CSM 166: Discrete Mathematics for Computer Science

BOOLEAN ALGEBRA AND BOOLEAN FUNCTIONS

Isaac Afari Addo <addoisaacafari@gmail.com>
National Institute for Mathematical Science (NIMS) - Ghana
Department of Mathematics, KNUST
Kumasi-Ghana.

29/07/2021

Content

Introduction

Definition 1 (Boolean Algebra)

A Boolean algebra is a set B with two binary operations $+$ and $\cdot$, elements 0 and 1, and a unary operation $\bar{\ }$ such that these properties hold for all $x, y, z \in B$:

$$(1) \quad \left.\begin{array}{l} x+0=x \\ x\cdot 1=x \end{array}\right\} \qquad \textit{Identity laws}$$

$$(2) \quad \left.\begin{array}{l} x+y=y+x \\ x\cdot y=y\cdot x \end{array}\right\} \qquad \textit{Commutative laws}$$

Introduction

(3) $$\left.\begin{aligned} x+(y\cdot z) &= (x+y)(x\cdot y) \\ x\cdot(y+z) &= (x\cdot y)+(x\cdot z) \end{aligned}\right\}$$ Distributive laws

(4) $$\left.\begin{aligned} x+\bar{x} &= 1 \\ x\cdot\bar{x} &= 0 \end{aligned}\right\}$$ Complement laws

(5) $$\left.\begin{aligned} (x+y)+z &= x+(y+z) \\ (x\cdot y)\cdot z &= x\cdot(y\cdot z) \end{aligned}\right\}$$ Associative laws

The element 0 is called the **zero** element, the element 1 is called the **unit** element, and $\bar{x}$ is called the **complement** of x.

Introduction

The **complement** of an element is defined by
$\bar{0} = 1$ and $\bar{1} = 0$

The results of the operations $+$ and $\cdot$ are called the Boolean **sum** and **product** respectively.

The elements 0 and 1 corresponds to False (F) and True(T)

The complement, Boolean sum, and Boolean product correspond to the logical operators, $\neg, \vee, \wedge$ respectively.

Note: Order of Precedence: $\bar{\ }, \cdot, +$

Example 1

(i) Find the value of $1 \cdot 0 + \overline{(0+1)}$.

Solution:

$$\begin{aligned} 1 \cdot 0 + \overline{(0+1)} &= 0 + \bar{1} \\ &0 + 0 \\ &= 0. \end{aligned}$$

(ii) Translate $1 \cdot 0 + (0+1) = 0$, into a logical equivalence.

Solution

$(T \wedge F) \vee \neg(T \vee F) \equiv F$

Boolean Expressions and Boolean Functions

Let $B = \{0, 1\}$. Then
$B^n = \{(x_1, x_2, \ldots, x_n) | x_i \in B \text{ for } 1 \leq i \leq n\}$ is the set of all possible n-tuples of 0s and 1s.

The variable x is called a **Boolean variable** if it assumes values only from B.

A function from B^n to B is called a **Boolean Function of degree** n

Boolean Expressions and Boolean Functions

Example 2

Find the values of the Boolean function represented by $F(x, y) = x\bar{y}$

Solution

$F(1,1) = 0$

$F(1,0) = 1$

$F(0,1) = 0$

$F(0,0) = 0$

x	y	F(x,y)
1	1	0
1	0	1
0	1	0
0	0	0

Example 3

Find the values of the Boolean function represented by $F(x, y, z) = xy + \bar{z}$.

Solution

x	y	z	xy	$\bar{z}$	$F(x, y, z) = xy + \bar{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Boolean Expressions and Boolean Functions

Boolean functions F and G of n variables are equal if and only if
$F(b_1, b_2, \ldots, b_n) = G(b_1, b_2, \ldots, b_n$ for all $b_1, b_2, \ldots b_n$

Two different Boolean expressions that represent the same function are called **equivalent**.

The **complement** of the Boolean function F is the function $\overline{F}$ where
$\bar{F}(F(x_1, x_2, \ldots, x_n \ldots b_n) = \overline{F(x_1, x_2, \ldots, x_n}$

Boolean Expressions and Boolean Functions

Let F and G be Boolean functions of degree n. The Boolean sum $F+G$ and the Boolean product FG are defined by

$$(F+G)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n) + G(x_1, x_2, \ldots, x_n)$$

$$(FG)(x_1, x_2, \ldots, x_n) = F(x_1, x_2, \ldots, x_n)\, G(x_1, x_2, \ldots, x_n)$$

If E_1 and E_2 are Boolean expressions, then E_1, $(E_1 E_2)$, and $(E_1 + E_2)$ are Boolean expressions.

Identities of Boolean Algebra

Identity	Name
$\bar{\bar{x}} = x$	Law of the double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 1 = x$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Domination laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws

Identities of Boolean Algebra

$x + yz = (x+y)(x+z)$ $x(y+z) = xy + xz$	Distributive laws
$\overline{(xy)} = \bar{x} + \bar{y}$ $\overline{(x+y)} = \bar{x}\bar{y}$	De Morgan's laws
$x + xy = x$ $x(x+y) = x$	Absorption laws
$x + \bar{x} = 1$	Unity
$x\bar{x} = 0$	Zero property

Example 4

Prove the absorption law $x(x+y) = x$ using the other identities of Boolean algebra

Solution

Proof.

$$
\begin{aligned}
x(x+y) &= (x+0)(x+y) && \textit{Identitylaw} \\
&= x + 0 \cdot y && \textit{Distributivelaw} \\
&= x + y \cdot 0 && \textit{Commutativelaw} \\
&= x + 0 && \textit{Dominationlaw} \\
&= x && \textit{Identitylaw}
\end{aligned}
$$

$\square$

Example 5

Show that the distributive law $x(y+z) = xy + xz$ is valid.

x	y	z	$y+x$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Duality

The dual of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0s and 1s.

Example 6

Find the duals of $x(y+0)$ and $\bar{x}\cdot 1+(\bar{y}+z)$.

Solution

The dual of $x(y+0)$ is $x+(y\cdot 1)$

The dual $\bar{x}\cdot 1+(\bar{y}+z)$ is $(\bar{x}+0)(\bar{x}z)$

Exercise A:

1. Find the values of these expressions
   a) $1 \cdot \bar{0}$ b) $1 + \bar{1}$ c) $\bar{0} \cdot 0$ d) $\overline{1+0}$

2. Find the values, if any, of the Boolean variable x that satisfy these equations.
   a) $x \cdot 1 = 0$ b) $x + x = 0$
   c) $x \cdot 1 = x$ d) $x \cdot \bar{x}$

3. Use a table to express the values of each of these Boolean functions
   a) $F(x, y, z) = \bar{z}$
   b) $F(x, y, z) = \bar{x}y + \bar{y}z$
   c) $F(x, y, z) = \overline{(xyz + x)}$
   d) $F(x, y, z) = \bar{y}(xz + \bar{x}\bar{z})$

Sum-of-Products Expansions

Definition 2

*A **literal** is a Boolean variable or its complement.*

*A **minterm** of the Boolean variables $x_1, x_2, \ldots, x_n$ is a Boolean product $y_1 y_2 \ldots y_n$, where $y_i = x_i$ or $y_i = \bar{x}_i$.*

*Hence, a **minterm** is a product of n literals, with one literal for each variable.*

***NB:** A minterm has the value 1 for one and only one combination of values of its variables.*

Sum-of-Products Expansion

- The sum of minterms that represents the function is called the **sum-of-products expansion** or the **disjunctive normal form** of the Boolean function.

Sum-of-Products Expansions

Find Boolean expressions that represent the functions $F(x, y, z)$ and $G(x, y, z)$, which are given in the table

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Solution:

An expression that has the value 1 when $x = z = 1$ and $y = 0$, and the value 0 otherwise, is needed to represent F.

Thus $F(x, y, z) = x\bar{y}z$

Similarly

$G(x, yz) = xy\bar{z} + \bar{x}y\bar{z}$

Example 7

Find the sum-of-products expansion for the function $F(x, y, z) = (x + y)z$.

Solution

$$\begin{aligned}
F(x, y, z) &= (x + y)\bar{z} \\
&= x\bar{z} + y\bar{z} && \text{Distributive law} \\
&= x1\bar{z} + 1y\bar{z} && \text{Identity law} \\
&= x(y + \bar{y})\bar{z} + (x + \bar{x})y\bar{z} && \text{Unity property} \\
&= xy\bar{z} + x\bar{y}\bar{z} + xy\bar{z} + \bar{x}y\bar{z} && \text{Distributive law} \\
&= xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z} && \text{Idempotent law}
\end{aligned}$$

Alternative Solution

x	y	z	x+y	$\bar{z}$	$(x+y)\bar{z}$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

$$F(x,y,z) = xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$$

Exercise B:

1. Find a Boolean product of the Boolean variables x, y, and z, or their complements, that has the value 1 if and only if

   (a) $x = y = 0,\ z = 1$ (b) $x = 0, y = 1 z = 0$

   (c) $x = 0, y = z = 1$ (d) $x = y = z = 0$

2. Find the sum-of-products expansions of these Boolean functions.

   (a) $F(x, y, z) = \bar{x} + y$ (b) $F(x, y) = x\bar{y}$

   (b) $F(x, y) = 1$ (d) $F(x, y) = \bar{y}$

3. Find the sum-of-products expansion of the Boolean function $F(w, x, y, z)$ that has the value 1 if and only if an odd number of w, x, y, and z have the value 1.

Functional Completeness

Every Boolean function can be expressed as a Boolean sum of minterms.

Each minterm is the Boolean product of Boolean variables or their complements.

Thus every Boolean function can be represented using $\cdot$, $+$, and $\bar{}$.

The set $\{\cdot, +, \bar{}\}$ is functionally complete.

Using De Morgan's laws, a smaller functionally complete set can be achieved by eliminating one of the three operators $\cdot$, $+$, and $\bar{}$.

Functional Completeness

All Boolean sums can be eliminated using the identity :

$$x + y = \overline{\bar{x}\bar{y}}$$

This means that the set $\{\cdot, \bar{}\ \}$ is **functionally complete**.

Similarly all Boolean products can be eliminated using the identity

$$xy = \overline{\bar{x} + \bar{y}}$$

Hence the set set $\{+, \bar{}\ \}$ is **functionally complete**

Functional Completeness

Note that the set $\{+, \cdot\}$ is not functionally complete, because it is impossible to express the Boolean function $F(x) = \bar{x}$ using these operators

Define two operators, the $|$ or **NAND** operator, defined by $1|1 = 0$ and $1|0 = 0|1 = 0|0 = 1$; and the $\downarrow$ or **NOR** operator, defined by
$1 \downarrow 1 = 1 \downarrow 0 = 0 \downarrow 1 = 0$
and $0 \downarrow 0 = 1$. Both of the sets $\{|\}$ and $\{\downarrow\}$ are functionally complete.

$\bar{x} = x|x$ and $xy = (x|y)|(x|y)$

Logic Gates

A computer, or any electronic device, is made up of circuits.

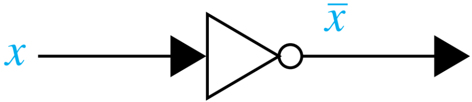
Each circuit is designed using the rules of **Boolean algebra**.

The basic elements of circuits are called **gates**, so that each type of gate implements a Boolean operation.

These circuits have no memory capabilities. Such circuits are called **combinational circuits** or **gating networks**.

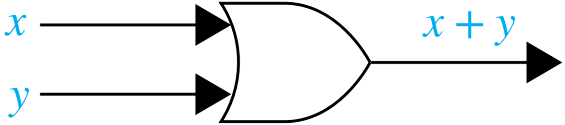
Elements of Combinatorial Circuts

An inverter: Accepts the value of a Boolean variable as an input and produces its complement as its output.



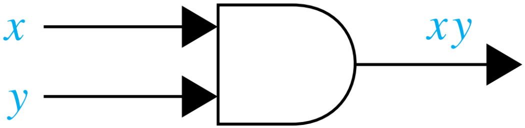
Elements of Combinatorial Circuts

OR gate: The inputs of this gate are two or more Boolean variables. The output is the Boolean sum of their values.

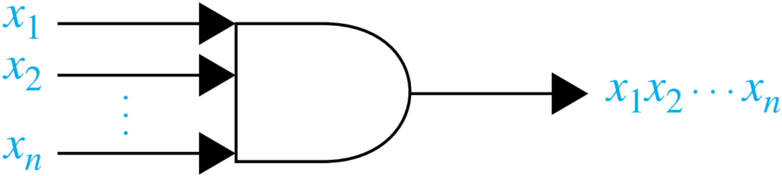


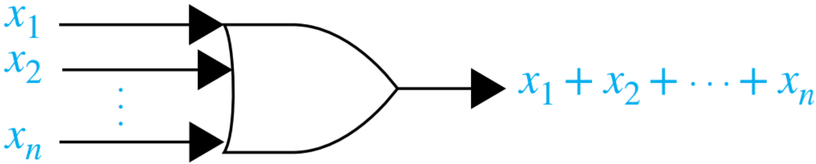
Elements of Combinatorial Circuts

AND gate: The inputs to this gate are of two or more Boolean variables. The output is the Boolean product of their values.



Gates with n inputs.





Combinations of Gates

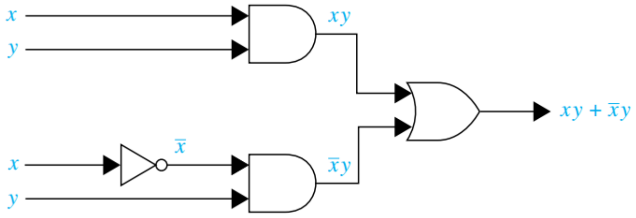
Combinational circuits can be constructed using a combination of **inverters**, **OR gates**, and **AND gates**.

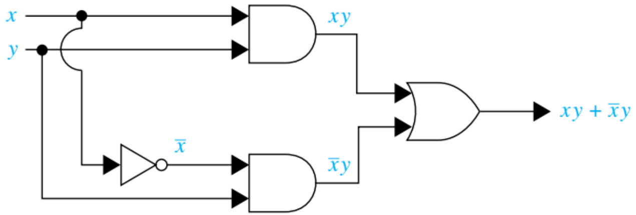
When combinations of circuits are formed, some gates may share inputs.

One method is to use branchings that indicate all the gates that use a given input.

The other method is to indicate this input separately for each gate.

Two methods of combing gates





Combinations of Gates

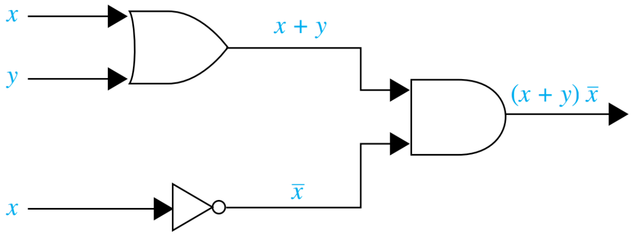
Example 8

Construct circuits that produce the following outputs:

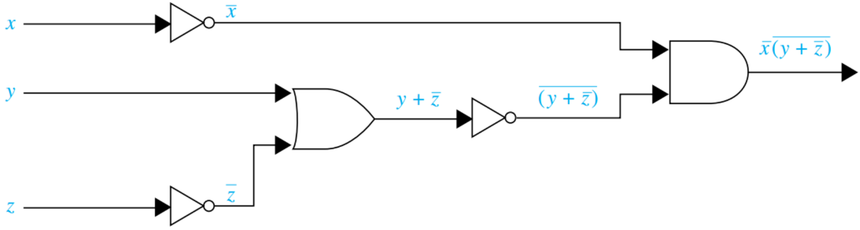
(a) $(x+y)\bar{x}$ (b) $\bar{x}\overline{(y+\bar{z})}$ (c) $(x+y+z)(\bar{x}\bar{y}\bar{z})$

Solution

(a)

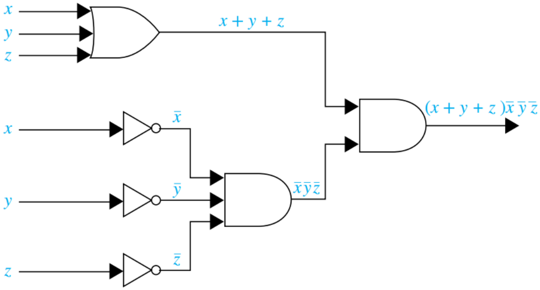


(b)



x
y
z
$\overline{x}$
$y + \overline{z}$
$\overline{z}$
$\overline{(y + \overline{z})}$
$\overline{x}\,\overline{(y + \overline{z})}$

(c)



x
y
z
$x + y + z$
x
y
z
$\overline{x}$
$\overline{y}$
$\overline{z}$
$\overline{x}\,\overline{y}\,\overline{z}$
$(x + y + z)\overline{x}\,\overline{y}\,\overline{z}$

Combinations of Gates

Example 9

A committee of three individuals decides issues for an organization. Each individual votes either yes or no for each proposal that arises. A proposal is passed if it receives at least two yes votes.

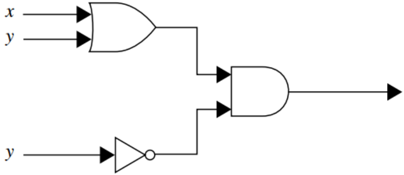
Combinations of Gates

Example 10

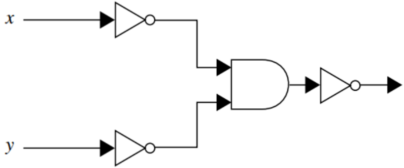
Sometimes light fixtures are controlled by more than one switch. Circuits need to be designed so that flipping any one of the switches for the fixture turns the light on when it is off and turns the light off when it is on.
Design circuits that accomplish this when there are two switches and when there are three switches.

Exercise C: Find the output of the given circuit

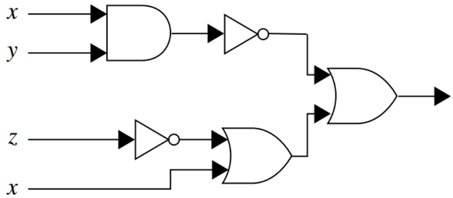
1.



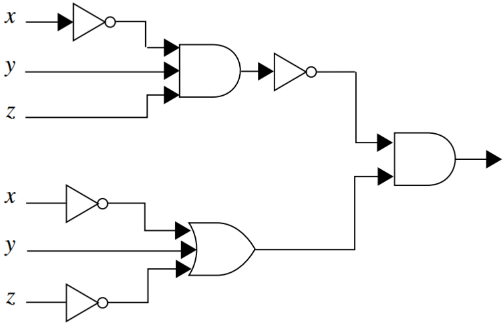
2.



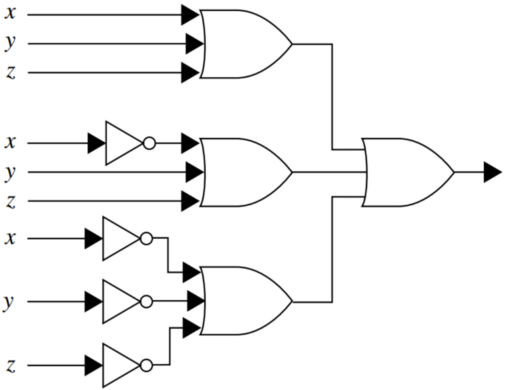
3.



4.



5.



Exercise D:

1. Construct circuits from inverters, **AND** gates, and **OR**gates to produce these outputs.

   (a) $\bar{x}+y$ (b) $\overline{(x+y)x}$

   (c) $xyz+\bar{x}\bar{y}\bar{z}$ (d) $\overline{(x+z)(y+\bar{z}}$

2. Design a circuit that implements majority voting for five individuals.

End of Lecture

Questions…???

Thanks

Reference Books

1. Kenneth H. Rosen, “Discrete Mathematics and Its Applications”, Tata Mcgraw Hill, New Delhi, India, seventh Edition, 2012.
2. H. Levy, F. Lessman Finite Difference Equations.Dover books on mathematices
3. Gary Chartrand. Ping Zhang. Discrete Mathematics 1^{th}
4. Oscar Leven. Discrete Mathematics: An open introduction. 2nd Edition. 2013