CSM 166: Discrete Mathematics for Computer Science

Complex Numbers

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17/06/2021

Content

Complex Numbers

Definition 1 (Complex Numbers)

A complex number is a number that can be expressed in the form $\mathbf{z} = \mathbf{a} + \mathbf{bi}$, *where a and b are real numbers and* $\mathbf{i}$ *is the imaginary unit, that satisfies the equation* $\mathbf{i}^2 = -1$.

Let

$$z = a + bi \tag{1}$$

Then **a** is the real part of z denoted by $a = Re(z)$ and **b** is the imaginary part of z denoted $b = Im(z)$

Complex Numbers

1. If Re(z) = 0 then z is purely imaginary:

$$z = bi$$

2. If $Im(z) = 0$ then z is a real number:

$$z = a$$

Conjugate Of A Complex Numbers

Definition 2

The complex conjugate of a complex number $\mathbf{z} = \mathbf{a} + \mathbf{bi}$ *is defined as*

$$\overline{z} = \mathbf{a} - b\mathbf{i} \tag{2}$$

Operations with Complex Numbers I

Let $z_1 = a_1 + b_1\mathbf{i}$ and $z_2 = a_2 + b_2\mathbf{i}$. Then

1. **Addition**:

$$\begin{aligned} z_1 + z_2 &= a_1 + b_1 i + a_2 + b_2\mathbf{i} \\ &= (a_1 + a_2) + (b_1 + b_2)\mathbf{i} \end{aligned}$$

2. **Subtraction**:

$$\begin{aligned} z_1 - z_2 &= a_1 + b_1 i - (a_2 + b_2)\mathbf{i} \\ &= (a_1 + a_2) + (b_1 - b_2)\mathbf{i} \end{aligned}$$

Operations with Complex Numbers II

3. **Multiplication**:

$$
\begin{aligned}
z_1 \cdot z_2 &= (a_1 + b_1\mathbf{i}) \cdot (a_2 + b_2)\mathbf{i} \\
&= a_1 a_2 + a_1 b_2\mathbf{i} + a_2 b_1\mathbf{i} + b_1 b_2\mathbf{i}^2 \\
&= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\mathbf{i}
\end{aligned}
$$

4. **Division**:

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{a_1 + b_1\mathbf{i}}{a_2 + b_2\mathbf{i}} = \frac{a_1 + b_1\mathbf{i}}{a_2 + b_2\mathbf{i}} \cdot \frac{a_2 - b_2\mathbf{i}}{a_2 - b_2\mathbf{i}} \\
&= \frac{1}{(a_2^2 + b_2^2)}[a_1 a_2 + b_1 b_2 + (a_2 b_1 - a_1 b_2)\mathbf{i}]
\end{aligned}
$$

Properties Of Complex Conjugates

1. $\overline{\overline{z}} = z$

2. $Re(z) = \dfrac{z + \overline{z}}{2}$ and $Im(z) = \dfrac{z - \overline{z}}{2\mathbf{i}}$

3. $\overline{z + w} = \overline{z} + \overline{w}$

4. $\overline{z - w} = \overline{z} - \overline{w}$

5. $\overline{zw} = \overline{z}\,\overline{w}$

6. $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

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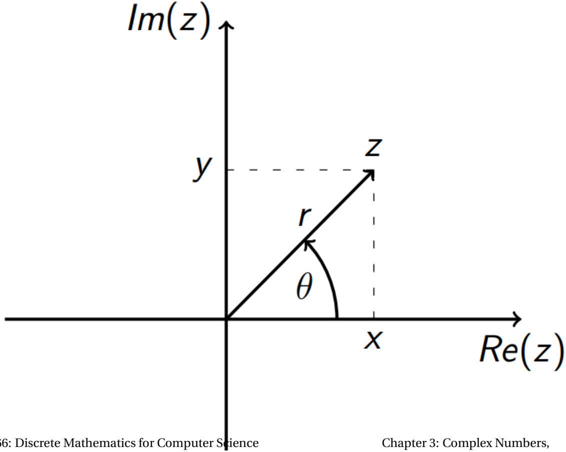
Geometric Representation Of A Complex Number (Argand Diagram)

Let $z = x + y\mathbf{i}$



Absolute Value Of A Complex Numbers

Definition 3

The absolute value of a complex number $z = x + y\mathbf{i}$ *is defined as*

$$|z| = \sqrt{x^2 + y^2}$$

Thus $|z|$ is the distance from the origin to the point z in the complex plane

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Argument Of A Complex Numbers

Definition 4

*The angle θ is called the **argument** of the complex number z.*

The Principal Argument is $0 < \theta \leq 2\pi$ or

$-\pi < \theta < \pi$

Deduction From The Argand Diagram

From the Argand diagram, we deduce the following

1. $x = r\cos\theta$
2. $y = r\sin\theta$
3. $z = r\cos\theta + \mathbf{i}\sin\theta$
4. $|z| = r$
5. $\theta = arg\, z = \tan^{-1}(\frac{y}{x})$

Thus z can be represented in (x, y) or (r, θ) coordinates.
(r, θ) is called **Polar Coordinate**.

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Theorem 1 (Euler)

$$e^{i\theta} = \cos\theta + \mathbf{i}\sin\theta \tag{3}$$

This leads to **De Moivre theorem.**

Theorem 2 (De Moivre)

$$(\cos\theta + \boldsymbol{i}\sin\theta)^n = \cos(n\theta) + \boldsymbol{i}\sin(n\theta)$$

which is the polar coordinate representation of z

Note: If $z = re^{i\theta}$ then $\bar{z} = re^{-i\theta}$

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Operations In Polar Coordinates I

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. Then

1. **Addition** :

$$
\begin{aligned}
z_1 + z_2 &= r_1 e^{i\theta_1} + r_2 e^{i\theta_2} \\
&= r_1 \cos\theta_1 + i r_1 \sin\theta_1 + r_2 \cos\theta_2 + i r_2 \sin\theta_2 \\
&= (r_1 \cos\theta_1 + r_2 \cos\theta_2) + i(r_1 \sin\theta_1 + r_2 \sin\theta_2)
\end{aligned}
$$

2. **Subtraction** :

$$
\begin{aligned}
z_1 - z_2 &= r_1 e^{i\theta_1} - r_2 e^{i\theta_2} \\
&= r_1 \cos\theta_1 + i r_1 \sin\theta_1 - (r_2 \cos\theta_2 + i r_2 \sin\theta_2) \\
&= (r_1 \cos\theta_1 - r_2 \cos\theta_2) + i(r_1 \sin\theta_1 - r_2 \sin\theta_2)
\end{aligned}
$$

Operations In Polar Coordinates II

3. **Multiplication** :

$$z_1 \cdot z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

4. **Division** :

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

End of Lecture

Questions…???

Thanks