CSM 165: Discrete Mathematics for Computer Science

Chapter 3: Matrices and Principle of Mathematical Induction

Isaac Afari Addo <addoisaacafari@gmail.com>
National Institute for Mathematical Science (NIMS) - Ghana
Department of Mathematics, KNUST
Kumasi-Ghana.

27/02/2021

Content

Matrices

Principle of Mathematical Induction

Matrices

Definition 1

A matrix is a rectangular array of numbers.

A matrix with m rows and n columns is called an $m \times n$ matrix.

*A matrix with the same number of rows as columns is called **square**.*

Two matrices are equal if they have the same number of rows and the same number of columns and the corresponding entries in every position are equal.

Matrices

Example 1

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & be_{32} & b_{33} \end{bmatrix}$$

A is a 3×2 and B is a 3×3 matrix

B is a square matrix (since it has equal number of rows and column.

Matrix Notation

Definition 2

Let m and n be positive integers and let

$$A = \left[a_{ij}\right]$$

$$= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

The ith row of A is the matrix $\left[a_{i1}, a_{i2}, \ldots, a_{in}\right]$.

Matrix Addition

Definition 3

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ matrices.
The sum of A and B, denoted by $A + B$, is the $m \times n$ matrix that has $a_{ij} + b_{ij}$ as its (i, j)th element.
.i.e.

$$A + B = [a_{ij} + b_{ij}]$$

Example 2

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & -3 \\ 3 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 3 & 4 & -1 \\ 1-3 & 0 & \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & -2 \\ 3 & -1 & -3 \\ 2 & 5 & 2 \end{bmatrix}$$

Multiplication of Matrices

Let **A** be an $m \times k$ matrix and **B** be a $k \times n$ matrix. The product of **A** and B, denoted by **AB**, is the $m \times n$ matrix with its $(i, j)th$ entry equal to the sum of the products of the corresponding elements from the ith row of A and the jth column of **B**

Or

If $\mathbf{AB} = [c_{ij}]$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj}$$

Multiplication of Matrices

Example 3

Let $A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 4 \\ 1 & 1 \\ 3 & 0 \end{bmatrix}$, find AB

Solution

$$AB = \begin{bmatrix} (1\times 2)+(0\times 1)+(4\times 3) & (1\times 4)+(0\times 1)+(4\times 0) \\ (2\times 2)+(1\times 1)+(1\times 3) & (2\times 4)+(1\times 1)+(1\times 0) \\ (3\times 2)+(1\times 1)+(0\times 3) & (3\times 4)+(1\times 1)+(0\times 0) \end{bmatrix}$$

$$= \begin{bmatrix} 2+0+12 & 4+0+0 \\ 4+1+3 & 8+1+0 \\ 6+1+0 & 12+1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 4 \\ 8 & 9 \\ 7 & 13 \end{bmatrix}$$

Transpose of Matrices

Definition 4

The identity matrix of order n is the $n \times n$ matrix $I_n = [\delta_{ij}]$, where

$$\delta_{ij} = \begin{cases} 1 \ if \, i = j \\ 0 \ if i \neq j. \end{cases}$$

i.e

$$I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Transpose of Matrices

Definition 5

Let $A = [a_{ij}]_{m\times n}$, the transpose of A denoted by A^T is the $n \times n$ matrix obtained by interchanging the rows and columns of A .i.e

$$A^T = [a_{ji}]_{n\times m}$$

Example 4

The transpose of the matrix $\begin{bmatrix} 1 & 3 & 5 \\ 4 & 6 & 9 \end{bmatrix}$ is $\begin{bmatrix} 1 & 4 \\ 3 & 6 \\ 5 & 9 \end{bmatrix}$

Symmetric metrix

A square matrix A is called symmetric if $A = A^T$.

Thus $A = [a_{ij}]$ is symmetric if $a_{ij} = a_{ji}\ \forall\, i$ and j with $1 \le i \le n$ and $1 \le j \le n$.

Example 5

The matrix following matrices are symmetric

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 4 & 5 \\ 4 & 3 & 6 \\ 5 & 6 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

Boolean Matrix (Zero-One Matrix)

A matrix A is said to be Boolean if its entries are 0's and 1's

Example 6

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Operations on Boolean Matrix

Definition 6 (Join and Meet)

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$ zero–one matrices.

*The **join** of A and B denoted by $A \vee B$ is defined by*

$$A \vee B = a_{ij} \vee b_{ij} = \begin{cases} 1, \ if\, a_{ij} = 1 \ or \ b_{ij} = 1 \\ 0 \ otherwise \end{cases}$$

*The **meet** of A and B is denoted by $A \wedge B$.*

$$A \wedge B = a_{ij} \wedge b_{ij} = \begin{cases} 1, \ if\, a_{ij} = 1 \ and \ b_{ij} = 1 \\ 0 \ otherwise \end{cases}$$

Example 7

Given $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

find $A \vee B$ and $A \wedge B$

Solution

$$A \vee B = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A \wedge B = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Boolean Product ($\odot$)

Definition 7

*Let $A = [a_{ij}]$ be an $m \times k$ **zero–one** matrix and $B = [b_{ij}]$ be a $k \times n$ zero–one matrix. Then the Boolean product of A and B, denoted by $A \odot B$, is defined by*

$$A \odot B = C = c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ik} \wedge b_{kj}).$$

Example 8

Find the Boolean product of A and B, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$A \odot B$

$$= \begin{bmatrix} (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \\ (0 \wedge 1) \vee (1 \wedge 0) & (0 \wedge 1) \vee (1 \wedge 1) & (0 \wedge 0) \vee (1 \wedge 1) \\ (1 \wedge 1) \vee (0 \wedge 0) & (1 \wedge 1) \vee (0 \wedge 1) & (1 \wedge 0) \vee (0 \wedge 1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 1 & 0 \vee 1 \\ 1 \vee 0 & 1 \vee 0 & 0 \vee 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Boolean Powers

For a square zero-one matrix A, and any $k \geq 0$, the $k-$th Boolean power of A is simply the Boolean product of k factors of A.

The kth Boolean product of A is denoted by $A^{[k]}$

$$A^k = \underbrace{A \odot A \odot A \odot \cdots \odot A}_{k \text{ times}}.$$

NB: We define $A^{[0]}$ to be I_n

Boolean Powers

Example 9

Find $A^{[n]}$ for all positive integers n if

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution

$$A^{[2]} = A \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A^{[3]} = A^{[2]} \odot A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{[4]} = A^{[3]} \odot A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A^{[5]} = A^{[4]} \odot A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$A^{[n]} = A^{[5]}$ for all positive integers n with $n \geq 5$.

Principle of Mathematical Induction

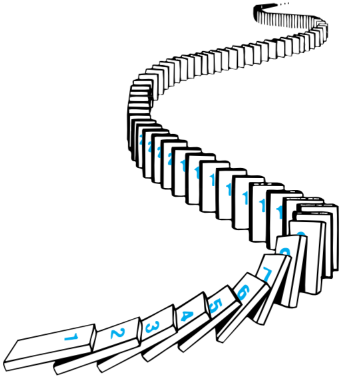

Figure 1 : The Domino effect

Mathematical Induction

Suppose there is a given statement $P(n)$ involving the natural number n such that

1. The statement is true for $n = 1$, i.e., P(1) is true, and

2. If the statement is true for $n = k \quad (k \in \mathbb{Z}^+)$, then the statement is also true for $n = k + 1$, i.e., We show that the conditional statement $P(k) \rightarrow P(k+1)$.

$$(P(1) \wedge \forall k(P(k) \rightarrow P(k+1))) \rightarrow \forall n P(n)$$

Principle of Mathematical Induction

Example 10

Show that if n is a positive integer, then

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Principle of Mathematical Induction

Solution:

1. Base Step: $n = 1$

$$P(1)\ \ 1 = \frac{1(1+1)}{2} = 1$$

$\therefore n = 1$ is true

2. Inductive Step: $n = k$

$$p(k): \sum_{i=1}^{k} = \frac{k(k+1)}{2}$$

Principle of Mathematical Induction

$$
\begin{aligned}
\sum_{i=1}^{k} i &= 1+2+\cdots+k+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)[(k+1)+1]}{2}
\end{aligned}
$$

Hence $p(k+1)$ is true for all k

Principle of Mathematical Induction

Example 11

Prove that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Principle of Mathematical Induction

Solution:

1. Base Step: n=1

$$P(1)\colon 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{2(3)}{6} = 1$$

2. Inductive Step: $n = k$
   Assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

   to be true

We show for $p(k+1)$

$$\sum_{i=1}^{k} i^2 = 1^2 + 2^2 + \cdots + k^2 + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{}$$

$$= \frac{(k+1)[2k(k+2) + 3(k+2)]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Hence $p(k+1)$ is true for all k.

Mathematical Induction

Exercise A: Prove that

1. $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$

2. $2^n > n$ for all positive integers n.

3. $(n^3 + 2n)$ is divisible by 3

4. $2^n < n!$ for $n \geq 4$

End of Lecture

Questions...???

Thanks

Reference Books

1. Kenneth H. Rosen, “Discrete Mathematics and Its Applications”, Tata Mcgraw Hill, New Delhi, India, seventh Edition, 2012.

2. J. P. Tremblay, R. Manohar, “Discrete Mathematical Structures with Applications to Computer Science”, Tata Mc Grforaw Hill, India, 1st Edition, 1997.