DETERMINANTS

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Outline

Outline of Presentation

Introduction

Definition

The determinant is a scalar value that is a function of the entries of a square matrix. It allows characterizing some properties of the matrix.

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For example, in terms of linear algebra, determinants can be used to

1. characterize nonsingular matrices,
2. Used to express solutions of nonsingular systems $Ax = b$
3. Used to express vector cross products.

Determinant of Matrix

Theorem

Let M_{ij} denote the determinant of the $(n-1) \times (n-1)$ submatrix of A formed by deleting the i_{th} row and j_{th} column of A. Assume that the determinant function has been defined for matrices of size $(n-1) \times (n-1)$. Then the determinant of the $n \times n$ matrix A is defined by what we call the first-row Laplace expansion:

$$|A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} M_{1j} \quad (1)$$

$$= a_{11}M_{11} - a_{12}M_{12} + \cdots + (-1)^{1+n} M_{1n} \quad (2)$$

The values M_{ij} are termed minors.

Determinant of 2×2 Matrix

1. The determinant of the general 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is

$$|A| = a_{11}(a_{22}) - a_{12}(a_{21})$$

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Example

The determinant of $Z = \begin{bmatrix} 11 & 2 \\ 3 & -1 \end{bmatrix}$ is

$$|Z| = 11(-1) - 2(3) = -17 \tag{3}$$

Determinant of 3×3 Matrix

- Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

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$$= a_{11}[a_{22}(a_{33}) - a_{23}a_{32}] - a_{12}[a_{21}(a_{33}) - a_{23}(a_{31})] + a_{13}[a_{21}(a_{32}) - a_{22}(a_{31})] \tag{5}$$

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Compute the determinant of the 3×3 matrix $C = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 1 \\ 3 & 5 & -9 \end{bmatrix}$

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$$|C| = 1\begin{vmatrix} 4 & 1 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 3 & -9 \end{vmatrix}$$

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$$\begin{aligned} |C| \quad &= 1\begin{vmatrix} 4 & 1 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 0 & 1 \\ 3 & -9 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 4 \\ 3 & 5 \end{vmatrix} && (6) \\ &= 1[4(-9) - 1(5)] - 2[0(-9) - 1(3)] - 1[0(5) - 3(4)] && (7) \\ &= 1(-41) - 2(-3) - 1(-12) && (8) \\ &= -23 && (9) \end{aligned}$$

The evaluation of an $n \times n$ matrix was presented in terms of the first-row expansion. Actually, we can expand the determinant along any row or column

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$$|A| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \tag{10}$$

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3. The expression $(-1)^{i+j}$ obeys the chessboard pattern of signs:

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ + & - & + & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Cofactors, Adjoint and Inverse of Matrices

Definition (Cofactor)

The (i, j) cofactor of A, denoted by C_{ij} is defined by

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If $A = [a_{ij}]$ is an $n \times n$ matrix, the adjoint of A, denoted by $adj\ A$, is the transpose of the matrix of cofactors.

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If $A = [a_{ij}]$ is an $n \times n$ matrix, the adjoint of A, denoted by $adj\ A$, is the transpose of the matrix of cofactors.

Definition (Inverse)

The inverse of a matrix is given by the relation

$$A^{-1} = \frac{1}{\det A} adj\ A$$

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Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{bmatrix}$

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2. The cofactor matrix is

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$$A^{-1} = -\frac{1}{3}\begin{bmatrix} -3 & 6 & -3 \\ 12 & -15 & 6 \\ -8 & 8 & -3 \end{bmatrix} \tag{14}$$

$$= \begin{bmatrix} 1 & -2 & 1 \\ -4 & 5 & -2 \\ 8/3 & -8/3 & 1 \end{bmatrix} \tag{15}$$

Outline of Presentation

1. Developing the Determinant of a Matrix
   - Introduction
   - Determinant of $n \times n$ Matrix
   - Cofactors, Adjoint, and Inverse of a Matrix
2. Some Properties of Determinant
3. Cramer's Rule

Definition

The product of a matrix A and its adjoint is a diagonal matrix whose diagonal entries are $\det(A)$.

For a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{bmatrix}$ it adjoint its $adj\ A = \begin{bmatrix} -3 & 6 & -3 \\ 12 & -15 & 6 \\ -8 & 8 & -3 \end{bmatrix}$ then

$$A \times adj\, A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 8 & 9 \end{bmatrix} \begin{bmatrix} -3 & 6 & -3 \\ 12 & -15 & 6 \\ -8 & 8 & -3 \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \tag{17}$$

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Note

$$|A| = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 8 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 8 & 8 \end{vmatrix} = -3 + 24 - 24 = -3 \tag{18}$$

Properties of Determinant

Definition

The determinant of a lower triangular matrix

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is the product of the diagonal elements

$$a_{11} \times a_{22} \times a_{33} \times \cdots \times a_{nn}$$

The same result applies to the determinant of an upper triangular matrix and a diagonal Matrix

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Expanding by the second column, we have

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or

$$|C| = a_{11} \times a_{22} \times a_{33} = 1 \times 1 \times -9 = -9 \qquad (22)$$

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A matrix and its transpose have equal determinants; that is

$$|A| = |A^T|$$

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If

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then

$$C^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & -9 \end{bmatrix} \tag{23}$$

Thus

$$|C^T| = a_{11} \times a_{22} \times a_{33} = 1 \times 1 \times -9 = -9 \tag{24}$$

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Expanding by the second column, we have

$$\begin{aligned} |C| &= -0[2(-9) - 1(5)] + 0[1(-9) - 1(3)] - 0[1(5) - 3(2)] && (25) \\ &= 0 + 0 + 0 && (26) \\ &= 0 && (27) \end{aligned}$$

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If $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $c = 10$ then $cA = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$ and such

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$$c^n \det(A) = 10^4(1 \times 1 \times 1 \times 1) = 10^4 \tag{30}$$

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$$\det(AB) = 10 \times 10 \times 10 \times 10 = 10^4 \tag{32}$$

and

$$\det(A)\det(B) = (1 \times 1 \times 1 \times 1)(10 \times 10 \times 10 \times 10) = 10^4 \tag{33}$$

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If two rows are added, with all other rows remaining the same, the determinants are added.

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$$\begin{vmatrix} 2 & 3 & 4 \\ -1 & -2 & -3 \\ -4 & -3 & -4 \end{vmatrix} = 2 \tag{34}$$

$$\begin{vmatrix} 5 & 6 & 7 \\ -1 & -2 & -3 \\ -4 & -3 & -4 \end{vmatrix} = 8 \tag{35}$$

$$\begin{vmatrix} 2+5 & 3+6 & 4+7 \\ -1 & -2 & -3 \\ -4 & -3 & -4 \end{vmatrix} = \begin{bmatrix} 7 & 9 & 11 \\ -1 & -2 & -3 \\ -4 & -3 & -4 \end{bmatrix} = 10 \tag{36}$$

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Scalar Multiplication

If a row of A is multiplied by a scalar t, then the determinant of the modified matrix is $t\det A$.

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$$\begin{vmatrix} 1 & 4 & 0 \\ 2 & 5 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 4 \tag{37}$$

Then multiplying row 2 by 7, we have

$$\begin{vmatrix} 1 & 4 & 0 \\ 14 & 35 & 7 \\ 1 & 0 & 0 \end{vmatrix} = 28 = 4(7) \tag{38}$$

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$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 5 & 1 \\ 1 & 4 & 0 \end{vmatrix} = -4 \tag{39}$$

Interchanging rows 2 and 3

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 5 & 1 \end{vmatrix} = 4 \tag{40}$$

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$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ 2 & 5 & 1 \end{vmatrix} \overrightarrow{NR_3 = R_3 - 8R_2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 4 & 0 \\ -6 & -27 & 1 \end{vmatrix} = 4 \tag{41}$$

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When two rows of a matrix are equal, the determinant is zero.

$$\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 8 \\ 1 & 0 & 1 \end{vmatrix} = 0 \tag{42}$$

Theorem

1. *A matrix* A *is nonsingular if and only if* $\det A \neq 0$
2. A *is singular if and only if* $\det A = 0$
3. *The homogeneous system* $Ax = 0$ *has a nontrivial solution if and only if* $\det A = 0$.

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3. *The homogeneous system $Ax = 0$ has a nontrivial solution if and only if* $\det A = 0$.

Example

Find numbers a for which the following homogeneous system has a nontrivial solution and solve the system for these values of a:

$$\begin{aligned} x - 2y + 3z &= 0, \\ ax + 3y + 2z &= 0, \\ 6x + y + az &= 0 \end{aligned}$$

A coefficient matrix say $D = \begin{bmatrix} 1 & -2 & 3 \\ a & 3 & 2 \\ 6 & 1 & a \end{bmatrix}$

Then

$$|D| = (3a-2) + 2(a^2-12) + 3(a-18) \tag{43}$$
$$= 2a^2 + 6a - 80 \tag{44}$$

This

$$|D| = 0 \implies 2a^2 + 6a - 80 = 0 \implies a = -8 \text{ or } a = 5$$

These values of a are the only values for which the given homogeneous system has a nontrivial solution.

1. When $a = -8$ we obtain $D = \begin{bmatrix} 1 & -2 & 3 \\ -8 & 3 & 2 \\ 6 & 1 & -8 \end{bmatrix}$
2. ERO will lead to the upper triangular (that is the augmented matrix)

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{45}$$

3. Solving this system gives the nontrivial solution

$$x = z \text{ and } y = 2z$$

Exercise

The case of $a = 5$ is left as an exercise.

Outline of Presentation

Cramer's Rule

Definition (Cramer's Rule)

Let A be an $n \times n$ invertible matrix, and let b be a column vector with n components. Let A_i be the matrix obtained by replacing the i_{th} column of A with b.

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix}$ is the unique solution to the linear system $Ax = b$, then

$$x_i = \frac{\det(A_i)}{\det(A)}; \qquad i = 1, 2, \cdots, n \tag{46}$$

Example

Use Cramer's rule to solve the linear system.

$$2x + 3y = 2 \tag{47}$$

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$$y = \frac{\begin{vmatrix} 2 & 2 \\ -5 & 3 \end{vmatrix}}{29} = \frac{6 - (-10)}{29} = \frac{16}{29} \tag{51}$$

Example

Solve the linear system

$$
\begin{aligned}
2x + 3y - z &= 2 && (52)\\
3x - 2y + z &= -1 && (53)\\
-5x - 4y + 2z &= 3 && (54)\\
& && (55)
\end{aligned}
$$

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$$3x - 2y + z = -1 \quad (53)$$
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$$(55)$$

The determinant of the coefficient matrix is given by

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 1 \\ -5 & -4 & 2 \end{vmatrix} = -11 \quad (56)$$

By Cramer's rule the solution to the system is

$$x = -\frac{1}{11}\begin{vmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ 3 & -4 & 2 \end{vmatrix} = -\frac{5}{11} \tag{57}$$

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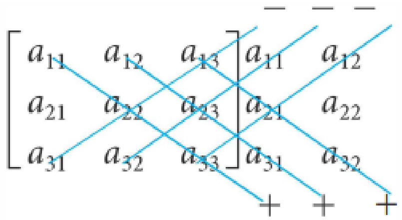
$$z = -\frac{1}{11}\begin{vmatrix} 2 & 3 & 2 \\ 3 & -2 & -1 \\ -5 & -4 & 3 \end{vmatrix} = -\frac{76}{11} \tag{59}$$

Alternative Method for Solving Determinant

Another method for calculating the determinant of a 3×3 matrix A is follows

1. Copy the first two columns of A to the right of the matrix
2. Take the products of the elements on the six diagonals shown below.
3. Attach plus signs to the products from the downward-sloping diagonals
4. Attach minus signs to the products from the upward-sloping diagonals.



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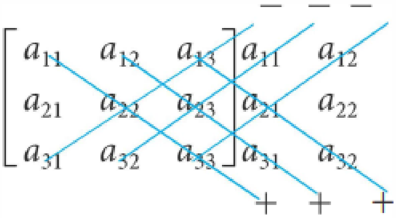

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

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Calculate the determinant of the matrix $A = \begin{bmatrix} 5 & -3 & 2 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{bmatrix}$

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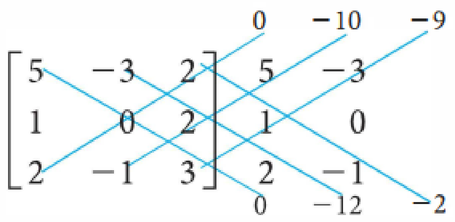
We adjoin to A its first two columns and compute the six indicated products:



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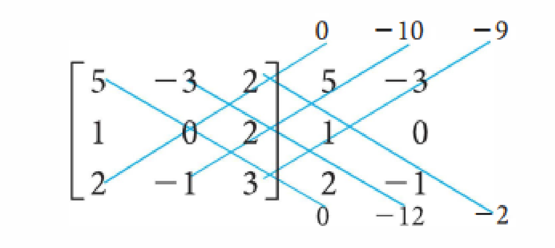

Adding the three products at the bottom and subtracting the three products at the top gives

$$\det(A) = 0 + (-12) + (-2) - 0 - (-10) - (-9) = 5 \tag{60}$$

Exercises

1. Given the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & 4 \\ 5 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -3 & -3 & -1 & -1 \\ -1 & -1 & -3 & 2 \\ -1 & -2 & 2 & 1 \end{bmatrix}$ Compute
   a. The determinant
   b. The cofactor matrix
   c. The inverse
2. Show that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = (1-r)^3$$

END OF LECTURE

THANK YOU