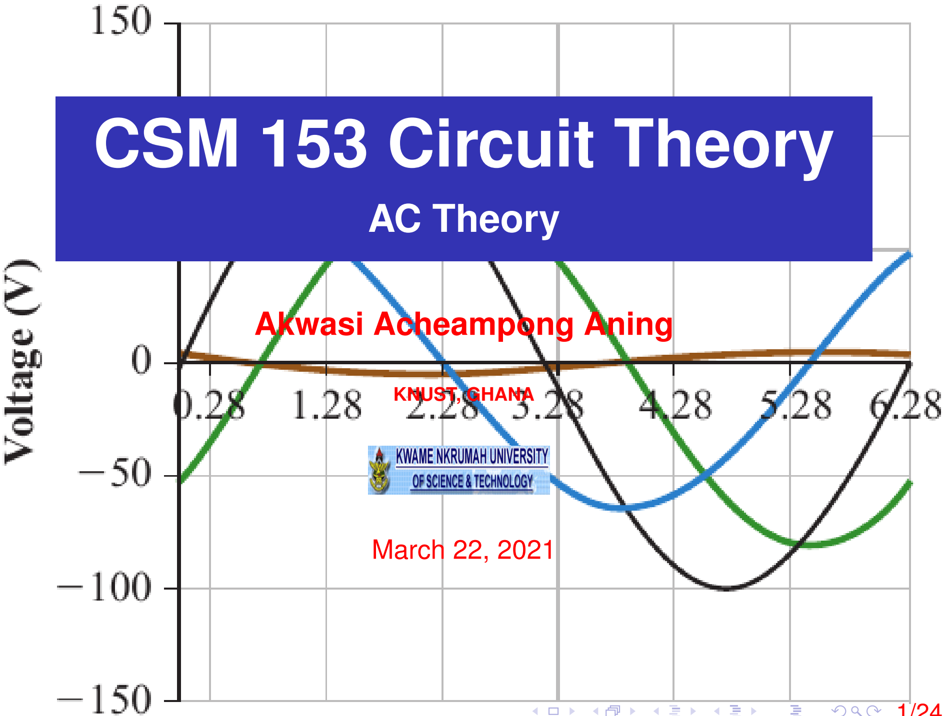

CSM 153 Circuit Theory

AC Theory

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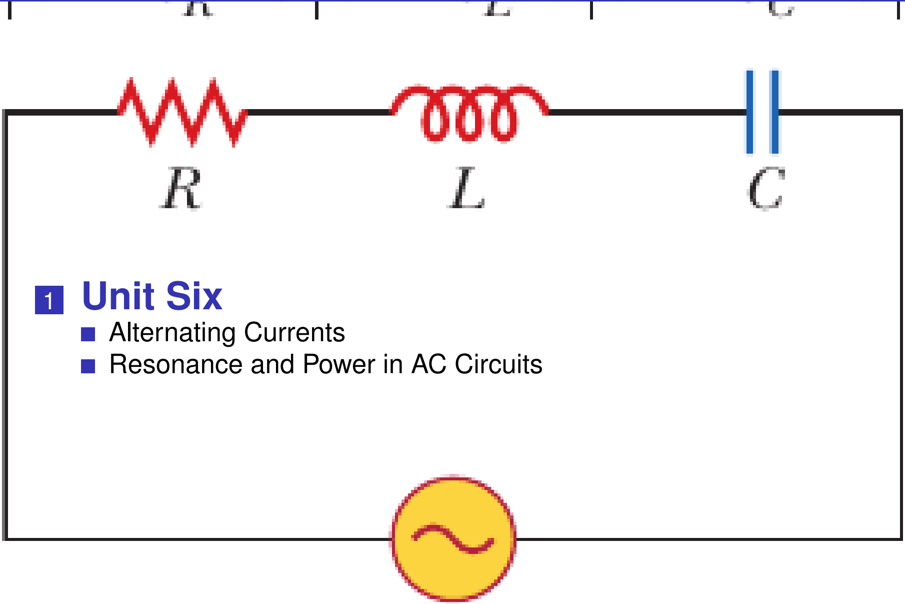
Outline I



1. **Unit Six**
   - Alternating Currents
   - Resonance and Power in AC Circuits

UNIT SIX
A. C. THEORY

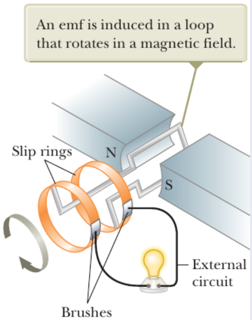
- Alternating Currents
- Resonance and Power in AC Circuits

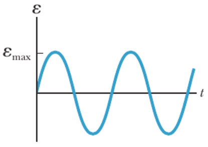
Alternating Currents

Alternating Currents

- Electrical energy or power is usually supplied to homes, offices and industries by means of oscillating emfs and currents
- The current is said to be an alternating current (ac)
- The alternating or oscillating emfs and currents vary in sinusoidal form with respect to time
- The fundamental advantage of ac is that, as the voltage or current changes (alternates), so does the magnetic field surrounding the wire
- This makes the application of Faraday's law of induction possible
- That is, we can step up (increase) or step down (decrease) the magnitude of the alternating voltage using a transformer
- In addition, ac can be readily adapted to rotating machinery such as electric generators (dynamos) and motors than direct current (dc)
- In North America, the frequency of AC oscillation is 60 Hz; in the European Union and many other countries including Ghana, the AC frequency is 50 Hz.

Alternating Currents





- The AC generator consists of a loop of wire rotated by some external means in a magnetic field

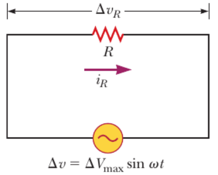
- The ends of the loop are connected to two slip rings that rotate with the loop. Connections from these slip rings, which act as output terminals of the generator, to the external circuit are made by stationary metallic brushes in contact with the slip rings
- In commercial power plants, the energy required to rotate the loop can be derived from a variety of sources. For example, in a hydroelectric plant, falling water directed against the blades of a turbine produces the rotary motion
- As the conducting coil or loop rotates in the external magnetic field, induced sinusoidal emf ε in the loop is given by

$$\varepsilon = \varepsilon_{max} \sin \omega t \tag{1}$$

where ε_{max} is the amplitude of emf ε, ω is the angular frequency of the emf and ωt is the phase of the induced emf

Alternating Currents

Resistors in an AC Circuit



- For an ac source of emf ε connected to a resistor, R the current increases and decreases with the alternating potential difference (voltage) according Ohm's law. From Kirchoff's law

$$\Delta v_R = i_R R = \Delta V_{max} \sin \omega t \quad (2)$$

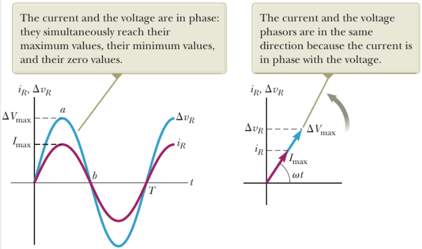
- If we rearrange eqn. 2

$$i_R = \frac{\Delta v}{R} = \frac{\Delta V_{max}}{R} \sin \omega t = I_{max} \sin \omega t \quad (3)$$

where I_{max} is the maximum current

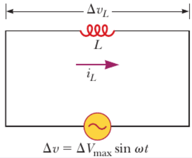
$$I_{max} = \frac{\Delta v}{R} = \frac{\Delta V_{max}}{R} \quad (4)$$

Resistors in an AC Circuit



- The current and voltage are in step with each other because they both vary as $\sin \omega t$ and reach their maximum values at the same time as shown in figure above. They are said to be **in phase**
- This means their corresponding maxima and minima occur at the same times
- A phasor is a vector whose length is proportional to the maximum value of the variable it represents (ΔV_{max} for voltage and I_{max} for current in figure above).

Inductors in an AC Circuit



- From Kirchoff's loop rule

$$\Delta v = \Delta v_L = L\frac{di_L}{dt} = \Delta V_{max} \sin \omega t \quad (5)$$

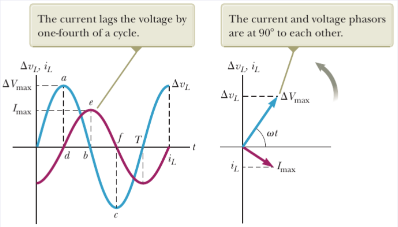
$$\therefore di_L = \frac{\Delta V_{max}}{L} \sin \omega t \quad (6)$$

$$\therefore i_L = \frac{\Delta V_{max}}{L} \int \sin \omega t dt = -\frac{\Delta V_{max}}{\omega L} \cos \omega t \quad (7)$$

Using the trigonometric identity $\cos \omega t = -\sin(\omega t - \frac{\pi}{2})$, we can express eqn 7

$$\therefore i_L = \frac{\Delta V_{max}}{\omega L}\left(\sin \omega t - \frac{\pi}{2}\right) \quad (8)$$

Inductors in an AC Circuit



- Comparing equations 5 and eqn 8 shows that the instantaneous current i_L in the inductor and the instantaneous voltage Δv_L across the inductor are out of phase by $\frac{\pi}{2}$ rad = 90°as shown in figure above
- Equation 7 shows that the current in an inductive circuit reaches its maximum value when $\cos \omega t = \pm 1$

$$I_{max} = \frac{\Delta V_{max}}{\omega L} \quad (9)$$

Inductors in an AC Circuit

- ωL has the same units as resistance and is related to current and voltage in the same way as resistance and we define ωL as the inductive reactance X_L

$$X_L \equiv \omega L = 2\pi f L \tag{10}$$

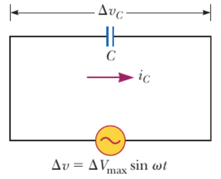
where f is the linear frequency
We can write Equation 9 as

$$I_{max} = \frac{\Delta V_{max}}{X_L} \tag{11}$$

- The instantaneous voltage across the inductor is

$$\Delta v_L = -L\frac{di_L}{dt} = -\Delta V_{max}\sin\omega t = -I_{max}X_L\sin\omega t \tag{12}$$

Capacitors in an AC Circuit



- From Kirchoff's loop rule

$$\Delta v = \frac{q}{C} \tag{13}$$

- Substituting $\Delta V_{max}\sin\omega t$ and rearranging gives

$$q = C\Delta V_{max}\sin\omega t \tag{14}$$

- Differentiating equation 14 with respect to time gives the instantaneous current in the circuit:

$$i_C = \frac{dq}{dt} = \omega C\Delta V_{max}\cos\omega t \tag{15}$$

Capacitors in an AC Circuit

- Using the trigonometric identity $\cos\omega t = \sin(\omega t + \frac{\pi}{2})$, we can express eqn 15 in the form

$$\therefore\, i_C = \omega C \Delta V_{max}\left(\sin\omega t + \frac{\pi}{2}\right) \qquad (16)$$

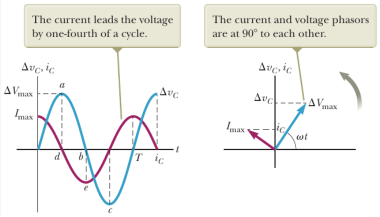
- Comparing equations 5 and eqn 16 shows that the instantaneous current i_L in the capacitor and the instantaneous voltage Δv_L across the capacitor are out of phase by $\frac{\pi}{2}$ rad = 90°as shown in figure on the right
- Equation 15 shows that the current in the circuit reaches its maximum value when $\cos\omega t = \pm 1$

$$I_{max} = \omega C \Delta V_{max} \qquad (17)$$

- The capacitive reactance X_C

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} \qquad (18)$$

Capacitors in an AC Circuit



- We can now write equation 17 as

$$I_{max} = \frac{\Delta V_{max}}{X_C} \qquad (19)$$

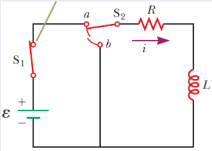
- The instantaneous voltage across the capacitor is

$$\Delta v_C = \Delta V_{max}\sin\omega t = I_{max} X_C \sin\omega t \qquad (20)$$

- Equations 17 and 19 indicate that as the frequency of the voltage source increases, the capacitive reactance decreases and the maximum current therefore increases

Alternating Currents

RL Circuit



- This circuit is an RL circuit because the elements connected to the battery are a resistor and an inductor
- The inductance of an inductor results in a back emf and as a result an inductor in a circuit opposes changes in the current in that circuit
- If the battery voltage in the circuit is increased so that the current rises, the inductor opposes this change and the rise is not instantaneous. If the battery voltage is decreased, the inductor causes a slow drop in the current rather than an immediate drop

RL Circuit

- When switch S_1 is open, there is no current anywhere in the circuit.
- When switch S_1 is closed, the current increases and an emf that opposes the increasing current is induced in the inductor.
- From Kirchhoff's loop rule,

$$\varepsilon - iR - L\frac{di}{dt} = 0 \tag{21}$$

  where iR is the voltage drop across the resistor and $L\frac{di}{dt}$ is the voltage drop across the inductor
- The solution to equation 21 is

$$i = \frac{\varepsilon}{R}(1 - e^{-Rt\backslash L}) \tag{22}$$

- This expression shows how the inductor affects the current.

Alternating Currents

RL Circuit

- Equation 22 shows that the current does not increase instantly to its final equilibrium value when the switch is closed, but instead increases according to an exponential function
- We can also write equation 22

$$i = \frac{\varepsilon}{R}(1 - e^{-t\backslash\tau}) \tag{23}$$

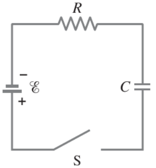
where the constant τ is the time constant of the RL circuit and τ is

$$\tau = \frac{L}{R} \tag{24}$$

and is τ the time constant of the circuit

- The larger the self-inductance L or the smaller the resistance R, the longer it takes for the current to reach any specified fraction of its final current

RC Circuit



- Capacitors and resistors are often found together in a circuit.
- They are used to control the speed of a car's windshield wipers and the timing of traffic lights; they are used in camera flashes, in heart pacemakers, and in many other electronic devices
- When the switch S is closed, current immediately begins to flow through the circuit
- Electrons will flow out from the negative terminal of the battery, through the resistor R, and accumulate on the upper plate of the capacitor

Alternating Currents

RC Circuit

- As charge accumulates on the capacitor, the potential difference across it increases $V_C = Q\backslash C$, and the current is reduced until eventually the voltage across the capacitor equals the emf of the battery, ε
- There is then no further current flow, and no potential difference across the resistor.
- From Kirchhoff's loop rule,

$$\varepsilon - iR - Q\backslash C = 0 \qquad (25)$$

where iR is the voltage drop across the resistor and $Q\backslash C$ is the voltage drop across the capacitor

- Substitute $i = dq\backslash dt$ equation 25 and divide by R, we get

$$dq\backslash dt = \varepsilon\backslash R - q\backslash RC \qquad (26)$$

RC Circuit

- The solution to equation 26 is

$$q(t) = C\varepsilon(1 - e^{-t\backslash RC}) = Q_{max}(1 - e^{-t\backslash RC}) \qquad (27)$$

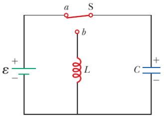
- This expression shows how the capacitor affects the current.
- The quantity RC, which appears in the exponents of equation 27 is called the time constant τ of the circuit:

$$\tau = RC \qquad (28)$$

- If the resistance is much smaller, the time constant is much smaller and the capacitor becomes charged much more quickly. This makes sense, because a lower resistance will retard the flow of charge less.
- All circuits contain some resistance (if only in the connecting wires), so a capacitor can never be charged instantaneously when connected to a battery

Alternating Currents

LC Circuit



- The charging process is essentially instantaneous because we assume no resistance in the circuit
- The absence of resistance means that no energy in the circuit is transformed to internal energy. We also assume an idealized situation in which energy is not radiated away from the circuit
- With the switch at position a, the inductor is not in the circuit, so no energy is stored in the inductor
- The capacitor begins to discharge with b closed; the rate at which charges leave the capacitor plates (which is also the rate at which the charge on the capacitor changes) is equal to the current in the circuit.

LC Circuit

- The energy stored in the electric field of the capacitor decreases. Some energy is now stored in the magnetic field of the inductor because of the current in the circuit
- Therefore, energy is transferred from the electric field of the capacitor to the magnetic field of the inductor. When the capacitor is fully discharged, it stores no energy. At this time, the current reaches its maximum value and all the energy in the circuit is stored in the inductor
- The current continues in the same direction, decreasing in magnitude, with the capacitor eventually becoming fully charged again but with the polarity of its plates now opposite the initial polarity
- At that point, the current stops and there is no energy stored in the inductor
- The energy continues to transfer back and forth between inductor and capacitor

Alternating Currents

LC Circuit

- Telecommunication (radios, televisions, cell phones) - is based on electromagnetic signals that oscillate at a well-defined frequency.
- Let's apply Kirchhoff's law around the loop containing the capacitor and the inductor

$$q \backslash C + Ldi \backslash dt = 0 \tag{29}$$

where $Ldi \backslash dt$ is the voltage drop across the inductor and $q \backslash C$ is the voltage drop across the capacitor

- Substitute $i = dq \backslash dt$ in equation 29 and divide by R, we get

$$q \backslash C + Ld^2 q \backslash dt^2 = 0 \tag{30}$$

$$\frac{d^2 q}{dt^2} = -\frac{q}{LC} \tag{31}$$

- The solution of this equation is

$$q = Q_{max} \cos(\omega t + \phi) \tag{32}$$

LC Circuit

- where Q_{max} is the maximum charge on the capacitor and the angular frequency ω is the square root of the coefficient of q in equation 31

$$\omega = \frac{1}{\sqrt{LC}} \tag{33}$$

- The charge on the capacitor undergoes a simple harmonic oscillation, alternating between polarities
- An LC circuit is an electric oscillator, oscillating at frequency $f = \omega \backslash 2\pi$.
- Note that the angular frequency of the oscillations depends solely on the inductance and capacitance of the circuit
- Equation 33 gives the natural frequency of oscillation of the LC circuit.
- We also have two kinds of energy in the LC circuit - electric energy and magnetic energy

Alternating Currents

LC Circuit

- The electric energy stored in the capacitor is

$$U_e = \frac{1}{2} Q V_C = \frac{1}{2} \frac{Q^2}{C} \tag{34}$$

Substituting $Q_{max} \cos \omega t$ for Q

$$U_e = \frac{1}{2C} Q_{max}^2 \cos^2 \omega t \tag{35}$$

- The magnetic energy stored in the inductor is

$$U_m = \frac{1}{2} L I^2 \tag{36}$$

Substituting $I = -\omega Q_{max} \sin \omega t$

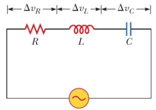
$$U_m = \frac{1}{2} L \omega^2 Q_{max}^2 \sin^2 \omega t = \frac{1}{2C} Q_{max}^2 \sin^2 \omega t \tag{37}$$

- The sum $U_e + U_m$ is a constant and is equal to the total energy $\frac{Q_{max}^2}{2C}$, or $\frac{1}{2} L I_{max}^2$

Thus,

$$\frac{1}{2} \frac{Q_{max}^2}{C} = \frac{1}{2} L I_{max}^2 \tag{38}$$

RLC Circuit



- The instantaneous applied voltage is

$$\Delta v = \Delta V_{max} \sin \omega t \tag{39}$$

- The current in the circuit is given by

$$i = I_{max} \sin \omega t \tag{40}$$

- The current everywhere in the circuit must be the same at any instant because the circuit elements are in series
- That is, the current at all points in a series AC circuit has the same amplitude and phase
- The voltage across each element has a different amplitude and phase

Alternating Currents

RLC Circuit

- The voltage across the resistor is in phase with the current, the voltage across the inductor leads the current by $90°$, and the voltage across the capacitor lags behind the current by $90°$

$$\Delta v_R = I_{max} R \sin \omega t = \Delta V_R \sin \omega t \quad (41)$$

$$\Delta v_L = I_{max} X_L sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t \quad (42)$$

$$\Delta v_C = I_{max} X_C sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_C \cos \omega t \quad (43)$$

- The sum of these three voltages must equal the instantaneous voltage Δv from the ac sources
- The three voltages cannot be added directly because of their different phase relationships with the current

RLC Circuit

- The maximum applied voltage is

$$V_{max} = I_{max} \sqrt{R^2 + (X_L - X_C)^2} \quad (44)$$

- Therefore, we can express the maximum current as

$$I_{max} = \frac{V_{max}}{\sqrt{R^2 + (X_L - X_C)^2}} \quad (45)$$

- The denominator of the fraction plays the role of resistance and is called the impedance Z of the circuit:

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \quad (46)$$

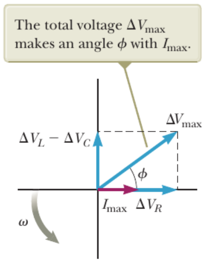
- Therefore equation 45 can be written in the form

$$I_{max} = \frac{V_{max}}{Z} \quad (47)$$

method

Alternating Currents

RLC Circuit



- The inductance and capacitance phasors are added together and then added vectorially to the resistance phasor
- From the right triangle in the phasor diagram in Figure above, the phase angle ϕ between the current and the voltage is found as follows:

$$\phi = \tan^{-1}\left(\frac{\Delta V_L - \Delta V_C}{\Delta V_R}\right) = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \tag{48}$$

RLC Circuit

- When $X_L > X_C$ (which occurs at high frequencies), the phase angle is positive, signifying that the current lags the applied voltage. Then the circuit is more inductive than capacitive
- When $X_L < X_C$ the phase angle is negative, signifying that the current leads the applied voltage, and the circuit is more capacitive than inductive.
- When $X_L = X_C$, the phase angle is zero and the circuit is purely resistive

Resonance and Power in AC Circuits

Resonance and Power in AC Circuits

Resonance

- **Resonance** is the state of a system in which an abnormally large vibration is produced in response to an external stimulus occurring when the frequency of the stimulus is the same, or nearly the same, as the natural vibration frequency of the system
- The rms current in an LRC series circuit is given by

$$I_{rms} = \frac{V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}} \quad (49)$$

- Because the reactance of inductors and capacitors depends on the frequency f of the source, the current in an LRC circuit depends on frequency. From equation 49 we see that the current will be maximum at a frequency that satisfies the condition

$X_L - X_C = 0$

Resonance

- $$2\pi f L - \frac{1}{2\pi f C} = 0 \quad (50)$$
- We solve this for f, and call the solution f_0

$$f_0 = \frac{1}{2\pi}\frac{1}{\sqrt{LC}} \quad (51)$$

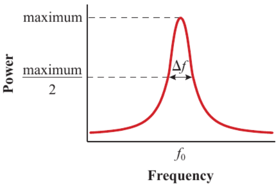
- When $f = f_0$, the circuit is in resonance, and f_0 is the resonant frequency of the circuit
- At this frequency, $X_C = X_L$ and the impedance is purely resistive.
- This frequency matches the natural frequency of oscillation of an LC circuit
- The rms current in a series RLC circuit has its maximum value when the frequency of the applied voltage matches the natural oscillator frequency, which depends only on L and C
- The current is in phase with the applied voltage at the resonance frequency

Resonance and Power in AC Circuits

Resonance

- If a capacitor is discharged through an inductor, the charge and the voltage on the capacitor oscillate with angular frequency $\omega = \dfrac{1}{\sqrt{LC}}$
- The current in the inductor oscillates with the same frequency, but it is out of phase with the charge by 90°.
- The energy oscillates between electric energy in the capacitor and magnetic energy in the inductor
- The oscillations are damped in the RLC circuit because energy is dissipated in the resistor

Resonance



- The quality factor (**Q factor**) is the ratio of a resonator's centre frequency to its bandwidth when subject to an oscillating driving force

$$Q = \frac{f_0}{\Delta f} \quad (52)$$

where f_0 is the centre frequency of the peak and Δf is the half-power bandwidth of the peak
- The half-power bandwidth is the difference between the two frequencies at which the power in the circuit is equal to half the power at the centre frequency (see figure above)

Resonance and Power in AC Circuits

Power in AC Circuit

- A resistor dissipates energy at the rate

$$P_R = i_R v_R = i_R^2 R \tag{53}$$

Let $i_R = I_R \cos\omega t$

$$P_R = i_R^2 R = I_R^2 R cos^2 \omega t \tag{54}$$

- The average power P is the total energy dissipated per second

Using the identity $\cos^2(x) = \frac{1}{2}(1 + \cos 2x)$

we can write $P_R = I_R^2 R cos^2 \omega t =$
$I_R^2 R \left[\frac{1}{2}(1 + \cos 2\omega t)\right] = \frac{1}{2} I_R^2 R + \frac{1}{2} I_R^2 R \cos 2\omega t$

- The $\cos 2\omega t$ term oscillates positive and negative twice during each cycle of the emf. Its average, over one cycle, is zero.

$$P_R = \frac{1}{2} I_R^2 R \tag{55}$$

This is the average power loss in a resistor

Power in AC Circuit

- From equation 55

$$P_R = \left(\frac{I_R}{\sqrt{2}}\right)^2 R = I_{rms}^2 R \tag{56}$$

$$\therefore I_{rms} = \frac{I_R}{\sqrt{2}} \tag{57}$$

This is the root-mean-square current. For a sinusoidal oscillation, the rms value turns out to be the peak value divided by $\sqrt{2}$

- Similarly, we can define the root-mean-square voltage as

$$V_{rms} = \frac{V_R}{\sqrt{2}} \tag{58}$$

- The resistor's average power loss in terms of the rms quantities is

$$P_R = I_{rms} V_{rms} \tag{59}$$

Resonance and Power in AC Circuits

Power in AC Circuit

- Energy is transferred into the capacitor when it is being charged, and the energy is stored as potential energy in the capacitor's electric field
- The energy is given back to the circuit when the capacitor discharges, as a result a capacitor's average power loss is zero: $P_C = 0$
- An inductor alternately stores energy in the magnetic field, as the current is increasing and then transfers energy back to the circuit as the current decreases
- The instantaneous power oscillates between positive and negative, but an inductor's average power loss is zero: $P_C = 0$

Power in AC Circuit

- **NOTE**: We're assuming ideal capacitors and inductors. Real capacitors and inductors inevitably have a small amount of resistance and dissipate a small amount of energy. However, their energy dissipation is negligible compared to that of the resistors in most practical circuits
- In an RLC circuit, energy is supplied by the emf and dissipated by the resistor. The RLC circuit unlike a purely resistive circuit, the current is not in phase with the potential difference of the emf, and the average power loss is

$$P_{av} = I_{rms} V_{rms} \cos\phi \tag{60}$$

The factor $\cos\phi$ is referred to as the power factor of the circuit

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method

THANK YOU FOR YOUR ATTENTION