

# MATH 162

## INTRODUCTION TO PURE MATHEMATICS

Dr. Ebenezer Bonya

KWAME NKRUMAH UNIVERSITY OF SCIENCE AND  
TECHNOLOGY



COLLEGE OF SCIENCE  
DEPARTMENT OF MATHEMATICS

March 17, 2019

## ❶ INTEGRATION

# Course Content

① INTEGRATION

② COORDINATE GEOMETRY

# Course Content

- 1 INTEGRATION
- 2 COORDINATE GEOMETRY
- 3 CIRCLES

# Course Content

- 1 INTEGRATION
- 2 COORDINATE GEOMETRY
- 3 CIRCLES
- 4 CONIC SECTION

# Integration

$$\int f(x) dx$$

- The differential  $dx$  is written next to the function to be integrated in order to indicate the independent variable used for the original differentiation, and the variable which is to be used for the integration.
- Thus,  $\int f(x) dx$  means that  $f(x)$  is to be integrated with respect to  $x$ .
- It is important to remember that the variables in the function to be integrated and the variables in the differential must be the same.
- Thus,  $\int \cos y dx$  cannot be integrated as it stands. We may first have to express  $\cos y$  as a function of  $x$ .

# Cont.

- We also need to note that, we can use other variables besides  $x$  to indicate the independent variable.
- Thus,  $\int t \, dt$  indicates that  $t$  is the independent variable, and that we need to integrate with respect to  $t$ .

# Integrating $x^n$

By inspection:

$$\int x dx = \frac{1}{2}x^2 + C$$

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

$$\int x^3 dx = \frac{1}{4}x^4 + C$$

$$\int x^4 dx = \frac{1}{5}x^5 + C$$



# Integrating $x^n$

By inspection:

$$\int x dx = \frac{1}{2}x^2 + C$$

$$\int x^2 dx = \frac{1}{3}x^3 + C$$

$$\int x^3 dx = \frac{1}{4}x^4 + C$$

$$\int x^4 dx = \frac{1}{5}x^5 + C$$

From the above, we define that

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

# Integrating $x^n$ (cont.)

**NB:** The formula for integrating  $x^n$  holds for all values of  $n$  except  $n = -1$  which may be treated later.

# Integrating $x^n$ (cont.)

**NB:** The formula for integrating  $x^n$  holds for all values of  $n$  except  $n = -1$  which may be treated later.

Also,

$$\begin{aligned}\int ax^n dx &= a \int x^n dx \\ &= \frac{a}{n+1} x^{n+1} + C\end{aligned}$$

# Integrating $x^n$ (cont.)

**NB:** The formula for integrating  $x^n$  holds for all values of  $n$  except  $n = -1$  which may be treated later.

Also,

$$\begin{aligned}\int ax^n dx &= a \int x^n dx \\ &= \frac{a}{n+1} x^{n+1} + C\end{aligned}$$

Again,

$$\begin{aligned}\int (ax+b)^n dx &= \frac{1}{n+1} (ax+b)^{n+1} \times \frac{1}{\frac{d}{dx}(ax+b)} \\ &= \frac{1}{a(n+1)} (ax+b)^{n+1} + C\end{aligned}$$

# Integrating $x^n$ - Examples

**E.g. 1**

$$\begin{aligned}\int 5x^7 dx &= 5 \int x^7 dx = 5 \cdot \frac{x^8}{8} + C \\ &= \frac{5}{8}x^8 + C\end{aligned}$$

# Integrating $x^n$ - Examples

**E.g. 1**

$$\begin{aligned}\int 5x^7 dx &= 5 \int x^7 dx = 5 \cdot \frac{x^8}{8} + C \\ &= \frac{5}{8}x^8 + C\end{aligned}$$

**E.g. 2**

$$\begin{aligned}\int 4\sqrt{x} dx &= 4 \int x^{\frac{1}{2}} dx = 4 \cdot \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C \\ &= 4 \cdot \frac{2}{3}x^{\frac{2}{3}} + C \\ &= \frac{8}{3}x^{\frac{2}{3}} + C\end{aligned}$$

# Integrating $x^n$ - Examples (cont.)

**E.g. 3**

$$\begin{aligned}\int \frac{1}{\sqrt{x}} dx &= \int x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= 2x^{\frac{1}{2}} + C \\ &= 2\sqrt{x} + C\end{aligned}$$

# Integrating $x^n$ - Examples (cont.)

**E.g. 3**

$$\begin{aligned}\int \frac{1}{\sqrt{x}} dx &= \int x^{-\frac{1}{2}} dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\ &= 2x^{\frac{1}{2}} + C \\ &= 2\sqrt{x} + C\end{aligned}$$

**E.g. 4**

$$\begin{aligned}\int \frac{2}{\sqrt{x}} dx &= 2 \int x^{-\frac{1}{2}} dx = 2 \int x^{-\frac{1}{2}} dx \\ &= 2 \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C \\ &= 2 \cdot 2 \cdot x^{\frac{1}{2}} + C \\ &= 4\sqrt{x} + C\end{aligned}$$



# Integrating $x^n$ - Examples (cont.)

**E.g. 5 - Case For  $n = 0$**

$$\begin{aligned}\int dx &= \int x^0 dx = \frac{x^{0+1}}{0+1} + C \\ &= x + C\end{aligned}$$

# Integrating $x^n$ - Examples (cont.)

**E.g. 5 - Case For  $n = 0$**

$$\begin{aligned}\int dx &= \int x^0 dx = \frac{x^{0+1}}{0+1} + C \\ &= x + C\end{aligned}$$

**E.g. 6 (Case For  $n < 0$ )**

$$\begin{aligned}\int x^{-3} dx &= \frac{x^{-3+1}}{-3+1} + C \\ &= \frac{x^{-2}}{-2} + C \\ &= -\frac{1}{2x^2} + C\end{aligned}$$

# Integrating $x^n$ - Examples (cont.)

**NB:**

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

# Integrating $x^n$ - Examples (cont.)

**NB:**

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

**E.g. 7**

$$\begin{aligned}\int (5x^2 + 4x - 3) dx &= \int 5x^2 dx + \int 4x dx - \int 3 dx \\ &= 5 \int x^2 dx + 4 \int x dx - 3 \int dx \\ &= 5 \cdot \frac{x^3}{3} + 4 \cdot \frac{x^2}{2} - 3 \cdot x + C \\ &= \frac{5}{3}x^3 + 2x^2 - 3x + C\end{aligned}$$

# Integrating $x^n$ - Examples (cont.)

**E.g. 8**

$$\begin{aligned}\int (x^3 - 5x^2 + 7x - 11) dx &= \int x^3 dx - 5 \int x^2 dx + 7 \int x dx - 11 \int dx \\ &= \frac{x^4}{4} - \frac{5}{3}x^3 + \frac{7}{2}x^2 - 11x + C\end{aligned}$$

# Integrating $x^n$ - Examples (cont.)

## E.g. 8

$$\begin{aligned}\int (x^3 - 5x^2 + 7x - 11) dx &= \int x^3 dx - 5 \int x^2 dx + 7 \int x dx - 11 \int dx \\ &= \frac{x^4}{4} - \frac{5}{3}x^3 + \frac{7}{2}x^2 - 11x + C\end{aligned}$$

## E.g. 9

$$\begin{aligned}\int \left( \sqrt[3]{x} - \frac{1}{\sqrt[3]{x}} \right) dx &= \int x^{\frac{1}{3}} dx - \int x^{-\frac{1}{3}} dx \\ &= \frac{1}{\frac{1}{3} + 1} x^{\frac{1}{3} + 1} - \frac{1}{-\frac{1}{3} + 1} x^{-\frac{1}{3} + 1} + C \\ &= \frac{3}{4} x^{\frac{4}{3}} - \frac{3}{2} x^{\frac{2}{3}} + C\end{aligned}$$

# Integrating $\frac{1}{x}$

- If we apply the rule for integrating  $x^n$  to  $\frac{1}{x}$ , that is, to  $x^{-1}$ , we find

$$\int \frac{1}{x} dx = \int x^{-1} dx = \frac{x^{-1+1}}{-1+1} + C = \frac{x^0}{0} + C$$

which involves a division by zero.

- This shows that the rule

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

does not work in the case where  $n = -1$ .

# Integrating $\frac{1}{x}$

- We know however, that the rule for differentiating the logarithm function is

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \quad \text{for } x > 0$$

Hence we conclude that

$$\int \frac{1}{x} dx = \ln x + C \quad \text{for } x > 0 \quad (1)$$

**NB:** This rule only applies when  $x > 0$ .

- However, when we differentiate  $\ln(-x)$  which is defined only for  $x < 0$ , we find that

$$\begin{aligned} \frac{d}{dx}(\ln(-x)) &= -\frac{1}{-x} = \frac{1}{x} \quad \text{for } x < 0 \\ \implies \int \frac{1}{x} dx &= \ln(-x) + C \quad \text{for } x < 0 \end{aligned} \quad (2)$$



# Integrating $\frac{1}{x}$ (cont.)

- We thus combine the two equations (1) and (2) for integrating  $\frac{1}{x}$  by writing

$$\int \frac{1}{x} dx = \ln |x| + C$$

where  $|x|$  is the modulus of  $x$ , and is defined by

$$|x| = x \text{ if } x \geq 0 \text{ and } |x| = -x \text{ if } x < 0$$

By combining the rule for differentiating a function of a function with the last result, we find that if  $y = \ln |f(x)|$  then

$$\frac{dy}{dx} = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$$

Consequently 
$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)|$$

# Integrating $\frac{1}{x}$ (cont.)

- Thus, when integrating a rational (or fractional) function in which, after a suitable adjustment of constant, if necessary, you see that the numerator is the derivative of the denominator, then the integral is the logarithm of the modulus of the denominator.
- Thus, you can use this rule to integrate all rational functions of  $x$  in which the denominator is a linear function, by a suitable adjustment of constants.
- **Example:**

$$\begin{aligned}\int \frac{1}{ax} dx &= \frac{1}{a} \int \frac{a}{ax} dx \\ &= \frac{1}{a} \ln|ax| + C\end{aligned}$$

# Natural Logarithm

**E.g. 1**

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C$$

# Natural Logarithm

**E.g. 1**

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C$$

**E.g. 2**

$$\int \frac{1}{x} dx = \ln |x| + C$$

# Natural Logarithm

**E.g. 1**

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| + C$$

**E.g. 2**

$$\int \frac{1}{x} dx = \ln |x| + C$$

**E.g. 3**

$$\int \frac{1}{3x} dx = \frac{1}{3} \int \frac{1}{x} dx = \frac{1}{3} \ln |x| + C$$

# Natural Logarithm (cont.)

**E.g. 4**

$$\int \frac{1}{7x-4} dx = ?$$

Let  $u = 7x - 4$  and  $du = 7 dx$

$$\begin{aligned}\Rightarrow \int \frac{1}{7x-4} dx &= \int \frac{1}{u} \cdot \frac{1}{7} du = \frac{1}{7} \int \frac{1}{u} du \\ &= \frac{1}{7} \ln|u| + C \\ &= \frac{1}{7} \ln|7x-4| + C\end{aligned}$$

# Natural Logarithm (cont.)

E.g. 5

$$\int \frac{x^3}{x^4 - 1} dx = ?$$

$$\text{Let } u = x^4 - 1 \text{ and } du = 4x^3 dx$$

$$\implies x^3 dx = \frac{du}{4}$$

$$\begin{aligned}\implies \int \frac{x^3}{x^4 - 1} dx &= \int \frac{1}{u} \cdot \frac{1}{4} du = \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{4} \ln |u| + C \\ &= \frac{1}{4} \ln |x^4 - 1| + C\end{aligned}$$

# Natural Logarithm (cont.)

**E.g. 6**

$$\int \cot x dx = ?$$

Let  $u = \sin x$  and  $du = \cos x dx$

$$\begin{aligned}\Rightarrow \int \cot x dx &= \int \frac{\cos x}{\sin x} dx = \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |\sin x| + C\end{aligned}$$



# Natural Logarithm (cont.)

**E.g. 7**

$$\int \frac{1}{x \ln x} dx = ?$$

$$\text{Let } u = \ln x \text{ and } du = \frac{1}{x} dx$$

$$\begin{aligned} \Rightarrow \int \frac{1}{x \ln x} dx &= \int \frac{1}{u} du &&= \ln |u| + C \\ &= \ln |\ln(x)| + C \end{aligned}$$

# Natural Logarithm (cont.)

E.g. 8

$$\int \frac{\cos 2x}{1 - \sin 2x} dx = ?$$

Let  $u = 1 - \sin 2x$  and  $du = -2 \cos 2x dx$

$$\begin{aligned} \Rightarrow \int \frac{\cos 2x}{1 - \sin 2x} dx &= -\frac{1}{2} \int \frac{1}{u} du \\ &= -\frac{1}{2} \ln |u| + C \\ &= -\frac{1}{2} \ln |1 - \sin 2x| + C \end{aligned}$$

# Natural Logarithm (cont.)

**E.g. 9**

$$\int \frac{\sec^2 x}{\tan x} dx = ?$$

Let  $u = \tan x$  and  $du = \sec^2 x dx$

$$\begin{aligned}\Rightarrow \int \frac{\sec^2 x}{\tan x} dx &= \int \frac{1}{u} du \\ &= \ln |u| + C \\ &= \ln |\tan x| + C\end{aligned}$$

# Natural Logarithm (cont.)

**E.g. 10**

$$\int \frac{\ln x}{x} dx = ?$$

$$\text{Let } u = \ln x \text{ and } du = \frac{1}{x} dx$$

$$\begin{aligned} \Rightarrow \int \frac{\ln x}{x} dx &= \int (\ln x) \frac{1}{x} dx = \int u du \\ &= \frac{1}{2} u^2 + C \\ &= \frac{1}{2} (\ln x)^2 + C \end{aligned}$$

# Natural Logarithm (cont.)

**E.g. 11**

$$\int \tan x dx = ?$$

$$\begin{aligned}\text{But } \int \tan x dx &= \int \frac{\sin x}{\cos x} dx = - \int \frac{-\sin x}{\cos x} \\ &= -\ln |\cos x| + C \\ &= \ln |\cos^{-1} x| + C \\ &= \ln |\sec x| + C\end{aligned}$$

# Natural Logarithm (cont.)

**E.g. 12**

$$\int \frac{x}{2x^2 + 3} dx = ?$$

Let  $u = 2x^3 + 3$  and  $du = 4x dx$

$$\begin{aligned}\Rightarrow \int \frac{x}{2x^2 + 3} dx &= \frac{1}{4} \int \frac{4x}{2x^2 + 3} dx = \frac{1}{4} \int \frac{1}{u} du \\ &= \frac{1}{4} \ln|2x^2 + 3| + C\end{aligned}$$

# Natural Logarithm (cont.)

**E.g. 13**

$$\int \frac{2(x+1)}{x^2+2x+7} dx = ?$$
$$\Rightarrow \int \frac{2x+2}{x^2+2x+7} dx = \ln|x^2+2x+7| + C$$

# Natural Logarithm (cont.)

**E.g. 13**

$$\int \frac{2(x+1)}{x^2+2x+7} dx = ?$$
$$\Rightarrow \int \frac{2x+2}{x^2+2x+7} dx = \ln|x^2+2x+7| + C$$

**E.g. 14**

$$\int \frac{3x^5 + 2x^2 - 3}{x^3} dx = ?$$
$$\int \left( \frac{3x^5}{x^3} + \frac{2x^2}{x^3} - \frac{3}{x^3} \right) dx = \int \left( 3x^2 + \frac{2}{x} - \frac{3}{x^3} \right) dx$$
$$= 3 \int x^2 dx + 2 \int \frac{1}{x} dx - 3 \int x^{-3} dx$$
$$= x^3 + 2 \ln|x| + \frac{3}{2} x^{-2} + C$$



# Try the following

Find the integrals of the following

i.  $\int \frac{1}{3-2x} dx$

ii.  $\int \frac{1}{ax+b} dx$

iii.  $\int \frac{2x}{x^2+4} dx$

iv.  $\int \left( \frac{3}{x-1} - \frac{4}{x-2} \right) dx$

v.  $\int \frac{1}{2} x^{-\frac{1}{2}} dx$

vi.  $\int \frac{1}{2\sqrt{2x^3}} dx$

vii.  $\int [(x+3)(x-3)] dx$

viii.  $\int \left( 2 - \frac{1}{3}x^2 - \frac{1}{2\sqrt{x}} \right) dx$

ix.  $\int \left( \frac{1}{x^3} - \frac{1}{x^2} + \frac{1}{x} - 1 \right) dx$

x.  $\int \left( x^{\frac{2}{3}} + 1 + x^{-\frac{2}{3}} \right) dx$

xi.  $\int \frac{x^3-7}{x} dx$

xii.  $\int \frac{1}{\sqrt[3]{x}} dx$

# Definite Integrals - Examples

1. Evaluate  $\int_1^2 \frac{x-1}{x^2-2x+2} dx$

# Definite Integrals - Examples

1. Evaluate  $\int_1^2 \frac{x-1}{x^2-2x+2} dx$

**Solution**

$$\begin{aligned}\int_1^2 \frac{x-1}{x^2-2x+2} dx &= \frac{1}{2} \int_1^2 \frac{2x-2}{x^2-2x+2} dx \\ &= \frac{1}{2} \ln |x^2-2x+2| \Big|_1^2 \\ &= \frac{1}{2} (\ln 2 - \ln 1) \\ &= \frac{1}{2} \ln 2\end{aligned}$$

# Definite Integrals - Examples (cont.)

2. Find  $\int_1^2 \frac{x}{3x^2 - 2} dx$

# Definite Integrals - Examples (cont.)

2. Find  $\int_1^2 \frac{x}{3x^2 - 2} dx$

**Solution**

$$\begin{aligned}\int_1^2 \frac{x}{3x^2 - 2} dx &= \frac{1}{6} \int_1^2 \frac{6x}{3x^2 - 2} dx \\ &= \frac{1}{6} \ln |3x^2 - 2| \Big|_1^2 \\ &= \frac{1}{6} (\ln 10 - \ln 1) \\ &= \frac{1}{6} \ln 10\end{aligned}$$

# Definite Integrals - Examples (cont.)

2. Evaluate  $\int_{\pi/6}^{\pi/2} \frac{\cos x}{2 - \sin x} dx$

# Definite Integrals - Examples (cont.)

2. Evaluate  $\int_{\pi/6}^{\pi/2} \frac{\cos x}{2 - \sin x} dx$

**Solution**

$$\begin{aligned}\int_{\pi/6}^{\pi/2} \frac{\cos x}{2 - \sin x} dx &= - \int_{\pi/6}^{\pi/2} \frac{-\cos x}{2 - \sin x} dx \\ &= - \ln |2 - \sin x| \Big|_{\pi/6}^{\pi/2} \\ &= - \left( \ln 1 - \ln \frac{3}{2} \right) \\ &= \ln \frac{3}{2} = \ln 3 - \ln 2\end{aligned}$$

# Exponential Functions

$$\int e^x dx = e^x + C$$

$$\int ae^x dx = ae^x + C$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax} + C$$



# Exponential Functions - Example

1.

$$\int e^{3x} dx = \frac{1}{3}e^{3x} + C$$

# Exponential Functions - Example

1.

$$\int e^{3x} dx = \frac{1}{3}e^{3x} + C$$

2.

$$\begin{aligned}\int 2e^{5x} dx &= 2 \int e^{5x} + C \\ &= \frac{2}{5}e^{5x} + C\end{aligned}$$

# Integrals of Standard Forms

There are a number of integrals known as STANDARD FORMS. They are derivatives of functions which could be applied or used as such.

# Integrals of Standard Forms

There are a number of integrals known as STANDARD FORMS. They are derivatives of functions which could be applied or used as such.

## Algebraic Functions

$$1. \int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

$$2. \int \frac{1}{x} dx = \ln |x| + C$$

$$3. \int a^x dx = \frac{1}{\ln a} a^x + C$$

$$4. \int e^x dx = e^x + C$$

# Trigonometric Functions

$$1. \int \sin x dx = -\cos x + C$$

$$2. \int \cos x dx = \sin x + C$$

$$3. \int \tan x dx = -\ln |\cos x| + C = \ln |\sec x|$$

$$4. \int \cot x dx = \ln |\sin x| + C$$

# Trigonometric Functions

$$1. \int \sin x dx = -\cos x + C$$

$$2. \int \cos x dx = \sin x + C$$

$$3. \int \tan x dx = -\ln |\cos x| + C = \ln |\sec x|$$

$$4. \int \cot x dx = \ln |\sin x| + C$$

**NB:**

The derivatives of  $\sec x$  and  $\operatorname{cosec} x$  namely  $\sec x \tan x$  and  $-\operatorname{cosec} x \cot x$  do not give rise to standard forms, but to products of them. The integrals of  $\sec x$  and  $\operatorname{cosec} x$  do not arise by direct differentiation.

# Additional Standard Integrals

The following are additional standard forms which are useful for integration. Some of them are results of differentiating standard forms.

$$1. \int \sec x \tan x dx = \sec x + C$$

$$2. \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$$

$$3. \int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$4. \int \sec^2 x dx = \tan x + C$$

# Inverse Trigonometric Functions

$$1. \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

$$2. \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \left( \frac{x}{a} \right) + C$$

$$3. - \int \frac{1}{\sqrt{1-x^2}} dx = \cos^{-1} x + C$$

$$4. - \int \frac{1}{\sqrt{a^2-x^2}} dx = \cos^{-1} \left( \frac{x}{a} \right) + C$$

$$5. \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$6. \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C$$



# Hyperbolic Functions

$$1. \int \sinh x dx = \cosh x + C$$

$$2. \int \cosh x dx = \sinh x + C$$

$$3. \int \tanh x dx = \ln |\cosh x| + C$$

$$4. \int \coth x dx = \ln |\sinh x| + C$$

# Inverse Hyperbolic Functions

$$1. \int \frac{1}{\sqrt{x^2 + a^2}} dx = \sinh^{-1} \left( \frac{x}{a} \right) + C$$

$$2. \int \frac{1}{\sqrt{x^2 - a^2}} dx = \cosh^{-1} \left( \frac{x}{a} \right) + C$$

$$3. \int \frac{1}{a^2 - x^2} dx = \frac{1}{a} \tanh^{-1} \left( \frac{x}{a} \right) + C$$

# Try the following

Find the following integrals

1.  $\int e^{3x-1} dx$

2.  $\int e^{\frac{x}{a}} dx$

3.  $\int \frac{x^2 - x + 1}{x^3} dx$

4.  $\int \sqrt{ax+b} dx$

5.  $\int \frac{x}{x^2-1} dx$

6.  $\int \sqrt{1+\frac{x}{2}} dx$

7.  $\int x(1+x)(1+x^2) dx$

8.  $\int \frac{e^{3x}}{e^{3x}+6} dx$

9.  $\int \sin(\alpha - 3x) dx$

10.  $\int \sin \frac{1}{3}x dx$

11.  $\int 2^x dx$

12.  $\int \frac{\sin ax}{1+\cos ax} dx$

13.  $\int \cos(2x + \beta) dx$

14.  $\int \frac{1}{\sqrt{1-x}} dx$

# Integration by Substitution - Example

1. Find  $\int (3x - 5)^{12} dx$

# Integration by Substitution - Example

1. Find  $\int (3x - 5)^{12} dx$

## Solution

Let  $u = 3x - 5$  and  $du = 3dx$

$$\implies dx = \frac{1}{3} du$$

$$\begin{aligned}\implies \int (3x - 5)^{12} dx &= \int u^{12} \frac{1}{3} du = \frac{1}{3} \int u^{12} du \\ &= \left(\frac{1}{3}\right) \left(\frac{1}{13}\right) u^{13} + C \\ &= \frac{1}{39} u^{13} + C \\ &= \frac{1}{49} (3x - 5)^{12} + C\end{aligned}$$

# Integration by Substitution - Example

2. Find  $\int \sqrt{7x+4} dx$

# Integration by Substitution - Example

2. Find  $\int \sqrt{7x+4} dx$

## Solution

Let  $u = 7x + 4$  and  $du = 7dx$

$$\implies dx = \frac{1}{7} du$$

$$\begin{aligned}\implies \int \sqrt{7x+4} dx &= \int \sqrt{u} \frac{1}{7} du = \frac{1}{7} \int u^{\frac{1}{2}} du \\ &= \left(\frac{1}{7}\right) \left(\frac{2}{3}\right) u^{\frac{3}{2}} + C \\ &= \frac{2}{21} u^{\frac{3}{2}} + C \\ &= \frac{2}{21} (7x+4)^{\frac{3}{2}} + C\end{aligned}$$

# Integration by Substitution - Example

3. Find  $\int (4 - 2t^2)^7 t \, dt$



# Integration by Substitution - Example

3. Find  $\int (4 - 2t^2)^7 t \, dt$

## Solution

Let  $u = 4 - 2t^2$  and  $du = -4t \, dt$

$$\implies dt = -\frac{1}{4t} du$$

$$\begin{aligned}\implies \int (4 - 2t^2)^7 t \, dt &= \int u^7 \left( -\frac{du}{4t} \right) t = -\frac{1}{4} \int u^7 \, du \\ &= -\left( \frac{1}{4} \right) \left( \frac{1}{8} \right) u^8 + C \\ &= -\frac{1}{32} (4 - 2t^2)^8 + C\end{aligned}$$

# Integration by Substitution - Example

4. Find  $\int x^2 \sqrt[3]{x^3 + 5} dx$

# Integration by Substitution - Example

4. Find  $\int x^2 \sqrt[3]{x^3 + 5} dx$

## Solution

Let  $u = x^3 + 5$  and  $du = 3x^2 dx$

$$\implies dx = \frac{1}{3x^2} du \implies x^2 dx = \frac{du}{3}$$

$$\implies \int x^2 \sqrt[3]{x^3 + 5} dx = \int \sqrt[3]{u} \cdot \frac{du}{3} = \frac{1}{3} \int u^{\frac{1}{3}} du$$

$$= \left(\frac{1}{3}\right) \left(\frac{3}{4}\right) u^{\frac{4}{3}} + C$$

$$= \frac{1}{4} (x^3 + 5)^{\frac{4}{3}} + C$$

$$= \frac{1}{4} \left( \sqrt[3]{x^3 + 5} \right)^4 + C$$

# Integration by Substitution - Example

5. Find  $\int \frac{x}{\sqrt{x+1}} dx$

# Integration by Substitution - Example

5. Find  $\int \frac{x}{\sqrt{x+1}} dx$

## Solution

$$\text{Let } u = x + 1 \text{ and } du = dx$$

$$\implies x = u - 1$$

$$\begin{aligned}\implies \int \frac{x}{\sqrt{x+1}} dx &= \int \frac{u-1}{\sqrt{u}} du = \int (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) du \\ &= \frac{2}{3} u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C \\ &= \frac{2}{3} (\sqrt{x+1})^3 - 2\sqrt{x+1} + C \\ &= 2\sqrt{x+1} \left[ \frac{1}{3} (x+1) - 1 \right] + C\end{aligned}$$

# Integration by Substitution - Example

6. Find  $\int \sqrt[3]{x^2 - 2x + 1} dx$

# Integration by Substitution - Example

6. Find  $\int \sqrt[3]{x^2 - 2x + 1} dx$

## Solution

$$\text{Let } u = x^2 - 2x + 1 = (x - 1)^2 \text{ and } u = x - 1 \\ \implies du = dx$$

$$\begin{aligned} \implies \int \sqrt[3]{x^2 - 2x + 1} dx &= \int \sqrt[3]{(x - 1)^2} dx = \int \sqrt[3]{u^2} du \\ &= \int u^{\frac{2}{3}} du = \frac{3}{5} u^{\frac{5}{3}} + C \\ &= \frac{3}{5} (\sqrt[3]{u})^5 + C \\ &= \frac{3}{5} \left( \sqrt[3]{x - 1} \right)^5 + C \end{aligned}$$

# Integration by Substitution - Example

7. Find  $\int (x^4 + 1)^{\frac{1}{3}} x^7 dx$



# Integration by Substitution - Example

7. Find  $\int (x^4 + 1)^{\frac{1}{3}} x^7 dx$

## Solution

$$\text{Let } u = x^4 + 1 \text{ and } du = 4x^3 dx$$

$$\implies x^4 = u - 1 \text{ and } x^3 dx = \frac{du}{4}$$

$$\text{Also } x^7 = x^4 x^3 dx = (u - 1) \frac{du}{4}$$

$$\begin{aligned} \implies \int (x^4 + 1)^{\frac{1}{3}} x^7 dx &= \int u^{\frac{1}{3}} (u - 1) \frac{du}{4} \\ &= \frac{1}{4} \int (u^{\frac{4}{3}} - u^{\frac{1}{3}}) du \\ &= \frac{1}{4} \left( \frac{3}{7} u^{\frac{7}{3}} - \frac{3}{4} u^{\frac{4}{3}} \right) + C \end{aligned}$$

# Integration by Substitution - Example (cont.)

$$\begin{aligned} &= \frac{3}{4} u^{\frac{4}{3}} \left( \frac{1}{7} u - \frac{1}{4} \right) + C \\ &= \frac{3}{4} (x^4 + 1)^{\frac{4}{3}} \left[ \frac{1}{7} (x^4 + 1) - \frac{1}{4} \right] + C \\ &= \frac{3}{4} (x^4 + 1)^{\frac{4}{3}} \left[ \frac{4(x^4 + 1) - 7}{28} \right] + C \\ &= \frac{3}{4} (x^4 + 1)^{\frac{4}{3}} \left[ \frac{4x^4 + 4 - 7}{28} \right] + C \\ &= \frac{3}{112} (x^4 + 1)^{\frac{4}{3}} (4x^4 - 3) + C \end{aligned}$$

# Integration by Substitution - Example

8. Find  $\int \frac{x}{\sqrt{1+5x^2}} dx$

# Integration by Substitution - Example

8. Find  $\int \frac{x}{\sqrt{1+5x^2}} dx$

## Solution

Let  $u = 1 + 5x^2$  and  $du = 10x dx$

$$\begin{aligned}\implies x dx &= \frac{du}{10} \\ \implies \int \frac{x}{\sqrt{1+5x^2}} dx &= \int \frac{1}{\sqrt{u}} \frac{du}{10} \\ &= \frac{1}{10} \int u^{-\frac{1}{2}} du \\ &= \frac{1}{10} \cdot 2u^{\frac{1}{2}} + C \\ &= \frac{1}{5} \sqrt{1+5x^2} + C\end{aligned}$$

# Integration by Substitution - Example

9. Find  $\int x\sqrt{ax+b}dx$  when  $a \neq 0$

# Integration by Substitution - Example

9. Find  $\int x\sqrt{ax+b}dx$  when  $a \neq 0$

## Solution

Let  $u = ax + b$  and  $du = a dx$

$$\implies x = \frac{u-b}{a} \text{ and } dx = \frac{du}{a}$$

$$\begin{aligned}\implies \int x\sqrt{ax+b}dx &= \int \frac{u-b}{a} \sqrt{u} \frac{du}{a} \\ &= \frac{1}{a^2} \int (u^{\frac{3}{2}} - bu^{\frac{1}{2}}) du \\ &= \frac{1}{a^2} \left( \frac{2}{5} u^{\frac{5}{2}} - \frac{2b}{3} u^{\frac{3}{2}} \right) + C\end{aligned}$$

# Integration by Substitution - Example (cont.)

$$\begin{aligned} &= \frac{2}{a^2} u^{\frac{3}{2}} \left( \frac{1}{5} u - \frac{b}{3} \right) + C \\ &= \frac{2}{a^2} \left( \sqrt{ax+b} \right)^3 \left[ \frac{1}{5} (ax+b) - \frac{b}{3} \right] + C \\ &= \frac{2}{a^2} \left( \sqrt{ax+b} \right)^3 \left[ \frac{a}{5} x - \frac{2}{15} b \right] + C \\ &= \frac{2}{15a^2} \left( \sqrt{ax+b} \right)^3 (3ax - 2b) + C \end{aligned}$$

# Using Substitution For Trig. Functions - Example

1. Find  $\int \sin(3x - 1) dx$



# Using Substitution For Trig. Functions - Example

1. Find  $\int \sin(3x - 1) dx$

## Solution

Let  $u = 3x - 1$  and  $du = 3 dx$

$$\implies dx = \frac{du}{3}$$

$$\begin{aligned}\implies \int \sin(3x - 1) dx &= \int \sin u \frac{du}{3} \\ &= \frac{1}{3} \int \sin u du \\ &= \frac{1}{3} (-\cos u) + C \\ &= -\frac{1}{3} \cos(3x - 1) + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

2. Find  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

# Using Substitution For Trig. Functions - Example

2. Find  $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$

## Solution

$$\text{Let } u = \sqrt{x} = x^{1/2} \text{ and } du = \frac{1}{2} \left( \frac{1}{\sqrt{x}} \right) dx$$

$$\implies 2du = \frac{1}{\sqrt{x}} dx$$

$$\implies \int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = \int \cos u \cdot 2 du$$

$$= 2 \int \cos u du$$

$$= 2(\sin u) + C$$

$$= 2 \sin(\sqrt{x}) + C$$

# Using Substitution For Trig. Functions - Example

3. Find  $\int \sec^2\left(\frac{x}{2}\right) dx$

# Using Substitution For Trig. Functions - Example

3. Find  $\int \sec^2\left(\frac{x}{2}\right) dx$

## Solution

$$\text{Let } u = \frac{x}{2} \text{ and } du = \frac{1}{2} dx$$

$$\implies 2du = dx$$

$$\implies \int \sec^2\left(\frac{x}{2}\right) dx = \int \sec^2 u \cdot 2 du$$

$$= 2 \int \sec^2 u du$$

$$= 2(\tan u) + C$$

$$= 2 \tan\left(\frac{x}{2}\right) + C$$

# Using Substitution For Trig. Functions - Example

4. Find  $\int \frac{\cos 3x}{\sin^2 3x} dx$

# Using Substitution For Trig. Functions - Example

4. Find  $\int \frac{\cos 3x}{\sin^2 3x} dx$

## Solution

Let  $u = \sin 3x$  and  $du = 3 \cos 3x dx$

$$\Rightarrow \frac{du}{3} = \cos 3x dx$$

$$\Rightarrow \int \frac{\cos 3x}{\sin^2 3x} dx = \int \frac{1}{u^2} \frac{du}{3} = \frac{1}{3} \int u^{-2} du$$

$$= \frac{1}{3}(-u^{-1}) + C$$

$$= -\frac{1}{3u} + C$$

$$= -\frac{1}{3 \sin 3x} + C = -\frac{1}{3} \operatorname{cosec} 3x + C$$

# Using Substitution For Trig. Functions - Example

5. Find  $\int \sin^4 x \cos x \, dx$



# Using Substitution For Trig. Functions - Example

5. Find  $\int \sin^4 x \cos x \, dx$

## Solution

Let  $u = \sin x$  and  $du = \cos x \, dx$

$$\begin{aligned}\Rightarrow \int \sin^4 x \cos x \, dx &= \int u^4 \, du \\ &= \frac{1}{5} u^5 + C \\ &= \frac{1}{5} \sin^5 x + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

6. Find  $\int \sin 3x \cos 3x \, dx$

# Using Substitution For Trig. Functions - Example

6. Find  $\int \sin 3x \cos 3x dx$

## Solution

Let  $u = \sin 3x$  and  $du = 3 \cos 3x dx$

$$\implies \frac{du}{3} = \cos 3x dx$$

$$\begin{aligned}\implies \int \sin 3x \cos 3x dx &= \int u \frac{du}{3} \\ &= \frac{1}{3} \int u du \\ &= \frac{1}{3} \cdot \frac{1}{2} u^2 + C \\ &= \frac{1}{6} \sin^2 3x + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

7. Find  $\int \sec^2 4x \tan 4x \, dx$

# Using Substitution For Trig. Functions - Example

7. Find  $\int \sec^2 4x \tan 4x \, dx$

## Solution

Let  $u = \tan 4x$  and  $du = 4 \sec^2 4x \, dx$

$$\implies \frac{du}{4} = \sec^2 4x \, dx$$

$$\begin{aligned}\implies \int \sec^2 4x \tan 4x \, dx &= \int u \frac{du}{4} \\ &= \frac{1}{4} \int u \, du \\ &= \frac{1}{4} \cdot \frac{1}{2} u^2 + C \\ &= \frac{1}{8} \tan^2 4x + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

8. Find  $\int \cos x \sin x \sqrt{1 + \sin^2 x} dx$

# Using Substitution For Trig. Functions - Example

8. Find  $\int \cos x \sin x \sqrt{1 + \sin^2 x} dx$

## Solution

Let  $u = 1 + \sin^2 x$  and  $du = 2 \sin x \cos x dx$

$$\implies \frac{du}{2} = \sin x \cos x dx$$

$$\begin{aligned}\implies \int \cos x \sin x \sqrt{1 + \sin^2 x} dx &= \int \sqrt{u} \frac{du}{2} \\ &= \frac{1}{2} \int u^{\frac{1}{2}} du \\ &= \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C \\ &= \frac{1}{3} (1 + \sin^2 x)^{3/2} + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

9. Find  $\int \sec^2 x \sqrt{\tan^3 x} dx$



# Using Substitution For Trig. Functions - Example

9. Find  $\int \sec^2 x \sqrt{\tan^3 x} dx$

## Solution

Let  $u = \tan x$  and  $du = \sec^2 x dx$

$$\begin{aligned}\Rightarrow \int \sec^2 x \sqrt{\tan^3 x} dx &= \int u^{\frac{3}{2}} du \\ &= \frac{2}{5} u^{\frac{5}{2}} + C \\ &= \frac{2}{5} \left( \sqrt{\tan^5 x} \right) + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

10. Find  $\int \tan^2 \theta \sec^4 \theta d\theta$

# Using Substitution For Trig. Functions - Example

10. Find  $\int \tan^2 \theta \sec^4 \theta d\theta$

## Solution

Let  $u = \tan \theta$  and  $du = \sec^2 \theta d\theta$

Also  $\sec^2 \theta = 1 + \tan^2 \theta = 1 + u^2$

$$\begin{aligned}\Rightarrow \int \tan^2 \theta \sec^4 \theta d\theta &= \int u^2(1 + u^2) du \\ &= \int (u^2 + u^4) du \\ &= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\ &= \frac{1}{3}\tan^3 \theta + \frac{1}{5}\tan^5 \theta + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

11. Find  $\int \cos^3 5x \sin^2 5x \, dx$

# Using Substitution For Trig. Functions - Example

11. Find  $\int \cos^3 5x \sin^2 5x dx$

## Solution

Let  $u = \sin 5x$  and  $du = 5 \cos 5x dx$

Also  $\cos^2 5x = 1 - \sin^2 5x = 1 - u^2$

$$\begin{aligned}\Rightarrow \int \cos^3 5x \sin^2 5x dx &= \int (1 - u^2) u^2 \frac{du}{5} \\ &= \frac{1}{5} \int (u^2 - u^4) du \\ &= \frac{1}{5} \left( \frac{1}{3} u^3 - \frac{1}{5} u^5 \right) + C \\ &= \frac{1}{15} \sin^3 5x - \frac{1}{25} \sin^5 5x + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

12. Find  $\int \sec^5 x \tan x \, dx$

# Using Substitution For Trig. Functions - Example

12. Find  $\int \sec^5 x \tan x \, dx$

## Solution

Let  $u = \sec x$  and  $du = \sec x \tan x \, dx$

$$\begin{aligned}\Rightarrow \int \sec^5 x \tan x \, dx &= \int \sec^4 x \sec x \tan x \, dx \\ &= \int u^4 \, du \\ &= \frac{1}{5} u^5 + C \\ &= \frac{1}{5} \sec^5 x + C\end{aligned}$$

# Using Substitution For Trig. Functions - Example

13. Find  $\int \left( \frac{\sec x}{1 + \tan x} \right)^2 dx$



# Using Substitution For Trig. Functions - Example

13. Find  $\int \left( \frac{\sec x}{1 + \tan x} \right)^2 dx$

## Solution

Let  $u = 1 + \tan x$  and  $du = \sec^2 x dx$

$$\begin{aligned} \Rightarrow \int \left( \frac{\sec x}{1 + \tan x} \right)^2 dx &= \int \frac{\sec^2 x}{(1 + \tan x)^2} dx \\ &= \int \frac{1}{u^2} du = \int u^{-2} du \\ &= -u^{-1} + C \\ &= -\frac{1}{1 + \tan x} + C \end{aligned}$$

# Example

1. Find  $\int \frac{1}{1 + \cos x} dx$

# Example

1. Find  $\int \frac{1}{1 + \cos x} dx$

## Solution

$$\begin{aligned}\frac{1}{1 + \cos x} &= \frac{1}{1 + \cos x} \cdot \frac{1 - \cos x}{1 - \cos x} = \frac{1 - \cos x}{\sin^2 x} \\ &= \operatorname{cosec}^2 x - \cot x \operatorname{cosec} x\end{aligned}$$

$$\begin{aligned}\Rightarrow \int \frac{1}{1 + \cos x} dx &= \int (\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x) dx \\ &= -\cot x + \csc x + C\end{aligned}$$

# Example

2. Find  $\int \frac{1}{1 + \sin x} dx$

# Example

2. Find  $\int \frac{1}{1 + \sin x} dx$

## Solution

$$\begin{aligned}\frac{1}{1 + \sin x} &= \frac{1}{1 + \sin x} \cdot \frac{1 - \sin x}{1 - \sin x} = \frac{1 - \cos x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} - \tan x \sec x \\ &= \sec^2 x - \sec x \tan x\end{aligned}$$

$$\begin{aligned}\Rightarrow \int \frac{1}{1 + \sin x} dx &= \int (\sec^2 x - \sec x \tan x) dx \\ &= \tan x - \sec x + C\end{aligned}$$

# Using Identities And Definite Limits

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\sin 2x = 2 \sin x \cos x$$

$$\sin^2 x + \cos^2 x = 1$$

# Example

1. Find  $\int \cos^2 x \, dx$

# Example

1. Find  $\int \cos^2 x \, dx$

## Solution

$$\begin{aligned}\int \cos^2 x \, dx &= \frac{1}{2} \int (1 + \cos 2x) \, dx \\ &= \frac{1}{2} \left( x + \frac{1}{2} \sin 2x \right) + C \\ &= \frac{1}{2} (x + \sin x \cos x) + C\end{aligned}$$



# Example

2. Find  $\int \sin^2 x \, dx$

# Example

2. Find  $\int \sin^2 x \, dx$

## Solution

$$\begin{aligned}\int \sin^2 x \, dx &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\ &= \frac{1}{2} \left( x - \frac{1}{2} \sin 2x \right) + C \\ &= \frac{1}{2} (x - \sin x \cos x) + C\end{aligned}$$

# Example

3. Find  $\int_0^{\pi/4} \cos x \, dx$

# Example

3. Find  $\int_0^{\pi/4} \cos x \, dx$

## Solution

$$\begin{aligned}\int_0^{\pi/4} \cos x \, dx &= \sin x \Big|_0^{\pi/4} \\ &= \sin \frac{\pi}{4} - \sin 0 \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

# Example

4. Find  $\int_0^{\pi/3} \sec^2 x \, dx$

# Example

4. Find  $\int_0^{\pi/3} \sec^2 x \, dx$

**Solution**

$$\begin{aligned}\int_0^{\pi/3} \sec^2 x \, dx &= \tan x \Big|_0^{\pi/3} \\ &= \tan \frac{\pi}{3} - \tan 0 \\ &= \sqrt{3}\end{aligned}$$

# Example

5. Find  $\int_{\pi/6}^{\pi/3} \sin x \, dx$

# Example

5. Find  $\int_{\pi/6}^{\pi/3} \sin x \, dx$

## Solution

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \sin x \, dx &= -\cos x \Big|_{\pi/6}^{\pi/3} \\&= -\cos \frac{\pi}{3} + \cos \frac{\pi}{6} \\&= -\frac{1}{2} + \frac{\sqrt{3}}{2} \\&= \frac{\sqrt{3}-1}{2}\end{aligned}$$



# Example

6. Find  $\int_0^{\pi/2} \sin^2 x \cos x \, dx$

# Example

6. Find  $\int_0^{\pi/2} \sin^2 x \cos x \, dx$

## Solution

$$\begin{aligned}\int_0^{\pi/2} \sin^2 x \cos x \, dx &= \frac{1}{3} \sin^3 x \Big|_0^{\pi/2} \\ &= \frac{1}{3} \left[ \sin^3 \frac{\pi}{2} - \sin^3 0 \right] \\ &= \frac{1}{3}\end{aligned}$$

# Example

7. Find  $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

# Example

7. Find  $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

## Solution

$$\begin{aligned}\int_0^{\pi/4} \tan x \sec^2 x \, dx &= \frac{1}{2} \tan^2 x \Big|_0^{\pi/4} \\ &= \frac{1}{2} \left[ \tan^2 \frac{\pi}{4} - \tan^2 0 \right] \\ &= \frac{1}{2} (1 - 0) = \frac{1}{2}\end{aligned}$$

# Example

8. Find  $\int_0^{\pi/2} \sqrt{\sin x + 1} \cos x \, dx$

# Example

8. Find  $\int_0^{\pi/2} \sqrt{\sin x + 1} \cos x \, dx$

## Solution

$$\begin{aligned} \int_0^{\pi/2} \sqrt{\sin x + 1} \cos x \, dx &= \int_0^{\pi/2} (\sin x + 1)^{1/2} \cos x \, dx \\ &= \frac{2}{3} (\sin x + 1)^{\frac{3}{2}} \Big|_0^{\frac{\pi}{2}} \\ &= \frac{2}{3} \left[ \left( \sin \left( \frac{\pi}{2} \right) + 1 \right)^{\frac{3}{2}} - (\sin(0) + 1)^{3/2} \right] \\ &= \frac{2}{3} (2\sqrt{2} - 1) \end{aligned}$$

# Try

1. Find  $\int \sqrt{1-x} \cdot x^2 dx$

## Try

1. Find  $\int \sqrt{1-x} \cdot x^2 dx$

**Solution**

$$\text{Let } u = 1 - x \text{ and } du = -dx$$

$$\implies x = 1 - u$$

$$\begin{aligned}\implies \int \sqrt{1-x} \cdot x^2 dx &= \int \sqrt{u}(1-u^2)(-1) du \\ &= - \int \sqrt{u}(1-2u+u^2) du \\ &= - \int (u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}}) du \\ &= - \left[ \frac{2}{3}u^{\frac{3}{2}} - 2 \left( \frac{2}{5} \right) u^{\frac{5}{2}} + \frac{2}{7}u^{\frac{7}{2}} \right] + C\end{aligned}$$



# Try (cont.)

$$\begin{aligned} &= -2u^{\frac{3}{2}} \left( \frac{1}{3} - \frac{2}{5}u + \frac{1}{7}u^2 \right) + C \\ &= -2(\sqrt{1-x})^3 \left[ \frac{1}{3} - \frac{2}{5}(1-x) + \frac{1}{7}(1-x)^2 \right] + C \\ &= -\frac{2}{105}(\sqrt{1-x})^3(8 + 12x + 15x^2) + C \end{aligned}$$

# Try (cont.)

2. Find  $\int \frac{x^3}{\sqrt{x^2 + 25}} dx$

# Try (cont.)

2. Find  $\int \frac{x^3}{\sqrt{x^2 + 25}} dx$

## Solution

Let  $u = x^2 + 25$  and  $du = 2x dx$

$$\implies x^2 = u - 25 \text{ and } \frac{du}{2} = x dx$$

$$\begin{aligned} \implies \int \frac{x^3}{\sqrt{x^2 + 25}} dx &= \int \frac{u - 25}{\sqrt{u}} \cdot \frac{du}{2} \\ &= \frac{1}{2} \int (u^{\frac{1}{2}} - 25u^{-\frac{1}{2}}) du \\ &= \frac{1}{2} \left( \frac{2}{3} u^{\frac{3}{2}} - 50u^{\frac{1}{2}} \right) + C \\ &= \frac{1}{3} u^{\frac{1}{2}} (u - 75) + C = \frac{1}{3} \sqrt{x^2 + 25} (x^2 - 50) + C \end{aligned}$$

# Try (cont.)

3. Find  $\int (2x^3 + x)(x^4 + x^2 + 1)^{49} dx$

## Try (cont.)

3. Find  $\int (2x^3 + x)(x^4 + x^2 + 1)^{49} dx$

### Solution

Let  $u = x^4 + x^2 + 1$  and  $du = (4x^3 + 2) dx = 2(2x^3 + x) dx$

$$\implies \frac{du}{2} = (2x^3 + x) dx$$

$$\begin{aligned}\implies \int (2x^3 + x)(x^4 + x^2 + 1)^{49} dx &= \int u^{49} \frac{du}{2} \\ &= \frac{1}{2} \left( \frac{1}{50} \right) u^{50} + C \\ &= \frac{1}{100} u^{50} + C \\ &= \frac{1}{100} (x^4 + x^2 + 1)^{50} + C\end{aligned}$$

# Standard Integrals Of Exponential Functions

## Special Case

$$\int a^x dx = \frac{1}{\ln a} a^x$$

# Example

1. Find  $\int a^x dx$

# Example

1. Find  $\int a^x dx$

## Solution

Let  $a^x = e^{x \ln a}$  ( $a^x = e^{x \log_e a} = e^{\log_e a^x}$ )

and  $u = x \ln a$

$$\implies du = \ln a \, dx \text{ and } \frac{du}{\ln a} = dx$$

$$\implies \int a^x dx = \int e^{x \ln a} dx = \int e^u \frac{du}{\ln a}$$

$$= \frac{1}{\ln a} e^u + C$$

$$= \frac{1}{\ln a} e^{x \ln a} + C$$

$$= \frac{1}{\ln a} a^x + C$$



# Example

2. Find  $\int 3^{2x} dx$

# Example

2. Find  $\int 3^{2x} dx$

## Solution

$$\text{Let } 3^{2x} = e^{2x \ln 3} \text{ and } u = 2x \ln 3$$

$$\implies du = 2 \ln 3 \, dx \text{ and } \frac{du}{2 \ln 3} = dx$$

$$\begin{aligned} \implies \int 3^{2x} dx &= \int e^u \frac{du}{2 \ln 3} = \frac{1}{2 \ln 3} \int e^u du \\ &= \frac{1}{2 \ln 3} e^u + C \\ &= \frac{1}{2 \ln 3} e^{2x \ln 3} + C \\ &= \frac{1}{2 \ln 3} 3^{2x} + C \end{aligned}$$

# Example

1. Find  $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

# Example

1. Find  $\int \frac{1}{\sqrt{a^2 - x^2}} dx$

## Solution

$$\text{Let } x = a \sin \theta \text{ and } dx = a \cos \theta d\theta$$

$$\Rightarrow \sin \theta = \frac{x}{a} \text{ and } \theta = \sin^{-1} \left( \frac{x}{a} \right)$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{a^2 - x^2}} dx &= \int \frac{a \cos \theta}{\sqrt{a^2 - a^2 \sin^2 \theta}} d\theta \\ &= \int \frac{a \cos \theta}{\sqrt{a^2(1 - \sin^2 \theta)}} d\theta = \int \frac{a \cos \theta}{\sqrt{a^2 \cos^2 \theta}} d\theta \\ &= \int \frac{a \cos \theta}{a \cos \theta} d\theta = \int d\theta \\ &= \theta + C = \sin^{-1} \left( \frac{x}{a} \right) + C \end{aligned}$$

# Example

2. Find  $\int \frac{1}{a^2 + x^2} dx$

# Example

2. Find  $\int \frac{1}{a^2 + x^2} dx$

## Solution

Let  $x = a \tan \theta$  and  $dx = a \sec^2 \theta d\theta$

$$\implies \tan \theta = \frac{x}{a} \text{ and } \theta = \tan^{-1} \left( \frac{x}{a} \right)$$

$$\begin{aligned} \implies \int \frac{1}{a^2 + x^2} dx &= \int \frac{a \sec^2 \theta}{a^2 + a^2 \tan^2 \theta} d\theta \\ &= \int \frac{a \sec^2 \theta}{a^2 (1 + \tan^2 \theta)} d\theta = \int \frac{a \sec^2 \theta}{a^2 \sec^2 \theta} d\theta \\ &= \frac{1}{a} \int d\theta = \frac{1}{a} \theta + C \\ &= \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C \end{aligned}$$

# Example

3. Find  $\int \sqrt{16 - 9x^2} \, dx$

# Example

3. Find  $\int \sqrt{16 - 9x^2} dx$

## Solution

$$\text{Let } x = \frac{4}{3} \sin \theta \text{ and } dx = \frac{4}{3} \cos \theta d\theta$$

$$\Rightarrow \sin \theta = \frac{3x}{4} \text{ and } \theta = \sin^{-1} \left( \frac{3x}{4} \right)$$

$$\text{Also } \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \left( \frac{3x}{4} \right)^2}$$

$$= \frac{1}{4} \sqrt{16 - 9x^2}$$

$$\Rightarrow \int \sqrt{16 - 9x^2} dx = \int \sqrt{16 - 9 \left( \frac{16}{9} \sin^2 \theta \right)} \cdot \frac{4}{3} \cos \theta d\theta$$



# Example (cont.)

$$\begin{aligned} &= \int \sqrt{16 - 16 \sin^2 \theta} \cdot \frac{4}{3} \cos \theta \, d\theta \\ &= \int 4 \cos \theta \cdot \frac{4}{3} \cos \theta \, d\theta \\ &= \frac{16}{3} \int \cos^2 \theta \, d\theta \\ &= \frac{16}{3} \int \frac{1 + \cos 2\theta}{2} \, d\theta = \frac{8}{3} \left( \theta + \frac{1}{2} \sin 2\theta \right) + C \\ &= \frac{8}{3} (\theta + \sin \theta \cos \theta) + C \\ &= \frac{8}{3} \left[ \sin^{-1} \left( \frac{3}{4}x \right) + \left( \frac{3x}{4} \right) \left( \frac{1}{4} \sqrt{16 - 9x^2} \right) \right] + C \\ &= \frac{8}{3} \sin^{-1} \left( \frac{3}{4}x \right) + \frac{1}{2}x \sqrt{16 - 9x^2} + C \end{aligned}$$

# Example

4. Find  $\int \frac{1}{x^2 \sqrt{1+x^2}} dx$

# Example

4. Find  $\int \frac{1}{x^2 \sqrt{1+x^2}} dx$

## Solution

Let  $x = \tan \theta$  and  $dx = \sec^2 \theta d\theta$

$$\begin{aligned} \Rightarrow \int \frac{1}{x^2 \sqrt{1+x^2}} dx &= \int \frac{\sec^2 \theta}{\tan^2 \theta \sqrt{1+\tan^2 \theta}} \\ &= \int \frac{\sec^2 \theta}{\tan^2 \theta \sec \theta} d\theta \\ &= \int \frac{\sec \theta}{\tan^2 \theta} d\theta \\ &= \int \frac{1}{\cos \theta} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} d\theta = \int \frac{\cos \theta}{\sin^2 \theta} d\theta \end{aligned}$$

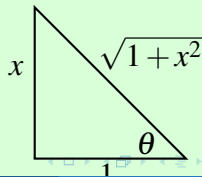
# Example (cont.)

Now, let  $u = \sin \theta$  and  $du = \cos \theta d\theta$

$$\begin{aligned}\Rightarrow \int \frac{\cos \theta}{\sin^2 \theta} d\theta &= \int \frac{1}{u^2} du \\ &= \int u^{-2} du = -\frac{1}{u} + C \\ &= -\frac{1}{\sin \theta} + C \\ &= -\frac{\sqrt{1+x^2}}{x} + C\end{aligned}$$

**NB:**

If  $\tan \theta = x$  then  $\sin \theta = \frac{x}{\sqrt{1+x^2}}$



# Example

5. Find  $\int \frac{1}{\sqrt{2-3x^2}} dx$

# Example

5. Find  $\int \frac{1}{\sqrt{2-3x^2}} dx$

## Solution

$$\text{Let } x = \sqrt{\frac{2}{3}} \sin \theta \text{ and } dx = \sqrt{\frac{2}{3}} \cos \theta d\theta$$

$$\begin{aligned} \Rightarrow \int \frac{1}{\sqrt{2-3x^2}} dx &= \int \frac{\sqrt{\frac{2}{3}} \cos \theta}{\sqrt{2-2\sin^2 \theta}} d\theta \\ &= \int \frac{\sqrt{\frac{2}{3}} \cos \theta}{\sqrt{2} \cos \theta} d\theta \\ &= \frac{1}{\sqrt{3}} \int d\theta = \frac{1}{\sqrt{3}} \theta + C \\ &= \frac{1}{\sqrt{3}} \sin^{-1} \sqrt{\frac{2}{3}} + C \end{aligned}$$

# Try the following

i.  $\int \sqrt{25 - x^2} \, dx$

ii.  $\int \sqrt{9 - x^2} \, dx$

iii.  $\int \sqrt{1 - 4x^2} \, dx$

iv.  $\int \sqrt{9 - 4x^2} \, dx$

v.  $\int \frac{1}{x^2 \sqrt{1 - x^2}} \, dx$

vi.  $\int \frac{1}{x^2 \sqrt{a^2 + x^2}} \, dx$

vii.  $\int \frac{1}{1 + \cos x} \, dx$

viii.  $\int \frac{1}{1 + \sin x} \, dx$

ix.  $\int \frac{1}{1 - \sin x} \, dx$

x.  $\int \frac{1}{5 + 3 \cos x} \, dx$

xi.  $\int \frac{1}{5 - 3 \cos x} \, dx$

xii.  $\int x^3 \sqrt{1 - x^2} \, dx$

# Using The Substitution $t = \tan \frac{1}{2}x$

- An occasionally useful trigonometrical substitution is given by the following formulae, in which  $\sin x$  and  $\cos x$  are expressed in terms of  $\tan \frac{1}{2}x$

$$\sin x = \frac{2 \tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}; \quad \cos x = \frac{1 - \tan^2 \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}$$

Letting  $t = \tan \frac{1}{2}x$ , we have

$$dt = \frac{1}{2} \sec^2 \frac{1}{2}x dx \tag{1}$$

$$\implies \sin x = \frac{2t}{1+t^2} \quad \text{and} \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\text{From (1) } dx = \frac{2}{\sec^2 \frac{1}{2}x} = \frac{2}{1+t^2}$$



# Example

1. Find  $\int \operatorname{cosec} x \, dx$

# Example

1. Find  $\int \operatorname{cosec} x \, dx$

## Solution

$$\int \operatorname{cosec} x \, dx = \int \frac{1}{\sin x} \, dx$$

$$\text{Let } \sin x = \frac{2t}{1+t^2} \text{ and } dx = \frac{2}{1+t^2} dt$$

$$\begin{aligned} \Rightarrow \operatorname{cosec} x \, dx &= \int \frac{1}{\sin x} \, dx = \int \frac{1}{\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt \\ &= \int \frac{2}{2t} \, dt = \int \frac{1}{t} \, dt \\ &= \ln |t| + C = \ln \left| \tan \frac{1}{2}x \right| + C \end{aligned}$$

# Example

2. Find  $\int \frac{1}{5+4\cos x} dx$

# Example

2. Find  $\int \frac{1}{5+4\cos x} dx$

## Solution

Let  $t = \tan \frac{1}{2}x$  and  $dx = \frac{2}{1+t^2} dt \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$

$$\begin{aligned} \Rightarrow \int \frac{1}{5+4\cos x} dx &= \int \frac{2}{1+t^2} dt \div \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \\ &= \int \frac{2}{5(1+t^2)+4(1-t^2)} dt \\ &= \int \frac{2}{9+t^2} dt = \frac{2}{3} \tan^{-1} \frac{t}{3} + C \\ &= \frac{2}{3} \tan^{-1} \left( \frac{1}{3} \tan \frac{x}{2} \right) + C \end{aligned}$$

# Integration by Parts

- This method for integration comes from the rule for differentiating a product using the product rule:

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

where  $u$  and  $v$  are functions of  $x$ .

- Integrating throughout with respect to  $x$ , we have

$$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx$$

- Rearranging, we have

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

- Since  $u$  and  $v$  are functions of  $x$ , we can write the relation in the form

$$\int u dv = uv - \int v du$$

# Example

1. Find  $\int x \sin x \, dx$

# Example

1. Find  $\int x \sin x \, dx$

## Solution

Let  $u = x$  and  $dv = \sin x \, dx$

$$du = dx \text{ and } v = \int \sin x \, dx = -\cos x$$

$$\begin{aligned} \Rightarrow \int x \sin x \, dx &= -x \cos x + \int \cos x \, dx \\ &= -x \cos x + \sin x + C \end{aligned}$$

**NB:** It is the final stage that we put in the constant of integration.

# Example

2. Find  $\int x \cos x \, dx$



# Example

2. Find  $\int x \cos x \, dx$

## Solution

Let  $u = x$  and  $dv = \cos x \, dx$

$$du = dx \text{ and } v = \int \cos x \, dx = \sin x$$

$$\begin{aligned} \Rightarrow \int x \cos x \, dx &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C \end{aligned}$$

# Example

3. Find  $\int x^2 \sin x \, dx$

# Example

3. Find  $\int x^2 \sin x \, dx$

## Solution

Let  $u = x^2$  and  $dv = \sin x \, dx$

$$du = 2x \, dx \quad \text{and} \quad v = \int \sin x \, dx = -\cos x$$

$$\begin{aligned} \Rightarrow \int x^2 \sin x \, dx &= x(-\cos x) - \int (-\cos x) 2x \, dx \\ &= -x^2 \cos x + 2 \int x \cos x \, dx \end{aligned}$$

Applying integration by parts again to the latter integral, we have

$$\int x \cos x \, dx = ?$$

# Example (cont.)

Let  $u = x$  and  $dv = \cos x dx$

$$du = dx \text{ and } v = \int \cos x dx = \sin x$$

$$\begin{aligned}\Rightarrow \int x \cos x dx &= x \sin x - \int \sin x dx \\ &= x \sin x + \cos x + C\end{aligned}$$

$$\begin{aligned}\therefore \int x^2 \sin x dx &= -x^2 \cos x + 2(x \sin x + \cos x) + C \\ &= -x^2 \cos x + 2x \sin x + 2 \cos x + C\end{aligned}$$

# Example

4. Find  $\int \ln x \, dx$

# Example

4. Find  $\int \ln x \, dx$

## Solution

Since  $\ln x$  produces a simple expression when we differentiate it, we use the substitution for  $\ln x$ .

$$\text{Let } u = \ln x \text{ and } dv = dx$$

$$du = \frac{1}{x} dx \text{ and } v = \int dx = x$$

$$\begin{aligned} \Rightarrow \int \ln x \, dx &= x(\ln x) - \int x \cdot \frac{1}{x} \, dx \\ &= x \ln x - \int dx \\ &= x \ln x - x + C \end{aligned}$$

# Example

5. Find  $\int x^2 e^{-x} dx$

# Example

5. Find  $\int x^2 e^{-x} dx$

## Solution

Let  $u = x^2$  and  $dv = e^{-x} dx$

$$du = 2x dx \text{ and } v = \int e^{-x} dx = -e^{-x}$$

$$\implies \int x^2 e^{-x} dx = -x^2 e^{-x} + 2 \int x e^{-x} dx$$

Applying integration by parts again to the latter integral, we have

$$\int x e^{-x} dx = ?$$



# Example (cont.)

Let  $u = x$  and  $dv = e^{-x} dx$

$$du = dx \text{ and } v = \int e^{-x} dx = -e^{-x}$$

$$\begin{aligned}\Rightarrow \int x e^{-x} dx &= -x e^{-x} + \int e^{-x} dx \\ &= -x e^{-x} + (-e^{-x}) + C \\ &= -x e^{-x} - e^{-x} + C = -e^{-x}(x+1) + C\end{aligned}$$

$$\begin{aligned}\therefore \int x^2 e^{-x} dx &= -x^2 e^{-x} + 2[-e^{-x}(x+1)] + C \\ &= -e^{-x}(x^2 + 2x + 2) + C\end{aligned}$$

# Example

6. Find  $\int e^x \sin x \, dx$

# Example

6. Find  $\int e^x \sin x \, dx$

## Solution

Let  $u = \sin x$  and  $dv = e^x \, dx$

$$du = \cos x \, dx \text{ and } v = \int e^x \, dx = e^x$$

$$\implies \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx \quad (1)$$

Applying integration by parts again to (1), we have

Let  $u = \cos x$  and  $dv = e^x \, dx$

$$du = -\sin x \, dx \text{ and } v = \int e^x \, dx = e^x$$

# Example (cont.)

$$\implies \int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx \quad (2)$$

Substituting (2) into (1), we have

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - \left( e^x \cos x + \int e^x \sin x \, dx \right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + C \end{aligned}$$

$$\implies 2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) + C$$

$$\therefore \int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$$

# Example

7. Find  $\int x \cos(5x - 1) dx$

# Example

7. Find  $\int x \cos(5x - 1) dx$

## Solution

Let  $u = x$  and  $dv = \cos(5x - 1) dx$

$$du = dx \text{ and } v = \int \cos(5x - 1) dx = \frac{1}{5} \sin(5x - 1)$$

$$\begin{aligned} \Rightarrow \int x \cos(5x - 1) dx &= \frac{1}{5} x \sin(5x - 1) - \frac{1}{5} \int \sin(5x - 1) dx \\ &= \frac{1}{5} x \sin(5x - 1) + \frac{1}{5} \left( \frac{1}{5} \right) \cos(5x - 1) + C \\ &= \frac{1}{25} [5x \sin(5x - 1) + \cos(5x - 1)] + C \end{aligned}$$

# Try the following

a.  $\int x^3 e^x dx$

b.  $\int x e^{3x} dx$

c.  $\int \frac{\ln x}{x^2} dx$

d.  $\int x^2 \ln x dx$

e.  $\int \ln(x^2 + 1) dx$

f.  $\int 3x^4 \ln x dx$

g.  $\int \frac{\ln 3x^2}{x^5} dx$

# Try the following

a.  $\int x^3 e^x dx$

**Ans:**  $e^x(x^3 - 3x^2 + 6x - 6) + C$

b.  $\int x e^{3x} dx$

**Ans:**  $\frac{1}{9}e^{3x}(3x - 1) + C$

c.  $\int \frac{\ln x}{x^2} dx$

**Ans:**  $-\frac{1}{x}(\ln x + 1) + C$

d.  $\int x^2 \ln x dx$

**Ans:**  $\frac{1}{9}x^3(3 \ln x - 1) + C$

e.  $\int \ln(x^2 + 1) dx$

**Ans:**  $x \ln(x^2 + 1) - 2x + 2 \tan^{-1} x + C$

f.  $\int 3x^4 \ln x dx$

**Ans:**  $\frac{1}{25}x^5(15 \ln x - 3) + C$

g.  $\int \frac{\ln 3x^2}{x^5} dx$

**Ans:**  $-\frac{1}{8x^4}(2 \ln 3x^2 + 1) + C$



# Try the following (cont.)

**h.**  $\int x(\ln 8x)^2 dx$

**i.**  $\int 9x^8 \ln 6x^5 dx$

# Try the following (cont.)

**h.**  $\int x(\ln 8x)^2 dx$

**Ans:**

$$-\frac{1}{4}x^2 [2(\ln 8x)^2 - 2(\ln 8x - 1)] + C$$

**i.**  $\int 9x^8 \ln 6x^5 dx$

**Ans:**  $\frac{1}{9}x^9(9\ln 6x^5 - 5) + C$

# Inverse Trig. Functions - Example

1. Find  $\int \sin^{-1} x \, dx$

# Inverse Trig. Functions - Example

1. Find  $\int \sin^{-1} x \, dx$

## Solution

Let  $u = \sin^{-1} x$  and  $dv = dx$

$$du = \frac{1}{\sqrt{1-x^2}} dx \text{ and } v = \int dx = x$$

$$\Rightarrow \int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx \quad (1)$$

Now, we have  $\int \frac{x}{\sqrt{1-x^2}} dx$  to integrate

Let  $u = 1 - x^2$  and  $du = -2x \, dx$

$$\Rightarrow -\frac{du}{2} = x \, dx$$

# Inverse Trig. Functions - Example (cont.)

$$\begin{aligned}\Rightarrow \int \frac{x}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{u}} \cdot -\frac{du}{2} \\ &= -\frac{1}{2} \int u^{-\frac{1}{2}} du \\ &= -\frac{1}{2} \cdot 2u^{\frac{1}{2}} = -\sqrt{1-x^2}\end{aligned}\tag{2}$$

Substitute (2) into (1), we have

$$\begin{aligned}\Rightarrow \int \sin^{-1} x dx &= x \sin^{-1} x - \left(-\sqrt{1-x^2}\right) + C \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

# Inverse Trig. Functions - Example

2. Find  $\int \cos^{-1} x \, dx$

# Inverse Trig. Functions - Example

2. Find  $\int \cos^{-1} x \, dx$

## Solution

Let  $u = \cos^{-1} x$  and  $dv = dx$

$$du = -\frac{1}{\sqrt{1-x^2}} dx \text{ and } v = \int dx = x$$

$$\Rightarrow \int \cos^{-1} x \, dx = x \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} dx \quad (1)$$

Now, we have  $\int \frac{x}{\sqrt{1-x^2}} dx$  to integrate

Let  $u = 1 - x^2$  and  $du = -2x \, dx$

$$\Rightarrow -\frac{du}{2} = x \, dx$$

# Inverse Trig. Functions - Example (cont.)

$$\begin{aligned}\Rightarrow \int \frac{x}{\sqrt{1-x^2}} dx &= \int \frac{1}{\sqrt{u}} \cdot -\frac{du}{2} \\ &= -\frac{1}{2} \int u^{-\frac{1}{2}} du \\ &= -\frac{1}{2} \cdot 2u^{\frac{1}{2}} = -\sqrt{1-x^2}\end{aligned}\tag{2}$$

Substitute (2) into (1), we have

$$\begin{aligned}\Rightarrow \int \cos^{-1} x dx &= x \cos^{-1} x + \left(-\sqrt{1-x^2}\right) + C \\ &= x \cos^{-1} x - \sqrt{1-x^2} + C\end{aligned}$$



# Inverse Trig. Functions - Example

3. Find  $\int \tan^{-1} x \, dx$

# Inverse Trig. Functions - Example

3. Find  $\int \tan^{-1} x \, dx$

## Solution

Let  $u = \tan^{-1} x$  and  $dv = dx$

$$\implies du = \frac{1}{1+x^2} dx \text{ and } v = \int dx = x$$

$$\begin{aligned}\implies \int \tan^{-1} x \, dx &= x \tan^{-1} x - \int \frac{x}{1+x^2} \, dx \\ &= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx \\ &= x \tan^{-1} x - \frac{1}{2} \ln |1+x^2| + C\end{aligned}$$

# Inverse Trig. Functions - Example

4. Find  $\int x^2 \tan^{-1} x \, dx$

# Inverse Trig. Functions - Example

4. Find  $\int x^2 \tan^{-1} x \, dx$

## Solution

Let  $u = \tan^{-1} x$  and  $dv = x^2 \, dx$

$$\Rightarrow du = \frac{1}{1+x^2} dx \text{ and } v = \int x^2 \, dx = \frac{1}{3}x^3$$

$$\Rightarrow \int x^2 \tan^{-1} x \, dx = \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3} \int \left( x - \frac{x}{1+x^2} \right) \, dx$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3} \left[ \frac{1}{2}x^2 - \frac{1}{2} \ln |1+x^2| \right] + C$$

$$= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6} \ln |1+x^2| + C$$

# Inverse Trig. Functions - Example

5. Find  $\int (\sin^{-1} x)^2 dx$

# Inverse Trig. Functions - Example

5. Find  $\int (\sin^{-1} x)^2 dx$

## Solution

$$\text{Let } u = \sin^{-1} x \text{ and } du = \frac{1}{\sqrt{1-x^2}} dx$$

$$\implies x = \sin u \text{ and } \sqrt{1-x^2} du = dx$$

$$\begin{aligned} \implies \int (\sin^{-1} x)^2 dx &= \int u^2 \sqrt{1-x^2} du \\ &= \int u^2 \sqrt{1-\sin^2 u} du \\ &= \int u^2 \cos u du \end{aligned} \quad (1)$$

# Inverse Trig. Functions - Example (cont.)

Applying integration by parts to (1), we have

$$\begin{aligned}\int u^2 \cos u \, du &= (u^2 - 2) \sin u + 2u \cos u + C \\ &= [(\sin^{-1} x)^2 - 2] x + 2(\sin^{-1} x) \sqrt{1 - x^2} + C \\ \therefore \int (\sin^{-1} x)^2 \, dx &= [(\sin^{-1} x)^2 - 2] x + 2(\sin^{-1} x) \sqrt{1 - x^2} + C\end{aligned}$$

# Exercise

i.  $\int x \sin x dx$

ii.  $\int x \sin 3x dx$

iii.  $\int x^2 \cos x dx$

iv.  $\int x \ln x dx$

v.  $\int x e^{-ax} dx$

vi.  $\int \tan^{-1} x dx$

vii.  $\int x \tan^{-1} x dx$

viii.  $\int \sin^{-1} x dx$



# Try the following

a.  $\int e^x \cos x dx$

b.  $\int e^{ax} \cos bx dx$

c.  $\int e^{ax} \sin bx dx$

d.  $\int e^{ax} \cos (bx + c) dx$

e.  $\int x^2 e^{3x} dx$

# Try the following

a.  $\int e^x \cos x dx$

$$\frac{1}{2}e^x(\sin x + \cos x) + C$$

b.  $\int e^{ax} \cos bx dx$

$$\frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + C$$

c.  $\int e^{ax} \sin bx dx$

$$\frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + C$$

d.  $\int e^{ax} \cos (bx + c) dx$

$$\frac{e^{ax}}{a^2 + b^2}[a \cos(bx + c) + b \sin(bx + c)] + D$$

e.  $\int x^2 e^{3x} dx$

$$\frac{1}{27}e^{3x}[9x^2 - 6x + 2] + C$$

# Integrating Rational Functions - Example

1. Find  $\int \frac{1}{x^2 - 9} dx$

# Integrating Rational Functions - Example

1. Find  $\int \frac{1}{x^2 - 9} dx$

## Solution

$$\frac{1}{x^2 - 9} = \frac{1}{(x - 3)(x + 3)} = \frac{A}{x - 3} + \frac{B}{x + 3}$$

$$\implies 1 = A(x + 3) + B(x - 3)$$

$$\text{Let } x = 3, \text{ then } 1 = 6A \implies A = \frac{1}{6}$$

$$\text{Let } x = -3, \text{ then } 1 = -6B \implies B = -\frac{1}{6}$$

$$\text{So } \frac{1}{x^2 - 9} = \frac{1}{6} \left[ \frac{1}{x - 3} \right] - \frac{1}{6} \left[ \frac{1}{x + 3} \right]$$

# Example (cont.)

$$\begin{aligned}\text{Hence } \int \frac{1}{x^2 - 9} dx &= \frac{1}{6} \ln|x - 3| - \frac{1}{6} \ln|x + 3| + C \\ &= \frac{1}{6} \ln \left| \frac{x - 3}{x + 3} \right| + C\end{aligned}$$

# Example

2. Find  $\int \frac{x}{(x+2)(x+3)} dx$

# Example

2. Find  $\int \frac{x}{(x+2)(x+3)} dx$

## Solution

$$\frac{x}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$\implies x = A(x+3) + B(x+2)$$

$$\text{Let } x = -3, \text{ then } -3 = -B \implies B = 3$$

$$\text{Let } x = -2, \text{ then } -2 = A \implies A = -2$$

$$\text{So } \frac{x}{(x+2)(x+3)} = -\frac{2}{x+2} + \frac{3}{x+3}$$

# Example (cont.)

$$\begin{aligned}\text{Hence } \int \frac{x}{(x+2)(x+3)} dx &= -2 \ln|x+2| + 3 \ln|x+3| + C \\ &= \frac{1}{6} \ln \left| \frac{(x+3)^3}{(x+2)^2} \right| + C\end{aligned}$$



# Example

3. Find  $\int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} dx$

# Example

3. Find  $\int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} dx$

## Solution

$$\frac{x^4 - 4x^2 + x + 1}{x^2 - 4} = x^2 + \frac{x + 1}{x^2 - 4}$$

$$\begin{aligned} \Rightarrow \int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} dx &= \int x^2 dx + \int \frac{x + 1}{x^2 - 4} dx \\ &= \frac{1}{3}x^3 + \int \frac{x + 1}{x^2 - 4} dx \end{aligned}$$

$$\text{But } \frac{x + 1}{x^2 - 4} = \frac{x + 1}{(x + 2)(x - 2)} = \frac{A}{x + 2} + \frac{B}{x - 2}$$

$$\Rightarrow x + 1 = A(x - 2) + B(x + 2)$$

$$\text{If } x = 2 \text{ then } 3 = 4B \Rightarrow B = \frac{3}{4}$$

# Example (cont.)

$$\text{If } x = -2 \text{ then } -1 = -4A \Rightarrow A = \frac{1}{4}$$

$$\text{Thus } \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} = \frac{1}{4(x+2)} + \frac{3}{4(x-2)}$$

$$\begin{aligned} \text{Hence } \int \frac{x}{(x+2)(x+3)} dx &= \frac{1}{4} \ln|x+2| + \frac{3}{4} \ln|x-2| + C \\ &= \frac{1}{4} \ln \left| \frac{x+2}{(x-2)^3} \right| + C \end{aligned}$$

# Example

4. Find  $\int \frac{x^3 + 1}{x(x+3)(x+2)(x-1)} dx$

# Example

4. Find  $\int \frac{x^3 + 1}{x(x+3)(x+2)(x-1)} dx$

**Solution**

$$\frac{x^3 + 1}{x(x+3)(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x+2} + \frac{D}{x-1}$$

$$\begin{aligned} \implies x^3 + 1 &= A(x+3)(x+2)(x-1) + Bx(x+2)(x-1) \\ &\quad + Cx(x+3)(x-1) + Dx(x+3)(x+2) \end{aligned}$$

$$\text{Let } x = 0 \implies 1 = -6A \implies A = -\frac{1}{6}$$

$$\text{Let } x = -3 \implies -26 = -12B \implies B = \frac{13}{6}$$

$$\text{Let } x = -2 \implies -7 = 6C \implies C = -\frac{7}{6}$$

$$\text{Let } x = 1 \implies 2 = 12D \implies D = \frac{1}{6}$$

# Example (cont.)

$$\Rightarrow \frac{x^3 + 1}{x(x+3)(x+2)(x-1)} = -\frac{1}{6x} + \frac{13}{6(x+3)} - \frac{7}{6(x+2)} + \frac{1}{6(x-1)}$$

$$\begin{aligned} \text{Hence } \int \frac{x^3 + 1}{x(x+3)(x+2)(x-1)} dx &= -\frac{1}{6} \ln|x| + \frac{13}{6} \ln|x+3| - \\ &\quad \frac{7}{6} \ln|x+2| + \frac{1}{6} \ln|x-1| + C \\ &= \frac{1}{6} (\ln - \ln|x| + 13 \ln|x+3| - \\ &\quad 7 \ln|x+2| + \ln|x-1|) + C \\ &= \frac{1}{6} \ln \left| \frac{(x+3)^{13}(x-1)}{x(x+2)^7} \right| + C \end{aligned}$$

# Example

5. Find  $\int \frac{x}{x^4 - 13x^2 + 36} dx$

# Example

5. Find  $\int \frac{x}{x^4 - 13x^2 + 36} dx$

## Solution

$$\begin{aligned}\frac{x}{x^4 - 13x^2 + 36} &= (x^2 - 9)(x^2 - 4) \\ &= (x - 3)(x + 3)(x + 2)(x - 2)\end{aligned}$$

$$\text{Let } \frac{x}{(x - 3)(x + 3)(x + 2)(x - 2)} = \frac{A}{x - 3} + \frac{B}{x + 3} + \frac{C}{x + 2} + \frac{D}{x - 2}$$

$$\begin{aligned}\implies x &= A(x + 3)(x + 2)(x - 2) + B(x - 3)(x + 2)(x - 2) \\ &\quad + C(x - 3)(x + 3)(x - 2) + D(x - 3)(x + 3)(x + 2)\end{aligned}$$

$$\text{Let } x = 3 \implies 3 = 30A \implies A = \frac{1}{10}$$

$$\text{Let } x = -3 \implies -3 = -30B \implies B = \frac{1}{10}$$



# Example (cont.)

$$\text{Let } x = -2 \Rightarrow -2 = 20C \Rightarrow C = -\frac{1}{10}$$

$$\text{Let } x = 1 \Rightarrow 2 = -20D \Rightarrow D = -\frac{1}{10}$$

$$\Rightarrow \frac{x}{x^4 - 13x^2 + 36} = \frac{1}{10(x-3)} + \frac{1}{10(x+3)} - \frac{1}{10(x+2)} - \frac{1}{10(x-2)}$$

# Example (cont.)

$$\begin{aligned}\text{Hence } \int \frac{x}{x^4 - 13x^2 + 36} dx &= \frac{1}{10} \ln|x-3| + \frac{1}{10} \ln|x+3| - \\ &\quad \frac{1}{10} \ln|x+2| - \frac{1}{10} \ln|x-2| + C \\ &= \frac{1}{10} (\ln|x-3| + \ln|x+3| - \\ &\quad \ln|x+2| - \ln|x-2|) + C \\ &= \frac{1}{10} \ln \left| \frac{(x+3)(x-3)}{(x+2)(x-2)} \right| + C\end{aligned}$$

# Example

6. Find  $\int \frac{x-5}{x^2(x+1)} dx$

# Example

6. Find  $\int \frac{x-5}{x^2(x+1)} dx$

## Solution

$$\frac{x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\implies x-5 = Ax(x+1) + B(x+1) + Cx^2$$

$$\text{Let } x=0 \Rightarrow -5 = B$$

$$\text{Let } x=-1 \Rightarrow -6 = C$$

To find  $A$ , compare coefficients of  $x^2$  on both sides of the equation:

$$0 = A + C \Rightarrow A = -C = 6$$

# Example (cont.)

$$\implies \frac{x-5}{x^2(x+1)} = \frac{6}{x} - \frac{5}{x^2} - \frac{6}{x+1}$$

Hence 
$$\begin{aligned} \int \frac{x-5}{x^2(x+1)} dx &= \int \left[ \frac{6}{x} - \frac{5}{x^2} - \frac{6}{x+1} \right] dx \\ &= 6 \ln|x| + \frac{5}{x} - 6 \ln|x+1| + C \\ &= 6 \ln \left| \frac{x}{x+1} \right| + \frac{5}{x} + C \end{aligned}$$

# Example

7. Find  $\int \frac{2x}{(x-2)^2(x+2)} dx$

# Example

7. Find  $\int \frac{2x}{(x-2)^2(x+2)} dx$

## Solution

$$\frac{2x}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2}$$

$$\implies 2x = A(x-2)(x+2) + B(x+2) + C(x-2)^2$$

$$\text{Let } x = 2 \implies 4B = B \implies B = 1$$

$$\text{Let } x = -2 \implies -4 = 16C \implies C = -\frac{1}{4}$$

To find  $A$ , compare coefficients of  $x^2$  on both sides of the equation:

$$0 = A + C \implies A = -C = \frac{1}{4}$$

# Example (cont.)

$$\Rightarrow \frac{2x}{(x-2)^2(x+2)} = \frac{1}{4(x-2)} + \frac{1}{(x-2)^2} - \frac{1}{4(x+2)}$$

Hence 
$$\int \frac{2x}{(x-2)^2(x+2)} dx = \int \left[ \frac{1}{4(x-2)} + \frac{1}{(x-2)^2} - \frac{1}{4(x+2)} \right] dx$$

$$= \frac{1}{4} \ln|x-2| - \frac{1}{x-2} - \frac{1}{4} \ln|x+2| + C$$
$$= \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| - \frac{1}{x-2} + C$$



# Example

8. Find  $\int \frac{x+4}{x^3+6x^2+9x} dx$

# Example

8. Find  $\int \frac{x+4}{x^3+6x^2+9x} dx$

## Solution

$$\frac{x+4}{x^3+6x^2+9x} = \frac{x+4}{x(x^2+6x+9)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

$$\implies x+4 = A(x+3)^2 + Bx(x+3) + Cx$$

$$\text{Let } x=0 \implies 4=9A \implies A=\frac{4}{9}$$

$$\text{Let } x=-3 \implies 1=-3C \implies C=-\frac{1}{3}$$

To find  $B$ , compare coefficients of  $x^2$  on both sides of the equation:

$$0 = A + B \implies B = -A = -\frac{4}{9}$$

# Example (cont.)

$$\Rightarrow \frac{x+4}{x^3+6x^2+9x} = \frac{4}{9x} - \frac{4}{9(x+3)} - \frac{1}{3(x+3)^2}$$

Hence 
$$\begin{aligned} \int \frac{x+4}{x^3+6x^2+9x} dx &= \int \left[ \frac{4}{9x} - \frac{4}{9(x+3)} - \frac{1}{3(x+3)^2} \right] dx \\ &= \frac{4}{9} \ln|x| - \frac{4}{9} \ln|x+3| + \frac{1}{3(x+3)} + C \\ &= \frac{4}{9} \ln \left| \frac{x}{x+3} \right| + \frac{1}{3(x+3)} + C \end{aligned}$$

# Note

- When the denominator of the algebraic fraction is the sum of two squares, we cannot factorize it and had to resort to a different strategy for integration.
- The method is to use two standard forms of integration as shown below:

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{\tan^{-1} \frac{x}{a}} + C \quad \text{and}$$

$$\int \frac{x}{a^2 + x^2} dx = \frac{1}{2} \ln(a^2 + x^2) + C$$

# Example

1. Find  $\int \frac{2x+3}{4+x^2} dx$

# Example

1. Find  $\int \frac{2x+3}{4+x^2} dx$

## Solution

Here we could split the numerator into two parts or we could proceed with the method of partial fractions. Thus

$$\int \frac{2x+3}{4+x^2} dx = \int \frac{2x}{4+x^2} dx + \int \frac{3}{4+x^2} dx$$

Using standard forms, we have

$$\int \frac{2x}{4+x^2} dx = \ln|x^2+4| + C_1$$

and  $\int \frac{3}{4+x^2} dx = 3 \ln \frac{1}{4+x^2} dx = \frac{3}{2} \tan^{-1} \frac{x}{2} + C_2$

# Example (cont.)

$$\Rightarrow \int \frac{2x+3}{4+x^2} dx = \ln|x^2+4| + \frac{3}{2} \tan^{-1} \frac{x}{2} + C$$

# Example

2. Find  $\int \frac{8-x}{x^2+x-1} dx$



# Example

2. Find  $\int \frac{8-x}{x^2+x-1} dx$

## Solution

$$\begin{aligned}\int \frac{8-x}{x^2+x-1} dx &= \int \frac{(8-x)}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx \\&= \int \frac{(-x - \frac{1}{2}) + \frac{17}{2}}{(x+\frac{1}{2})^2 + \frac{3}{4}} \\&= - \int \frac{x + \frac{1}{2}}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx + \frac{17}{2} \int \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx \\&= -\frac{1}{2} \ln \left| \left( x + \frac{1}{2} \right) \right| + \frac{17}{2} \cdot \frac{2}{\sqrt{3}} \tan^{-1} \frac{(x+\frac{1}{2})}{\sqrt{3}/2}\end{aligned}$$

# Note

Where the integral is of the form

$$\int \frac{Ax + B}{\sqrt{px^2 + qx + r}} dx$$

**Example:**

$$\int \frac{x+1}{\sqrt{2x^2+x-3}} dx = ?$$

$$\begin{aligned} \text{Here we rewrite } 2x^2 + x - 3 &= 2 \left( x + \frac{1}{4} \right)^2 - \frac{25}{8} \\ &= \frac{25}{8} \left[ \frac{2(8)}{25} \left( x + \frac{1}{4} \right)^2 - 1 \right] \end{aligned}$$

$$\text{and let } \sqrt{2} \left( x + \frac{1}{4} \right) = \frac{5}{2\sqrt{2}} u$$

cont.

$$\Rightarrow x = \frac{5}{4}u - \frac{1}{4} \Rightarrow dx = \frac{5}{4}du$$

$$\Rightarrow \int \frac{x+1}{\sqrt{2x^2+x-3}} dx = \frac{\left[\left(\frac{5}{4}u - \frac{1}{4}\right) + 1\right]}{\sqrt{\frac{25}{8}u^2 - \frac{25}{8}}} \times \frac{5}{4}du$$

$$= \frac{5}{4} \cdot \frac{5}{4} \sqrt{\frac{8}{25}} \int \frac{u}{\sqrt{u^2-1}} du +$$

$$\frac{3}{4} \cdot \frac{5}{4} \cdot \sqrt{\frac{8}{25}} \int \frac{1}{\sqrt{u^2-1}} du$$

$$= \frac{5}{4\sqrt{2}} \sqrt{u^2-1} + \frac{3}{4\sqrt{2}} \ln|u + \sqrt{u^2-1}| + C$$

cont.

After re-substituting for  $u$  and some simplification, we find that

$$\int \frac{x+1}{\sqrt{2x^2+x-3}} dx = \frac{1}{2} \sqrt{2x^2+x-3} + \frac{3}{4\sqrt{2}} \ln \left| \frac{4x+1}{5} + \sqrt{\frac{8(2x^2+x-3)}{25}} \right|$$

# Try the following

a.

$$\int \frac{x^4}{x^3 - 2x^2 - 7x - 4} dx$$

b.

$$\int \frac{1}{x(x^2 + 5)} dx$$

c.

$$\int \frac{1}{(x^2 + 1)(x^2 + 4)} dx$$

d.

$$\int \frac{1 - 9x^2}{x(x^2 + 9)} dx$$

# Try the following

**a.**

$$\int \frac{x^4}{x^3 - 2x^2 - 7x - 4} dx$$

$$\frac{1}{2}x^2 + 2x + \frac{256}{25} \ln|x - 4| + \frac{19}{25} \ln|x + 1| + \frac{1}{5(x + 1)} + C$$

**b.**

$$\int \frac{1}{x(x^2 + 5)} dx$$

$$\frac{1}{5} \ln \left| \frac{x}{\sqrt{x^2 + 5}} \right| + C$$

**c.**

$$\int \frac{1}{(x^2 + 1)(x^2 + 4)} dx$$

$$\frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C$$

**d.**

$$\int \frac{1 - 9x^2}{x(x^2 + 9)} dx$$

$$\frac{1}{9} \ln|x| - \frac{41}{9} \ln|x^2 + 9| + C$$

# Try the following

a.  $\int \frac{1}{x(x^2 + 1)^2} dx$

b.  $\int \frac{x^2}{(x - 1)(x^2 + 4)^2} dx$

# Try the following

a.  $\int \frac{1}{x(x^2 + 1)^2} dx$

$$\ln|x| - \frac{3}{2} \tan^{-1} x - \frac{1}{2} \left( \frac{x}{x^2 + 1} \right) + C$$

b.  $\int \frac{x^2}{(x-1)(x^2+4)^2} dx$

$$\frac{1}{25} \ln \frac{x-1}{\sqrt{x^2+4}} + \frac{3}{100} \tan^{-1} \frac{x}{2} + \frac{x-4}{10(x^2+4)} + C$$



# Exercise A

i.  $\int \frac{x-1}{x^3+2x^2-x-2} dx$

ii.  $\int \frac{x^2+2}{x(x^2+5x+6)} dx$

iii.  $\int \frac{1}{x^2-1} dx$

iv.  $\int \frac{x^2}{x^2-4} dx$

v.  $\int \frac{2x-1}{(x+2)^2} dx$

vi.  $\int \frac{x^2-2}{x^2-x-12} dx$

vii.  $\int \frac{1+x}{(1-x)^2} dx$

viii.  $\int \frac{2x+1}{(2x+3)^2} dx$

ix.  $\int \frac{x-1}{(x+1)(x^2+1)} dx$

x.  $\int \frac{x^2}{(x-1)(x^2+4x+5)} dx$

xi.  $\int \frac{8x+1}{2x^2-9x-35} dx$

xii.  $\int \frac{7x-8}{4x^2+3x-1} dx$

# Exercise B

i.  $\int \frac{1}{x^2 + 6x + 17} dx$

ii.  $\int \frac{1}{x^2 + 4x + 6} dx$

iii.  $\int \frac{1}{2x^2 + 2x + 7} dx$

iv.  $\int \frac{2x + 5}{x^2 + 4x + 5} dx$

v.  $\int \frac{1}{x^3 + 1} dx$

vi.  $\int \frac{(x - 1)^2}{x^2 + 2x + 2} dx$

vii.  $\int \frac{1}{\sqrt{x^2 + 2x + 4}} dx$

viii.  $\int \frac{1}{x^2 - 4x + 2} dx$

ix.  $\int \frac{x}{\sqrt{x^2 - x + 1}} dx$

x.  $\int \frac{x + 2}{\sqrt{x^2 + 2x - 1}} dx$

xi.  $\int \frac{1}{\sqrt{1 - x - x^2}} dx$

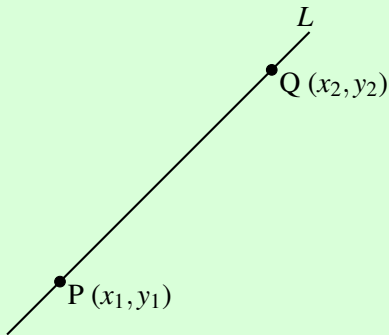
xii.  $\int \frac{2x - 3}{\sqrt{x^2 - 2x + 5}} dx$

# Gradient Of Straight Lines

- Given two points  $P(x, y)$  and  $Q(x, y)$ , the gradient of the line  $\overline{PQ}$ , that passess through  $P$  and  $Q$  is given by

# Gradient Of Straight Lines

- Given two points  $P(x_1, y_1)$  and  $Q(x_2, y_2)$ , the gradient of the line  $\overline{PQ}$ , that passess through  $P$  and  $Q$  is given by



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{or } m = \frac{y_1 - y_2}{x_1 - x_2}$$

# Equation Of A Straight Line

A straight line has a constant gradient. Hence, if  $P(x, y)$  is any general point on the line,  $L$ . Then, the gradient of  $L$  is calculated using  $P(x, y)$  and either of  $A(x_1, y_1)$  or  $B(x_2, y_2)$ . Thus

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \quad (1)$$

$$\implies y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1) \quad (2)$$

Thus, (2) defines the equation of a line.

# Example

Find the equation of a line joining  $A(3, -5)$  and  $B(1, 1)$ .

# Example

Find the equation of a line joining  $A(3, -5)$  and  $B(1, 1)$ .

## Solution

Let  $P(x, y)$  be any point on the line. Then

$$\frac{y - (-5)}{x - 3} = \frac{-5 - 1}{3 - 1}$$
$$\implies y + 5 = \frac{-5 - 1}{3 - 1}(x - 3)$$

$$y + 5 = -3(x - 3)$$

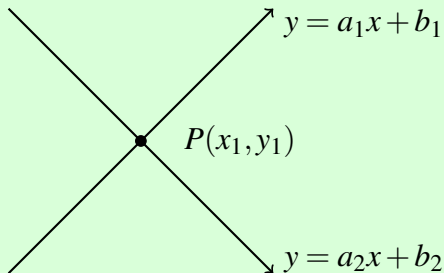
$$\therefore 3x + y - 4 = 0$$

# Intersection of Two Straight Lines

$$\text{Given } y = a_1x + b_1 \quad (3)$$

$$\text{and } y = a_2x + b_2 \quad (4)$$

The point of intersection of these two lines is given by solving the two equations simultaneously.





# Example

Find the point of intersection of the lines

$$2y - 3x + 4 = 0 \quad \text{and}$$

$$y + 2x - 3 = 0$$

# Example

Find the point of intersection of the lines

$$2y - 3x + 4 = 0 \quad \text{and}$$

$$y + 2x - 3 = 0$$

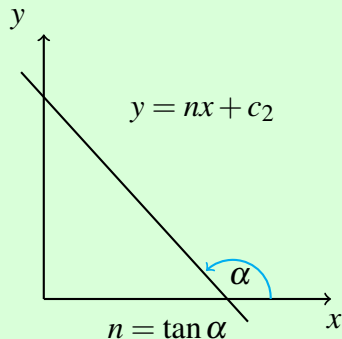
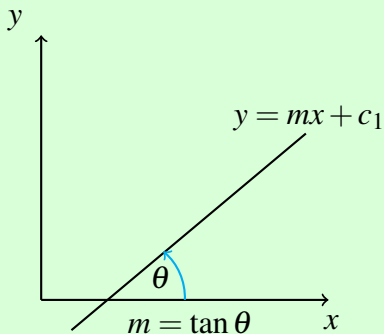
## Solution

Solving the two equations simultaneously, we have

$$x = \frac{10}{7} \quad \text{and}$$

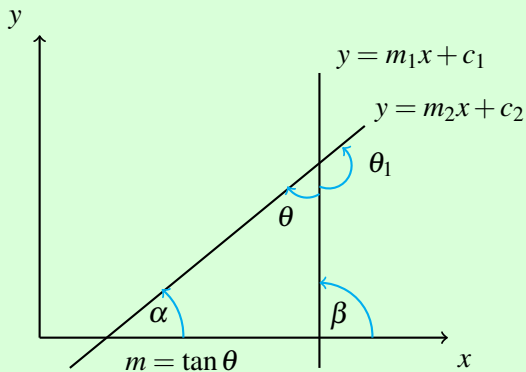
$$y = \frac{1}{7}$$

# Relationship of The Gradient $m$ of a Line To The Angle Made by The Line With The $x$ -Axis



# Angle Between Two Lines

- Suppose that  $\theta$  is the angle between the two lines whose gradients are  $m_1$  and  $m_2$  then



# Angle Between Two Lines (cont.)

Now  $m_1 = \tan \beta$  and  $m_2 = \tan \alpha$

But  $\beta = \theta + \alpha$  (sum of opposite interior angles = exterior angle)

$$\implies \theta = \beta - \alpha$$

Thus  $\tan(\beta - \alpha) = \tan \theta$

$$\begin{aligned}\implies \tan \theta &= \frac{\tan \beta - \tan \alpha}{1 + \tan \beta \tan \alpha} \\ &= \frac{m_1 - m_2}{1 - m_1 m_2}\end{aligned}$$

# Angle Between Two Lines (cont.)

**NB:**

$$\begin{aligned}\theta + \theta_1 &= 180^\circ \Rightarrow \theta_1 = 180^\circ - \theta \\ \Rightarrow \tan \theta_1 &= \tan(180^\circ - \theta) \\ &= -\tan \theta \\ &= -\left(\frac{m_1 - m_2}{1 - m_1 m_2}\right)\end{aligned}$$

Thus, in general if  $\theta$  is the angle between any two given lines, then

$$\tan \theta = \pm \left(\frac{m_1 - m_2}{1 - m_1 m_2}\right)$$

# Example

Find the acute angle between the lines  $3y + 2x = 6$  and  $x - y = 5$

## Example

Find the acute angle between the lines  $3y + 2x = 6$  and  $x - y = 5$

### Solution

$$y = \frac{6 - 2x}{3} = 2 - \frac{2}{3}x \Rightarrow m_1 = -\frac{2}{3} \text{ and}$$

$$y = x - 5 \Rightarrow m_2 = 1$$

If  $\theta$  is the angle between the two lines, then

$$\begin{aligned}\tan \theta &= \pm \left( \frac{m_1 - m_2}{1 - m_1 m_2} \right) \\ &= \pm \left( \frac{-\frac{2}{3} - 1}{1 + (-\frac{2}{3})(1)} \right) \\ &= \pm(-5)\end{aligned}$$

$$\therefore \tan \theta = +5 = 78.7^\circ \approx 79^\circ \text{ (acute angle)}$$



# Parallel Lines

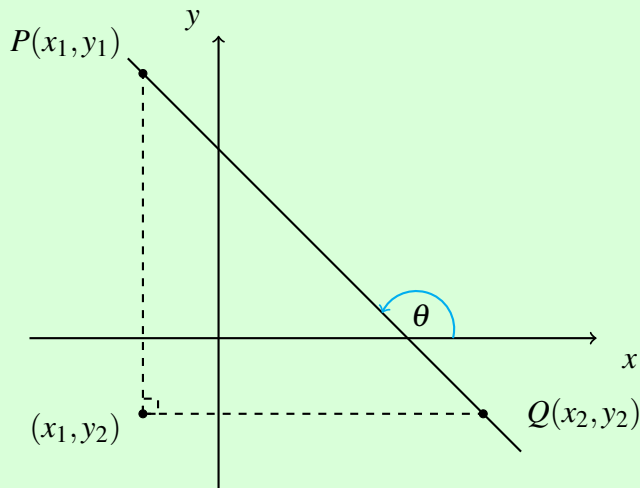
If two lines are parallel, then  $\theta = 0$  and that

$$\tan \theta = \frac{m_1 - m_2}{1 - m_1 m_2} = 0$$

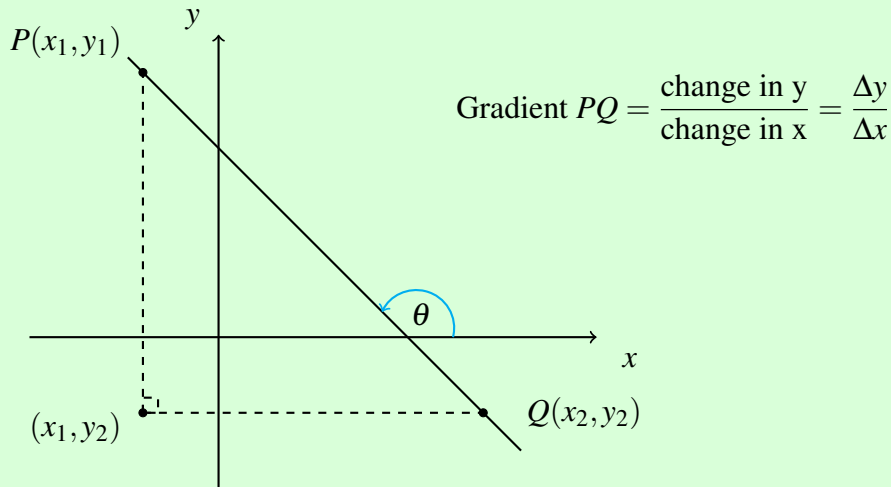
$$\implies m_1 - m_2 = 0$$

Thus  $m_1 = m_2$  (i.e. two parallel lines have equal gradients)

# The Gradient And Inclination of a Straight Line



# The Gradient And Inclination of a Straight Line



# The Gradient And Inclination of a Straight Line (cont.)

- If  $P$  is the point  $(x_1, y_1)$  and  $Q$  is the point  $(x_2, y_2)$ , then

$$\text{Grad } PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = m$$

where  $x_1 \neq x_2$

- **Definition:**

We define the inclination of  $PQ$  to be the angle  $\theta$  ( $0^\circ < \theta < 180^\circ$ ) made by the line with positive direction of the  $x$ -axis, measured in an anticlockwise direction.

Inclination of  $PQ = \theta$

$$\text{where } \tan \theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = m$$

$$\theta \neq 90^\circ$$

# Example

Given the points **(a)**  $(-3, 5)$  and  $(2, 0)$ ; **(b)**  $(-2, -3)$  and  $(3, -1)$ , determine

- i. the gradient and
- ii. the inclination of the line joining the points

# Example

Given the points **(a)**  $(-3, 5)$  and  $(2, 0)$ ; **(b)**  $(-2, -3)$  and  $(3, -1)$ , determine

- i. the gradient and
- ii. the inclination of the line joining the points

## Solution

(a)

i. Gradient =  $\frac{5-0}{-3-2} = -1$       ii.

$$\tan \theta = -1$$

$$\theta = 180^\circ - 45^\circ$$

$$\therefore \text{inclination} = 135^\circ$$

## Example (cont.)

### Solution

(b)

i. Gradient =  $\frac{-3 - (-1)}{-2 - 3} = \frac{2}{5} = 0.4$

ii.

$$\tan \theta = 0.4$$

$$\theta = \tan^{-1}(0.4) = 21.8^\circ$$

$$\therefore \text{Angle of inclination} = 21.8^\circ$$

# Perpendicular Lines

- If two lines are perpendicular, then

$$\theta = \frac{\pi}{2}$$

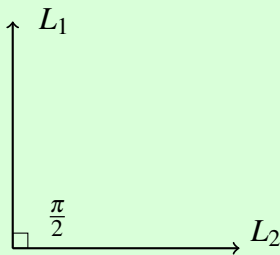
$$\text{Thus } \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \infty$$

$$\implies \tan\left(\frac{\pi}{2}\right) = \frac{m_1 - m_2}{1 + m_1 m_2} = \infty$$

$$\text{Now } \frac{1 + m_1 m_2}{m_1 - m_2} = \frac{1}{\infty}$$

$$\implies 1 + m_1 m_2 = 0$$

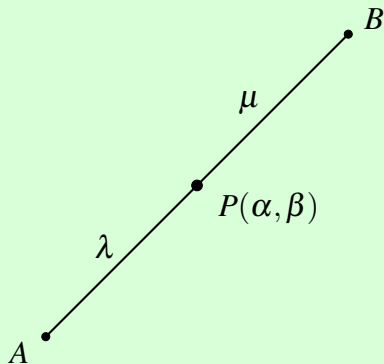
$$\implies m_1 m_2 = -1$$



Thus, the product of the gradients of two perpendicular lines  
 $(m_1 \cdot m_2) = -1$



# Midpoint



$$\alpha = \frac{\mu x_1 + \lambda x_2}{\mu + \lambda} = \frac{x_1 + x_2}{2}$$

$$\beta = \frac{\mu y_1 + \lambda y_2}{\mu + \lambda} = \frac{y_1 + y_2}{2}$$

$$\text{Midpoint } P(\alpha, \beta) = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

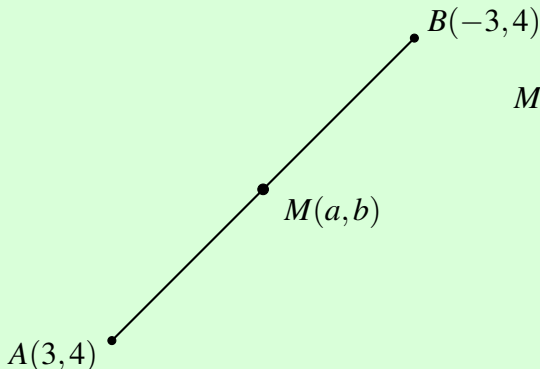
# Example

Find the coordinates of the point  $M(a, b)$  that divide the line  $A(3, 4)$  and  $B(-3, 4)$  into two equal parts.

## Example

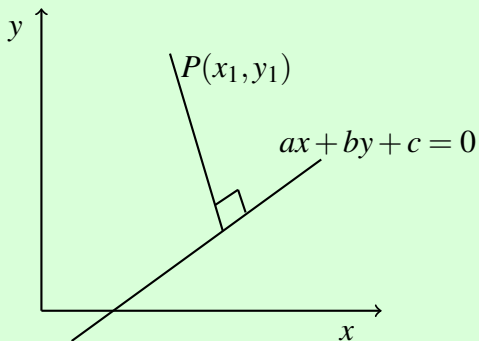
Find the coordinates of the point  $M(a, b)$  that divide the line  $A(3, 4)$  and  $B(-3, 4)$  into two equal parts.

### Solution



$$\begin{aligned} M(a, b) &= \left[ \frac{3 + (-3)}{2}, \frac{4 + 4}{2} \right] \\ &= \left( \frac{0}{2}, \frac{8}{2} \right) \\ &= (0, 4) \end{aligned}$$

# Perpendicular Distance of $P(x, y)$ From The Line $ax + by + c = 0$



$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

# Example

Find the distance of the point  $(5,6)$  from the line  $3x - 4y - 5 = 0$ .

# Example

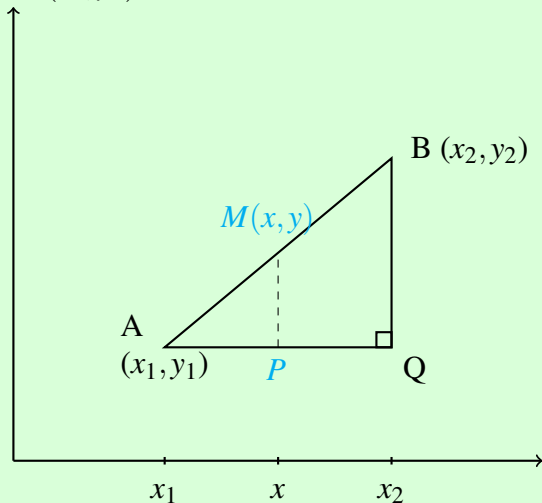
Find the distance of the point (5,6) from the line  $3x - 4y - 5 = 0$ .

## Solution

$$\begin{aligned} d &= \left| \frac{3(5) + (-4)(6) - 5}{\sqrt{3^2 + (-4)^2}} \right| \\ &= \left| \frac{15 - 24 - 5}{\sqrt{25}} \right| = \frac{14}{5} \text{ units} \end{aligned}$$

## Method 2 - The Midpoint of a Line Segment

- To find the midpoint coordinates  $M$  of the line segment joining  $A(x_1, y_1)$  and  $B(x_2, y_2)$ , we have



## Method 2 - The Midpoint of a Line Segment (cont.)

- Let the coordinates of  $M$  be  $(x, y)$ .
- $APQ$  is parallel to the  $x$ -axis and  $MP$  and  $BQ$  are parallel to the  $y$ -axis. Hence, the coordinates of  $P(x, y_1)$  and the coordinates of  $Q$  are  $(x_2, y)$

$$\text{If } AM = MB$$

$$\text{then } AP = PQ \quad (MP \parallel BQ \text{ in } \triangle AQB)$$

$$x - x_1 = x_2 - x \quad \Rightarrow \quad 2x = x_1 + x_2$$

$$\Rightarrow x = \frac{x_1 + x_2}{2}$$

- Similarly, by drawing lines perpendicular to the  $y$ -axis through  $M$  and  $B$ , it can be seen that  $y = \frac{y_1 + y_2}{2}$ .

Thus, the coordinates of the midpoint of the line joining  $(x_1, y_1)$  and  $(x_2, y_2)$  are  $\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$



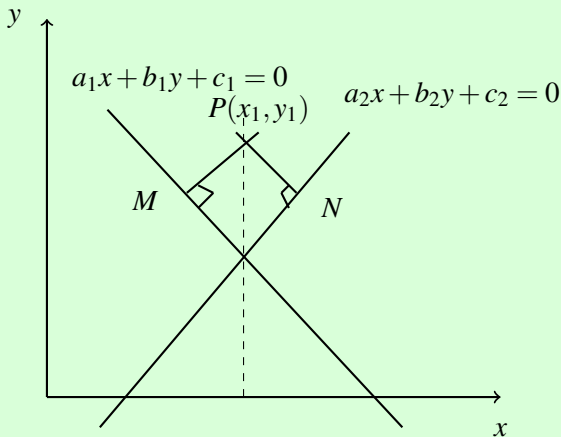
# Exercises

1. Calculate the coordinates of the midpoint of the line joining the points
  - (i)  $(-2, 3)$  and  $(6, 3)$
  - (ii)  $(a, b)$  and  $(-a, \sqrt{2b})$
2. Determine the values of  $x$  and  $y$  if
  - (i)  $(-3, 2)$  is the midpoint of the line segment joining  $(-1, 5)$  and  $(x, y)$ .
  - (ii)  $(-1, y)$  is the midpoint of the line segment joining  $(0, -2)$  and  $(x, 8)$ .
3. Calculate the lengths of the medians of  $\triangle ABC$  in which the coordinates of the vertices are

$$A(-3, 1), \quad B(-5 - 3) \text{ and } C(1, -5)$$

# Equations of The Bisectors of The Given Lines:

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0$$



cont.

$$\begin{aligned} |PM| &= |PN| \\ \Rightarrow \pm \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} &= \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} \end{aligned}$$

# Example

Find the locus of the point which moves so that its perpendicular distance from two lines  $4x - 3y = 0$  and  $5x + 12y = 0$  are equal.

## Example

Find the locus of the point which moves so that its perpendicular distance from two lines  $4x - 3y = 0$  and  $5x + 12y = 0$  are equal.

### Solution

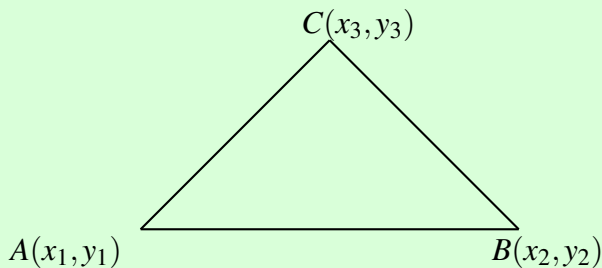
The distance from  $A$  to  $P$ , i.e.  $\overline{AP}$  and  $B$  to  $P$ , i.e.  $\overline{BP}$  are respectively

$$\left| \frac{4x - 3y}{5} \right| \quad \text{and} \quad \left| \frac{5x + 12y}{13} \right|$$

Now, the locus of the point  $P(x, y)$  is

$$\begin{aligned} |\overline{AP}| &= |\overline{BP}| \\ \Rightarrow \frac{4x - 3y}{5} &= \frac{5x + 12y}{13} \\ \therefore 3x - 11y &= 0 \end{aligned}$$

# Area of a Triangle With Given Vertices



$$\text{Area} = \frac{1}{2} [x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)]$$

# Area of a Triangle With Given Vertices (cont.)

Using the determinant method, we have

$$\text{Area of the } \triangle = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\text{OR } \triangle = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & 1 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

# Example

Find the area of a triangle  $ABC$  whose vertices are

$$A(3,4), \quad B(1,3) \text{ and } C(5,-3)$$



## Example

Find the area of a triangle  $ABC$  whose vertices are

$$A(3,4), \quad B(1,3) \text{ and } C(5,-3)$$

### Solution

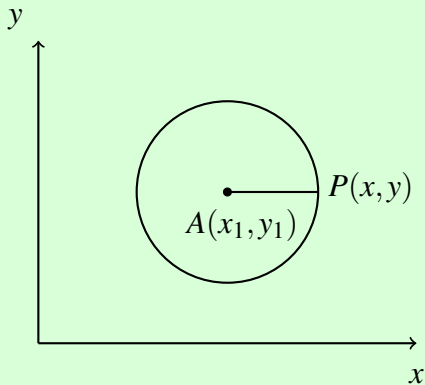
$$\begin{aligned}\text{Area} &= \frac{1}{2} \begin{vmatrix} 3 & 1 & 5 \\ 4 & 3 & -3 \\ 1 & 1 & 1 \end{vmatrix} \\ &= \frac{1}{2} [3(3 - -3) - 1(4 - -3) + 5(4 - 3)] \\ &= \frac{1}{2} [18 - 7 + 5] \\ &= \frac{1}{2} (16) \\ &= 8 \text{ square units}\end{aligned}$$

# Exercise

1. Find the distance between the following pairs of points
  - i.  $(2, 3)$  and  $(4, 5)$
  - ii.  $(6, 1)$  and  $(-6, 6)$
  - iii.  $(3, -7)$  and  $(-1, 3)$
2. Calculate the coordinates of the midpoint of the line joining the points
  - i.  $(-3, 1)$  and  $(1, 5)$
  - ii.  $(\sqrt{3}, 1)$  and  $(3\sqrt{3}, -1)$

# Equation of a Circle With Center $A(x_1, y_1)$ And Radius $r$

- A circle is defined as the locus of points in a plane which is equidistant from a fixed point, that is the centre.



# Equation of a Circle (cont.)

- Let  $A(x_1, y_1)$  be the centre of the circle and  $P(x, y)$  be any point on the circle

$$\begin{aligned}\text{Thus } |AP| &= r \text{ and } |AP|^2 = r^2 \\ \implies |AP|^2 &= (x - x_1)^2 + (y - y_1)^2\end{aligned}$$

Thus the equation of a circle with centre  $A(x_1, y_1)$  and radius,  $r$  is given by

$$|AP|^2 = r^2 = (x - x_1)^2 + (y - y_1)^2$$

- If  $x_1 = y_1 = 0$ , the equation of the circle with centre at the origin  $(0, 0)$  and radius,  $r$  is given by  $x^2 + y^2 = r^2$ .

# Example

1. Find the equation of the circle with a centre of  $(3, 1)$  and radius of 2 units.

# Example

1. Find the equation of the circle with a centre of  $(3, 1)$  and radius of 2 units.

## Solution

Using the formula  $(x - x_1)^2 + (y - y_1)^2 = r^2$ , the required equation is

$$\begin{aligned}(x - 3)^2 + (y - 1)^2 &= 2^2 \\ x^2 - 6x + 9 + y^2 - 2y + 1 &= 4 \\ \implies x^2 + y^2 - 6x - 2y + 6 &= 0\end{aligned}$$

# Example

2. A circle has its centre  $(5, -4)$  and radius 5 units. Find the equation of the circle.

# Example

2. A circle has its centre  $(5, -4)$  and radius 5 units. Find the equation of the circle.

## Solution

The required equation is

$$\begin{aligned}(x - 5)^2 + [y - (-4)]^2 &= 5^2 \\ x^2 - 10x + 25 + y^2 + 8y + 16 &= 25 \\ \implies x^2 + y^2 - 10x + 8y + 16 &= 0\end{aligned}$$



# Example

3. Find the equation of the circle which has its centre at  $(3, 2)$  and passes through the point  $(-1, -2)$ .

# Example

3. Find the equation of the circle which has its centre at  $(3, 2)$  and passes through the point  $(-1, -2)$ .

## Solution

Let  $A = (3, 2)$  and  $B = (-1, -2)$

$$\begin{aligned}\text{Radius } r^2 &= |AB|^2 = (-1 - 3)^2 + (-2 - 2)^2 \\ &= (-4)^2 + (-4)^2 = 32\end{aligned}$$

Using the formula, we have

$$\begin{aligned}(x - 3)^2 + (y - 2)^2 &= 32 \\ x^2 - 6x + 9 + y^2 - 4y + 4 &= 32 \\ \implies x^2 + y^2 - 6x - 4y - 19 &= 0\end{aligned}$$

# Example

4. Find the equation of the circle whose centre is at the point  $(2, 1)$  and passes through the point  $(4, -3)$ .

# Example

4. Find the equation of the circle whose centre is at the point  $(2, 1)$  and passes through the point  $(4, -3)$ .

## Solution

Let  $A = (2, 1)$  and  $B = (4, -3)$

$$\begin{aligned}\text{Radius } r^2 &= |AB|^2 = (4 - 2)^2 + (-3 - 1)^2 \\ &= (2)^2 + (-4)^2 = 20\end{aligned}$$

Using the formula, we have

$$\begin{aligned}(x - 2)^2 + (y - 1)^2 &= 20 \\ x^2 - 4x + 4 + y^2 - 2y + 1 &= 20 \\ \implies x^2 + y^2 - 4x - 2y - 15 &= 0\end{aligned}$$

# Example

5. Find the equation of the circle with centre  $(4, 5)$  which passes through the point where the line  $5x - 2y + 6 = 0$  cuts the  $y$ -axis.

# Example

5. Find the equation of the circle with centre  $(4, 5)$  which passes through the point where the line  $5x - 2y + 6 = 0$  cuts the  $y$ -axis.

## Solution

Given that  $5x - 2y + 6 = 0$  and centre  $O = (4, 5)$  we have

On the  $y$ -axis  $x = 0$

$$\implies 5(0) - 2y + 6 = 0$$

$$-2y = -6 \implies y = 3$$

Thus, the coordinates on the  $y$ -axis is  $P = (0, 3)$ .

$$\text{Radius } r = |OP|$$

$$\implies r^2 = (0 - 4)^2 + (3 - 5)^2 = 20$$

$$\text{Thus } (x - 4)^2 + (y - 5)^2 = 20$$

$$\implies x^2 + y^2 - 8x - 10y + 21 = 0$$

# Example

6. Find the equation of the circle whose centre lies on the line  $x - 2y + 2 = 0$  and which touches the positive  $x$ -axis.

# Example

6. Find the equation of the circle whose centre lies on the line  $x - 2y + 2 = 0$  and which touches the positive  $x$ -axis.

## Solution

Given that  $x - 2y + 2 = 0$ , we have

On the  $x$ -axis  $y = 0$

$$\implies x - 2(0) + 2 = 0 \implies x = -2$$

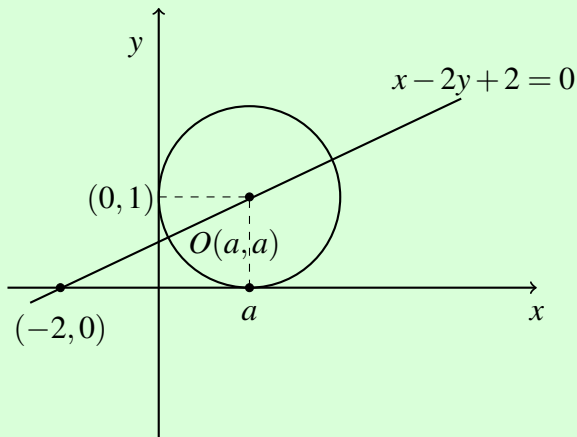
On the  $y$ -axis  $x = 0$

$$\implies 0 - 2y + 2 = 0 \implies y = 1$$

Hence, the intercepts are  $(-2, 0)$  and  $(0, 1)$ .



# Example (cont.)



**NB:**

The circle touches the positive axes at the same distance from the origin.

## Example (cont.)

Let the centre be  $(a, a)$ . Thus we have

$$\begin{aligned}x - 2y + 2 &= 0 \\ \implies a - 2a + 2 &= 0 \\ \implies a &= 2 \text{ [i.e. radius } a = 2 \text{ and centre } = (2, 2)]\end{aligned}$$

Thus, the equation of the circle with centre  $(2, 2)$  and radius,  $r = 2$  is

$$\begin{aligned}(x - 2)^2 + (y - 2)^2 &= 2^2 \\ x^2 - 4x + 4 + y^2 - 4y + 4 &= 4 \\ \implies x^2 + y^2 - 4x - 4y + 4 &= 0\end{aligned}$$

# The General Form of an Equation of a Circle

The equation of a circle with centre  $(x_1, y_1)$  and radius,  $r$  is given by

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

$$x^2 + y^2 - 2xx_1 - 2yy_1 + (x_1^2 + y_1^2 - r^2) = 0$$

$$\text{Let } x_1 = -g, \ y_1 = -f \text{ and } x_1^2 + y_1^2 - r^2 = c$$

$$\implies x^2 + y^2 + 2gx + 2fy + c = 0$$

Thus the general form of a circle with centre  $(-g, -f)$  and radius,  $r$  is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\text{where } c = x_1^2 + y_1^2 - r^2$$

$$\implies r^2 = g^2 + f^2 - c$$

$$r = \sqrt{g^2 + f^2 - c}$$

# The General Form of an Equation of a Circle (cont.)

**NB:**

- The coefficient of  $x^2$  and  $y^2$  are equal.
- There is no term in  $xy$ .

# The General Form of an Equation of a Circle (cont.)

**NB:**

- The coefficient of  $x^2$  and  $y^2$  are equal.
- There is no term in  $xy$ .

## Example

(i)  $x^2 + y^2 - 7x + 5y + 8 = 0$

(ii)  $x^2 + y^2 - 4x + 3y - 7 = 0$

(iii)  $x^2 + y^2 - 4x + 9y + 3 = 0$

(iv)  $x^2 + y^2 - 3x = 0$

# The General Form of an Equation of a Circle (cont.)

**NB:**

- The coefficient of  $x^2$  and  $y^2$  are equal.
- There is no term in  $xy$ .

## Example

- (i)  $x^2 + y^2 - 7x + 5y + 8 = 0$
- (ii)  $x^2 + y^2 - 4x + 3y - 7 = 0$
- (iii)  $x^2 + y^2 - 4x + 9y + 3 = 0$
- (iv)  $x^2 + y^2 - 3x = 0$

## Check the following

1.  $x^2 + 2y^2 + 7x - 5y + 3 = 0$  ?
2.  $x^2 + y^2 + 2x + 5xy + 4 = 0$  ?
3.  $x^2 - y^2 + 3x - 6y + 2 = 0$  ?

# Equation of a Circle Passing Through Three Points

- The equation of a circle is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

- Since the three points  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  lie on the circle, we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$$

Solving the 3 equations simultaneously, the required equation could be found.

# Example

Find the equation of the circle passing through the points  $(2, 1)$ ,  $(0, 2)$  and  $(1, 0)$ .



# Example

Find the equation of the circle passing through the points  $(2, 1)$ ,  $(0, 2)$  and  $(1, 0)$ .

## Solution

Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\begin{aligned}(2, 1) : \quad 2^2 + 1^2 + 2g(2) + 2f(1) + c &= 0 \\ \implies 4g + 2f + c &= -5\end{aligned}\tag{1}$$

$$\begin{aligned}(0, 2) : \quad 0^2 + 2^2 + 2g(0) + 2f(2) + c &= 0 \\ \implies 4f + c &= -4\end{aligned}\tag{2}$$

$$\begin{aligned}(1, 0) : \quad 1^2 + 0^2 + 2g(1) + 2f(0) + c &= 0 \\ \implies 2g + c &= -1\end{aligned}\tag{3}$$

## Example (cont.)

Solving (1), (2) and (3) simultaneously, we have

$$c = -1, \quad g = -\frac{5}{6} \quad \text{and} \quad f = -\frac{7}{6}$$

Thus, the equation of the circle is

$$\begin{aligned} x^2 + y^2 + 2\left(-\frac{5}{6}\right)x + 2\left(-\frac{7}{6}\right)y + \frac{2}{3} &= 0 \\ \implies 3x^2 + 3y^2 - 5x - 7y + 2 &= 0 \end{aligned}$$

# Finding The Radius And Centre of a Given Equation of a Circle

To find the centre and radius of a given equation

- First reduce the coefficients of  $x^2$  and  $y^2$  to a unit of 1.
- Then we complete the squares in  $x$  and  $y$ .

# Example

1. Find the centre and radius of the circle whose given equation is  $3x^2 + 3y^2 - 24x + 12y + 11 = 0$ .

## Example

1. Find the centre and radius of the circle whose given equation is  $3x^2 + 3y^2 - 24x + 12y + 11 = 0$ .

### Solution

$$3x^2 + 3y^2 - 24x + 12y + 11 = 0$$

$$x^2 + y^2 - 8x + 4y + \frac{11}{3} = 0$$

Completing the squares in  $x$  and  $y$ , we have

$$x^2 - 8x + 16 + y^2 + 4y + 4 + \left(\frac{11}{3} - 16 - 4\right) = 0$$

$$(x - 4)^2 + (y + 2)^2 - \frac{49}{3} = 0$$

$$(x - 4)^2 + (y + 2)^2 = \frac{49}{3}$$

Thus the centre =  $(4, -2)$  and the radius,  $r = \sqrt{49}/\sqrt{3}$

# Example

2. Find the centre and radius of the circle  $x^2 + y^2 - x + 4y - 2 = 0$ .

# Example

2. Find the centre and radius of the circle  $x^2 + y^2 - x + 4y - 2 = 0$ .

## Solution

$$x^2 + y^2 - x + 4y - 2 = 0$$

$$x^2 - x + \frac{1}{4} + y^2 + 4y + 4 + \left(-2 - \frac{1}{4} - 4\right) = 0$$

$$\left(x - \frac{1}{2}\right)^2 + (y + 2)^2 - \frac{25}{4} = 0$$

$$\left(x - \frac{1}{2}\right)^2 + (y + 2)^2 = \frac{25}{4}$$

Thus the centre =  $\left(\frac{1}{2}, -2\right)$  and the radius,  $r = \sqrt{\frac{25}{4}} = \frac{5}{2}$

# Example

2. Find the centre and radius of the circle  $x^2 + y^2 - x + 4y - 2 = 0$ .



# Example

2. Find the centre and radius of the circle  $x^2 + y^2 - x + 4y - 2 = 0$ .

## Alternative Solution

Using the general form:  $x^2 + y^2 + 2gx + 2fy + c = 0$ , where the center =  $(-g, -f)$  and radius,  $r = \sqrt{g^2 + f^2 - c}$ , we have

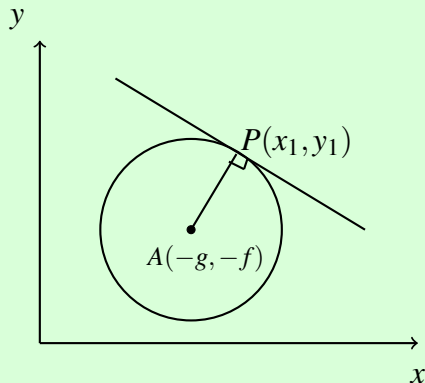
$$2g = -1 \Rightarrow g = -\frac{1}{2}$$

$$2f = 4 \Rightarrow f = 2$$

$$c = 2 \Rightarrow \text{centre} = \left(\frac{1}{2}, -2\right)$$

$$r = \sqrt{\left(-\frac{1}{2}\right)^2 + (2)^2 - (-2)} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

# Finding The Equation of The Tangent Line at a Point $(x_1, y_1)$ of The Circle



$$\begin{aligned}\text{Gradient of } AP &= \frac{y_1 - (-f)}{x_1 - (-g)} \\ &= \frac{y_1 + f}{x_1 + g}\end{aligned}$$

# Cont.

$\implies$  Gradient of the tangent at  $P$  is  $-\left(\frac{x_1 + g}{y_1 + f}\right)$

Thus, the equation of the tangent at  $P$  is given by

$$y - y_1 = -\left(\frac{x_1 + g}{y_1 + f}\right)(x - x_1)$$

# Example

Find the equation of the tangent at the point  $(-2, 3)$  to the circle  $x^2 + y^2 - 4x + 2y - 27 = 0$ .

## Example

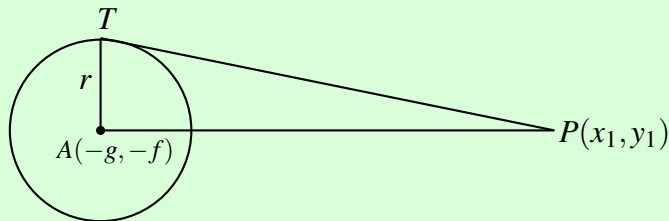
Find the equation of the tangent at the point  $(-2, 3)$  to the circle  $x^2 + y^2 - 4x + 2y - 27 = 0$ .

### Solution

Centre  $= (-g, -f) = (2, -1)$ . Thus the equation of the tangent at  $P(x_1, y_1)$  is given by

$$\begin{aligned}y - y_1 &= - \left( \frac{x_1 + g}{y_1 + f} \right) (x - x_1) \\ \implies y - 3 &= - \frac{(-2 + -2)}{3 + 1} (x - -2) \\ y - 3 &= \frac{4}{4} (x + 2) = x + 2 \\ \implies y &= x + 5 \\ \therefore x - y + 5 &= 0\end{aligned}$$

# Finding The Length of The Tangent From The Point $P(x_1, y_1)$ to The Circle



- Center  $(-g, -f)$
- $P(x_1, y_1)$  is the tangent from  $P$  to the circle
- $|AT|^2 = r^2 = g^2 + f^2 - c$
- Distance between  $A$  and  $P$  is given by

$$|AP|^2 = (y_1 + f)^2 + (x_1 + g)^2$$

# Cont.

- From Pythagoras theorem,

$$\begin{aligned}|PT|^2 &= |AP|^2 - |AT|^2 \\&= [(x_1 + g)^2 + (y_1 + f)^2] - (g^2 + f^2 - c) \\&= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c \\&= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \\ \implies |PT| &= \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}\end{aligned}$$

# Example

1. Find the length of the tangent from  $P(3, 5)$  to the circle  $x^2 + y^2 - 6x + 2y + 6 = 0$ .



# Example

1. Find the length of the tangent from  $P(3,5)$  to the circle  $x^2 + y^2 - 6x + 2y + 6 = 0$ .

## Solution

$$P(x_1, y_1) = P(3, 5) \Rightarrow x_1 = 3 \text{ and } y_1 = 5$$

$$2g = -6 \Rightarrow g = -3$$

$$2f = -2 \Rightarrow f = -1$$

$$c = 6$$

Thus, the length of the tangent from  $P(3,5)$  to the circle is

$$= \sqrt{3^2 + 5^2 + 2(3)(-3) + 2(-1)(5) + 6}$$

$$= \sqrt{9 + 25 + (-18) + (-10) + 6}$$

$$= \sqrt{12} = 2\sqrt{3}$$

# Example

2. Find the length of the tangent from  $(-3, -2)$  to the circle  $x^2 + y^2 - 24x + 95 = 0$ .

# Example

2. Find the length of the tangent from  $(-3, -2)$  to the circle  $x^2 + y^2 - 24x + 95 = 0$ .

## Solution

$$P(x_1, y_1) = P(-3, -2) \Rightarrow x_1 = -3 \text{ and } y_1 = -2$$

$$2g = -24 \Rightarrow g = -12$$

$$2f = 0 \Rightarrow f = 0$$

$$c = 95$$

Thus, the length of the tangent from  $(-3, -2)$  to the circle is

$$= \sqrt{(-3)^2 + (-2)^2 + 2(-12)(-3) + 2(0)(-2) + 95}$$

$$= \sqrt{180}$$

$$= 6\sqrt{5}$$

# Finding The Points of Intersection of a Straight Line And a Circle

- Let the equation of the line be

$$y = mx + c \quad (1)$$

- Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c_1 = 0 \quad (2)$$

- Since the points of intersection lie on both the line and the circle, the points of intersection will be given by the solutions of equations (1) and (2) considered as simultaneous equations in  $x$  and  $y$ .

# Cont.

- Substituting equation (1) into (2) becomes

$$x^2 + (mx + c)^2 + 2gx + 2f(mx + c) + c_1 = 0$$

$$x^2 + m^2x^2 + 2mxc + c^2 + 2gx + 2fmx + 2fc + c_1 = 0$$

$$(1 + m^2)x^2 + 2x(g + mf + mc) + c^2 + 2fc + c_1 = 0$$

is a quadratic equation in  $x$  and gives two values for  $x_1$ .

- From equation (2), the corresponding values for  $y$  may be found.

# Example

1. Find the points of intersection of the line  $y = 3x + 5$  and the circle  $x^2 + y^2 + 4x + 2y - 27 = 0$ .

# Example

1. Find the points of intersection of the line  $y = 3x + 5$  and the circle  $x^2 + y^2 + 4x + 2y - 27 = 0$ .

## Solution

$$y = 3x + 5 \quad (1)$$

$$x^2 + y^2 + 4x + 2y - 27 = 0 \quad (2)$$

Substituting (1) into (2), we have

$$x^2 + (3x + 5)^2 + 4x + 2(3x + 5) - 27 = 0$$

$$x^2 + 9x^2 + 30x + 25 + 4x + 6x + 10 - 27 = 0$$

$$10x^2 + 40x + 8 = 0$$

$$5x^2 + 20x + 4 = 0$$

## Example (cont.)

$$x = \frac{-20 \pm \sqrt{20^2 - 4(5)(4)}}{10}$$

$$\implies x = -0.221 \text{ or } x = -3.789$$

$$\text{When } x = -0.221 \quad y = 3(-0.221) + 5 = 4.367$$

$$\text{When } x = -3.789 \quad y = 3(-3.789) + 5 = -6.367$$

$\therefore$  The points of intersection are  $(-0.221, 4.367)$  and  $(-3.789, -6.367)$



# Example

2. Find the coordinates of the point of intersection of the line  $y = 2x$  and the circle  $x^2 + y^2 - 8x - y + 5 = 0$ .

# Example

2. Find the coordinates of the point of intersection of the line  $y = 2x$  and the circle  $x^2 + y^2 - 8x - y + 5 = 0$ .

## Solution

$$y = 2x \quad (1)$$

$$x^2 + y^2 - 8x - y + 5 = 0 \quad (2)$$

Substituting (1) into (2), we have

$$x^2 + (2x)^2 - 8x - (2x) + 5 = 0$$

$$x^2 + 4x^2 - 10x + 5 = 0$$

$$5x^2 - 10x + 5 = 0$$

$$x^2 - 2x + 1 = 0$$

## Example (cont.)

$$(x - 1)(x - 1) = 0$$

$$\implies x = 1 \text{ twice}$$

$$\text{When } x = 1, y = 2(1) = 2$$

$\therefore$  The point of contact is  $(1, 2)$ .

# Try the following

1.
  - (i) Find the equation of the circle whose centre lies on the line  $y = 4$ , and which passes through the points  $(2, 0)$  and  $(6, 0)$ .
  - (ii) The points  $(8, 4)$  and  $(2, 2)$  are the ends of a diameter of a circle. Find the coordinates of the centre and the radius and find the equation of the circle.
2.
  - (i) Find the the points of intersection of the line  $y = x + 1$  and the circle  $x^2 + y^2 - 8x - 2y + 9 = 0$ .
  - (ii) Find the coordinates of the point of contact of the line  $x - 2y + 12 = 0$  at which it touches the circle  $x^2 + y^2 - x - 31 = 0$ .
3.
  - (i) Find the the equation of the circle through  $(3, -2)$  and  $(5, -4)$  with center on  $x - y = 1$ .

## Try the following (cont.)

- (ii) Find the equation of the circle passing through  $(12, 4)$ ,  $(8, 12)$  and  $(-6, -2)$ .
4. (i) Determine the equation of a circle on  $AB$  as diameter if  $A$  and  $B$  are the points  $(-3, 2)$  and  $(1, 6)$ .
- (ii) Determine the center and radius of the following circles
- (a)  $(x + 1)^2 + (y - 2)^2 = 49$
  - (b)  $x^2 + (y + 2)^2 = 10$
  - (c)  $x^2 + 6x + y^2 - 4y - 12 = 0$
  - (d)  $x^2 + y^2 + 6x - 5y - \frac{3}{4} = 0$
  - (e)  $3x^2 + 3y^2 + 6x + 9y + 6 = 0$
  - (f)  $(x - 3)^2 + y^2 = 10$
  - (g)  $x^2 + y^2 - 3x + 4y + 5\frac{1}{4} = 0$

## Try the following (cont.)

5. (i) Determine the equation of the circle with
- (a) center  $(2, 1)$  and radius 4 units.
  - (b) center  $(-1, 1)$  and radius  $\sqrt{3}$  units.
  - (c) center  $(-3, 4)$  and radius 5 units.
  - (d) center  $(-2, -3)$  passing through the origin.
  - (e) center  $(a, b)$  and radius 3 units.
  - (f) center  $(3, -5)$  passing through the point  $(-1, 1)$ .
- (ii)  $A(4, a)$  and  $B(4, b)$  are two points on the circle  $(x - 1)^2 + (y + 1)^2 = 25$ . Find the possible values of  $a$  and  $b$  and also the equations of the tangents to the circle which are parallel to  $AB$ .

## Try the following (cont.)

6. (i)  $y = 7x$  cuts the circle  $x^2 + y^2 = 50$  at  $A$  and  $B$ . Find the coordinates of  $A$  and  $B$ .
- (ii) If  $x - y = 1$  cuts the circle  $x^2 + y^2 = 13$  at  $P$  and  $Q$ , find the following
- (a) the coordinates of  $P$  and  $Q$ .
  - (b) the length of the chord  $PQ$ .
  - (c) the midpoint,  $M$  of the chord  $PQ$
7. Find the equation of the tangent to
- (a)  $x^2 + y^2 = 5$  which has a gradient of  $-2$ .
  - (b)  $x^2 + y^2 = 18$  which has an inclination of  $135^\circ$ .
  - (c)  $(x + 1)^2 + y^2 = 20$  which is parallel to  $2y - x = 0$
  - (d)  $(x - 2)^2 + (y + 3)^2 = 16$  which is parallel to the  $y$ -axis.

# Conic Section

• ...