0.095310333\ldots$$

Compute $\ln(1.1)$ directly on your calculator and compare your answer.
This value is correct to six decimal places of accuracy.

Taylor Series

Taylor series of f at the point c

Generally, the Taylor series for a function f about the point c is given as:

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots \tag{19}$$

$$\approx \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{20}$$

$f', f'', \cdots f^{(n)}$ are the derivatives of the function f. Again the values of these successive derivatives should exist at the point c.

Maclaurin Series

In the special case where $c = 0$, then we obtain the **Maclaurin series** given by:

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots \tag{21}$$

$$\approx \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{22}$$

Example

What is the Taylor series of the function

$$f(x) = 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5$$

at the point $c = 2$?

Solution

To compute the coefficients in the series, we need the numerical values of $f^{(n)}(2)$ for $n \geq 0$. Here are the details of the computation:

$$
\begin{aligned}
f(x) &= 3x^5 - 2x^4 + 15x^3 + 13x^2 - 12x - 5 & f(2) &= 207 & (23)\\
f'(x) &= 15x^4 - 8x^3 + 45x^2 + 26x - 12 & f'(2) &= 396 & (24)\\
f''(x) &= 60x^3 - 24x^2 + 90x + 26 & f''(2) &= 590 & (25)\\
f'''(x) &= 180x^2 - 48x + 90 & f'''(2) &= 714 & (26)\\
f^{(4)}(x) &= 360x - 48 & f^{(4)}(2) &= 672 & (27)\\
f^{(5)}(x) &= 360 & f^{(5)}(2) &= 360 & (28)\\
f^{(6)}(x) &= 0 & f^{(6)}(2) &= 0 & (29)
\end{aligned}
$$

Therefore, we have

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \tag{30}$$

$$\frac{f^{(4)}(c)}{4!}(x-c)^4 + \frac{f^{(5)}(c)}{5!}(x-c)^5 + \frac{f^{(6)}(c)}{6!}(x-c)^6 \tag{31}$$

$$\approx f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3 +$$

$$\frac{f^{(4)}(2)}{4!}(x-2)^4 + \frac{f^{(5)}(2)}{5!}(x-2)^5 + \frac{f^{(6)}(2)}{6!}(x-2)^6 \tag{32}$$

$$\approx 207 + 396(x-2) + 295(x-2)^2 + 119(x-2)^3 + 28(x-2)^4 + 3(x-2)^5$$

Example

Find the Taylor series and the Maclaurin series of the following function using the first four terms of the series.

$$f(x) = e^{2x}, \qquad \text{where } c = 1$$

Solution

i. For the Taylor series we obtain

$$f(x) = e^{2x} \qquad f(1) = e^2 \tag{33}$$

$$f'(x) = 2e^{2x} \qquad f'(1) = 2e^2 \tag{34}$$

$$f''(x) = 4e^{2x} \qquad f''(1) = 4e^2 \tag{35}$$

$$f'''(x) = 8e^{2x} \qquad f'''(1) = 8e^2 \tag{36}$$

$$
\begin{aligned}
f(x) &\approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 && (37)\\
&\approx f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 && (38)\\
&= e^2 + 2e^2(x-1) + \frac{4e^2}{2}(x-1)^2 + \frac{8e^2}{6}(x-1)^3 && (39)\\
&= e^2\left[1 + 2x - 2 + 2x^2 - 4x + 2 + \frac{4}{3}(x^3 - 3x^2 + 3x - 1)\right] && (40)\\
&= e^2\left(\frac{7}{3} + 2x - 2x^2 + \frac{4}{3}x^3\right) && (41)
\end{aligned}
$$

ii. The Maclaurin series is obtained with $c = 0$. Thus,

$$f(x) = e^{2x} \qquad f(0) = e^0 = 1 \tag{42}$$
$$f'(x) = 2e^{2x} \qquad f'(0) = 2e^0 = 2 \tag{43}$$
$$f''(x) = 4e^{2x} \qquad f''(0) = 4e^0 = 4 \tag{44}$$
$$f'''(x) = 8e^{2x} \qquad f'''(0) = 8e^0 = 8 \tag{45}$$

$$f(x) \approx f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 \tag{46}$$
$$\approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 \tag{47}$$
$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 \tag{48}$$

Taylor's Theorem for $f(x)$

Definition

If the function f possesses continuous derivatives of order $0, 1, 2, \cdots, (n+1)$ in a closed interval $I = [a, b]$, then for any c and x in I,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(x-c)^k + E_{n+1} \tag{49}$$

where the error term E_{n+1} can be given in the form

$$E_{n+1} = \frac{f^{n+1}(\epsilon)}{(n+1)!}(x-c)^{n+1} \tag{50}$$

for some ϵ between c and x and depends on both.

In practical computations with Taylor series, it is usually necessary to truncate the series because it is not possible to carry out an infinite number of additions.

Example

Derive the formal Taylor series for $f(x) = \ln(1+x)$ at $c = 0$

$$
\begin{aligned}
f(x) &= ln(1+x) \implies & f(0) &= 0 && (51)\\
f'(x) &= (1+x)^{-1} \implies & f'(0) &= 1 && (52)\\
f''(x) &= -(1+x)^{-2} \implies & f''(0) &= -1 && (53)\\
f'''(x) &= 2(1+x)^{-3} \implies & f'''(0) &= 2 && (54)\\
f^{(4)}(x) &= -6(1+x)^{-4} \implies & f^{(4)}(0) &= -6 && (55)\\
&\vdots & &\vdots && (56)\\
f^{(k)}(x) &= (-1)^{k-1}(k-1)!(1+x)^{-k} \implies & f^{(k)}(0) &= (-1)^{k-1}(k-1)!
\end{aligned}
$$

Hence by Taylor's Theorem, we obtain

$$
\begin{aligned}
\ln(1+x) &= \sum_{k=1}^{n}(-1)^{k-1}\frac{(k-1)!}{k!}x^k + \frac{(-1)^n n!(1+\epsilon)^{-n-1}}{(n+1)!}x^{n+1} && (57)\\
&= \sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} + \frac{(-1)^n}{n+1}(1+\epsilon)x^{n+1} && (58)
\end{aligned}
$$

Taylor's Theorem for $f(x+h)$

Definition

If the function f possesses continuous derivatives of order $0, 1, 2, \cdots, (n+1)$ in a closed interval $I = [a, b]$, then for any x in I,

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + E_{n+1} \tag{59}$$

where h is any value such that $x + h$ is in I and where

$$E_{n+1} = \frac{f^{n+1}(\epsilon)}{(n+1)!} h^{n+1} \tag{60}$$

for some ϵ between x and $x + h$.

Big O notation

1. The error term E_{n+1} depends on h in two ways:
   1. First, h^{n+1} is explicitly present;
   2. Second, the point ϵ generally depends on h.
2. So as h converges to zero, E_{n+1} converges to zero with essentially the same rapidity with which h^{n+1} converges to zero. quite rapid.
3. To express this qualitative fact, we write

$$E_{n+1} = \mathcal{O}(h^{n+1}) \tag{61}$$

as $h \to 0$. This is called big O notation, and it is shorthand for the inequality

$$|E_{n+1}| \leq C|h|^{n+1} \tag{62}$$

where C is a constant.

Derived Equations from Equation(59)

$n = 0$

$$f(x+h) = f(x) + f'(\epsilon_1)h \tag{63}$$
$$= f(x) + \mathcal{O}(h) \tag{64}$$

$n = 1$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(\epsilon_2)h^2 \tag{65}$$
$$= f(x) + f'(x)h + \mathcal{O}(h^2) \tag{66}$$

$n = 2$

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + \frac{1}{3!}f'''(\epsilon_3)h^3 \tag{67}$$
$$= f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + \mathcal{O}(h^3) \tag{68}$$

Exercise

Find the Taylor series and the Maclaurin series of the following function using the first four terms of the series.

1. $$f(x) = \sin(2x), \qquad \textit{where } c = \pi$$

2. $$f(x) = \cosh(3x), \qquad \textit{where } c = 2$$

END OF LECTURE

THANK YOU