NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS: MULTI-STEP METHODS

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Lecture Outline

Outline of Presentation

Multi-step Methods

1. A k-step multi-step method requires a previous k number of values to start the iteration process.
2. The k values that are required for starting the iteration are obtained using some single-step schemes.
3. The chosen single-step scheme should be of the same or lower order than the order of the multi-step method.
4. A classical example of the explicit multi-step method is the Adams-Bashforth scheme.
5. For the implicit schemes, we have the Adams-Moulton scheme and the Milne-Simpson scheme.

Single-Step and Multi-step Methods

Explicit schemes are predictor method, while implicit schemes are corrector methods.

Single-step predictor methods

Euler, modified Euler, Euler-Cauchy and Runge-Kutta methods

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Backward Euler, and the trapezium methods.

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Adams-Moulton, and Milne-Simpson Method.

However, multi-step schemes (explicit and implicit) are not self starting. They always require the assistance of some single-step methods to start the iteration.

Outline of Presentation

1. Introduction
2. Multi-step Predictor Method
   - Adams-Bashforth Scheme
3. Multi-step Corrector Methods
   - Adams-Moulton Method
   - Milne-Simpson Method
4. Predictor-Corrector Methods

Adams-Bashforth Scheme

A general Adams-Bashforth (AB) method is given by the equation

$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\Delta f_i + \frac{5}{12}\Delta^2 f_i + \frac{3}{8}\Delta^3 f_i + \frac{251}{720}\Delta^4 f_i + \cdots\right] \tag{1}$$

Equation (1) is an infinite series, each truncation yields a different iterative scheme.

Note

$$\Delta f_i = f_i - f_{i-1} \tag{2}$$

again

$$\begin{aligned}
\Delta^2 f_i &= \Delta f_i - \Delta f_{i-1} && (3)\\
&= (f_i - f_{i-1}) - (f_{i-1} - f_{i-2}) && (4)\\
&= f_i - 2f_{i-1} + f_{i-2} && (5)
\end{aligned}$$

Higher-order changes in f_i can be deduced by the same continuous iterations.

Note

The following are synonymous

$$\begin{aligned}
f_i &= f(x_i,\ y_i) && (6)\\
f_{i-1} &= f(x_{i-1},\ y_{i-1}) && (7)\\
f_{i-2} &= f(x_{i-2},\ y_{i-2}) && (8)
\end{aligned}$$

Case 1: AB1 or Euler Method

$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\Delta f_i + \frac{5}{12}\Delta^2 f_i + \frac{3}{8}\Delta^3 f_i + \frac{251}{720}\Delta^4 f_i + \cdots\right] \qquad (9)$$

This first-order AB method or simply AB1 method is obtained by chopping eq. (9) after the first term; considering the terms in the square bracket. This yields:

$$y_{i+1} = y_i + h(f_i) \qquad (10)$$
$$= y_i + hf(x_i, y_i) \qquad (11)$$

This is the same as the Euler method.

Case 2: AB2

$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\Delta f_i + \frac{5}{12}\Delta^2 f_i + \frac{3}{8}\Delta^3 f_i + \frac{251}{720}\Delta^4 f_i + \cdots\right] \tag{12}$$

The second-order AB method or simply AB2 method is obtained by chopping eq. (12) after the second term; considering the terms in the square bracket. This yields:

$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\Delta f_i\right] \tag{13}$$

$$= y_i + h\left[f_i + \frac{1}{2}\left(f_i - f_{i-1}\right)\right] \tag{14}$$

$$= y_i + \frac{h}{2}\left[3f_i - f_{i-1}\right] \tag{15}$$

Case 3: AB3

$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\Delta f_i + \frac{5}{12}\Delta^2 f_i + \frac{3}{8}\Delta^3 f_i + \frac{251}{720}\Delta^4 f_i + \cdots\right] \tag{16}$$

The third-order AB method or simply AB3 method is obtained by chopping eq. (16) after the third term; considering the terms in the square bracket. This yields:

$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\Delta f_i + \frac{5}{12}\Delta^2 f_i\right] \tag{17}$$

Thus, (17) can be simplified as

$$y_{i+1} = y_i + \frac{h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}] \tag{18}$$

Case 4: AB4

$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\Delta f_i + \frac{5}{12}\Delta^2 f_i + \frac{3}{8}\Delta^3 f_i + \frac{251}{720}\Delta^4 f_i + \cdots\right] \tag{19}$$

The fourth-order AB method or simply AB4 method is obtained by chopping eq. (19) after the fourth term ; considering the terms in the square bracket. This yields:

$$y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\Delta f_i + \frac{5}{12}\Delta^2 f_i + \frac{3}{8}\Delta^3 f_i\right] \tag{20}$$

Thus, (20) can be simplified as

$$y_{i+1} = y_i + \frac{h}{24}[55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}] \tag{21}$$

Example

Solve the following IVP

$$yy' = x, \quad x \in [0,\ 1], \quad y(0) = 1, \quad h = 0.2$$

using the Adams-Bashforth method of third-order.
Compute all previous values of y using Runge-Kutta of fourth-order.

Solution

Given the step size $h = 0.2$. Then the x values are given by the **interval table**

$$x_0 = 0, \quad x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6, \quad x_4 = 0.8, \quad x_5 = 1 \tag{22}$$

We know that

$$y' = f(x, y) = \frac{x}{y} \tag{23}$$

Iteratively,

$$f(x_i, y_i) = \frac{x_i}{y_i} \tag{24}$$

AB3 is given by the formula

$$y_{i+1} = y_i + \frac{h}{12}[23f_i - 16f_{i-1} + 5f_{i-2}] \tag{25}$$

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AB3 is given by the formula

$$y_{i+1} = y_i + \frac{h}{12}[23 f_i - 16 f_{i-1} + 5 f_{i-2}] \tag{25}$$

To begin the iteration, we identify an ith value when substituted into eq. (25) the first f value will be f_0. So we begin the iteration with $i = 2$.

Iteration 1: when i=2

Equation (25) reduces

$$y_3 = y_2 + \frac{h}{12}[23f_2 - 16f_1 + 5f_0] \tag{26}$$

1. We are required to find the values of f_0, f_1 and f_2 before a solution could be obtained. From the question, these values are to obtained using the Runge-Kutta scheme.
2. We will skip the details of the computation here, since it has been solved under the single-step scheme in the previous lecture.
3. We can recall that, the nodal points of the RK4 solution are:

$$(x_0, y_0) = (0, 1); \quad (x_1, y_1) = (0.2, 1.02); \quad (x_2, y_2) = (0.4, 1.079); \quad (x_3, y_3) = (0.6, 1.168) \quad (2$$

Only the first three point is needed for this computation.

Iteration 1: when i=2

Thus

$$\begin{aligned} f_0 &= f(x_0, y_0) = \frac{x_0}{y_0} = \frac{0}{1} = 0 \\ f_1 &= f(x_1, y_1) = \frac{x_1}{y_1} = \frac{0.2}{1.02} = 1.923 \\ f_2 &= f(x_2, y_2) = \frac{x_2}{y_2} = \frac{0.4}{1.079} = 0.3707 \end{aligned} \tag{28}$$

Iteration 1: when i=2

Thus

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Now we can substitute eq. (28) into eq. (26):

$$y_3 = y_2 + \frac{h}{12}[23f_2 - 16f_1 + 5f_0] \tag{29}$$

$$= 1.079 + \frac{0.2}{12}[23(0.3707) - 16(1.923) + 5(0)] \tag{30}$$

$$= 1.079 + 0.167(5.47) \tag{31}$$

$$= 1.99 \tag{32}$$

Iteration 2: when i=3

Equation (25) reduces

$$y_4 = y_3 + \frac{h}{12}[23f_3 - 16f_2 + 5f_1] \tag{33}$$

The new f value that is to estimated is f_3, since the other values f_2 and f_1 are known.

Iteration 2: when i=3

Equation (25) reduces

$$y_4 = y_3 + \frac{h}{12}[23 f_3 - 16 f_2 + 5 f_1] \tag{33}$$

The new f value that is to estimated is f_3, since the other values f_2 and f_1 are known. Therefore

$$\begin{aligned} f_1 &= f(x_1, y_1) = \frac{x_1}{y_1} = \frac{0.2}{1.02} = 1.923 \\ f_2 &= f(x_2, y_2) = \frac{x_2}{y_2} = \frac{0.4}{1.079} = 0.3707 \\ f_3 &= f(x_3, y_3) = \frac{x_3}{y_3} = \frac{0.6}{1.99} = 0.3 \end{aligned} \tag{34}$$

Iteration 2: when i=3

Note: The values for f_3 are from the interval table (22), $x_3 = 0.6$, and the previous solution $y_3 = 1.99$.

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Now we can substitute eq. (34) into eq. (33)

$$y_4 = y_3 + \frac{h}{12}[23 f_3 - 16 f_2 + 5 f_1] \tag{35}$$

$$= 1.99 + \frac{0.2}{12}[23(0.3) - 16(0.37) + 5(0.19)] \tag{36}$$

$$= 1.99 + 0.167(1.95) \tag{37}$$

$$= 2.31 \tag{38}$$

Iteration 3: when i=4

Equation (25) reduces

$$y_5 = y_4 + \frac{h}{12}[23f_4 - 16f_3 + 5f_2] \tag{39}$$

The new f value that is to estimated is f_4, since the other values f_3 and f_2 are known.

Iteration 3: when i=4

Equation (25) reduces

$$y_5 = y_4 + \frac{h}{12}[23f_4 - 16f_3 + 5f_2] \tag{39}$$

The new f value that is to estimated is f_4, since the other values f_3 and f_2 are known. Therefore

$$\begin{aligned} f_2 &= f(x_2, y_2) = \frac{x_2}{y_2} = \frac{0.4}{1.079} = 0.3707 \\ f_3 &= f(x_3, y_3) = \frac{x_3}{y_3} = \frac{0.6}{1.99} = 0.3 \\ f_4 &= f(x_4, y_4) = \frac{x_4}{y_4} = \frac{0.8}{2.31} = 0.346 \end{aligned} \tag{40}$$

Iteration 3: when i=4

Now we can substitute eq. (40) into eq. (39)

$$y_5 = y_4 + \frac{h}{12}[23f_4 - 16f_3 + 5f_2] \tag{41}$$

$$= 2.31 + \frac{0.2}{12}[23(0.346) - 16(0.3) + 5(0.3707)] \tag{42}$$

$$= 2.31 + 0.167(5.008) \tag{43}$$

$$= 3.146 \tag{44}$$

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$$= 2.31 + 0.167(5.008) \tag{43}$$

$$= 3.146 \tag{44}$$

Thus, the nodal points are

$$(x_0, y_0) = (0, 1); \quad (x_1, y_1) = (0.2, 1.02); \quad (x_2, y_2) = (0.4, 1.079); \tag{45}$$

$$(x_3, y_3) = (0.6, 1.99), \quad (x_4, y_4) = (0.8, 2.31), \quad (x_5, y_5) = (1, 3.146) \tag{46}$$

The first three y points were computed using RK4, whiles the last three value were computed using AB3. Thus, the multi-step method (AB3) was started with a single-step method (RK4).

Outline of Presentation

Adams-Moulton Method

The two corrector methods considered here are, Adams-Moulton method and Milne-Simpson method.

The general formula for the Adams-Moulton (AM) method is given by

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\Delta f_{i+1} - \frac{1}{12}\Delta^2 f_{i+1} - \frac{1}{24}\Delta^3 f_{i+1} - \frac{19}{720}\Delta^4 f_{i+1} - \cdots\right] \tag{47}$$

Equation (47) is also an infinite series, so each truncation yields a different iterative scheme.

Case 1: AM1

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\Delta f_{i+1} - \frac{1}{12}\Delta^2 f_{i+1} - \frac{1}{24}\Delta^3 f_{i+1} - \frac{19}{720}\Delta^4 f_{i+1} - \cdots\right] \tag{48}$$

The first-order AM method or AM1 is obtained by truncating eq. (48) after the first term; considering the terms in the square bracket. Thus

$$y_{i+1} = y_i + hf_{i+1} \tag{49}$$

$$= y_i + hf(x_{i+1}, y_{i+1}) \tag{50}$$

This is the same as the backward Euler method.

Case 2: AM2

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\Delta f_{i+1} - \frac{1}{12}\Delta^2 f_{i+1} - \frac{1}{24}\Delta^3 f_{i+1} - \frac{19}{720}\Delta^4 f_{i+1} - \cdots\right] \tag{51}$$

The second-order AM method or AM2 is obtained by truncating eq. (51) after the second term; considering the terms in the square bracket. Thus

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\Delta f_{i+1}\right] \tag{52}$$

$$= y_i + h\left[f_{i+1} - \frac{1}{2}(f_{i+1} - f_i)\right] \tag{53}$$

$$= y_i + \frac{h}{2}\left[f_{i+1} - f_i\right] \tag{54}$$

This is also the same as the trapezium method.

Case 3: AM3

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\Delta f_{i+1} - \frac{1}{12}\Delta^2 f_{i+1} - \frac{1}{24}\Delta^3 f_{i+1} - \frac{19}{720}\Delta^4 f_{i+1} - \cdots\right] \tag{55}$$

The third-order AM method or AM3 is obtained by truncating eq. (55) after the third term; considering the terms in the square bracket. Thus

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\Delta f_{i+1} - \frac{1}{12}\Delta^2 f_{i+1}\right] \tag{56}$$

$$= y_i + h\left[f_{i+1} - \frac{1}{2}(f_{i+1} - f_i) - \frac{1}{12}(f_{i+1} - 2f_i + f_{i-1})\right] \tag{57}$$

$$= y_i + \frac{h}{12}\left[5f_{i+1} + 8f_i - f_{i-1}\right] \tag{58}$$

Case 4: AM4

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\Delta f_{i+1} - \frac{1}{12}\Delta^2 f_{i+1} - \frac{1}{24}\Delta^3 f_{i+1} - \frac{19}{720}\Delta^4 f_{i+1} - \cdots\right] \tag{59}$$

The fourth-order AM method or AM4. This is obtained by truncating eq. (59) after the fourth term; considering the terms in the square bracket. Thus

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{1}{2}\Delta f_{i+1} - \frac{1}{12}\Delta^2 f_{i+1} - \frac{1}{24}\Delta^3 f_{i+1}\right] \tag{60}$$

$$= y_i + \frac{h}{24}\left[9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}\right] \tag{61}$$

Milne-Simpson Method

The general formula for the Milne-Simpson (MS) method is given by

$$y_{i+1} = y_{i-1} + h\left[2f_{i+1} - 2\Delta f_{i+1} + \frac{1}{3}\Delta^2 f_{i+1} - 0 \times \Delta^3 f_{i+1} - \frac{1}{90}\Delta^4 f_{i+1} - \cdots\right] \tag{62}$$

We will skip the derivation of MS1, MS2 and MS3.

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We will skip the derivation of MS1, MS2 and MS3.

At the fourth truncation, we obtain the MS4 or fourth-order Milne-Simpson method as

$$y_{i+1} = y_{i-1} + h\left[2f_{i+1} - 2\Delta f_{i+1} + \frac{1}{3}\Delta^2 f_{i+1} + (0)\Delta^3 f_{i+1}\right] \tag{63}$$

This is simplified as:

$$y_{i+1} = y_{i-1} + \frac{h}{3}\left[f_{i+1} + 4f_i + f_{i-1}\right] \tag{64}$$

Outline of Presentation

Predictor-Corrector Methods

We have derived explicit single-step methods (Euler, Modified Euler, Euler-Cauchy, Runge-Kutta), explicit multi-step method (Adams-Bashforth), implicit single-step methods (Backward Euler, Trapezium) and implicit multi-step methods (Adams-Moulton, Milne-Simpson) for solving initial value problems of the form

$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{65}$$

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$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{65}$$

Explicit Schemes

If we perform analysis for numerical stability of these methods, we find that all explicit methods require very small step lengths to be used for convergence. If the solution of the problem is required over a large interval, we may need to use the method thousands or even millions of steps, which is computationally very expensive.

Predictor-Corrector Methods

Implicit Schemes

However, most implicit methods have strong stability properties, that is, we can use sufficiently large step lengths for computations, and we can obtain convergence. But, we need to solve a non-linear algebraic equation for the solution at each nodal point. This procedure may also be computationally expensive.

Predictor-Corrector Methods

Implicit Schemes

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PC Methods

Therefore, we combine the explicit methods (which have weak stability properties) and implicit methods (which have strong stability properties) to obtain new methods. Such methods are called Predictor-Corrector methods or PC methods.

1. The order of the predictor should be less than or equal to the order of the corrector.
2. If the orders of the predictor and corrector are same, then we may require only one or two corrector iterations.
3. For example, if the predictor and corrector are both of fourth order, then the combination (PC method) is also of fourth-order, and we may require one or two corrector iterations.
4. If the order of the predictor is less than the order of the corrector, then we require more iterations of the corrector.
5. For example, if we use a first-order predictor and a second-order corrector, then one application of the combination gives a result of first order.
6. If corrector is iterated once more, then the order of the combination increases by one, that is the result is now of second-order.
7. If we iterate a third time, then the truncation error of the combination reduces, that is, we may get a better result. Further iterations may not change the results.

Predictor-Corrector Methods

In the computations we will denote P for predictor and C for corrector.

Simple possible combinations of the predictor-corrector method

Predictor: Euler

$$y_{i+1}^{(p)} = y_i + hf(x_i, y_i) \tag{66}$$

Corrector: Backward Euler

$$y_{i+1}^{(c)} = y_i + hf(x_{i+1}, y_{i+1}^{(p)}) \tag{67}$$

Both are of first-order.

Complex possible combinations of the predictor-corrector method

Predictor: Adams-Bashforth of fourth-order

$$y_{i+1}^{(p)} = y_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}] \tag{68}$$

Corrector: Adams-Moulton of fourth-order

$$y_{i+1}^{(c)} = y_i + \frac{h}{24}\left[9f(x_{i+1}, y_{i+1}^{(p)}) + 19f_i - 5f_{i-1} + f_{i-2}\right] \tag{69}$$

Example

Using the fourth-order Adams-Bashforth-Moulton (ABM) predictor-corrector method evaluate $y(0.8)$ if

$$yy' = x, \quad x \in [0,\ 0.8], \quad y(0) = 1, \quad h = 0.2, \quad \epsilon = 0.001$$

Compute the necessary previous values using the Euler method.

Solution

Given the step size $h = 0.2$. Then the x values are given by the **interval table**

$$x_0 = 0, \quad x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6, \quad x_4 = 0.8 \tag{70}$$

We know that

$$y' = f(x, y) = \frac{x}{y} \tag{71}$$

Iteratively,

$$f(x_i, y_i) = \frac{x_i}{y_i} \tag{72}$$

AB4 is given by the formula

$$y_{i+1}^{(p)} = y_i + \frac{h}{24}[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}] \tag{73}$$

Solution

1. To begin the iteration, we identify an ith value when substituted into eq. (73) the first f value will be f_0. So we begin with $i = 3$
2. When $i = 3$, eq. (73) reduces
$$y_4^{(p)} = y_3 + \frac{h}{24}[55 f_3 - 59 f_2 + 37 f_1 - 9 f_0] \tag{74}$$
3. We are required to find the values of f_0, f_1, f_2 and f_3 before a solution would be obtained.
4. From the question, these values are to obtained using the Euler method.
5. We will skip the details of the computation here, since it has been solved under the single-step scheme in the previous lecture.

We recall the that, nodal points of the Euler solution are

$$(x_0, y_0) = (0, 1); \quad (x_1, y_1) = (0.2, 1); \quad (x_2, y_2) = (0.4, 1.04); \quad (x_3, y_3) = (0.6, 1.117)$$

All these points are needed for this computation.
Thus

$$\begin{aligned}
f_0 = f(x_0, y_0) &= \frac{x_0}{y_0} = \frac{0}{1} = 0 \\
f_1 = f(x_1, y_1) &= \frac{x_1}{y_1} = \frac{0.2}{1} = 0.2 \\
f_2 = f(x_2, y_2) &= \frac{x_2}{y_2} = \frac{0.4}{1.04} = 0.385 \\
f_3 = f(x_3, y_3) &= \frac{x_3}{y_3} = \frac{0.6}{1.117} = 0.537
\end{aligned} \tag{75}$$

Now we can substitute eq. (75) into eq. (74)

Predictor

$$y_4^{(p)} = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0] \tag{76}$$

$$= 1.117 + \frac{0.2}{24}[55(0.537) - 59(0.385) + 37(0.2) - 9(0)] \tag{77}$$

$$= 1.23611 \tag{78}$$

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Corrector: The Adams-Moulton of fourth-order is

$$y_{i+1}^{(c)} = y_i + \frac{h}{24}\left[9f(x_{i+1}, y_{i+1}^{(p)}) + 19f_i - 5f_{i-1} + f_{i-2}\right]$$

From here, we can begin the iterations. The $y_{i+1}^{(p)}$ values are coming from the predictor solution.

Iteration 1

When $i = 3$, the AM4 is given us

$$y_4^{(c_1)} = y_3 + \frac{h}{24}\left[9f(x_4, y_4^{(p)}) + 19f_3 - 5f_2 + f_1\right] \tag{79}$$

$$= 1.117 + \frac{0.2}{24}\left[9\left(\frac{0.8}{1.23611}\right) + 19(0.537) - 5(0.385) + 0.2\right] \tag{80}$$

$$= 1.236 \tag{81}$$

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$$= 1.236 \tag{81}$$

1. We will stop the iteration procedure when the stopping criterion is satisfied.
2. Since is the very first iteration, we will need the next iteration to make comparison.
3. In the next step, we replace $y_{i+1}^{(p)}$ with $y_{i+1}^{(c_1)}$ to obtain an iterative scheme in terms of a corrector function.

Iteration 2

$$y_4^{(c_2)} = y_3 + \frac{h}{24}\left[9f(x_4, y_4^{(c_1)}) + 19f_3 - 5f_2 + f_1\right] \tag{82}$$

$$= 1.117 + \frac{0.2}{24}\left[9\left(\frac{0.8}{1.236}\right) + 19(0.537) - 5(0.385) + 0.2\right] \tag{83}$$

$$= 1.236 \tag{84}$$

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$$y_4^{(c_2)} = y_3 + \frac{h}{24}\left[9f(x_4, y_4^{(c_1)}) + 19f_3 - 5f_2 + f_1\right] \tag{82}$$

$$= 1.117 + \frac{0.2}{24}\left[9\left(\frac{0.8}{1.236}\right) + 19(0.537) - 5(0.385) + 0.2\right] \tag{83}$$

$$= 1.236 \tag{84}$$

1. Check stopping criterion: $|y_4^{(c_2)} - y_4^{(c_1)}| = |1.236 - 1.236| = 0 \implies < \epsilon$. Hence stop iterations.
2. Therefore

$$y(0.8) = 1.236 \tag{85}$$

Exercise

1. Using the Runge-Kutta method of order 4, find the $y's$ for $x = 0.1, 0.2, 0.3$ given that
$$\frac{dy}{dx} = xy + y^2, \qquad y(0) = 1$$
Hence find the solution at $x = 0.4$ using the Milne-Simpson method with $\epsilon = 0.05$

2. Given that
$$\frac{dy}{dx} = x^2(1+y), \quad y(1) = 1,\ y(1.1) = 1.233,\ y(1.2) = 1.548,\ y(1.3) = 1.979$$
Evaluate $y(1.4)$ using Adams-Bashforth-Moulton of fourth-order with $\epsilon = 0.05$.

END OF LECTURE

THANK YOU