POLYNOMIAL APPROXIMATION AND INTERPOLATION I

(APPROXIMATION WITH UNEVENLY SPACED POINTS)

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Lecture Outline

Outline of Presentation

Introduction

We now discuss the problem of approximating a given function by polynomials.

Why approximating polynomials

1. To reconstruct the function $f(x)$ when it is not given explicitly and only values of $f(x)$ and/or its certain order derivatives are given at a set of distinct points called nodes or tabular points.

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The deviation of $P(x)$ from $f(x)$, that is $f(x) - P(x)$, is called the error of approximation.

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The polynomial $P_n(x)$ is called the interpolating polynomial. The conditions given in eq. (3) are called the interpolating conditions.

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3. That is, through three distinct points, we can construct a unique polynomial of degree ≤ 2.
4. In general, through $n+1$ distinct points, we can construct a unique polynomial of degree $\leq n$.

Outline of Presentation

1. Introduction
2. Lagrange Interpolation
3. Divided Difference Interpolation
   - Difference Method
   - Newton's Divided Difference Interpolation
4. Inverse Interpolation

Lagrange Interpolation

x	x_0	x_1	x_2	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\cdots$	$f(x_n)$

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where $l_i(x);\ i = 0, 1, 2, \cdots, n$ are polynomials of degree n defined as

$$l_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)} \tag{6}$$

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Equation (5) is called the Lagrange interpolating polynomial and eq. (6) are called the Lagrange fundamental polynomials.

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Equation (5) is called the Lagrange interpolating polynomial and eq. (6) are called the Lagrange fundamental polynomials. Note that the denominator is obtained by setting $x = x_i$ in the numerator.

Linear interpolation

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$$P_1(x) = l_0(x) f_0 + l_1(x) f_1 \tag{8}$$

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Two Data Values

Example

Using the data $\sin(0.1) = 0.09983$ and $\sin(0.2) = 0.19867$, find an approximate value of $\sin(0.15)$ by Lagrange interpolation.

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$$P_1(0.15) = (0.5)(0.09983) + (0.5)(0.19867) = 0.14925. \tag{14}$$

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Example

Given that $f(0) = 1,\ f(1) = 3,\ f(3) = 55$, find the unique polynomial of degree 2 or less, which fits the given data.

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$$P_2(x) = l_0(x)f_0 + l_1(x)f_1 + l_2(x)f_2 \tag{18}$$

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Example

Construct the Lagrange interpolation polynomial for the data below and hence, interpolate at $x = 5$.

x	-1	1	4	7
$f(x)$	-2	0	63	342

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$$l_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{(x+1)(x-1)(x-4)}{(7+1)(7-1)(7-4)} = \frac{1}{144}(x^3-4x^2-x+4) \tag{24}$$

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Hence,

$$
f(5) = P_3(5) = 53 - 1 = 124 \qquad (29)
$$

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Remarks

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Note

The Lagrange interpolating polynomial does not have the permanence property.

Outline of Presentation

Divided differences

Let the data, $(x_i, f(x_i)),\ i = 0, 1, 2, \cdots, n$, be given. We define the divided differences as follows.

First divided difference

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

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$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

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Example

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}, \ f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1},$$

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Example

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0},\ f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1},\ \text{ and } f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} \tag{31}$$

Second divided difference

Consider any three consecutive data values $(x_i, f(x_i)),\ (x_{i+1}, f(x_{i+1})),\ (x_{i+2}, f(x_{i+2})),$

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Example

A 2_{nd} divided difference can be defined as:

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A 2_{nd} divided difference can be defined as:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \quad \text{and} \quad f[x_6, x_7, x_8] = \frac{f[x_7, x_8] - f[x_6, x_7]}{x_8 - x_6} \tag{34}$$

n_{th} Divided Difference

The n_{th} divided difference using all the data values in the table, is defined as

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, x_2, \cdots, x_n] - f[x_0, x_1, \cdots, x_{n-1}]}{x_n - x_0} \tag{35}$$

Table of Divided differences

x	$f(x)$	First	Second	Third
x_0	f_0			
		$f[x_0, x_1]$		
x_1	f_1		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	f_2		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	f_3			

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x	$f(x)$	First	Second	Third
x_0	f_0			
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x_1	f_1		$f[x_0,x_1,x_2]$	
		$f[x_1,x_2]$		$f[x_0,x_1,x_2,x_3]$
x_2	f_2		$f[x_1,x_2,x_3]$	
		$f[x_2,x_3]$		
x_3	f_3			

Example

Obtain the divided difference table for the data

x	-1	0	2	3
$f(x)$	-8	3	1	12

x	$f(x)$	First	Second	Third
-1	-8			
		$\frac{3+8}{0+1} = 11$		
0	3		$\frac{-1-11}{2+1} = -4$	
		$\frac{1-3}{2-0} = -1$		$\frac{4+4}{3+1} = 2$
2	1		$\frac{11+1}{3-0} = 4$	
		$\frac{12-1}{3-2} = 11$		
3	12			

Newton's Divided Difference Interpolation

The Newton's divided difference interpolating polynomial is defined as

$$\begin{aligned} f(x) =& P_n(x) \\ =& f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &+ \cdots + (x - x_0)(x - x_1), \cdots, (x - x_{n-1}) f[x_0, x_1, \cdots, x_n] \end{aligned} \tag{36}$$

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Note

Newton's divided difference interpolating polynomial possesses the permanence property.

Example

Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula

x	-4	-1	0	2	5
$f(x)$	1245	33	5	9	1335

x	$f(x)$	First	Second	Third	Fourth
-4	1245				
		$\frac{33-1245}{-1+4}=-404$			
-1	33		$\frac{-28+404}{4}=94$		
		$\frac{5-33}{0+1}=-28$		$\frac{10-94}{2+4}=-14$	
0	5		$\frac{2+28}{2+1}=10$		$\frac{13+14}{5+4}=3$
		$\frac{9-5}{2-0}=2$		$\frac{88-10}{5+1}=13$	
2	9		$\frac{442-2}{5-0}=88$		
		$\frac{1335-9}{5-2}=442$			
5	1335				

$$f(x) = f(x_0)$$

$$f(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$

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(x - x_0)(x - x_1)(x - x_2)(x - x_3) f[x_0, x_1, x_2, x_3, x_4]
\end{aligned}
\tag{37}
$$

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$$f(x) = 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + (x + 4)(x + 1)x(-14) + (x + 4)(x + 1)x(x - 2)(3) \tag{38}$$

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$$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5 \quad (40)$$

Example

Find $f(x)$ as a polynomial in x for the following data by Newton's divided difference formula

x	1	3	4	5	7	10
$f(x)$	3	31	69	131	351	1011

Hence,

1. Interpolate at $x = 3.5$ and $x = 8.0$
2. Find, $f'(3)$ and $f''(1.5)$

x	$f(x)$	First	Second	Third	Fourth
1	3				
		14			
3	31		8		
		38		1	
4	69		12		0
		62		1	
5	131		16		0
		110		1	
7	351		22		
		220			
10	1011				

Since, the fourth order differences are zeros, the data represents a third degree polynomial.

Newton's divided difference formula gives the polynomial as

$$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] \tag{41}$$

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$$\begin{aligned} f(x) &= 3 + (x-1)(14) + (x-1)(x-3)(8) + (x-1)(x-3)(x-4)(1) && (42)\\ &= 3 + 14x - 14 + 8x^2 - 32x + 24 + x^3 - 8x^2 + 19x - 12 && (43)\\ &= x^3 + x + 1 && (44) \end{aligned}$$

$$f(3.5) \approx P_3(3.5) = (3.5)^3 + 3.5 + 1 = 47.375 \tag{45}$$

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$$f'(3) \approx P'(3) = 3(9) + 1 = 28 \tag{49}$$

$$f''(1.5) \approx P''(1.5) = 6(1.5) = 9 \tag{50}$$

Outline of Presentation

Inverse Interpolation

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4. For other problem, we consider the given data as $(f(x_i), x_i)$; $i = 0, 1, 2, \cdots, n$ and construct the interpolation polynomial.

5. That is, we consider $f(x)$ as the independent variable and x as the dependent variable. This procedure is called inverse interpolation

Exercise

1. Using Lagrange interpolation, find the unique polynomial $P(x)$ of degree 2 or less such that

$$P(1) = 1, \qquad P(3) = 27, \qquad P(4) = 64$$

2. A third degree polynomial passes through the points $(0,-1), (1,1), (2,1)$, and $(3,2)$. Determine this polynomial using Lagrange's interpolation. Hence, find the value at 1.5.

3. Using Lagrange interpolation, find y(10) given that

$$y(5) = 12,\ y(6) = 13,\ y(9) = 14,\ y(11) = 16.$$

4. Using Newton's divided difference method, find $f(1.5)$ using the data $f(1.0) = 0.7651977$, $f(1.3) = 0.6200860$, $f(1.6) = 0.4554022$, $f(1.9) = 0.2818186$, and $f(2.2) = 0.1103623$.

END OF LECTURE

THANK YOU