NUMERICAL SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS: SINGLE-STEP METHODS

Dr. Gabriel Obed Fosu

Department of Mathematics
Kwame Nkrumah University of Science and Technology
Google Scholar: https://scholar.google.com/citations?user=ZJfCMyQAAAAJ&hl=en&oi=ao
ResearchGate ID: https://www.researchgate.net/profile/Gabriel_Fosu2

Lecture Outline

1. Introduction to ODE
2. Single-Step Methods and Multi-Step Schemes
3. Solution To IVP's Using Single-Step Methods

Outline of Presentation

Introduction

1. The general form of an nth order ODE is

$$f(x,\ y',\ y'',\ y''',\ \cdots, y^{(n)}) = 0 \tag{1}$$

2. The order of an ODE is the order of its highest derivative.
3. A differential equation together with some initial conditions is called an Initial Value Problem(IVP).
4. A first order IVP can be written as:

$$y' = f(x, y); \qquad y(x_0) = y_0 \tag{2}$$

5. The methods for solving this IVP's can be classified into single-step methods and multi-step methods.

Recall

1. Given an interval $[x_0,\ b]$ in which a solution is desired.
2. The interval is divided into finite number of sub-intervals by points

$$x_0 < x_1 < x_2 < \cdots < x_n; \qquad x_n = b \tag{3}$$

3. These points are called mesh points or grid points.
4. The spacing between the points are given by

$$h_i = x_i - x_{i-1}, \qquad i = 1, 2, 3, \cdots, n \tag{4}$$

Outline of Presentation

Single-Step Scheme

Definition

For single step method's the solution at any point is obtained by using the solution at the previous point.

Single-step methods can be classified as

- Implicit single-step schemes
- Explicit single-step schemes

Implicit Single-Step Schemes

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A general single step implicit method can be written as

$$y_{i+1} = y_i + hf(x_{i+1},\ x_i,\ y_{i+1},\ y_i,\ h) \tag{5}$$

2. The function f is called the increment function.

Explicit Single-Step Scheme

1. In the explicit case, the right-hand side does not depend on y_{i+1}. For the explicit scheme eq. (5) reduces to:

$$y_{i+1} = y_i + hf(x_i,\ y_i,\ h) \tag{6}$$

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2. Given an initial value y_0, then the other values of y are computed successively as

$$
\begin{aligned}
y_1 &= y_0 + hf(x_0,\ y_0,\ h), && \text{when } i = 0 && (7)\\
y_2 &= y_1 + hf(x_1,\ y_1,\ h), && \text{when } i = 1 && (8)\\
y_3 &= y_2 + hf(x_2,\ y_2,\ h), && \text{when } i = 2 && (9)\\
\vdots & \qquad\quad \vdots && \qquad \vdots \\
y_n &= y_{n-1} + hf(x_{n-1},\ y_{n-1},\ h), && \text{when } i = n-1 && (10)
\end{aligned}
$$

Single-Step Scheme

1. The solution of y_1 requires only one previous point y_0.
2. The solution of y_2 requires only one previous point y_1.
3. The solution of y_3 requires only one previous point y_2.
4. The solution of y_n requires only one previous point y_{n-1}.

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Note

All single-step methods are self starting, that is, they do not require values of y or it's derivatives beyond the immediate previous point.

Multi-Step Methods

With this method, the solution at point y_{i+1} is obtained using the solution at a number of previous points, $y_i,\ y_{i-1},\ y_{i-2},\ y_{i-3}, \cdots$.

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Multi-Step Schemes

1. A classical example of a two-step implicit method can be written as

$$y_{i+1} = y_i + hf(x_{i+1},\ x_i,\ x_{i-1},\ y_{i+1},\ y_i,\ y_{i-1},\ h) \tag{11}$$

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2. A classical example of a three-step explicit method can be written as

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3. A general k-step explicit method can be written as

$$y_{i+1} = y_i + hf(x_{i-k+1},\ \cdots,\ x_{i-1},\ x_i,\ y_{i-k+1},\ \cdots,\ y_{i-1},\ y_i,\ h) \tag{13}$$

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Note

The following are equivalent

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$$f_i = f[x_i,\ y(x_i)], \qquad f_{i+1} = f[x_{i+1},\ y(x_{i+1})], \quad \cdots$$

Outline of Presentation

Solution To IVP's Using Single-Step Methods

Some numerical techniques used for solving IVP's include:

1. Euler or Taylor series Method
2. Backward Euler
3. Modified Euler or Midpoint Method
4. Trapezium Method
5. Heun's Method or Euler-Cauchy Method
6. Runge-Kutta (RK) Methods

Solution To IVP's Using Single-Step Methods

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6. Runge-Kutta (RK) Methods

All these methods are derived using Taylor series. Given the Taylor series:

$$y(x_{i+1}) = y(x_i) + hf[(x_i + \theta h), y(x_i + \theta h)]; \qquad 0 \leq \theta \leq 1 \tag{15}$$

These schemes can be deduced from the above depending on the value of θ.

Taylor Series of Order 1 or Euler Method

1. Given the general Taylor series:

$$y(x_{i+1}) = y(x_i) + hf[(x_i+\theta h), y(x_i+\theta h)]; \qquad 0 \leq \theta \leq 1 \tag{16}$$

2. The Euler Method is obtained from eq. (16) be letting

$$\theta = 0$$

3. The scheme is given by the formula

$$y(x_{i+1}) = y(x_i) + hf[x_i,\ y(x_i)] \tag{17}$$

4. This is an explicit scheme.

2. Backward Euler

1. Given the general Taylor series:

$$y(x_{i+1}) = y(x_i) + hf[(x_i + \theta h), y(x_i + \theta h)]; \qquad 0 \le \theta \le 1 \tag{18}$$

2. The Backward Euler is obtained from eq. (18) be letting

$$\theta = 1$$

.

3. The scheme is given by the formula

$$y(x_{i+1}) = y(x_i) + hf[(x_i + h),\ y(x_i + h)] \tag{19}$$
$$= y(x_i) + hf[x_{i+1},\ y(x_{i+1})] \tag{20}$$

4. This is an implicit scheme.

3. Modified Euler or Midpoint Method

1. Given the general Taylor series:

$$y(x_{i+1}) = y(x_i) + hf[(x_i + \theta h), y(x_i + \theta h)]; \qquad 0 \leq \theta \leq 1 \tag{21}$$

2. The Modified Euler method is obtained from eq. (21) be letting

$$\theta = \frac{1}{2}$$

3. The scheme is given by the formula

$$y(x_{i+1}) = y(x_i) + hf\left[\left(x_i + \frac{h}{2}\right),\ y\left(x_i + \frac{h}{2}\right)\right] \tag{22}$$

4. However, $x_i + \frac{h}{2}$ is not a nodal point, hence we approximate $y\left(x_i + \frac{h}{2}\right)$ using the Euler method with spacing $\frac{h}{2}$.

Modified Euler or Midpoint Method

1. The Euler approximation is given by eq. (23).

$$y\left(x_i + \frac{h}{2}\right) = y_i + \frac{h}{2} f(x_i,\ y_i) \tag{23}$$

2. Substituting eq. (23) into eq. (22), we obtain the Modified Euler as

$$y(x_{i+1}) = y(x_i) + hf\left[\left(x_i + \frac{h}{2}\right),\ y_i + \frac{h}{2} f(x_i,\ y_i)\right] \tag{24}$$

3. This is an explicit scheme

4. Trapezium Method

1. If the continuously varied slope in x_i and x_{i+1} is approximated by the mean of the slope, then the trapezium method is deduced as

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}\{f[x_i,\ y(x_i)] + f[x_{i+1},\ y(x_{i+1})]\} \tag{25}$$

$$= y_i + \frac{h}{2}[f_i + f_{i+1}] \tag{26}$$

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2. This is an implicit scheme.
3. When this is converted to an explicit scheme we obtain the Heun's method.

5. Heun's Method or Euler-Cauchy Method

1. This is the explicit form of the trapezium method. This conversion is made possible by using the approximation

$$y(x_{i+1}) = y(x_i) + hf[x_i,\ y(x_i)] \tag{27}$$

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2. Substituting eq. (27) into eq. (25) we obtain the Euler-Cauchy iterative scheme as:

$$\begin{aligned} y(x_{i+1}) &= y(x_i) + \frac{h}{2}\left\{f[x_i,\ y(x_i)] + f[x_{i+1},\ y(x_{i+1})]\right\} && (28)\\ &= y(x_i) + \frac{h}{2}\left\{f[x_i,\ y(x_i)] + f[x_{i+1},\ y(x_i) + hf[x_i,\ y(x_i)]]\right\} && (29) \end{aligned}$$

Example

Solve the following IVP

$$yy' = x, \quad y(0) = 1, \quad 0 \le x \le 0.6, \quad h = 0.2$$

using

1. Euler method
2. Modified Euler method
3. Euler-Cauchy method

In each case compute the absolute error at $x = 0.6$

Euler Method

Given the step size $h = 0.2$. Then the x values are given by the interval table

$$x_0 = 0, \quad x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6 \tag{30}$$

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The Euler formula is given as

$$y(x_{i+1}) = y(x_i) + hf[x_i,\ y(x_i)] \tag{31}$$

$$y_{i+1} = y_i + hf(x_i, y_i) \tag{32}$$

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From eq. (2), $y' = f(x, y)$. Therefore making y' the subject from the question we have

$$y' = f(x, y) = \frac{x}{y} \tag{33}$$

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Iteratively,

$$f(x_i, y_i) = \frac{x_i}{y_i} \tag{34}$$

Euler Method: Iteration 1: when i=0

The formula reduces to

$$y(x_1) = y(x_0) + hf[x_0,\ y(x_0)] \tag{35}$$

$$y_1 = y_0 + hf(x_0, y_0) \tag{36}$$

Euler Method: Iteration 1: when i=0

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$$y_1 = y_0 + hf(x_0, y_0) \tag{36}$$

From the question, that is $y(0) = 1$, the initial conditions can be deduced as

$$x_0 = 0, \quad \text{and } y_0 = 1$$

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Therefore

$$f(x_0, y_0) = \frac{x_0}{y_0} = \frac{0}{1} = 0$$

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Therefore

$$f(x_0, y_0) = \frac{x_0}{y_0} = \frac{0}{1} = 0$$

Hence

$$y_1 = y_0 + hf(x_0, y_0) \tag{37}$$

$$= 1 + 0.2(0) \tag{38}$$

$$= 1 \tag{39}$$

Euler Method: Iteration 2: when i=1

The formula reduces to

$$y_2 = y_1 + hf(x_1, y_1)$$

Euler Method: Iteration 2: when i=1

The formula reduces to

$$y_2 = y_1 + hf(x_1, y_1)$$

From the interval table (30), $x_1 = 0.2$, and from the previous solution $y_1 = 1$

Euler Method: Iteration 2: when i=1

The formula reduces to

$$y_2 = y_1 + hf(x_1, y_1)$$

From the interval table (30), $x_1 = 0.2$, and from the previous solution $y_1 = 1$
Therefore

$$f(x_1, y_1) = \frac{x_1}{y_1} = \frac{0.2}{1} = 0.2$$

Euler Method: Iteration 2: when i=1

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$$y_2 = y_1 + hf(x_1, y_1)$$

From the interval table (30), $x_1 = 0.2$, and from the previous solution $y_1 = 1$
Therefore

$$f(x_1, y_1) = \frac{x_1}{y_1} = \frac{0.2}{1} = 0.2$$

Hence

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) && (40) \\ &= 1 + 0.2(0.2) && (41) \\ &= 1.04 && (42) \end{aligned}$$

Euler Method: Iteration 3: when i=2

The formula reduces to

$$y_3 = y_2 + hf(x_2, y_2)$$

From the interval table (30), $x_2 = 0.4$, and from the previous solution $y_2 = 1.04$
Therefore

$$y_3 = y_2 + hf(x_2, y_2) \tag{43}$$

$$= 1.04 + 0.2\left(\frac{0.4}{1.04}\right) \tag{44}$$

$$= 1.117 \tag{45}$$

Euler Method: Iteration 3: when i=2

The formula reduces to

$$y_3 = y_2 + hf(x_2, y_2)$$

From the interval table (30), $x_2 = 0.4$, and from the previous solution $y_2 = 1.04$
Therefore

$$\begin{aligned} y_3 &= y_2 + hf(x_2, y_2) && (43)\\ &= 1.04 + 0.2\left(\frac{0.4}{1.04}\right) && (44)\\ &= 1.117 && (45) \end{aligned}$$

Therefore, the nodal points are:

$$(x_0, y_0) = (0, 1); \quad (x_1, y_1) = (0.2, 1); \quad (x_2, y_2) = (0.4, 1.04); \quad (x_3, y_3) = (0.6, 1.117)$$

Euler vs Analytical Solution

The differential equation is solved using separation of variables

$$y' = \frac{dy}{dx} = \frac{x}{y} \implies \int dy\, y = \int x\, dx \implies y^2 = x^2 + c$$

Implementing the initial condition to find c

$$1^2 = 0^2 + c \implies c = 1$$

Therefore the analytical solution is

$$y = \sqrt{x^2 + 1}$$

Hence at point $x_3 = 0.6$

$$y_3 = y(0.6) = \sqrt{0.6^2 + 1} = 1.166$$

The absolute error

$$AE = |ES - AS| = |1.166 - 1.117| = 0.049$$

Modified Euler

The formula is given as

$$y(x_{i+1}) = y(x_i) + hf\left[\left(x_i + \frac{h}{2}\right),\ y_i + \frac{h}{2}f(x_i,\ y_i)\right] \tag{46}$$

$$y_{i+1} = y_i + hf\left[x_i + \frac{h}{2},\ y_i + \frac{h}{2}f(x_i, y_i)\right] \tag{47}$$

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Again

$$f(x_i, y_i) = \frac{x_i}{y_i} \tag{48}$$

The initial conditions can be deduced as

$$x_0 = 0, \quad \text{and } y_0 = 1$$

Modified Euler Iteration 1: when i=0

The formula reduces to

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2},\; y_0 + \frac{h}{2}f(x_0, y_0)\right] \tag{49}$$

Modified Euler Iteration 1: when i=0

The formula reduces to

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2},\ y_0 + \frac{h}{2} f(x_0, y_0)\right] \tag{49}$$

$$= 1 + 0.2f\left[0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}\left(\frac{0}{1}\right)\right] \tag{50}$$

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$$= 1 + 0.2f(0.1,\ 1) \tag{51}$$

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$$= 1 + 0.2f(0.1,\ 1) \tag{51}$$

$$= 1 + 0.2\left(\frac{0.1}{1}\right) \tag{52}$$

$$= 1.02 \tag{53}$$

Modified Euler Iteration 2: when i=1

From the interval table (30), $x_1 = 0.2$, and from the previous solution $y_1 = 1.02$
The solution is as follows

$$y_2 = y_1 + hf\left[x_1 + \frac{h}{2},\ y_1 + \frac{h}{2} f(x_1, y_1)\right] \tag{54}$$

$$= 1.02 + 0.2f\left[0.2 + \frac{0.2}{2},\ 1.02 + \frac{0.2}{2}\left(\frac{0.2}{1.02}\right)\right] \tag{55}$$

$$= 1.02 + 0.2f(0.3,\ 1.04) \tag{56}$$

$$= 1.02 + 0.2\left(\frac{0.3}{1.04}\right) \tag{57}$$

$$= 1.03 \tag{58}$$

Modified Euler Iteration 3: when i=2

From the interval table (30), $x_2 = 0.4$, and from the previous solution $y_2 = 1.03$
The solution is as follows

$$y_3 = y_2 + hf\left[x_2 + \frac{h}{2},\ y_2 + \frac{h}{2} f(x_2, y_2)\right] \tag{59}$$

$$= 1.03 + 0.2f\left[0.4 + \frac{0.2}{2},\ 1.03 + \frac{0.2}{2}\left(\frac{0.4}{1.03}\right)\right] \tag{60}$$

$$= 1.03 + 0.2f(0.5,\ 1.07) \tag{61}$$

$$= 1.03 + 0.2\left(\frac{0.5}{1.07}\right) \tag{62}$$

$$= 1.123 \tag{63}$$

Modified Euler vs Analytical Solution

Therefore the nodal points are:

$$(x_0, y_0) = (0, 1); \quad (x_1, y_1) = (0.2, 1.02); \quad (x_2, y_2) = (0.4, 1.03); \quad (x_3, y_3) = (0.6, 1.123)$$

Modified Euler vs Analytical Solution

Therefore the nodal points are:

$$(x_0, y_0) = (0, 1); \quad (x_1, y_1) = (0.2, 1.02); \quad (x_2, y_2) = (0.4, 1.03); \quad (x_3, y_3) = (0.6, 1.123)$$

Hence

$$AE = |ES - AS| = |1.167 - 1.123| = 0.043$$

6. Runge-Kutta Methods

1. This is also a single-step method used for solving IVPs.
2. A Runge-Kutta method of second-order uses two slopes, that is

$$k_1 \text{ and } k_2 \tag{64}$$

3. whereas the fourth-order Runge-Kutta uses four slopes;

$$k_1, k_2, k_3, \text{ and } k_4 \tag{65}$$

Second-Order Runge-Kutta

1. A general second-order Runge-Kutta (RK2) is of the form

$$y_{i+1} = y_i + \left(1 - \frac{1}{2\theta}\right) k_1 + \frac{k_2}{2\theta} \tag{66}$$

   where
   $k_1 = hf(x_i, y_i)$ and
   $k_2 = hf(x_i + \theta h,\ y_i + \theta k_1)$

2. The value of θ is arbitrary such that $0 \leq \theta \leq 1$.
3. This lead to a myriad number of solution schemes

Second-Order Runge-Kutta

1. **When $\theta = 1$**

$$y_{i+1} = y_i + \frac{1}{2}k_1 + \frac{1}{2}k_2 \tag{67}$$

where $k_1 = hf(x_i, y_i)$ and
$k_2 = hf(x_i + h,\ y_i + k_1)$
This is the same as the Heun's method.

Second-Order Runge-Kutta

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$k_2 = hf(x_i + h,\ y_i + k_1)$
This is the same as the Heun's method.

2. **When** $\theta = 1/2$

$$y_{i+1} = y_i + k_2 \tag{68}$$

where $k_1 = hf(x_i, y_i)$ and
$k_2 = hf(x_i + h/2,\ y_i + k_1/2)$
This is the same as the Modified Euler

Fourth-Order Runge-Kutta

In the case of RK4, the iterative scheme is given by

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{69}$$

where

1. $k_1 = hf(x_i, y_i)$,
2. $k_2 = hf\left(x_i + \frac{h}{2},\ y_i + \frac{k_1}{2}\right)$
3. $k_3 = hf\left(x_i + \frac{h}{2},\ y_i + \frac{k_2}{2}\right)$
4. $k_4 = hf\left(x_i + h,\ y_i + k_3\right)$

Example

Solve the following IVP

$$yy' = x, \quad y(0) = 1, \quad 0 \leq x \leq 0.6, \quad h = 0.2$$

using Runge-Kutta fourth-order scheme.
Hence determine the absolute error at $x = 0.6$

Solution

Given the step size $h = 0.2$. Then the x values are given by the interval table

$$x_0 = 0, \quad x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6 \tag{70}$$

Solution

Given the step size $h = 0.2$. Then the x values are given by the interval table

$$x_0 = 0, \quad x_1 = 0.2, \quad x_2 = 0.4, \quad x_3 = 0.6 \tag{70}$$

We know that

$$y' = f(x, y) = \frac{x}{y} \tag{71}$$

Solution

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We know that

$$y' = f(x, y) = \frac{x}{y} \tag{71}$$

Iteratively,

$$f(x_i, y_i) = \frac{x_i}{y_i} \tag{72}$$

The initial conditions can be deduced as

$$x_0 = 0, \quad \text{and } y_0 = 1$$

Iteration 1: when i=0

The formula reduces to

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{73}$$

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where

$$k_1 = hf(x_0, y_0) = 0.2\left(\frac{0}{1}\right) = 0 \tag{74}$$

Iteration 1: when i=0

The formula reduces to

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where

$$k_1 = hf(x_0, y_0) = 0.2\left(\frac{0}{1}\right) = 0 \tag{74}$$

$$\begin{aligned} k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2}\right) &= 0.2f\left(0 + \frac{0.2}{2},\ 1 + \frac{0}{2}\right) \\ &= 0.2f(0.1,\ 1) \\ &= 0.2\left(\frac{0.1}{1}\right) = 0.02 \end{aligned} \tag{75}$$

Iteration 1: when i=0

$$\begin{aligned} k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) &= 0.2f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.02}{2}\right) \\ &= 0.2f(0.1,\ 1.01) \\ &= 0.2\left(\frac{0.1}{1.01}\right) \\ &= 0.02 \end{aligned} \tag{76}$$

Iteration 1: when i=0

$$
\begin{aligned}
k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right) &= 0.2f\left(0 + \frac{0.2}{2},\ 1 + \frac{0.02}{2}\right) \\
&= 0.2f(0.1,\ 1.01) \\
&= 0.2\left(\frac{0.1}{1.01}\right) \\
&= 0.02
\end{aligned} \tag{76}
$$

$$
\begin{aligned}
k_4 = hf\left(x_0 + h,\ y_0 + k_3\right) &= 0.2f(0 + 0.2,\ 1 + 0.02) \\
&= 0.2f(0.2,\ 1.02) \\
&= 0.2\left(\frac{0.2}{1.02}\right) \\
&= 0.04
\end{aligned} \tag{77}
$$

Iteration 1: when i=0

Now let substitute eqs. (74) to (77) into the main formula eq. (73). Hence

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{78}$$

$$= 1 + \frac{1}{6}[0 + 2(0.02) + 2(0.02) + 0.04] \tag{79}$$

$$= 1.02 \tag{80}$$

Iteration 2: when i=1

From the interval table (70), $x_1 = 0.2$, and from the previous solution $y_1 = 1.02$
The formula reduces to

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{81}$$

Iteration 2: when i=1

From the interval table (70), $x_1 = 0.2$, and from the previous solution $y_1 = 1.02$
The formula reduces to

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{81}$$

where

$$k_1 = hf(x_1, y_1) = 0.2\left(\frac{0.2}{1.02}\right) = 0.04 \tag{82}$$

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From the interval table (70), $x_1 = 0.2$, and from the previous solution $y_1 = 1.02$
The formula reduces to

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{81}$$

where

$$k_1 = hf(x_1, y_1) = 0.2\left(\frac{0.2}{1.02}\right) = 0.04 \tag{82}$$

$$\begin{aligned} k_2 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_1}{2}\right) &= 0.2f\left(0.2 + \frac{0.2}{2},\ 1.02 + \frac{0.04}{2}\right) \\ &= 0.2f(0.3,\ 1.04) \\ &= 0.2\left(\frac{0.3}{1.04}\right) = 0.06 \end{aligned} \tag{83}$$

Iteration 2: when i=1

$$\begin{aligned}
k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right) &= 0.2f\left(0.2 + \frac{0.2}{2},\ 1.02 + \frac{0.06}{2}\right) \\
&= 0.2f(0.3,\ 1.05) \\
&= 0.2\left(\frac{0.3}{1.05}\right) \\
&= 0.06
\end{aligned} \tag{84}$$

Iteration 2: when i=1

$$\begin{aligned}
k_3 = hf\left(x_1 + \frac{h}{2},\ y_1 + \frac{k_2}{2}\right) &= 0.2f\left(0.2 + \frac{0.2}{2},\ 1.02 + \frac{0.06}{2}\right) \\
&= 0.2f(0.3,\ 1.05) \\
&= 0.2\left(\frac{0.3}{1.05}\right) \\
&= 0.06
\end{aligned} \tag{84}$$

$$\begin{aligned}
k_4 = hf\left(x_1 + h,\ y_1 + k_3\right) &= 0.2f(0.2 + 0.2,\ 1.02 + 0.06) \\
&= 0.2f(0.4,\ 1.08) \\
&= 0.2\left(\frac{0.4}{1.08}\right) \\
&= 0.079
\end{aligned} \tag{85}$$

Iteration 2: when i=1

Now let substitute eqs. (82) to (85) into the main formula eq. (81). Hence

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{86}$$

$$= 1.02 + \frac{1}{6}[0.04 + 2(0.06) + 2(0.06) + 0.079] \tag{87}$$

$$= 1.079 \tag{88}$$

Iteration 3: when i=2

From the interval table (70), $x_2 = 0.4$, and from the previous solution $y_2 = 1.079$
The formula reduces to

$$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{89}$$

where

$$k_1 = hf(x_2, y_2) = 0.2\left(\frac{0.4}{1.079}\right) = 0.074 \tag{90}$$

$$\begin{aligned} k_2 = hf\left(x_2 + \frac{h}{2},\ y_2 + \frac{k_1}{2}\right) &= 0.2f\left(0.4 + \frac{0.2}{2},\ 1.079 + \frac{0.074}{2}\right) \\ &= 0.2f(0.5,\ 1.116) \\ &= 0.2\left(\frac{0.5}{1.116}\right) = 0.09 \end{aligned} \tag{91}$$

Iteration 3: when i=2

$$\begin{aligned} k_3 = hf\left(x_2 + \frac{h}{2},\ y_2 + \frac{k_2}{2}\right) &= 0.2f\left(0.4 + \frac{0.2}{2},\ 1.079 + \frac{0.09}{2}\right) \\ &= 0.2f(0.5,\ 1.125) \\ &= 0.2\left(\frac{0.5}{1.125}\right) \\ &= 0.089 \end{aligned} \tag{92}$$

$$\begin{aligned} k_4 = hf\left(x_2 + h,\ y_2 + k_3\right) &= 0.2f(0.4 + 0.2,\ 1.079 + 0.089) \\ &= 0.2f(0.6,\ 1.168) \\ &= 0.2\left(\frac{0.6}{1.168}\right) \\ &= 0.103 \end{aligned} \tag{93}$$

Iteration 3: when i=2

Now let substitute eqs. (90) to (93) into the main formula eq. (89). Hence

$$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{94}$$

$$= 1.079 + \frac{1}{6}[0.07 + 2(0.09) + 2(0.089) + 0.103] = 1.168 \tag{95}$$

Iteration 3: when i=2

Now let substitute eqs. (90) to (93) into the main formula eq. (89). Hence

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$$= 1.079 + \frac{1}{6}[0.07 + 2(0.09) + 2(0.089) + 0.103] = 1.168 \tag{95}$$

Therefore the nodal points are;

$(x_0, y_0) = (0, 1); \quad (x_1, y_1) = (0.2, 1.02); \quad (x_2, y_2) = (0.4, 1.079); \quad (x_3, y_3) = (0.6, 1.168)$

Iteration 3: when i=2

Now let substitute eqs. (90) to (93) into the main formula eq. (89). Hence

$$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{94}$$

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Therefore the nodal points are;

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Hence $$AE = |ES - AS| = |1.167 - 1.168| = 0.001 \tag{96}$$

Iteration 3: when i=2

Now let substitute eqs. (90) to (93) into the main formula eq. (89). Hence

$$y_3 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{94}$$

$$= 1.079 + \frac{1}{6}[0.07 + 2(0.09) + 2(0.089) + 0.103] = 1.168 \tag{95}$$

Therefore the nodal points are;

$$(x_0, y_0) = (0, 1); \quad (x_1, y_1) = (0.2, 1.02); \quad (x_2, y_2) = (0.4, 1.079); \quad (x_3, y_3) = (0.6, 1.168)$$

Hence $$AE = |ES - AS| = |1.167 - 1.168| = 0.001 \tag{96}$$

Comparing the absolute error, we may conclude that, the fourth-order Runge-Kutta scheme gives a more approximate solution than the other single-step schemes.

Exercise

Solve the initial value problem

$$y' = 2x + 3y, \quad y(0) = 1, \quad x \in [0,\ 0.4], \quad h = 0.1$$

using

1. Euler method, hence determine the relative error at $x = 0.1$
2. Modified Euler method, hence determine the relative error at $x = 0.2$
3. Euler-Cauchy method, hence determine the relative error at $x = 0.3$
4. Fourth-Order Runge-Kutta, hence determine the relative error at $x = 0.4$

END OF LECTURE

THANK YOU