SYSTEM OF LINEAR EQUATION

Dr. Gabriel Obed Fosu
Department of Mathematics
Kwame Nkrumah University of Science and Technology

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Outline

- Introduction

1. Gaussian Elimination
2. Gauss-Jordan Elimination Method
3. Inverse Approach
   - Computing Inverse

Introduction to linear systems

- This lecture introduces the basics of solving linear equations using some elimination methods.
- A system of n linear equations in n unknowns $x_1, x_2, \cdots, x_n$ is a family of equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 && (1)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 && (2)\\
\vdots \quad &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n && (3)
\end{aligned}
$$

- We wish to determine if such a system has a solution, that is to find out if there exist numbers $x_1, x_2, \cdots, x_n$ that satisfy each of the equations simultaneously.

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- We wish to determine if such a system has a solution, that is to find out if there exist numbers $x_1, x_2, \cdots, x_n$ that satisfy each of the equations simultaneously.
- We say that the system is consistent if it has a solution. Otherwise, the system is called inconsistent.

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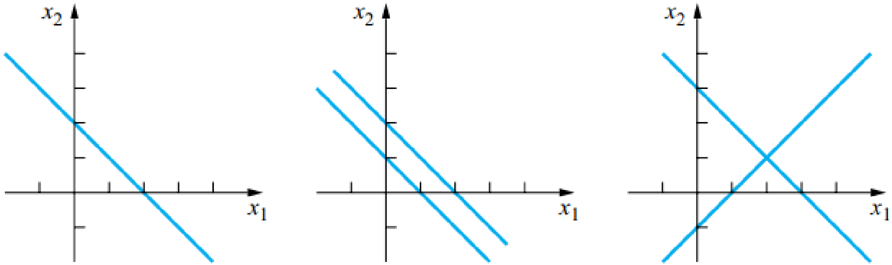
- We wish to determine if such a system has a solution, that is to find out if there exist numbers $x_1, x_2, \cdots, x_n$ that satisfy each of the equations simultaneously.
- We say that the system is consistent if it has a solution. Otherwise, the system is called inconsistent.
- More generally, the above is called a homogeneous system of linear equations when $b_1 = b_2 = \cdots = b_n = 0$

Geometrically, solving a system of linear equations in two (or three) unknowns is equivalent to determining whether or not a family of lines (or planes) has a common point of intersection.



Coincident lines
Infinitely many solutions

(a)

Parallel lines
No solution

(b)

Intersecting lines
Unique solution

(c)

(a) Consistent with dependent solutions (b) Inconsistent (c) Consistent with independent solution.

Example

Solve the system

$$2x + 3y = 6$$
$$x - y = 2$$

By the elimination or substitution method, the solution to the system is

$$x = 12/5, \quad y = 2/5 \tag{4}$$

This approach would be tedious and virtually unworkable for a large number of equations. We will develop a means of solving systems by using the matrix form of the equation.

Solving Square Linear Systems

1. Given the system

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \quad (5)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \quad (6)\\
\vdots \qquad\qquad &= \vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \quad (7)
\end{aligned}
$$

Solving Square Linear Systems

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$$\vdots \quad = \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \tag{7}$$

2. This can be recast in matrices as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \tag{8}$$

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3. Which is of the general from

$$Ax = b \tag{9}$$

The matrix A is appended by b to form what we call the **Augmented Matrix**. This is denoted by

$$A|b = \left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & \cdots a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots a_{nn} & b_n \end{array}\right] \tag{10}$$

- The matrix is really just a compact way of writing the system of equations.

Elementary row operations

Three main types of elementary row operations can be performed on matrices:

1. Interchanging two rows: $R_i \leftrightarrow R_j$ interchanges rows i and j (Partial Pivoting).

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Note

The above operations are implemented on the augmented matrix during elimination approach, and will not change the solution to the system.

Note that performing these operations on the matrix is equivalent to performing the same operations directly on the equations.

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Definition (Row equivalence)

Matrix A is row-equivalent to matrix B if B is obtained from A by a sequence of elementary row operations.

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Gaussian Elimination

Gaussian elimination performs row operations on the augmented matrix until the portion corresponding to the coefficient matrix is reduced to upper-triangular form.

Definition (Upper-Triangular Matrix)

An $n \times n$ matrix A whose entries are of the form

$$U_{ij} = \begin{cases} a_{ij} & i \leq j \\ 0 & i > j \end{cases}$$

is called an upper triangular matrix. Specifically:

$$U = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

The method starts with

$$A|b = \left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & \cdots a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots a_{nn} & b_n \end{array}\right] \tag{11}$$

then using row elimination produces a matrix in upper triangular form

$$\left[\begin{array}{cccc|c} c_{11} & c_{12} & c_{13} & \cdots c_{1n} & d_1 \\ 0 & c_{22} & c_{23} & \cdots c_{2n} & d_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots c_{nn} & d_n \end{array}\right] \tag{12}$$

which is easy to solve using back substitution.

Gaussian Elimination Procedure

1. Begin at element a_{11}. If $a_{11} = 0$, exchange rows so $a_{11} \neq 0$.

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4. Repeat this process until the matrix is in upper-triangular form.
5. Then execute back substitution to compute the solution.

Example

Solve the following system of equations using the Gaussian elimination method

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 3x_2 - x_3 &= 6 \\ 3x_1 + 5x_2 + 3x_3 &= 4 \end{aligned}$$

Solution

This problem is first recast into matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \tag{13}$$

The augmented matrix is deduced from (13) as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right] \tag{14}$$

Now, let reduce (14) to an upper triangular matrix using elementary row operations.

Iteration 1

The first pivot point is 1 in the first column (first term).

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right] \tag{15}$$

We are to reduce the values beneath 1, that is 4 and 3 to zeros using elementary row operations. The following manipulations are used here.

$NR_2 = 4R_1 - R_2, \qquad \Longrightarrow\ 4 \to 0$

$NR_3 = 3R_1 - R_3, \qquad \Longrightarrow\ 3 \to 0$

Note that these computations affect the entire row.

Note

NR is used to denote New Row, such that NR_2 is read 'new row 2'.
R is used to denote Row, such that R_1 is read 'row 1'.

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Therefore the new matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & -2 & 0 & -1 \end{array}\right] \tag{16}$$

Iteration 2

The second pivot point is 1 in the second column (diagonal value).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 5 & -2 \\ 0 & -2 & 0 & -1 \end{array}\right] \tag{17}$$

We are to reduce the value beneath 1, that is -2 to zero using elementary row operations. The following manipulations are used here.

$NR_3 = 2R_2 + R_3, \qquad \Longrightarrow \ -2 \to 0$

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Therefore the new matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 10 & -5 \end{array}\right] \quad (18)$$

Now we have an upper triangular matrix. So the solution could be finally obtained using back substitution. That is substitution and solving from the last row. We have

$$10x_3 = -5 \implies x_3 = -0.5$$

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$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + 0.5 - 0.5 &= 1 \\ x_1 &= 1 \end{aligned}$$

These are the solutions to the given problem.

Dimension

Suppose a system of n linear equations in n unknowns has augmented matrix C and that C is row-equivalent to a matrix D in upper-triangular form. Then C and D have dimension $n \times (n+1)$.

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Inconsistent System

If we perform elementary row operations on the augmented matrix of the system and get a matrix with one of its rows equal to

$$[0\ 0\ 0\cdots 0\ b],\ \text{where } b \neq 0 \tag{19}$$

or a row of the form

$$[0\ 0\ 0\cdots 0] \tag{20}$$

then there may be no solution or infinitely many solutions.

Homogeneous Systems

An $n \times n$ system of homogeneous linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\vdots \quad &= \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0
\end{aligned}
$$

is always consistent since $x_1 = 0, \cdots, x_n = 0$ is a solution. This solution is called the trivial solution, and any other solution is called a nontrivial solution.

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If the homogeneous system $Ax = 0$ has only the trivial solution, then A is nonsingular; that is A^{-1} exists.

Outline of Presentation

Gauss-Jordan Elimination Method

1. This method is based on the idea of reducing the given system of equation

$$Ax = b$$

   to a **diagonal matrix** of the form

$$Dx = z$$

   D is the diagonal matrix, and z is the new column vector on the right hand side.

2. All solution techniques of the Gaussian elimination method do apply here. While a given problem is reduced to an upper triangular matrix in the case of Gaussian elimination, it is reduced to a diagonal matrix in the case of Gauss-Jordan elimination.

Example

Solve the following system of equations using the Gauss-Jordan elimination method

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 3x_2 - x_3 &= 6 \\ 3x_1 + 5x_2 + 3x_3 &= 4 \end{aligned}$$

This problem is first recast into matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \tag{21}$$

The augmented matrix is deduced from (41) as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right] \tag{22}$$

Note that we are to reduce value above and beneath the leading diagonals to zeros. Now, let reduce (22) to an upper triangular matrix using elementary row operations.

Iteration 1

The first pivot point is 1 in the first column (first term).

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 1 \\ 4 & 3 & -1 & 6 \\ 3 & 5 & 3 & 4 \end{array}\right] \tag{23}$$

We are to reduce the values beneath 1, that is 4 and 3 to zeros using elementary row operations. The following manipulations are used here.

$NR_2 = 4R_1 - R_2, \quad \implies 4 \to 0$

$NR_3 = 3R_1 - R_3, \quad \implies 3 \to 0$

Therefore the new matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & -2 & 0 & -1 \end{array}\right] \tag{24}$$

Iteration 2

The second pivot point is 1 in the second column (diagonal value).

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & \boxed{1} & 5 & -2 \\ 0 & -2 & 0 & -1 \end{array}\right] \tag{25}$$

We are to reduce the value beneath 1, that is -2 to zero using elementary row operations. The following manipulations are used here.
$NR_3 = 2R_2 + R_3, \qquad \Longrightarrow \ -2 \to 0$
Therefore the new matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & 10 & -5 \end{array}\right] \tag{26}$$

Iteration 3

Since we need a diagonal matrix, we will start manipulating the other non-diagonal matrix to zero starting from the leading diagonal of the last column.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 5 & -2 \\ 0 & 0 & \boxed{10} & -5 \end{array}\right] \tag{27}$$

We are to reduce the values above 10, that is 5 and 1 to zeros elementary row operations. The following manipulations are used here.

$NR_2 = R_3 - 2R_2, \qquad \Longrightarrow \; 5 \rightarrow 0$

$NR_1 = R_3 - 10R_1, \qquad \Longrightarrow \; 1 \rightarrow 0$

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$NR_2 = R_3 - 2R_2, \qquad \implies 5 \to 0$

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The new matrix is

$$\left[\begin{array}{ccc|c} -10 & -10 & 0 & -15 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 10 & -5 \end{array}\right] \tag{28}$$

Iteration 4

The next pivot point is -2 in the second column (diagonal value).

$$\left[\begin{array}{ccc|c} -10 & -10 & 0 & -15 \\ 0 & \boxed{-2} & 0 & -1 \\ 0 & 0 & 10 & -5 \end{array}\right] \tag{29}$$

We are to reduce the value above -2, that is -10 to zero using elementary row operations. The following manipulations are used here.
$NR_1 = -5R_2 + R_1, \qquad \implies -10 \to 0$

Iteration 4

The next pivot point is -2 in the second column (diagonal value).

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We are to reduce the value above -2, that is -10 to zero using elementary row operations. The following manipulations are used here.
$NR_1 = -5R_2 + R_1, \qquad \Longrightarrow \; -10 \to 0$
The new matrix

$$\left[\begin{array}{ccc|c} -10 & 0 & 0 & -10 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 10 & -5 \end{array}\right] \tag{30}$$

Since we have reduced the system to a diagonal matrix, the values of x can be obtained using direct substitution. That is

$$-10x_1 = -10 \implies x_1 = 1$$
$$-2x_2 = -1 \implies x_2 = 0.5$$
$$10x_3 = -5 \implies x_3 = -0.5$$

Outline of Presentation

Computing Inverse

Begin by setting an augmented matrix of the form $A|I$. For 3×3 matrix we have

$$A|I = \left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right] \tag{31}$$

Then perform ERO on the coefficient matrix to obtain a diagonal matrix. All the while performing the row operations on the augmented matrix $A|I$. When the Gauss-Jordan procedure is completed, we obtain

$$[A|I] \longrightarrow [I|A^{-1}] \tag{32}$$

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Note

Partial pivoting can also be done using the augmented matrix $[A|I]$. However, we cannot first interchange the rows of A and then find the inverse. Then, we would be finding the inverse of a different matrix.

Example

Find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$

We perform the following elementary row transformations and do the eliminations

Iteration 1

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \longrightarrow \\ NR_2 = R_2 - 4R_1 \\ NR_3 = R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \tag{33}$$

Iteration 2

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \xrightarrow{NR_2 = -R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \tag{34}$$

Iteration 2

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -5 & -4 & 1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \underset{NR_2 = -R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \tag{34}$$

Iteration 3

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 2 & 0 & -3 & 0 & 1 \end{array}\right] \underset{\substack{NR_1 = R_1 - R_2 \\ NR_3 = R_3 - 2R_2}}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1-3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \tag{35}$$

Iteration 4

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1-3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & -10 & -11 & 2 & 1 \end{array}\right] \overrightarrow{NR_3 = R_3/(-10)}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 1-3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & 11/10 & -2/10 & -1/10 \end{array}\right]$$

Iteration 5

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & 11/10 & -2/10 & -1/10 \end{array}\right] \begin{array}{c} \longrightarrow \\ NR_1 = R_1 + 4R_3 \\ NR_2 = R_2 - 5R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14/10 & 2/10 & -4/10 \\ 0 & 1 & 0 & -15/10 & 0 & 5/10 \\ 0 & 0 & 1 & 11/10 & -2/10 & -1/10 \end{array}\right]$$

Iteration 5

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -3 & 1 & 0 \\ 0 & 1 & 5 & 4 & -1 & 0 \\ 0 & 0 & 1 & 11/10 & -2/10 & -1/10 \end{array}\right] \begin{array}{c} \longrightarrow \\ NR_1 = R_1 + 4R_3 \\ NR_2 = R_2 - 5R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 14/10 & 2/10 & -4/10 \\ 0 & 1 & 0 & -15/10 & 0 & 5/10 \\ 0 & 0 & 1 & 11/10 & -2/10 & -1/10 \end{array}\right]$$

Therefore, the inverse of the given matrix is given by

$$\begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix}$$

Solve the same problem by employing partial pivoting.

Solving A System of Linear Equation Equation using the Inverse Approach

1. Given the system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \quad (36)$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \quad (37)$$
$$\vdots \quad = \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \quad (38)$$

2. Which is of the general from

$$Ax = b \quad (39)$$

3. Pre-multiply both sides by A^{-1}

$$x = A^{-1}b \quad (40)$$

Note that $AA^{-1} = I$

Example

Solve the following system of equations using the inverse approach

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 4x_1 + 3x_2 - x_3 &= 6 \\ 3x_1 + 5x_2 + 3x_3 &= 4 \end{aligned}$$

This problem is first recast into matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \tag{41}$$

Then we are to find

$$x = A^{-1}b \tag{42}$$

Using A^{-1} as obtained from above we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/2 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} \tag{43}$$

$$x_1 = \frac{7}{5} + \frac{8}{5} - \frac{10}{5} = 1 \tag{44}$$

$$x_2 = -\frac{3}{2} + 0 + \frac{4}{2} = \frac{1}{2} \tag{45}$$

$$x_3 = \frac{11}{10} - \frac{6}{5} - \frac{4}{10} = -\frac{1}{2} \tag{46}$$

Exercise 1

1. Solve the homogeneous system

$$\begin{aligned} -3x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - 3x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - 3x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 - 3x_4 &= 0 \end{aligned}$$

2. For which rational numbers λ does the homogeneous system have a nontrivial solution?

$$\begin{aligned} x + (\lambda - 3)y &= 0 \\ (\lambda - 3)x + y &= 0 \end{aligned}$$

Exercise 2

Solving the following system of equations using

1. Gaussian elimination method without pivoting
2. Gaussian elimination method with partial pivoting
3. Gaussian elimination method with complete pivoting
4. Gauss-Jordan elimination method.
5. Inverse Approach

$$\begin{aligned} 10x + 4y - 2z &= 20 \\ 3x + 12y - z &= 28 \\ x + 4y + 7z &= 2 \end{aligned}$$

$$\begin{aligned} 2a + b + c + d &= 2 \\ 4a + 2c + d &= 3 \\ 3a + 2b + 2c &= -1 \\ a + 3b + 2c + 6d &= 2 \end{aligned}$$

END OF LECTURE

THANK YOU