EIGENVALUES AND EIGENVECTORS

Dr. Gabriel Obed Fosu
Department of Mathematics
Kwame Nkrumah University of Science and Technology

March 30, 2022

Outline

Outline of Presentation

Introduction

Computing eigenvalues and eigenvectors is one of the most important problems in numerical linear algebra.

Introduction

Computing eigenvalues and eigenvectors is one of the most important problems in numerical linear algebra.

1. Let A be an $n \times n$ matrix. For a large number of problems in engineering and science, it is necessary to find vector v such that Av is a multiple of λ of v

Introduction

Computing eigenvalues and eigenvectors is one of the most important problems in numerical linear algebra.

1. Let A be an $n \times n$ matrix. For a large number of problems in engineering and science, it is necessary to find vector v such that Av is a multiple of λ of v
2. In other words,

$$Av = \lambda v$$

Av is parallel to v, and λ either stretches or shrinks v.

Introduction

Computing eigenvalues and eigenvectors is one of the most important problems in numerical linear algebra.

1. Let A be an $n \times n$ matrix. For a large number of problems in engineering and science, it is necessary to find vector v such that Av is a multiple of λ of v
2. In other words,
$$Av = \lambda v$$
Av is parallel to v, and λ either stretches or shrinks v.
3. The value λ is an eigenvalue
4. v is an eigenvector associated with λ.

Definition

1. If A is an $n \times n$ matrix, in order to find eigenvalue λ and an associated eigenvector v, it must be the case that $Av = \lambda v$, and this is equivalent to the homogeneous system
$$(A - \lambda I)v = 0 \tag{1}$$

Definition

1. If A is an $n \times n$ matrix, in order to find eigenvalue λ and an associated eigenvector v, it must be the case that $Av = \lambda v$, and this is equivalent to the homogeneous system
$$(A - \lambda I)v = 0 \tag{1}$$

2. From (the homogeneous system $Ax = 0$ has a nontrivial solution if and only if $\det A = 0$) we deduce that
$$det(A - \lambda I) = 0 \tag{2}$$
in order that this system to have a nonzero solution.

Definition

1. If A is an $n \times n$ matrix, in order to find eigenvalue λ and an associated eigenvector v, it must be the case that $Av = \lambda v$, and this is equivalent to the homogeneous system
$$(A - \lambda I)v = 0 \tag{1}$$
2. From (the homogeneous system $Ax = 0$ has a nontrivial solution if and only if $\det A = 0$) we deduce that
$$det(A - \lambda I) = 0 \tag{2}$$
in order that this system to have a nonzero solution.
3. The determinant of $A - \lambda I$ is a polynomial of degree n, so Equation (2) is a problem of finding roots of the polynomial
$$p(\lambda) = det(A - \lambda I) \tag{3}$$

Definition

1. The polynomial

$$p(\lambda) = det(A - \lambda I) \tag{4}$$

is called the characteristic polynomial of A

Definition

1. The polynomial
$$p(\lambda) = det(A - \lambda I) \tag{4}$$
is called the characteristic polynomial of A
2. The equation
$$p(\lambda) = 0 \tag{5}$$
is termed the characteristic equation.

Definition

1. The polynomial
$$p(\lambda) = det(A - \lambda I) \tag{4}$$
is called the characteristic polynomial of A
2. The equation
$$p(\lambda) = 0 \tag{5}$$
is termed the characteristic equation.
3. If λ is a root of p, it is termed an eigenvalue of A,
4. If v is a nonzero column vector satisfying $Av = \lambda v$, it is an eigenvector of A. We say that v is an eigenvector corresponding to the eigenvalue λ.

Example

1. Let

$$A = \begin{bmatrix} -0.4707 & 0.7481 \\ 1.7481 & 1.4707 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0.2898 \\ 0.9571 \end{bmatrix}$$

2. Then

$$A\vec{v} = \begin{bmatrix} 0.5796 \\ 1.9142 \end{bmatrix}$$

3. Moreover

$$A\vec{v} = \begin{bmatrix} 0.5796 \\ 1.9142 \end{bmatrix} = 2\vec{v} \tag{6}$$

Thus $\vec{v}$ is an eigenvector of A corresponding to eigenvalue 2.

Example

1. Let

$$A = \begin{bmatrix} -0.4707 & 0.7481 \\ 1.7481 & 1.4707 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0.2898 \\ 0.9571 \end{bmatrix}$$

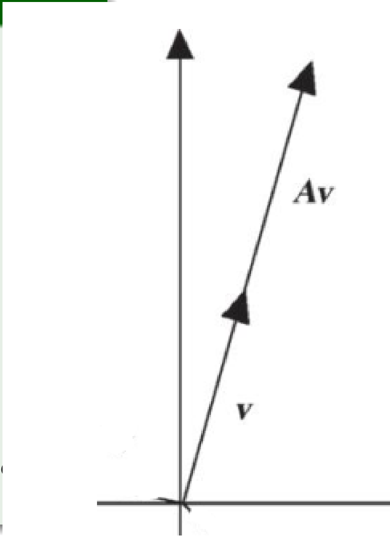
2. Then

$$A\vec{v} = \begin{bmatrix} 0.5796 \\ 1.9142 \end{bmatrix}$$

3. Moreover

$$A\vec{v} = \begin{bmatrix} 0.5796 \\ 1.9142 \end{bmatrix} = 2\vec{v}$$

Thus $\vec{v}$ is an eigenvector of A corresponding to eig
2.



Summary Steps To Finding Eigenvalues and Eigenvectors

Given an $n \times n$ matrix A

1. Find the characteristic polynomial

$$p(\lambda) = det(A - \lambda I) \tag{7}$$

Summary Steps To Finding Eigenvalues and Eigenvectors

Given an $n \times n$ matrix A

1. Find the characteristic polynomial
$$p(\lambda) = det(A - \lambda I) \tag{7}$$

2. Compute the n roots of
$$p(\lambda) = 0 \tag{8}$$
These are the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A.

Summary Steps To Finding Eigenvalues and Eigenvectors

Given an $n \times n$ matrix A

1. Find the characteristic polynomial

$$p(\lambda) = det(A - \lambda I) \tag{7}$$

2. Compute the n roots of

$$p(\lambda) = 0 \tag{8}$$

   These are the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A.

3. For each distinct λ_i, find an eigenvector x_i such that $Ax_i = \lambda_i x$ by solving

$$(A - \lambda I)v = 0 \tag{9}$$

Summary Steps To Finding Eigenvalues and Eigenvectors

Given an $n \times n$ matrix A

1. Find the characteristic polynomial
$$p(\lambda) = det(A - \lambda I) \tag{7}$$
2. Compute the n roots of
$$p(\lambda) = 0 \tag{8}$$
These are the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A.
3. For each distinct λ_i, find an eigenvector x_i such that $Ax_i = \lambda_i x$ by solving
$$(A - \lambda I)v = 0 \tag{9}$$

If λ_i is a multiple root, there may be only one associated eigenvector. If not, compute the distinct eigenvectors.

Definition

A polynomial

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

of degree n has exactly n roots; either being, real distinct roots, real repeated roots, or complex roots. Any complex roots occur in conjugate pairs.

Definition

A polynomial

$$p(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0$$

of degree n has exactly n roots; either being, real distinct roots, real repeated roots, or complex roots. Any complex roots occur in conjugate pairs.

Definition

Vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$ in R^n are said to be linearly dependent if there exist scalars $k_1, k_2, \cdots, k_n$ not all zero, such that

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \cdots + k_n\mathbf{v_n} = 0 \tag{10}$$

The vectors $\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n}$ are called linearly independent if they are not linearly dependent.

Example

Find the eigenvalues and all the eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Example

Find the eigenvalues and all the eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

First find the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) \quad = \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{11}$$

Example

Find the eigenvalues and all the eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

First find the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{11}$$

$$= \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \tag{12}$$

Example

Find the eigenvalues and all the eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

First find the characteristic polynomial

$$\begin{aligned} p(\lambda) = \det(A - \lambda I) \quad &= \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) && (11) \\ &= \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) && (12) \\ &= \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) && (13) \end{aligned}$$

Example

Find the eigenvalues and all the eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

First find the characteristic polynomial

$$\begin{aligned}
p(\lambda) = \det(A - \lambda I) &= \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) && (11) \\
&= \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) && (12) \\
&= \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) && (13) \\
&= (2-\lambda)^2 - 1 && (14)
\end{aligned}$$

Example

Find the eigenvalues and all the eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

First find the characteristic polynomial

$$\begin{aligned}
p(\lambda) = \det(A - \lambda I) &= \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) && (11)\\
&= \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) && (12)\\
&= \det\left(\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}\right) && (13)\\
&= (2-\lambda)^2 - 1 && (14)\\
&= \lambda^2 - 4\lambda + 3 && (15)
\end{aligned}$$

1. Next, compute the n roots of $p(\lambda) = 0$. That is

$$\lambda^2 - 4\lambda + 3 = 0 \tag{16}$$
$$(\lambda - 1)(\lambda - 3) = 0 \tag{17}$$

Thus the eigenvalues are

$$\lambda = 1 \text{ or } \lambda = 3$$

1. To find the eigenvector when $\lambda = 1$, we solve

$$(A - \lambda I)\vec{v} = 0 \tag{18}$$

1. To find the eigenvector when $\lambda = 1$, we solve

$$(A - \lambda I)\vec{v} = 0 \tag{18}$$

Let $\vec{v} = [x, \ y]^T$

1. To find the eigenvector when $\lambda = 1$, we solve
$$(A - \lambda I)\vec{v} = 0 \tag{18}$$
Let $\vec{v} = [x, \ y]^T$
2. Thus we solve the homogeneous system
$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

1. To find the eigenvector when $\lambda = 1$, we solve
$$(A - \lambda I)\vec{v} = 0 \tag{18}$$
Let $\vec{v} = [x,\ y]^T$
2. Thus we solve the homogeneous system
$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$
$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

1. To find the eigenvector when $\lambda = 1$, we solve

$$(A - \lambda I)\vec{v} = 0 \tag{18}$$

Let $\vec{v} = [x,\ y]^T$

2. Thus we solve the homogeneous system

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21}$$

1. To find the eigenvector when $\lambda = 1$, we solve
$$(A - \lambda I)\vec{v} = 0 \tag{18}$$
Let $\vec{v} = [x,\ y]^T$
2. Thus we solve the homogeneous system
$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$
$$\begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20}$$
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21}$$
3. Solving (21) using Gaussian elimination, the system reduce to
$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{22}$$

1. This implies that

$$x + y = 0 \implies x = -y$$

1. This implies that

$$x + y = 0 \implies x = -y$$

2. Let $y \in R$ but $y \neq 0$ then the eigenvectors corresponding to $\lambda = 1$ are the vectors of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

1. This implies that

$$x + y = 0 \implies x = -y$$

2. Let $y \in R$ but $y \neq 0$ then the eigenvectors corresponding to $\lambda = 1$ are the vectors of the form

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

3. Choose $y = 1$ to obtain a specific eigenvector

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{23}$$

1. To find the eigenvector when $\lambda = 3$, we solve

$$(A - \lambda I)\vec{v} = 0 \tag{24}$$

Let $\vec{v} = [x,\ y]^T$

1. To find the eigenvector when $\lambda = 3$, we solve

$$(A - \lambda I)\vec{v} = 0 \tag{24}$$

   Let $\vec{v} = [x, \ y]^T$

2. Thus we solve the homogeneous system

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{27}$$

1. To find the eigenvector when $\lambda = 3$, we solve

$$(A - \lambda I)\vec{v} = 0 \tag{24}$$

   Let $\vec{v} = [x, \ y]^T$

2. Thus we solve the homogeneous system

$$\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{25}$$

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{26}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{27}$$

3. Solving (27) using Gaussian elimination, the system reduce to

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{28}$$

1. This implies that

$$-x + y = 0 \implies x = y$$

1. This implies that
$$-x+y=0 \implies x=y$$

2. Let $y \in R$ but $y \neq 0$ then the eigenvectors corresponding to $\lambda = 3$ are the vectors of the form
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ y \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

3. Choose $y = 1$ to obtain a specific eigenvector
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{29}$$

Example

Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 4 & 8 & 3 \\ 0 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

Example

Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 4 & 8 & 3 \\ 0 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

1. The characteristic polynomial is

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 8 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix}\right) \tag{30}$$

$$= (-1-\lambda)(4-\lambda)(2-\lambda) \tag{31}$$

That is expanding by the second row.

1. Now $p(\lambda) = 0$

$$(-1-\lambda)(4-\lambda)(2-\lambda) = 0 \tag{32}$$

The roots of the characteristic polynomial are

$$\lambda_1 = 4, \quad \lambda_2 = -1, \quad \lambda_3 = 2$$

$\lambda = 4$

1. Now the eigenvector when $\lambda = 4$. We solve the homogeneous system $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 4-\lambda & 8 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (33)$$

$\lambda = 4$

1. Now the eigenvector when $\lambda = 4$. We solve the homogeneous system $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 4-\lambda & 8 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} 4-4 & 8 & 3 \\ 0 & -1-4 & 0 \\ 0 & -2 & 2-4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

$\lambda = 4$

1. Now the eigenvector when $\lambda = 4$. We solve the homogeneous system $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 4-\lambda & 8 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} 4-4 & 8 & 3 \\ 0 & -1-4 & 0 \\ 0 & -2 & 2-4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} 0 & 8 & 3 \\ 0 & -5 & 0 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

1. Solving using Gaussian elimination, reduces the system to

$$\begin{bmatrix} 0 & 8 & 3 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{36}$$

2. Thus

$$y = 0, \quad z = 0$$

1. Solving using Gaussian elimination, reduces the system to

$$\begin{bmatrix} 0 & 8 & 3 \\ 0 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{36}$$

2. Thus

$$y = 0, \quad z = 0$$

3. The first row specifies that

$$(0)x + 8(0) + 3(0) = 0 \tag{37}$$

The component x is not constrained. Any value of x will work. So we choose $x = 1$ to obtain the eigenvector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{38}$$

$\lambda = -1$

1. We solve the homogeneous system $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 4-\lambda & 8 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

$$\begin{bmatrix} 4+1 & 8 & 3 \\ 0 & -1+1 & 0 \\ 0 & -2 & 2+1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{40}$$

$$\begin{bmatrix} 5 & 8 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

1. Solving using Gaussian elimination (interchange row 2 and 3), reduces the system to

$$\begin{bmatrix} 5 & 8 & 3 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

2. Thus

$$y = \frac{3z}{2}, \quad x = -\frac{8(3z/2) + 3z}{5} = -3z$$

3. This gives a general eigenvector

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3z \\ 3z/2 \\ z \end{bmatrix} = z \begin{bmatrix} -3 \\ 3/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3/2 \\ 1 \end{bmatrix} \tag{43}$$

Note that z chosen as $z = 1$

$\lambda = 2$

1. We solve the homogeneous system $(A - \lambda I)\vec{v} = 0$

$$\begin{bmatrix} 4-\lambda & 8 & 3 \\ 0 & -1-\lambda & 0 \\ 0 & -2 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{44}$$

$$\begin{bmatrix} 4-2 & 8 & 3 \\ 0 & -1-2 & 0 \\ 0 & -2 & 2-2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{45}$$

$$\begin{bmatrix} 2 & 8 & 3 \\ 0 & -3 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{46}$$

1. Solving using Gaussian elimination reduces the system to
$$\begin{bmatrix} 2 & 8 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{47}$$
2. Thus
$$y = 0, \quad x = -3z/2$$
3. This gives a general eigenvector
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3z/2 \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} \tag{48}$$
4. If we choose $z = 1$, then the eigenvector is
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} \tag{49}$$

Example

Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 6 & 12 & 19 \\ -9 & -20 & -33 \\ 4 & 9 & 15 \end{bmatrix}$$

Example

Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 6 & 12 & 19 \\ -9 & -20 & -33 \\ 4 & 9 & 15 \end{bmatrix}$$

1. The characteristic polynomial is

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 6-\lambda & 12 & 19 \\ -9 & -20-\lambda & -33 \\ 4 & 9 & 15-\lambda \end{bmatrix}\right) \tag{50}$$

$$= (\lambda+1)(\lambda-1)^2 \tag{51}$$

1. The roots of the characteristic polynomial are

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

$\lambda = 1$ is a multiple root.

$\lambda = 1$

1. We solve the homogeneous system

$$\begin{bmatrix} 6-\lambda & 12 & 19 \\ -9 & -20-\lambda & -33 \\ 4 & 9 & 15-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{52}$$

$$\begin{bmatrix} 5 & 12 & 19 \\ -9 & -21 & -33 \\ 4 & 9 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{53}$$

$\lambda = 1$

1. We solve the homogeneous system

$$\begin{bmatrix} 6-\lambda & 12 & 19 \\ -9 & -20-\lambda & -33 \\ 4 & 9 & 15-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{52}$$

$$\begin{bmatrix} 5 & 12 & 19 \\ -9 & -21 & -33 \\ 4 & 9 & 14 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{53}$$

2. Solving using Gaussian elimination reduces the system to

$$\begin{bmatrix} 5 & 12 & 19 \\ 0 & 3/5 & 6/5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{54}$$

1. Thus

$$x = z, \quad y = -2z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \tag{55}$$

2. There is only one linearly independent eigenvector associated with $\lambda = 1$
3. The eigenvector when $\lambda = -1$ is left as an exercise.

Note

There are cases where an eigenvalue of multiplicity k does produce k linearly independent eigenvectors.

Symmetric Matrix

Whenever an $n \times n$ real matrix is symmetric, it has n linearly independent eigenvectors, even if its characteristic equation has roots of multiplicity 2 or more.

Example

Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Example

Find the eigenvalues and eigenvectors of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

1. The characteristic polynomial is

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix}\right) \tag{56}$$

$$= (\lambda - 3)\lambda^2 \tag{57}$$

1. The roots of the characteristic polynomial are

$$\lambda_1 = 0, \quad \lambda_2 = 3$$

$\lambda = 0$ is an eigenvalue of multiplicity 2.

$\lambda = 0$

1. We solve the homogeneous system

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{58}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{59}$$

$\lambda = 0$

1. We solve the homogeneous system

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (58)$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (59)$$

2. Solving using Gaussian elimination reduces the system to

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (60)$$

1. Thus
$$x = -y - z$$

1. Thus
$$x = -y - z$$
2. When we choose $z = 0$, then we have
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{61}$$

1. Thus
$$x = -y - z$$
2. When we choose $z = 0$, then we have
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{61}$$
3. Similarly, when we choose $y = 0$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{62}$$

1. Thus
$$x = -y - z$$
2. When we choose $z = 0$, then we have
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{61}$$
3. Similarly, when we choose $y = 0$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{62}$$
4. Thus we have two linearly independent eigenvectors
$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

1. Thus
$$x = -y - z$$
2. When we choose $z = 0$, then we have
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{61}$$
3. Similarly, when we choose $y = 0$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ 0 \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \tag{62}$$
4. Thus we have two linearly independent eigenvectors
$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$
5. The eigenvector when $\lambda = 3$ is left as an exercise.

Some Properties of Eigenvalues and Eigenvectors

Singular

An $n \times n$ matrix A is singular if and only if it has a 0 eigenvalue.

Some Properties of Eigenvalues and Eigenvectors

Singular

An $n \times n$ matrix A is singular if and only if it has a 0 eigenvalue.

Eigenspace

The eigenvectors corresponding to λ form a subspace called an eigenspace.

Some Properties of Eigenvalues and Eigenvectors

Singular

An $n \times n$ matrix A is singular if and only if it has a 0 eigenvalue.

Eigenspace

The eigenvectors corresponding to λ form a subspace called an eigenspace.

Determinant

If A is an $n \times n$ matrix, then

$$\det A = \prod_{i=1}^{n} \lambda_i$$

Outline of Presentation

Diagonalization

Definition (Diagonalization)

Diagonalization involves how we can use the eigenvectors of a matrix A to transform A into a diagonal matrix of eigenvalues.

Diagonalization

The procedure for diagonalizing a matrix A can be done with the following steps:

1. Form the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of A.

Diagonalization

The procedure for diagonalizing a matrix A can be done with the following steps:

1. Form the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of A.
2. Find the roots of p; the eigenvalues and its corresponding eigenvectors.

Diagonalization

The procedure for diagonalizing a matrix A can be done with the following steps:

1. Form the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of A.
2. Find the roots of p; the eigenvalues and its corresponding eigenvectors.
3. Form the matrix

$$X = [\vec{v_1},\ \vec{v_2}, \cdots \vec{v_{n-1}},\ \vec{v_n}]$$

whose columns are eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{n-1},\ \lambda_n$.

Diagonalization

The procedure for diagonalizing a matrix A can be done with the following steps:

1. Form the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of A.
2. Find the roots of p; the eigenvalues and its corresponding eigenvectors.
3. Form the matrix
$$X = [\vec{v_1},\ \vec{v_2}, \cdots \vec{v_{n-1}},\ \vec{v_n}]$$
whose columns are eigenvectors of A corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_{n-1},\ \lambda_n$.
4. Then,
$$D = X^{-1}AX \tag{63}$$
where D is the diagonal matrix with $\lambda_1, \lambda_2, \cdots, \lambda_{n-1},\ \lambda_n$ on its diagonal. Then the matrix A and D are called similar matrices.

Definition (Similar Matrices)

The matrix B is similar to matrix A if there exists a nonsingular matrix X such that

$$B = X^{-1}AX$$

Thus, two similar matrices have the same eigenvalues.

Definition (Similar Matrices)

The matrix B is similar to matrix A if there exists a nonsingular matrix X such that

$$B = X^{-1}AX$$

Thus, two similar matrices have the same eigenvalues.

Example

Show that the following matrix can be diagonalize

$$A = \begin{bmatrix} 4 & 8 & 3 \\ 0 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

From the previous example, we obtained the eigenvalues

$$\lambda_1 = 4, \quad \lambda_2 = -1, \quad \lambda_3 = 2 \tag{64}$$

From the previous example, we obtained the eigenvalues

$$\lambda_1 = 4, \quad \lambda_2 = -1, \quad \lambda_3 = 2 \tag{64}$$

These have the corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3 \\ 3/2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} \tag{65}$$

From the previous example, we obtained the eigenvalues

$$\lambda_1 = 4, \quad \lambda_2 = -1, \quad \lambda_3 = 2 \tag{64}$$

These have the corresponding eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -3 \\ 3/2 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} \tag{65}$$

The matrix of eigenvectors is given as

$$X = [\vec{v_1}, \ \vec{v_2}, \ \vec{v_3}] = \begin{bmatrix} 1 & -3 & -3/2 \\ 0 & 3/2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{66}$$

1. Then we can find $D = X^{-1}AX$

1. Then we can find $D = X^{-1}AX$
2. Clearly

$$X^{-1} = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 2/3 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \tag{67}$$

1. Then we can find $D = X^{-1}AX$
2. Clearly

$$X^{-1} = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 2/3 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \tag{67}$$

Then

$$X^{-1}A = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 2/3 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 & 3 \\ 0 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 6 \\ 0 & -2/3 & 0 \\ 0 & -4/3 & 2 \end{bmatrix} \tag{68}$$

1. Then we can find $D = X^{-1}AX$
2. Clearly

$$X^{-1} = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 2/3 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \tag{67}$$

Then

$$X^{-1}A = \begin{bmatrix} 1 & 1 & 3/2 \\ 0 & 2/3 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 8 & 3 \\ 0 & -1 & 0 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 6 \\ 0 & -2/3 & 0 \\ 0 & -4/3 & 2 \end{bmatrix} \tag{68}$$

and

$$D = X^{-1}AX = \begin{bmatrix} 4 & 4 & 6 \\ 0 & -2/3 & 0 \\ 0 & -4/3 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & -3/2 \\ 0 & 3/2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{69}$$

Note

1. If an $n \times n$ matrix A has distinct eigenvalues, then it can be diagonalized.

Note

1. If an $n \times n$ matrix A has distinct eigenvalues, then it can be diagonalized.
2. Again if the matrix have n linearly independent eigenvectors, it can be diagonalized.

Note

1. If an $n \times n$ matrix A has distinct eigenvalues, then it can be diagonalized.
2. Again if the matrix have n linearly independent eigenvectors, it can be diagonalized.
3. If there are complex roots, the matrix cannot be diagonalized in $R^{n\times n}$.

Note

1. If an $n \times n$ matrix A has distinct eigenvalues, then it can be diagonalized.
2. Again if the matrix have n linearly independent eigenvectors, it can be diagonalized.
3. If there are complex roots, the matrix cannot be diagonalized in $R^{n\times n}$.

Example

The matrix

$$A = \begin{bmatrix} 6 & 12 & 19 \\ -9 & -20 & -33 \\ 4 & 9 & 15 \end{bmatrix}$$

cannot be diagonalized, because we found that there is only one linearly independent eigenvector associated with $\lambda = 1$.

If A is a real symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

If A is a real symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Example

Let A be the symmetric matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$, solving the eigenvalues are

If A is a real symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Example

Let A be the symmetric matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$, solving the eigenvalues are

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 6 \tag{70}$$

If A is a real symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Example

Let A be the symmetric matrix $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$, solving the eigenvalues are

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 6 \tag{70}$$

The corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \tag{71}$$

These three eigenvectors are mutually orthogonal, and are linearly independent.

The matrix of eigenvectors is given as

$$X = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \tag{72}$$

The matrix of eigenvectors is given as

$$X = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \tag{72}$$

Thus

$$X^{-1}AX = D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \tag{73}$$

D is the diagonal matrix with the eigenvalues of A on the diagonal.

Power of Matrices

If a matrix A can be diagonalized, then is simple to compute A^n. That is

1. $$\text{If } D = X^{-1}AX, \implies A = XDX^{-1} \tag{74}$$

Power of Matrices

If a matrix A can be diagonalized, then is simple to compute A^n. That is

1.
$$\text{If } D = X^{-1}AX, \implies A = XDX^{-1} \tag{74}$$

then

$$A^2 = (XDX^{-1})(XDX^{-1}) = (XD)I(DX^{-1}) = XD^2X^{-1} \tag{75}$$

Power of Matrices

If a matrix A can be diagonalized, then is simple to compute A^n. That is

1. $$\text{If } D = X^{-1}AX, \implies A = XDX^{-1} \tag{74}$$

   then

   $$A^2 = (XDX^{-1})(XDX^{-1}) = (XD)I(DX^{-1}) = XD^2X^{-1} \tag{75}$$

   again

   $$A^3 = A^2A = (XD^2X^{-1})(XDX^{-1}) = XD^3X^{-1} \tag{76}$$

Power of Matrices

If a matrix A can be diagonalized, then is simple to compute A^n. That is

1. $$\text{If } D = X^{-1}AX, \implies A = XDX^{-1} \tag{74}$$

   then

   $$A^2 = (XDX^{-1})(XDX^{-1}) = (XD)I(DX^{-1}) = XD^2X^{-1} \tag{75}$$

   again

   $$A^3 = A^2A = (XD^2X^{-1})(XDX^{-1}) = XD^3X^{-1} \tag{76}$$

   This can be generalized as

   $$A^n = XD^nX^{-1} \tag{77}$$

Exercises

1. Find the eigenvalues and associated eigenvectors for the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 7 \\ 1 & -4 \end{bmatrix}$. Verify that AB and BA have the same eigenvalues.

3. Diagonalize

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

4. Obtain an orthogonal matrix out of the following symmetric matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

END OF LECTURE

THANK YOU