ITERATIVE METHODS FOR SOLVING LINEAR SYSTEM OF EQUATIONS

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Lecture Outline

1. Gauss-Jacobi Method
2. Gauss-Seidel Method
3. Successive Over-Relaxation (SOR)

The Methods

The method for solving system of equations can be classified into

1. Direct Method: This produce the exact solution after a finite number of steps. Some methods discussed earlier include but not limited to
   a. Gaussian Elimination Method
   b. Gauss-Jordan Elimination Method
2. Indirect/Iterative Method: This is based on the method of successive approximations. It start with an initial approximation to the solution to obtain a sequence of approximate solutions. Some example of the indirect methods are
   a. Gauss-Jacobi Method
   b. Gauss-Seidel Method
   c. Successive Over-Relaxation (SOR) method

Stopping criterion

1. Given an initial solution x_0 you can solve the system of equation $Ax = b$ to obtain the approximate solution $x_1, x_2, x_3, \cdots, x_k$.
2. We stop the iteration process when

$$|x_{k+1} - x_k| < \epsilon$$

where ϵ is the error term.

A matrix A can be decomposed into the sum of an upper triangular, a lower triangular, and a diagonal matrix. That is

Given the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1}$$

then

$$L = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} \qquad D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \qquad U = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \tag{2}$$

that is

$$L + D + U = A \tag{3}$$

Gauss-Jacobi Method

1. This method is sometimes called the Jacobi method. It is a form of fixed-point iteration the solves $Ax = b$.
2. Let D denote the main diagonal of A, L denote the lower triangle of A (entries below the main diagonal), and U denote the upper triangle (entries above the main diagonal).

$$Ax = b \tag{4}$$

$$(D+L+U)x = b \tag{5}$$

$$Dx = b-(L+U)x \tag{6}$$

$$x = D^{-1}[b-(L+U)x] \tag{7}$$

3. Iteratively

$$x^{(k+1)} = D^{-1}[b-(L+U)x^{(k)}]; \qquad k = 0,1,2,\cdots; \qquad x^{(k)} = \text{ initial vector} \tag{8}$$

Since D is a diagonal matrix, its inverse is the matrix of reciprocals of the diagonal entries of A.

Gauss-Jacobi Method

The following give explicit details of the Gauss-Jacobi-Method

1. The method assumes that the entries of the leading diagonal of a matrix A cannot be zero, that is $a_{ii} \neq 0$.

Given the following system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \tag{9}$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \tag{10}$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \tag{11}$$

Gauss-Jacobi Method

The Jacobi method makes the coefficient of the leading diagonal of each equation the subject. Therefore, equation (9) reduces to (12), (10) reduces to (13), and (11) reduces to (14).

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \tag{12}$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \tag{13}$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \tag{14}$$

Gauss-Jacobi Method

Iteratively, the eqs. (12) to (14) becomes

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right] \tag{15}$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}\right] \tag{16}$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k)} - a_{32}x_2^{(k)}\right] \tag{17}$$

Equations (15) to (17) are the iterative formula for the Gauss-Jacobi method for solving three equation with three unknowns.
The method can be generalized for higher order system of equations.

Convergence and Diagonal Dominance

The method is guaranteed to converge if the coefficient of the matrix A is diagonally dominant. That is

$$|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|, \quad i \neq j \tag{18}$$

If the system is not diagonally dominant, we may exchange the equations if possible.

Example

Determine whether the matrices

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 8 & 1 \\ 9 & 2 & -2 \end{bmatrix}$$

are strictly diagonally dominant

1. The matrix A is diagonally dominant because $|3| > |1| + |-1|$, $|-5| > |2| + |2|$, and $|8| > |1| + |6|$
2. B is not, because, for example, $|3| > |2| + |6|$ is not true.
3. However, if the first and third rows of B are exchanged, then B is strictly diagonally dominant and Jacobi is guaranteed to converge.

Example

Solve the following system of equation using the Gauss-Jacobi method taking the origin as the initial solution.

$$2x + y = 4 \tag{19}$$
$$x - y = 5 \tag{20}$$

Take $\epsilon = 0.05$, otherwise stop on the 5th iteration.

We first have to identify the leading diagonals based on the arrangement of the equations. Again, we inquire if the system is diagonal dominant. Yes, it is indeed diagonal dominant. So we a sure that the solution will converge.
The leading diagonals are x in eq. (19), and y in eq. (20).

$$2\boxed{x} + y = 4 \tag{21}$$
$$x - \boxed{y} = 5 \tag{22}$$

Making these variables the subject we have

$$x = \frac{1}{2}[4 - y] \tag{23}$$

$$y = x - 5 \tag{24}$$

Iteratively we have

$$x^{(k+1)} = \frac{1}{2}[4 - y^{(k)}] \tag{25}$$

$$y^{(k+1)} = x^{(k)} - 5 \tag{26}$$

Iteration 1: when k =0

$$x^{(1)} = \frac{1}{2}[4 - y^{(0)}] \tag{27}$$

$$y^{(1)} = x^{(0)} - 5 \tag{28}$$

1. The initial approximations are the values at the origin, therefore
$$x^{(0)} = 0, \quad y^{(0)} = 0$$
2. Substituting the initial values, we have
$$x^{(1)} = \frac{1}{2}[4 - 0] = 2 \tag{29}$$
$$y^{(1)} = 0 - 5 = -5 \tag{30}$$
3. Check stopping criterion: Since this is the first iteration, we will skip this step. There is no previous iterative value to make such comparison.
4. Hence, we continue the iteration.

Note

The following are not equivalent

$$x^0 \neq x^{(0)}, \quad x^1 \neq x^{(1)}, \quad \text{and } x^2 \neq x^{(2)}$$

The former are exponents, while the latter are iterative numbers.

Iteration 2: when k =1

$$x^{(2)} = \frac{1}{2}[4 - y^{(1)}] \tag{31}$$
$$y^{(2)} = x^{(1)} - 5 \tag{32}$$

1. From iteration one
$$x^{(1)} = 2, \quad y^{(1)} = -5$$
2. Substituting these values, we have
$$x^{(2)} = \frac{1}{2}[4 - (-5)] = 4.5 \tag{33}$$
$$y^{(2)} = 2 - 5 = -3 \tag{34}$$
3. Check stopping criterion:
$|x^{(2)} - x^{(1)}| = |4.5 - 2| = 2.5 \implies \nless \epsilon$
$|y^{(2)} - y^{(1)}| = |-3 - (-5)| = 2 \implies \nless \epsilon$
4. Hence, move to the next iteration.

Iteration 3: when k =2

1. We have
$$x^{(3)} = \frac{1}{2}[4 - y^{(2)}] = \frac{1}{2}[4 - (-3)] = 3.5 \tag{35}$$
$$y^{(3)} = x^{(2)} - 5 = 4.5 - 5 = -0.5 \tag{36}$$
2. Check stopping criterion: $|x^{(3)} - x^{(2)}| = |3.5 - 4.5| = 1 \implies \not< \epsilon$
$|y^{(3)} - y^{(2)}| = |-0.5 - (-3)| = 2.5 \implies \not< \epsilon$
3. Hence move to the next iteration.

Iteration 4: when k =3

1. We have

$$x^{(4)} = \frac{1}{2}[4 - y^{(3)}] = \frac{1}{2}[4 - (-0.5)] = 2.25 \tag{37}$$

$$y^{(4)} = x^{(3)} - 5 = 3.5 - 5 = -1.5 \tag{38}$$

2. Check stopping criterion:
   $|x^{(4)} - x^{(3)}| = |2.25 - 3.5| = 1.25 \implies \not< \epsilon$
   $|y^{(4)} - y^{(3)}| = |-1.5 - (-0.5)| = 1 \implies \not< \epsilon$
3. Hence, move to the next iteration.

Iteration 5: when k =4

1. We have

$$x^{(5)} = \frac{1}{2}[4 - y^{(4)}] = \frac{1}{2}[4 - (-1.5)] = 2.75 \tag{39}$$
$$y^{(5)} = x^{(4)} - 5 = 2.25 - 5 = -2.75 \tag{40}$$

2. Based on the otherwise condition we halt the iteration process here.
3. Therefore, the solution to the system is

$$x = 2.75, \quad y = -2.75 \tag{41}$$

Example

Solve the system of equation

$$4x_1 + x_2 + x_3 = 2 \tag{42}$$
$$x_1 + 5x_2 + 2x_3 = -6 \tag{43}$$
$$x_1 + 2x_2 + 3x_3 = -4 \tag{44}$$

using the Jacobi method by performing five iterations. Use the initial approximations $[x_1,\ x_2,\ x_3] = [0,\ 0,\ 0]$

We first have to identify the leading diagonals based on the arrangement of the equations. Again, we inquire if the system is diagonal dominant. Yes it is indeed diagonal dominant.

The leading diagonals are x_1 in eq. (42), x_2 in eq. (43), and x_3 in eq. (44).

$$4\boxed{x_1} + x_2 + x_3 = 2 \tag{45}$$
$$x_1 + 5\boxed{x_2} + 2x_3 = -6 \tag{46}$$
$$x_1 + 2x_2 + 3\boxed{x_3} = -4 \tag{47}$$

Making these variables the subject we have

$$x_1 = \frac{1}{4}[2 - (x_2 + x_3)] \tag{48}$$
$$x_2 = \frac{1}{5}[-6 - (x_1 + 2x_3)] \tag{49}$$
$$x_3 = \frac{1}{3}[-4 - (x_1 + 2x_2)] \tag{50}$$

Iteratively we have

$$x_1^{(k+1)} = 0.25[2 - (x_2^{(k)} + x_3^{(k)})] \tag{51}$$

$$x_2^{(k+1)} = 0.2[-6 - (x_1^{(k)} + 2x_3^{(k)})] \tag{52}$$

$$x_3^{(k+1)} = 0.33[-4 - (x_1^{(k)} + 2x_2^{(k)})] \tag{53}$$

Iteration 1: when k= 0

From the initial conditions we have $x_1^{(0)} = 0, x_2^{(0)} = 0, x_3^{(0)} = 0.$
Substituting

$$x_1^{(1)} = 0.25[2 - (x_2^{(0)} + x_3^{(0)})] = 0.25(2) = 0.5$$
$$x_2^{(1)} = 0.2[-6 - (x_1^{(0)} + 2x_3^{(0)})] = 0.2(-6) = -1.2$$
$$x_3^{(1)} = 0.3333[-4 - (x_1^{(0)} + 2x_2^{(0)})] = 0.33(-4) = -1.3333$$

Since we are required to perform five iterations there is no need checking the stopping criterion.

Iteration 2: when k= 1

$$x_1^{(2)} = 0.25[2-(x_2^{(1)}+x_3^{(1)})] = 0.25[2-(-1.2-1.33333)] \quad (54)$$
$$= 1.13333 \quad (55)$$

$$x_2^{(2)} = 0.2[-6-(x_1^{(1)}+2x_3^{(1)})] = 0.2[-6-(0.5+2(-1.33333))] \quad (56)$$
$$= -0.76668 \quad (57)$$

$$x_3^{(2)} = 0.33[-4-(x_1^{(1)}+2x_2^{(1)})] = 0.33333[-4-(0.5+2(-1.2))] \quad (58)$$
$$= -0.7 \quad (59)$$

Iteration 3: when k= 2

$$
\begin{aligned}
x_1^{(3)} &= 0.25[2-(x_2^{(2)}+x_3^{(2)})] = 0.25[2-(-0.76668-0.7)] && (60)\\
&= 0.86667 && (61)
\end{aligned}
$$

$$
\begin{aligned}
x_2^{(3)} &= 0.2[-6-(x_1^{(2)}+2x_3^{(2)})] = 0.2[-6-(1.13333+2(-0.7))] && (62)\\
&= -1.14667 && (63)
\end{aligned}
$$

$$
\begin{aligned}
x_3^{(3)} &= 0.33[-4-(x_1^{(2)}+2x_2^{(2)})] = 0.33333[-4-(1.13333+2(-0.76668))] && (64)\\
&= -1.19998 && (65)
\end{aligned}
$$

Iteration 4: when k= 3

$$x_1^{(4)} = 0.25[2-(x_2^{(3)}+x_3^{(3)})] = 0.25[2-(-1.14667-1.19999)] \quad (66)$$
$$= 1.08666 \quad (67)$$

$$x_2^{(4)} = 0.2[-6-(x_1^{(3)}+2x_3^{(3)})] = 0.2[-6-(0.86667+2(-1.19998))] \quad (68)$$
$$= -0.89334 \quad (69)$$

$$x_3^{(4)} = 0.33[-4-(x_1^{(3)}+2x_2^{(3)})] = 0.33333[-4-(0.86667+2(-1.14667))] \quad (70)$$
$$= -0.85777 \quad (71)$$

Iteration 5: when k= 4

$$x_1^{(5)} = 0.25[2-(x_2^{(4)}+x_3^{(4)})] = 0.25[2-(-0.89334-0.85777)] \quad (72)$$
$$= 0.93778 \quad (73)$$

$$x_2^{(5)} = 0.2[-6-(x_1^{(4)}+2x_3^{(4)})] = 0.2[-6-(1.08666+2(-0.85777))] \quad (74)$$
$$= -1.07422 \quad (75)$$

$$x_3^{(5)} = 0.33[-4-(x_1^{(4)}+2x_2^{(4)})] = 0.33333[-4-(1.08666+2(-0.89334))] \quad (76)$$
$$= -1.09998 \quad (77)$$

Therefore, the system has the solution

$$x_1 = 0.93778, \; x_2 = -1.07422, \; x_3 = -1.0998 \quad (78)$$

Gauss-Seidel Method

1. This method is similar to the Gauss-Jacobi method.
2. However, this uses a continual iterative technique to solve $Ax = b$.
3. Let D denote the main diagonal of A, L denote the lower triangle of A (entries below the main diagonal), and U denote the upper triangle (entries above the main diagonal).

$$Ax = b \tag{79}$$

$$(D + L + U)x = b \tag{80}$$

$$(L + D)x = b - Ux \tag{81}$$

$$(L + D)x_{k+1} = b - Ux_k \tag{82}$$

$$Dx_{k+1} = b - Ux_k - Lx_{k+1} \tag{83}$$

$$x_{k+1} = D^{-1}(b - Ux_k - Lx_{k+1}); \qquad k = 0, 1, 2, \cdots \tag{84}$$

where x_0 is a given initial vector.

Gauss-Seidel Method

The following give explicit details of the Gauss-Seidel Method

1. Given the system of equation below

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \quad (85)$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \quad (86)$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \quad (87)$$

Gauss-Seidel Method

1. The Gauss-Seidel method likewise makes the coefficient of the leading diagonal of each equation the subject.
2. Therefore, equation (85) reduces to (88), (86) reduces to (89), and (87) reduces to (90).

$$x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right] \tag{88}$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right] \tag{89}$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right] \tag{90}$$

Gauss-Seidel Method

1. However, the continual iterative method of the Gauss-Seidel approach is deduced from eqs. (88) to (90) as

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right] \tag{91}$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right] \tag{92}$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right] \tag{93}$$

2. Equations (91) to (93) are the iterative formula for the Gauss-Seidel method for solving three equation with three unknowns.
3. The method can be generalized for higher order system of equations.

Convergence and Diagonal Dominance

The method is guaranteed to converge if the coefficient of the matrix A is diagonally dominant. That is

$$|a_{ii}| \geq \sum_{j=1}^{n} |a_{ij}|, \quad i \neq j \tag{94}$$

1. If the system is not diagonally dominant, we may exchange the equations if possible.
2. If both methods converges, then the Gauss-Seidel method converges at least two times faster than the Gauss-Jacobi method.

Example

Solve the following system of equation

$$20x + y - 2z = 17 \tag{95}$$

$$3x + 20y - z = -18 \tag{96}$$

$$2x - 3y + 20z = 25 \tag{97}$$

using the Gauss-Seidel method with the initial solution

$$x^{(0)} = y^{(0)} = z^{(0)} = 1$$

Take $\epsilon = 0.01$, otherwise stop on the 5th iteration

We first have to identify the leading diagonals based on the arrangement of the equations. Again, we inquire if the system is diagonal dominant. Yes, it is indeed diagonal dominant

The leading diagonals are x in eq. (95), y in eq. (96) and z in eq. (96).

$$20\boxed{x} + y - 2z = 17 \tag{98}$$
$$3x + 20\boxed{y} - z = -18 \tag{99}$$
$$2x - 3y + 20\boxed{z} = 25 \tag{100}$$

Making these variables the subject we have

$$x = \frac{1}{20}[17 - y + 2z] \tag{101}$$
$$y = \frac{1}{20}[-18 - 3x + z] \tag{102}$$
$$z = \frac{1}{20}[25 - 2x + 3y] \tag{103}$$

Iteratively we have

$$x^{(k+1)} = \frac{1}{20}[17 - y^{(k)} + 2z^{(k)}] \tag{104}$$

$$y^{(k+1)} = \frac{1}{20}[-18 - 3x^{(k+1)} + z^{(k)}] \tag{105}$$

$$z^{(k+1)} = \frac{1}{20}[25 - 2x^{(k+1)} + 3y^{(k+1)}] \tag{106}$$

Iteration 1: when k =0

The initial solution is

$$x^{(0)} = 1, \; y^{(0)} = 1, \; z^{(0)} = 1$$

$$x^{(1)} = \frac{1}{20}[17 - y^{(0)} + 2z^{(0)}] = 0.05[17 - 1 - 2(1)] = 0.9 \tag{107}$$

$$y^{(1)} = \frac{1}{20}[-18 - 3x^{(1)} + z^{(0)}] = 0.05[-18 - 3(0.9) + 1] = -0.9895 \tag{108}$$

$$z^{(1)} = \frac{1}{20}[25 - 2x^{(1)} + 3y^{(1)}] = 0.05[25 - 2(0.9) + 3(-0.9895)] = 1.01225 \tag{109}$$

Note

Be meticulous on how this continual substitution is done.

Since this is the very first iteration, we may not worry ourselves checking the stopping criterion

Iteration 2: when k =1

$$x^{(2)} = \frac{1}{20}[17 - y^{(1)} + 2z^{(1)}] = 0.05[17 - 1(-0.9895) - 2(1.01225)] = 1.00475 \quad (110)$$

$$y^{(2)} = \frac{1}{20}[-18 - 3x^{(2)} + z^{(1)}] = 0.05[-18 - 3(1.00475) + 1.01225] = -0.999 \quad (111)$$

$$z^{(2)} = \frac{1}{20}[25 - 2x^{(2)} + 3y^{(2)}] = 0.05[25 - 2(1.00475) + 3(-0.999)] = 1.000 \quad (112)$$

1. Check stopping criterion:
   $|x^{(2)} - x^{(1)}| = |1.00475 - 0.9| = 0.10475 \implies \nless \epsilon$
   $|y^{(2)} - y^{(1)}| = |-0.999 - (-0.9895)| = 0.0095 \implies < \epsilon$
   $|z^{(2)} - z^{(1)}| = |1.01225 - 1.000| = 0.01225 \implies \nless \epsilon$
2. Since two of these variables do not satisfy the condition we move to the next iteration.

Iteration 3: when k =2

$$x^{(3)} = \frac{1}{20}[17 - y^{(2)} + 2z^{(2)}] = 0.05[17 - 1(-0.999) - 2(1.0)] = 1.00 \tag{113}$$

$$y^{(3)} = \frac{1}{20}[-18 - 3x^{(3)} + z^{(2)}] = 0.05[-18 - 3(1.00) + 1.00] = -1 \tag{114}$$

$$z^{(3)} = \frac{1}{20}[25 - 2x^{(3)} + 3y^{(3)}] = 0.05[25 - 2(1.00) + 3(-1)] = 1.000 \tag{115}$$

1. Check stopping criterion:
   $|x^{(3)} - x^{(2)}| = |1.00 - 1.00475| = 0.00475 \implies < \epsilon$
   $|y^{(3)} - y^{(2)}| = |-1 - (-0.999)| = 0.001 \implies < \epsilon$
   $|z^{(3)} - z^{(2)}| = |1.00 - 1.000| = 0.00 \implies < \epsilon$
   Since this criterion is satisfied we stop the iteration here.

Therefore, the solution to the system is

$$x = 1, \quad y = -1, \quad z = 1 \tag{116}$$

Successive Over-Relaxation (SOR)

1. The method of successive over-relaxation (SOR) is a variant of the Gauss–Seidel method for solving a linear system of equations, resulting in faster convergence.
2. The idea is to choose a value of a parameter ω that will accelerate the rate of convergence of the iterates to the solution.
3. The number ω is called the relaxation parameter, and $\omega > 1$ is referred to as over-relaxation.
4. When $\omega = 1$ we have the Gauss-Seidel method.
5. The parameter ω can also be allowed to be less than 1, in a method called Successive Under-Relaxation.
6. The relaxation parameter is most appropriate for $\omega \in (0, 2)$

Successive Over-Relaxation

1. Just as with Jacobi and Gauss–Seidel, an alternative derivation of SOR follows from treating the system as a fixed-point problem.
2. The problem $Ax = b$ can be written $(L + D + U)x = b$, and, upon multiplication by ω and rearranging, we have

$$(\omega L + \omega D + \omega U)x = \omega b \tag{117}$$

$$(\omega L + \omega D + \omega U)x + Dx - Dx = \omega b \tag{118}$$

$$\omega Lx + Dx = \omega b - \omega Dx - \omega Ux + Dx \tag{119}$$

$$(\omega L + D)x = \omega b - \omega Ux + (1-\omega)Dx \tag{120}$$

$$x = (\omega L + D)^{-1}[(1-\omega)Dx - \omega Ux] + \omega(D + \omega L)^{-1}b \tag{121}$$

$$x_{k+1} = (\omega L + D)^{-1}[(1-\omega)Dx_k - \omega Ux_k] + \omega(D + \omega L)^{-1}b \tag{122}$$

Example

Solve the following system of equation

$$20x + y - 2z = 17 \tag{123}$$

$$3x + 20y - z = -18 \tag{124}$$

$$2x - 3y + 20z = 25 \tag{125}$$

using the Successive over-relaxation method with the initial solution $[1, \ 1, \ 1]$ and $\omega = 1.25$. Take $\epsilon = 0.01$, otherwise stop on the 4th iteration

$$A = \begin{bmatrix} 20 & 1 & -2 \\ 3 & 20 & -1 \\ 2 & -3 & 20 \end{bmatrix}, \quad b = \begin{bmatrix} 17 \\ -18 \\ 25 \end{bmatrix}, \quad D = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 2 & -3 & 0 \end{bmatrix} \tag{126}$$

$$U = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad (\omega L + D)^{-1} = \begin{bmatrix} 0.0500 & 0 & 0 \\ -0.0094 & 0.0500 & 0 \\ -0.0080 & 0.0094 & 0.0500 \end{bmatrix} \tag{127}$$

$$(1-\omega)Dx_0 - \omega U x_0 = [-3.7500 \ \ -3.7500 \ \ -5.0000] \tag{128}$$

$$(\omega L + D)^{-1}[(1-\omega)Dx_k - \omega U x_k] + \omega(D + \omega L)^{-1} b = [0.8750 \ \ -1.4766 \ 0.9263] \tag{129}$$

Thus for the 1st iteration

$$[x \ \ y \ \ z] = [0.8750 \ \ -1.4766 \ 0.9263]$$

With the assistance of computer, the other iterations are given as

2nd Iteration

$$[x \ y \ z] = [1.0518 \ \ -0.8952 \ 1.0316]$$

3rd Iteration

$$[x \ y \ z] = [0.9844 \ \ -1.0213 \ 0.9900]$$

4th Iteration

$$[x \ y \ z] = [1.0040 \ \ -0.9960 \ 1.0027]$$

Exercise

Solving the following system of equations

$$\begin{aligned} 10x + 4y - 2z &= 20 \\ 3x + 12y - z &= 28 \\ x + 4y + 7z &= 2 \end{aligned}$$

$$\begin{aligned} 2a + b + c + d &= 2 \\ 4a + 2c + d &= 3 \\ 3a + 2b + 2c &= -1 \\ a + 3b + 2c + 6d &= 2 \end{aligned}$$

using the

1. Gauss-Seidel method
2. Gauss-Jacobi method
3. Successive over-relaxation method

taking the origin as the initial solution.
Take $\epsilon = 0.01$, otherwise stop on the 5th iteration

END OF LECTURE

THANK YOU