KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY

COLLEGE OF SCIENCE

DEPARTMENT OF MATHEMATICS

MATH 161

INTRODUCTORY TO PURE MATHEMATICS I

Contents

CHAPTER 1
INDICES

Given $a^n = a \times \times a \times \cdots \times a$ (n factors), n is called the index and a is called the base.
Example:

$$3^4 = 3 \times 3 \times 3 \times 3$$

where 4 is the index and 3 is the base and 3^4 is read as "3 to the power 4".
NB: index (*singular*) and indices (*plural*)

1.1 RULES OF INDICES

1. When numbers with the same base are multiplied, indices are added.

$$a^x \times a^y = a^{x+y}$$
$$\text{E.g}: \quad 2^4 \times 2^3 = 2^{4+3} = 2^7$$

2. When numbers with the same base are divided, indices are subtracted.

$$\frac{a^x}{a^y} = a^{x-y}$$
$$\text{E.g}: \quad \frac{3^6}{3^2} = 3^{6-2} = 3^4$$

3. When a number in index form is raised to another power, the indices are multiplied.

$$(a^x)^y = a^{xy}$$
$$\text{E.g}: \quad (2^4)^3 = 2^{4.3} = 2^{12} = 4096$$

4. When a number is raised to a zero index, the value is one.

$$a^0 = 1 \qquad \text{(zero index)}$$
$$\text{E.g}: \quad x^0 = 1; \quad 5^0 = 1$$

5. When a number is raised to a negative index, it can be converted to a number with positive index by inversion.

$$a^{-x} = \frac{1}{a} \qquad \text{(negative index)}$$

$$\text{E.g :} \quad 3^{-7} = \frac{1}{37}; \quad x^{-2} = \frac{1}{x^2}$$

6. When a number is raised to a fractional index, it is a root.

$$a^{\frac{x}{y}} = \left(\sqrt[y]{a}\right)^x \qquad \text{(fractional indices)}$$

$$\begin{aligned}\text{E.g :} \quad & x^{\frac{1}{m}} = \sqrt[m]{x}; \quad x^{\frac{1}{3}} = \sqrt[3]{x}\\ & x^{\frac{1}{2}} = \sqrt{x}; \quad x^{\frac{3}{5}} = (\sqrt[5]{x})^3\\ & 2^{\frac{1}{x}} = \sqrt[x]{2}\end{aligned}$$

1.2 Exercise

(a) (i) $\left(\frac{8x^3}{27y^6}\right)^{\frac{2}{3}} = \left(\frac{2^3x^3}{3^3y^6}\right)^{\frac{2}{3}} = \left(\frac{2x}{3y^2}\right)^2 = \frac{4x^2}{9y^4}$

(ii) $\left(\frac{81}{256}\right)^{3/2} = \left(\frac{3^4}{4^4}\right)^{3/2} = \left(\frac{3^2}{4^2}\right)^3 = \frac{729}{4096}$

(iii) $\left(\frac{25}{49}\right)^{-1/2} = \left(\frac{49}{25}\right)^{1/2} = \left(\frac{7^2}{5^2}\right)^{1/2} = \frac{7}{5}$

(b) (i)

$$\begin{aligned}(0.125)^{-1/3} &= \left(\frac{125}{1000}\right)^{-1/3} = \left(\frac{1000}{125}\right)^{1/3}\\ &= \left(\frac{10^3}{5^3}\right)^{1/3} = \frac{10}{5} = 2\end{aligned}$$

(ii) If $3^{5x+4} = 27^{x+2}$, find x.

Solution

$$\begin{aligned}3^{5x+4} &= 27^{x+2}\\ 3^{5x+4} &= 3^{3(x+2)}\\ \implies \quad 5x+4 &= 3(x+2)\\ \implies \quad 5x+4 &= 3x+6\\ 2x &= 2\\ x &= 1\end{aligned}$$

(iii) If $3^{3x} = 81$, find x.

Solution

$$\begin{aligned} 3^x &= 81 \\ \implies \quad 3^x &= 3^4 \\ x &= 4 \end{aligned}$$

(c) (i) If $3^{4x-2} = \left(\frac{1}{81}\right)^{2x+3}$, find x.

Solution

$$\begin{aligned} 3^{4x-2} &= \left(\frac{1}{81}\right)^{2x+3} \\ 3^{4x-2} &= 3^{-4(2x+3)} \\ \implies \quad 4x-2 &= -8x+12 \\ -4x &= 14 \\ x &= -\frac{14}{4} = -\frac{7}{2} = -3^{1/2} \end{aligned}$$

(ii) If $2^{7x-2} = 64\sqrt{2}$, find x.

Solution

$$\begin{aligned} 2^{7x-2} &= 64\sqrt{2} \\ 2^{7x-2} &= 3^6.2^{\frac{1}{2}} \\ 2^{7x-2} &= 2^{6+\frac{1}{2}} = 2^{\frac{13}{2}} \\ \implies \quad 7x-2 &= \frac{13}{2} \\ 14x-4 &= 13 \\ 14x &= 17 \\ x &= \frac{17}{14} = 1\frac{3}{14} \end{aligned}$$

(d) Simplify the following

(i) $3a^{\frac{1}{2}}b^2\sqrt{9a^{-3a}b^2}$

(ii) $2\left(4a^2\right)^{1/2}\sqrt[3]{a^6}$

(iii) $\left(x^{1/2}+2\right)\left(x-2x^{1/2}-1\right)$

(iv) $\left(a^{1/2}+b^{1/2}\right)\left(2a^{1/2}-3b^{1/2}\right)$

(v) $\dfrac{x}{x^{1/2}+y^{1/2}} + \dfrac{x}{x^{1/2}-y^{1/2}}$

(vi) If $4^{x+3} = \dfrac{1}{16}$, find x

(vii) Find x if $2x^{1/3} = 5$

(viii) Find x if $\left(\dfrac{1}{27}\right)^{x-1} = 81^{2-x}$

CHAPTER 2
LOGARITHMS

- **Common logarithm** (logarithm to base 10 usually written without indicating the base), i.e. log
- **Natural logarithm** (logarithm to base e), i.e. $\log_e$

Definition:
The logarithm of any number to a given base is the power to which th base must be raised to equal the given number. If $c = \log_b a$, then c is the log of a to base b.

NB: $\log_5 25$ – log 25 to base 5 and $\log_c a$ – log a to base c.

Exercise

1. If $x = \log_2 16$, find x.

Solution

$$\begin{aligned} 2^x &= 16 \\ 2^x &= 2^4 \\ \implies x &= 4 \end{aligned}$$

2. If $4 = \log_x 81$, find x.

Solution

$$\begin{aligned} x^4 &= 81 \\ x^4 &= 3^4 \\ \implies x &= 3 \end{aligned}$$

3. If $\log_7 x = 2$, find x.

Solution

$$\begin{aligned} \log_7 x &= 2 \\ \implies \quad x &= 7^2 = 49 \end{aligned}$$

2.1 RULES OF LOGARITHMS

Since logarithms are powers, the rules that govern the manipulation of logarithms closely follow the rules of powers:

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
3. $\log_a x^n = n\log_a x$
4. $\log_a a = 1$
5. $\log_a b = \frac{1}{\log_b a}$
6. $\log_a b = \frac{\log_c b}{\log_c a}$
7. $\log_{a^2} b^2 = \log_a b$
8. $e^{\log_e a} = e^{\ln a}$
9. If $p = \log_a b$ then $a^p = b$.

2.1.1 PROOFS

1.

$$
\begin{aligned}
&\log_a xy = \log_a x + \log_a y \\
&\text{Let } \log_a x = p \quad \text{and} \quad \log_a y = q \\
&\implies \quad x = a^p \quad \text{and} \quad y = a^q \\
&\qquad \implies \quad x.y = a^p.a^q = a^{p+q} \\
&\qquad \implies \quad x.y = a^{p+q} \\
&\qquad \log_a xy = \log_a a^{p+q} \quad \text{(Taking log to base } a \text{ on both sides)} \\
&\implies \quad \log_a xy = (p+q)\log_a a \\
&\qquad\qquad = p+q \quad (\text{since } \log_a a = 1) \\
&\qquad\qquad = \log_a x + \log_a y \\
&\therefore \quad \log_a xy = \log_a x + \log_a y
\end{aligned}
$$

2.

$$
\begin{aligned}
\log_a\left(\frac{x}{y}\right) &= \log_a x - \log_a y \\
\text{Let} \quad \log_a x = p \quad \text{and} \quad \log_a y = q \\
\implies \quad a^p = x \quad \text{and} \quad a^q = y \\
\implies \quad \frac{x}{y} &= \frac{a^p}{a^q} = a^{p-q} \\
\log_a\left(\frac{x}{y}\right) &= \log_a a^{p-q} \quad \text{(Taking log to base } a \text{ on both sides)} \\
&= (p-q)\log_a a \\
&= p-q \\
&= \log_a x - \log_a y \\
\therefore \quad log_a\left(\frac{x}{y}\right) &= \log_a x - \log_a y
\end{aligned}
$$

3.

$$
\begin{aligned}
\log_a b &= \frac{\log_c b}{\log_c a} \\
\text{Let} \quad \log_a b &= p \\
\implies \quad b &= a^p \\
\log_c b &= \log_c a^p \quad \text{(taking } \log_c \text{ of both sides)} \\
\log_c b &= p\log_c a \\
\implies \quad p &= \frac{\log_c b}{\log_c a} \\
\implies \quad \log_a b &= \frac{\log_c b}{\log_c a} \qquad (1) \\
\text{Deducing that } \log_a b &= \frac{1}{\log_b a} \\
\text{From (1), set } c &= b \text{ (on the RHS)} \\
\implies \quad \log_a b &= \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}
\end{aligned}
$$

4. To prove that $\log_a b = \dfrac{1}{\log_b a}$ directly without any deduction,

$$\begin{aligned}
\text{Let} \quad \log_a b &= p \\
\implies \quad b &= a^p \\
\log_b b &= \log_b a^p \\
1 &= p \log_b a \\
\implies \quad p &= \frac{1}{\log_b a} \\
\therefore \quad \log_a b &= \frac{1}{\log_b a}
\end{aligned}$$

2.1.2 Worked Examples

1. Simplify the following:

(i.) $2\log 5 + 5\log 2 + 3\log 3 - 3\log 6$

(ii.) $\log 280 + 2\log 15 - \log 63$

Solution

(i.)

$$\begin{aligned}
2\log 5 + 5\log 2 + 3\log 3 - 3\log 6 &= \log 5^2 + \log 2^5 + \log 3^3 - \log 6^3 \\
&= \log 25 + \log 32 + \log 27 - \log 216 \\
&= \log\left[\frac{25 \times 32 \times 27}{216}\right] \\
&= \log\left[\frac{21600}{216}\right] \\
&= \log 100 \\
&= 2\log 10 = 2
\end{aligned}$$

(ii.)

$$\begin{aligned}
\log 280 + 2\log 15 - \log 63 &= \log 280 + \log 15^2 - \log 63 \\
&= \log 280 + \log 225 - \log 63 \\
&= \log\left(\frac{280 \times 225}{63}\right) \\
&= \log\frac{63000}{63} \\
&= \log 1000 \\
&= \log 10^3 \\
&= 3\log 10 = 3
\end{aligned}$$

2.

$$\frac{\log_2 64+\log_2 4-\log_2 8}{\log_2 1024}$$

Solution

$$\begin{aligned}\frac{\log_2 64+\log_2 4-\log_2 8}{\log_2 1024} &= \frac{\log_2 2^6+\log_2 2^2-\log_2 2^3}{\log_2 2^{10}}\\ &= \frac{6\log_2 2+2\log_2 2-3\log_2 2}{10\log_2 2}\\ &= \frac{5\log_2 2}{10\log_2 2}\\ &= \frac{1}{2}\end{aligned}$$

3. Express the following in terms of $\log a$, $\log b$ and $\log c$

(i.) $\log\left(\frac{a^2b}{c^3}\right)$

(ii.) $\log ab^2c^3$

(iii.) $\log\left(\frac{a^5}{b^3c^2}\right)$

Solution

(i.)

$$\begin{aligned}\log\left(\frac{a^2b}{c^3}\right) &= \log a^2+\log b-\log c^3\\ &= 2\log a+\log b-3\log c\end{aligned}$$

(ii.)

$$\begin{aligned}\log\left(ab^2c^3\right) &= \log a+\log b^2-\log c^3\\ &= \log a+2\log b+3\log c\end{aligned}$$

(iii.)

$$\begin{aligned}\log\left(\frac{a^5}{b^3c^2}\right) &= \log a^5-\log b^3-\log c^2\\ &= 5\log a-3\log b-2\log c\end{aligned}$$

4. Rewrite the following equations in index form, clear of logarithms

(a) $\log\sqrt{x}=\frac{1}{2}\log y$

(b) $\log_x 3-\log_x y=2$

(c) $\log 8 + \log x = 2\log y$

Solution

(a)

$$\begin{aligned}
\log \sqrt{x} &= \frac{1}{2}\log y \\
\log x^{1/2} &= \frac{1}{2}\log y \\
\frac{1}{2}\log x &= \frac{1}{2}\log y \\
\log x &= \log y \\
\implies \quad x &= y
\end{aligned}$$

(b)

$$\begin{aligned}
\log_x 3 - \log_x y &= 2 \\
\log_x\left(\frac{3}{y}\right) &= 2 \\
x^2 &= \frac{3}{y} \\
\implies \quad x^2 y &= 3
\end{aligned}$$

(c)

$$\begin{aligned}
\log 8 + \log x &= 2\log y \\
\log 8 + \log x &= \log y^2 \\
\log 8x &= \log y^2 \\
\implies \quad 8x &= y^2 \\
\frac{x}{y^2} &= \frac{1}{8}
\end{aligned}$$

5. If $\log_b a + \log_c a = 2\log_b a \times \log_c a$, prove that $a^2 = bc$

Solution

$$\begin{aligned}
\log_b a + \log_c a &= 2\log_b a \times \log_c a \\
\frac{\log_b a + \log_c a}{\log_c a} &= 2\log_b a \quad \text{(Dividing both sides by } \log_c a) \\
\frac{\log_b a}{\log_c a} + 1 &= \log_b a^2
\end{aligned}$$

$$
\begin{aligned}
\text{But } \log_a c = \frac{\log_b c}{\log_b a} \quad \text{and} \quad \log_b b = 1 \\
\implies \quad \log_b a . \frac{\log_b c}{\log_b a} + \log_b b = \log_b a^2 \\
\log_b c + \log_b b = \log_b a^2 \\
\log_b bc = \log_b a^2 \\
\implies \quad bc = a^2
\end{aligned}
$$

2.1.3 Exercise 1

(i) $\log_3 x + 3\log_x 3 = 4$

(ii) $\log_x 3 + \log_x 27 = 2$

(iii) $\log(x+4) + \log(x-3) = 2\log(x-3)$

Solution

(i)

$$
\begin{aligned}
\log_3 x + 3\log_x 3 = 4 \\
\log_x 3 = \frac{\log_3 3}{\log_3 x} \\
\implies \quad \log_3 x + 3\frac{\log_3 3}{\log_3 x} = 4 \\
(\log_3 x)^2 + 3 = 4\log_3 x \\
\implies \quad (\log_3 x)^2 - 4\log_3 x + 3 = 0 \\
\text{Let } m = \log_3 x \\
\implies \quad m^2 - 4m + 3 = 0 \\
(m-3)(m-1) = 0 \\
m = 3 \text{ or } m = 1 \\
\implies \quad \log_3 x = 3 \text{ or } \log_3 x = 1 \\
\implies \quad x = 3^3 \text{ or } x = 3
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\log_x 3 + \log_x 27 = 2 \\
\log_x 3(27) = 2 \\
\log_x 81 = 2 \\
x^2 = 81 \\
x = 9
\end{aligned}
$$

(iii)

$$\begin{aligned}
\log(x+4)+\log(x-3) &= 2\log(x-3)\\
\log\,(x+4)(x-3) &= \log\,(x-3)^2\\
(x+4)(x-3) &= (x-3)^2\\
x^2+x-12 &= x^2-6x+9\\
7x &= 21\\
x &= 3
\end{aligned}$$

2.1.4 Exercise 2

Find the value(s) of x in the following equations

(i) $27^{\log_3 x} = 81$

(ii) $64^{\log_4 x} = 8$

(iii) $125^{\log_5 x} = 64$

(iv) $243^{\log_3 x} = 3125$

Solution

(i)

$$\begin{aligned}
27^{\log_3 x} &= 81\\
3^{3\log_3 x} &= 81\\
3^{\log_3 x^3} &= 3^4\\
\log_3 x^3 &= 4\\
x^3 &= 4^3\\
x &= 4
\end{aligned}$$

(ii)

$$\begin{aligned}
64^{\log_4 x} &= 8\\
4^{3\log_4 x} &= 8\\
4^{\log_4 x^3} &= 8\\
x^3 &= 2^3\\
x &= 2
\end{aligned}$$

(iii)

$$\begin{aligned}
125^{\log_5 x} &= 64 \\
5^{3\log_5 x} &= 64 \\
5^{\log_5 x^3} &= 4^3 \\
x^3 &= 4^3 \\
x &= 4
\end{aligned}$$

(iv)

$$\begin{aligned}
243^{\log_3 x} &= 3125 \\
3^{5\log_3 x} &= 5^5 \\
3^{\log_3 x^5} &= 5^5 \\
x^5 &= 5^5 \\
x &= 5
\end{aligned}$$

2.1.5 Exercise 3

1. Solve the following equations for x and y where possible

(i) $\log_2 4x = y + 4; \quad 3\log_2 x = y.$

(ii) $\log_a(x^2+3) - \log_a(x+7) = \log_a x - \log_a 2$

(iii) $\log_3 y - \log_y 3 = 2$

(iv) $\log_2 x + \log_2 y = 6; \quad \log_3 x - \log_3 y = 4$

(v) $\log_2(\log_3 x) = 4$

2. (i) If $\log_3(x-b) = 2$ and $\log_2(x-7) = 3$, prove that $x^2 - 13x + 42 = 72$

(ii) If $4x^2 + 9y^2 = 37xy$, show that

$$2\log\left(\frac{2x+3y}{7}\right) = \log x + \log y$$

(iii) If $25a^2 + 9b^2 = 66ab$, show that

$$2\log\left(\frac{5a-3b}{6}\right) = \log a + \log b$$

(iv) If $x^3 + y^3 = 5xy(x+y)$, show that

$$3\log\left(\frac{x+y}{2}\right) = \log x + \log y + \log(x+y)$$

3. Show that

(a) $\log_2 5 \,.\, \log_4 3 \,.\, \log_{25} 16 = \log_2 3$

(b) $\log_2 16 - 3\log_3 \frac{1}{3} + \log_{25} 5 = 7\frac{1}{2}$

(c) $\log_4 x = \frac{1}{2}\log_2 x$ and hence solve $\log_4 x + \frac{1}{2} = \log_2 x$.

4. Show that

(i) $\log\left(\frac{40^{\log 3}}{3^{\log 4}}\right) = \log 3$

(ii) $\log_5 x + \log_2 8 = 0$

5. Solve the following equations

(i) $\log_5 x = 16\log_x 5$

(ii) $3\log 2 - \log(x-1) = \log(x-3)$

(iii) $\log(2x-1) + \log(x+2) = 2\log(x-1)$

6. Given that $3^{x+2} = 6^{2x}$, show that $x = \log_{12} 9$, /

7. (a) If $\log_5 3 = 0.682$ and $\log_5 2 = 0.431$, find the value of

$$\log_5\left(\frac{3}{8}\right) + 2\log_5\left(\frac{4}{5}\right) - \log_5\left(\frac{2}{5}\right)$$

(b) Given that $\log_7 2 = x$, $\log_7 3 = y$ and $\log_7 5 = z$, write the following logarithm in terms of x, y and z.

i. $\log_7 24$

ii. $\log_7 0.14$

iii. $\log_7\left(1\frac{1}{20}\right)$

iv. $\log_7 \sqrt{180}$

8. (i) Given that $\log_2 x + 2\log_4 y = 4$, show that $xy = 16$. Hence solve for x and y in the following simultaneous equations

$$\log_{10}(x+y) = 1$$
$$\log_2 x + 2\log_4 y = 4$$

(ii) If $2^{2y} = 51 \times 3^{3x}$, prove that

$$x\log\frac{4}{27} = \log 51$$

and hence find x.

CHAPTER 3
SURDS

Numbers whose square roots give non-terminating decimals are called **surds**. Examples of surds are the irrational numbers of the form $\sqrt{2}$, $\sqrt{7}$, π, etc.

3.1 PROPERTIES OF SURDS

1. ADDITION AND SUBTRACTION

(i) $\sqrt{x}+\sqrt{y}=\sqrt{x}+\sqrt{y}$

(ii) $\sqrt{x}-\sqrt{y}=\sqrt{x}-\sqrt{y}$

(iii) $p\sqrt{x}+q\sqrt{x}=(p+q)\sqrt{x}$

(iv) $p\sqrt{x}-q\sqrt{x}=(p-q)\sqrt{x}$

Example:

(a) $4\sqrt{5}+3\sqrt{5}=(4+3)\sqrt{5}=7\sqrt{5}$

(b) $12\sqrt{5}-8\sqrt{7}=(12-7)\sqrt{7}=4\sqrt{7}$

NB:

(i) $\sqrt{x}+\sqrt{y}\neq\sqrt{x+y}$

(ii) $\sqrt{x}-\sqrt{y}\neq\sqrt{x-y}$

(iii) Surds are numbers whose values cannot be written down exactly, although it can be found to any desired degree of accuracy.

2. **MULTIPLICATION PROPERTY**

(i) $\sqrt{x} \times \sqrt{x} = (\sqrt{x})^2 = x$

(ii) $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$

Example:

(a) $\sqrt{3} \times \sqrt{7} = \sqrt{3 \times 7} = \sqrt{21}$

(b) $\sqrt{5} \times \sqrt{5} = (\sqrt{5})^2 = 5$

3. **DIVISION PROPERTY**

$$\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$$

$$\text{E.g} \quad \frac{\sqrt{21}}{\sqrt{7}} = \sqrt{\frac{21}{7}} = \sqrt{3}$$

NB:

(i)

$$x \times \sqrt{y} = x\sqrt{y}$$

$$\text{E.g.} \quad 5 \times \sqrt{3} = 5\sqrt{3}$$

(ii)

$$x\sqrt{y} = \sqrt{x^y}$$

$$\text{E.g.}\sqrt{20} \times \sqrt{4 \times 5} = \sqrt{2^2 \times 5} = 2\sqrt{5}$$

$$\sqrt{50} \times \sqrt{25 \times 2} = \sqrt{5^2 \times 2} = 5\sqrt{2}$$

$$\sqrt{32} \times \sqrt{16 \times 2} = \sqrt{4^2 \times 2} = 4\sqrt{2}$$

3.2 WORKED EXAMPLES

Simplify the following surds

(i) $\sqrt{28}+\sqrt{63}-\sqrt{112}$

(ii) $3\sqrt{507}+7\sqrt{147}-2\sqrt{27}-6\sqrt{192}$

Solution

(i)

$$\begin{aligned}\sqrt{28}+\sqrt{63}-\sqrt{112} &= \sqrt{4\times 7}+\sqrt{7\times 9}-\sqrt{7\times 16}\\ &= \sqrt{2^2\times 7}+\sqrt{7^2}-\sqrt{7\times 4^2}\\ &= 2\sqrt{7}+3\sqrt{7}-4\sqrt{7}\\ &= \sqrt{7}\end{aligned}$$

(ii)

$$\begin{aligned}3\sqrt{507}+7\sqrt{147}-2\sqrt{27}-6\sqrt{192} &= 3\sqrt{3\times 169}+7\sqrt{3\times 49}-2\sqrt{3\times 9}-6\sqrt{3\times 64}\\ &= 3\sqrt{3\times 13^2}+7\sqrt{3\times 7^2}-2\sqrt{3\times 3^2}-6\sqrt{3\times 8^2}\\ &= 3(13)\sqrt{3}+7(7)\sqrt{3}-2(3)\sqrt{3}-6(8)\sqrt{3}\\ &= 39\sqrt{3}+49\sqrt{3}-6\sqrt{3}-48\sqrt{3}\\ &= 34\sqrt{3}\end{aligned}$$

3.2.1 Exercise

1. Simplify the following surds

(i) $\dfrac{1}{6}\left(4\sqrt{18}-3\sqrt{2}\right)$

(ii) $5\sqrt{60}-2\sqrt{375}+3\sqrt{135}$

(iii) $9\sqrt{54}-7\sqrt{24}+\sqrt{216}$

(iv) $3\sqrt{125}-5\sqrt{20}+3\sqrt{80}$

2. Rationalize the following

(i) $\dfrac{\sqrt{7}-3\sqrt{3}}{2\sqrt{7}+\sqrt{3}}$

(ii) $\dfrac{4-\sqrt{x}}{1+2\sqrt{x}}$

(iii) $\dfrac{4\sqrt{x}-9}{2\sqrt{x}-3}$

(iv) $\dfrac{12}{\sqrt{24}-\sqrt{6}}$

(v) $\dfrac{\sqrt{5}-2}{\sqrt{5}}$

(vi) $\dfrac{4}{\sqrt{18}+\sqrt{2}}$

3.2.2 Exercise

Rationalize the following:

(i) $\dfrac{\sqrt{7}-3\sqrt{3}}{2\sqrt{7}+\sqrt{3}}$

(ii) $\dfrac{4-\sqrt{x}}{1+2\sqrt{x}}$

(iii) $\dfrac{4\sqrt{x}-9}{2\sqrt{x}-3}$

(iv) $\dfrac{\sqrt{5}-2}{\sqrt{5}}$

(v) $\dfrac{12}{\sqrt{24}-\sqrt{6}}$

(vi) $\dfrac{4}{\sqrt{18}+\sqrt{2}}$

3.3 SQUARE ROOTS OF SURDS

(a) Find the square root of $7\frac{1}{2}-5\sqrt{2}$

Solution

$$\begin{aligned}
\text{Let } \sqrt{7\frac{1}{2}-5\sqrt{2}} &= \pm\left(\sqrt{x}-\sqrt{y}\right); \quad x>y>0 \\
\frac{15}{2}-\sqrt{50} &= \left(\sqrt{x}-\sqrt{y}\right)^2 \\
\implies \frac{15}{2}-\sqrt{50} &= x+y-2\sqrt{xy} \\
&= x+y-\sqrt{4xy}
\end{aligned}$$

Comparing terms;

$$x+y=\frac{15}{2} \tag{1}$$

$$4xy=50 \tag{2}$$

$$\implies y=\frac{50}{4x}=\frac{25}{2x} \qquad \text{(making } y \text{ the subject from (2))}$$

Substituting (2) into (1) :

$$\begin{aligned}
x+\frac{25}{x} &= \frac{15}{2} \\
\implies 2x^2-15x+25 &= 0 \\
\implies (2x-5)(x-5) &= 0 \\
\implies x=\frac{5}{2} &\text{ or } x=5
\end{aligned}$$

$$\text{When } x = \frac{5}{2}, \; y = 5 \text{ and}$$
$$\text{When } x = 5, \; y = \frac{5}{2} \qquad (\text{Note } x > y)$$
$$\therefore \sqrt{7\frac{1}{2} - 5\sqrt{2}} = \pm\left(\sqrt{5} - \sqrt{\frac{5}{2}}\right)$$

(b) Find the square root of $14 + 6\sqrt{5}$

$$\begin{aligned}
\text{Let } \sqrt{14+6\sqrt{5}} &= \pm\left(\sqrt{x}+\sqrt{y}\right); \quad x > y > 0 \\
14+6\sqrt{5} &= \left(\sqrt{x}+\sqrt{y}\right)^2 \qquad (\text{squaring both sides}) \\
\Longrightarrow \; 14+6\sqrt{5} &= x+y+2\sqrt{xy} \\
&= x+y+\sqrt{4xy}
\end{aligned}$$

Comparing terms;

$$x + y = 14 \tag{1}$$
$$\sqrt{4xy} = 6\sqrt{5}$$
$$4xy = 36(5) \tag{2}$$
$$\Longrightarrow \; y = \frac{45}{x} \qquad (\text{making } y \text{ the subject from (2)})$$

Substituting (2) into (1) :

$$x + \frac{45}{x} = 14$$
$$\Longrightarrow \; x^2 - 14x + 45 = 0$$
$$\Longrightarrow \; (x-9)(x-5) = 0$$
$$\Longrightarrow \; x = 9 \text{ or } x = 5$$
$$\text{When } x = 9 \; y = 5$$
$$\text{When } x = 5, \; y = 9$$
$$\text{Since } x > y \Longrightarrow \; x = 9 \text{ and } y = 5$$
$$\begin{aligned}
\therefore \sqrt{14+6\sqrt{5}} &= \sqrt{9}+\sqrt{5} \\
&= 3+\sqrt{5}
\end{aligned}$$

3.3.1 Exercise

1. Find the square roots of the following:

   (a) $17 - 4\sqrt{15}$

   (b) $28 - 10\sqrt{3}$

   (c) $7 + 4\sqrt{3}$

2. Find $\sqrt{33+20\sqrt{2}}$

3. Find $\sqrt{5+2\sqrt{3}}$

CHAPTER 4

POLYNOMIAL AND RATIONAL FUNCTIONS

A polynomial in x is an expression involving powers of x, normally arranged in descending (or sometimes in ascending) powers. The degree of the polynomial is given by the highest power of x occurring in the expression.
E.g:

$$
\begin{aligned}
4x^6+5x^5+3x^4+2x^3+x^2+5 \;&-\; \text{is a polynomial of the } 6th \text{ degree}\\
5x^4+7x^3+3x+5 \;&-\; \text{is a polynomial of degree 4}\\
3x^3+2x^2+x+1 \;&-\; \text{is a polynomial of the 3rd degree}
\end{aligned}
$$

- A polynomial of the 1st degree is often referred to as a **Linear expression**.
  E.g. $3x+5$, $2x-8$.
- A polynomial of the 2nd degree is often referred to as a **Quadratic expression**.
- A polynomial of the 3rd degree is often referred to as a **Cubic expression**.
- A polynomial of the 4th degree is often referred to as a **Quartic expression**.

4.0.2 Exercise

1. Define the degrees of the following polynomials:

(i) $f(x)=5x^3+2x^2+x+8$

(ii) $f(x)=-2x^4+3x+2$

(iii) $f(x)=x^4-3x^3+2x-3$

4.1 THE REMAINDER THEOREM

The remainder theorem states that if a polynomial $f(x)$ is divided by $(x-a)$, then the quotient will be a polynomial $g(x)$ of one degree less than the degree of $f(x)$, together with a remainder R still to be divided by $(x-a)$. Thus

$$\begin{aligned}
\frac{f(x)}{x-a} &= g(x) + \frac{R}{x-a} \\
f(x) &= (x-a)g(x) + R \\
\text{when } \ x &= a \\
f(a) &= 0.g(x) + R \\
\implies \ f(a) &= R
\end{aligned}$$

Thus, if $f(x)$ were to be divided by $(x-a)$, then the remainder would be $f(a)$.

Example

1. Find the remainder when $f(x) = x^3 + 3x^2 - 13x - 10$ is divided by $x-3$.

<u>**Solution**</u>

If $f(x)$ is divisible by $x-3$, then $x-3=0 \implies x=3$

$$\implies f(3) = (3)^3 + 3(3)^2 - 13(3) - 10 = 5$$

Alternatively, we could perform the long division:

$$\begin{array}{r|l}
 & \quad\ \ x^2 \quad + 6x \quad + 5 \\
\hline
x-3 & x^3 + 3x^2 - 13x - 10 \\
 & -x^3 + 3x^2 \\
 & \qquad 6x^2 - 13x \\
 & \quad\ -6x^2 + 18x \\
 & \qquad\qquad 5x - 10 \\
 & \qquad\quad -5x + 15 \\
 & \qquad\qquad\qquad 5
\end{array}$$

$$\implies \quad \frac{x^3 + 3x^2 - 13x - 10}{x-3} = x^2 + 6x + 5 + \frac{5}{x-3} = (x-3)(x^2 + 6x + 5) + 5$$

2. Find the remainder when $f(x) = 5x^3 - 4x^2 - 3x + 6$ is divided by $x - 2$.

Solution

$$
\begin{array}{rrrrr}
 & 5x^2 & +6x & +9 & \\
\cline{2-5}
x-2) & 5x^3 & -4x^2 & -3x & +6 \\
 & -5x^3 & +10x^2 & & \\
\cline{2-3}
 & & 6x^2 & -3x & \\
 & & -6x^2 & +12x & \\
\cline{3-4}
 & & & 9x & +6 \\
 & & & -9x & +18 \\
\cline{4-5}
 & & & & 24
\end{array}
$$

$$\implies \quad \frac{5x^3 - 4x^2 - 3x + 6}{x-2} = 5x^2 + 6x + 9 + \frac{24}{x-2} = (x-2)(5x^2 + 6x + 9) + 24$$

Therefore, the remainder is 24.

Alternatively, if $f(x)$ is divisible by $x - 2$, then $x - 2 = 0 \implies x = 2$:

$$f(x) = 5x^3 - 4x^2 - 3x + 6$$
$$f(2) = 5(2)^3 - 4(2)^2 - 3(2) + 6$$
$$= 40 - 16 - 6 + 6 = 24$$

3. Find the remainder when $f(x) = 4x^3 - 3x^2 + 5x - 3$ is divided by $x - 4$.

Solution

$$
\begin{array}{rrrrr}
 & 4x^2 & +13x & +57 & \\
\cline{2-5}
x-4) & 4x^3 & -3x^2 & +5x & -3 \\
 & -4x^3 & +16x^2 & & \\
\cline{2-3}
 & & 13x^2 & +5x & \\
 & & -13x^2 & +52x & \\
\cline{3-4}
 & & & 57x & -3 \\
 & & & -57x & +228 \\
\cline{4-5}
 & & & & 225
\end{array}
$$

$$\implies \quad \frac{4x^3 - 3x^2 + 5x - 3}{x-4} = 4x^2 + 13x + 57 + \frac{225}{x-4} = (x-4)(4x^2 + 13x + 57) + 225$$

Therefore, the remainder is 225.

Alternatively, if $f(x)$ is divisible by $x - 4$, then $x - 4 = 0 \implies x = 4$:

$$f(x) = 4x^3 - 3x^2 + 5x - 3$$
$$f(4) = 4(4)^3 - 3(4)^2 + 5(4) - 3$$
$$= 256 - 48 + 20 - 3 = 225$$

4.1.1 FACTOR THEOREM

- If $f(x)$ is a polynomial and substituting $x = a$ gives a remainder of 0, i.e. $f(a) = 0$, then $(x-a)$ is a factor of $f(x)$.
- For example, if $f(x) = x^3 + 2x^2 - 14x + 12$ is divisible by $x - 2$, leaving a remainder of 0, then $(x-2)$ is a factor of $f(x)$.
- The remainder factor can be found by long division of $f(x)$ by $(x-2)$.

$$
\begin{array}{r}
x^2 \quad + 4x \quad - 6 \\
x - 2 \overline{\big) \; x^3 + 2x^2 - 14x + 12} \\
\underline{- x^3 + 2x^2 \qquad\qquad\quad} \\
4x^2 - 14x \qquad \\
\underline{- 4x^2 \; + 8x \qquad} \\
- 6x + 12 \\
\underline{6x - 12} \\
0
\end{array}
$$

$$\implies \quad f(x) = (x-2)(x^2 + 4x - 6)$$

E.g. 1 Find the remainder when $f(x) = 3x^3 - 11x^2 + 10x - 12$ is divided by $x - 3$.

Solution

$$
\begin{array}{r}
3x^2 \quad - 2x \quad + 4 \\
x - 3 \overline{\big) \; 3x^3 - 11x^2 + 10x - 12} \\
\underline{- 3x^3 \;\; + 9x^2 \qquad\qquad\qquad} \\
- 2x^2 + 10x \qquad \\
\underline{2x^2 \;\; - 6x \qquad} \\
4x - 12 \\
\underline{- 4x + 12} \\
0
\end{array}
$$

This follows that, when $f(x)$ is divided by $x - 3$, the remainder is zero.
Alternatively, if $f(x)$ is divisible by $x - 3$, then $x - 3 = 0 \implies x = 3$:

$$\implies f(3) = 3(3)^3 - 11(3)^2 + 10(3) - 12$$
$$= 81 - 99 + 30 - 12 = 0$$

E.g. 2 If $(x-3)$ is a factor of $f(x) = x^3 - 5x^2 - 2x + 24$, determine the remaining factors.

Solution

If $x-3$ is a factor of $f(x)$, then $x-3=0 \implies x=3 \implies f(3)=0$.

Using the long division, we have

$$\begin{array}{r r}
 & x^2 - 2x \quad - 8 \\
\cline{2-2}
x-3) & x^3 - 5x^2 - 2x + 24 \\
 & -x^3 + 3x^2 \qquad\qquad \\
\cline{2-2}
 & -2x^2 - 2x \qquad \\
 & 2x^2 - 6x \qquad \\
\cline{2-2}
 & -8x + 24 \\
 & 8x - 24 \\
\cline{2-2}
 & 0
\end{array}$$

$$f(x) = x^3 - 5x^2 - 2x + 24 = (x-3)(x^2 - 2x - 8)$$
$$= (x-3)(x+2)(x-4)$$

E.g. 3 Factorize completely if $f(x) = 2x^4 - x^3 - 8x^2 + x + 6$ is divisible by $x-1$.

Solution

$$\begin{array}{r r}
 & 2x^3 \quad + x^2 - 7x - 6 \\
\cline{2-2}
x-1) & 2x^4 \quad - x^3 - 8x^2 \quad + x + 6 \\
 & -2x^4 + 2x^3 \qquad\qquad\qquad\qquad \\
\cline{2-2}
 & x^3 - 8x^2 \qquad\qquad \\
 & -x^3 \quad + x^2 \qquad\qquad \\
\cline{2-2}
 & -7x^2 \quad + x \qquad \\
 & 7x^2 - 7x \qquad \\
\cline{2-2}
 & -6x + 6 \\
 & 6x - 6 \\
\cline{2-2}
 & 0
\end{array}$$

$$\implies f(x) = (x-1)(2x^3 + x^2 - 7x - 6)$$

Also, let us set $f(x) = 2x^3 + x^2 - 7x - 6$

Now, the possible factors of $f(x)$ are $\pm(1, 2, 3, 4, 6, 12)$

For $f(1) = 2(1)^3 + (1)^2 - 7(1) - 6$

$= 2 + 1 - 7 - 6 \neq 0$

but for $f(-1) = 2(-1)^3 + (-1)^2 - 7(-1) - 6$

$= -2 + 1 + 7 - 6 = 0$

Using the long division, we have

$$
\begin{array}{r}
2x^2 \quad - x - 6 \\
x+1 \overline{\big) \; 2x^3 \; + x^2 - 7x - 6} \\
\underline{-2x^3 - 2x^2} \qquad\qquad \\
-x^2 - 7x \quad\; \\
\underline{x^2 \;\; + x} \quad\; \\
-6x - 6 \\
\underline{6x + 6} \\
0
\end{array}
$$

$$
\begin{aligned}
\implies \quad 2x^3 + x^2 - 7x - 6 &= (x+1)(2x^2 - x - 6) \\
&= (x+1)(x-2)(2x+3) \\
\therefore \quad f(x) &= 2x^4 - x^3 - 8x^2 + x + 6 \\
&= (x-1)(2x^3 + x^2 - 7x - 6) \\
&= (x-1)(x+1)(2x^2 - x - 6) \\
&= (x-1)(x+1)(x-2)(2x+3)
\end{aligned}
$$

Trial Exercise

1. Find the values of a and b if $2x^2 - 5x + 2$ is a factor of $8x^4 - 14x^3 + ax^2 - 19x + b$.

2. When $ax^2 - 8x + 7$ is divided by $x - 1$, the remainder is the same when $x^2 + ax - 16$ is divided by $x - 3$. Find the value of a.

3. Determine the remainder if $x^4 - 2x^3 + 3x^2 - 4$ is divisible by $x - 2$.

4. If $f(x) = 6x^4 + x^3 - 25x^2 - 4x + 4$ is divisible by $x^2 - 4$, find the complete factors of $f(x)$.

5. Factorize the following polynomials:

(i) $2x^3 - 3x^2 - 3x + 2$

(ii) $2x^3 - 3x^2 - 39x + 20$

(iii) $x^4 + 2x^3 - 7x^2 - 8x + 12$

(iv) $x^3 - 2x^2 - 5x + 6$

6. Find the quotient and the remainder if $f(x)$ is divided by $x - c$

(i) $f(x) = x^3 + 3x^2 - x - 2; \quad x - 2$

(ii) $f(x) = -2x^4 + 3x^2 + 7x - 26; \quad x + 3$

(iii) $f(x) = 4x^3 - 2x^2 - 5x + 1; \quad x - 2/3$

4.2 PARTIAL FRACTIONS

A. DENOMINATOR WITH ONLY LINEAR FACTORS

$$\frac{x+4}{(x+1)(x-1)(2x-3)} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{2x-3}$$
$$\frac{2x^2+5x-6}{(x-3)(x+5)(2x-1)} = \frac{A}{x-3} + \frac{B}{x+5} + \frac{C}{2x-1}$$

E.g.1

$$\frac{6}{(x+3)(x-3)} = \frac{A}{x+3} + \frac{B}{x-3}$$

multiplying through by (x+3)(x-3)

To get B, we make $A = 0$, by putting $x = 3$

$$\implies 6B = B$$

$$\implies B = 1$$

To get A, we make $B = 0$, by putting $x = 3$

$$\implies -6A = A$$

$$\implies A = -1$$

$$\therefore \frac{6}{(x+3)(x-3)} = \frac{-1}{x+3} + \frac{1}{x-3} = \frac{1}{x-3} - \frac{1}{x+3}$$

E.g.2

$$\frac{11x+12}{(x+2)(x-3)(2x+3)} = \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{2x+3}$$

multiplying through by (x+2)(x-3)(2x+3)

$$11x+12 = A(x-3)(2x+3) + B(x+2)(2x+3) + C(x+2)(x-3)$$

$$\text{when } x = 3, \ 45B = 45 \implies B = 1$$

$$\text{when } x = -2, \ 5A = -10 \implies A = -2$$

$$\text{when } x = \frac{3}{2}, \ -\frac{9}{4}C = -\frac{9}{2} \implies C = 2$$

$$\therefore \frac{11x+12}{(x+2)(x-3)(2x+3)} = \frac{-2}{x+2} + \frac{1}{x-3} + \frac{2}{2x+3}$$
$$= \frac{1}{x-3} - \frac{2}{x+2} + \frac{2}{2x+3}$$

B. DENOMINATOR WITH LINEAR FACTORS REPEATED

E.g.1

$$\frac{4}{(x-1)^2(x-3)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-3}$$

$$\implies 4 = A(x-1)(x-3) + B(x-3) + C(x-1)^2$$

$$\text{when } x = 1,\ -2B = 4 \implies B = -2$$

$$\text{when } x = 3,\ 4C = 4 \implies C = 1$$

Comparing coefficient of x^2;

$$\implies (A+C) = 0$$

$$A + 1 = 0$$

$$A = -1$$

$$\therefore \frac{4}{(x-1)^2(x-3)} = \frac{-1}{x-1} + \frac{-2}{(x-1)^2} + \frac{1}{x-3}$$

$$= \frac{1}{x-3} - \frac{1}{x-1} - \frac{2}{(x-1)^2}$$

E.g.2

$$\frac{1}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\implies 1 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$$\text{when } x = 1,\ 3B = 1 \implies B = \frac{1}{3}$$

$$\text{when } x = -2,\ 9C = 1 \implies C = \frac{1}{9}$$

Comparing coefficient of x^2;

$$\implies (A+C) = 0$$

$$A + \frac{1}{9} = 0$$

$$A = -\frac{1}{9}$$

$$\therefore \frac{1}{(x-1)^2(x+2)} = \frac{-\frac{1}{9}}{x-1} + \frac{\frac{1}{3}}{(x-1)^2} + \frac{\frac{1}{9}}{x+2}$$

$$= \frac{1}{9(x+2)} + \frac{1}{3(x-1)^2} - \frac{1}{9(x-1)}$$

C. DENOMINATOR WITH QUADRATIC FACTORS

(i) $\dfrac{16x^2}{(x^2-4)(x^2+4)} = \dfrac{16x^2}{(x+4)(x-4)(x^2+4)} = \dfrac{A}{x+4} + \dfrac{B}{x-4} + \dfrac{Cx+D}{x^2+4}$

(ii) $\dfrac{2x^2+3x}{(x^2+4)^2} = \dfrac{Ax+B}{x^2+4} + \dfrac{Cx+D}{(x^2+4)^2}$

(iii) $\dfrac{3x+1}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

(iv) $\dfrac{4}{(x+1)(2x^2+x+3)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{2x^2+x+3}$

E.g.1

$$\begin{aligned}\frac{5x^2-6x+10}{(x+2)(x^2-3x+4)} &= \frac{A}{x+2} + \frac{Bx+C}{x^2-3x+4}\\ \implies 5x^2-6x+10 &= A(x^2-3x+4)+(Bx+C)(x+2)\\ &= A(x^2-3x+4)+Bx^2+2Bx+Cx+2C \qquad (1)\end{aligned}$$

$$\begin{aligned}\text{when } x=-2,\ 20+12+10 &= A(4+6+4)\\ \implies 42 &= 14A\\ \implies A &= 3\end{aligned}$$

Comparing coefficient of x^2 and x from (1)

$$\begin{aligned}x^2:\ 5 &= A+B\\ \implies 5 &= 3+B\\ \implies B &= 2\\ x:\ -6 &= -3A+2B+C\\ -6 &= -9+4+C\\ \implies C &= -6+9-4=-1\end{aligned}$$

$$\therefore\ \frac{5x^2-6x+10}{(x+2)(x^2-3x+4)} = \frac{3}{x+2} + \frac{2x-1}{x^2-3x+4}$$

E.g.2

$$
\begin{aligned}
\frac{4}{(x+1)(2x^2+x+3)} &= \frac{A}{x+1}+\frac{Bx+C}{2x^2+x+3} \\
\implies 4 &= A(2x^2+x+3)+(Bx+C)(x+1) \\
&= A(2x^2+x+3)+Bx^2+Bx+Cx+C \\
&= (2A+B)x^2+(A+B+C)x+3A+C \qquad (1)
\end{aligned}
$$

when $x=-1,\ 4A=1$

Comparing coefficient:

$$
\begin{aligned}
x^2:&\ \ 2A+B=0 \\
\implies&\ \ 2+B=0 \\
\implies&\ \ B=-2
\end{aligned}
$$

$$
\begin{aligned}
x:&\ \ A+B+C=0 \\
&\ \ 1-2+C=0 \\
\implies&\ \ C=1
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \frac{4}{(x+1)(2x^2+x+3)} &= \frac{1}{x+1}+\frac{-2x+1}{2x^2+x+3} \\
&= \frac{1}{x+1}+\frac{1-2x}{2x^2+x+3}
\end{aligned}
$$

D. NUMERATOR'S DEGREE $\geq$ DENOMINATOR'S DEGREE

Example: $$\frac{2x^4+3x^3-5x^2-34x-18}{x^3+x^2-6x} = 2x+1+\frac{6x^2-28x-18}{x^3+x^2-6x}$$

NB:

- To effect a partial fraction breakdown of a rational algebraic expression, it is necessary for the degree of the numerator to be less than the degree of the denominator.
- Thus if, in the original algebraic rational expression, the degree of the numerator is not less than the degree of the denominator, then first divide out by long division. This gives a polynomial with a rational remainder where the remainder has a degree of the numerator being less than the degree of the denominator. The remainder can be broken down into its partial fractions.

E.g.1 Express $\dfrac{x^2+3x-10}{x^2-2x-3}$ in partial fraction

Solution

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad 1 \\ \cline{2-2} x^2-2x-3) & \quad x^2+3x-10 \\ & -x^2+2x\ \ +3 \\ \cline{2-2} & \quad\quad\quad 5x\ \ -7 \end{array}$$

$$\implies \frac{x^2+3x-10}{x^2-2x-3} = 1+\frac{5x-7}{x^2-2x-3}$$

$$\implies \frac{5x-7}{x^2-2x-3} = \frac{5x-7}{(x+1)(x-3)} = \frac{A}{x+1}+\frac{B}{x-3}$$

$$\implies 5x-7 = A(x-3)+B(x+1)$$

When $x=3$; $4B=8 \implies B=2$

When $x=-1$; $-4A=-12 \implies A=3$

$$\implies \frac{5x-7}{x^2-2x-3} = \frac{3}{x+1}+\frac{2}{x-3}$$

$$\therefore \frac{x^2+3x-10}{x^2-2x-3} = 1+\frac{3}{x+1}+\frac{2}{x-3}$$

E.g.2 Express $\dfrac{3x^3-x^2-13x-13}{x^2-x-6}$ in partial fraction

Solution

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad 3x\ \ +2 \\ \cline{2-2} x^2-x-6) & \quad 3x^3\ \ -x^2-13x-13 \\ & -3x^3+3x^2+18x \\ \cline{2-2} & \quad\quad\quad 2x^2\ \ +5x-13 \\ & \quad\quad -2x^2\ \ +2x+12 \\ \cline{2-2} & \quad\quad\quad\quad\quad 7x\ \ -1 \end{array}$$

$$\implies \frac{3x^3-x^2-13x-13}{x^2-x-6} = 3x+2+\frac{7x-1}{x^2-x-6}$$

$$= 3x+2+\frac{7x-1}{(x+2)(x-3)}$$

$$\implies \frac{7x-1}{(x+2)(x-3)} = \frac{A}{x+2}+\frac{B}{x-3}$$

$$= \frac{3}{x+2}+\frac{4}{x-3}$$

$$\therefore \frac{3x^3-x^2-13x-13}{x^2-x-6} = 3x+2+\frac{3}{x+2}+\frac{4}{x-3}$$

Exercise

Express the following in partial fractions:

(a) $\dfrac{2x^3+3x^2-54x+50}{x^2+2x-24}$

(b) $\dfrac{2x^2+6x-35}{x^2-x-12}$

(c) $\dfrac{15x^2-x+2}{(x-5)(3x^2+4x-2)}$

(d) $\dfrac{10x^2+7x-42}{(x-2)(x+4)(x-1)}$

(e) $\dfrac{12x^2+36x+6}{x^3+6x^2+3x-10}$

(f) $\dfrac{35x+17}{(5x+2)^2}$

Try

Express the following in partial fractions

1. $\dfrac{1}{x^2-9}$
2. $\dfrac{x}{(x+2)(x+3)}$
3. $\dfrac{x^4-4x^2+x+1}{x^2-4}$
4. $\dfrac{2x^2+1}{(x-1)(x-2)(x-3)}$
5. $\dfrac{x^2-4}{x^3-3x^2-x+3}$
6. $\dfrac{x^3+1}{x(x+3)(x+2)(x-1)}$
7. $\dfrac{2x}{(x-2)^2(x+2)}$
8. $\dfrac{x+4}{x^3+6x^2+9x}$
9. $\dfrac{x^4}{x^3-2x^2-7x-4}$
10. $\dfrac{1}{x(x^2+9)}$
11. $\dfrac{1}{(x^2+1)(x^2+4)}$
12. $\dfrac{1-9x^2}{x(x^2+9)}$
13. $\dfrac{1}{x(x^2+1)^2}$
14. $\dfrac{x^2}{(x-1)(x^2+4)^2}$
15. $\dfrac{2x+1}{(2x+3)^2}$
16. $\dfrac{8x+1}{2x^2-9x-35}$

CHAPTER 5

THE BINOMIAL THEOREM

The factorial operation is the exclamation sign, !, and it is placed after an integer variable or number to indicate factorial function, which is defined as:

$$n! = n(n-1)(n-2)(n-3)...2 \times 1$$

From this definition, it follows that

$$n! = n[(n-1)!] = n(n-1)(n-2)! \quad \text{etc.}$$

Examples

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$3! = 3 \times 2 \times 1 = 6$$

$$2! = 2 \times 1 = 2$$

$$1! = 1$$

And by definition $0! = 1$. The binomial theorem or the binomial formula is a formula to expand a power of the binomial expression $a+b$, that is, it is a formula that enables us to expand $(a+b)^n$. Descriptions of the binomial theorem frequently use a shorthand notation for the coefficients of each term consisting of numbers over each other enclosed in parenthesis e.g.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

The notation $\binom{n}{k}$ is often called the binomial coefficient "n choose k".

In general, the binomial theorem is as follows:

$$(a+b)^n = \binom{n}{r}\sum_{r=0}^{n} a^{n-r}b^r =$$

$$\binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

Applying the binomial theorem to $(a+b)^3$, we obtain

$$\begin{aligned}(a+b)^3 &= \binom{3}{0}a^3b^0 + \binom{3}{1}a^2b^1 + \binom{3}{2}a^1b^2 + \binom{3}{3}a^0b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3\end{aligned}$$

E.g. 1 Expand $(x+2y)^5$

<u>Solution</u>

$$(x+2y)^5 =$$

$$\begin{aligned}&\binom{5}{0}x^5(2y)^0 + \binom{5}{1}x^4(2y)^1 + \binom{5}{2}x^3(2y)^2 + \binom{5}{3}x^2(2y)^3 + \binom{5}{4}x^1(2y)^4 + \binom{5}{5}x^0(2y)^5 \\ &= \frac{5!}{5!0!}x^5 + \frac{5!}{4!1!}x^4(2y) + \frac{5!}{3!2!}x^3(4y^2) + \frac{5!}{2!3!}x^2(8y^3) + \frac{5!}{1!4!}x(16y^4) + \frac{5!}{0!5!}(32y^5) \\ &= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5\end{aligned}$$

E.g. 2 Find the fifth term of $(a+2x^3)^{17}$

<u>Solution</u>

$$\begin{aligned}\text{The fifth term of } (a+2x^3)^{17} &= \binom{17}{4}a^{13}(2x^3)^4 \\ &= 38080a^{13}x^{12}\end{aligned}$$

E.g. 3 Find the term involving y^5 in the expansion of $(2x^2+y)^{10}$

<u>Solution</u>

$$\begin{aligned}\text{Thus, the sixth term } &= \binom{10}{5}(2x^2)^5y^5 \\ &= 8064x^{10}y^5\end{aligned}$$

E.g. 4 Find the constant term in the expansion of $(2x^2+\frac{1}{x})^9$.

Solution

$$\begin{aligned}\text{Thus, } 7^{th} \text{ term of this expansion } &= \binom{9}{6}(2x^2)^3\left(\frac{1}{x}\right)^6\\ &= 84(2)^3x^6\left(\frac{1}{x^6}\right)\\ &= 672\end{aligned}$$

NB:

$$(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\cdots+x^n$$

E.g. 1 Expand $(1+x)^4$.

Solution

$$\begin{aligned}(1+x)^4 &= 1+4x+\frac{4(3)}{2!}x^2+\frac{4(3)(2)}{3!}x^3+x^4\\ &= 1+4x+6x^2+4x^3+x^4\end{aligned}$$

E.g. 2 Expand $(1+2x)^3$ in ascending power of x.

Solution

$$\begin{aligned}(1+2x)^3 &= 1+3(2x)+\frac{3(2)(2x)^2}{2!}+(2x)^3\\ &= 1+6x+12x^2+8x^3\end{aligned}$$

E.g. 3 Expand $(1-2x)^5$ in ascending power of x.

Solution

$$\begin{aligned}(1-2x)^5 &= 1+5(-2x)+\frac{5(4)(-2x)^2}{2!}+\frac{5(4)(3)(-2x)^3}{3!}+\frac{5(4)(3)(2)(-2x)^4}{4!}+(-2x)^5\\ &= 1-10x+40x^2-80x^3+80x^4-32x^5\end{aligned}$$

E.g. 4 Expand $\left(1+\frac{x}{2}\right)^4$ in ascending powers of x.

Solution

$$
\begin{aligned}
\left(1+\frac{x}{2}\right)^4 &= 1+4\left(\frac{x}{2}\right)+\frac{4(3)\left(\frac{x}{2}\right)^2}{2!}+\frac{4(3)(2)\left(\frac{x}{2}\right)^3}{3!}+\left(\frac{x}{2}\right)^4 \\
&= 1+2x+\frac{4(3)}{4(2)(1)}x^2+\frac{4(3)(2)}{8(3)(2)(1)}x^3+\frac{1}{16}x^4 \\
&= 1+2x+\frac{3}{2}x^2+\frac{1}{2}x^3+\frac{1}{16}x^4
\end{aligned}
$$

E.g. 5 Expand $(2+x)^4$ in ascending powers of x.

Solution

$$
\begin{aligned}
(2+x)^4 &= 2^4\left(1+\frac{x}{2}\right)^4 \\
2^4\left(1+\frac{x}{2}\right)^4 &= 2^4\left[1+4\left(\frac{x}{4}\right)+\frac{4(3)\left(\frac{x}{2}\right)^2}{2!}+\frac{4(3)(2)\left(\frac{x}{2}\right)^3}{3!}+\left(\frac{x}{2}\right)^4\right] \\
&= 2^4\left[1+2x+\frac{4(3)x^2}{4(2)(1)}+\frac{4(3)(2)x^3}{8(3)(2)(1)}+\frac{x^4}{16}\right] \\
&= 16+32x+24x^2+8x^3+x^4
\end{aligned}
$$

E.g. 6 The first three terms of the binomial expansion $(1+px)^n$ are 1, $-10x$ and $40x^2$ respectively. Find the values of p and n.

Solution

$$(1+px)^n = 1+npx+\frac{n(n-1)(px)^2}{2!}$$

Comparing coefficients

$$np = -10 \tag{1}$$

$$\implies \frac{n(n-1)}{2!}p^2 = 40$$

$$n^2p^2 - np^2 = 80 \tag{2}$$

From (1) $n = \dfrac{-10}{p}$

Substituting the value of n in (2), we have

$$
\begin{aligned}
\left(-\frac{10}{p}\right)^2 p^2 - \left(-\frac{10}{p}\right) p^2 &= 80 \\
100 + 10p &= 80 \\
10p &= 80 - 100 \\
10p &= -20 \\
p &= -2
\end{aligned}
$$

Substituting the value of $p = -2$ in (1)

$$
\begin{aligned}
\Longrightarrow \quad -2n &= -10 \\
n &= \frac{-10}{-2} \\
n &= 5
\end{aligned}
$$

5.1 PASCAL TRIANGLE

$(a+b)^0$	1
$(a+b)^1$	1 1
$(a+b)^2$	1 2 1
$(a+b)^3$	1 3 3 1
$(a+b)^4$	1 4 6 4 1
$(a+b)^5$	1 5 10 10 5

That is $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5a^2b^3 + 5ab^4 + b^5$

CHAPTER 6

SERIES AND SEQUENCE

The series of

$$
\begin{aligned}
1+2+3+\cdots+n &= \sum_{r=1}^{n} r \\
\implies \sum_{r=1}^{n} r &= 1+2+3+\cdots+n \\
&= \frac{n(n+1)}{2}
\end{aligned}
$$

6.1 THEOREM

Given that c is a constant and n a positive integer, then

(a) $\sum_{r=1}^{n} 1 = n$

(b) $\sum_{r=1}^{n} c = nc$

(c) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

(d) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

E.g. 1

The sum of the first 100 natural numbers is given by

$$\sum_{r=1}^{100} r = \frac{100(101)}{2} = 5050$$

E.g. 2

Find the sum of the series $\sum_{r=1}^{5} r(3+2r)$

Solution

$$\begin{aligned}\sum_{r=1}^{5} r(3+2r) &= \sum_{r=1}^{5} \left(3r+2r^2\right)\\ &= \sum_{r=1}^{5} 3r + \sum_{r=1}^{5} 2r^2\\ &= 3\left[\frac{5(6)}{2}\right] + 2\left[\frac{5(6\times 11)}{6}\right]\\ &= 45+110 = 155\end{aligned}$$

E.g. 3

Find the sum of the series $\sum_{r=1}^{4} (2r+r^3)$

Solution

$$\begin{aligned}\sum_{r=1}^{4} \left(2r+r^3\right) &= \sum_{r=1}^{4} 2r + \sum_{r=1}^{4} r^3\\ &= 2\left[\frac{4(5)}{2}\right] + \left[\frac{4(5)}{2}\right]^2\\ &= 20+100 = 120\end{aligned}$$

6.1.1 Exercise

Evaluate the following

(i) The sum of the first n natural numbers is $\frac{n(n+1)}{2}$

(ii) The sum of the squares of the first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$

(iii) The sum of the cubes of the first n natural numbers is $\left[\frac{n(n+1)}{2}\right]^2$

6.2 SEQUENCES

A sequence is a set of quantities $u_1, u_2, u_3, ...$, stated in a definite order and each term formed according to a fixed pattern. For instance, in the sequence

$$\begin{array}{ccccccc} 1, & 2, & 3, & 4, & ..., & n, & ... \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ u_1, & u_2, & u_3, & u_4, & ..., & u_n, & ... \end{array}$$

1 is mapped unto u_1, 2 is mapped onto u_2, and so on. The numbers $u_1, u_2, u_3, ...u_n, ...$ are the **terms** of the sequence. The number u_n is the $\boldsymbol{n}$**th term** of the sequence.

6.2.1 Finite And Infinite Sequence

A **finite sequence** contains only a finite number of terms (E.g. $1, 3, 5, 7, ..., n$) while an **infinite sequence** is unending (E.g. $2, 6, 18, 54, ...$).
NB: A series is formed by the sum of the terms of the sequence.

6.2.2 ARITHMETIC SEQUENCE

This is a sequence in which the difference between any two successive terms is some constant d. For any arithmetic sequence

$$\begin{aligned} u_{n+1} &= u_n + d \\ d &= u_{n+1} - u_n \end{aligned}$$

The first term of an arithmetic sequence is denoted by $\boldsymbol{a}$. The general term or the $\boldsymbol{n}$**th term** of an arithmetic sequence is given by

$$\begin{aligned} u_n &= a + (n-1)d \\ \text{where } \ a &= \text{1st term} \\ d &= \text{common difference} \\ n &= \text{number of terms} \end{aligned}$$

The general arithmetic series can be written as

$$a+(a+d)+(a+2d)+(a+3d)+\cdots$$

$$\text{where } \ a = \text{1st term}$$

$$d = \text{common difference}$$

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right] \qquad \text{(Sum of } n \text{ terms)}$$

E.g. 1

Find the 15th term of the sequence $2, 5, 8$

Solution

$$a = 2, \ \ d = (5-2) = (8-5) = 3$$

$$u_n = a + (n-1)d$$

$$\implies \ u_{15} = 2 + (15-1)3$$

$$= 2 + 42$$

$$= 44$$

E.g. 2

The 7th term of an AP is 3 and the 12th term is -3. Find

(i) the first term and the common difference

(ii) the 10th term.

(i)

Solution

$$u_7 = a + 6d = 3 \qquad (1)$$

$$u_{12} = a + 11d = -3 \qquad (2)$$

$$(1)-(2): \ \ -5d = 6$$

$$\implies \ d = -\frac{6}{5} \qquad (3)$$

Substitute (3) into (1)

$$\implies \ a + 6\left(-\frac{6}{5}\right) = 3$$

$$5a - 36 = 15$$
$$5a = 51$$
$$a = \frac{51}{5}$$

(ii)

$$u_n = a + (n-1)d$$
$$\implies u_{10} = \frac{51}{5} + (10-1)\left(-\frac{6}{5}\right)$$
$$= \frac{51}{5} - \frac{54}{5} = -\frac{3}{5}$$

E.g. 3

Find the value of x if $2x+1$, $x-2$ and $3x+4$ are consecutive terms of an AP.

Solution

Here we apply the principle that

$$u_{n+1} - u_n = d$$
$$u_2 - u_1 = u_3 - u_2 = d$$
$$\implies (x-2) - (2x+1) = (3x+4) - (x-2)$$
$$x - 2 - 2x - 1 = 3x + 4 - x + 2$$
$$-x - 3 = 2x + 6$$
$$-3x = 9$$
$$\therefore \ x = -3$$

E.g. 4

If the 7th term of an AP is 22 and the 12th term is 37, find the series.

Solution

$$a + 6d = 22 \tag{1}$$
$$a + 11d = 37 \tag{2}$$

$$(2)\text{-}(1):\ \ 5d = 15$$

$$\implies\ \ d = 3$$

$$\text{From (1)}\ \ a + 6(3) = 22$$

$$\implies\ \ a + 18 = 22$$

$$\implies\ \ a = 22 - 18 = 4$$

$$\therefore\ \ \text{the series is given by}\ \ 4 + 7 + 10 + 13 + 16 + \cdots$$

6.2.3 ARITHMETIC MEAN

The arithmetic mean of x and y is given by $\dfrac{x+y}{2}$.
The arithmetic mean of two numbers is simply their average.

E.g.1 Insert five numbers between 5 and 17:

Solution

This means that we need to find 7 terms with $u_1 = 5$ and $u_7 = 17$

$$u_1 = a = 5 \tag{1}$$

$$u_7 = 5 + 6d = 17 \tag{2}$$

$$\text{From (2)}:\ \ 6d = 12$$

$$\implies\ \ d = 2$$

$$\therefore\ \ \text{the numbers are}\ \ 5, \mathbf{7}, \mathbf{9}, \mathbf{11}, \mathbf{13}, \mathbf{15}, 17$$

E.g.2 Insert 3 arithmetic means between 8 and 18.

Solution

Let the means be x, y and z

$$\implies\ \ 8 + x + y + z + 18\ \ \text{form an AP}$$

$$\begin{aligned}
a &= 8 \qquad \text{1st term} \\
a+4d &= 18 \qquad \text{5th term} \\
\text{From the 5th term } \; d &= \frac{5}{2} \\
\implies \; x &= 8+\frac{5}{2} = \frac{21}{2} \\
y &= \frac{21}{2}+\frac{5}{2} = \frac{26}{2} = 13 \\
z &= \frac{26}{2}+\frac{5}{2} = \frac{31}{2}
\end{aligned}$$

$$\therefore \quad 8, \frac{21}{2}, 13, \frac{31}{2}, 18 \; \text{ form an AP}$$

Exercise

1. Find 5 arithmetic means between 12 and 21.6.

2. Insert 5 arithmetic means between 3 and 15.

6.2.4 THE SUM, S_n OF AN ARITHMETIC SEQUENCE

$$\boldsymbol{S_n} = a+(a+d)+\cdots+a+(n-2)d+a+(n-1)d \tag{1}$$

Reversing :

$$\boldsymbol{S_n} = a+(n-1)d+[a+(n-2)d]+\cdots+(a+d)+a \tag{2}$$

Adding (1) and (2) :

$$\begin{aligned}
2\boldsymbol{S_n} &= n[2a+(n-1)d] \\
\boldsymbol{S_n} &= \frac{n}{2}[2a+(n-1)d]
\end{aligned}$$

E.g.1

Find the sum of the first twenty terms of an AP $1, 4, 7, 10, \cdots$

<u>**Solution**</u>

$$a = 1, \;\; d = 3 \;\; n = 20$$

$$\begin{aligned}\implies \boldsymbol{S_n} &= \frac{n}{2}[2a+(n-1)d] \\ &= \frac{20}{2}[2(1)+(20-1)3] \\ &= 10[2+19(3)] \\ &= 10(2+57) = 590\end{aligned}$$

E.g.2

If the 7th term of an AP is 22 and the 12th term is 37, find the sum of the first 10 terms

Solution

$$\text{7th term}: \quad a+6d = 22 \tag{1}$$

$$\text{12th term}: \quad a+11d = 37 \tag{2}$$

$$(2)-(1): \quad 5d = 15$$

$$\implies d = 3$$

$$\implies \text{from (1)}: \quad a+6(3) = 22$$

$$\implies a+18 = 22$$

$$\implies a = 22-18 = 4$$

The sum of n terms is given by

$$\begin{aligned}\boldsymbol{S_n} &= \frac{n}{2}[2a+(n-1)d] \\ \implies \boldsymbol{S}_{10} &= \frac{10}{2}[2(4)+(10-1)3] \\ &= 5[8+27] \\ &= 175\end{aligned}$$

E.g.3

The 6th term of an AP is –5 and the 10th term is –21. Find the sum of the first 30 terms.

Solution

$$\text{6th term}: \quad a+5d = -5 \tag{1}$$

$$\text{10th term}: \quad a+9d = -21 \tag{2}$$

$$(2)-(1): \quad 4d = -16$$

$$\implies d = -4$$

$$\implies \text{from (1)}: \quad a+5(-4) = -5$$
$$\implies a-20 = -5$$
$$\implies a = 20-5 = 15$$

The sum of n terms is given by

$$\boldsymbol{S_n} = \frac{n}{2}[2a+(n-1)d]$$
$$\implies \boldsymbol{S}_{30} = \frac{30}{2}\,[2(15)+(30-1)(-4)]$$
$$= 15[30-116]$$
$$= -1290$$

Exercise

1. The 7th term of an AP is 3 and the 12th term is –3. Find the first term and the common difference and the sum to 45 terms.

2. The 9th term of an AP is –21 and the 15th term is –45. Find the sum to 26 terms.

6.2.5 GEOMETRIC SEQUENCE (GP)

This is a sequence in which the ratio, $\boldsymbol{r}$ of any two successive terms is a constant. For any GP

$$r = \frac{u_{n+1}}{u_n}$$
$$\implies u_{n+1} = r\,u_n$$

A GP has the form $a+ar+ar^2+ar^3+\cdots$, where a = first term and r = common ratio.
The $\boldsymbol{n}$**th term** or the general term of a GP is given by

$$u_n = ar^{n-1}$$

where a = first term and r = common ratio.

E.g.1

Find the common ratio of the sequence $5-10+20-40+\cdots$

Solution

$$r = -\frac{10}{5} = -\frac{20}{10} = -\frac{40}{20} = -2$$

E.g.2

The 5th term of a GP is 162 and the 8th term is 4374, find the common ratio and the first term.

Solution

$$\text{5th term}: \quad ar^4 = 162 \qquad (1)$$

$$\text{12th term}: \quad ar^7 = 4374 \qquad (2)$$

$$\frac{(2)}{(1)}: \quad \frac{ar^7}{ar^4} = \frac{4373}{162}$$

$$\implies \quad r^3 = 27$$

$$\implies \quad r = 3$$

$$\text{From (1)}: \quad a(3)^4 = 162$$

$$\implies \quad a = \frac{162}{81} = 2$$

$\therefore$ first term = 2 and common ratio = 3

6.2.6 GEOMETRIC MEANS

If x, y and z form a geometric sequence, then

$$\frac{y}{x} = \frac{z}{y} \qquad \text{where } x, y, z \neq 0$$

$$\implies \quad y^2 = xz$$

$$\implies \quad y = \pm\sqrt{xz} \qquad \text{where } x, z > 0$$

Thus, the geometric mean, y of two positive numbers x and z is given by

$$y = \sqrt{xz}$$

. **E.g.1** Insert 4 geometric means between 5 and 1215.

Solution

Let the means be A, B, C and D

$$\Longrightarrow \quad 5, \text{A}, \text{B}, \text{C}, \text{D}, 1215 \text{ form a GP}$$

$$\text{1st term}: \quad a = 5 \qquad (1)$$

$$\text{6th term}: \quad ar^5 = 1215 \qquad (2)$$

$$\text{From (2)}: \quad r^5 = \frac{1215}{a} = \frac{1215}{5}$$

$$\Longrightarrow \quad r^5 = 243$$

$$\Longrightarrow \quad r = 3$$

$$\Longrightarrow \quad \text{A} = 5(3) = 15$$

$$\text{B} = 5(3)^2 = 45$$

$$\text{C} = 5(3)^3 = 135$$

$$\text{D} = 5(3)^4 = 405$$

$$\therefore \quad \text{the required geometric means are } 15, 45, 135, 405$$

E.g.2 Insert two geometric means between 5 and 8.64.

Solution

Let the means be A and B

$$\Longrightarrow \quad 5, \text{A}, \text{B}, 8.64 \text{ form a GP}$$

$$\text{1st term}: \quad a = 5 \qquad (1)$$

$$\text{4th term}: \quad ar^3 = 8.64 \qquad (2)$$

$$\text{From (2)}: \quad r^3 = \frac{8.64}{a} = \frac{8.64}{5}$$

$$\Longrightarrow \quad r^3 = 1.728$$

$$\Longrightarrow \quad r = 1.2$$

$$\Longrightarrow \quad \text{A} = 5(1.2) = 6.0$$

$$\text{B} = 5(1.2)^2 = 7.2$$

$$\therefore \quad \text{the required geometric means are } 6.0 \text{ and } 7.2$$

Exercise

1. The geometric sequence $1, \frac{3}{2}, \frac{9}{4}, \cdots$ has a term equal to $\frac{243}{32}$. Find the number of terms.

Solution

$$a = 1, \;\; r = \frac{3}{2}, \;\; u_n = \frac{243}{32}$$
$$\implies \;\; u_n = ar^{n-1} = \frac{243}{32}$$
$$1\left(\frac{3}{2}\right)^{n-1} = \frac{243}{32}$$
$$\left(\frac{3}{2}\right)^{n-1} = \left(\frac{3}{2}\right)^5$$
$$\implies \;\; n-1 = 5$$
$$n = 6$$

2. For which values of k are $2k$, $5k+2$ and $20k-4$ consecutive terms of a geometric sequence?

Solution

Here, we apply the principle $r = \dfrac{u_{n+1}}{u_n}$

$$\implies \;\; r = \frac{u_2}{u_1} = \frac{u_3}{u_2}$$
$$\implies \;\; r = u_2^2 = u_1 \times u_3$$
$$(5k+2)^2 = 2k(20k-4)$$
$$25K^2 + 20k + 4 = 40k^2 - 8k$$
$$15k^2 - 28k - 4 = 0$$
$$(15k+2)(k-2) = 0$$
$$15k+2 = 0 \;\text{ or }\; k-2 = 0$$
$$\implies \;\; k = -\frac{2}{15} \;\text{ or }\; k = 2$$
$$\text{When } \; k = -\frac{2}{15}, \;\text{ we have}$$
$$2k = -\frac{4}{15}$$
$$5k+2 = \frac{4}{3}$$
$$20k-4 = -\frac{20}{3}$$

When $k=2$, we have

$$2k=4$$

$$5k+2=12$$

$$20k-4=36$$

$$\therefore \text{ the values of } k = \left\{k : k=-\frac{2}{15}, 2\right\}$$

6.2.7 SUM TO N TERMS OF A GEOMETRIC SEQUENCE

$$\text{Given } a, ar^2, ar^3, \cdots$$

$$\boldsymbol{S_n} = a+ar+ar^2+\cdots+ar^{n-1} \qquad (1)$$

$$r\boldsymbol{S_n} = ar+ar^2+ar^3+\cdots+ar^{n-1}+ar^n \qquad (2)$$

$$(1)-(2): \quad \boldsymbol{S_n} - r\boldsymbol{S_n} = a-ar^n$$

$$\Longrightarrow \boldsymbol{S_n}(1-r) = a(1-r^n)$$

$$\Longrightarrow \boldsymbol{S_n} = \frac{a(1-r^n)}{1-r}$$

NB: The formula for $\boldsymbol{S_n}$ is useful when r is a fraction between +1 and –1 but for values of r outside this range, the alternative form

$$\boldsymbol{S_n} = \frac{a(r^n-1)}{r-1}$$

could be used.

E.g.1 Find the sum of the first 8 terms of the series $8+4+2+1+\frac{1}{2}$.

<u>Solution</u>

$$a=8; \;\; r=\frac{4}{8}=\frac{2}{4}=\frac{1}{2}; \;\; n=8$$

$$\Longrightarrow \boldsymbol{S_n} = \frac{a(1-r^n)}{1-r}$$

$$\Longrightarrow \boldsymbol{S_8} = \frac{8\left[1-\left(\frac{1}{2}\right)^8\right]}{1-\frac{1}{2}}$$

$$= \frac{8\left(1-\frac{1}{256}\right)}{\frac{1}{2}} = \frac{255}{16}$$

E.g.2 Find the sum of the first 6 terms of the GP $\frac{1}{4}, \frac{1}{2}, 1, \ldots$

Solution

$$\begin{aligned}
a = \frac{1}{4};\;\; r = \frac{1}{1/2} = \frac{1/2}{1/4} = 2;\;\; n = 6 \\
\implies \boldsymbol{S_n} &= \frac{a(r^n - 1)}{r-1} \quad (r > 1) \\
\implies \boldsymbol{S_6} &= \frac{\frac{1}{4}\left(2^6 - 1\right)}{2-1} \\
&= \frac{1}{4}(63) = 15\frac{3}{4}
\end{aligned}$$

E.g.3 Find the sum of the first 10 terms of the series $12, -9, 6\frac{3}{4}, \ldots$

Solution

$$\begin{aligned}
a = 12;\;\; r = -\frac{3}{4};\;\; n = 10 \\
\implies \boldsymbol{S_n} &= \frac{a(1-r^n)}{1-r} \\
\implies \boldsymbol{S}_{10} &= \frac{12\left[1 - \left(-\frac{3}{4}\right)^{10}\right]}{1 - \left(-\frac{3}{4}\right)} \\
&= \frac{12(0.9436)}{1.75} = 6.47
\end{aligned}$$

E.g.4 Find the sum of the series $2 + 4 + 8 + 16 + \cdots$ to 6 terms

Solution

$$\begin{aligned}
a = 2;\;\; r = 2;\;\; n = 6 \\
\implies \boldsymbol{S_n} &= \frac{a(r^n - 1)}{r-1} \quad (r > 1) \\
\implies \boldsymbol{S_6} &= \frac{2\left(2^6 - 1\right)}{2-1} \\
&= \frac{2(32-1)}{1} = 62
\end{aligned}$$

E.g.5 Find the sum to 7 terms of the series $3-6+12-24+\cdots$

<u>**Solution**</u>

$$a=3;\;\; r=\frac{-6}{3}=\frac{12}{-6}=\frac{-24}{12}=-2;\;\; n=7$$
$$\implies \boldsymbol{S_n}=\frac{a(r^n-1)}{r-1} \quad (r<-1)$$
$$\implies \boldsymbol{S_7}=\frac{3\left[(-2)^7-1\right]}{-2-1}$$
$$=\frac{3(-129)}{-3}=129$$

Exercise

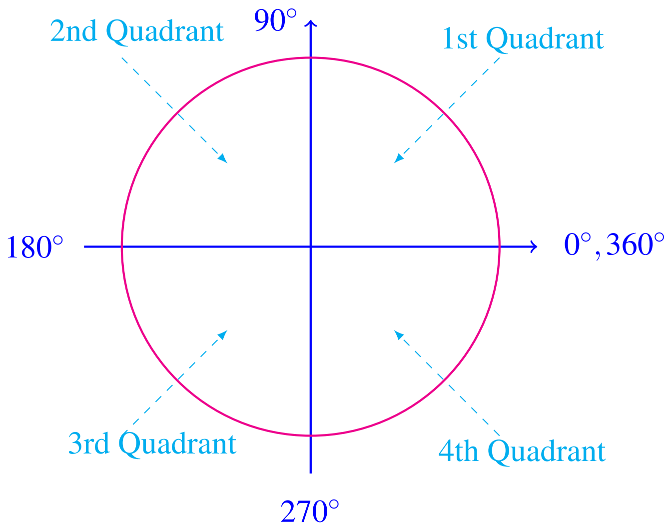
1. The 5th, 9th and 16th terms of a linear sequence are consecutive terms of an exponential sequence, GP. Find the common difference in terms of the first term.

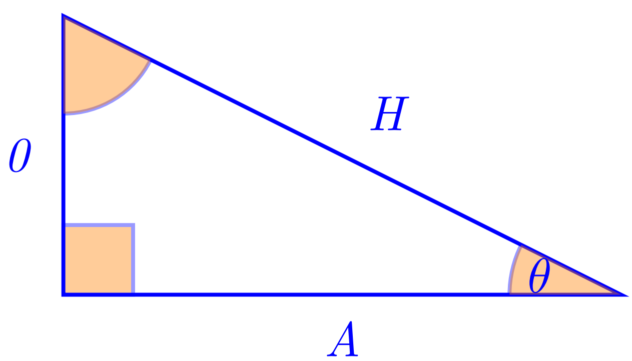
2. The 3rd and 6th terms of an exponential sequence are $\frac{1}{4}$ and $\frac{1}{32}$ respectively. Find the sum of the first 10 terms.

CHAPTER 7

TRIGONOMETRY

Trigonometry means "***measurement of triangles***". A positive angle measures a rotation in an anticlockwise direction.





7.1 TRIGONOMETRY FUNCTIONS

- $\sin\theta = \dfrac{0}{H}$
- $\cos\theta = \dfrac{A}{H}$
- $\tan\theta = \dfrac{0}{A} = \dfrac{\sin\theta}{\cos\theta}$
- $\cos^2\theta + \cos^2\theta = 1$

7.1.1 RADIANS AND ANGLES

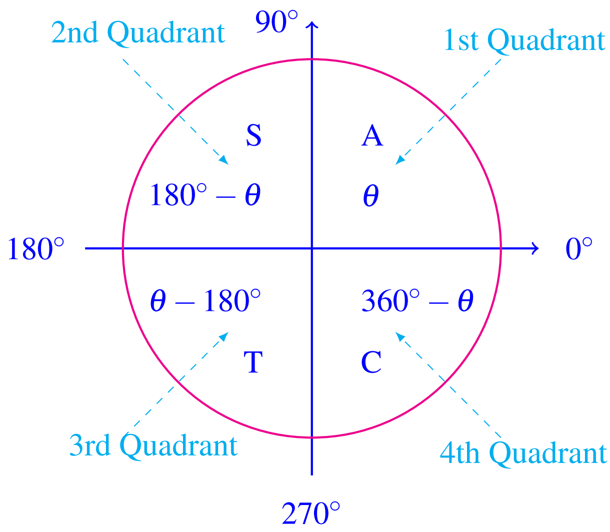
$$\pi\,\text{radian} = 180^\circ$$
$$1 \text{ radian} \;=\; \frac{180^\circ}{\pi} \;\text{ and }\; 1^\circ = \frac{\pi}{180^\circ}\text{radian}$$

To change from radian to degrees, multiply by $\frac{180}{\pi}$ and from degrees to radian multiply by $\frac{\pi}{180}$.

7.1.2 COMMON ANGLES

Angle in radians:	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Angle in degrees:	0	30°	45°	60°	90°	180°	270°	360°

7.1.3 EXPRESSING ALL OTHER ANGLES IN THE ACUTE ANGLE



- **1st Quadrant**: $+\sin\theta \quad +\cos\theta \quad +\tan\theta$
- **2nd Quadrant**:

$$\sin\theta = \sin(180^\circ - \theta)$$
$$\cos\theta = -\cos(180^\circ - \theta)$$
$$\tan\theta = -\tan(180^\circ - \theta)$$

- **3rd Quadrant**:

$$\sin\theta = -\sin(\theta - 180^\circ)$$
$$\cos\theta = -\cos(\theta - 180^\circ)$$
$$\tan\theta = +\tan(\theta - 180^\circ)$$

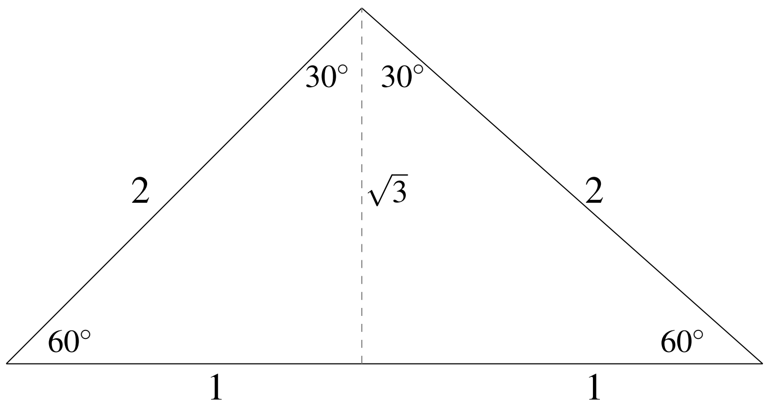
- **4th Quadrant**:

$$\sin\theta = -\sin(360^\circ - \theta)$$
$$\cos\theta = +\cos(360^\circ - \theta)$$
$$\tan\theta = -\tan(360^\circ - \theta)$$

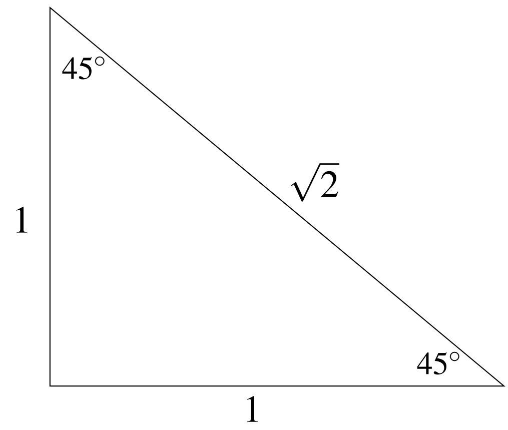
E.g. 1

$$\sin 150^\circ = \sin(180^\circ - 150^\circ) = \sin 30^\circ$$
$$\cos 150^\circ = -\cos(180^\circ - 150^\circ) = -\cos 30^\circ$$
$$\sin 240^\circ = -\sin(240^\circ - 180^\circ) = -\sin 60^\circ$$
$$\tan 300^\circ = -\tan(360^\circ - 300^\circ) = -\tan 60^\circ$$

7.1.4 TRIG RATIOS OF 30°, 45°, 60°



- $\sin 60^\circ = \frac{\sqrt{3}}{2}$
- $\sin 60^\circ = \frac{1}{2}$
- $\tan 60^\circ = \sqrt{3}$
- $\sin 30^\circ = \frac{1}{2}$
- $\cos 30^\circ = \frac{\sqrt{3}}{2}$
- $\tan 30^\circ = \frac{1}{\sqrt{3}}$



$\sin 45^\circ = \frac{1}{\sqrt{2}}$ $\qquad$ $\cos 45^\circ = \frac{1}{\sqrt{2}}$ $\qquad$ $\tan 45^\circ = 1$

7.1.5 SOME TRIG. IDENTITIES

NB : $\dfrac{1}{\sin\theta} = \csc\theta \quad \dfrac{1}{\cos\theta} = \sec\theta \quad \dfrac{1}{\tan\theta} = \cot\theta$

$$\sin^2\theta + \cos^2\theta = 1 \tag{1}$$

Dividing through (1) by $\sin^2\theta$ $\quad \left[\dfrac{\sin^2\theta}{\sin^2\theta} + \dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}\right]$

$$1 + \cot^2\theta = \csc^2\theta \tag{2}$$

Dividing through (1) by $\cos^2\theta$ $\quad \left[\dfrac{\sin^2\theta}{\cos^2\theta} + \dfrac{\cos^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}\right]$

$$\tan^2\theta + 1 = \sec^2\theta \tag{3}$$

7.1.6 DIFFERENCE OF TWO ANGLES

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

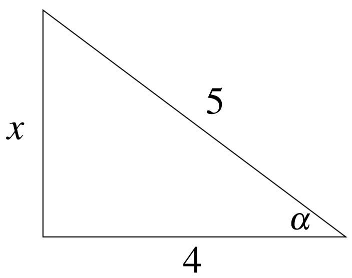
$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

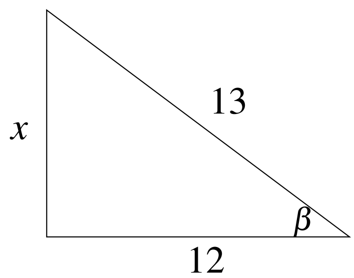
7.1.7 MULTIPLE ANGLES

$$
\begin{aligned}
\sin 2A = \sin(A+A) &= \sin A\cos A + \cos A\sin A \\
& 2\sin A\cos A \\
\cos 2A = \cos(A+A) &= \cos A\cos A - \sin A\sin A \\
&= \cos^2 A - \sin^2 A \qquad (\text{using } \sin^2\theta + \cos^2\theta = 1) \\
&= \cos^2 A - (1-\cos^2 A) \\
&= 2\cos^2 A - 1 \\
2\cos^2 A &= \cos 2A + 1 \\
\implies \quad \cos^2 A &= \frac{1}{2}[1+\cos 2A]
\end{aligned}
$$

$$
\begin{aligned}
\text{Also} \quad \cos 2A &= \cos^2 A - \sin^2 A \\
&= 1 - \sin^2 A - \sin^2 A \\
&= 1 - 2\sin^2 A \\
\implies \quad \sin^2 A &= \frac{1}{2}[1-\cos 2A]
\end{aligned}
$$

E.g. 1 If $\cos\alpha = \frac{4}{5}$ and $\cos\beta = \frac{12}{13}$, find the value of $\sin(\alpha-\beta)$





$$
\begin{aligned}
x^2 + 4^2 &= 5^2 \\
x^2 &= 25 - 26 \\
x^2 &= 9 \\
x &= 3
\end{aligned}
\qquad\qquad
\begin{aligned}
x^2 + 12^2 &= 13^2 \\
x^2 &= 169 - 144 \\
x^2 &= 25 \\
x &= 5
\end{aligned}
$$

$$
\begin{aligned}
\sin\alpha &= \frac{3}{5};\ \ \sin\beta = \frac{5}{13} \\
\Longrightarrow \quad \sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
&= \left[\frac{3}{5}\times\frac{12}{13}\right] - \left[\frac{4}{5}\times\frac{5}{13}\right] \\
&= \frac{16}{65}
\end{aligned}
$$

7.1.8 Exercise

1. Prove that

(i) $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{1-\cos\theta}{\sin\theta}$

(ii) $(1-\sin x)(1+\sin x) = \dfrac{1}{1+\tan^2 x}$

Solution

(i)

$$
\begin{aligned}
\frac{\sin\theta}{1+\cos\theta} &= \frac{\sin\theta}{1+\cos\theta}\times\frac{1-\cos\theta}{1-\cos\theta} \\
&= \frac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta} \\
&= \frac{\sin\theta(1-\cos\theta)}{\sin^2\theta} \\
&= \frac{1-\cos\theta}{\sin\theta}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
\text{LHS}: &\ \ (1-\sin x)(1+\sin x) = 1-\sin^2 x \\
\text{RHS}: &\ \ \frac{1}{1+\tan^2 x} = \frac{1}{\sec^2 x} = \cos^2 x
\end{aligned}
$$

2. Find the value of $\sin 15^\circ$, leaving answer in surd form.

Solution

$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$
$$= \left[\frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2}\right] - \left[\frac{1}{\sqrt{2}} \times \frac{\sqrt{1}}{2}\right]$$
$$\sin 15^\circ = \sin(45^\circ - 30^\circ) = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

7.1.9 TANGENTS OF COMPOUND ANGLES

1.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$$
$$= \frac{\sin A \cos A + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$
$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}}$$
$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}$$
$$= \frac{\sin A \cos A - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$
$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} - \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} + \frac{\sin A \sin B}{\cos A \cos B}}$$
$$= \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

2.

$$\sin 3A = \sin(2A + A)$$
$$= \sin 2A \cos A + \cos 2A \sin A$$
$$= (2\sin A \cos A)\cos A + (\cos^2 A - \sin^2 A)\sin A$$
$$= 2\sin A \cos^2 A + \sin A \cos^2 A - \sin^3 A$$
$$= 3\sin A \cos^2 A - \sin^3 A$$

3.

$$\begin{aligned}
\cos 3A &= \cos(2A+A) \\
&= \cos 2A \cos A - \sin 2A \sin A \\
&= (\cos^2 A - \sin^2 A)\cos A - (2\sin A \cos A)\sin A \\
&= \cos^3 A - \sin^2 A \cos A - 2\sin^2 A \cos A \\
&= \cos^3 A - 3\sin^2 A \cos A
\end{aligned}$$

4.

$$\begin{aligned}
\tan 3A &= \tan(2A+A) \\
&= \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A} \\
&= \frac{\left[\frac{2\tan A}{1-\tan^2 A}\right] + \tan A}{1 - \left[\frac{2\tan A}{1-\tan^2 A}\right]\tan A} \\
&= \frac{2\tan A + \tan A(1-\tan^2 A)}{1 - \tan^2 A - 2\tan A \tan A} \\
&= \frac{2\tan A + \tan A - \tan^3 A}{1 - \tan^2 A - 2\tan^2 A} \\
&= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}
\end{aligned}$$

7.1.10 Exercise

1. If $t = \tan\theta$, simplify the following

   (i) $\sqrt{1+t^2}$ (ii) $\frac{t}{\sqrt{1+t^2}}$ (iii) $t(1+t^2)$

Solution

(i)

$$\begin{aligned}\sqrt{1+t^2} &= \sqrt{1+\tan^2\theta}\\ &= \sqrt{\sec^2\theta}\\ &= \sec\theta\end{aligned}$$

(ii)

$$\begin{aligned}\frac{t}{\sqrt{1+t^2}} &= \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{\tan\theta}{\sqrt{\sec^2\theta}}\\ &= \frac{\tan\theta}{\sec\theta} = \frac{\sin\theta}{\cos\theta}.\frac{1}{\sec\theta}\\ &= \frac{\sin\theta}{\cos\theta}.\cos\theta\\ &= \sin\theta\end{aligned}$$

2. If $x = a\sin\theta$ and $y = b\sec\theta$, simplify

(i) $\dfrac{1}{\sqrt{a^2-x^2}}$ (ii) $\dfrac{\sqrt{y^2-b^2}}{y}$

Solution

(i)

$$\begin{aligned}\frac{1}{\sqrt{a^2-x^2}} &= \frac{1}{\sqrt{a^2-a^2\sin^2\theta}} = \frac{1}{\sqrt{a^2(1-\sin^2\theta)}}\\ &= \frac{1}{a\cos\theta}\end{aligned}$$

(ii)

$$\begin{aligned}\frac{\sqrt{y^2-b^2}}{y} &= \frac{\sqrt{b^2\sec^2\theta-b^2}}{b\sec\theta} = \frac{\sqrt{b^2(\sec^2\theta-1)}}{b\sec\theta}\\ &= \frac{\sqrt{\sec^2\theta-1}}{\sec\theta} \qquad (1)\\ \text{but}\quad \sec^2\theta &= 1+\tan^2\theta \quad\text{and}\quad \frac{1}{\sec\theta} = \cos\theta\\ \sec^2\theta-1 &= \tan^2\theta\\ \implies \frac{b\sqrt{\sec^2\theta-1}}{\sec\theta} &= b\tan\theta.\cos\theta\\ &= b\frac{\sin\theta}{\cos\theta}.\cos\theta\\ &= b\sin\theta\end{aligned}$$

3. If $x = \sin\theta$, $y = \cos\theta$ and $z = \tan\theta$, find:

(i) $\dfrac{1}{\sqrt{1-x^2}}$ (ii) $-\dfrac{1}{\sqrt{1-y^2}}$ (iii) $\dfrac{1}{1+z^2}$

Solution

$$\text{(i)}\quad \frac{1}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-\sin^2\theta}}=\frac{1}{\sqrt{\cos^2\theta}}=\frac{1}{\cos\theta}=\sec\theta$$

$$\text{(ii)}\quad -\frac{1}{\sqrt{1-y^2}}=-\frac{1}{\sqrt{1-\cos^2\theta}}=-\frac{1}{\sqrt{\sin^2\theta}}=-\frac{1}{\sin\theta}=-\csc\theta$$

$$\text{(iii)}\quad \frac{1}{1+z^2}=\frac{1}{1+\tan^2\theta}=\frac{1}{\sec^2\theta}=\cos^2\theta$$

4. Solve the equation below

$$2\sin\theta-3\cos\theta=0 \qquad 0^\circ<\theta<360^\circ$$

Solution

$$\begin{aligned}
2\sin\theta-3\cos\theta&=0\\
2\frac{\sin\theta}{\cos\theta}-3\frac{\cos\theta}{\cos\theta}&=\frac{0}{\cos\theta}\\
2\tan\theta-3&=0\\
\implies \tan\theta&=\frac{3}{2}\\
\therefore\quad \theta&=56.3^\circ,\ 236.3^\circ
\end{aligned}$$

5. If $\tan(A-B)=\frac{1}{5}$ and $\tan A=2$, find the value of $\tan B$.

Solution

$$\begin{aligned}
\tan(A-B)&=\frac{\tan A-\tan B}{1+\tan A\tan B}\\
&=\frac{2-\tan B}{1+2\tan B}=\frac{1}{5}\\
\implies 5(2-\tan B)&=1+2\tan B\\
10-5\tan B&=1+2\tan B\\
7\tan B&=9\\
\tan B&=\frac{9}{7}
\end{aligned}$$

7.1.11 HALF ANGLES

$$
\begin{aligned}
\sin 2A &= 2\sin A\cos A \\
\implies \quad \sin A &= 2\sin\frac{A}{2}\cos\frac{A}{2} \\
\cos 2A &= \cos^2 A - \sin^2 A \\
\implies \quad \cos A &= \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \\
&= 1 - 2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2} - 1 \\
\tan 2A &= \frac{2\tan A}{1-\tan^2 A}
\end{aligned}
$$

Expressing Half-Angles In Terms of Tangent

$$
\begin{aligned}
\sin A &= 2\sin\frac{A}{2}\cos\frac{A}{2} \\
&= \frac{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}{\sin^2\left(\frac{A}{2}\right)+\cos^2\left(\frac{A}{2}\right)} \\
&= \frac{\frac{2\sin(\frac{A}{2})\cos(\frac{A}{2})}{\cos^2(\frac{A}{2})}}{\frac{\sin^2(\frac{A}{2})}{\cos^2(\frac{A}{2})}+\frac{\cos^2(\frac{A}{2})}{\cos^2(\frac{A}{2})}} \\
&= \frac{2\frac{\sin(\frac{A}{2})}{\cos(\frac{A}{2})}}{\tan^2(\frac{A}{2})+1} = \frac{2t}{1+t^2}
\end{aligned}
$$

$$
\begin{aligned}
\cos A &= \cos^2\frac{A}{2} - \sin^2\frac{A}{2} \\
&= \frac{\cos^2\left(\frac{A}{2}\right)-\sin^2\left(\frac{A}{2}\right)}{\sin^2\left(\frac{A}{2}\right)+\cos^2\left(\frac{A}{2}\right)} \\
&= \frac{\frac{\cos^2(\frac{A}{2})}{\cos^2(\frac{A}{2})}-\frac{\sin^2(\frac{A}{2})}{\cos^2(\frac{A}{2})}}{\frac{\sin^2(\frac{A}{2})}{\cos^2(\frac{A}{2})}+\frac{\cos^2(\frac{A}{2})}{\cos^2(\frac{A}{2})}} \\
&= \frac{1-t^2}{1+t^2}
\end{aligned}
$$

$$
\tan A = \frac{2\tan(\frac{A}{2})}{1-\tan^2\left(\frac{A}{2}\right)} = \frac{2t}{1-t^2}
$$

7.1.12 FACTOR FORMULA

$$
\begin{aligned}
\sin A+\sin B &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\
\sin A-\sin B &= 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)\\
\cos A+\cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)\\
\cos A-\cos B &= -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)
\end{aligned}
$$

E.g. Express $\sin 4\theta+\sin\theta$ as a factor.

Solution

$$
\begin{aligned}
\sin A+\sin B &= 2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)\\
\implies \quad \sin 4\theta+\sin\theta &= 2\sin\frac{1}{2}(5\theta)\cos\frac{1}{2}(3\theta)\\
&= 2\sin\frac{5\theta}{2}\cos\frac{3\theta}{2}
\end{aligned}
$$

7.1.13 PARAMETRIC EQUATIONS

E.g. Eliminate θ from the following equations:

(a) $x=3\cos(\theta)-5$ and $y=3+2\sin\theta$

(b) $x=a\tan\theta$ and $y=b\cos\theta$

Solution

(a)

$$
\begin{aligned}
\cos\theta &= \frac{x+5}{3} \text{ and } \sin\theta=\frac{y-3}{2}\\
\implies \quad \cos^2\theta+\sin^2\theta &= \left(\frac{x+5}{3}\right)^2+\left(\frac{y-3}{2}\right)^2=1
\end{aligned}
$$

(b)

$$
\begin{aligned}
\frac{x}{a} = \tan\theta \ \text{ and } \ \frac{y}{b} = \cos\theta & \\
\implies \frac{1}{\cos\theta} = \sec\theta = \frac{b}{y} & \\
\text{But } \ \tan^2\theta + 1 = \sec^2\theta & \\
\left(\frac{x}{a}\right)^2 + 1 = \left(\frac{b}{y}\right)^2 & \\
\implies \ \frac{x^2}{a^2} + 1 = \frac{b^2}{y^2} &
\end{aligned}
$$

CHAPTER 8

LIMITS

8.1 DEFINITION

The limit of a function $f(x)$ as x tends to a is L. This means that, as x gets closer and closer to a, $f(x)$ gets closer and closer to L. Thus

$$\lim_{x \to a} f(x) = L$$

If a real-valued function of a real variable is defined by a formula, and if no domain is stated explicitly, then it is to be understood that the domain consists of all real numbers for which the formula yields a real value called the ***natural domain*** of the function.

E.g.1: $f(x) = x^3$
The function f has real values for all real x, so its natural domain is the interval $(-\infty, +\infty)$.

E.g.2: $f(x) = \dfrac{1}{(x-1)(x-3)}$
The function f has real values for all real x, except $x = 1$ and $x = 3$ where divisions by zero occur. Therefore the natural domain = $\{x : x \neq 1 \text{ and } x \neq 3\}$.

E.g.3: $f(x) = \dfrac{x^2-4}{x-2}$
The function f consists of all real x, except $x = 2$. However, if we factorize the numerator and then cancel the common factor in the numerator and denominator, we obtain

$$f(x) = \frac{(x-2)(x+2)}{x-2} = x+2$$

which is defined at $x = 2$ since $f(2) = 4$ for the altered function f.
NB: The algebraic simplification of a given function (if possible) may alter the domain of the

function.

8.2 CONTINUITY

If $f(x)$ is continuous at a point $x = a$, then

(i) $\lim_{x \to a} f(x)$ exists and

(ii) $\lim_{x \to a} f(x) = f(a)$

E.g. Check the following points of discontinuities:

(a) $f(x) = \dfrac{x^2-4}{x+2} \qquad (x \neq -2)$

(b) $f(x) = \dfrac{x^2-1}{x-1} \qquad (f(x) \text{ is not defined at } x = 1)$

(c) $f(x) = \dfrac{3x+3}{x^2-3x-4} = \dfrac{3x+3}{(x-4)(x+1)} \qquad (f(x) \text{ is discontinuous at } x = 4 \text{ and } x = -1)$

(d) $\displaystyle\int_1^5 \frac{1}{x-2} = ?$

(e) $f(x)\dfrac{x^2-16}{x-4} = ?$

(f) $f(x)\dfrac{\sqrt{7x+2}-\sqrt{6x+4}}{x-2} = ?$

(g) $f(x)\dfrac{3}{x-4} = ?$

(h) $f(x)\dfrac{x^2-b^2}{x-b} = ?$

8.2.1 Exercise

Find the domain of the following functions:

(i) $f(x) = \dfrac{1}{x-3}$

(ii) $f(x) = \dfrac{x^2-1}{x+1}$

(iii) $g(x) = \dfrac{1}{1-\sin x}$

(iv) $f(x) = \dfrac{3}{2-\cos x}$

(v) $f(x) = \dfrac{x+1}{x-1}$

(vi) $f(x) = 3x^2 - 2$

(vii) $f(x) = \dfrac{3}{x}$

(viii) $f(x) = \dfrac{\sqrt{x^2-4}}{x-2}$

8.2.2 Worked Examples

1. Find $\lim_{x\to 2} \frac{x^2-4}{x-2}$

Solution

Since both the numerator and the denominator approaches 0, we have

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-4}{x-2} &= \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2}(x+2) = 4\end{aligned}$$

2. Find $\lim_{x\to 3/2} \frac{2x^2+5x-12}{2x-3}$

Solution

Since both the numerator and the denominator approaches 0, we have

$$\begin{aligned}\lim_{x\to 3/2} \frac{2x^2+5x-12}{2x-3} &= \lim_{x\to 3/2} \frac{(2x-3)(x+4)}{2x-3} \\ &= \lim_{x\to 3/2}(x+4) \\ &= \frac{11}{2} = 5\frac{1}{2}\end{aligned}$$

3. Find $\lim_{x\to 4} \frac{x^2-x-12}{x-4}$

Solution

Since both the numerator and the denominator approaches 0, we have

$$\begin{aligned}\lim_{x\to 4} \frac{x^2-x-12}{x-4} &= \lim_{x\to 4} \frac{(x+3)(x-4)}{x-4} \\ &= \lim_{x\to 4}(x+3) \\ &= 7\end{aligned}$$

4. Find $\lim_{x\to 2} \frac{x^3-5x^2+2x-4}{x^2-3x+3}$

Solution

Since neither the numerator nor the denominator approaches 0, we have

$$\lim_{x\to 2}\frac{x^3-5x^2+2x-4}{x^2-3x+3}=\frac{\lim_{x\to 2}x^3-5x^2+2x-4}{\lim_{x\to 2}x^2-3x+3}$$
$$=-12$$

5. Find $\lim_{x\to 0}\frac{5x^2-4x}{x}$

Solution

Since neither the numerator nor the denominator approaches 0, we have

$$\lim_{x\to 0}\frac{5x^2-4x}{x}=\lim_{x\to 0}\frac{x(5x-4)}{x}$$
$$=\lim_{x\to 0}(5x-4)=-4$$

6. Find $\lim_{x\to 0}\frac{6x^2-5}{x+2}$

Solution

As x approaches 0, we have

$$\lim_{x\to 0}\frac{6x^2-5}{x+2}=-\frac{5}{2}=-2\frac{1}{2}$$

7. Find $\lim_{x\to 0}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$

Solution

As x approaches 0, we have

$$\lim_{x\to 0}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)=\frac{1}{-2}-\frac{4}{-4}$$
$$=-\frac{1}{2}+1=\frac{1}{2}$$

8. Find $\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$

Solution

As x approaches 2, both $\dfrac{1}{x-2}$ and $\dfrac{4}{x^2-4}$ become infinitely large in magnitude and therefore we try to simplify the given function as follows:

$$\begin{aligned}
\frac{1}{x-2}-\frac{4}{x^2-4} &= \frac{x+2-4}{(x+2)(x-2)} = \frac{x-2}{(x+2)(x-2)} = \frac{1}{x+2} \\
\implies \lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right) &= \lim_{x\to 2}\frac{1}{x+2} \\
&= \frac{1}{4}
\end{aligned}$$

9. Find $\lim\limits_{x\to 4}\left(\dfrac{1}{x-4}-\dfrac{8}{x^2-16}\right)$

Solution

As x approaches 4, we have

$$\begin{aligned}
\lim_{x\to 4}\left(\frac{1}{x-4}-\frac{8}{x^2-16}\right) &= \lim_{x\to 4}\left[\frac{x+4-8}{(x+4)(x-4)}\right] \\
&= \lim_{x\to 4}\left[\frac{x-4}{(x+4)(x-4)}\right] = \frac{1}{8}
\end{aligned}$$

10. Find $\lim\limits_{x\to -3}\dfrac{x+2}{x-3}$

Solution

As x approaches 4, we have

$$\lim_{x\to -3}\frac{x+2}{x-3} = \frac{-3+2}{-3-3} = \frac{1}{6}$$

Now, let us consider situations in which we may have to factorize out the highest power of the given variable:

1. Evaluate $\lim\limits_{x\to +\infty}(2x^{11}-5x^6+3x^2+1)$

Solution

$$2x^{11} - 5x^6 + 3x^2 + 1 = x^{11}\left(2 - \frac{5}{x^5} + \frac{3}{x^9} + \frac{1}{x^{11}}\right)$$

$$\text{But } \frac{5}{x^5}, \frac{3}{x^9}, \text{and} \frac{1}{x^{11}} \text{ all approach 0 as } x+ \to \infty$$

$$\text{Thus as } x \to +\infty,\ 2 - \frac{5}{x^5} + \frac{3}{x^9} + \frac{1}{x^{11}} \text{ approach 2}$$

At the same time x approaches $+\infty$. Hence the limit is $+\infty$

$$\therefore \lim_{x \to +\infty}(2x^{11} - 5x^6 + 3x^2 + 1) = +\infty$$

2. Find $\lim_{x \to +\infty}(2x^3 - 12x^2 + x - 7)$

Solution

$$2x^3 - 12x^2 + x - 7 = x^3\left(2 - \frac{12}{x} + \frac{1}{x^2} - \frac{7}{x^3}\right) \tag{1}$$

$$\text{As } x \to +\infty\ \frac{12}{x}, \frac{1}{x^2}, \text{and } \frac{7}{x^3} \text{ all approach 0}$$

$$\text{Hence } \left(2 - \frac{12}{x} + \frac{1}{x^2} - \frac{7}{x^3}\right) \text{ approaches 2}$$

But from (1) x^3 approaches $+\infty$

$$\implies \lim_{x \to +\infty}(2x^3 - 12x^2 + x - 7) = +\infty$$

3. Evaluate $\lim_{x \to -\infty}(2x^3 - 12x^2 + x - 7)$

Solution

$$2x^3 - 12x^2 + x - 7 = x^3\left(2 - \frac{12}{x} + \frac{1}{x^2} - \frac{7}{x^3}\right) \tag{1}$$

$$\text{As } x \to -\infty\ \frac{12}{x}, \frac{1}{x^2}, \text{and } \frac{7}{x^3} \text{ all approach 0}$$

$$\text{Hence } \left(2 - \frac{12}{x} + \frac{1}{x^2} - \frac{7}{x^3}\right) \text{ approaches 2}$$

But from (1) x^3 approaches $-\infty$

$$\implies \lim_{x \to -\infty}(2x^3 - 12x^2 + x - 7) = -\infty$$

4. Find $\lim_{x \to -\infty}(3x^4 - x^2 + x - 7)$

Solution

$$3x^4 - x^2 + x - 7 = x^4\left(3 - \frac{1}{x^2} + \frac{1}{x^3} - \frac{7}{x^4}\right)$$

As $x \to -\infty$ $\frac{12}{x}, \frac{1}{x^2}$, and $\frac{7}{x^3}$ all approach 0

Hence $\left(3 - \frac{1}{x^2} + \frac{1}{x^3} - \frac{7}{x^4}\right)$ approaches 3

But x^4 approaches $-\infty$

$$\implies \lim_{x\to-\infty}(3x^4 - x^2 + x - 7) = -\infty$$

8.3 Another Type of Limit Evaluations

1. Evaluate $\lim_{x\to\infty} \frac{2x+5}{x^2-7x+3}$

Solution

Both the numerator and the denominator approach $+\infty$. Hence, we divide the numerator and the denominator by the highest power of x in the denominator. In this case the highest power of x in the denominator is x^2.

$$\begin{aligned}
\lim_{x\to+\infty} \frac{2x+5}{x^2-7x+3} &= \lim_{x\to+\infty}\left[\frac{\frac{2x}{x^2} + \frac{5}{x^2}}{\frac{x^2}{x^2} - \frac{7x}{x^2} + \frac{3}{x^2}}\right] \\
&= \lim_{x\to+\infty}\left[\frac{\frac{2}{x} + \frac{5}{x^2}}{1 - \frac{7}{x} + \frac{3}{x^2}}\right] \\
&= \lim_{x\to+\infty}\left(\frac{0+0}{1-0+0}\right) = 0
\end{aligned}$$

2. Find $\lim_{x\to+\infty} \frac{3x^3-4x+2}{7x^3+5}$

Solution

As $x \to \infty$, both the numerator and the denominator approach $+\infty$. So we divide both the numerator and denominator by the highest power of x in the denominator (i.e. x^3).

$$\begin{aligned}
\lim_{x\to\infty} \frac{3x^3-4x+2}{7x^3+5} &= \lim_{x\to+\infty}\left[\frac{\frac{3x^3}{x^3}-\frac{4x}{x^3}+\frac{2}{x^3}}{\frac{7x^3}{x^3}+\frac{5}{x^3}}\right] \\
&= \lim_{x\to+\infty}\left[\frac{3-\frac{4}{x^2}+\frac{2}{x^3}}{7+\frac{5}{x^3}}\right] \\
&= \lim_{x\to+\infty}\left(\frac{3-0+0}{7+0}\right) = \frac{3}{7}
\end{aligned}$$

3. Find $\lim\limits_{x\to+\infty} \dfrac{4x^3-1}{3x^3+7}$

Solution

As $x \to +\infty$, both the numerator and the denominator approach $+\infty$. So we divide both the numerator and denominator by the highest power of x in the denominator (i.e. x^3).

$$\begin{aligned}
\lim_{x\to+\infty} \frac{4x^3-1}{3x^3+7} &= \lim_{x\to+\infty}\left[\frac{\frac{4x^5}{x^3}-\frac{1}{x^3}}{\frac{3x^3}{x^3}+\frac{7}{x^3}}\right] \\
&= \lim_{x\to+\infty}\left[\frac{4x^2-\frac{1}{x^3}}{3+\frac{7}{x^3}}\right] \\
&= \lim_{x\to+\infty}\left(\frac{4x^2-0}{3+0}\right) \\
&= \frac{4x^2}{3} = +\infty
\end{aligned}$$

Now, let us consider cases in which the function contains a square root sign either in the denominator (***Type A***) or the numerator (***Type B***).

Type A: (When the square root is in the denominator)

1. Find $\lim\limits_{x\to\infty} \dfrac{4x-1}{\sqrt{x^2+2}}$

Solution

As $x \to +\infty$, both the numerator and the denominator approach $+\infty$. So we divide both the given expression by the highest power of x (i.e. $\sqrt{x^2}$).

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x-1}{\sqrt{x^2+2}} &= \lim_{x\to+\infty} \left[\frac{(4x-1)/\sqrt{x^2}}{(\sqrt{x^2+2})/\sqrt{x^2}} \right] \\
&= \lim_{x\to+\infty} \left[\frac{\frac{4x}{\sqrt{x^2}} - \frac{1}{\sqrt{x^2}}}{\sqrt{\frac{x^2+2}{x^2}}} \right] \\
&= \lim_{x\to+\infty} \left(\frac{4-\frac{1}{x}}{\sqrt{1+\frac{2}{x^2}}} \right) \\
&= \frac{4-0}{\sqrt{1+0}} = 4
\end{aligned}$$

2. Find $\lim\limits_{x\to\infty} \dfrac{7x-4}{\sqrt{x^3+5}}$

Solution

As $x \to +\infty$, both the numerator and the denominator approach $+\infty$. So we divide both the given expression by the highest power of x (i.e. $\sqrt{x^{3/2}}$).

$$\begin{aligned}
\lim_{x\to\infty} \frac{7x-4}{\sqrt{x^3+5}} &= \lim_{x\to+\infty} \left[\frac{(7x-4)/\sqrt{x^3}}{(\sqrt{x^3+5})/\sqrt{x^3}} \right] \\
&= \lim_{x\to+\infty} \left[\frac{\frac{7}{\sqrt{x}} - \frac{4}{\sqrt{x^3}}}{\sqrt{\frac{x^3+5}{x^3}}} \right] \\
&= \lim_{x\to+\infty} \left(\frac{\frac{7}{\sqrt{x}} - \frac{4}{\sqrt{x^3}}}{\sqrt{1+\frac{5}{x^3}}} \right) \\
&= \frac{0-0}{\sqrt{1+0}} = 0
\end{aligned}$$

3. Find $\lim\limits_{x\to-\infty} \dfrac{3x^3+2}{\sqrt{x^4-2}}$

Solution

As $x \to +\infty$, both the numerator and the denominator approach $+\infty$. So we divide both the given expression by the highest power of x (i.e. $\sqrt{x^4} = x^2$).

$$
\begin{aligned}
\lim_{x \to -\infty} \frac{3x^3+2}{\sqrt{x^4-2}} &= \lim_{x \to -\infty} \left[\frac{\frac{3x^3}{x^2} + \frac{2}{x^2}}{(\sqrt{x^4-2})/\sqrt{x^4}} \right] \\
&= \lim_{x \to -\infty} \left[\frac{3x + \frac{2}{x^2}}{\sqrt{\frac{x^4-2}{x^4}}} \right] \\
&= \lim_{x \to -\infty} \left(\frac{3x + \frac{2}{x^2}}{\sqrt{1 - \frac{2}{x^4}}} \right) \\
&= \lim_{x \to -\infty} \left(\frac{3x+0}{\sqrt{1-0}} \right) \\
&= \lim_{x \to -\infty} 3x = -\infty
\end{aligned}
$$

Type B: (When the square root is in the numerator)

1. Evaluate $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+5}}{3x^2-2}$.

Solution

As $x \to +\infty$, both the numerator and the denominator approach $+\infty$. So we divide both the given expression by the highest power of x (i.e. $\sqrt{x^4} = x^2$).

$$
\begin{aligned}
\lim_{x \to \infty} \frac{\sqrt{x^2+5}}{3x^2-2} &= \lim_{x \to +\infty} \left[\frac{(\sqrt{x^2+5})/\sqrt{x^4}}{\frac{3x^2}{x^2} - \frac{2}{x^2}} \right] \\
&= \lim_{x \to +\infty} \left[\frac{\sqrt{\frac{x^2+5}{x^4}}}{\frac{3x^2}{x^2} - \frac{2}{x^2}} \right]
\end{aligned}
$$

$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{3x^2-2} = \lim_{x\to+\infty}\left[\frac{\sqrt{\frac{1}{x^2}+\frac{5}{x^4}}}{\frac{3x^2}{x^2}-\frac{2}{x^2}}\right]$$

$$= \lim_{x\to+\infty}\left[\frac{\sqrt{\frac{1}{x^2}+\frac{5}{x^4}}}{3-\frac{2}{x^2}}\right]$$

$$= \lim_{x\to+\infty} \frac{\sqrt{0+0}}{3-0} = \frac{0}{3} = 0$$

2. Evaluate $\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{3x^2-2}$.

Solution

As $x \to +\infty$, both the numerator and the denominator approach $+\infty$. So we divide both the given expression by the highest power of x (i.e. $\sqrt{x^4} = x^2$).

$$\lim_{x\to\infty} \frac{\sqrt{x^2+5}}{3x^2-2} = \lim_{x\to+\infty}\left[\frac{(\sqrt{x^2+5})/\sqrt{x^4}}{\frac{3x^2}{x^2}-\frac{2}{x^2}}\right]$$

$$= \lim_{x\to+\infty}\left[\frac{\sqrt{\frac{x^2+5}{x^4}}}{\frac{3x^2}{x^2}-\frac{2}{x^2}}\right]$$

$$= \lim_{x\to+\infty}\left[\frac{\sqrt{\frac{1}{x^2}+\frac{5}{x^4}}}{\frac{3x^2}{x^2}-\frac{2}{x^2}}\right]$$

$$= \lim_{x\to+\infty}\left[\frac{\sqrt{\frac{1}{x^2}+\frac{5}{x^4}}}{3-\frac{2}{x^2}}\right]$$

$$= \lim_{x\to+\infty} \frac{\sqrt{0+0}}{3-0} = \frac{0}{3} = 0$$

3. Evaluate $\lim_{x\to\infty} \frac{\sqrt{x^2-8}}{5x^2+4}$.

Solution

As $x \to +\infty$, both the numerator and the denominator approach $+\infty$. So we divide both the given expression by the highest power of x (i.e. $\sqrt{x^4} = x^2$).

$$\lim_{x\to\infty} \frac{\sqrt{x^2-8}}{5x^2+4} = \lim_{x\to+\infty} \left[\frac{(\sqrt{x^2-8})/\sqrt{x^4}}{\frac{5x^2}{x^2} - \frac{4}{x^2}} \right]$$

$$= \lim_{x\to+\infty} \left[\frac{\sqrt{\frac{x^2-8}{x^4}}}{5-\frac{4}{x^2}} \right]$$

$$= \lim_{x\to+\infty} \left[\frac{\sqrt{\frac{1}{x^2}+\frac{8}{x^4}}}{5-\frac{4}{x^2}} \right]$$

$$= \lim_{x\to+\infty} \frac{\sqrt{0-0}}{5-0} = \frac{0}{5} = 0$$

Exercise

Evaluate the following

(i) $\lim\limits_{x\to-\infty} \dfrac{3x-4}{x^2-100x}$

(ii) $\lim\limits_{x\to-\infty} \dfrac{6x+2}{\sqrt{x^2-3}}$

(iii) $\lim\limits_{x\to\infty} \dfrac{2x+5}{\sqrt[3]{x^3-2}}$

(iv) $\lim\limits_{x\to\infty} \dfrac{8x-5}{\sqrt{x^3+4}}$

(v) $\lim\limits_{x\to-\infty} \dfrac{4x^3+6}{\sqrt{x^4-9}}$

(vi) $\lim\limits_{x\to 1} \dfrac{x^3-1}{x-1}$

(vii) $\lim\limits_{x\to\infty} \dfrac{\sqrt{7x+6}}{x^4-5}$

8.4 L'HOPITALS RULE

L'Hopitals' rule holds for type $\frac{0}{0}$ and type $\frac{\infty}{\infty}$. Thus if $f(x)$ and $g(x)$ both tend to ∞ or as x tends to a, then the ratio $\dfrac{f'(x)}{g'(x)}$ approaches a limit or tends to infinity. Thus

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

where $f(a)$ and $g(a)$ are both zero and $f'(x)$ and $g'(x)$ approach a limit or tends to infinity.

Generally,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}\cdots\lim_{x\to a}\frac{f^n(x)}{g^n(x)}$$

Example:

1. Evaluate $\lim_{x\to 1}\frac{x^2+5x-2}{x^2+1}$.

Solution

Using L'Hopitals' rule, we have

$$\lim_{x\to 1}\frac{x^2+5x-2}{x^2+1} = \lim_{x\to 1}\frac{2x+5}{2x} = \lim_{x\to 1}\frac{2}{2} = 1$$

2. Find $\lim_{x\to 1}\left\{\frac{x^3+x^2-x-1}{x^2+2x-3}\right\}$.

Solution

We note that if we substitute $x = 1$, we get the indeterminate form $\frac{0}{0}$. Therefore, we apply the L'Hopiital's rule.

$$\begin{aligned}\lim_{x\to 1}\left\{\frac{x^3+x^2-x-1}{x^2+2x-3}\right\} &= \lim_{x\to 1}\left\{\frac{3x^2+2x-1}{2x+2}\right\}\\ &= \frac{3(1)^2+2(1)-1}{2(1)+2}\\ &= \frac{3+2-1}{2+2} = \frac{4}{4} = 1\end{aligned}$$

3. Determine $\lim_{x\to 0}\left[\frac{x^2-\sin 3x}{x^2+4x}\right]$.

Solution

Since direct substitution gives $\frac{0}{0}$, so we apply the L'Hopital's rule.

$$\implies \lim_{x\to 0}\left[\frac{x^2-\sin 3x}{x^2+4x}\right] = \lim_{x\to 0}\left[\frac{2x-3\cos 3x}{2x+4}\right]$$
$$= \frac{0-3}{0+4}$$
$$= -\frac{3}{4}$$

4. Determine $\lim_{x\to 0}\left[\frac{x-\sin x}{x^2}\right]$.

Solution

Since direct substitution gives $\frac{0}{0}$, so we apply the L'Hopital's rule.

$$\implies \lim_{x\to 0}\left[\frac{x-\sin x}{x^2}\right] = \lim_{x\to 0}\left[\frac{1-\cos x}{2x}\right]$$

Substituting $x=0$ in the derivatives produces another indeterminate form $\frac{0}{0}$, so we apply the L'Hopital's rule a second time.

$$\implies \lim_{x\to 0}\left[\frac{x-\sin x}{x^2}\right] = \lim_{x\to 0}\left[\frac{1-\cos x}{2x}\right]$$
$$= \lim_{x\to 0}\left[\frac{\sin x}{2}\right] = 0$$

NB:

- For limiting values, when the indeterminate form $\frac{0}{0}$ exists, apply the L'Hopitals rule and continue to apply the rule until a stage is reached where the limits of the numerator and the denominator are not simultaneously zero.
- Thus, we need to apply L'Hopital's rule again and again until we reach a stage where the limits of the numerator and the denominator are not simultaneously zero to arrive at the definite limiting value of the function.

8.4.1 Exercise

1. Evaluate $\lim_{x\to 0}\left[\frac{x\cos x-\sin x}{x^3}\right]$.

Solution

Since direct substitution gives $\frac{0}{0}$, we apply the L'Hopital's rule.

$$\begin{aligned}\implies \lim_{x\to 0}\left[\frac{x\cos x-\sin x}{x^3}\right] &= \lim_{x\to 0}\left[\frac{-x\sin x+\cos x-\cos x}{3x^2}\right]\\ &= \lim_{x\to 0}\left[\frac{-x\sin x}{3x^2}\right]\\ &= \lim_{x\to 0}\left[\frac{-\sin x}{3x}\right]\\ &= \lim_{x\to 0}\left[\frac{-\cos x}{3}\right] = -\frac{1}{3}\end{aligned}$$

2. Evaluate $\lim_{x\to 0}\left[\frac{\tan x-x}{\sin x-x}\right]$.

Solution

Since direct substitution gives $\frac{0}{0}$, we apply the L'Hopital's rule.

$$\begin{aligned}\implies \lim_{x\to 0}\left[\frac{\tan x-x}{\sin x-x}\right] &= \lim_{x\to 0}\left[\frac{\sec^2 x-1}{\cos x-1}\right]\\ &= \lim_{x\to 0}\left[\frac{2\sec^2 x\tan^2 x}{-\sin x}\right]\\ &= \lim_{x\to 0}\left[\frac{2\sec^2 x\sec^2 x+4\sec^2 x\tan^2 x}{-\cos x}\right]\\ &= \frac{2+0}{-1} = -2\end{aligned}$$

3. Use L'Hopital rule if possible:

(a) $\lim_{x\to 0}\left[\frac{x^3-2x^2+4x-3}{4x^2-5x+1}\right]$

(b) $\lim_{x\to 0}\left[\frac{e^{x^2}}{x^2}\right]$

(c) $\lim_{x\to \infty}\left[\frac{4x}{3x^2-4}\right]$

(d) $\lim_{x\to 0}\left[\frac{2x\cos x}{x+1}\right]$

(e) $\lim\limits_{x\to 3}\left[\dfrac{1}{x-3}-\dfrac{6}{x^2-9}\right]$

(f) $\lim\limits_{x\to 2}\left[\dfrac{2}{x-2}-\dfrac{8}{x^2-4}\right]$

(g) $\lim\limits_{x\to\infty}\left[\dfrac{\sqrt{2x^4-1}}{x^3+2}\right]$

(h) $\lim\limits_{x\to 3}\left[\dfrac{x-3}{\sqrt{x^2-9}}\right]$

(i) $\lim\limits_{x\to 0}\left[\dfrac{1-e^{x^3}}{x^3}\right]$

(j) $\lim\limits_{x\to 0}\left[\dfrac{\sin x+x}{x+x^2}\right]$

(k) $\lim\limits_{x\to 0}\left[\dfrac{\tan x-x}{x-\sin x}\right]$

(l) $\lim\limits_{x\to 0}\left[\dfrac{\tan x-\sin x}{x^3}\right]$

(m) $\lim\limits_{x\to 0}\left[\dfrac{1-2\sin^2 x-\cos^3 x}{5x^2}\right]$

(n) $\lim\limits_{x\to 0}\left[\dfrac{\sin x-\cos x}{x^3}\right]$

(o) $\lim\limits_{x\to 0}\left[\dfrac{\sin x-x}{x^3}\right]$

(p) $\lim\limits_{x\to 0}\left[\dfrac{2x^3+3x^2-2x-3}{x^2-1}\right]$

(q) $\lim\limits_{x\to 3}\left[\dfrac{x-9}{\sqrt{x}-3}\right]$

(r) $\lim\limits_{x\to 2}\left[\dfrac{1-\frac{2}{x}}{x^2-4}\right]$

(s) $\lim\limits_{x\to\infty}\left[\dfrac{x+5}{\sqrt{7x^2-4}}\right]$

(t) $\lim\limits_{x\to 1}\left[\dfrac{\sqrt{x+1}}{2x+7}\right]$

(u) $\lim\limits_{x\to 0}\left[\dfrac{x-\sin x}{x-\tan x}\right]$

(v) $\lim\limits_{x\to\infty}\left[\dfrac{3x+7}{\sqrt[3]{x^3-5}}\right]$

(w) $\lim\limits_{x\to-\infty}\left[\dfrac{3x+2}{\sqrt{x^2-5}}\right]$

(x) $\lim\limits_{x\to\infty}\left[\dfrac{4x^3+20x^2}{x^4+8}\right]$

(y) $\lim\limits_{x\to\infty}\left[\dfrac{7x^3+2x^2}{4x^3-x}\right]$

(z) $\lim\limits_{x\to\infty}\left[\dfrac{3x-4}{\sqrt[3]{x^2+20}}\right]$

CHAPTER 9

DIFFERENTIATION

9.1 Standard Derivatives

1. $y = x^n$; $\dfrac{dy}{dx} = nx^{n-1}$

2. $y = e^x$; $\dfrac{dy}{dx} = e^x$

3. $y = e^{kx}$; $\dfrac{dy}{dx} = ke^{kx}$

4. $y = \ln x$; $\dfrac{dy}{dx} = \dfrac{1}{x}$

5. $y = \sin x$; $\dfrac{dy}{dx} = \cos x$

6. $y = \cos x$; $\dfrac{dy}{dx} = \sin x$

7. $y = \tan x$; $\dfrac{dy}{dx} = \sec^2 x$

8. $y = \sec x$; $\dfrac{dy}{dx} = \sec x \tan x$

9. $y = \cot x$; $\dfrac{dy}{dx} = -\csc^2 x$

10. $y = \csc x$; $\dfrac{dy}{dx} = -\csc x \cot x$

9.1.1 THE POWER RULE

If n is a positive integer, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

E.g. 1

1. $\frac{d}{dx}(x^5) = 5x^4$
2. $\frac{d}{dx}(x^{12}) = 12x^{11}$
3. $\frac{d}{dx}(x) = 1.x^0 = 1$
4. $\frac{d}{dx}(4x^8) = 4\frac{d}{dx}(x^8) = 4(8x^7) = 32x^7$
5. $\frac{d}{dx}(x^4 + 6x^{11}) = \frac{d}{dx}(x^4) - \frac{d}{dx}(6x^{11}) = 4x^3 + 6(11)x^{10} = 4x^3 + 66x^{10}$
6. $\frac{d}{dx}(c) = 0;\ \frac{d}{dx}(12) = 0$

Using the power rule to find derivatives of functions

$$\frac{d}{dx}(x^n) = nx^{n-1}\frac{d}{dx}, \quad \text{where } x \text{ is a fraction}$$

$$\text{Given } f(u) = u^n$$

$$\text{then } \frac{d}{dx}(u^n) = \frac{d}{du}(u^n)\frac{du}{dx}$$

$$= nu^{n-1}\frac{du}{dx}$$

E.g. 1 Find $\frac{d}{dx}(x^2-1)^{50}$

<u>Solution</u>

$$\frac{d}{dx}(x^2-1)^{50} = 50(x^2-1)^{49}\frac{d}{dx}(x^2-1)$$

$$= 50(x^2-1)(2x)$$

$$= 100x(x^2-1)$$

E.g. 2 Find $\dfrac{d}{dx}(x^2-1)^4$

Solution

$$\begin{aligned}\frac{d}{dx}(x^2-1)^4 &= 4(x^2-1)^3\frac{d}{dx}(x^2-1)\\ &= 4(x^2-1)(2x)\\ &= 8x(x^2-1)\end{aligned}$$

E.g. 3 Find th derivative of $y = \dfrac{1}{(4x^2-3)^5}$

Solution

$$\begin{aligned}y &= \frac{1}{(4x^2-3)^5} = (4x^2-3)^{-5}\\ \frac{dy}{dx} &= -5(4x^2-3)^{-6}\frac{d}{dx}(4x^2-3)\\ &= -5(4x^2-3)^{-6}(8x)\\ &= -40x(4x^2-3)^{-6}\\ &= -\frac{40x}{(4x^2-3)^6}\end{aligned}$$

E.g. 4 Find the derivative of $y = \sqrt[3]{3x-2}$

Solution

$$\begin{aligned}y &= \sqrt[3]{3x-2} = (3x-2)^{\frac{1}{3}}\\ \frac{dy}{dx} &= \frac{1}{3}(3x-2)^{-2/3}\frac{d}{dx}(3x-2)\\ &= \frac{1}{3}(3x-2)^{-2/3}(3)\\ &= (3x-2)^{-2/3}\\ &= \frac{1}{\sqrt[3]{(3x-2)^2}}\end{aligned}$$

E.g. 5 Find the derivative of $y = \dfrac{3}{\sqrt{x}} - 2\sqrt{x}$

Solution

$$\begin{aligned}
y &= \frac{3}{\sqrt{x}} - 2\sqrt{x} = 3x^{-1/2} - 2x^{1/2} \\
\frac{dy}{dx} &= 3\left(-\frac{1}{2}\right)x^{-3/2} - 2\left(\frac{1}{2}\right)x^{-1/2} \\
&= -\frac{3}{2}x^{-3/2} - x^{-1/2} \\
&= \frac{-3}{2x^{3/2}} - \frac{1}{x^{1/2}} \\
&= \frac{-3}{2\sqrt{x^3}} - \frac{1}{\sqrt{x}}
\end{aligned}$$

E.g. 6 Find the derivative of $y = (2x^5 - 4x^3 - x)^3$

Solution

$$\begin{aligned}
y &= (2x^5 - 4x^3 - x)^3 \\
\frac{dy}{dx} &= 3\left(2x^5 - 4x^3 - x\right)^2 \frac{d}{dx}\left(2x^5 - 4x^3 - x\right) \\
&= 3\left(2x^5 - 4x^3 - x\right)^2\left(10x^4 - 12x^2 - 1\right) \\
&= 3\left(10x^4 - 12x^2 - 1\right)\left(2x^5 - 4x^3 - x\right)^2
\end{aligned}$$

E.g. 7 Find the derivative of $y = (6x^5 - 4x^3 - 5)^7$

Solution

$$\begin{aligned}
y &= (6x^5 - 4x^3 - 5)^7 \\
\frac{dy}{dx} &= 7\left(6x^5 - 4x^3 - 5\right)^6 \frac{d}{dx}\left(6x^5 - 4x^3 - 5\right) \\
&= 7\left(6x^5 - 4x^3 - 5\right)^6\left(30x^4 - 12x^2\right) \\
&= 7\left(30x^4 - 12x^2\right)\left(6x^5 - 4x^3 - 5\right)^6 \\
&= 7(6x^2)\left(5x^2 - 2\right)\left(6x^5 - 4x^3 - 5\right) = 42x^2\left(5x^2 - 2\right)\left(6x^5 - 4x^3 - 5\right)
\end{aligned}$$

E.g. 8 Find the derivative of $y = \dfrac{2}{\sqrt[4]{x^3 - x^2 - x}}$

Solution

$$
\begin{aligned}
y &= \frac{2}{\sqrt[4]{x^3 - x^2 - x}} = 2\left(x^3 - x^2 - x\right)^{-1/4} \\
\frac{dy}{dx} &= 2\left(-\frac{1}{4}\right)\left(x^3 - x^2 - x\right)^{-5/4} \frac{d}{dx}\left(x^3 - x^2 - x\right) \\
&= -\frac{1}{2}\left(x^3 - x^2 - x\right)^{-5/4}\left(3x^2 - 2x - 1\right) \\
&= -\frac{(3x^2 - 2x - 1)}{2(x^3 - x^2 - x)^{\frac{5}{4}}} \\
&= \frac{1 + 2x - 3x^2}{2\left(\sqrt[4]{x^3 - x^2 - x}\right)^5}
\end{aligned}
$$

E.g. 9 Find the derivative of $\dfrac{d}{dx}\left(x + \dfrac{1}{x}\right)^{-3}$

Solution

$$
\begin{aligned}
\frac{d}{dx}\left(x + \frac{1}{x}\right)^{-3} &= -3\left(x + \frac{1}{x}\right)^{-4} \frac{d}{dx}\left(x + \frac{1}{x}\right) \\
&= -3\left(x + \frac{1}{x}\right)^{-4}\left(1 - \frac{1}{x^2}\right)
\end{aligned}
$$

9.2 THE PRODUCT RULE

If u and v are differentiable functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

This follows that if we are asked to differentiate the product of a function:

(i) We may have to keep the first function constant and differentiate the second function.

(ii) And also keep the second function constant and differentiate the first function

NB: If u, v and w are differentiable functions of x, then

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

Proof:

$$\begin{aligned}\frac{d}{dx}(uvw) = \frac{d}{dx}\left[(uv)w\right] &= uv\frac{dw}{dx} + w\frac{dv}{dx}(uv)\\ &= uv\frac{dw}{dx} + w\left[u\frac{dv}{dx} + v\frac{du}{dx}\right]\\ &= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}\end{aligned}$$

WORKED EXAMPLES

1. Differentiate $y = (x^2+1)(x^4-2x)$

Solution

$$\begin{aligned} y &= (x^2+1)(x^4-2x)\\ \frac{dy}{dx} &= (x^2+1)\frac{d}{dx}(x^4-2x) + (x^4-2x)\frac{d}{dx}(x^2+1)\\ &= (x^2+1)(4x^3-2) + (x^4-2x)(2x)\\ &= 4x^5 - 2x^2 + 4x^3 - 2 + 2x^5 - 2\\ &= 6x^5 + 4x^3 - 6x^2 - 2\end{aligned}$$

2. Differentiate $\frac{d}{dx}\left(x+\frac{1}{x}\right)^{-3}$

Solution

$$\begin{aligned}\frac{d}{dx}\left(x+\frac{1}{x}\right)^{-3} &= -3\left(x+\frac{1}{x}\right)^{-4}\frac{d}{dx}\left(x+\frac{1}{x}\right)\\ &= -3\left(x+\frac{1}{x}\right)^{-4}\left(1-\frac{1}{x^2}\right)\end{aligned}$$

3. Find $\dfrac{dy}{dx}$ if $y = x^5(3x^3 - 2x + 1)^3$

Solution

$$
\begin{aligned}
y &= x^5(3x^3 - 2x + 1)^3 \\
\frac{dy}{dx} &= x^5 \frac{dy}{dx}(3x^3 - 2x + 1)^3 + (3x^3 - 2x + 1)^3 \frac{d}{dx}(x^5) \\
&= x^5(3)(3x^3 - 2x + 1)^2 \frac{d}{dx}(3x^3 - 2x + 1) + (3x^3 - 2x + 1)^3(5x^4) \\
&= 3x^5(3x^3 - 2x + 1)^2(9x^2 - 2) + 5x^4(3x^3 - 2x + 1)^3 \\
&= x^4(3x^3 - 2x + 1)^2\left[3x(9x^2 - 2) + 5(3x^3 - 2x + 1)\right] \\
&= x^4(3x^3 - 2x + 1)^2(27x^3 - 6x + 15x^3 - 10x + 5) \\
&= x^4(3x^3 - 2x + 1)^2(42x^3 - 16x + 5)
\end{aligned}
$$

4. Find $\dfrac{dy}{dx}$ if $y = (2x^4 - 1)^4(x^8 - 3x^2)^7$

Solution

$$
\begin{aligned}
y &= (2x^4 - 1)^4(x^8 - 3x^2)^7 \\
\frac{dy}{dx} &= (2x^4 - 1)^4 \frac{d}{dx}(x^8 - 3x^2)^7 + (x^8 - 3x^2)^7 \frac{d}{dx}(2x^4 - 1)^4 \\
&= (2x^4 - 1)^4(7)(x^8 - 3x^2)^6 \frac{d}{dx}(x^8 - 3x^2) + (x^8 - 3x^2)^7(4)(2x^4 - 1)^3 \\
&\quad \frac{d}{dx}(2x^4 - 1) \\
&= 7(2x^4 - 1)^4(x^8 - 3x^2)^6(8x^7 - 6x) + 4(2x^4 - 1)^3(x^8 - 3x^2)^7(8x^3) \\
&= (2x^4 - 1)^3(x^8 - 3x^2)^6\left[7(2x^4 - 1)(8x^7 - 6x) + 32x^3(x^8 - 3x^2)\right] \\
&= (2x^4 - 1)^3(x^8 - 3x^2)^6(112x^{11} - 84x^5 - 56x^7 + 42x + 32x^{11} - 96x^5) \\
&= (2x^4 - 1)^3(x^8 - 3x^2)^6(144x^{11} - 56x^7 - 180x^5 + 42x)
\end{aligned}
$$

$$\frac{dy}{dx} = (2x^4-1)^3(x^8-3x^2)^6x(144x^{10}-56x^6-180x^4+4)$$
$$= x(2x^4-1)^3(x^8-3x^2)^6(144x^{10}-56x^6-180x^4+42)$$

5. Find $\dfrac{dy}{dx}$ if $y = (2x^3-1)^5\sqrt{x^2-1}$

<u>**Solution**</u>

$$\begin{aligned}
y &= (2x^3-1)^5\sqrt{x^2-1}\\
\frac{dy}{dx} &= (2x^3-1)^5\frac{d}{dx}(x-1)^{\frac{1}{2}} + (x-1)^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\frac{d}{dx}(2x^3-1)^5\\
&= (2x^3-1)^5\frac{1}{2}(x-1)^{-\frac{1}{2}}\frac{d}{dx}(x-1) + (x-1)^{\frac{1}{2}}(5)(2x^3-1)^4\frac{d}{dx}(2x^3-1)\\
&= (2x^3-1)^5\frac{1}{2}(x-1)^{-\frac{1}{2}}(1) + (x-1)^{\frac{1}{2}}(5)(2x^3-1)^4(6x^2)\\
&= \frac{1}{2}(2x^3-1)^5(x-1)^{-\frac{1}{2}} + (x-1)^{\frac{1}{2}}(2x^3-1)^4(30x^2)\\
&= \frac{\frac{1}{2}(2x^3-1)^5}{(x-1)^{\frac{1}{2}}} + 30x^2(x-1)^{\frac{1}{2}}(2x^3-1)^4\\
&= \frac{(2x^3-1)^5}{2\sqrt{x-1}} + 30x^2(2x^3-1)^4\sqrt{x-1}
\end{aligned}$$

6. Find $\dfrac{dy}{dx}$ if $y = x^2(2x-1)(6x+5)$

<u>**Solution**</u>

$$\begin{aligned}
y &= x^2(2x-1)(6x+5)\\
\frac{dy}{dx} &= x^2(2x-1)\frac{d}{dx}(6x+5) + x^2(6x+5)\frac{d}{dx}(2x-1) + (2x-1)(6x+5)\frac{d}{dx}(x^2)\\
&= x^2(2x-1)(6) + x^2(6x+5)(2) + (2x-1)(6x+5)(2x)\\
&= 6x^2(2x-1) + 2x^2(6x+5) + (2x-1)(12x^2+10x)
\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= 12x^3 - 6x^2 + 12x^3 + 10x^2 + 24x^3 + 20x^2 - 12x^2 - 10x \\ &= 48x^3 + 12x^2 - 10x\end{aligned}$$

7. Given that $y = (3x-1)^2(2x^2-3)(x^3+4)^5$, find $\frac{dy}{x}$

Solution

$$\begin{aligned}
y &= (3x-1)^2(2x^2-3)(x^3+4)^5 \\
\frac{dy}{dx} &= (3x-1)^2(2x^2-3)\frac{d}{dx}(x^3+4)^5 + (3x-1)^2(x^3+4)^5\frac{d}{dx}(2x^2-3) + \\
&\quad (2x^2-3)(x^3+4)^5\frac{d}{dx}(3x-1)^2 \\
&= (3x-1)^2(2x^2-3)5(x^3+4)^4\frac{d}{dx}(x^3+4) + (3x-1)^2(x^3+4)^5(4x) + \\
&\quad (2x^2-3)(x^3+4)^5 2(3x-1)\frac{d}{dx}(3x-1) \\
&= (3x-1)^2(2x^2-3)5(x^3+4)^4(3x^2) + 4x(3x-1)^2(x^3+4)^5 + \\
&\quad 2(2x^2-3)(x^3+4)^5(3x-1)(3) \\
&= 15x^2(3x-1)^2(2x^2-3)(x^3+4)^4 + 4x(3x-1)^2(x^3+4)^5 + \\
&\quad 6(3x-1)(2x^2-3)(x^3+4)^5 \\
&= (3x-1)(x^3+4)^4\Big[15x^2(3x-1)(2x^2-3) + 4x(3x-1)(x^3+4) + \\
&\quad 6(2x^2-3)(x^3+4)\Big]
\end{aligned}$$

8. Calculate y' when $y = (4x^2-x)^3(x^3-2x^2)^4(x-1)^6$

Solution

$$
\begin{aligned}
y &= (4x^2-x)^3(x^3-2x^2)^4(x-1)^6 \\
\frac{dy}{dx} &= (4x^2-x)^3(x^3-2x^2)^4\frac{d}{dx}(x-1)^6+(4x^2-x)^3(x-1)^6\frac{d}{dx}(x^3-2x^2)^4+ \\
&\quad (x^3-2x^2)^4(x-1)^6\frac{d}{dx}(4x^2-x)^3 \\
&= (4x^2-x)^3(x^3-2x^2)^4(6)(x-1)^5\frac{d}{dx}(x-1)+(4x^2-x)^3(x-1)^6(4)(x^3-2x^2)^3 \\
&\quad \frac{d}{dx}(x^3-2x^2)+(x^3-2x^2)^4(x-1)^6(3)(4x^2-x)^2\frac{d}{dx}(4x^2-x) \\
&= (4x^2-x)^3(x^3-2x^2)^4(6)(x-1)^5(1)+(4x^2-x)^3(x-1)^6(4)(x^3-2x^2)^3 \\
&\quad (3x^2-4x)+(x^3-2x^2)^4(x-1)^6(3)(4x^2-x)^2(8x-1) \\
&= 6(4x^2-x)^3(x^3-2x^2)^4(x-1)^5+4(4x^2-x)^3(x-1)^6(x^3-2x^2)^3(3x^2-4x)+ \\
&\quad 3(x^3-2x^2)^4(x-1)^6(4x^2-x)^2(8x-1) \\
&= (4x^2-x)^2(x-1)^5(x^3-2x^2)^3\Big[6(4x^2-x)(x^3-2x^2)+4(4x^2-x)(x-1) \\
&\quad (3x^2-4x)+3(x^3-2x^2)(x-1)(8x-1)\Big]
\end{aligned}
$$

9. Find $\dfrac{dy}{dx}$ if $y=(8x^3-x^2)^5(7x^2-3x)^2(x^5-2)^{10}$

Solution

$$
\begin{aligned}
y &= (8x^3-x^2)^5(7x^2-3x)^2(x^5-2)^{10} \\
\frac{dy}{dx} &= (8x^3-x^2)^5(7x^2-3x)^2\frac{d}{dx}(x^5-2)^{10}+(8x^3-x^2)^5(x^5-2)^{10}\frac{d}{dx}(7x^2-3x)^2+ \\
&\quad (7x^2-3x)^2(x^5-2)^{10}\frac{d}{dx}(8x^3-x^2)^5 \\
&= (8x^3-x^2)^5(7x^2-3x)^2(10)(x^5-2)^9\frac{d}{dx}(x^5-2)+ \\
&\quad (8x^3-x^2)^5(x^5-2)^{10}(2)(7x^2-3x)\frac{d}{dx}(7x^2-3x)+ \\
&\quad (7x^2-3x)^2(x^5-2)^{10}(5)(8x^3-x^2)^4\frac{d}{dx}(8x^3-x^2)
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy}{dx} &= (8x^3-x^2)^5(7x^2-3x)^2(10)(x^5-2)^9(5x^4)+\\
&\quad (8x^3-x^2)^5(x^5-2)^{10}(2)(7x^2-3x)(14x-3)+\\
&\quad (7x^2-3x)^2(x^5-2)^{10}(5)(8x^3-x^2)^4(24x^2-2x)\\
&= 50x^4(8x^3-x^2)^5(7x^2-3x)^2(x^5-2)^9+\\
&\quad 2(8x^3-x^2)^5(x^5-2)^{10}(7x^2-3x)(14x-3)+\\
&\quad 5(7x^2-3x)^2(x^5-2)^{10}(8x^3-x^2)^4(24x^2-2x)\\
&= (7x^2-3x)(8x^3-x^2)^4(x^5-2)^9\left[50x^4(8x^3-x^2)(7x^2-3x)+\right.\\
&\quad \left.2(8x^3-x^2)(x^5-2)(14x-3)+5(7x^2-3x)(x^5-2)(24x^2-2x)\right]\\
&= (7x^2-3x)(8x^3-x^2)^4(x^5-2)^9\left[(400x^7-50x^6)(7x^2-3x)+\right.\\
&\quad \left.(2x^5-4)(8x^3-x^2)(14x-3)+(5x^5-10)(7x^2-3x)(24x^2-2x)\right]
\end{aligned}
$$

10. Find y' when $y=(2x^5-3x^3)^4(5x-4x^2)^3(x^6-6x^2)^7$

Solution

$$
\begin{aligned}
y &= (2x^5-3x^3)^4(5x-4x^2)^3(x^6-6x^2)^7\\
\frac{dy}{dx} &= (2x^5-3x^3)^4(5x-4x^2)^3\frac{d}{dx}(x^6-6x^2)^7+\\
&\quad (2x^5-3x^3)^4(x^6-6x^2)^7\frac{d}{dx}(5x-4x^2)^3+\\
&\quad (5x-4x^2)^3(x^6-6x^2)^7\frac{d}{dx}(2x^5-3x^3)^4
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy}{dx} &= 7(2x^5-3x^3)^4(5x-4x^2)^3(x^6-6x^2)^6(6x^5-12x)+\\
&\quad 3(2x^5-3x^3)^4(x^6-6x^2)^7(5x-4x^2)^2(5-8x)+\\
&\quad 4(5x-4x^2)^3(x^6-6x^2)^7(2x^5-3x^3)^3(10x^4-9x^2)\\
&= 7(2x^5-3x^3)^4(5x-4x^2)^3(x^6-6x^2)^6(6x^5-12x)+\\
&\quad 3(2x^5-3x^3)^4(x^6-6x^2)^7(5x-4x^2)^2(5-8x)+\\
&\quad 4(5x-4x^2)^3(x^6-6x^2)^7(2x^5-3x^3)^3(10x^4-9x^2)\\
&= 7(6x^5-12x)(2x^5-3x^3)^4(5x-4x^2)^3(x^6-6x^2)^6+\\
&\quad 3(5-8x)(2x^5-3x^3)^4(5x-4x^2)^2(x^6-6x^2)^7+\\
&\quad 4(10x^4-9x^2)(2x^5-3x^3)^3(5x-4x^2)^3(x^6-6x^2)^7\\
&= (2x^5-3x^3)^3(5x-4x^2)^2(x^6-6x^2)^6\Big[7(6x^5-12x)(2x^5-3x^3)(5x-4x^2)+\\
&\quad 3(5-8x)(2x^5-3x^3)(x^6-6x^2)+4(10x^4-9x^2)(5x-4x^2)(x^6-6x^2)\Big]
\end{aligned}
$$

9.3 QUOTIENT RULE

If u and v are differentiable functions of x and $y=\frac{u}{v}$, where $v\neq 0$, then

$$\frac{dy}{dx}=y'=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Worked Examples

1. Differentiate $y=\frac{x+1}{x-1}$ with respect to x.

Solution

$$y=\frac{x+1}{x-1}$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{(x-1)\frac{d}{dx}(x+1)-(x+1)\frac{d}{dx}(x-1)}{(x-1)^2} \\
&= \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} \\
&= \frac{(x-1)-(x+1)}{(x-1)^2} \\
&= \frac{x-1-x-1}{(x-1)^2} \\
&= \frac{-2}{(x-1)^2}
\end{aligned}
$$

2. Find $\dfrac{dy}{dx}$ if $y=\dfrac{x^3}{x^2+1}$

Solution

$$
\begin{aligned}
y &= \frac{x^3}{x^2+1} \\
\frac{dy}{dx} &= \frac{(x^2+1)\frac{d}{dx}(x^3)-x^3\frac{d}{dx}(x^2+1)}{(x^2+1)^2} \\
&= \frac{(x^2+1)(3x^2)-x^3(2x)}{(x^2+1)^2} \\
&= \frac{3x^4+3x^2-2x^4}{(x^2+1)^2} \\
&= \frac{x^4+3x^2}{(x^2+1)^2} \\
&= \frac{x^2(x^2+3)}{(x^2+1)^2}
\end{aligned}
$$

3. Calculate y' when $y=\dfrac{2x^5-x^3}{3x^4-2x}$

Solution

$$
\begin{aligned}
y &= \frac{2x^5 - x^3}{3x^4 - 2x} \\
\frac{dy}{dx} &= \frac{(3x^4 - 2x)\frac{d}{dx}(2x^5 - x^3) - (2x^5 - x^3)\frac{d}{dx}(3x^4 - 2x)}{(3x^4 - 2x)^2} \\
&= \frac{(3x^4 - 2x)(10x^4 - 3x^2) - (2x^5 - x^3)(12x^3 - 2)}{(3x^4 - 2x)^2} \\
&= \frac{30x^8 - 9x^6 - 20x^5 + 6x^3 - 24x^3 + 4x^5 + 12x^6 - 2x^3}{(3x^4 - 2x)^2} \\
&= \frac{6x^8 + 3x^6 - 16x^5 + 4x^3}{(3x^4 - 2x)^2} \\
&= \frac{x^3(6x^5 + 3x^3 - 16x^2 + 4)}{(3x^4 - 2x)^2}
\end{aligned}
$$

4. Find $\frac{dy}{dx}$ if $y = \frac{(3x^2 - x)^4}{(x^3 - 2x^4)^3}$

Solution

$$
\begin{aligned}
y &= \frac{(3x^2 - x)^4}{(x^3 - 2x^4)^3} \\
\frac{dy}{dx} &= \frac{(x^3 - 2x^4)^3\frac{d}{dx}(3x^2 - x)^4 - (3x^2 - x)^4\frac{d}{dx}(x^3 - 2x^4)^3}{((x^3 - 2x^4)^3)^2} \\
&= \frac{(x^3 - 2x^4)^3(4)(3x^2 - x)^3\frac{d}{dx}(3x^2 - x) - (3x^2 - x)^4(3)(x^3 - 2x^4)^2\frac{d}{dx}(x^3 - 2x^4)}{(x^3 - 2x^4)^6} \\
&= \frac{4(x^3 - 2x^4)^3(3x^2 - x)^3(6x - 1) - 3(3x^2 - x)^4(x^3 - 2x^4)^2(3x^2 - 8x^3)}{(x^3 - 2x^4)^6} \\
&= \frac{(x^3 - 2x^4)^2(3x^2 - x)^3\left[4(x^3 - 2x^4)(6x - 1) - 3(3x^2 - x)(3x^2 - 8x^3)\right]}{(x^3 - 2x^4)^6} \\
&= \frac{(3x^2 - x)^3\left[24x^4 - 48x^5 - 4x^3 + 8x^4 - 27x^4 + 9x^3 + 72x^5 - 24x^4\right]}{(x^3 - 2x^4)^4} \\
&= \frac{(3x^2 - x)^3\left[24x^5 - 19x^4 + 5x^3\right]}{(x^3 - 2x^4)^4}
\end{aligned}
$$

5. Calculate $\dfrac{dy}{dx}$ if $y = \left(\dfrac{2x-3}{5x^2+1}\right)^7$

Solution

$$\begin{aligned}
y &= \left(\frac{2x-3}{5x^2+1}\right)^7 \\
&= 7\left(\frac{2x-3}{5x^2+1}\right)^6 \frac{d}{dx}\left(\frac{2x-3}{5x^2+1}\right) \\
\frac{dy}{dx} &= 7\left(\frac{2x-3}{5x^2+1}\right)^6 \left[\frac{(5x^2+1)\frac{d}{dx}(2x-3)-(2x-3)\frac{d}{dx}(5x^2+1)}{(5x^2+1)^2}\right] \\
&= 7\left(\frac{2x-3}{5x^2+1}\right)^6 \left[\frac{(5x^2+1)(2)-(2x-3)(10x)}{(5x^2+1)^2}\right] \\
&= 7\left(\frac{2x-3}{5x^2+1}\right)^6 \left[\frac{10x^2+2-20x^2+30x}{(5x^2+1)^2}\right] \\
&= 7\left(\frac{2x-3}{5x^2+1}\right)^6 \left[\frac{2+30x-10x^2}{(5x^2+1)^2}\right] \\
&= \frac{7(2x-3)^6(2+30x-10x^2)}{(5x^2+1)^6(5x^2+1)^2} \\
&= \frac{14(1+15x-5x^2)(2x-3)^6}{(5x^2+1)^8}
\end{aligned}$$

6. Calculate $\dfrac{dy}{dx}\left[\dfrac{x^3-1}{\sqrt[5]{4x^2+3}}\right]$

Solution

$$\begin{aligned}
\frac{dy}{dx}\left[\frac{x^3-1}{\sqrt[5]{4x^2+3}}\right] &= \frac{\sqrt[5]{4x^2+3}\,\frac{d}{dx}(x^3-1)-(x^3-1)\frac{d}{dx}(4x^2+3)^{\frac{1}{5}}}{(\sqrt[5]{4x^2+3})^2} \\
&= \frac{\sqrt[5]{4x^2+3}(3x^2)-(x^3-1)\frac{1}{5}(4x^2+3)^{-\frac{4}{5}}\frac{d}{dx}(4x^2+3)}{\sqrt[5]{(4x^2+3)^2}} \\
&= \frac{3x^2\sqrt[5]{4x^2+3}-\frac{1}{5}(x^3-1)(4x^2+3)^{-\frac{4}{5}}(8x)}{\sqrt[5]{(4x^2+3)^2}}
\end{aligned}$$

$$\frac{dy}{dx}\left[\frac{x^3-1}{\sqrt[5]{4x^2+3}}\right]=\frac{3x^2\sqrt[5]{4x^2+3}}{\sqrt[5]{(4x^2+3)^2}}-\frac{8x(x^3-1)}{5}\cdot\frac{1}{\sqrt[5]{(4x^2+3)^2}}\cdot\frac{1}{(4x^2+3)^{\frac{4}{5}}}$$

$$=\frac{3x^2}{\sqrt[5]{(4x^2+3)}}-\frac{8x(x^3-1)}{5}\cdot\frac{1}{\sqrt[5]{(4x^2+3)^2}}\cdot\frac{1}{\sqrt[5]{(4x^2+3)^4}}$$

$$=\frac{3x^2}{\sqrt[5]{(4x^2+3)}}-\frac{8x(x^3-1)}{5\sqrt[5]{(4x^2+3)^6}}$$

7. Find y' when $\frac{\sqrt{x-1}}{x^3}$

Solution

$$y=\frac{\sqrt{x-1}}{x^3}$$

$$\frac{dy}{dx}=\frac{x^3\frac{d}{dx}(x-1)^{\frac{1}{2}}-(x-1)^{\frac{1}{2}}\frac{d}{dx}(x^3)}{(x^3)^2}$$

$$=\frac{x^3\frac{1}{2}(x-1)^{-\frac{1}{2}}\frac{d}{dx}(x-1)-(x-1)^{\frac{1}{2}}(3x^2)}{x^6}$$

$$=\frac{x^3\frac{1}{2}(x-1)^{-\frac{1}{2}}(1)-(x-1)^{\frac{1}{2}}(3x^2)}{x^6}$$

$$=\frac{\frac{1}{2}x^3(x-1)^{-\frac{1}{2}(1)}-(x-1)^{\frac{1}{2}}(3x^2)}{x^6}$$

$$=\frac{2(x-1)^{\frac{1}{2}}}{2(x-1)^{\frac{1}{2}}}\left[\frac{\frac{1}{2}x^3(x-1)^{-\frac{1}{2}}-3x^2(x-1)^{\frac{1}{2}}}{x^6}\right]$$

$$=\frac{x^3-6x^2(x-1)}{2x^6(x-1)^{\frac{1}{2}}}=\frac{x^3-6x^3+6x^2}{2x^6\sqrt{x-1}}$$

$$=\frac{6x^2-5x^3}{2x^6\sqrt{x-1}}=\frac{x^2(6-5x)}{2x^6\sqrt{x-1}}$$

$$=\frac{6-5x}{2x^4\sqrt{x-1}}$$

NB: Because $y = \frac{1}{x^n}$ can be expressed as $y = x^{-n}$, all problems where quotient rule is applicable can be converted to products so that you apply product rule. Here, it is a matter of choice or preference.

Worked Examples:

1. Differentiate

   (a) $(x^2-2)(x+3)^{-2}$ as a product

   (b) $\frac{x^-2}{(x+3)^2}$ as a quotient

Solution

(a)

$$\text{Let } y = (x^2-2)(x+3)^{-2}$$

$$\frac{dy}{dx} = (x^2-2)\frac{d}{dx}(x+3)^{-2} + (x+3)^{-2}\frac{d}{dx}(x^2-2) \qquad \text{[Product rule]}$$

$$= (x^2-2)(-2)(x+3)^{-3}\frac{d}{dx}(x+3) + (x+3)^{-2}(2x)$$

$$= \frac{-2(x^2-2)}{(x+3)^3}(1) + \frac{(2x)}{(x+3)^2}$$

$$= \frac{-2x^2+4}{(x+3)^3} + \frac{2x}{(x+3)^2}$$

$$= \frac{-2x^2+4+2x(x+3)}{(x+3)^3}$$

$$= \frac{-2x^2+4+2x^2+6x}{(x+3)^3}$$

$$= \frac{4+6x}{(x+3)^3} = \frac{2(2+3x)}{(x+3)^3}$$

Solution

(b)

$$\text{Let } y = \frac{x^2-2}{(x+3)^2}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(x+3)^2\frac{d}{dx}(x^2-2)-(x^2-2)\frac{d}{dx}(x+3)^2}{[(x+3)^2]^2}\\
&= \frac{(x+3)^2(2x)-(x^2-2)(2)(x+3)\frac{d}{dx}(x+3)}{(x+3)^4}\\
&= \frac{2x(x+3)^2-2(x^2-2)(x+3)(1)}{(x+3)^4}\\
&= \frac{(x+3)\left[2x(x+3)-2(x^2-2)\right]}{(x+3)^4}\\
&= \frac{2x^2+6x-2x^2+4}{(x+3)^3}\\
&= \frac{6x+4}{(x+3)^3} = \frac{2(2+3x)}{(x+3)^3}
\end{aligned}$$

NB: On comparing $\frac{dy}{dx}$ for (i) and (ii), you will notice that the final results are the same. Hence,

$$y = \frac{x^2-2}{(x+3)^2} = (x^2-2)(x+3)^{-2}$$

2. Differentiate

(a) $(x-1)^3(x^3-1)^{-1}$ as a product

(b) $\dfrac{(x-1)^3}{(x^3-1)}$ as a quotient

Solution

(a)

$$\begin{aligned}
y &= (x-1)^3(x^3-1)^{-1}\\
\frac{dy}{dx} &= (x-1)^3\frac{d}{dx}(x^3-1)^{-1}+(x^3-1)^{-1}\frac{d}{dx}(x-1)^3\\
&= (x-1)^3(-1)(x^3-1)^{-2}\frac{d}{dx}(x^3-1)+(x^3-1)^{-1}(3)(x-1)^2\frac{d}{dx}(x-1)
\end{aligned}$$

$$
\begin{aligned}
\frac{dy}{dx} &= -(x-1)^3(x^3-1)^{-2}(3x^2) + 3(x^3-1)^{-1}(x-1)^2(1) \\
&= \frac{-3x^2(x-1)^3}{(x^3-1)^2} + \frac{3(x-1)^2}{(x^3-1)} \\
&= \frac{-3x^2(x-1)^3 + 3(x-1)^2(x^3-1)}{(x^3-1)^2} \\
&= \frac{(x-1)^2\left[-3x^2(x-1) + 3(x^3-1)\right]}{(x^3-1)^2} \\
&= \frac{(x-1)^2(-3x^3+3x^2+3x^3-3)}{(x^3-1)^2} \\
&= \frac{(x-1)^2(3x^2-3)}{(x^3-1)^2} \\
&= \frac{3(x-1)^2(x^2-1)}{(x^3-1)^2} = \frac{3(x+1)(x-1)^3}{(x^3-1)^2}
\end{aligned}
$$

<u>Solution</u>

(b)

$$
\begin{aligned}
y &= \frac{(x-1)^3}{x^3-1} \\
\frac{dy}{dx} &= \frac{(x^3-1)\frac{d}{dx}(x-1)^3 - (x-1)^3\frac{d}{dx}(x^3-1)}{(x^3-1)^2} \\
&= \frac{(x^3-1)(3)(x-1)^2\frac{d}{dx}(x-1) - (x-1)^3(3x^2)}{(x^3-1)^2} \\
&= \frac{3(x^3-1)(x-1)^2(1) - 3x^2(x-1)^3}{(x^3-1)^2} \\
&= \frac{3(x-1)^2\left[(x^3-1) - x^2(x-1)\right]}{(x^3-1)^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{3(x-1)^2(x^3-1-x^3+x^2)}{(x^3-1)^2} \\
&= \frac{3(x-1)^2(x^2-1)}{(x^3-1)^2} \\
&= \frac{3(x-1)^2(x-1)(x+1)}{(x^3-1)^2} \\
&= \frac{3(x-1)^3(x+1)}{(x^3-1)^2}
\end{aligned}
$$

3. Differentiate

   (a) $(x^2-1)^3x^{-4}$ as a product

   (b) $\dfrac{(x^2-1)^3}{x^4}$ as a quotient

Solution

(a)

$$
\begin{aligned}
y &= (x^2-1)^3x^{-4} \\
\frac{dy}{dx} &= (x^2-1)^3\frac{d}{dx}(x^{-4}) + x^{-4}\frac{d}{dx}(x^2-1)^3 \\
&= (x^2-1)^3(-4)x^{-5} + x^{-4}(3)(x^2-1)^2\frac{d}{dx}(x^2-1) \\
&= \frac{-4(x^2-1)^3}{x^5} + 3x^{-4}(x^2-1)^2(2x) \\
&= \frac{-4(x^2-1)^3}{x^5} + \frac{6x(x^2-1)^2}{x^4} \\
&= \frac{-4(x^2-1)^3 + 6x^2(x^2-1)^2}{x^6} \\
&= \frac{(x^2-1)^2\left[-4(x^2-1)+6x^2\right]}{x^5} \\
&= \frac{(x^2-1)^2(-4x^2+4+6x^2)}{x^5} \\
&= \frac{(x^2-1)^2(2x^2+4)}{x^5} = \frac{2(x^2+2)(x^2-1)^2}{x^5}
\end{aligned}
$$

Solution

(b)

$$
\begin{aligned}
y &= \frac{(x^2-1)^3}{x^4}\\
\frac{dy}{dx} &= \frac{x^4\frac{d}{dx}(x^2-1)^3-(x^2-1)^3\frac{d}{dx}(x^4)}{(x^4)^2}\\
&= \frac{x^4(3)(x^2-1)^2\frac{d}{dx}(x^2-1)-(x^2-1)^3(4x^3)}{x^8}\\
&= \frac{3x^4(x^2-1)^2(2x)-4x^3(x^2-1)^3}{x^8}\\
&= \frac{6x^5(x^2-1)^2-4x^3(x^2-1)^3}{x^8}\\
&= \frac{2x^3(x^2-1)^2\left[3x^2-2(x^2-1)\right]}{x^8}\\
&= \frac{2(x^2-1)^2(3x^2-2x^2+2)}{x^5}\\
&= \frac{2(x^2-1)^2(x^2+2)}{x^5}
\end{aligned}
$$

9.4 HIGHER-ORDER DERIVATIVES

A given function $f(x)$ can be differentiated twice, thrice etc. If the given function $f(x)$ is differentiated twice, we get its second derivative. It is denoted by $f''(x)$ or

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

When $f(x)$ is differentiated thrice, we get its third derivative. It is denoted by $f'''(x)$ or

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}\left[\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$$

Worked Examples

1. Given the function $y=x^2-x-\dfrac{1}{5x^6}$, find $\dfrac{d^2y}{dx^2}$.

Solution

$$
\begin{aligned}
y &= x^2 - x - \frac{1}{5x^6} = x^2 - x - \frac{1}{5}x^{-6} \\
\frac{dy}{dx} &= 2x - 1 - \frac{1}{5}(-6)x^{-7} \\
&= 2x - 1 + \frac{6}{5}x^{-7} \\
\text{but } \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x - 1 + \frac{6}{5}x^{-7}\right) \\
\implies \frac{d^2y}{dx^2} &= 2 + \frac{6}{5}(-7)x^{-8} \\
&= 2 - \frac{42}{5}x^{-8} = 2 - \frac{42}{5x^8}
\end{aligned}
$$

2. Find y'' when $y = 3\sqrt{x} - \dfrac{2}{\sqrt{x}}$.

Solution

$$
\begin{aligned}
y &= 3\sqrt{x} - \frac{2}{\sqrt{x}} = 3x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} \\
\frac{dy}{dx} &= 3.\frac{1}{2}x^{-\frac{1}{2}} - 2\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} \\
&= \frac{3}{2}x^{-\frac{1}{2}} + x^{-\frac{3}{2}} \\
\text{but } \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{3}{2}x^{-\frac{1}{2}} + x^{-\frac{3}{2}}\right) \\
\implies \frac{d^2y}{dx^2} &= \frac{3}{2}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}} + \left(-\frac{3}{2}\right)x^{-\frac{5}{2}} \\
&= -\frac{3}{4}x^{-\frac{3}{2}} - \frac{3}{2}x^{-\frac{5}{2}} \\
&= -\frac{3}{4x^{3/2}} - \frac{3}{2x^{5/2}} \\
&= -\frac{3}{4\sqrt{x^3}} - \frac{3}{2\sqrt{x^5}}
\end{aligned}
$$

3. iF $y=(3x^2-2x)^3$, find $\dfrac{d^2y}{dx^2}$.

Solution

$$
\begin{aligned}
y &= (3x^2-2x)^3\\
\frac{dy}{dx} &= 3(3x^2-2x)^2\frac{d}{dx}(3x^2-2x)\\
&= 3(3x^2-2x)^2(6x-2)\\
\text{but } \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[3(3x^2-2x)^2(6x-2)\right]\\
\Longrightarrow \frac{d^2y}{dx^2} &= 3(3x^2-2x)^2\frac{d}{dx}(6x-2)+3(6x-2)\frac{d}{dx}(3x^2-2x)^2\\
&= 3(3x^2-2x)^2(6)+3(6x-2)(2)(3x^2-2x)\frac{d}{dx}(3x^2-2x)\\
&= 18(3x^2-2x)^2+6(6x-2)(2)(3x^2-2x)(6x-2)\\
&= 6(3x^2-2x)\left[3(3x^2-2x)+(6x-2)^2\right]\\
&= 6(3x^2-2x)(9x^2-6x+36x^2-24x+4)\\
&= 6(3x^2-2x)(45x^2-30x+4)
\end{aligned}
$$

4. Find $\dfrac{d^2y}{dx^2}$ when $y=(x^2-1)(x^3+2)$.

Solution

$$
\begin{aligned}
y &= (x^2-1)(x^3+2)\\
\frac{dy}{dx} &= (x^2-1)\frac{d}{dx}(x^3+2)+(x^3+2)\frac{d}{dx}(x^2-1)\\
&= \frac{dy}{dx} = (x^2-1)(3x^2)+(x^3+2)(2x)\\
&= \frac{dy}{dx} = 3x^4-3x^2+2x^4+4x = 5x^4-3x^2+4x
\end{aligned}
$$

$$\text{But } \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(5x^4 - 3x^2 + 4x)$$
$$= \frac{d^2y}{dx^2} = 20x^4 - 6x + 4$$

5. calculate y'' when $y = \dfrac{1}{\sqrt[3]{x-1}}$

<u>**Solution**</u>

$$y = \frac{1}{\sqrt[3]{x-1}} = \frac{1}{(x-1)^{\frac{1}{3}}} = (x-1)^{-\frac{1}{3}}$$
$$\frac{dy}{dx} = -\frac{1}{3}(x-1)^{-\frac{4}{3}}\frac{d}{dx}(x-1)$$
$$= -\frac{1}{3}(x-1)^{-\frac{4}{3}}(1) = -\frac{1}{3}(x-1)^{-\frac{4}{3}}$$
$$\text{But } \frac{d^2y}{dx^2} = \frac{dy}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[-\frac{1}{3}(x-1)^{-\frac{4}{3}}\right]$$
$$\Longrightarrow \quad \frac{d^2y}{dx^2} = \left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)(x-1)^{-\frac{7}{3}}\frac{d}{dx}(x-1)$$
$$= \frac{4}{9}(x-1)^{-\frac{7}{3}}(1)$$
$$= \frac{4}{9(x-1)^{\frac{7}{3}}} = \frac{4}{9\sqrt[3]{(x-1)^7}}$$

9.5 IMPLICIT DIFFERENTIATION

This is a method for finding the derivative of a function without first solving the equation explicitly for y in terms of x.

WORKED EXAMPLES

1. Find $\dfrac{dy}{dx}$ if $3x^3y^2 - x^2 = y$

Solution

$$3x^3y^2 - x^2 = y$$

$$3x^3\frac{d}{dx}(y^2) + 3y^2\frac{d}{dx}(x^3) - \frac{d}{dx}(x^2) = \frac{d}{dx}(y)$$

$$3x^3 . 2y\frac{dy}{dx} + 3y^2 . 3x^2 - 2x = \frac{dy}{dx}$$

$$6x^3y\frac{dy}{dx} + 9x^2y^2 - 2x = \frac{dy}{dx}$$

$$6x^3y\frac{dy}{dx} - \frac{dy}{dx} = 2x - 9x^2y^2$$

$$(6x^3y - 1)\frac{dy}{dx} = 2x - 9x^2y^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x - 9x^2y^2}{6x^3y - 1}$$

2. Find y' when $x^3 + y^2 = 5$.

Solution

$$x^3 + y^2 = 5$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^2) = \frac{d}{dx}(5)$$

$$3x^2 + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -3x^2$$

$$\therefore \quad \frac{dy}{dx} = -\frac{3x^2}{2y}$$

3. Given that $x^2 - 4xy + 3y^2 = 7y + 5x$, find $\frac{dy}{dx}$

Solution

$$x^2 - 4xy + 3y^2 = 7y + 5x$$

$$\frac{d}{dx}(x^2) - 4\frac{d}{dx}(xy) + 3\frac{d}{dx}(y^2) = 7\frac{d}{dx}(y) + 5\frac{d}{dx}(x)$$

$$2x - 4\left[x\frac{dy}{dx} + y\right] + 3(2y)\frac{dy}{dx} = 7\frac{dy}{dx} + 5$$

$$2x - 4x\frac{dy}{dx} - 4y + 6y\frac{dy}{dx} = 7\frac{dy}{dx} + 5$$

$$-4x\frac{dx}{dy} + 6y\frac{dy}{dx} - 7\frac{dy}{dx} = 5 - 2x + 4y$$

$$(6y - 4x - 7)\frac{dy}{dx} = 5 - 2x + 4y$$

$$\frac{dy}{dx} = \frac{5 - 2x + 4y}{6y - 4x - 7}$$

4. Find $\frac{dy}{dx}$ when $y^3 - x^3 = 4xy$

Solution

$$y^3 - x^3 = 4xy$$

$$\frac{d}{dy}(y^3) - \frac{d}{dy}(x^3) = 4\frac{d}{dx}(xy)$$

$$3y^2\frac{dy}{dx} - 3x^2 = 4\left[x\frac{dy}{dx} + y\frac{dy}{dx}\right]$$

$$3y^2\frac{dy}{dx} - 3x^2 = 4x\frac{dy}{dx} + 4y$$

$$3y^2\frac{dy}{dx} - 4x\frac{dy}{dx} = 3x^2 + 4y$$

$$(3y^2 - 4x)\frac{dy}{dx} = 3x^2 + 4y$$

$$\frac{dy}{dx} = \frac{3x^2 + 4y}{3y^2 - 4x}$$

5. If $x+y=xy$, show that $\dfrac{d^2y}{dx^2}=\dfrac{2y^3}{x^3}$

Solution

$$x+y=xy$$

$$1+\frac{dy}{dx}=x\frac{dy}{dx}+y$$

$$\frac{dy}{dx}-x\frac{dy}{dx}=y-1$$

$$(1-x)\frac{dy}{dx}=y-1$$

Using the product rule to the above

$$(1-x)\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\left[\frac{d}{dx}(1-x)\right]=\frac{d}{dx}(y-1)$$

$$(1-x)\frac{d^2y}{dx^2}+(-1)\frac{dy}{dx}=\frac{dy}{dx}$$

$$(1-x)\frac{d^2y}{dx^2}=\frac{dy}{dx}+\frac{dy}{dx}=2\frac{dy}{dx}$$

From $x+y=xy$, $y-1=\dfrac{y}{x}$ and $1-x=-\dfrac{x}{y}$

Hence, $\dfrac{dy}{dx}=\dfrac{y-1}{1-x}=\dfrac{\frac{y}{x}}{-\frac{x}{y}}=-\dfrac{y^2}{x^2}$

$$\implies \left(-\frac{x}{y}\right)\frac{d^2y}{dx^2}=2\left(-\frac{y^2}{x^2}\right)$$

$$\implies \frac{d^2y}{dx^2}=-\frac{2y^2}{x^2}\left(-\frac{y}{x}\right)$$

$$\therefore \frac{d^2y}{dx^2}=\frac{2y^3}{x^3}$$

9.6 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

1. $\dfrac{d}{dx}(\sin x) = \cos x$
2. $\dfrac{d}{dx}(\cos x) = -\sin x$
3. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
4. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
5. $\dfrac{d}{dx}(\tan x) = \dfrac{\sin x}{\cos x}$
6. $\dfrac{d}{dx}(\sec x) = \dfrac{1}{\cos x}$
7. $\dfrac{d}{dx}(\csc x) = \dfrac{1}{\sin x}$
8. $\dfrac{d}{dx}(\cot x) = \dfrac{1}{\tan x} = \dfrac{\cos x}{\sin x}$

WORKED EXAMPLES

1. Calculate $\dfrac{d}{dx}(\cos 3x)$

Solution

$$\begin{aligned}\frac{d}{dx}(\cos 3x) &= -\sin 3x \frac{d}{dx}(3x)\\ &= -\sin 3x(3)\\ &= -3\sin 3x\end{aligned}$$

2. Find $\dfrac{d}{dx}(x^2 \sin x)$

Solution

$$\begin{aligned}\frac{d}{dx}(x^2 \sin x) &= x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)\\ &= x^2 \cos x + \sin x(2x)\\ &= x^2 \cos x + 2x \sin x\end{aligned}$$

3. Find $\dfrac{d}{dx}$ if $y = \sin^3(8x^2 - 3x + 1)$

Solution

$$
\begin{aligned}
y &= \sin^3(8x^2-3x+1) = \left[\sin(8x^2-3x+1)\right]^3 \\
&= 3\left[\sin(8x^2-3x+1)\right]^2 \frac{d}{dx}\sin(8x^2-3x+1) \\
&= 3\left(\sin^2(8x^2-3x+1)\right)\cos(8x^2-3x+1)\frac{d}{dx}(8x^2-3x+1) \\
&= 3\left(\sin^2(8x^2-3x+1)\right)\cos(8x^2-3x+1)(16x-3) \\
&= 3(16x-3)\sin^2(8x^2-3x+1)\cos(8x^2-3x+1)
\end{aligned}
$$

4. Given that $y = (x^2+1)\tan(2x-3)$

Solution

$$
\begin{aligned}
y &= (x^2+1)\tan(2x-3) \\
&= (x^2+1)\frac{d}{dx}\left[\tan(2x-3)\right] + \tan(2x-3)\frac{d}{dx}(x^2+1) \\
&= (x^2+1)\sec^2(2x-3)\frac{d}{dx}(2x-3) + \tan(2x-3)(2x) \\
&= (x^2+1)\sec^2(2x-3)(2) + 2x\tan(2x-3) \\
&= 2(x^2+1)\sec^2(2x-3) + 2x\tan(2x-3)
\end{aligned}
$$

5. Show that $\frac{d}{dx}(\tan x) = \sec^2 x$

Solution

$$
\begin{aligned}
\frac{d}{dx}(\tan x) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{(\cos x)^2} \\
&= \frac{\cos x(\cos x) - \sin x(-\sin x)}{(\cos x)^2} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}
$$

6. Calculate $\dfrac{d}{dx}\left(\sqrt{\cos^5 4x^3}\right)$

Solution

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{\cos^5 4x^3}\right) &= \frac{d}{dx}\left(\cos 4x^3\right)^{5/2} \\
&= \frac{5}{2}\left(\cos 4x^3\right)^{3/2}\frac{d}{dx}(\cos 4x^3) \\
&= \frac{5}{2}\left(\cos 4x^3\right)^{3/2}(-\sin 4x^3) \\
&= -\frac{5}{2}\sin 4x^3\sqrt{(\cos 4x^3)^3}(12x^2) \\
&= -\frac{5}{2}(12x^2)\sin 4x^3\sqrt{\cos^3 4x^3} \\
&= -30x^2\sin 4x^3\sqrt{\cos^3 4x^3}
\end{aligned}$$

7. Calculate y' when $y = \tan^4(2x^3-1)^2$

Solution

$$\begin{aligned}
y = \tan^4(2x^3-1)^2 &= \left[\tan(2x^3-1)^2\right]^4 \\
\frac{dy}{dx} &= 4\left[\tan(2x^3-1)^2\right]^3\frac{d}{dx}\left[\tan(2x^3-1)^2\right] \\
&= 4\left[\tan(2x^3-1)^2\right]^3\sec^2(2x^3-1)^2\frac{d}{dx}\left[(2x^3-1)^2\right] \\
&= 4\tan^3(2x^3-1)^2\sec^2(2x^3-1)^2(2)(2x^3-1)\frac{d}{dx}\left[(2x^3-1)\right] \\
&= 8(2x^3-1)\tan^3(2x^3-1)^2\sec^2(2x^3-1)^2(6x^2) \\
&= 48x^2(2x^3-1)\tan^3(2x^3-1)^2\sec^2(2x^3-1)^2
\end{aligned}$$

8. Find y' when $y = (3x^5-1)\sin 7x^2$

Solution

$$
\begin{aligned}
y &= (3x^5 - 1)\sin 7x^2 \\
\frac{dy}{dx} &= (3x^5 - 1)\frac{d}{dx}(\sin 7x^2) + \sin 7x^2 \frac{d}{dx}(3x^5 - 1) \\
&= (3x^5 - 1)(\cos 7x^2)\frac{d}{dx}(7x^2) + (\sin 7x^2)(15x^4) \\
&= (3x^5 - 1)(\cos 7x^2)(14x) + 15x^4 \sin 7x^2 \\
&= 14x(3x^5 - 1)\cos 7x^2 + 15x^4 \sin 7x^2
\end{aligned}
$$

9.7 DERIVATIVES OF EXPONENTIAL FUNCTIONS

Given that $f(x) = e^x$, then $f'(x) = e^x \frac{d}{dx}(x)$ and if $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}\frac{d}{dx}(x)$

WORKED EXAMPLES

1. If $y = e^{-2x^3}$, find y'.

Solution

$$
\begin{aligned}
y &= e^{-2x^3} \\
\frac{dy}{dx} &= e^{-2x^3}\frac{d}{dx}(-2x^3) \\
&= e^{-2x^3}(-6x^2) \\
&= -6x^2 e^{-2x^3}
\end{aligned}
$$

2. Given that $y = e^{3x - x^5}$, find y'.

Solution

$$
\begin{aligned}
y &= e^{3x-x^5} \\
\frac{dy}{dx} &= e^{3x-x^5}\frac{d}{dx}(3x-x^5) \\
&= e^{3x-x^5}(3-5x^4) \\
&= (3-5x^4)e^{3x-x^5}
\end{aligned}
$$

3. Find y' when $y = e^{3(1-x^2)^8}$

Solution

$$
\begin{aligned}
y &= e^{3(1-x^2)^8} \\
\frac{dy}{dx} &= e^{3(1-x^2)^8}\frac{d}{dx}\left[3(1-x^2)^8\right] \\
&= e^{3(1-x^2)^8}(3)(8)(1-x^2)^7\frac{d}{dx}(1-x^2) \\
&= e^{3(1-x^2)^8}.24(1-x^2)^7(-2x) \\
&= -48x(1-x^2)^7e^{3(1-x^2)^8}
\end{aligned}
$$

4. Calculate y' when $y = \ln(x^2-1)^3$

Solution

$$
\begin{aligned}
y &= \ln(x^2-1)^3 \\
\frac{dy}{dx} &= \frac{1}{(x^2-1)^3}\frac{d}{dx}(x^2-1)^3 \\
&= \frac{1}{(x^2-1)^3}(3)(x^2-1)^2\frac{d}{dx}(x^2-1) \\
&= \frac{3(x^2-1)^2}{(x^2-1)^3}(2x)
\end{aligned}
$$

$$\frac{dy}{dx} = \frac{6x(x^2-1)^2}{(x^2-1)^3}$$
$$= \frac{6x}{(x^2-1)}$$

5. Calculate $\frac{d}{dx}\left[\ln(2x^4-3x^2-1)^4\right]$

Solution

$$\frac{d}{dx}\left[\ln(2x^4-3x^2-1)^4\right] = \frac{1}{(2x^4-3x^2-1)^4}\frac{d}{dx}(2x^4-3x^2-1)^4$$
$$= \frac{1}{(2x^4-3x^2-1)^4}(4)(2x^4-3x^2-1)^3\frac{d}{dx}(2x^4-3x^2-1)$$
$$= \frac{4}{(2x^4-3x^2-1)}(8x^3-6x)$$
$$= \frac{4(8x^3-6x)}{2x^4-3x^2-1} = \frac{8x(4x^2-3)}{2x^4-3x^2-1}$$

6. Find y' when$y = x^5e^{-4x^2}$

Solution

$$y = x^5e^{-4x^2}$$
$$\frac{dy}{dx} = x^5\frac{d}{dx}(e^{-4x^2}) + e^{-4x^2}\frac{d}{dx}(x^5)$$
$$= x^5e^{-4x^2}\frac{d}{dx}(-4x^2) + e^{-4x^2}(5x^4)$$
$$= x^5e^{-4x^2}(-8x) + 5x^4e^{-4x^2}$$
$$= -8x^6e^{-4x^2} + 5x^4e^{-4x^2}$$
$$= x^4e^{-4x^2}(5-8x^2)$$

7. Calculate $\dfrac{d}{dx}\left[e^{2x}\cos 5x^3\right]$

Solution

$$\begin{aligned}
\frac{d}{dx}\left[e^{2x}\cos 5x^3\right] &= e^{2x}\frac{d}{dx}(\cos 5x^3) + (\cos 5x^3)e^{2x}\frac{d}{dx}(2x) \\
&= e^{2x}(-\sin 5x^3)\frac{d}{dx}(5x^3) + (\cos 5x^3)\frac{d}{dx}(e^{2x}) \\
&= -e^{2x}\sin 5x^3(15x^2) + \cos 5x^3(e^{2x})(2) \\
&= -15x^2e^{2x}\sin 5x^3 + 2e^{2x}\cos 5x^3 \\
&= e^{2x}(2\cos 5x^3 - 15x^2\sin 5x^3)
\end{aligned}$$

9.8 INVERSE TRIGONOMETRY FUNCTIONS

1. Find $\dfrac{dy}{dx}$ when $y = \sin^{-1}x$

Solution

$$\begin{aligned}
y &= \sin^{-1}x \\
\implies \sin y &= x \\
\implies (\cos y)\frac{dy}{dx} &= 1 \\
\frac{dy}{dx} &= \frac{1}{\cos y}
\end{aligned}$$

But $\cos^2 y + \sin^2 y = 1$

$$\begin{aligned}
\implies \cos^2 y &= 1 - \sin^2 y = 1 - x^2 \\
\implies \cos y &= \sqrt{1-x^2} \\
\text{Hence } \frac{dy}{dx} &= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

2. Given that $y = \cos^{-1}2x$ findy'

Solution

$$y=\cos^{-1}2x$$

$$\implies \cos y=2x$$

$$\implies (-\sin y)\frac{dy}{dx}=2$$

$$\frac{dy}{dx}=\frac{-2}{\sin y}$$

But $\cos^2 y+\sin^2 y=1$

$$\implies \sin^2 y=1-\cos^2 y=1-(2x)^2=1-4x^2$$

$$\implies \sin y=\sqrt{1-4x^2}$$

Hence $\dfrac{dy}{dx}=\dfrac{-2}{\sqrt{1-4x^2}}$

3. Calculate $\dfrac{dy}{dx}$ when $y=\tan^{-1}\left(\dfrac{x}{1-x}\right)$ findy'

Solution

$$y=\tan^{-1}\left(\frac{x}{1-x}\right)$$

$$\implies \tan y=\frac{x}{1-x}$$

$$\implies (\sec^2 y)\frac{dy}{dx}=\frac{(1-x)\frac{d}{dx}(x)-x\frac{d}{dx}(1-x)}{(1-x)^2}$$

$$=\frac{(1-x)(1)-x(-1)}{(1-x)^2}$$

$$=\frac{1-x+x}{(1-x)^2}=\frac{1}{(1-x)^2}$$

$$\frac{dy}{dx}=\frac{1}{\sec^2 y}\cdot\frac{1}{(1-x)^2}$$

$$\text{But } \sec^2 y = \tan^2 y + 1 = \left(\frac{x}{1-x}\right)^2 + 1 = \frac{x^2+(1-x)^2}{(1-x)^2}$$

$$\implies \frac{dy}{dx} = \frac{(1-x^2)}{x^2+(1-x^2)} \cdot \frac{1}{(1-x)^2} = \frac{1}{x^2+(1-x)^2}$$

$$= \frac{1}{x^2+1-2x+x^2} = \frac{1}{2x^2-2x+1}$$

NB:

(i) $\dfrac{d}{dx}(\sin^{-1}x) = \dfrac{1}{\sqrt{1-x^2}}\dfrac{d}{dx}(x)$

(ii) $\dfrac{d}{dx}(\cos^{-1}x) = -\dfrac{1}{\sqrt{1-x^2}}\dfrac{d}{dx}(x)$

(iii) $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1+x^2}\dfrac{d}{dx}(x)$

Example:

1. Calculate $\dfrac{d}{dx}(x\sin^{-1}x^2)$

Solution

$$\begin{aligned}
\frac{d}{dx}(x\sin^{-1}x^2) &= x\frac{d}{dx}(\sin^{-1}x^2) + (\sin^{-1}x^2)\frac{d}{dx}(x) \\
&= x.\frac{1}{\sqrt{1-(x^2)^2}}\frac{d}{dx}(x^2) + (\sin^{-1}x^2)(1) \\
&= \frac{x}{\sqrt{1-x^4}}.(2x) + \sin^{-1}x^2 \\
&= \frac{2x^2}{\sqrt{1-x^4}} + \sin^{-1}x^2
\end{aligned}$$

2. Find y' if $y = x\tan^{-1}\sqrt{x}$

Solution

$$\begin{aligned}
y &= x\tan^{-1}\sqrt{x} \\
\frac{dy}{dx} &= x\frac{d}{dx}(\tan^{-1}\sqrt{x}) + (\tan^{-1}\sqrt{x})\frac{d}{dx}(x)
\end{aligned}$$

$$\frac{dy}{dx} = x.\frac{1}{1+(\sqrt{x})^2}\frac{d}{dx}(x^{1/2}) + (\tan^{-1}\sqrt{x})(1)$$
$$= \frac{x}{1+x}.\frac{1}{2}x^{-1/2} + \tan^{-1}\sqrt{x}$$
$$= \frac{x^{1/2}}{2(1+x)} + \tan^{-1}\sqrt{x} = \frac{\sqrt{x}}{2(1+x)} + \tan^{-1}\sqrt{x}$$

9.8.1 Trial Exercise

A. Differentiate the following with respect to x.

(i) $y = (5x^4 - 2x)^3(6x^7 - x^4)^2$

(ii) $y = x^7(3x^3 - 2x^2 - x)^5$

(iii) $y = (3x^5 - 2)^6(x^8 - 4x^3)^3$

(iv) $y = (x-1)^2(2x^5 - x^3)^4(7x - x^6)^7$

(v) $y = x^4(1 - 2x^5)^6(5 - 8x^3)^2$

(vi) $y = e^{-6x^2}\sin 6x^5$

(vii) $y = \dfrac{1}{(5x^3 - 1)^5}$

(viii) $y = \dfrac{1}{x + \sqrt{x}}$

(ix) $y = (2x^7 - 3x^5 - 4x^3 - 3)^8$

(x) $y = \left(\dfrac{3x^4 - 2}{x^3 + 1}\right)^6$

(xi) $y = \dfrac{(x^2 - 5)^6}{(2x^3 + 3)^4}$

(xii) $y = \sin^6(3x^2 - x + 1)$

(xiii) $y = \dfrac{x}{\sqrt{x+1}}$

(xiv) $y = e^{2x-1}\tan 3x$

(xv) $y = \dfrac{\sin x^2}{x^2}$

B. Use implicit differentiation to solve the following:

(i) $y = y^2 - \cos xy = x^2$

(ii) $2x^3y^2 - 5y = x^2 - 3$

(iii) Show that $\dfrac{d^2y}{dx^2} = \dfrac{2}{(x+2y)}$ when $xy + y^3 = 1$

(iv) Prove that $\dfrac{d^2y}{dx^2} = \dfrac{2}{(x+2y)^3}$ when $xy + y^3 = 3$.

(v) Show that when $x^2 + y^2 = a^2$, then $\dfrac{d^2y}{dx^2} = -\dfrac{a}{y^3}$.

C. (i) Show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

(ii) Differentiate the following from first principle:

(α) $3x^2 - 7x + 2$

(β) $\dfrac{1}{\sqrt{x}}$

(γ) $\dfrac{x-1}{x^2}$

(ω) $\dfrac{1}{(x+1)^2}$

D. Differentiate the following:

(i) $y = 4x^9 e^{-2x+x^3}$

(ii) $y = x^2 \ln(x-1)^3$

(iii) $y = (x^2+1)\sqrt{\tan 2x}$

(iv) $y = x^3 \tan^{-1} x^5$

(v) $y = (x+1)^3 \sin^{-1}(3x-1)$

(vi) $y = (x^2-5)\cos^{-1} 4x$

(vii) $y = \left(\dfrac{\sin 2x}{e^{-x}}\right)^6$

(viii) $y = \sqrt{\dfrac{x-1}{x+1}}$

(ix) $y = x^3\sqrt{1+x^2}$

9.9 THE CHAIN RULE

Suppose y is a differentiable function of u, and u in turn is a differentiable function of x;

$$\text{Thus} \quad y = f(u)$$

$$\text{and} \quad u = g(x)$$

Then y is a composite function of x

$$\implies \quad \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$$

E.g. 1 If $y = 2u$ and $u = 3x$ then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}.\frac{du}{dx}\\ &= (2)(3) = 6\end{aligned}$$

E.g. 2 If $y = u^3$ and $u = x^2$ then

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}.\frac{du}{dx}\\ &= 3u^2(2x)\\ &= 3(x^2)(2x) = 6x^5\end{aligned}$$

E.g. 3 Find $\dfrac{du}{dx}$ given that $y = \dfrac{u-1}{u+1}$ and $u = x^2$

Solution

$$y=\frac{u-1}{u+1};\quad u=x^2$$

$$\frac{dy}{du}=\frac{(u+1)(1)-(u-1)(1)}{(u+1)^2}=\frac{2}{(u+1)^2}$$

$$\frac{du}{dx}=2x$$

but $$\frac{du}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

$$=\left(\frac{2}{(u+1)^2}\right)(2x)=\frac{4x}{(x^2+1)^2}$$

E.g. 4 Find $\frac{dy}{ds}$ given that $y=3u+1,\ u=\frac{1}{x^2},\ x=1-s$

Solution

$$\frac{dy}{du}=3;\quad \frac{du}{dx}=-2x^{-3};\quad \frac{dx}{ds}=-1$$

But $$\frac{dy}{ds}=\frac{dy}{du}\cdot\frac{du}{dx}\cdot\frac{dx}{ds}$$

$$=(3)(-2x^{-3})(-1)$$

$$=6x^{-3}$$

$$=6(1-s)^{-3}=\frac{6}{(1-s)^3}$$

E.g. 5 Find $\frac{dy}{dt}$ at $t=9$ given that $\frac{u+2}{u-1},\ u=(3s-7)^2$ and $s=\sqrt{t}$

Solution

$$\frac{dy}{du}=-\frac{3}{(u-1)^2};\quad \frac{du}{ds}=6(3s-7);\quad \frac{ds}{dt}=-\frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt}=\frac{dy}{du}\cdot\frac{du}{ds}\cdot\frac{ds}{dt}$$

$$=-\left[\frac{3}{(u-1)^2}\right][6(3s-7)]\left[\frac{1}{2\sqrt{t}}\right]$$

At $t=9$ we have $s=3$ and $u=4$

$$
\begin{aligned}
\implies \frac{dy}{dt} &= -\left[\frac{3}{(4-1)^2}\right][6(9-7)]\left[\frac{1}{2\sqrt{9}}\right] \\
&= \left(-\frac{1}{3}\right)(12)\left(-\frac{1}{6}\right) \\
&= -\frac{2}{3}
\end{aligned}
$$

9.9.1 Exercise

1. Find $\dfrac{dy}{dx}$ given that $y = \dfrac{1}{1+u^2}$; $\quad u = 2x+1$
2. Find $\dfrac{dy}{dx}$ given that $y = \dfrac{2u}{1-4u}$; $\quad u = (3x+1)^4$
3. Find $\dfrac{dy}{dt}$ if $y = 1+u^2$; $\quad u = \dfrac{1-7x}{1+x^2}$; $x = 5t+2$

9.10 PARAMETRIC EQUATIONS

In some cases, it is more convenient to represent a function by expressing x and y separately in terms of a third independent variable called a ***parameter***. E.g. $y = \cos 2t$, $\sin t$, the two equations are parametric equations and the parameter is t.

E.g. 1 Given $y = \cos 2t$ and $x = \sin t$, find $\dfrac{dy}{dx}$.

Solution

$$
\begin{aligned}
\frac{dy}{dt} = -2\sin 2t; \quad \frac{dx}{dt} &= \cos t \\
\implies \frac{dy}{dx} &= \frac{dy}{dt}.\frac{dt}{dx} \\
&= \frac{dy}{dt}.\frac{1}{\frac{dx}{dt}} = (-2\sin 2t)\left(\frac{1}{\cos t}\right) \\
&= -4\sin t\cos t.\frac{1}{\cos t} \\
&= -4\sin t
\end{aligned}
$$

E.g. 2 If $x = \dfrac{2-3t}{1+t}$ and $y = \dfrac{3+2t}{1+t}$, find $\dfrac{dy}{dx}$.

Solution

$$
\begin{aligned}
x &= \frac{2-3t}{1+t} \\
\implies \quad \frac{dx}{dt} &= \frac{(1+t)(-3)-(2-3t)(1)}{(1+t)^2} \\
&= \frac{-3-3t-2+3t}{(1+t)^2} = \frac{-5}{(1+t)^2} \\
y &= \frac{3+2t}{1+t} \\
\frac{dy}{dt} &= \frac{(1+t)(2)-(3+2t)(1)}{(1+t)^2} \\
&= \frac{2+2t-3-2t}{(1+t)^2} = \frac{-1}{(1+t)^2} \\
\implies \quad \frac{dy}{dx} &= \frac{dy}{dt} . \frac{dt}{dx} \\
&= \frac{dy}{dt} . \frac{1}{\frac{dx}{dt}} = \frac{-1}{(1+t)^2}\left(\frac{(1+t)^2}{-5}\right) = \frac{1}{5}
\end{aligned}
$$

CHAPTER 10

MATHEMATICAL INDUCTION

- This is a set pattern in which natural induction can be carried out. It is a technique for proving the validity of statements about the integer n that are suspected to be true for all integers greater than or equal to some starting integer n.
- Suppose we have a sequence of identities or theorems or rules and we wish to prove that they are all true. Let us call the nth term of the sequence P_n. For the simple form of mathematical induction, the pattern is that we have to show that

  (i) If P_1 is true and

  (ii) P_n is true for any n, then P_{n+1} is true.

- For a general form of mathematical induction, the pattern is that we have to show that

  (i) P_1 is true and

  (ii) If $P_1, P_2, ..., P_n$ for any n, are true then P_{n+1} is also true.

E.g. 1 Prove by mathematical induction that $1+3+5+\cdots+(2n-1)=n^2$

Solution

Let $P_n = 1+3+5+\cdots+(2n-1)=n^2$

For $n=1$ P_1 : LHS $=(2(1)-1)=1$; RHS $=1^2=1$ which is true

For $n=2$ P_2 : LHS $=1+(2(2)-1)=4$; RHS $=2^2=4$ which is true

$\vdots \qquad \vdots$

$\Longrightarrow$ P_n is trueIf P_n is true then P_{n+1} is also true

$$
\begin{aligned}
\implies\ P_{n+1} &= P_n + (n+1)th \text{ term} \\
&= 1+3+5+\cdots+(2n-1)+(2[n+1]-1) = n^2+2[n+1]-1 \\
&= 1+3+5+\cdots+(2n-1)+2n+(2n+1) = n^2+2n+1-1
\end{aligned}
$$

From the RHS $n^2+2[n+1]-1 = (n+1)(n+1)$

$$= (n+1)^2$$

$\therefore$ The formula holds for all positive integral values of n.

E.g. 2 Prove by mathematical induction that $1^3+2^2+3^3+\cdots+n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

<u>**Solution**</u>

Let $P_n = 1^3+2^2+3^3+\cdots+n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

For $n=1$ P_1 : LHS $= 1^3 = 1$; RHS $= \left[\dfrac{1(1+1)}{2}\right]^2 = 1$ which is true

For $n=2$ P_2 : LHS $= 1^3+2^3 = 1+8 = 9$; RHS $= \left[\dfrac{2(2+1)}{2}\right]^3 = 3^2 = 9$ which is true

$\vdots \qquad \vdots$

$\implies$ P_n is true

If P_n is true then P_{n+1} is also true

$$
\begin{aligned}
\implies\ P_{n+1} &= P_n + (n+1)th \text{ term} \\
&= 1^3+2^2+3^3+\cdots+n^3(n+1)^3 = \left[\frac{n(n+1)}{2}\right]^2 + (n+1)^3 \\
&= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\
&= \frac{n^2(n+1)^2+4(n+1)^3}{4} \\
&= \frac{(n+1)^2}{4}\left[n^2+4(n+1)\right] \\
&= \frac{(n+1)^2}{4}\left(n^2+4n+4\right) \\
&= \frac{(n+1)^2}{4}(n+2)^2 \\
&= \left[\frac{(n+1)(n+2)}{2}\right]^2
\end{aligned}
$$

$\therefore$ The formula holds for all positive integral values of n.

E.g. 3 Prove by mathematical induction that $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

<u>Solution</u>

$$\sum_{r=1}^{n} r = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\text{Let } P_n = 1+2+3+\cdots n = \frac{n(n+1)}{2}$$

$$\text{For } P_1: \text{ LHS} = 1; \text{ RHS} = \frac{1(1+1)}{2} = 1 \text{ which is true}$$

$$\text{For } P_2: \text{ LHS} = 1+2 = 3; \text{ RHS} = \frac{2(2+1)}{2} = 3 \text{ which is true}$$

$$\vdots \qquad \vdots$$

$$\Longrightarrow \quad P_n \text{ is true}$$

If P_n is true then P_{n+1} is also true

$$\Longrightarrow P_{n+1} = P_n + (n+1)th \text{ term}$$

$$= 1+2+3+\cdots+n+(n+1) = \frac{n(n+1)}{2} + n + 1$$

$$= \frac{n(n+1)+2(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

E.g. 4 Prove by mathematical induction that $1+7+13+\cdots(6n-5) = n(3n-2)$

<u>Solution</u>

$$\text{Let } P_n = 1+7+13+\cdots+6n-5 = n(3n-2)$$

$$\text{For n =1, } P_1: \text{ LHS} = 1$$

$$\text{RHS} = 1(3.1-2) = 3-2 = 1 \text{ which is true}$$

$$\text{For n=2 } P_2: \text{ LHS} = 1+7 = 8;$$

$$\text{RHS} = 2(3.2-2) = 2(6-2) = 8 \text{ which is true}$$

$$\vdots \qquad \vdots$$

$$\Longrightarrow \quad P_n \text{ is true}$$

If P_n is true then P_{n+1} is also true

$$\Longrightarrow P_{n+1} = P_n + (n+1)th \text{ term}$$

$$\implies 1+7+13+\cdots+(6n-5)+[6(n+1)-5] = n(3n-2)+[6(n+1)-5]$$

$$\begin{aligned}
\implies 1+7+13+\cdots+(6n-5)+(6n+1) &= n(3n-2)+6n+1 \\
&= 3n^2-2n+6n+1 \\
&= 3n^2+4n+1 \\
&= (n+1)(3n+1) \\
&= (n+1)\,[3(n+1)-2]
\end{aligned}$$

$\therefore$ The formula holds for all positive integral values of n.

E.g. 5 Prove by mathematical induction that $1.2+2.3+3.4+\cdots+n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

<u>**Solution**</u>

$$\text{Let } P_n = 1.2+2.3+3.4+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3}$$

$$\text{For n =1, } P_1: \ \text{LHS} = 1(2) = 2;$$

$$\text{RHS} = \frac{1(2)(3)}{3} = 2 \ \text{ which is true}$$

$$\text{For n=2 } P_2: \ \text{LHS} = 1(2)+2(3) = 8$$

$$\text{RHS} = \frac{2(3)(4)}{3} = 8 \ \text{ which is true}$$

$$\vdots \qquad \vdots$$

$$\implies P_n \text{ is true}$$

$$\text{If } P_n \text{ is true then } P_{n+1} \text{ is also true}$$

$$\implies P_{n+1} = P_n + (n+1)th \ \text{ term}$$

$$\begin{aligned}
\implies 1.2+2.3+3.4+\cdots+n(n+1)+(n+1)(n+2) &= \frac{n(n+1)(n+2)}{3}+(n+1)(n+2) \\
&= \frac{n(n+1)(n+2)+3(n+1)(n+2)}{3} \\
&= \frac{(n+1)(n+2)}{3}[n+3] \\
&= \frac{(n+1)(n+2)(n+3)}{3}
\end{aligned}$$

$\therefore$ The formula holds for all positive integral values of n.

E.g. 6 Prove by mathematical induction that the sum of the first n terms of an AP is

$$S_n = \frac{1}{2}n\,[2a+(n-1)d]$$

Solution

$$\text{Let } S_n = a+(a+d)+(a+2d)+\cdots+[a+(n-1)d] = \frac{n}{2}[2a+(n-1)d]$$

$$\text{For n =1, } S_1: \text{ LHS} = a; \text{ RHS} = \frac{1}{2}[2a+(1-1)d] = a \text{ which is true}$$

$$\text{For n=2 } S_2: \text{ LHS} = a+(a+d) = 2a+d = 8; \text{ RHS} = \frac{2}{2}[2a+(2-1)d] = 2a+d \text{ which is true}$$

$$\implies \quad P_n \text{ is true}$$

$$\text{If } P_n \text{ is true then } P_{n+1} \text{ is also true}$$

$$\implies P_{n+1} = P_n + (n+1)th \text{ term}$$

$$\begin{aligned}
\implies a+(a+d)+\cdots+[a+(n-1)d]+[a+([n+1]-1)d] &= \frac{n}{2}[2a+(n-1)d]+a+dn \\
&= \frac{n}{2}[2a+dn-d]+a+dn \\
&= \frac{n(2a+dn-d)+2(a+dn)}{2} \\
&= \frac{2an+dn^2-dn+2a+2dn}{2} \\
&= \frac{2an+dn^2+2a+dn}{2} \\
&= \frac{(2an+2a)+(dn^2+dn)}{2} \\
&= \frac{2a(n+1)+dn(n+1)}{2} \\
&= \frac{n+1}{2}[2a+dn] \\
&= \frac{n+1}{2}[2a+[(n+1)-1]d]
\end{aligned}$$

$\therefore$ This holds for all positive integral values of n.

E.g. 7 Prove by mathematical induction that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{4.5}+\cdots+\frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{3}{2}-\frac{1}{n+1}-\frac{1}{n+2}\right]$$

Solution

$$\text{Let } P_n = \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{4.5} + \cdots + \frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$\text{For n =1, } P_1: \ \text{LHS} = \frac{1}{1.3} = \frac{1}{3};$$

$$\text{RHS} = \frac{1}{2}\left[\frac{3}{2} - \frac{1}{2} - \frac{1}{3}\right]$$

$$= \frac{1}{2}\left(\frac{9-3-2}{6}\right) = \frac{2}{6} = \frac{1}{3} \ \text{ which is true}$$

$$\text{For n =2, } P_1: \ \text{LHS} = \frac{1}{1.3} + \frac{1}{2.4} = \frac{1}{3} + \frac{1}{8} = \frac{8+3}{24} = \frac{11}{24};$$

$$\text{RHS} = \frac{1}{2}\left[\frac{3}{2} - \frac{1}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{2}\left(\frac{18-4-3}{12}\right) = \frac{1}{2}\left[\frac{11}{12}\right] = \frac{11}{24} \ \text{ which is true}$$

$$\vdots$$

$$\implies \quad P_n \ \text{ is true}$$

If P_n is true then P_{n+1} is also true

$$\implies P_{n+1} = P_n + (n+1)th \ \text{ term}$$

$$\frac{1}{1.3} + \frac{1}{2.4} + \cdots + \frac{1}{n(n+2)} + \frac{1}{(n+1)(n+3)} = \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] + \frac{1}{(n+1)(n+3)}$$

Expressing $\frac{1}{(n+1)(n+3)}$ in partial fractions, we have

$$\frac{1}{(n+1)(n+3)} = \frac{\frac{1}{2}}{n+1} - \frac{\frac{1}{2}}{n+3} = \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+3}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right] + \frac{1}{2}\left[\frac{1}{n+1} - \frac{1}{n+3}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+1} - \frac{1}{n+3}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{1}{n+2} - \frac{1}{n+3}\right] \quad \text{which is true for all } n$$

10.0.1 Exercise

Prove by mathematical induction the following statements

1. $\sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

2. $\sum_{r=1}^{n} r(2r-5) = \frac{n(n+1)(4n-13)}{6}$

3. $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$

4. $\sum_{r=1}^{n} \frac{1}{r(r+1)} = \frac{n}{n+1}$

5. $\sum_{r=1}^{n} r(r+1) = \frac{n(n+1)(n+2)}{3}$

6. $\sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)} = \frac{n}{6n+4}$

7. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = 2n^4 - n^2$

8. $1 + 3 + 5 + \cdots + (2n-1) = n^2$

9. $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{n}{4}(n+1)(n+2)(n+3)$

10. $\frac{5}{1.2.3} + \frac{6}{2.3.4} + \frac{7}{3.4.5} + \cdots + \frac{n+4}{n(n+1)(n+2)} = \frac{n(3n+7)}{2(n+1)(n+2)}$