PARTIAL FRACTIONS

A. DENOMINATOR WITH ONLY LINEAR FACTORS

$$Eg.\frac{x+4}{(x+1)(x-1)(2x-3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x-3}$$

$$\frac{2x^2+5x-6}{(x-3)(x+5)(2x-1)} = \frac{A}{x-3} + \frac{B}{x+5} + \frac{C}{2x-1}$$

WORKED EXAMPLES

1. $$\frac{6}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

Multiplying through by $(x+3)(x-3)$

$$6 = A(x-3) + B(x+3)$$

To get B, we make $A = 0$ by putting $x = 3$

$\Rightarrow 6 = 6B \Rightarrow B = 1$

To get A we make $B = 0$, by putting $x = -3$

$\Rightarrow 6 = -6A \Rightarrow A = -1$

Thus

$$\frac{6}{(x+3)(x-3)} = -\frac{1}{x+3} + \frac{1}{x-3}$$

$$\frac{6}{(x+3)(x-3)} = \frac{1}{x-3} - \frac{1}{x+3}$$

B. DENOMINATOR WITH LINEAR FACTORS REPEATED

EXAMPLES

$$(i) \quad \frac{4}{(x-1)^2(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-3}$$

$$(ii) \quad \frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$(iii) \quad \frac{x+1}{(x+3)^2} = \frac{A}{x+3} + \frac{B}{(x+3)^2}$$

SOLUTIONS

(i) $$\frac{4}{(x-1)^2(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-3}$$

$$4 = A(x-1)(x-3) + B(x-3) + C(x-1)^2$$ When

$x = 1, \Rightarrow -2B = 4 \Rightarrow B = -2$

When $x = 3, \Rightarrow 4C = 4 \Rightarrow C = 1$

Comparing coefficient of x^2;

$$(A + C) = 0$$

$$A + 1 = 0, \Rightarrow A = -1$$

$$\Rightarrow \frac{4}{(x-1)^2(x-3)} = \frac{-1}{x-1} + \frac{-2}{(x-1)^2} + \frac{1}{x-3}$$

$$\Rightarrow \frac{4}{(x-1)^2(x-3)} = \frac{1}{x-3} - \frac{1}{x-1} - \frac{2}{(x-1)^2}$$

$$(ii) \qquad \frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

When $x = 1, \Rightarrow 3B = 1 \Rightarrow B = \dfrac{1}{3}$

When $x = -2, \Rightarrow 9C = 1 \Rightarrow C = \dfrac{1}{9}$

Comparing coefficient of x^2 yields

$A + C = 0 \Rightarrow A + \dfrac{1}{9} = 0$

$\Rightarrow A = -\dfrac{1}{9}$

$$\Rightarrow \frac{1}{(x-1)^2(x+2)} = \frac{\left(-\frac{1}{9}\right)}{x-1} + \frac{\left(\frac{1}{3}\right)}{(x-1)^2} + \frac{\left(\frac{1}{9}\right)}{x-2}$$

$$= \frac{1}{9(x+2)} + \frac{1}{3(x-1)^2} - \frac{1}{9(x-1)}$$

C. DENOMINATOR WITH QUADRATIC FACTORS

EXAMPLES

$$(i)\quad \frac{16x^2}{(x^2-16)(x^2+4)} = \frac{16x^2}{(x+4)(x-4)(x^2+4)} = \frac{A}{x+4} + \frac{B}{x-4} + \frac{Cx+D}{x^2+4}$$

$$(ii)\quad \frac{2x^2+3x}{(x^2+4)^2} = \frac{Ax+B}{x^2+4} + \frac{Cx+D}{(x^2+4)^2}$$

$$(iii)\quad \frac{3x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$(iv)\quad \frac{4}{(x+1)(2x^2+x+3)} = \frac{A}{x+1} + \frac{Bx+C}{2x^2+x+3}$$

SOLUTIONS

1. $\dfrac{5x^2-6x+10}{(x+2)(x^2-3x+4)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2-3x+4}$

Multiply through by $(x+2)(x^2-3x+4)$;

$5x^2-6x+10 = A(x^2-3x+4) + (Bx+C)(x+2)$

When $x=-2$; $20+12+10 = A(4+6+4)$

$\Rightarrow 14A = 42 \Rightarrow A = 3$

Comparing coefficient of x^2 and x from

$$\begin{aligned}
5x^2-6x+10 &= A(x^2-3x+4) + Bx^2 + 2Bx + Cx + 2C \\
x^2 : 5 &= A + B \Rightarrow 5 = 3 + B \Rightarrow B = 2 \\
x : -6 &= -3A + 2B + C \\
-x &= -9 + 4 + C \\
C &= -6 + 9 - 4 \\
&= -1
\end{aligned}$$

$$\Rightarrow \frac{5x^2-6x+10}{(x+2)(x^2-3x+4)} = \frac{3}{x+2} + \frac{2x-1}{x^2-3x+4}$$

2. $\dfrac{4}{(x+1)(2x^2+x+3)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{2x^2+x+3}$

Multiply by $(x+1)(2x^2+x+3)$

$4 = A(2x^2+x+3) + (Bx+C)(x+1)$

When $x = -1, 4A = 4 \Rightarrow A = 1$

$4 = 2Ax^2 + Ax + 3A + Bx^2 + Bx + Cx + C$

$4 = (2A+B)x^2 + (A+B+C)x + 3A + C$

Comparing coefficients

$$x^2 : 2A + B = 0 \Rightarrow 2 + B = 0 \Rightarrow B = -2$$
$$x : A + B + C = 0 \Rightarrow -2 + C = 0 \Rightarrow C = 1$$
$$3A + C = 4$$
$$Thus$$
$$\frac{4}{(x+1)(2x^2+x+3)} = \frac{1}{x+1} + \frac{-2x+1}{2x^2+x+3}$$
$$= \frac{1}{x+1} + \frac{1-2x}{2x^2+x+3}$$

D. DENOMINATOR'S DEGREE EQUAL OR GREATER THAN NUMERATOR'S DEGREE

EXAMPLES

(*i*) $$\frac{2x^4+3x^3-5x^2-34x-18}{x^3+x^2-6x} = 2x+1+\frac{6x^2-28x-18}{x^3+x^2-6x}$$

(*ii*) $$\frac{x^2+3x-10}{x^2-2x-3}$$

(*iii*) $$\frac{2x^2+18x+31}{x^2+5x+6}$$

Note

To effect a partial fraction breakdown of a rational algebraic expression it is necessary for the degree of the numerator to be less than the degree of the denominator.
Thus if, in the original algebraic rational expression, the degree of the numerator is not less than the degree of the denominator then first divide out by long division.
This gives a polynomial with a rational remainder where the remainder has a degree of the numerator being less than the degree of the denominator. The remainder can then be broken down into its partial fractions.

SOLUTIONS

1. Express $\dfrac{x^2+3x-10}{x^2-2x-3}$ in a partial fractions

$$
\begin{array}{r|l}
 & \;\;1 \\
\cline{2-2}
x^2-2x-3 & \;\;x^2+3x-10 \\
 & -x^2+2x\;+3 \\
\cline{2-2}
 & \qquad\;\; 5x\;-7
\end{array}
$$

$$\frac{x^2+3x-10}{x^2-2x-3} = 1 + \frac{5x-7}{x^2-2x-3}$$

$$\frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$$

$5x-7 = A(x-3) + B(x+1)$

When $x = 3; 4B = 8 \Rightarrow B = 2$

When $x = -1; -4A = -12 \Rightarrow A = 3$

$$\Rightarrow \frac{5x-7}{x^2-2x-3} = \frac{3}{x+1} + \frac{2}{x-3}$$

$$\frac{x^2+3x-10}{x^2-2x-3} = 1 + \frac{3}{x+1} + \frac{2}{x-3}$$