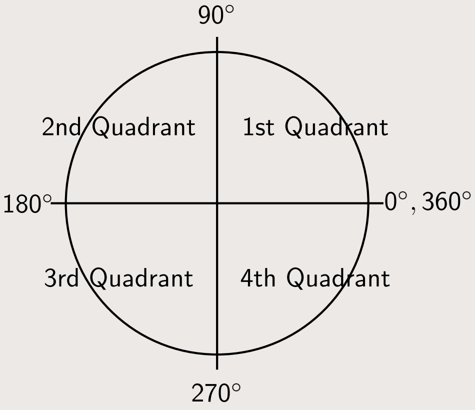
Outline

TRIGONOMETRY

DEFINITION

Trigonometry means "measurement of triangles".A positive angle measures a rotation in an anticlockwise direction.



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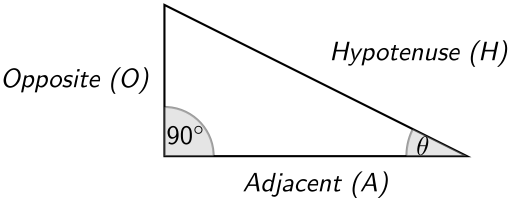
TRIGONOMETRICAL FUNCTIONS



$$sin\theta = \frac{O}{H}$$

$$cos\theta = \frac{A}{H}$$

$$tan\theta = \frac{O}{A}$$

$$tan\theta = \frac{sin\theta}{cos\theta}$$

$$sin^2\theta + cos^2\theta = 1$$

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RADIANS AND ANGLES

$\Pi \ Radian = 180^\circ$

$1 Radian = \dfrac{180^\circ}{\Pi}$ *and* $1^\circ = \dfrac{\Pi}{180} radians$

To change from radians to degrees, multiply by $\dfrac{180}{\Pi}$ and from degrees to radian multiply by $\dfrac{\Pi}{180}$

COMMON ANGLES

Angles in radians	0	$\frac{\Pi}{6}$	$\frac{\Pi}{4}$	$\frac{\Pi}{3}$	$\frac{\Pi}{2}$	Π	$\frac{3\Pi}{2}$	2Π
Angles in degrees	0	30°	45°	60°	90°	180°	270°	360°

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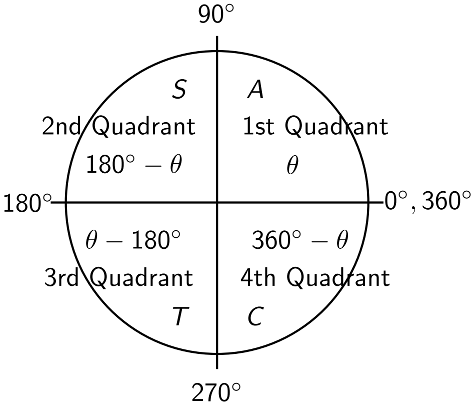
EXPRESSING ALL OTHER ANGLES IN THE ACUTE ANGLE, θ



1^{st} *Quadrant*

$$+sin\theta, +cos\theta, +tan\theta$$

2^{nd} *Quadrant*

$$sin\theta = sin(180^\circ - \theta)$$
$$cos\theta = -cos(180 - \theta)$$
$$tan\theta = -tan(180 - \theta)$$

3^{rd} *Quadrant*

$$sin\theta = -sin(\theta - 180^\circ)$$
$$cos\theta = -cos(\theta - 180^\circ)$$
$$tan\theta = +tan(\theta - 180^\circ)$$

4^{th} *Quadrant*

$$sin\theta = -sin(\theta - 180^\circ)$$
$$cos\theta = cos(\theta - 180^\circ)$$
$$tan\theta = -tan(360^\circ - \theta)$$

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EXAMPLES

$$\begin{aligned} sin150^\circ &= sin(180^\circ - 150^\circ) = sin30^\circ \\ cos150^\circ &= -cos(180^\circ - 150^\circ) = -cos30^\circ \\ sin240^\circ &= -sin(240^\circ - 180^\circ) = -sin60^\circ \\ tan300^\circ &= -tan(360^\circ - 300^\circ) = -tan60^\circ \end{aligned}$$

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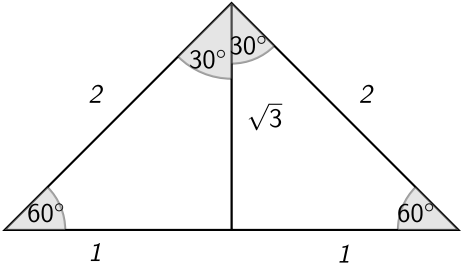
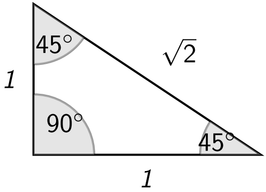
TRIG RATIOS OF $30^\circ, 45^\circ, 60^\circ$



$$sin60^\circ = \frac{\sqrt{3}}{2} \qquad sin30^\circ = \frac{1}{2}$$

$$cos60^\circ = \frac{1}{2} \qquad cos30^\circ = \frac{\sqrt{3}}{2}$$

$$tan60^\circ = \sqrt{3} \qquad tan30^\circ = \frac{1}{\sqrt{3}}$$



$$sin45^\circ = \frac{1}{\sqrt{2}}$$

$$cos45^\circ = \frac{1}{\sqrt{2}}$$

$$tan45^\circ = 1$$

Outline

SOME TRIG IDENTITIES

1. $sin^2\theta + cos^2\theta = 1$
   Dividing through by $sin^2\theta$
2. $1 + cot^2\theta = cosec^2\theta$ $\left[\frac{sin^2\theta}{sin^2\theta} + \frac{cos^2\theta}{sin^2\theta} = \frac{1}{sin^2\theta}\right]$

   Dividing through by $cos^2\theta$
3. $tan^2\theta + 1 = sec^2\theta$ $\left[\frac{sin^2\theta}{cos^2\theta} + \frac{cos^2\theta}{cos^2\theta} = \frac{1}{cos^2\theta}\right]$
4. $\frac{1}{sin\theta} = cosec\theta$; $\frac{1}{cos\theta} = sec\theta$; $\frac{1}{tan\theta} = cot\theta$

DIFFERENCE BETWEEN TWO ANGLES

$$sin(A+B) = sinAcosB + cosAsinB$$
$$sin(A-B) = sinAcosB - cosAsinB$$
$$cos(A+B) = cosAcosB - sinAsinB$$
$$cos(A-B) = cosAcosB + sinAsinB$$

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MULTIPLE ANGLES

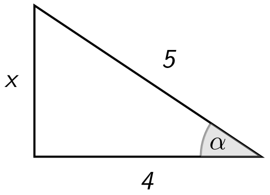
$$\begin{aligned}
sin2A = sin(A+A) &= sinAcosA + cosAsinA \\
&= 2sinAcosA \\
cos2A = cos(A+A) &= cosAcosA - sinAsinA \\
&= cos^2A - sin^2A \\
Using \;\; sin^2A + cos^2A = 1 & \\
cos2A &= cos^2A - (1 - cos^2A) \\
&= 2cos^2A - 1 \\
\implies 2cos^2A &= cos2A + 1 \\
cos^2A &= \frac{1}{2}\left[1 + cos2A\right] \\
Also, & \\
cos2A &= sin^2A - cos^2A \\
&= 1 - sin^2A - sin^2A \\
&= 1 - 2sin^2A
\end{aligned}$$

$$\implies sin^2 A = \frac{1}{2}\left[1 - cos2A\right]$$

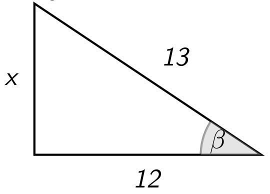
Example

if $cos\alpha = \dfrac{4}{5}$ and $cos\beta = \dfrac{12}{13}$,find the value of $sin(\alpha - \beta)$

Solution



$x^2 + 4^2 = 5^2$
$x^2 = 25 - 16$
$x^2 = 9$
$x = 3$



$x^2 + 12^2 = 13^2$
$x^2 = 169 - 144$
$x^2 = 25$

$x = 5$

$$\begin{aligned} sin\alpha &= \frac{3}{5}, \qquad sin\beta = \frac{5}{13} \\ sin(\alpha - \beta) &= sin\alpha cos\beta - cos\alpha sin\beta \\ &= \frac{3}{5} * \frac{12}{13} - \frac{4}{5} * \frac{5}{13} \\ &= \frac{16}{65} \end{aligned}$$

Prove that

(i) $\dfrac{sin\theta}{1 + cos\theta} = \dfrac{1 - cos\theta}{sin\theta}$

(ii) Find the value of $sin 15°$, leaving answer in surd form.

Solution

(i)

$$\begin{aligned}\frac{sin\theta}{1+cos\theta} &= \frac{sin\theta}{1+cos\theta} * \frac{1-cos\theta}{1-cos\theta}\\ &= \frac{sin\theta(1-cos\theta)}{1-cos^2\theta} = \frac{sin\theta(1-cos\theta)}{sin^2\theta}\\ &= \frac{1-cos\theta}{sin\theta}\end{aligned}$$

(ii)

$$\begin{aligned}sin15^\circ = sin(45^\circ - 30^\circ) &= sin45^\circ cos30^\circ - cos45^\circ sin30^\circ\\ &= \frac{1}{\sqrt{2}} * \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} * \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} * \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Outline

TANGENTS OF COMPOUND ANGLES

$$
\begin{aligned}
tan(A+B) &= \frac{sin(A+B)}{cos(A+B)} \\
&= \frac{sinAcosB + cosAsinB}{cosAcosB - sinAsinB} \\
&= \frac{\frac{sinAcosB}{cosAcosB} + \frac{cosAsinB}{cosAcosB}}{\frac{cosAcosB}{cosAcosB} - \frac{sinAsinB}{cosAcosB}} \\
&= \frac{tanA + tanB}{1 - tanAtanB}
\end{aligned}
$$

$$\begin{aligned} tan(A-B) &= \frac{sin(A-B)}{cos(A-B)} \\ &= \frac{sinAcosB - cosAsinB}{cosAcosB + sinAsinB} \\ &= \frac{\frac{sinAcosB}{cosAcosB} - \frac{cosAsinB}{cosAcosB}}{\frac{cosAcosB}{cosAcosB} + \frac{sinAsinB}{cosAcosB}} \\ &= \frac{tanA - tanB}{1 + tanAtanB} \end{aligned}$$

$$
\begin{aligned}
sin3A &= sin(2A + A) \\
&= sin2AcosA + cos2AsinA \\
&= (2sinAcosA)cosA + (cos^2 A - sin^2 A)sinA \\
&= 2sinAcos^2 A + sinAcos^2 A - sin^3 A \\
&= 3sinAcos^2 A - sin^3 A
\end{aligned}
$$

$$
\begin{aligned}
cos3A &= sin(2A + A) \\
&= cos2AcosA - sin2AsinA \\
&= (cos^2 A - sin^2 A)cosA - (2sinAcosA)sinA \\
&= cos^3 A - sin^2 AcosA - 2sin^2 AcosA \\
&= cos^3 A - 3sin^2 AcosA
\end{aligned}
$$

$$\begin{aligned}
tan3A &= tan(2A + A) \\
&= \frac{tan2A + tanA}{1 - tan2AtanA} \\
&= \frac{\left[\dfrac{2tanA}{1 - tan^2A}\right] + tanA}{1 - \left[\dfrac{2tanA}{1 - tan^2A}\right]tanA} \\
&= \frac{2tanA + tanA(1 - tan^2A)}{1 - tan^2A - 2tanAtanA} \\
&= \frac{2tanA + tanA - tan^3A}{1 - tan^2A - 2tan^2A} \\
&= \frac{3tanA - tan^3A}{1 - 3tan^2A}
\end{aligned}$$

Solving Problems

$t = tan\theta$, simplify the following

(i) $\sqrt{1+t^2}$

(ii) $\dfrac{t}{\sqrt{1+t^2}}$

solution

$$(i) \qquad \begin{aligned} \sqrt{1+t^2} &= \sqrt{1+tan^2\theta} \\ &= \sqrt{sec^2\theta} \\ &= sec\theta \end{aligned}$$

$$(ii) \qquad \begin{aligned} &= \frac{t}{\sqrt{1+t^2}} = \frac{tan\theta}{\sqrt{1+tan^2\theta}} = \frac{tan\theta}{\sqrt{sec^2\theta}} \\ &= \frac{tan\theta}{sec\theta} = \frac{sin\theta}{cos\theta} * \frac{1}{sec\theta} = \frac{sin\theta}{cos\theta} * cos\theta \\ &= sin\theta \end{aligned}$$

Outline

HALF ANGLES

$$sin2A = 2sinAcosB$$

$$\implies sinA = 2sin\frac{A}{2}cos\frac{A}{2}$$

$$cos2A = cos^2A - sin^2A$$

$$\implies cosA = cos^2\frac{A}{2} - sin^2\frac{A}{2}$$

$$= 1 - 2sin^2\frac{A}{2} = 2cos^2\frac{A}{2} - 1$$

$$tan2A = \frac{2tanA}{1 - tan^2A} = tanA = \frac{2tan\frac{A}{2}}{1 - tan^2\frac{A}{2}}$$

EXPRESSING HALF ANGLES IN TERMS OF TANGENTS $\left(t = tan\frac{A}{B}\right)$

$$sinA = 2sin\frac{A}{2}cos\frac{A}{2}$$

$$= \frac{2sin\frac{A}{2}cos\frac{A}{2}}{sin^2\frac{A}{2} + cos^2\frac{A}{2}} = \frac{2\frac{sin\frac{A}{2}cos\frac{A}{2}}{cos^2\frac{A}{2}}}{\frac{sin^2\frac{A}{2}}{cos^2\frac{A}{2}} + \frac{cos^2\frac{A}{2}}{cos^2\frac{A}{2}}} = \frac{\frac{2sin\frac{A}{2}}{cos\frac{A}{2}}}{tan^2\frac{A}{2} + 1}$$

$$= \frac{2t}{1 + t^2}$$

$$cosA = 2cos^2\frac{A}{2} - sin^2\frac{A}{2}$$

$$= \frac{cos^2\frac{A}{2} - sin^2\frac{A}{2}}{sin^2\frac{A}{2} + cos^2\frac{A}{2}} = \frac{\frac{cos^2\frac{A}{2}}{cos^2\frac{A}{2}} - \frac{sin^2\frac{A}{2}}{cos^2\frac{A}{2}}}{\frac{sin^2\frac{A}{2}}{cos^2\frac{A}{2}} + \frac{cos^2\frac{A}{2}}{cos^2\frac{A}{2}}} = \frac{1-t^2}{1+t^2}$$

$$tanA = \frac{2tan\frac{A}{2}}{1 - tan^2\frac{A}{2}} = \frac{2t}{1-t^2}$$

Outline

FACTOR FOMULAE

$$sinA + sinB = 2sin\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

$$sinA - sinB = 2cos\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

$$cosA + cosB = 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

$$cosA - cosB = -2sin\left(\frac{A+B}{2}\right)sin\left(\frac{A-B}{2}\right)$$

Example

Exx 1. Express $sin4\theta + sin\theta$ as a factor

Solution

$$sinA + sinB = 2sin\frac{1}{2}(A+B)cos\frac{1}{2}(A-B)$$

$$\implies sin\theta + sin\theta = 2sin\frac{1}{2}5\theta cos\frac{1}{2}3\theta$$

$$= 2sin\frac{5\theta}{2}cos\frac{3}{2}\theta$$

Outline

PARAMETRIC EQUATIONS

Exx Eliminate θ from the following equations.

(i) $x = 3cos\theta$ -5 and $y = 3 + 2sin\theta$

(ii) $x = atan\theta$ and $y = bcos\theta$

Soln,

$$(i) \qquad cos\theta = \frac{x+5}{3} \quad and \quad sin\theta = \frac{y-3}{2}$$

$$\left(\frac{x+5}{3}\right)^2 + \left(\frac{y-3}{2}\right)^2 = cos^2\theta + sin^2\theta = 1$$

$$(ii) \qquad tan\theta = \frac{x}{a} \quad and \quad cos\theta = \frac{y}{b}$$

$$\implies \frac{1}{cos\theta} = sec\theta = \frac{b}{y}$$

$$But \quad tan^2\theta + 1 = sec^2\theta$$

$$\left(\frac{x}{a}\right)^2 + 1 = \left(\frac{b}{y}\right)^2 \implies \frac{x^2}{a^2} + 1 = \frac{b^2}{y^2}$$