CSM 166: Discrete Mathematics for Computer Science

Complex Numbers

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Content

Complex Numbers

Definition 1 (Complex Numbers)

A complex number is a number that can be expressed in the form $\mathbf{z} = \mathbf{a} + \mathbf{bi}$, *where a and b are real numbers and* $\mathbf{i}$ *is the imaginary unit, that satisfies the equation* $\mathbf{i}^2 = -1$.

Let

$$z = a + bi \tag{1}$$

Then **a** is the real part of z denoted by $a = Re(z)$ and **b** is the imaginary part of z denoted $b = Im(z)$

Complex Numbers

1. If Re(z) = 0 then z is purely imaginary:

$$z = bi$$

2. If $Im(z) = 0$ then z is a real number:

$$z = a$$

Conjugate Of A Complex Numbers

Definition 2

The complex conjugate of a complex number $\mathbf{z} = \mathbf{a} + \mathbf{bi}$ *is defined as*

$$\overline{z} = \mathbf{a} - b\mathbf{i} \tag{2}$$

Operations with Complex Numbers I

Let $z_1 = a_1 + b_1\mathbf{i}$ and $z_2 = a_2 + b_2\mathbf{i}$. Then

1. **Addition**:

$$\begin{aligned} z_1 + z_2 &= a_1 + b_1 i + a_2 + b_2\mathbf{i} \\ &= (a_1 + a_2) + (b_1 + b_2)\mathbf{i} \end{aligned}$$

2. **Subtraction**:

$$\begin{aligned} z_1 - z_2 &= a_1 + b_1 i - (a_2 + b_2)\mathbf{i} \\ &= (a_1 + a_2) + (b_1 - b_2)\mathbf{i} \end{aligned}$$

Operations with Complex Numbers II

3. **Multiplication**:

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + b_1\mathbf{i}) \cdot (a_2 + b_2)\mathbf{i} \\ &= a_1 a_2 + a_1 b_2\mathbf{i} + a_2 b_1\mathbf{i} + b_1 b_2\mathbf{i}^2 \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\mathbf{i} \end{aligned}$$

4. **Division**:

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + b_1\mathbf{i}}{a_2 + b_2\mathbf{i}} = \frac{a_1 + b_1\mathbf{i}}{a_2 + b_2\mathbf{i}} \cdot \frac{a_2 - b_2\mathbf{i}}{a_2 - b_2\mathbf{i}} \\ &= \frac{1}{(a_2^2 + b_2^2)}[a_1 a_2 + b_1 b_2 + (a_2 b_1 - a_1 b_2)\mathbf{i}] \end{aligned}$$

Properties Of Complex Conjugates

1. $\overline{\overline{z}} = z$

2. $Re(z) = \dfrac{z + \overline{z}}{2}$ and $Im(z) = \dfrac{z - \overline{z}}{2\mathbf{i}}$

3. $\overline{z + w} = \overline{z} + \overline{w}$

4. $\overline{z - w} = \overline{z} - \overline{w}$

5. $\overline{zw} = \overline{z}\,\overline{w}$

6. $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

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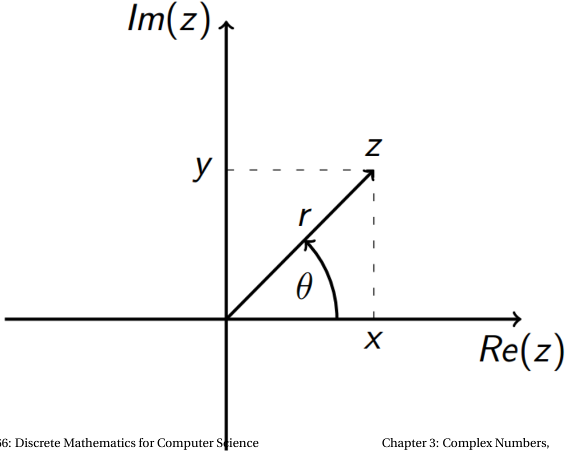
Geometric Representation Of A Complex Number (Argand Diagram)

Let $z = x + y\mathbf{i}$



Absolute Value Of A Complex Numbers

Definition 3

The absolute value of a complex number $z = x + y\mathbf{i}$ is defined as

$$|z| = \sqrt{x^2 + y^2}$$

Thus $|z|$ is the distance from the origin to the point z in the complex plane

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Argument Of A Complex Numbers

Definition 4

*The angle θ is called the **argument** of the complex number z.*

The Principal Argument is $0 < \theta \leq 2\pi$ or
$-\pi < \theta < \pi$

Deduction From The Argand Diagram

From the Argand diagram, we deduce the following

1. $x = r\cos\theta$
2. $y = r\sin\theta$
3. $z = r[\cos\theta + \mathbf{i}\sin\theta]$
4. $|z| = r$
5. $\theta = arg\, z = \tan^{-1}(\frac{y}{x})$

Thus z can be represented in (x, y) or (r, θ) coordinates.
(r, θ) is called **Polar Coordinate**.

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Theorem 1 (Euler)

$$e^{i\theta} = \cos\theta + \mathbf{i}\sin\theta \tag{3}$$

This leads to **De Moivre theorem.**

Theorem 2 (De Moivre)

$$(\cos\theta + \boldsymbol{i}\sin\theta)^n = \cos(n\theta) + \boldsymbol{i}\sin(n\theta)$$

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Operations In Polar Coordinates I

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

1. **Addition** :

$$\begin{aligned} z_1 + z_2 &= r_1 e^{i\theta_1} + r_2 e^{i\theta_2} \\ &= r_1 \cos\theta_1 + i r_1 \sin\theta_1 + r_2 \cos\theta_2 + i r_2 \sin\theta_2 \\ &= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + i(r_1 \sin\theta_1 + r_2 \sin\theta_2) \end{aligned}$$

2. **Subtraction** :

$$\begin{aligned} z_1 - z_2 &= r_1 e^{i\theta_1} - r_2 e^{i\theta_2} \\ &= r_1 \cos\theta_1 + i r_1 \sin\theta_1 - (r_2 \cos\theta_2 + i r_2 \sin\theta_2) \\ &= (r_1 \cos\theta_1 - r_2 \cos\theta_2) + i(r_1 \sin\theta_1 - r_2 \sin\theta_2) \end{aligned}$$

Operations In Polar Coordinates II

3. **Multiplication** :

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

4. **Division** :

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$$

Powers And Roots Of Complex Numbers

From

$$z = re^{i\theta}$$

It implies

$$z = re^{i(\theta+2\pi k)}$$

for $k = 0, \pm1, \pm2, \pm3, \ldots$

Hence $z^n = r^n e^{in(\theta+2\pi k)}$ and $z^{\frac{p}{q}} = r^{\frac{p}{q}} e^{\frac{ip(\theta+2\pi k)}{q}}$
for $k = 0, \pm1, \pm2, \pm3, \ldots$

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$\sin\theta$ And $\cos\theta$

From Euler's Theorem:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$\sinh\theta$ **And** $\cosh\theta$

From Euler's Theorem:

$$sin(i\theta) = \frac{e^{i(i\theta)} - e^{-i(i\theta)}}{2i} = \frac{e^{-\theta} - e^{\theta}}{2i} = \frac{i(e^{\theta} - e^{-\theta})}{2}$$

$$\Rightarrow \sin(i\theta) = i\sinh\theta$$

And

$$\cos(i\theta) = \frac{e^{i(i\theta)} + e^{-i(i\theta)}}{2} = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh\theta$$

Inverse Of A Complex Number

Note:

$$z \cdot \bar{z} = re^{i\theta} \cdot re^{-i\theta} = r^2 = |z|^2$$

The Inverse Of A Complex Number z is given by

$$z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{re^{-i\theta}}{r^2} = \frac{e^{-i\theta}}{r}$$

$\sin^n\theta$ and $\cos^n\theta$

Assumption:

Let $z = e^{i\theta}$ where $r = 1$.

$\cos^n\theta$

$$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

and

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

Thus

$$\cos^n\theta = \frac{1}{2^n}\left(z + \frac{1}{z}\right)^n$$

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$\sin^n\theta$ and $\cos^n\theta$

Assumption:

Let $z = e^{i\theta}$ where $r = 1$.

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and

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Thus

$$\sin^n\theta = \frac{1}{(2i)^n}\left(z - \frac{1}{z}\right)^n$$

$\sin^n\theta$ and $\cos^n\theta$

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and

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Thus

$$\sin^n\theta = \frac{1}{(2i)^n}\left(z - \frac{1}{z}\right)^n$$

Complex Numbers

Example 1

Find the all solutions to the following equations and plot the results on an Argand diagram

i) $Z^3 = 1$

ii) $Z^4 = 1$

iii) $Z^3 = 8$

iv) $z^3 = i$

End of Lecture

Questions...???

Thanks