MATH 162

INTRODUCTION TO PURE MATHEMATICS II

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Course Content

1. INTEGRATION

Course Content

1. INTEGRATION
2. COORDINATE GEOMETRY

Course Content

1. INTEGRATION
2. COORDINATE GEOMETRY
3. CIRCLES

Course Content

1. INTEGRATION
2. COORDINATE GEOMETRY
3. CIRCLES
4. CONIC SECTION

Integration

$$\int f(x)\,dx$$

- The differential dx is written next to the function to be integrated in order to indicate the independent variable used for the original differentiation, and the variable which is to be used for the integration.
- Thus, $\displaystyle\int f(x)\,dx$ means that $f(x)$ is to be integrated with respect to x.
- It is important to remember that the variables in the function to be integrated and the variables in the differential must be the same.
- Thus, $\displaystyle\int \cos y\,dx$ cannot be integrated as it stands. We may first have to express $\cos y$ as a function of x.

Cont.

- We also need to note that, we can use other variables besides x to indicate the independent variable.
- Thus, $\int t\,dt$ indicates that t is the independent variable, and that we need to integrate with respect to t.

Integrating x^n

By inspection:

$$\int x\,dx = \frac{1}{2}x^2 + C$$
$$\int x^2\,dx = \frac{1}{3}x^3 + C$$
$$\int x^3\,dx = \frac{1}{4}x^4 + C$$
$$\int x^4\,dx = \frac{1}{5}x^5 + C$$

Integrating x^n

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$$\int x^2\,dx = \frac{1}{3}x^3 + C$$
$$\int x^3\,dx = \frac{1}{4}x^4 + C$$
$$\int x^4\,dx = \frac{1}{5}x^5 + C$$

From the above, we define that

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

Integrating x^n (cont.)

NB: The formula for integrating x^n holds for all values of n except $n = -1$ which may be treated later.

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Also,

$$\begin{aligned} \int ax^n\,dx &= a\int x^n\,dx \\ &= \frac{a}{n+1}x^{n+1} + C \end{aligned}$$

Integrating x^n (cont.)

NB: The formula for integrating x^n holds for all values of n except $n = -1$ which may be treated later.

Also,

$$\begin{aligned}\int ax^n\,dx &= a\int x^n\,dx \\ &= \frac{a}{n+1}x^{n+1} + C\end{aligned}$$

Again,

$$\begin{aligned}\int (ax+b)^n\,dx &= \frac{1}{n+1}(ax+b)^{n+1} \times \frac{1}{\frac{d}{dx}(ax+b)} \\ &= \frac{1}{a(n+1)}(ax+b)^{n+1} + C\end{aligned}$$

Integrating x^n - Examples

E.g. 1

$$\int 5x^7\,dx = 5\int x^7\,dx = 5.\frac{x^8}{8} + C$$
$$= \frac{5}{8}x^8 + C$$

Integrating x^n - Examples

E.g. 1

$$\int 5x^7\,dx = 5\int x^7\,dx = 5.\frac{x^8}{8} + C$$
$$= \frac{5}{8}x^8 + C$$

E.g. 2

$$\int 4\sqrt{x}\,dx = 4\int x^{\frac{1}{2}}\,dx = 4.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$$
$$= 4.\frac{2}{3}x^{\frac{2}{3}} + C$$
$$= \frac{8}{3}x^{\frac{2}{3}} + C$$

Integrating x^n - Examples (cont.)

E.g. 3

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= 2x^{\frac{1}{2}} + C$$

$$= 2\sqrt{x} + C$$

Integrating x^n - Examples (cont.)

E.g. 3

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-\frac{1}{2}}\,dx = \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$
$$= 2x^{\frac{1}{2}} + C$$
$$= 2\sqrt{x} + C$$

E.g. 4

$$\int \frac{2}{\sqrt{x}}\,dx = 2\int x^{-\frac{1}{2}}\,dx = 2\int x^{-\frac{1}{2}}\,dx$$
$$= 2.\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$
$$= 2.2.x^{\frac{1}{2}} + C$$
$$= 4\sqrt{x} + C$$

Integrating x^n - Examples (cont.)

E.g. 5 - Case For $n = 0$

$$\int dx = \int x^0\, dx = \frac{x^{0+1}}{0+1} + C$$
$$= x + C$$

Integrating x^n - Examples (cont.)

E.g. 5 - Case For $n = 0$

$$\int dx = \int x^0\,dx = \frac{x^{0+1}}{0+1} + C$$
$$= x + C$$

E.g. 6 (Case For $n < 0$)

$$\int x^{-3}\,dx = \frac{x^{-3+1}}{-3+1} + C$$
$$= \frac{x^{-2}}{-2} + C$$
$$= -\frac{1}{2x^2} + C$$

Integrating x^n - Examples (cont.)

NB:

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Integrating x^n - Examples (cont.)

NB:

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

E.g. 7

$$\begin{aligned}
\int (5x^2+4x-3)\,dx &= \int 5x^2\,dx + \int 4x\,dx - \int 3\,dx \\
&= 5\int x^2\,dx + 4\int x\,dx - 3\int dx \\
&= 5.\frac{x^3}{3} + 4.\frac{x^2}{2} - 3.x + C \\
&= \frac{5}{3}x^3 + 2x^2 - 3x + C
\end{aligned}$$

Integrating x^n - Examples (cont.)

E.g. 8

$$\int (x^3 - 5x^2 + 7x - 11)\,dx = \int x^3\,dx - 5\int x^2\,dx + 7\int x\,dx - 11\int dx$$
$$= \frac{x^4}{4} - \frac{5}{3}x^3 + \frac{7}{2}x^2 - 11x + C$$

Integrating x^n - Examples (cont.)

E.g. 8

$$\int (x^3 - 5x^2 + 7x - 11)\,dx = \int x^3\,dx - 5\int x^2\,dx + 7\int x\,dx - 11\int dx$$
$$= \frac{x^4}{4} - \frac{5}{3}x^3 + \frac{7}{2}x^2 - 11x + C$$

E.g. 9

$$\int \left(\sqrt[3]{x} - \frac{1}{\sqrt[3]{x}}\right) dx = \int x^{\frac{1}{3}}\,dx - \int x^{-\frac{1}{3}}\,dx$$
$$= \frac{1}{\frac{1}{3}+1}x^{\frac{1}{3}+1} - \frac{1}{-\frac{1}{3}+1}x^{-\frac{1}{3}+1} + C$$
$$= \frac{3}{4}x^{\frac{4}{3}} - \frac{3}{2}x^{\frac{2}{3}} + C$$

Integrating $\frac{1}{x}$

- If we apply the rule for integrating x^n to $\dfrac{1}{x}$, that is, to x^{-1}, we find

$$\int \frac{1}{x}\,dx = \int x^{-1}\,dx = \frac{x^{-1+1}}{-1+1} + C = \frac{x^0}{0} + C$$

which involves a division by zero.

- This shows that the rule

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$

does not work in the case where $n = -1$.

Integrating $\frac{1}{x}$

- We know however , that the rule for differentiating the logarithm function is
$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \text{for } \; x > 0$$
Hence we conclude that
$$\int \frac{1}{x}\,dx = \ln x + C \qquad \text{for } \; x > 0 \tag{1}$$
NB: This rule only applies when $x > 0$.

- However, when we differentiate $\ln(-x)$ which is defined only for $x < 0$, we find that
$$\frac{d}{dx}(\ln(-x)) = -\frac{1}{-x} = \frac{1}{x} \qquad \text{for } \; x < 0$$
$$\implies \int \frac{1}{x}\,dx = \ln(-x) + C \qquad \text{for } \; x < 0 \tag{2}$$

Integrating $\frac{1}{x}$ (cont.)

- We thus combine the two equations (1) and (2) for integrating $\dfrac{1}{x}$ by writing

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

where $|x|$ is the modulus of x, and is defined by

$$|x| = x \text{ if } x \geq 0 \text{ and } |x| = -x \text{ if } x < 0$$

By combining the rule for differentiating a function of a function with the last result, we find that if $y = \ln|f(x)|$ then

$$\frac{dy}{dx} = \frac{1}{f(x)} \times f'(x) = \frac{f'(x)}{f(x)}$$

Consequently $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|$

Integrating $\frac{1}{x}$ (cont.)

- Thus, when integrating a rational (or fractional) function in which, after a suitable adjustment of constant, if necessary, you see that the numerator is the derivative of the denominator, then the integral is the logarithm of the modulus of the denominator.
- Thus, you can use this rule to integrate all rational functions of x in which the denominator is a linear function, by a suitable adjustment of constants.
- **Example**:

$$\int \frac{1}{ax}\,dx = \frac{1}{a}\int \frac{a}{ax}\,dx$$
$$= \frac{1}{a}\ln|ax| + C$$

Natural Logarithm

E.g. 1

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

Natural Logarithm

E.g. 1

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

E.g. 2

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

Natural Logarithm

E.g. 1

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C$$

E.g. 2

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

E.g. 3

$$\int \frac{1}{3x}\,dx = \frac{1}{3}\int \frac{1}{x}\,dx = \frac{1}{3}\ln|x| + C$$

Natural Logarithm (cont.)

E.g. 4

$$\int \frac{1}{7x-4}\,dx = ?$$

$$\text{Let } u = 7x-4 \text{ and } du = 7\,dx$$

$$\Longrightarrow \int \frac{1}{7x-4}\,dx = \int \frac{1}{u}.\frac{1}{7}\,du = \frac{1}{7}\int \frac{1}{u}\,du$$

$$= \frac{1}{7}\ln|u| + C$$

$$= \frac{1}{7}\ln|7x-4| + C$$

Natural Logarithm (cont.)

E.g. 5

$$\int \frac{x^3}{x^4-1}\,dx = ?$$

$$\text{Let } u = x^4 - 1 \text{ and } du = 4x^3\,dx$$

$$\implies x^3\,dx = \frac{du}{4}$$

$$\implies \int \frac{x^3}{x^4-1}\,dx = \int \frac{1}{u}.\frac{1}{4}\,du = \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|u| + C$$

$$= \frac{1}{4}\ln|x^4-1| + C$$

Natural Logarithm (cont.)

E.g. 6

$$\int \cot x\, dx = ?$$

$$\text{Let } u = \sin x \text{ and } du = \cos x\, dx$$

$$\begin{aligned}
\implies \int \cot x\, dx &= \int \frac{\cos x}{\sin x}\, dx = \int \frac{1}{u}\, du \\
&= \ln|u| + C \\
&= \ln|\sin x| + C
\end{aligned}$$

Natural Logarithm (cont.)

E.g. 7

$$\int \frac{1}{x\ln x}\,dx = ?$$

$$\text{Let } \; u = \ln x \;\text{ and }\; du = \frac{1}{x}\,dx$$

$$\Longrightarrow \int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|\ln(x)| + C$$

Natural Logarithm (cont.)

E.g. 8

$$\int \frac{\cos 2x}{1-\sin 2x}\,dx = ?$$

Let $u = 1-\sin 2x$ and $du = -2\cos 2x\,dx$

$$\implies \int \frac{\cos 2x}{1-\sin 2x}\,dx = -\frac{1}{2}\int \frac{1}{u}\,du$$

$$= -\frac{1}{2}\ln|u| + C$$

$$= -\frac{1}{2}\ln|1-\sin 2x| + C$$

Natural Logarithm (cont.)

E.g. 9

$$\int \frac{\sec^2 x}{\tan x}\, dx = ?$$

$$\text{Let } \; u = \tan x \; \text{ and } \; du = \sec^2 x \cos 2x\, dx$$

$$\implies \int \frac{\sec^2 x}{\tan x}\, dx = \int \frac{1}{u}\, du$$

$$= \ln|u| + C$$

$$= \ln|\tan x| + C$$

Natural Logarithm (cont.)

E.g. 10

$$\int \frac{\ln x}{x}\,dx = ?$$

$$\text{Let } u = \ln x \text{ and } du = \frac{1}{x}\,dx$$

$$\begin{aligned}
\Longrightarrow \int \frac{\ln x}{x}\,dx &= \int (\ln x)\frac{1}{x}\,dx = \int u\,du \\
&= \frac{1}{2}u^2 + C \\
&= \frac{1}{2}(\ln x)^2 + C
\end{aligned}$$

Natural Logarithm (cont.)

E.g. 11

$$\int \tan x\,dx = ?$$

But
$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\\ &= -\ln|\cos x| + C\\ &= \ln|\cos^{-1} x| + C\\ &= \ln|\sec x| + C\end{aligned}$$

Natural Logarithm (cont.)

E.g. 12

$$\int \frac{x}{2x^2+3}\,dx = ?$$

Let $u = 2x^3+3$ and $du = 4x\,dx$

$$\implies \int \frac{x}{2x^2+3}\,dx = \frac{1}{4}\int \frac{4x}{2x^2+3}\,dx = \frac{1}{4}\int \frac{1}{u}\,du$$

$$= \frac{1}{4}\ln|2x^2+3| + C$$

Natural Logarithm (cont.)

E.g. 13

$$\int \frac{2(x+1)}{x^2+2x+7}\,dx = ?$$

$$\Longrightarrow \int \frac{2x+2}{x^2+2x+7}\,dx = \ln|x^2+2x+7| + C$$

Natural Logarithm (cont.)

E.g. 13

$$\int \frac{2(x+1)}{x^2+2x+7}\,dx = ?$$

$$\implies \int \frac{2x+2}{x^2+2x+7}\,dx = \ln|x^2+2x+7| + C$$

E.g. 14

$$\int \frac{3x^5+2x^2-3}{x^3}\,dx = ?$$

$$\int \left(\frac{3x^5}{x^3} + \frac{2x^2}{x^3} - \frac{3}{x^3}\right) dx = \int \left(3x^2 + \frac{2}{x} - \frac{3}{x^3}\right) dx$$

$$= 3\int x^2\,dx + 1\int \frac{1}{x}\,dx - 3\int x^{-3}\,dx$$

$$= x^3 + 2\ln|x| + \frac{3}{2}x^{-2} + C$$

Try the following

Find the integrals of the following

i. $\displaystyle\int \frac{1}{3-2x}\,dx$

ii. $\displaystyle\int \frac{1}{ax+b}\,dx$

iii. $\displaystyle\int \frac{2x}{x^2+4}\,dx$

iv. $\displaystyle\int \left(\frac{3}{x-1}-\frac{4}{x-2}\right)dx$

v. $\displaystyle\int \frac{1}{2}x^{-\frac{1}{2}}\,dx$

vi. $\displaystyle\int \frac{1}{2\sqrt{2x^3}}\,dx$

vii. $\displaystyle\int [(x+3)(x-3)]\,dx$

viii. $\displaystyle\int \left(2-\frac{1}{3}x^2-\frac{1}{2\sqrt{x}}\right)dx$

ix. $\displaystyle\int \left(\frac{1}{x^3}-\frac{1}{x^2}+\frac{1}{x}-1\right)dx$

x. $\displaystyle\int \left(x^{\frac{2}{3}}+1+x^{-\frac{2}{3}}\right)dx$

xi. $\displaystyle\int \frac{x^3-7}{x}\,dx$

xii. $\displaystyle\int \frac{1}{\sqrt[3]{x}}\,dx$

Definite Integrals - Examples

1. Evaluate $\displaystyle\int_1^2 \frac{x-1}{x^2-2x+2}\,dx$

Definite Integrals - Examples

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Solution

$$\begin{aligned}
\int_1^2 \frac{x-1}{x^2-2x+2}\,dx &= \frac{1}{2}\int_1^2 \frac{2x-2}{x^2-2x+2}\,dx \\
&= \frac{1}{2}\ln|x^2-2x+2|\Big|_1^2 \\
&= \frac{1}{2}(\ln 2 - \ln 1) \\
&= \frac{1}{2}\ln 2
\end{aligned}$$

Definite Integrals - Examples (cont.)

2. Find $\displaystyle\int_1^2 \frac{x}{3x^2 - 2}\,dx$

Definite Integrals - Examples (cont.)

2. Find $\displaystyle\int_1^2 \frac{x}{3x^2-2}\,dx$

Solution

$$\begin{aligned}
\int_1^2 \frac{x}{3x^2-2}\,dx &= \frac{1}{6}\int_1^2 \frac{6x}{3x^2-2}\,dx \\
&= \frac{1}{6}\ln|3x^2-2|\Big|_1^2 \\
&= \frac{1}{6}(\ln 10 - \ln 1) \\
&= \frac{1}{6}\ln 10
\end{aligned}$$

Definite Integrals - Examples (cont.)

2. Evaluate $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{2-\sin x}\,dx$

Definite Integrals - Examples (cont.)

2. Evaluate $\displaystyle\int_{\pi/6}^{\pi/2} \frac{\cos x}{2-\sin x}\,dx$

Solution

$$\begin{aligned}
\int_{\pi/6}^{\pi/2} \frac{\cos x}{2-\sin x}\,dx &= -\int_{\pi/6}^{\pi/2} \frac{-\cos x}{2-\sin x}\,dx \\
&= \quad -\ln|2-\sin x|\Big]_{\pi/6}^{\pi/2} \\
&= -\left(\ln 1 - \ln\frac{3}{2}\right) \\
&= \ln\frac{3}{2} = \ln 3 - \ln 2
\end{aligned}$$

Exponential Functions

$$\int e^x\,dx = e^x + C$$

$$\int ae^x\,dx = ae^x + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

Exponential Functions - Example

1.

$$\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$$

Exponential Functions - Example

1.
$$\int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$$

2.
$$\begin{aligned}\int 2e^{5x}\,dx &= 2\int e^{5x} + C \\ &= \frac{2}{5}e^{5x} + C\end{aligned}$$

Integrals of Standard Forms

There are a number of integrals known as STANDARD FORMS. They are derivatives of functions which could be applied or used as such.

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There are a number of integrals known as STANDARD FORMS. They are derivatives of functions which could be applied or used as such.

Algebraic Functions

1. $$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$
2. $$\int \frac{1}{x}\,dx = \ln|x| + C$$
3. $$\int a^x\,dx = \frac{1}{\ln a}a^x + C$$
4. $$\int e^x\,dx = e^x + C$$

Trigonometric Functions

1. $\int \sin x\, dx = -\cos x + C$

2. $\int \cos x\, dx = \sin x + C$

3. $\int \tan x\, dx = -\ln|\cos x| + C = \ln|\sec x|$

4. $\int \cot x\, dx = \ln|\sin x| + C$

Trigonometric Functions

1. $\displaystyle\int \sin x\,dx = -\cos x + C$

2. $\displaystyle\int \cos x\,dx = \sin x + C$

3. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x|$

4. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

NB:
The derivatives of $\sec x$ and $\operatorname{cosec} x$ namely $\sec x \tan x$ and $-\operatorname{cosec} x \cot x$ do not give rise to standard forms, but to products of them. The integrals of $\sec x$ and $\operatorname{cosec} x$ do not arise by direct differentiation.

Additional Standard Integrals

The following are additional standard forms which are useful for integration. Some of them are results of differentiating standard forms.

1. $\int \sec x \tan x \, dx = \sec x + C$
2. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + C$
3. $\int \operatorname{cosec}^2 x \, dx = -\cot x + C$
4. $\int \sec^2 x \, dx = \tan x + C$

Inverse Trigonometric Functions

1. $$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

2. $$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

3. $$-\int \frac{1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x + C$$

4. $$-\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + C$$

5. $$\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

6. $$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Hyperbolic Functions

1. $\int \sinh x\,dx = \cosh x + C$

2. $\int \cosh x\,dx = \sinh x + C$

3. $\int \tanh x\,dx = \ln|\cosh x| + C$

4. $\int \coth x\,dx = \ln|\sinh x| + C$

Inverse Hyperbolic Functions

1. $$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)+C$$

2. $$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right)+C$$

3. $$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)+C$$

Try the following

Find the following integrals

1. $\displaystyle\int e^{3x-1}\,dx$
2. $\displaystyle\int e^{\frac{x}{a}}\,dx$
3. $\displaystyle\int \frac{x^2-x+1}{x^3}\,dx$
4. $\displaystyle\int \sqrt{ax+b}\,dx$
5. $\displaystyle\int \frac{x}{x^2-1}\,dx$
6. $\displaystyle\int \sqrt{1+\frac{x}{2}}\,dx$
7. $\displaystyle\int x(1+x)(1+x^2)\,dx$
8. $\displaystyle\int \frac{e^{3x}}{e^{3x}+6}\,dx$
9. $\displaystyle\int \sin(\alpha-3x)\,dx$
10. $\displaystyle\int \sin\frac{1}{3}x\,dx$
11. $\displaystyle\int 2^x\,dx$
12. $\displaystyle\int \frac{\sin ax}{1+\cos ax}\,dx$
13. $\displaystyle\int \cos(2x+\beta)\,dx$
14. $\displaystyle\int \frac{1}{\sqrt{1-x}}\,dx$

Integration by Substitution - Example

1. Find $\int (3x-5)^{12}\,dx$

Integration by Substitution - Example

1. Find $\int (3x-5)^{12}\,dx$

Solution

$$\text{Let } u = 3x-5 \text{ and } du = 3dx$$

$$\implies dx = \frac{1}{3}du$$

$$\implies \int (3x-5)^{12}\,dx = \int u^{12}\frac{1}{3}\,du = \frac{1}{3}\int u^{12}\,du$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{13}\right)u^{13} + C$$

$$= \frac{1}{39}u^{13} + C$$

$$= \frac{1}{49}(3x-5)^{12} + C$$

Integration by Substitution - Example

2. Find $\int \sqrt{7x+4}\,dx$

Integration by Substitution - Example

2. Find $\int \sqrt{7x+4}\,dx$

Solution

$$\text{Let } u = 7x+4 \text{ and } du = 7dx$$

$$\implies \quad dx = \frac{1}{7}du$$

$$\implies \int \sqrt{7x+4}\,dx = \int \sqrt{u}\frac{1}{7}\,du = \frac{1}{7}\int u^{\frac{1}{2}}\,du$$

$$= \left(\frac{1}{7}\right)\left(\frac{2}{3}\right)u^{\frac{3}{2}} + C$$

$$= \frac{2}{21}u^{\frac{3}{2}} + C$$

$$= \frac{2}{21}(7x+4)^{\frac{3}{2}} + C$$

Integration by Substitution - Example

3. Find $\int (4 - 2t^2)^7 t \, dt$

Integration by Substitution - Example

3. Find $\int (4-2t^2)^7 t\,dt$

Solution

$$\text{Let } u = 4-2t^2 \text{ and } du = -4t\,dt$$

$$\implies dt = -\frac{1}{4t}du$$

$$\implies \int (4-2t^2)^7 t\,dt = \int u^7\left(-\frac{du}{4t}\right)t = -\frac{1}{4}\int u^7\,du$$

$$= -\left(\frac{1}{4}\right)\left(\frac{1}{8}\right)u^8 + C$$

$$= -\frac{1}{32}(4-2t^2)^8 + C$$

Integration by Substitution - Example

4. Find $\int x^2 \sqrt[3]{x^3+5}\,dx$

Integration by Substitution - Example

4. Find $\int x^2 \sqrt[3]{x^3+5}\,dx$

Solution

$$\text{Let } u = x^3 + 5 \text{ and } du = 3x^2\,dx$$

$$\implies dx = \frac{1}{3x^2}du \Rightarrow x^2 dx = \frac{du}{3}$$

$$\implies \int x^2 \sqrt[3]{x^3+5}\,dx = \int \sqrt[3]{u}.\frac{du}{3} = \frac{1}{3}\int u^{\frac{1}{3}}\,du$$

$$= \left(\frac{1}{3}\right)\left(\frac{3}{4}\right)u^{\frac{4}{3}} + C$$

$$= \frac{1}{4}(x^3+5)^{\frac{4}{3}} + C$$

$$= \frac{1}{4}\left(\sqrt[3]{x^3+5}\right)^4 + C$$

Integration by Substitution - Example

5. Find $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx$

Integration by Substitution - Example

5. Find $\int \frac{x}{\sqrt{x+1}}\,dx$

Solution

$$\text{Let } u = x+1 \text{ and } du = 3\,dx$$

$$\implies x = u-1$$

$$\implies \int \frac{x}{\sqrt{x+1}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du = \int (u^{\frac{1}{2}} - u^{-\frac{1}{2}})\,du$$

$$= \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + C$$

$$= \frac{2}{3}(\sqrt{x+1})^3 - 2\sqrt{x+1} + C$$

$$= 2\sqrt{x+1}\left[\frac{1}{3}(x+1) - 1\right] + C$$

Integration by Substitution - Example

6. Find $\int \sqrt[3]{x^2 - 2x + 1}\, dx$

Integration by Substitution - Example

6. Find $\int \sqrt[3]{x^2 - 2x + 1}\, dx$

Solution

Let $u = x^2 - 2x + 1 = (x-1)^2$ and $u = x - 1$

$$\implies du = dx$$

$$\implies \int \sqrt[3]{x^2 - 2x + 1}\, dx = \int \sqrt[3]{(x-1)^2}\, dx = \int \sqrt[3]{u^2}\, du$$

$$= \int u^{\frac{2}{3}}\, du = \frac{3}{5} u^{\frac{5}{3}} + C$$

$$= \frac{3}{5} (\sqrt[3]{u})^5 + C$$

$$= \frac{3}{5} \left(\sqrt[3]{x-1}\right)^5 + C$$

Integration by Substitution - Example

7. Find $\int (x^4+1)^{\frac{1}{3}} x^7 \, dx$

Integration by Substitution - Example

7. Find $\int (x^4+1)^{\frac{1}{3}} x^7 \, dx$

Solution

$$\text{Let } u = x^4 + 1 \text{ and } du = 4x^3 \, dx$$

$$\implies x^4 = u - 1 \text{ and } x^3 \, dx = \frac{du}{4}$$

$$\text{Also } x^7 = x^4 x^3 \, dx = (u-1)\,\frac{du}{4}$$

$$\implies \int (x^4+1)^{\frac{1}{3}} x^7 \, dx = \int u^{\frac{1}{3}} (u-1)\,\frac{du}{4}$$

$$= \frac{1}{4}\int \left(u^{\frac{4}{3}} - u^{\frac{1}{3}}\right) du$$

$$= \frac{1}{4}\left(\frac{3}{7}u^{\frac{7}{3}} - \frac{3}{4}u^{\frac{4}{3}}\right) + C$$

Integration by Substitution - Example (cont.)

$$= \frac{3}{4}u^{\frac{4}{3}}\left(\frac{1}{7}u - \frac{1}{4}\right) + C$$

$$= \frac{3}{4}(x^4+1)^{\frac{4}{3}}\left[\frac{1}{7}(x^4+1) - \frac{1}{4}\right] + C$$

$$= \frac{3}{4}(x^4+1)^{\frac{4}{3}}\left[\frac{4(x^4+1)-7}{28}\right] + C$$

$$= \frac{3}{4}(x^4+1)^{\frac{4}{3}}\left[\frac{4x^4+4-7}{28}\right] + C$$

$$= \frac{3}{112}(x^4+1)^{\frac{4}{3}}(4x^4-3) + C$$

Integration by Substitution - Example

8. Find $\displaystyle\int \frac{x}{\sqrt{1+5x^2}}\,dx$

Integration by Substitution - Example

8. Find $\int \frac{x}{\sqrt{1+5x^2}}\,dx$

Solution

$$\text{Let } u = 1+5x^2 \text{ and } du = 10x\,dx$$

$$\implies x\,dx = \frac{du}{10}$$

$$\implies \int \frac{x}{\sqrt{1+5x^2}}\,dx = \int \frac{1}{\sqrt{u}}\frac{du}{10}$$

$$= \frac{1}{10}\int u^{-\frac{1}{2}}\,du$$

$$= \frac{1}{10}.2u^{\frac{1}{2}} + C$$

$$= \frac{1}{5}\sqrt{1+5x^2} + C$$

Integration by Substitution - Example

9. Find $\int x\sqrt{ax+b}\,dx$ when $a \neq 0$

Integration by Substitution - Example

9. Find $\int x\sqrt{ax+b}\,dx$ when $a \neq 0$

Solution

$$\text{Let } u = ax+b \text{ and } du = a\,dx$$
$$\implies x = \frac{u-b}{a} \text{ and } dx = \frac{du}{a}$$
$$\implies \int x\sqrt{ax+b}\,dx = \int \frac{u-b}{a}\sqrt{u}\,\frac{du}{a}$$
$$= \frac{1}{a^2}\int \left(u^{\frac{3}{2}} - bu^{\frac{1}{2}}\right)du$$
$$= \frac{1}{a^2}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2b}{3}u^{\frac{3}{2}}\right) + C$$

Integration by Substitution - Example (cont.)

$$
\begin{aligned}
&= \frac{2}{a^2} u^{\frac{3}{2}} \left(\frac{1}{5} u - \frac{b}{3} \right) + C \\
&= \frac{2}{a^2} \left(\sqrt{ax+b} \right)^3 \left[\frac{1}{5}(ax+b) - \frac{b}{3} \right] + C \\
&= \frac{2}{a^2} \left(\sqrt{ax+b} \right)^3 \left[\frac{a}{5}x - \frac{2}{15}b \right] + C \\
&= \frac{2}{15a^2} \left(\sqrt{ax+b} \right)^3 (3ax - 2b) + C
\end{aligned}
$$

Using Substitution For Trig. Functions - Example

1. Find $\int \sin(3x-1)\,dx$

Using Substitution For Trig. Functions - Example

1. Find $\int \sin(3x-1)\,dx$

Solution

$$\text{Let } u = 3x-1 \text{ and } du = 3\,dx$$

$$\implies dx = \frac{du}{3}$$

$$\implies \int \sin(3x-1)\,dx = \int \sin u \frac{du}{3}$$

$$= \frac{1}{3}\int \sin u\,du$$

$$= \frac{1}{3}(-\cos u) + C$$

$$= -\frac{1}{3}\cos(3x-1) + C$$

Using Substitution For Trig. Functions - Example

2. Find $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Using Substitution For Trig. Functions - Example

2. Find $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Solution

$$\text{Let } \ u = \sqrt{x} = x^{1/2} \ \text{ and } \ du = \frac{1}{2}\left(\frac{1}{\sqrt{x}}\right)dx$$

$$\implies \ 2du = \frac{1}{\sqrt{x}}\,dx$$

$$\begin{aligned}
\implies \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx &= \int \cos u\,.2\,du \\
&= 2\int \cos u\,du \\
&= 2(\sin u) + C \\
&= 2\sin(\sqrt{x}) + C
\end{aligned}$$

Using Substitution For Trig. Functions - Example

3. Find $\int \sec^2\left(\frac{x}{2}\right) dx$

Using Substitution For Trig. Functions - Example

3. Find $\int \sec^2\left(\frac{x}{2}\right) dx$

Solution

$$\text{Let } \; u = \frac{x}{2} \; \text{ and } \; du = \frac{1}{2}dx$$

$$\implies \; 2du = dx$$

$$\begin{aligned} \implies \int \sec^2\left(\frac{x}{2}\right) dx &= \int \sec^2 u \,.2\, du \\ &= 2\int \sec^2 u\, du \\ &= 2(\tan u) + C \\ &= 2\tan\left(\frac{x}{2}\right) + C \end{aligned}$$

Using Substitution For Trig. Functions - Example

4. Find $\displaystyle\int \frac{\cos 3x}{\sin^2 3x}\,dx$

Using Substitution For Trig. Functions - Example

4. Find $\int \frac{\cos 3x}{\sin^2 3x}\, dx$

Solution

$$\text{Let } u = \sin 3x \text{ and } du = 3\cos 3x\, dx$$

$$\implies \frac{du}{3} = \cos 3x\, dx$$

$$\implies \int \frac{\cos 3x}{\sin^2 3x}\, dx = \int \frac{1}{u^2}\frac{du}{3} = \frac{1}{3}\int u^{-2}\, du$$

$$= \frac{1}{3}(-u^{-1}) + C$$

$$= -\frac{1}{3u} + C$$

$$= -\frac{1}{3\sin 3x} + C = -\frac{1}{3}\operatorname{cosec} 3x + C$$

Using Substitution For Trig. Functions - Example

5. Find $\int \sin^4 x \cos x \, dx$

Using Substitution For Trig. Functions - Example

5. Find $\int \sin^4 x \cos x \, dx$

Solution

$$\text{Let } u = \sin x \text{ and } du = \cos x \, dx$$

$$\begin{aligned} \Longrightarrow \int \sin^4 x \cos x \, dx &= \int u^4 \, du \\ &= \frac{1}{5} u^5 + C \\ &= \frac{1}{5} \sin^5 x + C \end{aligned}$$

Using Substitution For Trig. Functions - Example

6. Find $\int \sin 3x \cos 3x \, dx$

Using Substitution For Trig. Functions - Example

6. Find $\int \sin 3x \cos 3x \, dx$

Solution

$$\text{Let } u = \sin 3x \text{ and } du = 3\cos 3x \, dx$$

$$\Longrightarrow \frac{du}{3} = \cos 3x \, dx$$

$$\begin{aligned}
\Longrightarrow \int \sin 3x \cos 3x \, dx &= \int u \frac{du}{3} \\
&= \frac{1}{3} \int u \, du \\
&= \frac{1}{3} \cdot \frac{1}{2} u^2 + C \\
&= \frac{1}{6} \sin^2 3x + C
\end{aligned}$$

Using Substitution For Trig. Functions - Example

7. Find $\int \sec^2 4x \tan 4x \, dx$

Using Substitution For Trig. Functions - Example

7. Find $\int \sec^2 4x \tan 4x \, dx$

Solution

$$\text{Let } u = \tan 4x \text{ and } du = 4\sec^2 4x \, dx$$

$$\Longrightarrow \frac{du}{4} = \sec^2 4x \, dx$$

$$\begin{aligned}
\Longrightarrow \int \sec^2 4x \tan 4x \, dx &= \int u \, \frac{du}{4} \\
&= \frac{1}{4}\int u \, du \\
&= \frac{1}{4}.\frac{1}{2}u^2 + C \\
&= \frac{1}{8}\tan^2 4x + C
\end{aligned}$$

Using Substitution For Trig. Functions - Example

8. Find $\displaystyle\int \cos x \sin x \sqrt{1+\sin^2 x}\, dx$

Using Substitution For Trig. Functions - Example

8. Find $\int \cos x \sin x \sqrt{1+\sin^2 x}\, dx$

Solution

$$\text{Let } u = 1+\sin^2 x \text{ and } du = 2\sin x \cos x\, dx$$

$$\implies \frac{du}{2} = \sin x \cos x\, dx$$

$$\begin{aligned}\implies \int \cos x \sin x \sqrt{1+\sin^2 x}\, dx &= \int \sqrt{u}\, \frac{du}{2} \\ &= \frac{1}{2}\int u^{\frac{1}{2}}\, du \\ &= \frac{1}{2}.\frac{2}{3} u^{\frac{3}{2}} + C \\ &= \frac{1}{3}\left(1+\sin^2 x\right)^{3/2} + C\end{aligned}$$

Using Substitution For Trig. Functions - Example

9. Find $\int \sec^2 x \sqrt{\tan^3 x}\, dx$

Using Substitution For Trig. Functions - Example

9. Find $\int \sec^2 x \sqrt{\tan^3 x}\, dx$

Solution

$$\text{Let } u = \tan x \text{ and } du = \sec^2 x\, dx$$

$$\begin{aligned} \Longrightarrow \int \sec^2 x \sqrt{\tan^3 x}\, dx &= \int u^{\frac{3}{2}}\, du \\ &= \frac{2}{5} u^{\frac{5}{2}} + C \\ &= \frac{2}{5}\left(\sqrt{\tan^5 x}\right) + C \end{aligned}$$

Using Substitution For Trig. Functions - Example

10. Find $\int \tan^2\theta \sec^4\theta \, d\theta$

Using Substitution For Trig. Functions - Example

10. Find $\int \tan^2\theta \sec^4\theta \, d\theta$

Solution

$$\text{Let } u = \tan\theta \text{ and } du = \sec^2\theta \, d\theta$$

$$\text{Also } \sec^2\theta = 1 + \tan^2\theta = 1 + u^2$$

$$\begin{aligned}
\Longrightarrow \int \tan^2\theta \sec^4\theta \, d\theta &= \int u^2(1+u^2)\, du \\
&= \int (u^2 + u^4)\, du \\
&= \frac{1}{3}u^3 + \frac{1}{5}u^5 + C \\
&= \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta + C
\end{aligned}$$

Using Substitution For Trig. Functions - Example

11. Find $\int \cos^3 5x \sin^2 5x \, dx$

Using Substitution For Trig. Functions - Example

11. Find $\int \cos^3 5x \sin^2 5x\, dx$

Solution

$$\text{Let } u = \sin 5x \text{ and } du = 5\cos 5x\, dx$$

$$\text{Also } \cos^2 5x = 1 - \sin^2 5x = 1 - u^2$$

$$\begin{aligned}
\Longrightarrow \int \cos^3 5x \sin^2 5x\, dx &= \int (1-u^2)u^2\, \frac{du}{5} \\
&= \frac{1}{5}\int (u^2 - u^4)\, du \\
&= \frac{1}{5}\left(\frac{1}{3}u^3 - \frac{1}{5}u^5\right) + C \\
&= \frac{1}{15}\sin^3 5x - \frac{1}{25}\sin^5 5x + C
\end{aligned}$$

Using Substitution For Trig. Functions - Example

12. Find $\int \sec^5 x \tan x \, dx$

Using Substitution For Trig. Functions - Example

12. Find $\int \sec^5 x \tan x \, dx$

Solution

$$\text{Let } u = \sec x \text{ and } du = \sec x \tan x \, dx$$

$$\begin{aligned}
\Longrightarrow \int \sec^5 x \tan x \, dx &= \int \sec^4 x \sec x \tan x \, dx \\
&= \int u^4 \, du \\
&= \frac{1}{5} u^5 + C \\
&= \frac{1}{5} \sec^5 x + C
\end{aligned}$$

Using Substitution For Trig. Functions - Example

13. Find $\int \left(\frac{\sec x}{1+\tan x} \right)^2 dx$

Using Substitution For Trig. Functions - Example

13. Find $\int \left(\frac{\sec x}{1+\tan x}\right)^2 dx$

Solution

Let $u = 1+\tan x$ and $du = \sec^2 x\, dx$

$$\implies \int \left(\frac{\sec x}{1+\tan x}\right)^2 dx = \int \frac{\sec^2 x}{(1+\tan x)^2}\, dx$$

$$= \int \frac{1}{u^2}\, du = \int u^{-2}\, du$$

$$= -u^{-1} + C$$

$$= -\frac{1}{1+\tan x} + C$$

Example

1. Find $\displaystyle\int \frac{1}{1+\cos x}\,dx$

Example

1. Find $\displaystyle\int \frac{1}{1+\cos x}\,dx$

Solution

$$
\begin{aligned}
\frac{1}{1+\cos x} &= \frac{1}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x} = \frac{1-\cos x}{\sin^2 x}\\
&= \operatorname{cosec}^2 x - \cot x \operatorname{cosec} x\\
\Longrightarrow \int \frac{1}{1+\cos x}\,dx &= \int\left(\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x\right)dx\\
&= -\cot x + \csc x + C
\end{aligned}
$$

Example

2. Find $\displaystyle\int \frac{1}{1+\sin x}\,dx$

Example

2. Find $\displaystyle\int \frac{1}{1+\sin x}\,dx$

Solution

$$
\begin{aligned}
\frac{1}{1+\sin x} &= \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x} = \frac{1-\cos x}{\cos^2 x}\\
&= \frac{1}{\cos^2 x} - \tan x\ \sec x\\
&= \sec^2 x - \sec x\ \tan x\\
\implies \int \frac{1}{1+\sin x}\,dx &= \int (\sec^2 x - \sec x\ \tan x)\,dx\\
&= \tan x - \sec x + C
\end{aligned}
$$

Using Identities And Definite Limits

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\sin 2x = 2\sin x \cos x$$

$$\sin^2 x + \cos^2 x = 1$$

Example

1. Find $\int \cos^2 x \, dx$

Example

1. Find $\int \cos^2 x \, dx$

Solution

$$\begin{aligned}\int \cos^2 x \, dx &= \frac{1}{2}\int (1+\cos 2x) \, dx \\ &= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C \\ &= \frac{1}{2}(x + \sin x \cos x) + C\end{aligned}$$

Example

2. Find $\int \sin^2 x \, dx$

Example

2. Find $\int \sin^2 x \, dx$

Solution

$$\begin{aligned}
\int \sin^2 x \, dx &= \frac{1}{2} \int (1 - \cos 2x) \, dx \\
&= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C \\
&= \frac{1}{2} (x - \sin x \cos x) + C
\end{aligned}$$

Example

3. Find $\displaystyle\int_0^{\pi/4} \cos x \, dx$

Example

3. Find $\displaystyle\int_0^{\pi/4} \cos x\, dx$

Solution

$$\begin{aligned}\int_0^{\pi/4} \cos x\, dx &= \left. \sin x \right]_0^{\frac{\pi}{4}} \\ &= \sin\frac{\pi}{4} - \sin 0 \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

Example

4. Find $\int_0^{\pi/3} \sec^2 x \, dx$

Example

4. Find $\int_0^{\pi/3} \sec^2 x\, dx$

Solution

$$\begin{aligned}\int_0^{\pi/3} \sec^2 x\, dx &= \left.\tan x\right|_0^{\frac{\pi}{3}} \\ &= \tan\frac{\pi}{3} - \tan 0 \\ &= \sqrt{3}\end{aligned}$$

Example

5. Find $\int_{\pi/6}^{\pi/3} \sin x \, dx$

Example

5. Find $\displaystyle\int_{\pi/6}^{\pi/3} \sin x\, dx$

Solution

$$\begin{aligned}\int_{\pi/6}^{\pi/3} \sin x\, dx &= -\cos x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= -\cos\frac{\pi}{3} + \cos\frac{\pi}{6} \\ &= -\frac{1}{2} + \frac{\sqrt{3}}{2} \\ &= \frac{\sqrt{3}-1}{2}\end{aligned}$$

Example

6. Find $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x \, dx$

Example

6. Find $\int_0^{\pi/2} \sin^2 x \cos x \, dx$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sin^2 x \cos x \, dx &= \frac{1}{3} \sin^3 x \Big|_0^{\frac{\pi}{2}} \\
&= \frac{1}{3} \left[\sin^3 \frac{\pi}{2} - \sin^3 0 \right] \\
&= \frac{1}{3}
\end{aligned}$$

Example

7. Find $\displaystyle\int_0^{\pi/4} \tan x \sec^2 x \, dx$

Example

7. Find $\int_0^{\pi/4} \tan x \sec^2 x \, dx$

Solution

$$\begin{aligned}\int_0^{\pi/4} \tan x \sec^2 x \, dx &= \frac{1}{2}\tan^2 x\Big|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\left[\tan^2\frac{\pi}{4} - \tan^2 0\right] \\ &= \frac{1}{2}(1-0) = \frac{1}{2}\end{aligned}$$

Example

8. Find $\displaystyle\int_0^{\pi/2} \sqrt{\sin x + 1}\,\cos x\,dx$

Example

8. Find $\displaystyle\int_0^{\pi/2} \sqrt{\sin x + 1}\cos x\,dx$

Solution

$$\begin{aligned}
\int_0^{\pi/2} \sqrt{\sin x + 1}\cos x\,dx &= \int_0^{\pi/2} (\sin x + 1)^{1/2}\cos x\,dx \\
&= \frac{2}{3}(\sin x + 1)^{\frac{3}{2}}\Big|_0^{\frac{\pi}{2}} \\
&= \frac{2}{3}\left[\left(\sin\left(\frac{\pi}{2}\right) + 1\right)^{\frac{3}{2}} - (\sin(0) + 1)^{3/2}\right] \\
&= \frac{2}{3}(2\sqrt{2} - 1)
\end{aligned}$$

Try

1. Find $\int \sqrt{1-x}\,.x^2\,dx$

Try

1. Find $\int \sqrt{1-x}\,.x^2\,dx$

Solution

$$\text{Let } u = 1-x \text{ and } du = -dx$$
$$\implies x = 1-u$$
$$\implies \int \sqrt{1-x}\,.x^2\,dx = \int \sqrt{u}(1-u^2)(-1)\,du$$
$$= -\int \sqrt{u}(1-2u+u^2)\,du$$
$$= -\int (u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}})\,du$$
$$= -\left[\frac{2}{3}u^{\frac{3}{2}} - 2\left(\frac{2}{5}\right)u^{\frac{5}{2}} + \frac{2}{7}u^{\frac{7}{2}}\right] + C$$

Try (cont.)

$$= -2u^{\frac{3}{2}}\left(\frac{1}{3}-\frac{2}{5}u+\frac{1}{7}u^2\right)+C$$

$$= -2(\sqrt{1-x})^3\left[\frac{1}{3}-\frac{2}{5}(1-x)+\frac{1}{7}(1-x)^2\right]+C$$

$$= -\frac{2}{105}(\sqrt{1-x})^3(8+12x+15x^2)+C$$

Try (cont.)

2. Find $\displaystyle\int \frac{x^3}{\sqrt{x^2+25}}\,dx$

Try (cont.)

2. Find $\int \frac{x^3}{\sqrt{x^2+25}}\,dx$

Solution

$$\text{Let } u = x^2+25 \text{ and } du = 2x\,dx$$

$$\implies x^2 = u-25 \text{and } \frac{du}{2} = x\,dx$$

$$\begin{aligned}
\implies \int \frac{x^3}{\sqrt{x^2+25}}\,dx &= \int \frac{u-25}{\sqrt{u}}\cdot\frac{du}{2} \\
&= \frac{1}{2}\int \left(u^{\frac{1}{2}} - 25u^{-\frac{1}{2}}\right) du \\
&= \frac{1}{2}\left(\frac{2}{3}u^{\frac{3}{2}} - 50u^{\frac{1}{2}}\right) + C \\
&= \frac{1}{3}u^{\frac{1}{2}}(u-75) + C = \frac{1}{3}\sqrt{x^2+25}(x^2-50) + C
\end{aligned}$$

Try (cont.)

3. Find $\int (2x^3 + x)(x^4 + x^2 + 1)^{49}\, dx$

Try (cont.)

3. Find $\int (2x^3+x)(x^4+x^2+1)^{49}\,dx$

Solution

Let $u = x^4+x^2+1$ and $du = (4x^3+2)\,dx = 2(2x^3+x)\,dx$

$$\implies \frac{du}{2} = (2x^3+x)\,dx$$

$$\begin{aligned}\implies \int (2x^3+x)(x^4+x^2+1)^{49}\,dx &= \int u^{49}\,\frac{du}{2} \\ &= \frac{1}{2}\left(\frac{1}{50}\right)u^{50}+C \\ &= \frac{1}{100}u^{50}+C \\ &= \frac{1}{100}(x^4+x^2+1)^{50}+C\end{aligned}$$

Standard Integrals Of Exponential Functions

Special Case

$$\int a^x \, dx = \frac{1}{\ln a} a^x$$

Example

1. Find $\int a^x \, dx$

Example

1. Find $\int a^x \, dx$

Solution

$$\text{Let } a^x = e^{x\ln a} \quad \left(a^x = e^{x\log_e a} = e^{\log_e a^x}\right)$$

$$\text{and } u = x\ln a$$

$$\implies du = \ln a \, dx \text{ and } \frac{du}{\ln a} = dx$$

$$\implies \int a^x \, dx = \int e^{x\ln a} \, dx = \int e^u \frac{du}{\ln a}$$

$$= \frac{1}{\ln a} e^u + C$$

$$= \frac{1}{\ln a} e^{x\ln a} + C$$

$$= \frac{1}{\ln a} a^x + C$$

Example

2. Find $\int 3^{2x}\, dx$

Example

2. Find $\int 3^{2x}\,dx$

Solution

$$\text{Let } 3^{2x} = e^{2x\ln 3} \text{ and } u = 2x\ln 3$$

$$\implies du = 2\ln 3\,dx \text{ and } \frac{du}{2\ln 3} = dx$$

$$\implies \int 3^{2x}\,dx = \int e^u \frac{du}{2\ln 3} = \frac{1}{2\ln 3}\int e^u\,du$$

$$= \frac{1}{2\ln 3}e^u + C$$

$$= \frac{1}{2\ln 3}e^{2x\ln 3} + C$$

$$= \frac{1}{2\ln 3}3^{2x} + C$$

Example

1. Find $\int \frac{1}{\sqrt{a^2 - x^2}}\, dx$

Example

1. Find $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$

Solution

$$\text{Let } x = a\sin\theta \text{ and } dx = a\cos\theta\,d\theta$$

$$\implies \sin\theta = \frac{x}{a} \text{ and } \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\implies \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \int \frac{a\cos\theta}{\sqrt{a^2-a^2\sin^2\theta}}\,d\theta$$

$$= \int \frac{a\cos\theta}{\sqrt{a^2(1-\sin^2\theta)}}\,d\theta = \int \frac{a\cos\theta}{\sqrt{a^2\cos^2\theta}}\,d\theta$$

$$= \int \frac{a\cos\theta}{a\cos\theta}\,d\theta = \int d\theta$$

$$= \theta + C = \sin^{-1}\left(\frac{x}{a}\right) + C$$

Example

2. Find $\displaystyle\int \frac{1}{a^2 + x^2}\, dx$

Example

2. Find $\int \frac{1}{a^2+x^2}\,dx$

Solution

$$\text{Let } x = a\tan\theta \text{ and } dx = a\sec^2\theta\,d\theta$$

$$\implies \tan\theta = \frac{x}{a} \text{ and } \theta = \tan^{-1}\left(\frac{x}{a}\right)$$

$$\implies \int \frac{1}{a^2+x^2}\,dx = \int \frac{a\sec^2\theta}{a^2+a^2\tan^2\theta}\,d\theta$$

$$= \int \frac{a\sec^2\theta}{a^2(1+\tan^2\theta)}\,d\theta = \int \frac{a\sec^2\theta}{a^2\sec^2\theta}\,d\theta$$

$$= \frac{1}{a}\int d\theta = \frac{1}{a}\theta + C$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

Example

3. Find $\int \sqrt{16 - 9x^2}\, dx$

Example

3. Find $\int \sqrt{16-9x^2}\, dx$

Solution

$$\text{Let } \; x = \frac{4}{3}\sin\theta \;\; \text{and} \;\; dx = \frac{4}{3}\cos\theta\, d\theta$$

$$\implies \;\; \sin\theta = \frac{3x}{4} \;\; \text{and} \;\; \theta = \sin^{-1}\left(\frac{3x}{4}\right)$$

$$\text{Also} \;\; \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{3x}{4}\right)^2}$$

$$= \frac{1}{4}\sqrt{16-9x^2}$$

$$\implies \;\; \int \sqrt{16-9x^2}\, dx = \int \sqrt{16-9\left(\frac{16}{9}\sin^2\theta\right)} \cdot \frac{4}{3}\cos\theta\, d\theta$$

Example (cont.)

$$
\begin{aligned}
&= \int \sqrt{16 - 16\sin^2\theta} \,.\, \frac{4}{3}\cos\theta \, d\theta \\
&= \int 4\cos\theta \,.\, \frac{4}{3}\cos\theta \, d\theta \\
&= \frac{16}{3}\int \cos^2\theta \, d\theta \\
&= \frac{16}{3}\int \frac{1+\cos 2\theta}{2} \, d\theta = \frac{8}{3}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C \\
&= \frac{8}{3}(\theta + \sin\theta\cos\theta) + C \\
&= \frac{8}{3}\left[\sin^{-1}\left(\frac{3}{4}x\right) + \left(\frac{3x}{4}\right)\left(\frac{1}{4}\sqrt{16-9x^2}\right)\right] + C \\
&= \frac{8}{3}\sin^{-1}\left(\frac{3}{4}x\right) + \frac{1}{2}x\sqrt{16-9x^2} + C
\end{aligned}
$$

Example

4. Find $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$

Example

4. Find $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$

Solution

$$\text{Let } x = \tan\theta \text{ and } dx = \sec^2\theta\,d\theta$$

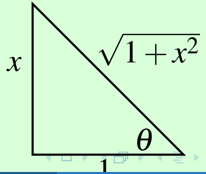
$$\begin{aligned}
\Longrightarrow \int \frac{1}{x^2\sqrt{1+x^2}}\,dx &= \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}} \\
&= \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\,d\theta \\
&= \int \frac{\sec\theta}{\tan^2\theta}\,d\theta \\
&= \int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta
\end{aligned}$$

Example (cont.)

$$\text{Now, let } u = \sin\theta \text{ and } du = \cos\theta\, d\theta$$

$$\begin{aligned}
\implies \int \frac{\cos\theta}{\sin^2\theta}\, d\theta &= \int \frac{1}{u^2}\, du \\
&= \int u^{-2}\, du = -\frac{1}{u} + C \\
&= -\frac{1}{\sin\theta} + C \\
&= -\frac{\sqrt{1+x^2}}{x} + C
\end{aligned}$$

NB:
If $\tan\theta = x$ then $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$



Example

5. Find $\displaystyle\int \frac{1}{\sqrt{2-3x^2}}\,dx$

Example

5. Find $\displaystyle\int \frac{1}{\sqrt{2-3x^2}}\,dx$

Solution

$$\text{Let } x = \sqrt{\frac{2}{3}}\sin\theta \text{ and } dx = \sqrt{\frac{2}{3}}\cos\theta\,d\theta$$

$$\implies \int \frac{1}{\sqrt{2-3x^2}}\,dx = \int \frac{\sqrt{\frac{2}{3}}\cos\theta}{\sqrt{2-2\sin^2\theta}}\,d\theta$$

$$= \int \frac{\sqrt{\frac{2}{3}}\cos\theta}{\sqrt{2}\cos\theta}\,d\theta$$

$$= \frac{1}{\sqrt{3}}\int d\theta = \frac{1}{\sqrt{3}}\theta + C$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\sqrt{\frac{2}{3}} + C$$

Try the following

i. $\int \sqrt{25 - x^2}\, dx$

ii. $\int \sqrt{9 - x^2}\, dx$

iii. $\int \sqrt{1 - 4x^2}\, dx$

iv. $\int \sqrt{9 - 4x^2}\, dx$

v. $\int \frac{1}{x^2\sqrt{1 - x^2}}\, dx$

vi. $\int \frac{1}{x^2\sqrt{a^2 + x^2}}\, dx$

vii. $\int \frac{1}{1 + \cos x}\, dx$

viii. $\int \frac{1}{1 + \sin x}\, dx$

ix. $\int \frac{1}{1 - \sin x}\, dx$

x. $\int \frac{1}{5 + 3\cos x}\, dx$

xi. $\int \frac{1}{5 - 3\cos x}\, dx$

xii. $\int x^3\sqrt{1 - x^2}\, dx$

Using The Substitution $t = \tan \frac{1}{2}x$

- An occasionally useful trigonometrical substitution is given by the following formulae, in which $\sin x$ and $\cos x$ are expressed in terms of $\tan \frac{1}{2}x$

$$\sin x = \frac{2\tan \frac{1}{2}x}{1+\tan^2 \frac{1}{2}x}; \quad \cos x = \frac{1-\tan^2 \frac{1}{2}x}{1+\tan^2 \frac{1}{2}x}$$

$$\text{Letting } \ t = \tan \frac{1}{2}x, \ \text{ we have}$$

$$dt = \frac{1}{2}\sec^2 \frac{1}{2}x\, dx \tag{1}$$

$$\implies \ \sin x = \frac{2t}{1+t^2} \ \text{ and } \ \cos x = \frac{1-t^2}{1+t^2}$$

$$\text{From (1)} \ \ dx = \frac{2}{\sec^2 \frac{1}{2}x} = \frac{2}{1+t^2}$$

Example

1. Find $\int \operatorname{cosec} x \, dx$

Example

1. Find $\displaystyle\int \operatorname{cosec} x \, dx$

Solution

$$\int \operatorname{cosec} x \, dx = \int \frac{1}{\sin x} \, dx$$

$$\text{Let } \sin x = \frac{2t}{1+t^2} \text{ and } dx = \frac{2}{1+t^2} \, dt$$

$$\implies \operatorname{cosec} x \, dx = \int \frac{1}{\sin x} \, dx = \int \frac{1}{\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt$$

$$= \int \frac{2}{2t} \, dt = \int \frac{1}{t} \, dt$$

$$= \ln|t| + C = \ln\left|\tan \frac{1}{2} x\right| + C$$

Example

2. Find $\displaystyle\int \frac{1}{5+4\cos x}\,dx$

Example

2. Find $\displaystyle\int \frac{1}{5+4\cos x}\,dx$

Solution

$$\text{Let } t = \tan\frac{1}{2}x \text{ and } dx = \frac{2}{1+t^2}\,dt \quad \Rightarrow \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\begin{aligned}
\Longrightarrow \int \frac{1}{5+4\cos x}\,dx &= \int \frac{2}{1+t^2}\,dt \div \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \\
&= \int \frac{2}{5(1+t^2)+4(1-t^2)}\,dt \\
&= \int \frac{2}{9+t^2}\,dt = \frac{2}{3}\tan^{-1}\frac{t}{3} + C \\
&= \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + C
\end{aligned}$$

Integration by Parts

- This method for integration comes from the rule for differentiating a product using the product rule:
$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$
where u and v are functions of x.
- Integrating throughout with respect to x, we have
$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$
- Rearranging, we have
$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$
- Since u and v are functions of x, we can write the relation in the form
$$\int u\,dv = uv - \int v\,du$$

Example

1. Find $\int x \sin x \, dx$

Example

1. Find $\int x \sin x \, dx$

Solution

$$\text{Let } \; u = x \; \text{ and } \; dv = \sin x \, dx$$

$$du = dx \; \text{ and } \; v = \int \sin x \, dx = -\cos x$$

$$\implies \int x \sin x \, dx = -x\cos x + \int \cos x \, dx$$

$$= -x\cos x + \sin x + C$$

NB: It is the final stage that we put in the constant of integration.

Example

2. Find $\int x \cos x \, dx$

Example

2. Find $\int x\cos x\,dx$

Solution

$$\text{Let } u = x \text{ and } dv = \cos x\,dx$$

$$du = dx \text{ and } v = \int \cos x\,dx = \sin x$$

$$\implies \int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

Example

3. Find $\displaystyle\int x^2 \sin x \, dx$

Example

3. Find $\int x^2 \sin x\, dx$

Solution

$$\begin{aligned} \text{Let } u = x^2 \ \text{ and } \ dv = \sin x\, dx \\ du = 2x\, dx \ \text{ and } \ v = \int \sin x\, dx = -\cos x \end{aligned}$$

$$\begin{aligned} \implies \int x^2 \sin x\, dx &= x(-\cos x) - \int (-\cos x) 2x\, dx \\ &= -x^2 \cos x + 2\int x \cos x\, dx \end{aligned}$$

Applying integration by parts again to the latter integral, we have

$$\int x \cos x\, dx = ?$$

Example (cont.)

$$\text{Let } u = x \text{ and } dv = \cos x\,dx$$

$$du = dx \text{ and } v = \int \cos x\,dx = \sin x$$

$$\implies \int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$= x\sin x + \cos x + C$$

$$\therefore \int x^2 \sin x\,dx = -x^2\cos x + 2(x\sin x + \cos x) + C$$

$$= -x^2\cos x + 2x\sin x + 2\cos x + C$$

Example

4. Find $\int \ln x \, dx$

Example

4. Find $\int \ln x \, dx$

Solution

Since $\ln x$ produces a simple expression when we differentiate it, we use the substitution for $\ln x$.

$$\text{Let } u = \ln x \text{ and } dv = dx$$

$$du = \frac{1}{x}\,dx \text{ and } v = \int dx = x$$

$$\implies \int \ln x \, dx = x(\ln x) - \int x \,.\, \frac{1}{x}\, dx$$

$$= x \ln x - \int dx$$

$$= x \ln x - x + C$$

Example

5. Find $\int x^2 e^{-x}\, dx$

Example

5. Find $\int x^2 e^{-x}\, dx$

Solution

$$\text{Let } u = x^2 \text{ and } dv = e^{-x}\, dx$$

$$du = 2x\, dx \text{ and } v = \int e^{-x}\, dx = -e^{-x}$$

$$\implies \int x^2 e^{-x}\, dx = -x^2 e^{-x} + 2\int x e^{-x}\, dx$$

Applying integration by parts again to the latter integral, we have

$$\int x e^{-x}\, dx = ?$$

Example (cont.)

$$\text{Let } u = x \text{ and } dv = e^{-x}\,dx$$

$$du = dx \text{ and } v = \int e^{-x}\,dx = -e^{-x}$$

$$\Longrightarrow \int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} + (-e^{-x}) + C$$

$$= -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C$$

$$\therefore \int x^2e^{-x}\,dx = -x^2e^{-x} + 2\left[-e^{-x}(x+1)\right] + C$$

$$= -e^{-x}(x^2 + 2x + 2) + C$$

Example

6. Find $\int e^x \sin x \, dx$

Example

6. Find $\int e^x \sin x\, dx$

Solution

$$\text{Let } u = \sin x \text{ and } dv = e^x\, dx$$

$$du = \cos x\, dx \text{ and } v = \int e^x\, dx = e^x$$

$$\implies \int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx \tag{1}$$

Applying integration by parts again to (1), we have

$$\text{Let } u = \cos x \text{ and } dv = e^x\, dx$$

$$du = -\sin x\, dx \text{ and } v = \int e^x\, dx = e^x$$

Example (cont.)

$$\implies \int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx \tag{2}$$

Substituting (2) into (1), we have

$$\begin{aligned}\int e^x \sin x \, dx &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right) \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + C\end{aligned}$$

$$\implies 2\int e^x \sin x \, dx = e^x(\sin x - \cos x) + C$$

$$\therefore \int e^x \sin x \, dx = \frac{1}{2} e^x(\sin x - \cos x) + C$$

Example

7. Find $\int x\cos(5x-1)\,dx$

Example

7. Find $\int x\cos(5x-1)\,dx$

Solution

$$\text{Let } u = x \text{ and } dv = \cos(5x-1)\,dx$$

$$du = dx \text{ and } v = \int \cos(5x-1)\,dx = \frac{1}{5}\sin(5x-1)$$

$$\implies \int x\cos(5x-1)\,dx = \frac{1}{5}x\sin(5x-1) - \frac{1}{5}\int \sin(5x-1)\,dx$$

$$= \frac{1}{5}x\sin(5x-1) + \frac{1}{5}\left(\frac{1}{5}\right)\cos(5x-1) + C$$

$$= \frac{1}{25}\left[5x\sin(5x-1) + \cos(5x-1)\right] + C$$

Try the following

a. $\displaystyle\int x^3 e^x \, dx$

b. $\displaystyle\int x e^{3x} \, dx$

c. $\displaystyle\int \frac{\ln x}{x^2} \, dx$

d. $\displaystyle\int x^2 \ln x \, dx$

e. $\displaystyle\int \ln(x^2+1) \, dx$

f. $\displaystyle\int 3x^4 \ln x \, dx$

g. $\displaystyle\int \frac{\ln 3x^2}{x^5} \, dx$

Try the following

a. $\int x^3 e^x \, dx$ **Ans:** $e^x(x^3 - 3x^2 + 6x - 6) + C$

b. $\int x e^{3x} \, dx$ **Ans:** $\frac{1}{9} e^{3x}(3x - 1) + C$

c. $\int \frac{\ln x}{x^2} \, dx$ **Ans:** $-\frac{1}{x}(\ln x + 1) + C$

d. $\int x^2 \ln x \, dx$ **Ans:** $\frac{1}{9} x^3(3 \ln x - 1) + C$

e. $\int \ln(x^2 + 1) \, dx$ **Ans:** $x \ln(x^2 + 1) - 2x + 2 \tan^{-1} x + C$

f. $\int 3x^4 \ln x \, dx$ **Ans:** $\frac{1}{25} x^5(15 \ln x - 3) + C$

g. $\int \frac{\ln 3x^2}{x^5} \, dx$ **Ans:** $-\frac{1}{8x^4}(2 \ln 3x^2 + 1) + C$

Try the following (cont.)

h. $\displaystyle\int x(\ln 8x)^2\,dx$

i. $\displaystyle\int 9x^8 \ln 6x^5\,dx$

Try the following (cont.)

h. $\displaystyle\int x(\ln 8x)^2\,dx$

Ans: $-\frac{1}{4}x^2\left[2(\ln 8x)^2 - 2(\ln 8x - 1)\right] + C$

i. $\displaystyle\int 9x^8 \ln 6x^5\,dx$

Ans: $\frac{1}{9}x^9(9\ln 6x^5 - 5) + C$

Inverse Trig. Functions - Example

1. Find $\int \sin^{-1} x \, dx$

Inverse Trig. Functions - Example

1. Find $\int \sin^{-1} x\, dx$

Solution

$$\text{Let } u = \sin^{-1} x \text{ and } dv = dx$$

$$du = \frac{1}{\sqrt{1-x^2}}\,dx \text{ and } v = \int dx = x$$

$$\implies \int \sin^{-1} x\, dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx \qquad (1)$$

Now, we have $\int \frac{x}{\sqrt{1-x^2}}\,dx$ to integrate

$$\text{Let } u = 1 - x^2 \text{ and } du = -2x\,dx$$

$$\implies -\frac{du}{2} = x\,dx$$

Inverse Trig. Functions - Example (cont.)

$$\Longrightarrow \int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{u}}\,.-\frac{du}{2}$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{2}\,.2u^{\frac{1}{2}} = -\sqrt{1-x^2} \qquad (2)$$

Substitute (2) into (1), we have

$$\Longrightarrow \int \sin^{-1}x\,dx = x\sin^{-1}x - \left(-\sqrt{1-x^2}\right) + C$$

$$= x\sin^{-1}x + \sqrt{1-x^2} + C$$

Inverse Trig. Functions - Example

2. Find $\int \cos^{-1} x \, dx$

Inverse Trig. Functions - Example

2. Find $\int \cos^{-1} x\, dx$

Solution

$$\text{Let } u = \cos^{-1} x \text{ and } dv = dx$$

$$du = -\frac{1}{\sqrt{1-x^2}}\, dx \text{ and } v = \int dx = x$$

$$\implies \int \cos^{-1} x\, dx = x\cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}}\, dx \qquad (1)$$

Now, we have $\int \frac{x}{\sqrt{1-x^2}}\, dx$ to integrate

$$\text{Let } u = 1 - x^2 \text{ and } du = -2x\, dx$$

$$\implies -\frac{du}{2} = x\, dx$$

Inverse Trig. Functions - Example (cont.)

$$\implies \int \frac{x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{u}} \cdot -\frac{du}{2}$$

$$= -\frac{1}{2}\int u^{-\frac{1}{2}}\,du$$

$$= -\frac{1}{2}\,.\,2u^{\frac{1}{2}} = -\sqrt{1-x^2} \qquad (2)$$

Substitute (2) into (1), we have

$$\implies \int \cos^{-1} x\,dx = x\cos^{-1} x + \left(-\sqrt{1-x^2}\right) + C$$

$$= x\cos^{-1} x - \sqrt{1-x^2} + C$$

Inverse Trig. Functions - Example

3. Find $\displaystyle\int \tan^{-1} x \, dx$

Inverse Trig. Functions - Example

3. Find $\int \tan^{-1} x\, dx$

Solution

$$\text{Let } u = \tan^{-1} x \text{ and } dv = dx$$

$$\implies du = \frac{1}{1+x^2}\, dx \text{ and } v = \int dx = x$$

$$\begin{aligned}
\implies \int \tan^{-1} x\, dx &= x \tan^{-1} x - \int \frac{x}{1+x^2}\, dx \\
&= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2}\, dx \\
&= x \tan^{-1} x - \frac{1}{2} \ln|1+x^2| + C
\end{aligned}$$

Inverse Trig. Functions - Example

4. Find $\int x^2 \tan^{-1} x \, dx$

Inverse Trig. Functions - Example

4. Find $\int x^2 \tan^{-1} x \, dx$

Solution

Let $u = \tan^{-1} x$ and $dv = x^2\, dx$

$$\implies du = \frac{1}{1+x^2}\, dx \text{ and } v = \int x^2\, dx = \frac{1}{3}x^3$$

$$\begin{aligned}
\implies \int x^2 \tan^{-1} x\, dx &= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2}\, dx \\
&= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\int \left(x - \frac{x}{1+x^2} \right) dx \\
&= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{3}\left[\frac{1}{2}x^2 - \frac{1}{2}\ln|1+x^2| \right] \\
&= \frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}\ln|1+x^2| + C
\end{aligned}$$

Inverse Trig. Functions - Example

5. Find $\int \left(\sin^{-1} x\right)^2 \, dx$

Inverse Trig. Functions - Example

5. Find $\int \left(\sin^{-1} x\right)^2 \, dx$

Solution

$$\text{Let } \; u = \sin^{-1} x \;\text{ and }\; du = \frac{1}{\sqrt{1-x^2}}\, dx$$

$$\Longrightarrow \quad x = \sin u \;\text{ and }\; \sqrt{1-x^2}\, du = dx$$

$$\begin{aligned} \Longrightarrow \quad \int \left(\sin^{-1} x\right)^2 \, dx &= \int u^2 \sqrt{1-x^2}\, du \\ &= \int u^2 \sqrt{1-\sin^2 u}\, du \\ &= \int u^2 \cos u \, du \qquad (1) \end{aligned}$$

Inverse Trig. Functions - Example (cont.)

Applying integration by parts to (1), we have

$$\int u^2 \cos u \, du = (u^2 - 2)\sin u + 2u\cos u + C$$
$$= \left[(\sin^{-1} x)^2 - 2\right] x + 2(\sin^{-1} x)\sqrt{1-x^2} + C$$
$$\therefore \int (\sin^{-1} x)^2 \, dx = \left[(\sin^{-1} x)^2 - 2\right] x + 2(\sin^{-1} x)\sqrt{1-x^2} + C$$

Exercise

i. $\int x\sin x\,dx$

ii. $\int x\sin 3x\,dx$

iii. $\int x^2\cos x\,dx$

iv. $\int x\ln x\,dx$

v. $\int xe^{-ax}\,dx$

vi. $\int \tan^{-1}x\,dx$

vii. $\int x\tan^{-1}x\,dx$

viii. $\int \sin^{-1}x\,dx$

Try the following

a. $\int e^x \cos x \, dx$

b. $\int e^{ax} \cos bx \, dx$

c. $\int e^{ax} \sin bx \, dx$

d.
$$\int e^{ax} \cos(bx+c) \, dx$$

e. $\int x^2 e^{3x} \, dx$

Try the following

a. $\int e^x \cos x \, dx$ $\quad \frac{1}{2}e^x(\sin x + \cos x) + C$

b. $\int e^{ax} \cos bx \, dx$ $\quad \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

c. $\int e^{ax} \sin bx \, dx$ $\quad \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

d. $\int e^{ax} \cos(bx+c) \, dx$ $\quad \frac{e^{ax}}{a^2+b^2}[a\cos(bx+c) + b\sin(bx+c)] + D$

e. $\int x^2 e^{3x} \, dx$ $\quad \frac{1}{27}e^{3x}\left[9x^2 - 6x + 2\right] + C$

Integrating Rational Functions - Example

1. Find $\displaystyle\int \frac{1}{x^2 - 9}\, dx$

Integrating Rational Functions - Example

1. Find $\displaystyle\int \frac{1}{x^2-9}\,dx$

Solution

$$\frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$$
$$\Longrightarrow \quad 1 = A(x+3) + B(x-3)$$

$$\text{Let } x = 3, \text{ then } 1 = 6A \Rightarrow A = \frac{1}{6}$$

$$\text{Let } x = -3, \text{ then } 1 = -6B \Rightarrow B = -\frac{1}{6}$$

$$\text{So } \frac{1}{x^2-9} = \frac{1}{6}\left[\frac{1}{x-3}\right] - \frac{1}{6}\left[\frac{1}{x+3}\right]$$

Example (cont.)

$$\text{Hence} \quad \int \frac{1}{x^2-9}\,dx = \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + C$$
$$= \frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C$$

Example

2. Find $\displaystyle\int \frac{x}{(x+2)(x+3)}\,dx$

Example

2. Find $\displaystyle\int \frac{x}{(x+2)(x+3)}\,dx$

Solution

$$\frac{x}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3}$$

$$\implies \quad x = A(x+3) + B(x+2)$$

$$\text{Let } \; x = -3, \;\; \text{then} \;\; -3 = -B \;\; \Rightarrow \;\; B = 3$$

$$\text{Let } \; x = -2, \;\; \text{then} \;\; -2 = A \;\; \Rightarrow \;\; A = -2$$

$$\text{So} \;\; \frac{x}{(x+2)(x+3)} = -\frac{2}{x+2} + \frac{3}{x+3}$$

Example (cont.)

$$\text{Hence} \int \frac{x}{(x+2)(x+3)}\,dx = -2\ln|x+2| + 3\ln|x+3| + C$$
$$= \frac{1}{6}\ln\left|\frac{(x+3)^3}{(x+2)^2}\right| + C$$

Example

3. Find $\displaystyle\int \frac{x^4 - 4x^2 + x + 1}{x^2 - 4}\, dx$

Example

3. Find $\displaystyle\int \frac{x^4-4x^2+x+1}{x^2-4}\,dx$

Solution

$$\frac{x^4-4x^2+x+1}{x^2-4} = x^2 + \frac{x+1}{x^2-4}$$

$$\implies \int \frac{x^4-4x^2+x+1}{x^2-4}\,dx = \int x^2\,dx + \int \frac{x+1}{x^2-4}\,dx$$

$$= \frac{1}{3}x^3 + \int \frac{x+1}{x^2-4}\,dx$$

$$\text{But } \frac{x+1}{x^2-4} = \frac{x+1}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}$$

$$\implies x+1 = A(x-2)+B(x+2)$$

$$\text{If } x=2 \text{ then } 3 = 4B \Rightarrow B = \frac{3}{4}$$

Example (cont.)

$$\text{If } x = -2 \text{ then } -1 = -4A \Rightarrow A = \frac{1}{4}$$

$$\text{Thus } \frac{x^4 - 4x^2 + x + 1}{x^2 - 4} = \frac{1}{4(x+2)} + \frac{3}{4(x-2)}$$

$$\text{Hence } \int \frac{x}{(x+2)(x+3)}\, dx = \frac{1}{4}\ln|x+2| + \frac{3}{4}\ln|x-2| + C$$

$$= \frac{1}{4}\ln\left|\frac{x+2}{(x-2)^3}\right| + C$$

Example

4. Find $\displaystyle\int \frac{x^3+1}{x(x+3)(x+2)(x-1)}\,dx$

Example

4. Find $\displaystyle\int \frac{x^3+1}{x(x+3)(x+2)(x-1)}\,dx$

Solution

$$\frac{x^3+1}{x(x+3)(x+2)(x-1)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{x+2} + \frac{D}{x-1}$$

$$\implies x^3+1 = A(x+3)(x+2)(x-1) + Bx(x+2)(x-1) + Cx(x+3)(x-1) + Dx(x+3)(x+2)$$

$$\text{Let } x=0 \;\Rightarrow\; 1=-6A \;\Rightarrow\; A=-\frac{1}{6}$$

$$\text{Let } x=-3 \;\Rightarrow\; -26=-12B \;\Rightarrow\; B=\frac{13}{6}$$

$$\text{Let } x=-2 \;\Rightarrow\; -7=6C \;\Rightarrow\; C=-\frac{7}{6}$$

$$\text{Let } x=1 \;\Rightarrow\; 2=12D \;\Rightarrow\; D=\frac{1}{6}$$

Example (cont.)

$$\implies \frac{x^3+1}{x(x+3)(x+2)(x-1)} = -\frac{1}{6x} + \frac{13}{6(x+3)} - \frac{7}{6(x+2)} + \frac{1}{6(x-1)}$$

Hence
$$\int \frac{x^3+1}{x(x+3)(x+2)(x-1)}\,dx = -\frac{1}{6}\ln|x| + \frac{13}{6}\ln|x+3| - \frac{7}{6}\ln|x+2| + \frac{1}{6}\ln|x-1| + C$$
$$= \frac{1}{6}(\ln - \ln|x| + 13\ln|x+3| - 7\ln|x+2| + \ln|x-1|) + C$$
$$= \frac{1}{6}\ln\left|\frac{(x+3)^{13}(x-1)}{x(x+2)^7}\right| + C$$

Example

5. Find $\displaystyle\int \frac{x}{x^4 - 13x^2 + 36}\,dx$

Example

5. Find $\displaystyle\int \frac{x}{x^4 - 13x^2 + 36}\, dx$

Solution

$$\begin{aligned} \frac{x}{x^4 - 13x^2 + 36} &= (x^2 - 9)(x^2 - 4) \\ &= (x-3)(x+3)(x+2)(x-2) \end{aligned}$$

Let $\displaystyle \frac{x}{(x-3)(x+3)(x+2)(x-2)} = \frac{A}{x-3} + \frac{B}{x+3} + \frac{C}{x+2} + \frac{D}{x-2}$

$$\Longrightarrow \quad x = A(x+3)(x+2)(x-2) + B(x-3)(x+2)(x-2)$$

$$+ C(x-3)(x+3)(x-2) + D(x-3)(x+3)(x+2)$$

Let $x = 3 \;\Rightarrow\; 3 = 30A \;\Rightarrow\; A = \dfrac{1}{10}$

Let $x = -3 \;\Rightarrow\; -3 = -30B \;\Rightarrow\; B = \dfrac{1}{10}$

Example (cont.)

$$\text{Let } x=-2 \;\Rightarrow\; -2=20C \;\Rightarrow\; C=-\frac{1}{10}$$

$$\text{Let } x=1 \;\Rightarrow\; 2=-20D \;\Rightarrow\; D=-\frac{1}{10}$$

$$\Longrightarrow \frac{x}{x^4-13x^2+36}=\frac{1}{10(x-3)}+\frac{1}{10(x+3)}-\frac{1}{10(x+2)}-\frac{1}{10(x-2)}$$

Example (cont.)

Hence $$\int \frac{x}{x^4 - 13x^2 + 36}\,dx = \frac{1}{10}\ln|x-3| + \frac{1}{10}\ln|x+3| - \frac{1}{10}\ln|x+2| - \frac{1}{10}\ln|x-2| + C$$
$$= \frac{1}{10}\left(\ln|x-3| + \ln|x+3| - \ln|x+2| - \ln|x-2|\right) + C$$
$$= \frac{1}{10}\ln\left|\frac{(x+3)(x-3)}{(x+2)(x-2)}\right| + C$$

Example

6. Find $\displaystyle\int \frac{x-5}{x^2(x+1)}\,dx$

Example

6. Find $\int \frac{x-5}{x^2(x+1)}\,dx$

Solution

$$\frac{x-5}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$
$$\Longrightarrow \quad x-5 = Ax(x+1) + B(x+1) + Cx^2$$
$$\text{Let } \; x = 0 \quad \Rightarrow \quad -5 = B$$
$$\text{Let } \; x = -1 \quad \Rightarrow \quad -6 = C$$

To find A, compare coefficients of x^2 on both sides of the equation:

$$0 = A + C \quad \Rightarrow \quad A = -C = 6$$

Example (cont.)

$$\implies \frac{x-5}{x^2(x+1)} = \frac{6}{x} - \frac{5}{x^2} - \frac{6}{x+1}$$

Hence
$$\int \frac{x-5}{x^2(x+1)}\,dx = \int \left[\frac{6}{x} - \frac{5}{x^2} - \frac{6}{x+1}\right] dx$$
$$= 6\ln|x| + \frac{5}{x} - 6\ln|x+1| + C$$
$$= 6\ln\left|\frac{x}{x+1}\right| + \frac{5}{x} + C$$

Example

7. Find $\displaystyle\int \frac{2x}{(x-2)^2(x+2)}\,dx$

Example

7. Find $\displaystyle\int \frac{2x}{(x-2)^2(x+2)}\,dx$

Solution

$$\frac{2x}{(x-2)^2(x+2)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{x+2}$$

$$\implies 2x = A(x-2)(x+2) + B(x+2) + C(x-2)^2$$

Let $x = 2 \;\Rightarrow\; 4B = B \;\Rightarrow B = 1$

Let $x = -2 \;\Rightarrow\; -4 = 16C \;\Rightarrow C = -\frac{1}{4}$

To find A, compare coefficients of x^2 on both sides of the equation:

$$0 = A + C \;\Rightarrow\; A = -C = \frac{1}{4}$$

Example (cont.)

$$\implies \frac{2x}{(x-2)^2(x+2)} = \frac{1}{4(x-2)} + \frac{1}{(x-2)^2} - \frac{1}{4(x+2)}$$

$$\text{Hence } \int \frac{2x}{(x-2)^2(x+2)}\,dx = \int \left[\frac{1}{4(x-2)} + \frac{1}{(x-2)^2} - \frac{1}{4(x+2)}\right] dx$$
$$= \frac{1}{4}\ln|x-2| - \frac{1}{x-2} - \frac{1}{4}\ln|x+2| + C$$
$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| - \frac{1}{x-2} + C$$

Example

8. Find $\displaystyle\int \frac{x+4}{x^3+6x^2+9x}\,dx$

Example

8. Find $\displaystyle\int \frac{x+4}{x^3+6x^2+9x}\,dx$

Solution

$$\frac{x+4}{x^3+6x^2+9x} = \frac{x+4}{x(x^2+6x+9)} = \frac{A}{x} + \frac{B}{x+3} + \frac{C}{(x+3)^2}$$

$$\implies \quad x+4 = A(x+3)^2 + Bx(x+3) + Cx$$

$$\text{Let } \; x=0 \quad \Rightarrow \quad 4 = 9A \quad \Rightarrow A = \frac{4}{9}$$

$$\text{Let } \; x=-3 \quad \Rightarrow \quad 1 = -3C \quad \Rightarrow C = -\frac{1}{3}$$

To find B, compare coefficients of x^2 on both sides of the equation:

$$0 = A + B \quad \Rightarrow \quad B = -A = -\frac{4}{9}$$

Example (cont.)

$$\implies \frac{x+4}{x^3+6x^2+9x} = \frac{4}{9x} - \frac{4}{9(x+3)} - \frac{1}{3(x+3)^2}$$

Hence
$$\int \frac{x+4}{x^3+6x^2+9x}\,dx = \int \left[\frac{4}{9x} - \frac{4}{9(x+3)} - \frac{1}{3(x+3)^2}\right] dx$$
$$= \frac{4}{9}\ln|x| - \frac{4}{9}\ln|x+3| + \frac{1}{3(x+3)} + C$$
$$= \frac{4}{9}\ln\left|\frac{x}{x+3}\right| + \frac{1}{3(x+3)} + C$$

Note

- When the denominator of the algebraic fraction is the sum of two squares, we cannot factorize it and had to resort to a different strategy for integration.

- The method is to use two standard forms of integration as shown below:

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{\tan^{-1}\frac{x}{a}} + C \qquad \text{and}$$

$$\int \frac{x}{a^2+x^2}\,dx = \frac{1}{2}\ln(a^2+x^2) + C$$

Example

1. Find $\displaystyle\int \frac{2x+3}{4+x^2}\,dx$

Example

1. Find $\int \frac{2x+3}{4+x^2}\,dx$

Solution

Here we could split the numerator into two parts or we could proceed with the method of partial fractions. Thus

$$\int \frac{2x+3}{4+x^2}\,dx = \int \frac{2x}{4+x^2}\,dx + \frac{3}{4+x^2}\,dx$$

Using standard forms, we have

$$\int \frac{2x}{4+x^2}\,dx = \ln|x^2+4| + C_1$$

$$\text{and} \int \frac{3}{4+x^2}\,dx = 3\ln\frac{1}{4+x^2}\,dx = \frac{3}{2}\tan^{-1}\frac{x}{2} + C_2$$

Example (cont.)

$$\implies \int \frac{2x+3}{4+x^2}\,dx = \ln|x^2+4| + \frac{3}{2}\tan^{-1}\frac{x}{2} + C$$

Example

2. Find $\displaystyle\int \frac{8-x}{x^2+x-1}\,dx$

Example

2. Find $\displaystyle\int \frac{8-x}{x^2+x-1}\,dx$

Solution

$$\begin{aligned}
\int \frac{8-x}{x^2+x-1}\,dx &= \int \frac{(8-x)}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx \\
&= \int \frac{(-x-\frac{1}{2})+\frac{17}{2}}{(x+\frac{1}{2})^2+\frac{3}{4}} \\
&= -\int \frac{x+\frac{1}{2}}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx + \frac{17}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx \\
&= -\frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)\right| + \frac{17}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{(x+\frac{1}{2})}{\sqrt{3}/2}
\end{aligned}$$

Note

Where the integral is of the form

$$\int \frac{Ax+B}{\sqrt{px^2+qx+r}}\,dx$$

Example:

$$\int \frac{x+1}{\sqrt{2x^2+x-3}}\,dx = ?$$

Here we rewrite $2x^2+x-3 = 2\left(x+\frac{1}{4}\right)^2 - \frac{25}{8}$

$$= \frac{25}{8}\left[\frac{2(8)}{25}\left(x+\frac{1}{4}\right)^2 - 1\right]$$

and let $\sqrt{2}\left(x+\frac{1}{4}\right) = \frac{5}{2\sqrt{2}}u$

cont.

$$\Longrightarrow \quad x = \frac{5}{4}u - \frac{1}{4} \quad \Rightarrow dx = \frac{5}{4}du$$

$$\Longrightarrow \int \frac{x+1}{\sqrt{2x^2+x-3}}\,dx = \frac{\left[\left(\frac{5}{4}u - \frac{1}{4}\right)+1\right]}{\sqrt{\frac{25}{8}u^2 - \frac{25}{8}}} \times \frac{5}{4}du$$

$$= \frac{5}{4}.\frac{5}{4}\sqrt{\frac{8}{25}}\int \frac{u}{\sqrt{u^2-1}}\,du +$$

$$\frac{3}{4}.\frac{5}{4}.\sqrt{\frac{8}{25}}\int \frac{1}{\sqrt{u^2-1}}\,du$$

$$= \frac{5}{4\sqrt{2}}\sqrt{u^2-1} + \frac{3}{4\sqrt{2}}\ln\left|u+\sqrt{u^2-1}\right| + C$$

cont.

After re-substituting for u and some simplification, we find that

$$\int \frac{x+1}{\sqrt{2x^2+x-3}}\,dx = \frac{1}{2}\sqrt{2x^2+x-3}+$$
$$\frac{3}{4\sqrt{2}}\ln\left|\frac{4x+1}{5}+\sqrt{\frac{8(2x^2+x-3)}{25}}\right|$$

Try the following

a.

$$\int \frac{x^4}{x^3 - 2x^2 - 7x - 4}\,dx$$

b. $$\int \frac{1}{x(x^2+5)}\,dx$$

c.

$$\int \frac{1}{(x^2+1)(x^2+4)}\,dx$$

d. $$\int \frac{1-9x^2}{x(x^2+9)}\,dx$$

Try the following

a. $$\int \frac{x^4}{x^3-2x^2-7x-4}\,dx$$ $$\frac{1}{2}x^2+2x+\frac{256}{25}\ln|x-4|+\frac{19}{25}\ln|x+1|+\frac{1}{5(x+1)}+C$$

b. $$\int \frac{1}{x(x^2+5)}\,dx$$ $$\frac{1}{5}\ln\left|\frac{x}{\sqrt{x^2+5}}\right|+C$$

c. $$\int \frac{1}{(x^2+1)(x^2+4)}\,dx$$ $$\frac{1}{3}\tan^{-1}x-\frac{1}{6}\tan^{-1}\frac{x}{2}+C$$

d. $$\int \frac{1-9x^2}{x(x^2+9)}\,dx$$ $$\frac{1}{9}\ln|x|-\frac{41}{9}\ln|x^2+9|+C$$

Try the following

a. $$\int \frac{1}{x(x^2+1)^2}\,dx$$

b.
$$\int \frac{x^2}{(x-1)(x^2+4)^2}\,dx$$

Try the following

a. $\displaystyle\int \frac{1}{x(x^2+1)^2}\,dx$ $\qquad \ln|x| - \frac{3}{2}\tan^{-1}x - \frac{1}{2}\left(\frac{x}{x^2+1}\right) + C$

b.

$\displaystyle\int \frac{x^2}{(x-1)(x^2+4)^2}\,dx$ $\qquad \frac{1}{25}\ln\frac{x-1}{\sqrt{x^2+4}} + \frac{3}{100}\tan^{-1}\frac{x}{2} + \frac{x-4}{10(x^2+4)} + C$

Exercise A

i. $\displaystyle\int \frac{x-1}{x^3+2x^2-x-2}\,dx$

ii. $\displaystyle\int \frac{x^2+2}{x(x^2+5x+6)}\,dx$

iii. $\displaystyle\int \frac{1}{x^2-1}\,dx$

iv. $\displaystyle\int \frac{x^2}{x^2-4}\,dx$

v. $\displaystyle\int \frac{2x-1}{(x+2)^2}\,dx$

vi. $\displaystyle\int \frac{x^2-2}{x^2-x-12}\,dx$

vii. $\displaystyle\int \frac{1+x}{(1-x)^2}\,dx$

viii. $\displaystyle\int \frac{2x+1}{(2x+3)^2}\,dx$

ix. $\displaystyle\int \frac{x-1}{(x+1)(x^2+1)}\,dx$

x. $\displaystyle\int \frac{x^2}{(x-1)(x^2+4x+5)}\,dx$

xi. $\displaystyle\int \frac{8x+1}{2x^2-9x-35}\,dx$

xii. $\displaystyle\int \frac{7x-8}{4x^2+3x-1}\,dx$

Exercise B

i. $\displaystyle\int \frac{1}{x^2+6x+17}\,dx$

ii. $\displaystyle\int \frac{1}{x^2+4x+6}\,dx$

iii. $\displaystyle\int \frac{1}{2x^2+2x+7}\,dx$

iv. $\displaystyle\int \frac{2x+5}{x^2+4x+5}\,dx$

v. $\displaystyle\int \frac{1}{x^3+1}\,dx$

vi. $\displaystyle\int \frac{(x-1)^2}{x^2+2x+2}\,dx$

vii. $\displaystyle\int \frac{1}{\sqrt{x^2+2x+4}}\,dx$

viii. $\displaystyle\int \frac{1}{x^2-4x+2}\,dx$

ix. $\displaystyle\int \frac{x}{\sqrt{x^2-x+1}}\,dx$

x. $\displaystyle\int \frac{x+2}{\sqrt{x^2+2x-1}}\,dx$

xi. $\displaystyle\int \frac{1}{\sqrt{1-x-x^2}}\,dx$

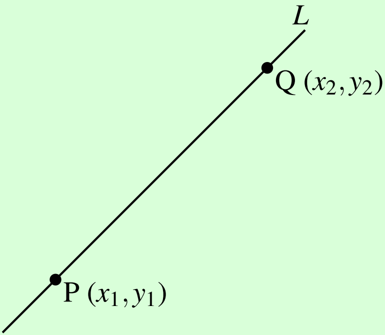
xii. $\displaystyle\int \frac{2x-3}{\sqrt{x^2-2x+5}}\,dx$

Gradient Of Straight Lines

- Given two points $P(x,y)$ and $Q(x,y)$, the gradient of the line $\overline{PQ}$, that passess through P and Q is given by

Gradient Of Straight Lines

- Given two points $P(x,y)$ and $Q(x,y)$, the gradient of the line $\overline{PQ}$, that passess through P and Q is given by



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$\text{or} \quad m = \frac{y_1 - y_2}{x_1 - x_2}$$

Equation Of A Straight Line

A straight line has a constant gradient. Hence, if $P(x,y)$ is any general point on the line, L. Then, the gradient of L is calculated using $P(x,y)$ and either of $A(x_1,y_1)$ or $B(x_2,y_2)$. Thus

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1} \tag{1}$$

$$\implies \quad y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1) \tag{2}$$

Thus, (2) defines the equation of a line.

Example

Find the equation of a line joining $A(3, -5)$ and $B(1, 1)$.

Example

Find the equation of a line joining $A(3,-5)$ and $B(1,1)$.

Solution

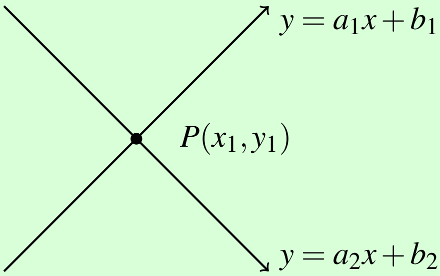
Let $P(x,y)$ be any point on the line. Then

$$\begin{aligned} \frac{y-(-5)}{x-3} &= \frac{-5-1}{3-1} \\ \implies \quad y+5 &= \frac{-5-1}{3-1}(x-3) \\ y+5 &= -3(x-3) \\ \therefore \quad 3x+y-4 &= 0 \end{aligned}$$

Intersection of Two Straight Lines

$$\text{Given } \; y = a_1 x + b_1 \tag{3}$$
$$\text{and } \; y = a_2 x + b_2 \tag{4}$$

The point of intersection of these two lines is given by solving the two equations simultaneously.



Example

Find the point of intersection of the lines

$$\begin{aligned} 2y - 3x + 4 &= 0 \quad \text{and} \\ y + 2x - 3 &= 0 \end{aligned}$$

Example

Find the point of intersection of the lines

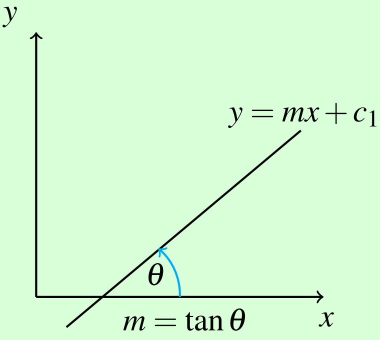
$$2y - 3x + 4 = 0 \quad \text{and}$$
$$y + 2x - 3 = 0$$

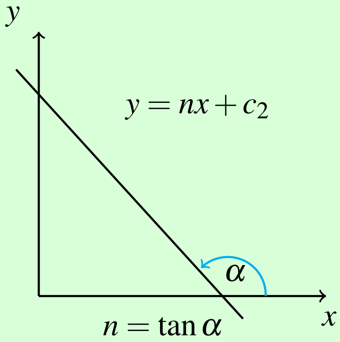
Solution

Solving the two equations simultaneously, we have

$$x = \frac{10}{7} \quad \text{and}$$
$$y = \frac{1}{7}$$

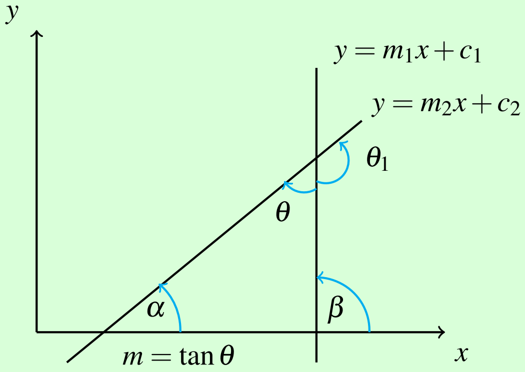
Relationship of The Gradient m of a Line To The Angle Made by The Line With The x-Axis





Angle Between Two Lines

- Suppose that θ is the angle between the two lines whose gradients are m_1 and m_2 then



Angle Between Two Lines (cont.)

Now $m_1 = \tan\beta$ and $m_2 = \tan\alpha$

But $\beta = \theta + \alpha$ (sum of opposite interior angles = exterior angle)

$\implies \theta = \beta - \alpha$

$$\text{Thus} \quad \tan(\beta - \alpha) = \tan\theta$$
$$\implies \tan\theta = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$
$$= \frac{m_1 - m_2}{1 - m_1 m_2}$$

Angle Between Two Lines (cont.)

NB:

$$
\begin{aligned}
\theta + \theta_1 &= 180^\circ \quad \Rightarrow \quad \theta_1 = 180^\circ - \theta \\
\Longrightarrow \quad \tan\theta_1 &= \tan(180^\circ - \theta) \\
&= -\tan\theta \\
&= -\left(\frac{m_1 - m_2}{1 - m_1 m_2}\right)
\end{aligned}
$$

Thus, in general if θ is the angle between any two given lines, then

$$
\tan\theta = \pm\left(\frac{m_1 - m_2}{1 - m_1 m_2}\right)
$$

Example

Find the acute angle between the lines $3y + 2x = 6$ and $x - y = 5$

Example

Find the acute angle between the lines $3y+2x=6$ and $x-y=5$

Solution

$$y=\frac{6-2x}{3}=2-\frac{2}{3}x \;\Rightarrow\; m_1=-\frac{2}{3} \;\text{ and}$$

$$y=x-5 \;\Rightarrow\; m_2=1$$

If θ is the angle between the two lines, then

$$\begin{aligned}\tan\theta &= \pm\left(\frac{m_1-m_2}{1-m_1m_2}\right)\\ &= \pm\left(\frac{-\frac{2}{3}-1}{1+(-\frac{2}{3})(1)}\right)\\ &= \pm(-5)\\ \therefore\quad \tan\theta &= +5 = 78.7^\circ \approx 79^\circ \;\;\text{(acute angle)}\end{aligned}$$

Parallel Lines

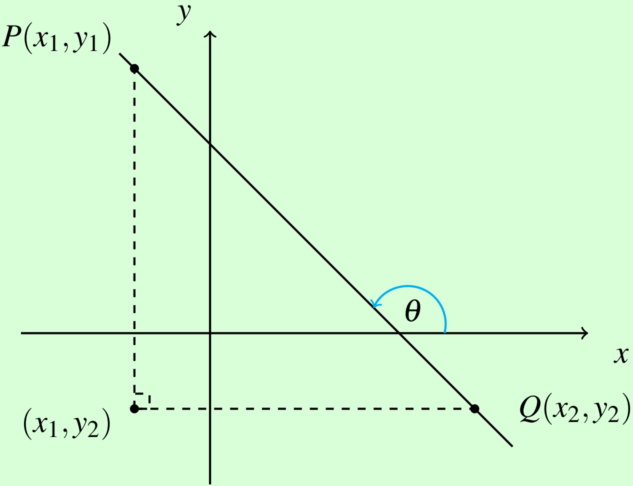
If two lines are parallel, then $\theta = 0$ and that

$$\tan\theta = \frac{m_1 - m_2}{1 - m_1 m_2} = 0$$

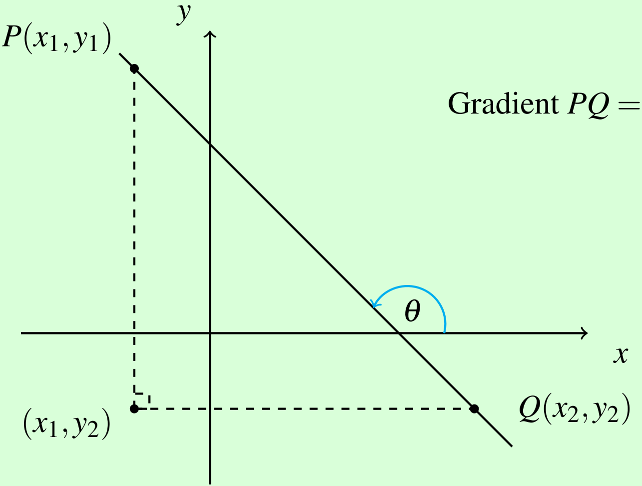
$$\implies \; m_1 - m_2 = 0$$

$$\text{Thus } \; m_1 = m_2 \; \text{ (i.e. two parallel lines have equal gradients)}$$

The Gradient And Inclination of a Straight Line



The Gradient And Inclination of a Straight Line



$$\text{Gradient } PQ = \frac{\text{change in y}}{\text{change in x}} = \frac{\Delta y}{\Delta x}$$

The Gradient And Inclination of a Straight Line (cont.)

- If P is the point (x_1, y_1) and Q is the point (x_2, y_2), then

$$\text{Grad}\, PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = m$$

  where $x_1 \neq x_2$

- **Definition:**
  We define the inclination of PQ to be the angle θ $(0^\circ < \theta < 180^\circ)$ made by the line with positive direction of the $x-$axis, measured in an anticlockwise direction.

$$\text{Inclination of } PQ = \theta$$

$$\text{where} \quad \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = m$$

$$\theta \neq 90^\circ$$

Example

Given the points **(a)** $(-3,5)$ and $(2,0)$; **(b)** $(-2,-3)$ and $(3,-1)$, determine

i. the gradient and
ii. the inclination of the line joining the points

Example

Given the points **(a)** $(-3,5)$ and $(2,0)$; **(b)** $(-2,-3)$ and $(3,-1)$, determine

i. the gradient and
ii. the inclination of the line joining the points

Solution

(a)

i. Gradient $= \dfrac{5-0}{-3-3} = -1$

ii.

$$\tan\theta = -1$$
$$\theta = 180^\circ - 45^\circ$$
$$\therefore \quad \text{inclination} = 135^\circ$$

Example (cont.)

Solution

(b)

i. Gradient $= \dfrac{-3-(-1)}{-2-3} = \dfrac{2}{5} = 0.4$

ii.

$$\begin{aligned} \tan\theta &= 0.4 \\ \theta &= \tan^{-1}(0.4) = 21.8^\circ \\ \therefore \quad \text{Angle of inclination} &= 21.8^\circ \end{aligned}$$

Perpendicular Lines

- If two lines are perpendicular, then

$$\theta = \frac{\pi}{2}$$

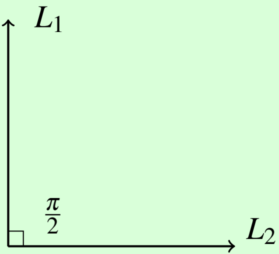
$$\text{Thus} \quad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \infty$$

$$\implies \quad \tan\left(\frac{\pi}{2}\right) = \frac{m_1 - m_2}{1 + m_1 m_2} = \infty$$

$$\text{Now} \quad \frac{1 + m_1 m_2}{m_1 - m_2} = \frac{1}{\infty}$$

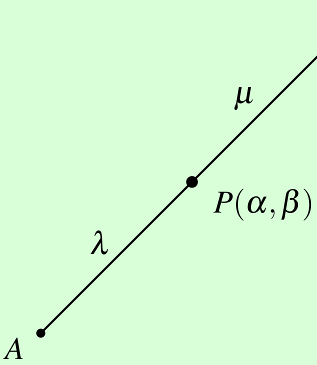
$$\implies \quad 1 + m_1 m_2 = 0$$

$$\implies \quad m_1 m_2 = -1$$



Thus, the product of the gradients of two perpendicular lines $(m_1 . m_2) = -1$

Midpoint



$$\alpha = \frac{\mu x_1 + \lambda x_2}{\mu + \lambda} = \frac{x_1 + x_2}{2}$$

$$\beta = \frac{\mu y_1 + \lambda y_2}{\mu + \lambda} = \frac{y_1 + y_2}{2}$$

$$\text{Midpoint } P(\alpha, \beta) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$$

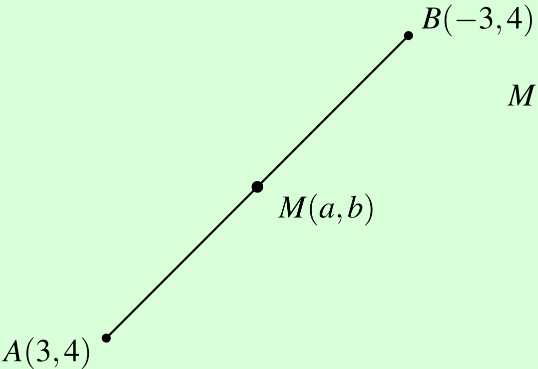
Example

Find the coordinates of the point $M(a,b)$ that divide the line $A(3,4)$ and $B(-3,4)$ into two equal parts.

Example

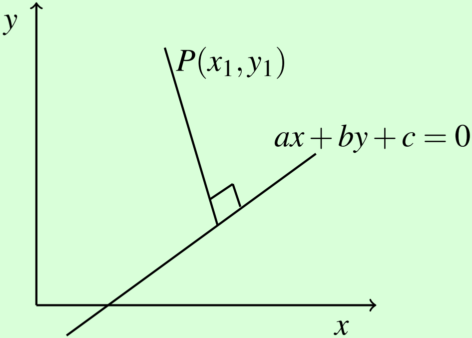
Find the coordinates of the point $M(a,b)$ that divide the line $A(3,4)$ and $B(-3,4)$ into two equal parts.

Solution



$$M(a,b) = \left[\frac{3+(-3)}{2}, \frac{6+4}{2}\right]$$
$$= \left(\frac{0}{2}, \frac{10}{2}\right)$$
$$= (0,5)$$

Perpendicular Distance of $P(x,y)$ From The Line $ax+by+c=0$



$$d=\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$$

Example

Find the distance of the point (5,6) from the line $3x-4y-5=0$.

Example

Find the distance of the point (5,6) from the line $3x - 4y - 5 = 0$.

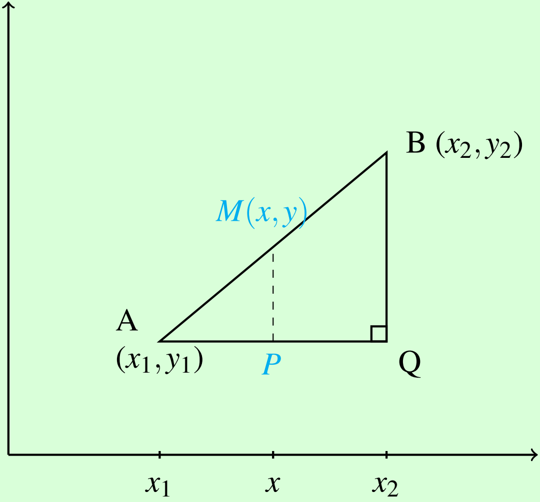
Solution

$$d = \left| \frac{3(5) + (-4)(6) - 5}{\sqrt{3^2 + (-4)^2}} \right|$$

$$= \left| \frac{15 - 24 - 5}{\sqrt{25}} \right| = \frac{14}{5} \text{ units}$$

Method 2 - The Midpoint of a Line Segment

- To find the midpoint coordinates M of the line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$, we have



Method 2 - The Midpoint of a Line Segment (cont.)

- Let the coordinates of M be (x, y).
- APQ is parallel to the $x-$axis and MP and BQ are parallel to the $y-$axis. Hence, the coordinates of $P(x, y_1)$ and the coordinates of Q are (x_2, y)

$$\begin{aligned} \text{If } \ AM &= MB \\ \text{then } \ AP &= PQ \quad (MP//BQ \text{ in } \Delta AQB) \\ x - x_1 &= x_2 - x \quad \Rightarrow \quad 2x = x_1 + x_2 \\ \Longrightarrow \quad x &= \frac{x_1 + x_2}{2} \end{aligned}$$

- Similarly, by drawing lines perpendicular to the $y-$axis through M and B, it can be seen that $y = \dfrac{y_1 + y_2}{2}$.
  Thus, the coordinates of the midpoint of the line joining (x_1, y_1) and (x_2, y_2) are $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Exercises

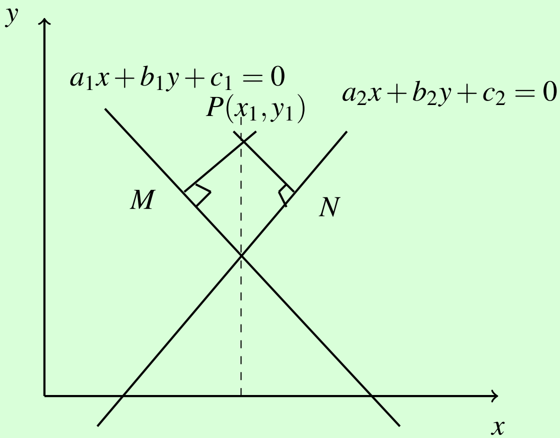
1. Calculate the coordinates of the midpoint of the line joining the points
   (i) $(-2,3)$ and $(6,3)$
   (ii) (a,b) and $(-a,\sqrt{2b})$

2. Determine the values of x and y if
   (i) $(-3,2)$ is the midpoint of the line segment joining $(-1,5)$ and (x,y).
   (ii) $(-1,y)$ is the midpoint of the line segment joining $(0,-2)$ and $(x,8)$.

3. Calculate the lengths of the medians of ΔABC in which the coordinates of the vertices are

$$A(-3,1), \quad B(-5-3) \;\text{ and }\; C(1,-5)$$

Equations of The Bisectors of The Given Lines: $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$



cont.

$$|PM| = |PN|$$

$$\implies \pm\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

Example

Find the locus of the point which moves so that its perpendicular distance from two lines $4x-3y=0$ and $5x+12y=0$ are equal.

Example

Find the locus of the point which moves so that its perpendicular distance from two lines $4x-3y=0$ and $5x+12y=0$ are equal.

Solution

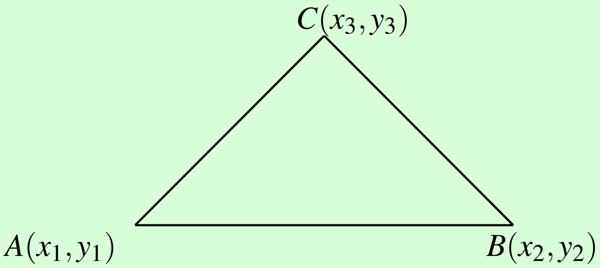
The distance from A to P, i.e. $\overline{AP}$ and B to P, i.e. $\overline{BP}$ are respectively

$$\left|\frac{4x-3y}{5}\right| \quad \text{and} \quad \left|\frac{5x+12y}{13}\right|$$

Now, the locus of the point $P(x,y)$ is

$$\begin{aligned} |\overline{AP}| &= |\overline{BP}| \\ \Longrightarrow \quad \frac{4x-3y}{5} &= \frac{5x+12y}{13} \\ \therefore \quad 3x-11y &= 0 \end{aligned}$$

Area of a Triangle With Given Vertices



$$\text{Area} = \frac{1}{2}\left[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)\right]$$

Area of a Triangle With Given Vertices (cont.)

Using the determinant method, we have

$$\text{Area of the } \triangle = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\text{OR } \triangle = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 \\ 1 & 1 & 1 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Example

Find the area of a triangle ABC whose vertices are

$$A(3,4), \quad B(1,3) \text{ and } C(5,-3)$$

Example

Find the area of a triangle ABC whose vertices are

$$A(3,4), \quad B(1,3) \text{ and } C(5,-3)$$

Solution

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\begin{vmatrix} 3 & 1 & 5 \\ 4 & 3 & -3 \\ 1 & 1 & 1 \end{vmatrix} \\
&= \frac{1}{2}[3(3--3)-1(4--3)+5(4-3)] \\
&= \frac{1}{2}[18-7+5] \\
&= \frac{1}{2}(16) \\
&= 8 \text{ square units}
\end{aligned}$$

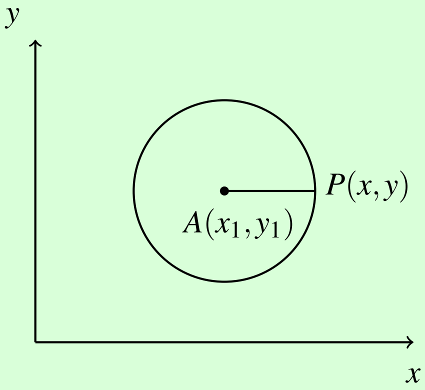
Exercise

1. Find the distance between the following pairs of points
   i. $(2,3)$ and $(4,5)$
   ii. $(6,1)$ and $(-6,6)$
   iii. $(3,-7)$ and $(-1,3)$

2. Calculate the coordinates of the midpoint of the line joining the points
   i. $(-3,1)$ and $(1,5)$
   ii. $(\sqrt{3},1)$ and $(3\sqrt{3},-1)$

Equation of a Circle With Center $A(x_1, y_1)$ And Radius r

- A circle is defined as the locus of points in a plane which is equidistant from a fixed point, that is the centre.



Equation of a Circle (cont.)

- Let $A(x_1, y_1)$ be the centre of the circle and $P(x, y)$ be any point on the circle

$$\text{Thus } |AP| = r \text{ and } |AP|^2 = r^2$$
$$\implies |AP|^2 = (x - x_1)^2 + (y - y_1)^2$$

Thus the equation of a circle with centre $A(x_1, y_1)$ and radius, r is given by

$$|AP|^2 = r^2 = (x - x_1)^2 + (y - y_1)^2$$

- If $x_1 = y_1 = 0$, the equation of the circle with centre at the origin $(0, 0)$ and radius, r is given by $x^2 + y^2 = r^2$.

Example

1. Find the equation of the circle with a centre of $(3,1)$ and radius of 2 units.

Example

1. Find the equation of the circle with a centre of $(3,1)$ and radius of 2 units.

Solution

Using the formula $(x-x_1)^2+(y-y_1)^2=r^2$, the required equation is

$$\begin{aligned}(x-3)^2+(y-1)^2&=2^2\\ x^2-6x+9+y^2-2y+1&=4\\ \implies x^2+y^2-6x-2y+6&=0\end{aligned}$$

Example

2. A circle has its centre $(5, -4)$ and radius 5 units. Find the equation of the circle.

Example

2. A circle has its centre $(5, -4)$ and radius 5 units. Find the equation of the circle.

Solution

The required equation is

$$(x-5)^2 + [y-(-4)]^2 = 5^2$$
$$x^2 - 10x + 25 + y^2 + 8y + 16 = 25$$
$$\implies x^2 + y^2 - 10x + 8y + 16 = 0$$

Example

3. Find the equation of the circle which has its centre at $(3, 2)$ and passes through the point $(-1, -2)$.

Example

3. Find the equation of the circle which has its centre at $(3,2)$ and passes through the point $(-1,-2)$.

Solution

Let $A = (3,2)$ and $B = (-1,-2)$

$$\begin{aligned} \text{Radius}\, r^2 = |AB|^2 &= (-1-3)^2 + (-2-2)^2 \\ &= (-4)^2 + (-4)^2 = 32 \end{aligned}$$

Using the formula, we have

$$\begin{aligned} (x-3)^2 + (y-2)^2 &= 32 \\ x^2 - 6x + 9 + y^2 - 4y + 4 &= 32 \\ \implies \quad x^2 + y^2 - 6x - 4y - 19 &= 0 \end{aligned}$$

Example

4. Find the equation of the circle whose centre is at the point $(2, 1)$ and passes through the point $(4, -3)$.

Example

4. Find the equation of the circle whose centre is at the point $(2,1)$ and passes through the point $(4,-3)$.

Solution

Let $A = (2,1)$ and $B = (4,-3)$

$$\begin{aligned} \text{Radius } r^2 = |AB|^2 &= (4-2)^2 + (-3-1)^2 \\ &= (2)^2 + (-4)^2 = 20 \end{aligned}$$

Using the formula, we have

$$\begin{aligned} (x-2)^2 + (y-1)^2 &= 20 \\ x^2 - 4x + 4 + y^2 - 2y + 1 &= 20 \\ \implies \quad x^2 + y^2 - 4x - 2y - 15 &= 0 \end{aligned}$$

Example

5. Find the equation of the circle with centre $(4,5)$ which passes through the point where the line $5x-2y+6=0$ cuts the $y-$axis.

Example

5. Find the equation of the circle with centre $(4,5)$ which passes through the point where the line $5x-2y+6=0$ cuts the $y-$axis.

Solution

Given that $5x-2y+6=0$ and centre $O=(4,5)$ we have

$$\begin{aligned} \text{On the y-axis } \; x &= 0 \\ \Longrightarrow \quad 5(0)-2y+6 &= 0 \\ -2y = -6 \quad \Rightarrow y &= 3 \end{aligned}$$

Thus, the coordinates on the $y-$axis is $P=(0,3)$.

$$\begin{aligned} \text{Radius } r &= |OP| \\ \Longrightarrow \quad r^2 &= (0-4)^2+(3-5)^2 = 20 \\ \text{Thus } \; (x-4)^2+(y-5)^2 &= 20 \\ \Longrightarrow \quad x^2+y^2-8x-10y+21 &= 0 \end{aligned}$$

Example

6. Find the equation of the circle whose centre lies on the line $x - 2y + 2 = 0$ and which touches the positive $x-$axis.

Example

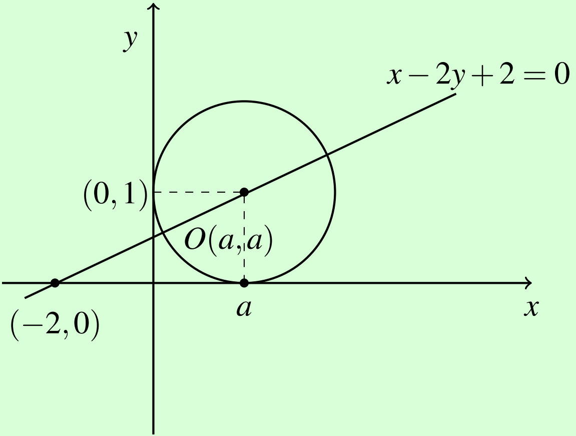
6. Find the equation of the circle whose centre lies on the line $x - 2y + 2 = 0$ and which touches the positive $x-$axis.

Solution

$$\begin{aligned}
\text{Given that } \; x - 2y + 2 = 0, & \quad \text{we have} \\
\text{On the x-axis } \; y = 0 & \\
\Longrightarrow \quad x - 2(0) + 2 = 0 & \quad \Rightarrow \; x = -2 \\
\text{On the y-axis } \; x = 0 & \\
\Longrightarrow \quad 0 - 2y + 2 = 0 & \quad \Rightarrow \; y = 1
\end{aligned}$$

Hence, the intercepts are $(-2, 0)$ and $(0, 1)$.

Example (cont.)



NB:
The circle touches the positive axes at the same distance from the origin.

Example (cont.)

Let the centre be (a,a). Thus we have

$$
\begin{aligned}
& x-2y+2=0 \\
\Longrightarrow\quad & a-2a+2=0 \\
& \Longrightarrow\quad a=2\ \ [\text{i.e. radius}\, a=2\ \text{ and }\ \text{centre}=(2,2)]
\end{aligned}
$$

Thus, the equation of the circle with centre $(2,2)$ and radius, $r=2$ is

$$
\begin{aligned}
(x-2)^2+(y-2)^2 &= 2^2 \\
x^2-4x+4+y^2-4y+4 &= 4 \\
\Longrightarrow\quad x^2+y^2-4x-4y+4 &= 0
\end{aligned}
$$

The General Form of an Equation of a Circle

The equation of a circle with centre (x_1, y_1) and radius, r is given by

$$(x-x_1)^2+(y-y_1)^2=r^2$$
$$x^2+y^2-2xx_1-2yy_1+(x_1^2+y_1^2-r^2)=0$$
$$\text{Let } x_1=-g,\ \ y_1=-f \ \text{ and } \ x_1^2+y_1^2-r^2=c$$
$$\implies \ x^2+y^2+2gx+2fy+c=0$$

Thus the general form of a circle with centre $(-g,-f)$ and radius, r is given by

$$x^2+y^2+2gx+2fy+c=0$$
$$\text{where } \ c=x_1^2+y_1^2-r^2$$
$$\implies \ r^2=g^2+f^2-c$$
$$r=\sqrt{g^2+f^2-c}$$

The General Form of an Equation of a Circle (cont.)

NB:

- The coefficient of x^2 and y^2 are equal.
- There is no term in xy.

The General Form of an Equation of a Circle (cont.)

NB:

- The coefficient of x^2 and y^2 are equal.
- There is no term in xy.

Example

(i) $x^2+y^2-7x+5y+8=0$

(ii) $x^2+y^2-4x+3y-7=0$

(iii) $x^2+y^2-4x+9y+3=0$

(iv) $x^2+y^2-3x=0$

The General Form of an Equation of a Circle (cont.)

NB:

- The coefficient of x^2 and y^2 are equal.
- There is no term in xy.

Example

(i) $x^2+y^2-7x+5y+8=0$

(ii) $x^2+y^2-4x+3y-7=0$

(iii) $x^2+y^2-4x+9y+3=0$

(iv) $x^2+y^2-3x=0$

Check the following

1. $x^2+2y^2+7x-5y+3=0$?
2. $x^2+y^2+2x+5xy+4=0$?
3. $x^2-y^2+3x-6y+2=0$?

Equation of a Circle Passing Through Three Points

- The equation of a circle is given by

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

- Since the three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) lie on the circle, we have

$$\begin{aligned} x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c &= 0 \\ x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c &= 0 \\ x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c &= 0 \end{aligned}$$

Solving the 3 equations simultaneously, the required equation could be found.

Example

Find the equation of the circle passing through the points $(2,1)$, $(0,2)$ and $(1,0)$.

Example

Find the equation of the circle passing through the points $(2,1)$, $(0,2)$ and $(1,0)$.

Solution

Let the equation of the circle be

$$
\begin{aligned}
& x^2+y^2+2gx+2fy+c=0 \\
(2,1):\quad & 2^2+1^2+2g(2)+2f(1)+c=0 \\
& \implies 4g+2f+c=-5 \qquad (1) \\
(0,2):\quad & 0^2+2^2+2g(0)+2f(2)+c=0 \\
& \implies 4f+c=-4 \qquad (2) \\
(1,0):\quad & 1^2+0^2+2g(1)+2f(0)+c=0 \\
& \implies 2g+c=-1 \qquad (3)
\end{aligned}
$$

Example (cont.)

Solving (1), (2) and (3) simultaneously, we have

$$c = -1, \;\; g = -\frac{5}{6} \;\text{ and }\; f = -\frac{7}{6}$$

Thus, the equation of the circle is

$$x^2 + y^2 + 2\left(-\frac{5}{6}\right)x + 2\left(-\frac{7}{6}\right)y + \frac{2}{3} = 0$$

$$\implies \;\; 3x^2 + 3y^2 - 5x - 7y + 2 = 0$$

Finding The Radius And Centre of a Given Equation of a Circle

To find the centre and radius of a given equation

- First reduce the coefficients of x^2 and y^2 to a unit of 1.
- Then we complete the squares in x and y.

Example

1. Find the centre and radius of the circle whose given equation is $3x^2 + 3y^2 - 24x + 12y + 11 = 0$.

Example

1. Find the centre and radius of the circle whose given equation is $3x^2 + 3y^2 - 24x + 12y + 11 = 0$.

Solution

$$3x^2 + 3y^2 - 24x + 12y + 11 = 0$$

$$x^2 + y^2 - 8x + 4y + \frac{11}{3} = 0$$

Completing the squares in x and y, we have

$$x^2 - 8x + 16 + y^2 + 4y + 4 + \left(\frac{11}{3} - 16 - 4\right) = 0$$

$$(x-4)^2 + (y+2)^2 - \frac{49}{3} = 0$$

$$(x-4)^2 + (y+2)^2 = \frac{49}{3}$$

Thus the centre $= (4, -2)$ and the radius, $r = \sqrt{49}/\sqrt{3}$

Example

2. Find the centre and radius of the circle $x^2 + y^2 - x + 4y - 2 = 0$.

Example

2. Find the centre and radius of the circle $x^2+y^2-x+4y-2=0.$

Solution

$$x^2+y^2-x+4y-2=0$$
$$x^2-x+\frac{1}{4}+y^2+4y+4+\left(-2-\frac{1}{4}-4\right)=0$$
$$\left(x-\frac{1}{2}\right)^2+(y+2)^2-\frac{25}{4}=0$$
$$\left(x-\frac{1}{2}\right)^2+(y+2)^2=\frac{25}{4}$$

Thus the centre $=(\frac{1}{2},-2)$ and the radius, $r=\sqrt{\frac{25}{4}}=\frac{5}{2}$

Example

2. Find the centre and radius of the circle $x^2 + y^2 - x + 4y - 2 = 0$.

Example

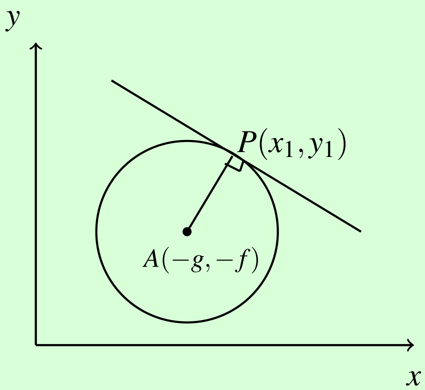
2. Find the centre and radius of the circle $x^2+y^2-x+4y-2=0$.

Alternative Solution

Using the general form: $x^2+y^2+2gx+2fy+c=0$, where the center $=(-g,-f)$ and radius, $r=\sqrt{g^2+f^2-c}$, we have

$$2g=-1 \;\Rightarrow\; g=-\frac{1}{2}$$
$$2f=4 \;\Rightarrow\; f=2$$
$$c=2 \;\Rightarrow\; \text{centre}=(\frac{1}{2},-2)$$
$$r=\sqrt{\left(-\frac{1}{2}\right)^2+(2)^2-(-2)}=\sqrt{\frac{25}{4}}=\frac{5}{2}$$

Finding The Equation of The Tangent Line at a Point (x_1, y_1) of The Circle



$$\text{Gradient of } AP = \frac{y_1 - (-f)}{x_1 - (-g)} = \frac{y_1 + f}{x_1 + g}$$

Cont.

$\Longrightarrow$ Gradient of the tangent at P is $-\left(\dfrac{x_1+g}{y_1+f}\right)$

Thus, the equation of the tangent at P is given by

$$y-y_1=-\left(\frac{x_1+g}{y_1+f}\right)(x-x_1)$$

Example

Find the equation of the tangent at the point $(-2,3)$ to the circle $x^2+y^2-4x+2y-27=0$.

Example

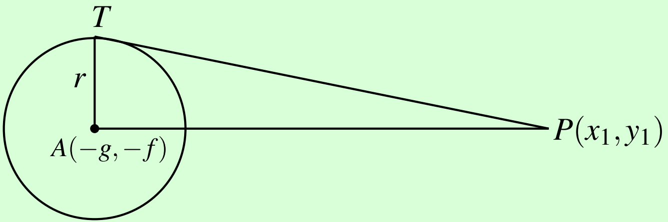
Find the equation of the tangent at the point $(-2,3)$ to the circle $x^2+y^2-4x+2y-27=0$.

Solution

Centre $=(-g,-f)=(2,-1)$. Thus the equation of the tangent at $P(x_1,y_1)$ is given by

$$
\begin{aligned}
y-y_1 &= -\left(\frac{x_1+g}{y_1+f}\right)(x-x_1) \\
\implies \quad y-3 &= -\frac{(-2+-2)}{3+1}(x--2) \\
y-3 &= \frac{4}{4}(x+2) = x+2 \\
\implies \quad y &= x+5 \\
\therefore \quad x-y+5 &= 0
\end{aligned}
$$

Finding The Length of The Tangent From The Point $P(x_1, y_1)$ to The Circle



- Center $(-g, -f)$
- $P(x_1, y_1)$ is the tangent from P to the circle
- $|AT|^2 = r^2 = g^2 + f^2 - c$
- Distance between A and P is given by

$$|AP|^2 = (y_1 + f)^2 + (x_1 + g)^2$$

Cont.

- From Pythagoras theorem,

$$\begin{aligned}
|PT|^2 &= |AP|^2 - |AT|^2 \\
&= \left[(x_1+g)^2 + (y_1+f)^2\right] - (g^2+f^2-c) \\
&= x_1^2 + 2gx_1 + g^2 + y_1^2 + 2fy_1 + f^2 - g^2 - f^2 + c \\
&= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c
\end{aligned}$$

$$\implies \quad |PT| = \sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Example

1. Find the length of the tangent from $P(3,5)$ to the circle $x^2+y^2-6x+2y+6=0$.

Example

1. Find the length of the tangent from $P(3,5)$ to the circle $x^2+y^2-6x+2y+6=0$.

Solution

$$P(x_1,y_1)=P(3,5) \Rightarrow x_1=3 \text{ and } y_1=5$$
$$2g=-6 \Rightarrow g=-3$$
$$2f=-2 \Rightarrow f=-1$$
$$c=6$$

Thus, the length of the tangent from $P(3,5)$ to the circle is

$$=\sqrt{3^2+5^2+2(3)(-3)+2(-1)(5)+6}$$
$$=\sqrt{9+25+(-18)+(-10)+6}$$
$$=\sqrt{12}=2\sqrt{3}$$

Example

2. Find the length of the tangent from $(-3,-2)$ to the circle $x^2+y^2-24x+95=0$.

Example

2. Find the length of the tangent from $(-3,-2)$ to the circle $x^2+y^2-24x+95=0$.

Solution

$$\begin{aligned} P(x_1,y_1)=P(-3,-2) &\Rightarrow x_1=-3 \text{ and } y_1=-2 \\ 2g=-24 &\Rightarrow g=-12 \\ 2f=0 &\Rightarrow f=0 \\ & \quad c=95 \end{aligned}$$

Thus, the length of the tangent from $(-3,-2)$ to the circle is

$$= \sqrt{(-3)^2+(-2)^2+2(-12)(-3)+2(0)(-2)+95}$$
$$= \sqrt{180}$$
$$= 6\sqrt{5}$$

Finding The Points of Intersection of a Straight Line And a Circle

- Let the equation of the line be

$$y = mx + c \tag{1}$$

- Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c_1 = 0 \tag{2}$$

- Since the points of intersection lie on both the line and the circle, the points of intersection will be given by the solutions of equations (1) and (2) considered as simultaneous equations in x and y.

Cont.

- Substituting equation (1) into (2) becomes

$$\begin{aligned} x^2+(mx+c)^2+2gx+2f(mx+c)+c_1 &= 0 \\ x^2+m^2x^2+2mxc+c^2+2gx+2fmx+2fc+c_1 &= 0 \\ (1+m^2)x^2+2x(g+mf+mc)+c^2+2fc+c_1 &= 0 \end{aligned}$$

  is a quadratic equation in x and gives two values for x_1.

- From equation (2), the corresponding values for y may be found.

Example

1. Find the points of intersection of the line $y = 3x + 5$ and the circle $x^2 + y^2 + 4x + 2y - 27 = 0$.

Example

1. Find the points of intersection of the line $y = 3x + 5$ and the circle $x^2 + y^2 + 4x + 2y - 27 = 0$.

Solution

$$y = 3x + 5 \tag{1}$$
$$x^2 + y^2 + 4x + 2y - 27 = 0 \tag{2}$$

Substituting (1) into (2), we have

$$\begin{aligned} x^2 + (3x+5)^2 + 4x + 2(3x+5) - 27 &= 0 \\ x^2 + 9x^2 + 30x + 25 + 4x + 6x + 10 - 27 &= 0 \\ 10x^2 + 40x + 8 &= 0 \\ 5x^2 + 20x + 4 &= 0 \end{aligned}$$

Example (cont.)

$$x = \frac{-20 \pm \sqrt{20^2 - 4(5)(4)}}{10}$$

$$\implies x = -0.221 \text{ or } x = -3.789$$

When $x = -0.221$ $y = 3(-0.221) + 5 = 4.367$

When $x = -3.789$ $y = 3(-3.789) + 5 = -6.367$

$\therefore$ The points of intersection are $(-0.221, 4.367)$ and $(-3.789, -6.367)$

Example

2. Find the coordinates of the point of intersection of the line $y = 2x$ and the circle $x^2 + y^2 - 8x - y + 5 = 0$.

Example

2. Find the coordinates of the point of intersection of the line $y = 2x$ and the circle $x^2 + y^2 - 8x - y + 5 = 0$.

Solution

$$y = 2x \tag{1}$$

$$x^2 + y^2 - 8x - y + 5 = 0 \tag{2}$$

Substituting (1) into (2), we have

$$\begin{aligned} x^2 + (2x)^2 - 8x - (2x) + 5 &= 0 \\ x^2 + 4x^2 - 10x + 5 &= 0 \\ 5x^2 - 10x + 5 &= 0 \\ x^2 - 2x + 1 &= 0 \end{aligned}$$

Example (cont.)

$$
\begin{aligned}
(x-1)(x-1) &= 0 \\
\implies x &= 1 \;\; \text{twice} \\
\text{When } \; x = 1, \;\; y &= 2(1) = 2
\end{aligned}
$$

$\therefore$ The point of contact is $(1,2)$.

Try the following

1. (i) Find the equation of the circle whose centre lies on the line $y = 4$, and which passes through the points $(2,0)$ and $(6,0)$.

   (ii) The points (8,4) and (2,2) are the ends of a diameter of a circle. Find the coordinates of the centre and the radius and find the equation of the circle.

2. (i) Find the the points of intersection of the line $y = x+1$ and the circle $x^2 + y^2 - 8x - 2y + 9 = 0$.

   (ii) Find the coordinates of the point of contact of the line $x - 2y + 12 = 0$ at which it touches the circle $x^2 + y^2 - x - 31 = 0$.

3. (i) Find the the equation of the circle through $(3,-2)$ and $(5,-4)$ with center on $x - y = 1$.

Try the following (cont.)

(ii) Find the equation of the circle passing through $(12,4)$, $(8,12)$ and $(-6,-2)$.

4. **(i)** Determine the equation of a circle on AB as diameter if A and B are the points $(-3,2)$ and $(1,6)$.

(ii) Determine the center and radius of the following circles

- **(a)** $(x+1)^2+(y-2)^2=49$
- **(b)** $x^2+(y+2)^2=10$
- **(c)** $x^2+6x+y^2-4y-12=0$
- **(d)** $x^2+y^2+6x-5y-\frac{3}{4}=0$
- **(e)** $3x^2+3y^2+6x+9y+6=0$
- **(f)** $(x-3)^2+y^2=10$
- **(g)** $x^2+y^2-3x+4y+5\frac{1}{4}=0$

Try the following (cont.)

5. **(i)** Determine the equation of the circle with

- **(a)** center $(2,1)$ and radius 4 units.
- **(b)** center $(-1,1)$ and radius $\sqrt{3}$ units.
- **(c)** center $(-3,4)$ and radius 5 units.
- **(d)** center $(-2,-3)$ passing through the origin.
- **(e)** center (a,b) and radius 3 units.
- **(f)** center $(3,-5)$ passing through the point $(-1,1)$.

(ii) $A(4,a)$ and $B(4,b)$ are two points on the circle $(x-1)^2+(y+1)^2=25$. Find the possible values of a and b and also the equations of the tangents to the circle which are parallel to AB.

Try the following (cont.)

6. **(i)** $y = 7x$ cuts the circle $x^2 + y^2 = 50$ at A and B. Find the coordinates of A and B.

(ii) If $x - y = 1$ cuts the circle $x^2 + y^2 = 13$ at P and Q, find the following

(a) the coordinates of P and Q.

(b) the length of the chord PQ.

(c) the midpoint, M of the chord PQ

7. Find the equation of the tangent to

(a) $x^2 + y^2 = 5$ which has a gradient of -2.

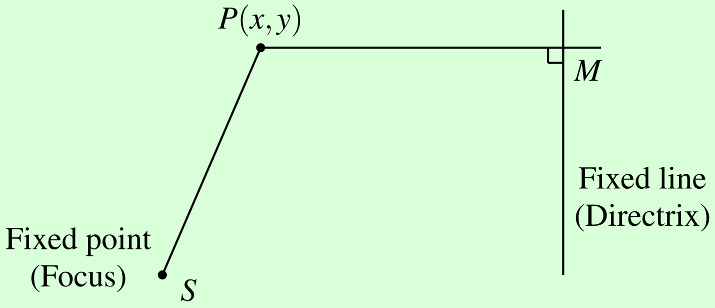
(b) $x^2 + y^2 = 18$ which has an inclination of 135°.

(c) $(x+1)^2 + y^2 = 20$ which is parallel to $2y - x = 0$

(d) $(x-2)^2 + (y+3)^2 = 16$ which is parallel to the $y-$axis.

Conic Section

- **Conic**:- The locus of a point which moves such that the ratio of its distance from a fixed point to its distance from a fixed line is constant.
- The constant (usually denoted by e) is called the ***eccentricity***.
- The fixed point is called the **focus** and the fixed line is called the **Directrix**

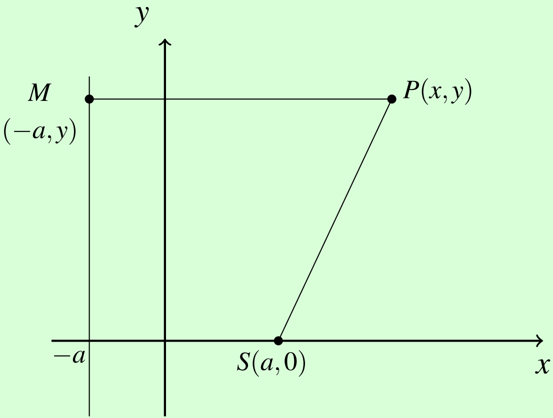


Conic Section (cont.)

- The set of all points P in the plane such that the ratio of the distance $|PS|$ (from P to the focus S) to the distance $|PM|$ (from P to the directrix M) is the constant e is a conic section.

$$\begin{array}{lll} \text{When } \; e<1 & \rightarrow & \text{an ellipse} \\ \text{When } \; e=1 & \rightarrow & \text{a parabola} \\ \text{When } \; e>1 & \rightarrow & \text{a hyperbola} \end{array}$$

Parabola



A parabola is the set of points P that are equidistant from the directrix M and the focus S.

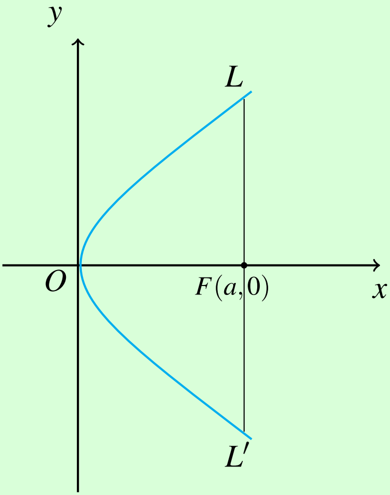
Parabola (cont.)

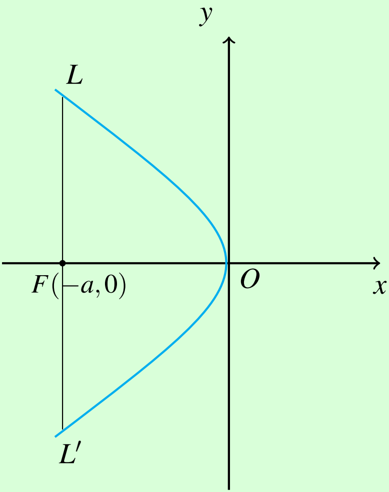
$$\text{Thus } \frac{|PS|}{|PM|} = e = 1$$
$$\implies |PS|^2 = |PM|^2$$
$$|PS|^2 = (y-0)^2 + (x-a)^2$$
$$= y^2 + x^2 - 2ax + a^2 \tag{1}$$
$$|PM|^2 = (y-y)^2 + [x-(-a)]^2$$
$$= (x+a)^2 = x^2 + 2ax + a^2 \tag{2}$$

Equating (1) and (2), we have

$$y^2 + x^2 - 2ax + a^2 = x^2 + 2ax + a^2$$
$$\implies y^2 - 2ax = 2ax$$
$$\therefore \; y^2 = 4ax \quad \text{is the general equation of a parabola}$$

Sketch of a Parabola Curve





Sketch of a Parabola Curve (cont.)

$$y^2 = 4ax \quad \Rightarrow \quad y = \pm\sqrt{4ax}$$

- y may take all values of x. Since $y = \pm\sqrt{4ax}$, the curve will not exist in the plane for which $x < 0$.
- If y is replaced by $-y$, the curve remains the same, which follows that there is symmetry about the $x-$axis.

$$\begin{aligned} \text{when } & y = 0, \;\; x = 0 \\ \text{when } & x = 0, \;\; y^2 = 0 \\ \Longrightarrow \; & y = 0 \;\; \text{twice} \\ \text{As } x \to +\infty, \; & y = \pm\infty \end{aligned}$$

The Axis of Symmetry

- The axis of symmetry has equation $y = 0$. Where the axis meets the curve is called the **vertex** of the curve.
- In the curves in the previous slides, the vertex is the point O.
- From the figures in the previous slides, the line LL' passing through the focus and parallel to the directrix is called the **Latus Rectum** of the parabola.

Translation of The Parabola by The Vector

$$\underline{d} = li + kj \quad \text{or} \quad \underline{d} = \begin{pmatrix} l \\ k \end{pmatrix}$$

- The translation is represented by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} l \\ k \end{pmatrix}$$

$$x' = x + l \;\Rightarrow\; x = x' - l$$

$$y' = y + k \;\Rightarrow\; y = y' - k$$

$$\therefore \; y^2 = 4ax \;\rightarrow\; (y' - l) = 4a(x' - k)$$

Example

1. Find the equation of the parabola whose directrix is $2x - y + 1 = 0$ and whose focus has the coordinates $(3, 2)$.

Example

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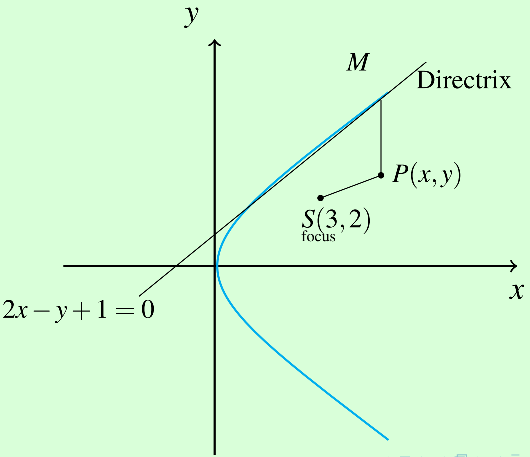
Solution



Example (cont.)

$$\frac{|PS|}{|PM|} = e = 1 \quad \Rightarrow \quad |PS| = |PM|$$

$$|PS|^2 = |PM|^2$$

$$\Longrightarrow \quad (3-x)^2 + (2-y)^2 = \left|\frac{2x-y+1}{\sqrt{2^2+1^2}}\right|^2$$

$$(9-6x+x^2) + (4-4y+y^2) = \left(\frac{2x-y+1}{\sqrt{5}}\right)^2 = \frac{(2x-y+1)^2}{5}$$

$$5(9-6x+x^2+4-4y+y^2) = (2x-y+1)(2x-y+1)$$

$$5(13-6x-4y+x^2+y^2) = 4x^2-2xy+2x-2xy+y^2-y+2x-y+1$$

$$65-30x-20y+5x^2+5y^2 = 4x^2+y^2-4xy+4x-2y+1$$

$$x^2+4y^2+4xy-34x-18y+64 = 0$$

Example

2. Find the focus and directrix for each of the following parabolas.

(a) $y^2 = 8x$

(b) $(y-2)^2 = 4x$

(c) $y^2 + 6y = 10x$

(d) $x^2 + 2x = 6y$

Example

2. Find the focus and directrix for each of the following parabolas.

(a) $y^2 = 8x$

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(c) $y^2 + 6y = 10x$

(d) $x^2 + 2x = 6y$

Solution

(a)

$$y^2 = 8x \text{ is in standard form } (y^2 = 4ax)$$

$$\Longrightarrow \quad 4a = 8 \quad \Rightarrow \quad a = 2$$

$$\therefore \quad \text{The focus is } (2, 0) \text{ and the directrix is } x = -2$$

Example (cont.)

Solution

Example (cont.)

Solution

(b)

$$(y-2)^2 = 4x \text{ the vertex is } (2,0)$$
$$\text{In standard form } Y^2 = 4x \text{ where } Y = y-2 \text{ and } X = x$$
$$\text{Focus } 4a = 4 \Rightarrow a = 1$$
$$\text{when } Y = 0 \text{ and } X = 0$$
$$\Longrightarrow y = 2 \text{ and } x = 0$$
$$\therefore \text{ Vertex } = (0,2); \text{ Focus } = (1,2) \text{ and Directrix, } x = -1$$

Example (cont.)

Solution

Example (cont.)

Solution

(c)

$$y^2 + 6y = 10x$$

$$\implies (y+3)^2 = 10x + 9 = 10\left(x + \frac{9}{10}\right)$$

In standard form : $(y+3)^2 = 10\left(x + \frac{9}{10}\right) = 4\left(\frac{5}{2}\right)\left(x + \frac{9}{10}\right)$

The parent equation has been translated by the vector $\begin{pmatrix} -\frac{9}{10} \\ -3 \end{pmatrix}$

$\therefore$ The vertex $= \left(-\frac{9}{10}, -3\right)$; the directrix, $x = -\frac{5}{2} - \frac{9}{10} = -\frac{34}{10}$

and the focus $= \left(\frac{5}{2} - \frac{9}{10}, 0 - 3\right) = \left(\frac{8}{5}, -3\right)$

Example (cont.)

Solution

Example (cont.)

Solution

(d)

$$x^2 + 2x = 6y$$

$$\Longrightarrow \quad (x+1)^2 = 6y - 1 = 6\left(y - \frac{1}{6}\right)$$

$$= 4\,.\,\frac{3}{2}\left(y - \frac{1}{6}\right)$$

In standard form : $X^2 = 4\left(\frac{3}{2}\right)Y$ where $X = x + 1$ and $Y = y - \frac{1}{6}$

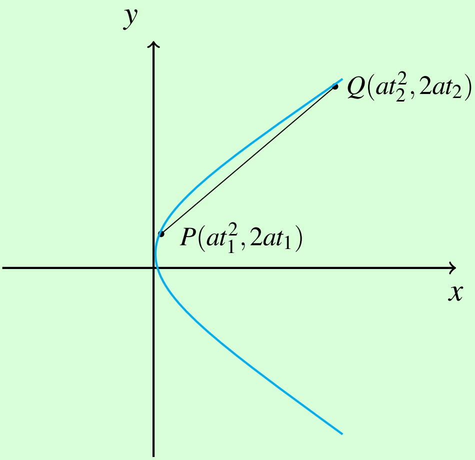
$\therefore$ The vertex $= \left(-1, \frac{1}{6}\right)$; the directrix, $y = -\frac{3}{2} + \frac{1}{6} = -\frac{4}{3}$

and the focus $= \left(-1, \frac{3}{2} + \frac{1}{6}\right) = \left(-1, \frac{5}{3}\right)$

Parametric Form of The Parabola

- The equations $x = at^2$ and $y = 2at$ where t is a parameter are called the parametric equations of the parabola.
- The point $(at^2, 2at)$ lies on the parabola since
$$y^2 = (2at)^2 = 4a^2t^2 = 4a(at^2) = 4ax$$
Consider $P(at^2, 2at)$ where t is a parameter
$$x = at^2 \tag{1}$$
$$y = 2at \tag{2}$$
$$\text{From (2)} \;\; t = \frac{y}{2a} \implies x = a\left(\frac{y}{2a}\right)^2$$
$$ay^2 = (2a)^2x$$
$$\implies y^2 = 4ax \;\; \text{which is the equation of the parabola}$$
- Thus, the equations $x = at^2$ and $y = 2at$ represent the parametric form of the parabola $y^2 = 4ax$.

Equation of The Chord Joining Two Points



The equation of the chord joining the points P and Q whose parameters are t_1 and t_2 is

$$P(at_1^2, 2at_1)$$

and

$$Q(at_2^2, 2at_2)$$

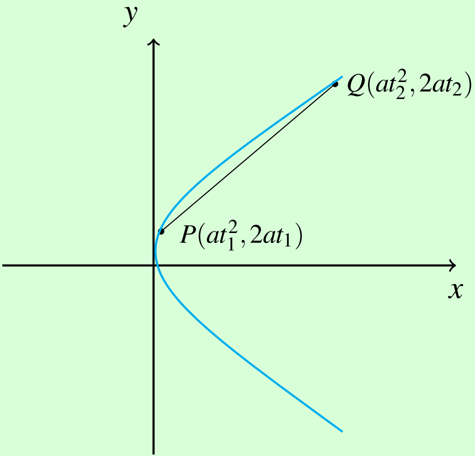
Equation of The Chord Joining Two Points (cont.)

- Finding the equation:

$$\text{Gradient of } \overline{PQ} = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} = \frac{2a(t_2 - t_1)}{a(t_2 + t_1)(t_2 - t_1)}$$
$$= \frac{2}{t_1 + t_2}$$
$$\text{Equation of } \overline{PQ} = \frac{y - 2at_1}{x - at_1^2} = \frac{2}{t_1 + t_2}$$
$$\implies \quad y - 2at_1 = \frac{2}{t_1 + t_2}(x - at_1^2)$$
$$(t_1 + t_2)y - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$$
$$(t_1 + t_2)y = 2x + 2at_1t_2$$

which is the equation of the chord $\overline{PQ}$

Deducing The Equation of The Tangent At P



As the point, Q approaches the point P and the chord $\overline{PQ}$ turns to the tangent at P. At that instant, the two points P and Q coincide

$$\implies t_1 = t_2 = t$$

Deducing The Equation of The Tangent At P (cont.)

- Using the equation of the chord,

$$
\begin{aligned}
(t_1 + t_2)y &= 2x + 2at_1t_2 \\
\text{Since } \ t_1 = t_2 &= t, \ \ \text{we have} \\
(t + t)y &= 2x + 2at\,.t \\
2ty &= 2x + 2at^2 \\
ty &= x + at^2
\end{aligned}
$$

Thus, the equation of the tangent at any point of a parabola is given by

$$ty = x + at^2$$

Deducing The Equation of The Tangent At P (cont.)

- Using the parametric form $(at^2, 2at)$ of the parabola to find the tangent at the point P with parameter t, we have
  Coordinates of P are $(at^2, 2at)$

$$x = at^2 \quad \text{and} \quad y = 2at$$

$$\implies \quad \frac{dx}{dt} = 2at \quad \text{and} \quad \frac{dy}{dt} = 2a$$

$$\implies \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$$

Thus, the equation of the tangent is

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty = x + at^2$$

Using The General Form of a Parabola to Find The Tangent at a Point (x_1, y_1)

$$y^2 = 4ax$$
$$2y\frac{dy}{dx} = 4a$$
$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

Thus, the equation of the tangent at (x_1, y_1) is

$$y - y_1 = \frac{2a}{y_1}(x - x_1)$$
$$yy_1 - y_1^2 = 2ax - 2ax_1$$
$$\text{Since } \ y_1^2 = 4ax_1, \ \text{ we have}$$
$$yy_1 - 4ax_1 = 2ax - 2ax_1$$

cont.

$$\implies yy_1 = 2ax + 2ax_1$$

$$yy_1 = 2a(x + x_1)$$

At point $(at^2, 2at)$ $y^2 = 4ax$

$$2y\frac{dy}{dx} = 4a \quad \Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

At point $(at^2, 2at)$ $\dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$

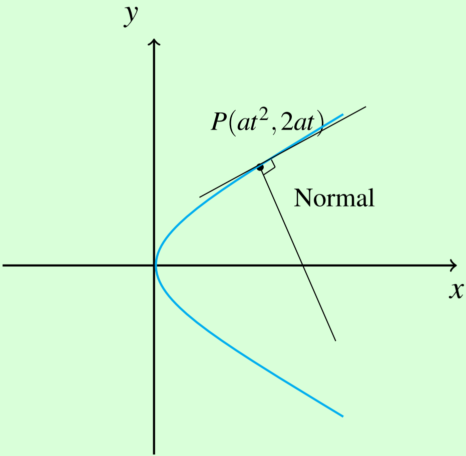
Equation of the tangent: $y - 2at = \dfrac{1}{t}(x - at^2)$

$$t(y - 2at) = x - at^2$$

$$ty - 2at^2 = x - at^2$$

$$\therefore \quad ty = x + at^2$$

Equation of Normal at $P(at^2, 2at)$



- Gradient of the tangent at $P = \frac{1}{t}$
- Gradient of the normal at $P = -t$

Equation of Normal at $P(at^2, 2at)$ (cont.)

- Equation of the normal at P is given by

$$y - 2at = -t(x - at^2)$$
$$y - 2at = -tx + at^3$$
$$y + tx = 2at + at^3$$

Using the general form:

$$y^2 = 4ax$$
$$2y\frac{dy}{dx} = 4a$$
$$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$$

Equation of Normal at $P(at^2, 2at)$ (cont.)

$$\text{At } (at^2, 2at), \quad \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$\implies \text{Gradient of the tangent} = \frac{1}{t}$$

$$\text{Gradient of the tangent} = -t$$

Thus, the equation of the normal at $(at^2, 2at)$ is

$$y - 2at = -t(x - at^2)$$

$$y - 2at = -tx + at^3$$

$$y + tx = 2at + at^3$$

Example

1. The normal at the point $(2,4)$ on the parabola $y^2 = 8x$ meets the parabola again at the point, T. Find the coordinates of the point T.

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1. The normal at the point $(2,4)$ on the parabola $y^2 = 8x$ meets the parabola again at the point, T. Find the coordinates of the point T.

Solution

$$y^2 = 8x \quad \Rightarrow \quad 2y\frac{dy}{dx} = 8$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$$

$$\text{At} \;\; (2,4) \;\; \frac{dy}{dx} = \frac{4}{4} = 1$$

The gradient of normal at $(2,4)$ is -1

$$\Longrightarrow \quad y-4 = -1(x-2) = -x+2$$

$$y+x-6 = 0$$

Example (cont.)

To find where the normal intersects the parabola, we solve simultaneously the equations:

$$y + x - 6 = 0 \;\text{ and} \tag{1}$$

$$y^2 = 8x \tag{2}$$

$$\text{From (1)} \;\; y = 6 - x$$

$$\Longrightarrow \;\; (6-x)^2 = 8x$$

$$36 - 12x + x^2 = 8x$$

$$x^2 - 20x + 36 = 0$$

Example (cont.)

Since the line passes through $(2,4)$, $(x-2)$ is a factor

$$\begin{aligned} \implies \quad x^2 - 20x + 36 &= 0 \\ (x-2)(x-18) &= 0 \\ \implies \quad x = 2 \;\; \text{or} \;\; x &= 18 \\ \text{when} \;\; x = 2, \;\; y = 4 \;\; \text{and} & \\ \text{when} \;\; x = 18, \;\; y = 6 - 18 &= -12 \\ \therefore \quad \text{T has coordinates} \;\; (18, -12) & \end{aligned}$$

Example

2. A parabola has equation $y^2 = 8x$.
   (a) Find the focus of the parabola.
   (b) Show that the parabola passes through the point $(1, 2\sqrt{2})$. Obtain the equations of the tangent and normal at this point.

Example

2. A parabola has equation $y^2 = 8x$.

(a) Find the focus of the parabola.

(b) Show that the parabola passes through the point $(1, 2\sqrt{2})$. Obtain the equations of the tangent and normal at this point.

Solution

(a)

$$y^2 = 8x = 4(2)x$$

$$\therefore \quad \text{The focus is } \ (2,0)$$

Example (cont.)

(b)

$$y^2 = 2(\sqrt{2})^2 x = 4(2)x = 8x$$

$$\therefore \quad (1, 2\sqrt{2}) \text{ lies on the parabola}$$

$$\text{Now } \; 2y\frac{dy}{dx} = 8 \;\Rightarrow\; \frac{dy}{dx} = \frac{8}{2y} = \frac{4}{y}$$

$$\text{At point } \; (1, 2\sqrt{2}), \; \frac{dy}{dx} = \frac{4}{2\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Thus, the equation of the tangent at $(1, 2\sqrt{2})$ is given by

$$y - 2\sqrt{2} = \sqrt{2}(x-1)$$

$$y - \sqrt{2}x - 2\sqrt{2} + \sqrt{2} = 0$$

$$\Longrightarrow \; y - \sqrt{2}x - \sqrt{2} = 0$$

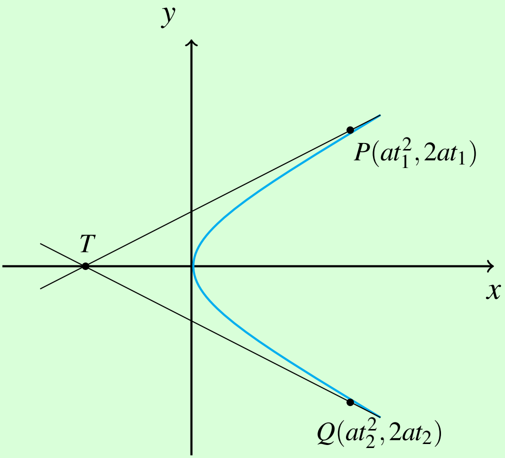
Example (cont.)

(b) The gradient of normal $= -\dfrac{1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2}$.

Thus, the equation of normal at $(1, 2\sqrt{2})$ is given by

$$
\begin{aligned}
y - 2\sqrt{2} &= -\frac{\sqrt{2}}{2}(x-1) \\
2y - 4\sqrt{2} &= -\sqrt{2}x + \sqrt{2} \\
\implies \quad 2y + \sqrt{2}x - 5\sqrt{2} &= 0
\end{aligned}
$$

Obtaining The Coordinates of The Point of Intersection of The Two Tangents Drawn at The Points P and Q Whose Parameters Are t_1 and t_2



cont.

$$\text{Equation of tangent at } P: \quad t_1 y = x + at_1^2 \tag{1}$$

$$\text{Equation of tangent at } Q: \quad t_2 y = x + at_2^2 \tag{2}$$

$$\begin{aligned} (1)-(2): \quad (t_1 - t_2)y &= a(t_1^2 - t_2^2) \\ (t_1 - t_2)y &= a(t_1 + t_2)(t_1 - t_2) \\ y &= a(t_1 + t_2) \end{aligned} \tag{3}$$

Substituting (3) into (1), we have

$$\begin{aligned} t_1\left(a(t_1 + t_2)\right) &= x + at_1^2 \\ at_1(t_1 + t_2) &= x + at_1^2 \\ at_1^2 + at_1t_2 - at_1^2 &= x \\ \implies \quad x &= at_1t_2 \end{aligned}$$

$\therefore$ the coordinates of the point of intersection at $T = (at_1t_2, a(t_1 + t_2))$

Obtaining The Point of Intersection of The Normals at T

$$\text{Equation of the normal at } P: \quad y + t_1 x = 2at_1 + at_1^3 \tag{1}$$

$$\text{Equation of the normal at } Q: \quad y + t_2 x = 2at_2 + at_2^3 \tag{2}$$

$$(1) - (2): \quad (t_1 - t_2)x = 2a(t_1 - t_2) + a(t_1^3 - t_2^3)$$

$$\implies \quad x = 2a + a\left(\frac{t_1^3 - t_2^3}{t_1 - t_2}\right) \tag{3}$$

$$\text{Now } \; t_1^3 - t_2^3 = (t_1 - t_2)(t_1^2 + t_1 t_2 + t_2^2)$$

$$\text{NB}: \quad a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

$$\text{Thus from (3)}: \quad x = 2a + a(t_1^2 + t_1 t_2 + t_2^2) \tag{4}$$

Obtaining The Point of Intersection of The Normals at T

Substituting (4) into (1), we have

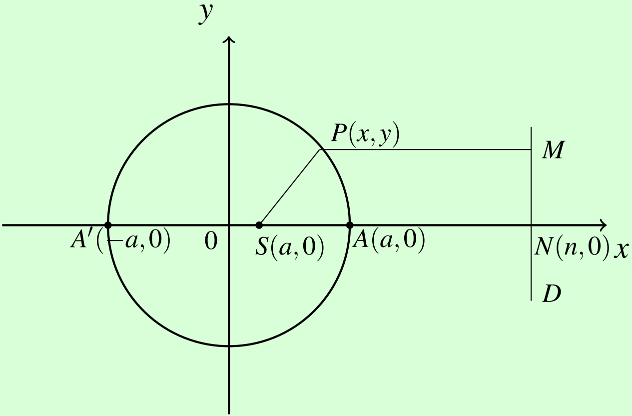
$$\begin{aligned} y &= at_1 + at_1^3 - 2at_1 - at_1^3 - at_1^2t_2 - at_1t_2 \\ &= at_1t_2(t_1 + t_2) \end{aligned}$$

Thus, the coordinates of the point of intersection of the normal at T is given by

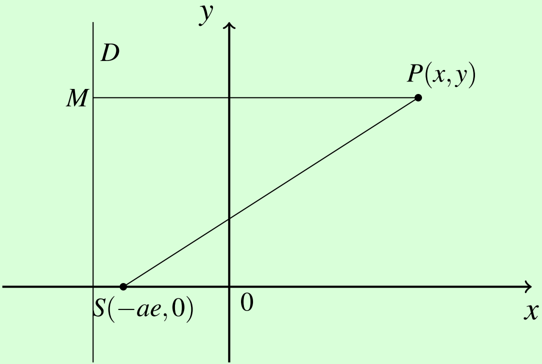
$$T\left[\left(2a + a(t_1^2 + t_1t_2 + t_2^2)\right),\ at_1t_2(t_1 + t_2)\right]$$

Ellipse

- The ellipse is the locus of a point in a plane such that its distance from a certain fixed point, F (the focus) is less than its distance from a certain fixed line, D (the directrix)



Ellipse (cont.)



From the definition:

$$\frac{|PS|}{|PM|} = e$$

$$\implies |PS|^2 = e^2|PM|^2 \quad (1)$$

$$\text{But } \frac{SA'}{AN'} = e$$

$$\text{and } \frac{SA}{AN} = e$$

$$\implies a+s = e(n+a) \quad (2)$$

$$a-s = e(n-a) \quad (3)$$

Ellipse (cont.)

$$\text{Solving}: \quad 2a = 2en \quad \Rightarrow \quad n = \frac{a}{e} \quad \text{and}$$

$$2s = 2ae \quad \Rightarrow \quad s = ae$$

$$\text{From (1)}, \quad |PS|^2 = e^2|PM|^2$$

$$\Longrightarrow \quad (x-ae)^2 + y^2 = e^2\left(\frac{a}{e} - x\right)^2$$

$$x^2 - 2ae + a^2e^2 + y^2 = a^2 - 2aex + e^2x^2$$

$$x^2(1-e^2) + y^2 = a^2(1-e^2)$$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

$$\Longrightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where} \quad b^2 = a^2(1-e^2)$$

Ellipse (cont.)

- **NB**: From definition:

$$\frac{|PS|}{|PM|} = e, \quad 0 < e < 1$$

- For convenience, the focus has coordinates $(-ae, 0)$ and the directrix has the equation $x = -\frac{a}{e}$.

$$\text{Thus } |PS| = e|PM|$$

$$\sqrt{(x+ae)^2 + y^2} = e\left(x + \frac{a}{e}\right)$$

$$(x+ae)^2 + y^2 = e^2\left(x + \frac{a}{e}\right)^2 = (ex+a)^2$$

$$\implies x^2 + 2aex + a^2e^2 + y^2 = e^2x^2 + 2aex + a^2$$

$$x^2(1-e^2) + y^2 = a^2(1-e^2)$$

Ellipse (cont.)

Dividing through by $a^2(1-e^2)$, we have

$$\frac{x^2(1-e^2)}{a^2(1-e^2)} + \frac{y^2}{a^2(1-e^2)} = 1$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where} \;\; b^2 = a^2(1-e^2)$$

Thus, the equation of an ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The Graph of The Ellipse

- Intercept on $x-$axis, set $y = 0$

$$\implies x^2 = a^2 \Rightarrow x = \pm a$$

Thus, the curve passes through $(-a, 0)$ and $(a, 0)$

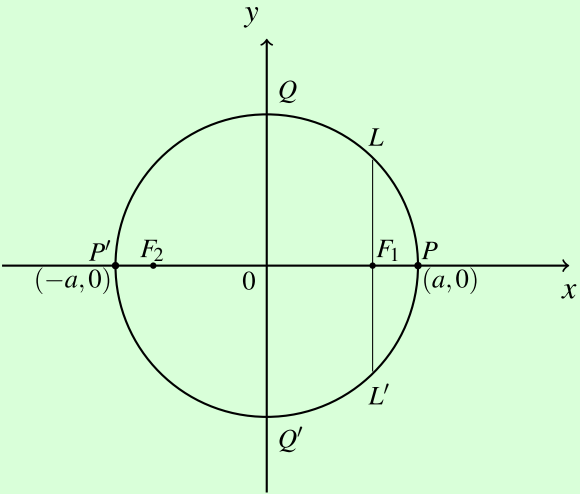
- Intercept on $y-$axis, set $x = 0$

$$\implies y^2 = b^2 \Rightarrow y = \pm b$$

Thus, the curve passes through $(0, -b)$ and $(0, b)$

Symmetry

The graph is symmetric about both the x-axis and the y-axis.



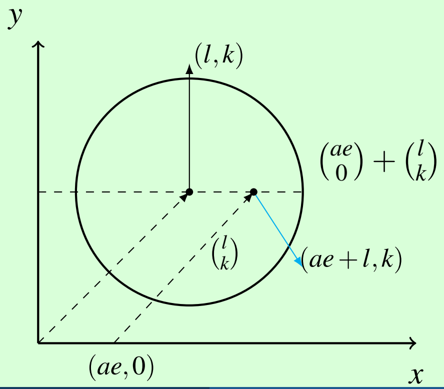
Symmetry (cont.)

- The equation obtained is the standard form of the equation of the ellipse with center $(0,0)$.
- The line LL' is the ***Latus Rectum*** of the ellipse. (The line LL' passing through the focus and parallel to the directrix is called the Latus Rectum of the ellipse).
- The ellipse has two axes
- The line PP' is called the major axis and has length $2a$. The number a is called the semi-major axis of the ellipse.
- The line QQ' is called the minor axis and has length $2b$. The number b is called the semi-minor axis of the ellipse

Translation of The Ellipse

- If the ellipse is translated by the vector $\begin{pmatrix} l \\ k \end{pmatrix}$, we obtain the equation

$$\frac{(x-l)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$



The foci are translated by the vector $\begin{pmatrix} l \\ k \end{pmatrix}$. The semi-major and semi-minor axes remain unchanged.

Finding Equations of Tangents And Normals At $P(x_1, y_1)$ of an Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\implies \frac{y}{b^2}\frac{dy}{dx} = \frac{-x}{a^2}$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \quad (1)$$

At point (x_1, y_1) on the ellipse, we have

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{b^2}{a^2}\frac{x_1}{y_1} \quad (2)$$

cont.

Thus, the equation of the tangent is given by

$$\begin{aligned} y - y_1 &= -\frac{b^2 x_1}{a^2 y_1}(x - x_1) \\ \implies \quad a^2 y_1 (y - y_1) &= -b^2 x_1 (x - x_1) \\ a^2 y y_1 - a^2 y_1^2 &= -b^2 x x_1 + b^2 x_1^2 \end{aligned}$$

Dividing through by $a^2 b^2$:

$$\begin{aligned} \frac{y y_1}{b^2} - \frac{y_1^2}{b^2} &= -\frac{x x_1}{a^2} + \frac{x_1^2}{a^2} \\ \frac{x x_1}{a^2} + \frac{y y_1}{b^2} &= \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1 \\ \implies \quad \frac{x x_1}{a^2} + \frac{y y_1}{b^2} &= 1 \end{aligned}$$

cont.

From (1) and (2), we have

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

$$\text{At point } (x_1, y_1) \quad \left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{b^2x_1}{a^2y_1}$$

Thus, the gradient of the normal at point (x_1, y_1) is given by

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{a^2y_1}{b^2x_1}$$

cont.

$$\text{Gradient of the chord } PQ = \frac{b\sin\theta_1 - b\sin\theta_2}{a\cos\theta_1 - a\cos\theta_2}$$
$$= \frac{b(\sin\theta_1 - \sin\theta_2)}{a(\cos\theta_1 - \cos\theta_2)}$$

Using the half angles, we have

$$\sin\theta_1 - \sin\theta_2 = 2\cos\frac{1}{2}(\theta_1+\theta_2)\sin\frac{1}{2}(\theta_1-\theta_2)$$
$$\cos\theta_1 - \cos\theta_2 = -2\sin\frac{1}{2}(\theta_1+\theta_2)\sin\frac{1}{2}(\theta_1-\theta_2)$$
$$\text{Thus Grad. } PQ = \frac{b}{a}\left[\frac{2\cos\frac{1}{2}(\theta_1+\theta_2)\sin\frac{1}{2}(\theta_1-\theta_2)}{-2\sin\frac{1}{2}(\theta_1+\theta_2)\sin\frac{1}{2}(\theta_1-\theta_2)}\right]$$
$$= -\frac{b}{a}\cot\frac{1}{2}(\theta_1+\theta_2)$$

cont.

Equation of the chord PQ of the ellipse is given by

$$y - b\sin\theta_1 = -\frac{b}{a}\cot\frac{1}{2}(\theta_1+\theta_2)(x - a\cos\theta_1)$$
$$ay - ab\sin\theta_1 = -b\cot\frac{1}{2}(\theta_1+\theta_2)(x - a\cos\theta_1)$$

Deducing The Equation of The Tangent at P

Using the equation of the chord of an ellipse:

$$ay - ab\sin\theta_1 = -b\cot\frac{1}{2}(\theta_1+\theta_2)(x - a\cos\theta_1)$$

As Q approaches the point P, the chord PQ turns to the tangent at P. Which follows that the equation of the normal at (x_1, y_1) is given by

$$\begin{aligned}
y - y_1 &= \frac{a^2 y_1}{b^2 x_1}(x - x_1)\\
b^2 x_1 (y - y_1) &= a^2 y_1 (x - x_1)\\
b^2 x_1 y - b^2 x_1 y_1 &= a^2 y_1 x - a^2 y_1 x_1\\
b^2 x_1 y - a^2 y_1 x &= b^2 x_1 y_1 - a^2 y_1 x_1\\
b^2 x_1 y - a^2 y_1 x &= (b^2 - a^2) x_1 y_1
\end{aligned}$$

Deducing The Equation of The Tangent at P (cont.)

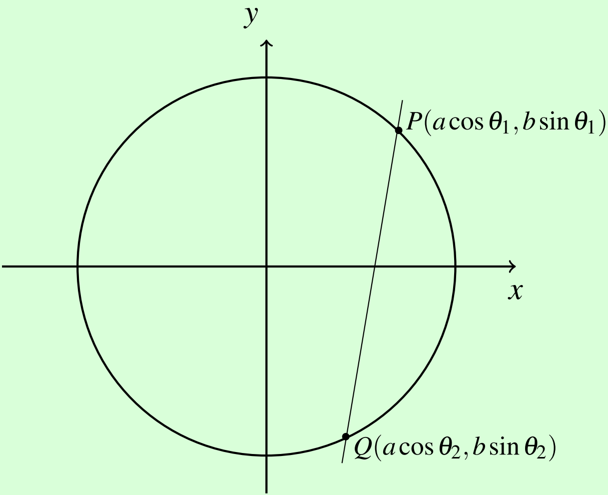
Dividing through by a^2b^2, we have

$$\frac{x_1 y}{a^2} - \frac{y_1 x}{b^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right) x_1 y_1$$

Thus, the equation of the normal at (x_1, y_1) of an ellipse is given by

$$\frac{x_1 y}{a^2} - \frac{y_1 x}{b^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right) x_1 y_1$$

Equation of The Chord Joining The Points P and Q of an Ellipse With Parameters θ_1 And θ_2



cont.

$$
\begin{aligned}
& ay - ab\sin\theta_1 = -b\cot\frac{1}{2}(2\theta_1)(x - a\cos\theta_1) \\
\Longrightarrow \quad & ay - ab\sin\theta_1 = -b\cot\theta_1(x - a\cos\theta_1)
\end{aligned}
$$

Example

1. Find the equation of (i) the tangent and (ii) the normal at the point $P(x_1, y_1)$ of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Example

1. Find the equation of (i) the tangent and (ii) the normal at the point $P(x_1, y_1)$ of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Solution

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\implies \quad \frac{dy}{dx} = -\frac{2x}{a^2}\frac{b^2}{2y} = -\frac{b^2 x}{a^2 y}$$

$$\text{At}\, P(x_1, y_1), \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{b^2 x_1}{a^2 y_1}$$

$$\implies \quad \text{Grad. of the tangent} = -\frac{b^2 x_1}{a^2 y_1}$$

Example (cont.)

The equation of the tangent is given by

$$
\begin{aligned}
(y-y_1) &= -\frac{b^2x_1}{a^2y_1}(x-x_1)\\
a^2y_1(y-y_1) &= -b^2x_1(x-x_1)\\
a^2y_1y-a^2y_1^2 &= -b^2x_1x+b^2x_1^2\\
b^2xx_1+a^2yy_1 &= b^2x_1+a^2y_1^2
\end{aligned}
$$

Dividing through by a^2b^2, we have

$$
\begin{aligned}
&\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2} = 1\\
\implies &\frac{xx_1}{a^2}+\frac{yy_1}{b^2} = 1
\end{aligned}
$$

Example (cont.)

(ii) Normal

$$\text{Gradient of the normal} = \frac{a^2y}{b^2x}$$

$$\text{At } (x_1, y_1), \text{ gradient of the normal} = \frac{a^2y_1}{b^2x_1}$$

Equation of the normal at (x_1, y_1) is given by

$$(y - y_1) = \frac{a^2y_1}{b^2x_1}(x - x_1)$$
$$b^2x_1(y - y_1) = a^2y_1(x - x_1)$$
$$b^2x_1y - b^2x_1y_1 = a^2y_1x - a^2y_1x_1$$
$$b^2x_1y - a^2y_1x = b^2x_1y_1 - a^2y_1x_1$$
$$b^2x_1y - a^2y_1x = (b^2 - a^2)x_1y_1$$

Example (cont.)

Dividing through by a^2b^2, we have

$$\frac{x_1y}{a^2} - \frac{y_1x}{b^2} = \left(\frac{1}{a^2} - \frac{1}{b^2}\right)x_1y_1$$

Example

2. Find the centre, the foci and the eccentricity of the following ellipses:

   (a) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

   (b) $64x^2 + 25y^2 = 1600$

   (c) $4x^2 + 9y^2 - 8x + 36y - 14 = 0$

Example (cont.)

Solution

(a)

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \text{ is in standard form}$$

$$\text{Semi-major axis} = 4$$

$$\text{Semi-minor axis} = 3$$

$$\text{The centre} = (0,0)$$

The eccentricity may be computed from

$$b^2 = a^2(1-e^2) \Rightarrow 9 = 16(1-e^2)$$

$$\Longrightarrow 1-e^2 = \frac{9}{16} \Rightarrow e = \sqrt{1-\frac{9}{16}} = \frac{\sqrt{7}}{4}$$

$$\Longrightarrow \text{Eccentricity} = \frac{\sqrt{7}}{4}$$

Example (cont.)

The foci are

$$
\begin{aligned}
&(-ae, 0) \ \text{ and } \ (ae, 0) \\
\text{i.e. } &\left(-\frac{\sqrt{7}}{4}.4,\ 0\right) \ \text{ and } \ \left(\frac{\sqrt{7}}{4}.4,\ 0\right) \\
&= (-\sqrt{7}, 0) \ \text{ and } \ (\sqrt{7}, 0)
\end{aligned}
$$

Example (cont.)

Solution

(b)

$$
\begin{aligned}
64x^2 + 25y^2 &= 1600 \\
\frac{64}{1600}x^2 + \frac{25}{1600}y^2 &= 1 \\
\frac{x^2}{25} + \frac{y^2}{64} &= 1 \\
\text{Semi-major axis } \; a &= 5 \\
\text{Semi-minor axis } \; b &= 8 \\
\text{The center } &= (0,0)
\end{aligned}
$$

The eccentricity is given by

$$
\begin{aligned}
b^2 = a^2(1-e^2) \;\; \Rightarrow \;\; 25 &= 64(1-e^2) \\
\Longrightarrow \;\; 1 - e^2 &= \frac{25}{64} \\
\Longrightarrow \;\; e = \sqrt{1 - \frac{25}{64}} &= \frac{\sqrt{39}}{8} \\
\Longrightarrow \;\; \text{Eccentricity } \; e &= \frac{\sqrt{39}}{8}
\end{aligned}
$$

Example (cont.)

The foci are

$$
\begin{aligned}
&(-ae,0) \text{ and } (ae,0)\\
\text{i.e. } &\left(0,\ -\frac{\sqrt{39}}{8}\right) \text{ and } \left(0,\ \frac{\sqrt{39}}{8}\right)\\
&= (0,-6.2) \text{ and } (0,6.2)
\end{aligned}
$$

Example (cont.)

Solution

(c)

$$4x^2 + 9y^2 - 8x + 36y - 14 = 0$$

Completing the squares, we have

$$\begin{aligned}
4(x^2 - 2x) + 9(y^2 + 4y) + 4 &= 0 \\
4\left[(x-1)^2 - 1\right] + 9\left[(y+2)^2 - 4\right] + 4 &= 0 \\
4(x-1)^2 - 4 + 9(y+2)^2 - 36 + 4 &= 0 \\
4(x-1)^2 + 9(y+2)^2 &= 36 \\
\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} &= 1
\end{aligned}$$

Thus, the ellipse has centre $(1, -2)$, semi-major axis is 3 and semi-minor axis is 2

Example (cont.)

The eccentricity is given by

$$
\begin{aligned}
b^2 &= a^2(1-e^2) \\
\implies \quad 3^2 &= 2^2(1-e^2) \\
\implies \quad 1-e^2 &= \frac{3^2}{2^2} \\
\implies \quad e &= \sqrt{1-\frac{9}{4}} = \sqrt{\frac{5}{9}} \\
\implies \quad \text{Eccentricity} \quad e &= \frac{\sqrt{5}}{3}
\end{aligned}
$$

Example (cont.)

The foci are

$$\begin{aligned} &(-ae, 0) \text{ and } (ae, 0) \\ \text{i.e. } &(-\sqrt{5}, 0) \text{ and } (\sqrt{5}, 0) \\ \Longrightarrow &\begin{pmatrix} -\sqrt{5} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1-\sqrt{5} \\ -2 \end{pmatrix} = \begin{pmatrix} -1.2 \\ -2 \end{pmatrix} \\ \text{and } &\begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1+\sqrt{5} \\ -2 \end{pmatrix} = \begin{pmatrix} 3.2 \\ -2 \end{pmatrix} \end{aligned}$$

Thus, the foci of the ellipse is given by $(-1.2, -2)$ and $(3.2, -2)$

Example

3. The tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the point $(a\cos t, b\sin t)$ and the normal at the same point meet the x-axis at the points T and S respectively. Find the coordinates of T and S.

Example

3. The tangent to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ at the point $(a\cos t, b\sin t)$ and the normal at the same point meet the x-axis at the points T and S respectively. Find the coordinates of T and S.

Solution

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\implies \quad \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$\text{At point } T(a\cos t, b\sin t): \quad \frac{dy}{dx}=\frac{-b^2(a\cos t)}{a^2(b\sin t)}=-\frac{b}{a}\cot(t)$$

Example (cont.)

Equation of the tangent at T is given by

$$y - b\sin t = -\frac{b}{a}\cot(t)(x - a\cos t)$$

where it meets the x-axis, $y = 0$

$$\begin{aligned}
\implies \quad -b\sin t &= -\frac{b}{a}\frac{\cos t}{\sin t}(x - a\cos t) \\
-ab\sin^2 t &= -b\cos t(x - a\cos t) \\
\implies \quad x - a\cos t &= \frac{ab\sin^2 t}{b\cos t} = \frac{a\sin^2 t}{\cos t} \\
x &= a\left(\cos t + \frac{\sin^2 t}{\cos t}\right) = \frac{a}{\cos t}
\end{aligned}$$

Thus, the coordinates of T are $\left(\frac{a}{\cos t}, 0\right)$ or $(a\sec t, 0)$

Example (cont.)

$$\text{Gradient of the tangent} = -\frac{b}{a}\cot(t)$$

$$\implies \text{Gradient of the normal} = \frac{a}{b}\frac{\sin t}{\cos t}$$

Equation of the normal at S is given by

$$y - b\sin t = \frac{a}{b}\frac{\sin t}{\cos t}(x - a\cos t)$$

$$\text{At } S,\ y = 0$$

$$\implies \quad -b\sin t = \frac{a}{b}\frac{\sin t}{\cos t}(x - a\cos t)$$

$$-b^2\sin t\cos t = a\sin t(x - a\cos t)$$

$$x - a\cos t = -\frac{b^2\sin t\cos t}{a\sin t} = -\frac{b^2}{a}\cos t$$

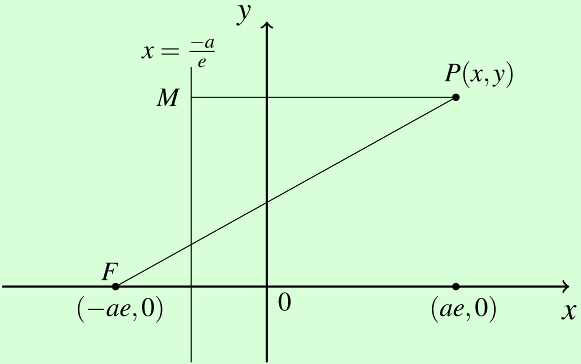
Example (cont.)

$$x = a\cos t - \frac{b^2}{a}\cos t$$
$$= \frac{(a^2 - b^2)}{a}\cos t$$

Thus, the coordinates of S are $\left(\dfrac{a^2 - b^2}{a}\cos t, 0\right)$

The Hyperbola

- The hyperbola is the locus of a point in a plane such that its distance from a certain fixed point F, the focus, is greater than its distance from a certain fixed line D, the directrix.



The Hyperbola (cont.)

In this case $\dfrac{|PF|}{|PM|} = e, \quad e > 1$

$$\begin{aligned}
\Longrightarrow \quad |PF|^2 &= e^2|PM|^2 \\
(x+ae)^2 + y^2 &= e^2(x+\tfrac{a}{e})^2 = (xe+a)^2 \\
x^2 + 2aex + a^2e^2 + y^2 &= x^2e^2 + 2aex + a^2 \\
x^2(1-e^2) + y^2 &= a^2(1-e^2) \\
x^2(e^2-1) - y^2 &= a^2(e^2-1) \;\; \text{(multiply through by } -1\text{)} \\
\Longrightarrow \quad \frac{x^2(e^2-1)}{a^2(e^2-1)} - \frac{y^2}{a^2(e^2-1)} &= 1 \\
\Longrightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \quad \text{where } b^2 = a^2(e^2-1)
\end{aligned}$$

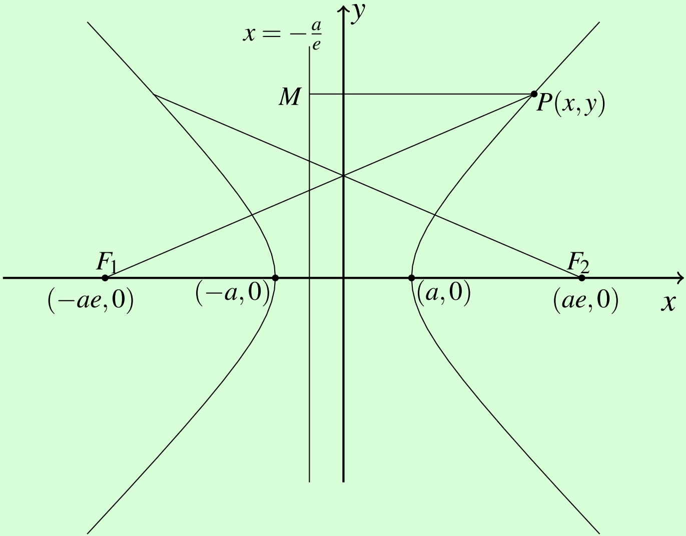
Graph of The Hyperbola

$$\text{Intercept}: \quad x = 0 \;\Rightarrow\; \frac{y^2}{b^2} = -1 \;\text{ which is impossible}$$

Thus, the curve does not cut the y-axis.

$$\text{Intercept}: \quad y = 0 \;\Rightarrow\; \frac{x^2}{a^2} = 1 \;\Rightarrow\; x = \pm a$$

Graph of The Hyperbola

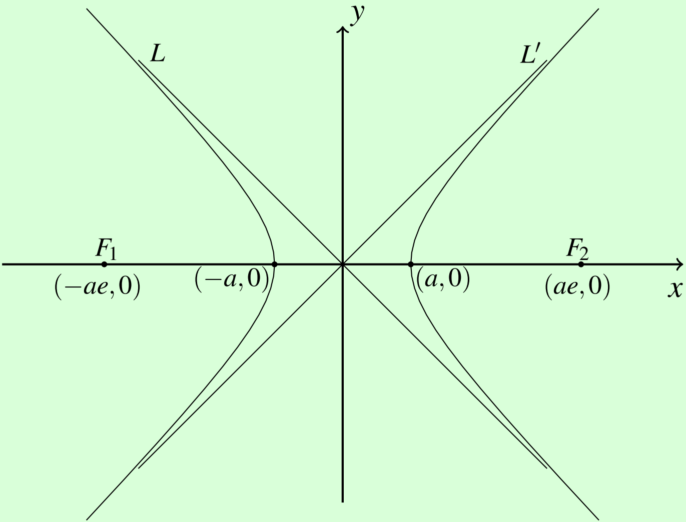


The curve does not pass through the origin

Symmetry

- The curve is symmetrical about both the x-axis and the y-axis since the equation is unchanged under the transformation $x \to -x$ and $y \to -y$.
- As $x \to \infty$, $y \to \pm\infty$ and as $x \to -\infty$, $y \to \pm\infty$

Asymptotes



The line L and L' are called the asymptotes of the hyperbola.

Asymptotes (cont.)

- To find the asymptotes' equation (or equations of the asymptotes), we write

$$
\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\
\implies \frac{y^2}{b^2} &= \frac{x^2}{a^2} - 1 \\
\implies y^2 &= b^2 \left(\frac{x^2}{a^2} - 1 \right) \\
&= \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2x^2}{a^2} \left(1 - \frac{a^2}{x^2} \right) \\
\implies y &= \pm \frac{b}{a} x \sqrt{\left(1 - \frac{a^2}{x^2} \right)}
\end{aligned}
$$

Asymptotes (cont.)

Now, we observe that as $n \to \pm\infty$, $y = \pm\frac{b}{a}x$ (i.e. for large values of x, the curve looks like the line $y = \frac{b}{a}x$).

$$\text{Thus } L \text{ has the equation } \; y = -\frac{b}{a}x$$
$$\text{and } L' \text{ has the equation } \; y = \frac{b}{a}x$$

Rectangular Hyperbola

- If the angle between L and L' is $90°$, then the curve is called a ***Rectangular*** hyperbola. In this case, the asymptotes will have the equation

$$y = \pm x \quad (\text{since } a = b \text{ in this case})$$

The equation of the rectangular hyperbola is given by

$$x^2 - y^2 = a^2$$

- A rectangular hyperbola has the general equation

$$xy = c^2$$

Parametric equations of the rectangular hyperbola is given by

$$x = ct; \quad y = \frac{c}{t}$$

Rectangular Hyperbola (cont.)

Thus, from $xy = c^2$, we have

$$(ct)\left(\frac{c}{t}\right) = c^2$$

$$\text{Also}\ \ xy = c^2$$

$$y + x\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

At the point $T\left(xt, \frac{c}{t}\right)$, we have

$$\frac{dy}{dx} = -\frac{\frac{c}{t}}{ct} = -\frac{1}{t^2}$$

Rectangular Hyperbola (cont.)

Equation of the tangent of the rectangular hyperbola:

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$
$$t^2y - ct = -x + ct$$
$$x + t^2y - 2ct = 0$$

Finding the equation of the normal at $T(ct, \frac{c}{t})$ of a rectangular hyperbola:

$$\text{Gradient of tangent} = -\frac{1}{t^2}$$
$$\text{Gradient of normal} = t^2$$

Rectangular Hyperbola (cont.)

Equation of the normal at T is given by

$$y - \frac{c}{t} = t^2(x - ct)$$
$$ty - c = t^3x - ct^4$$
$$t^3x - ty - ct^4 + c = 0$$

Thus, the equation of the normal at the point $T(ct, \frac{c}{t})$ of a rectangular hyperbola is given by

$$t^3x - ty - ct^4 + c = 0$$

Parametric Equations

The parametric equations for the hyperbola are

(A) $P(a\sec\theta, b\tan\theta)$

(B) $P\left[a\left(\frac{1+t^2}{1-t^2}\right), b\left(\frac{2t}{1-t^2}\right)\right]$, where θ and t are parameters.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$x = a\sec\theta; \quad y = b\tan\theta$$

$$\text{or} \quad x = a\sec t; \quad y = b\tan t$$

$$\Longrightarrow \quad \frac{(a\sec t)^2}{a^2} - \frac{(b\tan t)^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

Tangents And Normals

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\implies \frac{dy}{dx} = \frac{b^2x}{a^2y}$$

Equation of the tangent at point (x_1, y_1) on the hyperbola is given by

$$y - y_1 = \frac{b^2x_1}{a^2y_1}(x - x_1)$$
$$a^2yy_1 - a^2y_1^2 = b^2x_1x - b^2x_1^2$$
$$a^2yy_1 - b^2xx_1 = a^2y_1^2 - b^2x_1^2$$

Tangents And Normals (cont.)

$$\frac{yy_1}{b^2} - \frac{xx_1}{a^2} = \frac{y_1^2}{b^2} - \frac{x_1^2}{a^2}$$

$$\text{or} \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = 1$$

$$\Longrightarrow \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Equation of the normal at $T(x_1, y_1)$ on the hyperbola is given by

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$$b^2 x_1 y - b^2 x_1 y_1 = -a^2 y_1 x + a^2 y_1 x_1$$

$$a^2 y_1 x + b^2 x_1 y = (a^2 + b^2) x_1 y_1$$

Tangents And Normals (cont.)

At the point $T(a\sec t, b\tan t)$, we have

$$\frac{dy}{dx} = \frac{b^2(a\sec t)}{a^2(b\sin t)} = \frac{b\sec t}{a\tan t} = \frac{b}{a}\cdot\frac{1}{\cos t}\cdot\frac{\cos t}{\sin t} = \frac{b}{a\sin t}$$

Equation of the tangent at $T(a\sec t, b\tan t)$ is given by

$$\begin{aligned}
y - b\tan t &= \frac{b}{a\sin t}(x - a\sec t)\\
ay\sin t - ab\tan t\sin t &= bx - ab\sec t\\
(a\sin t)y - bx &= ab(\tan t\sin t - \sec t)\\
&= ab(\sin^2 t - 1)\\
&= -\frac{ab}{\cos t}\cos^2 t = -ab\cos t
\end{aligned}$$

Tangents And Normals (cont.)

$$\implies (a\sin t)y - bx = -ab\cos t$$
$$\text{or } \; bx - (a\sin t)y = ab\cos t$$

The normal at $T(a\sec t, b\tan t)$ has the equation

$$\begin{aligned}
y - b\tan t &= -\frac{a\sin t}{b}(x - a\sec t) \\
by - b^2\tan t &= -ax\sin t + a^2\sin t\sec t \\
by + (a\sin t)x &= a^2\sin t\sec t + b^2\tan t \\
&= a^2\tan t + b^2\tan t \\
&= (a^2+b^2)\tan t \\
\implies (a\sin t)x + by &= (a^2+b^2)\tan t
\end{aligned}$$

Example

1. Find the centre, foci and asymptotes of

   (i) $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$

   (ii) $4x^2 - y^2 + 24x + 4y + 28 = 0$

   (iii) $x^2 - y^2 = 1$

Example

1. Find the centre, foci and asymptotes of
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   (iii) $x^2 - y^2 = 1$

Solution

(i)

$$\frac{x^2}{9} - \frac{y^2}{4} = 1 \quad \Rightarrow \quad \text{center} = (0,0)$$

$$b^2 = a^2(e^2 - 1) \ \text{ where } a = 3 \text{ and } b = 2$$

$$\Longrightarrow \quad e^2 = \sqrt{\frac{b^2}{a^2} + 1} = \sqrt{\frac{4}{9} + 1} = \frac{\sqrt{13}}{3}$$

Example (cont.)

(i) The foci are

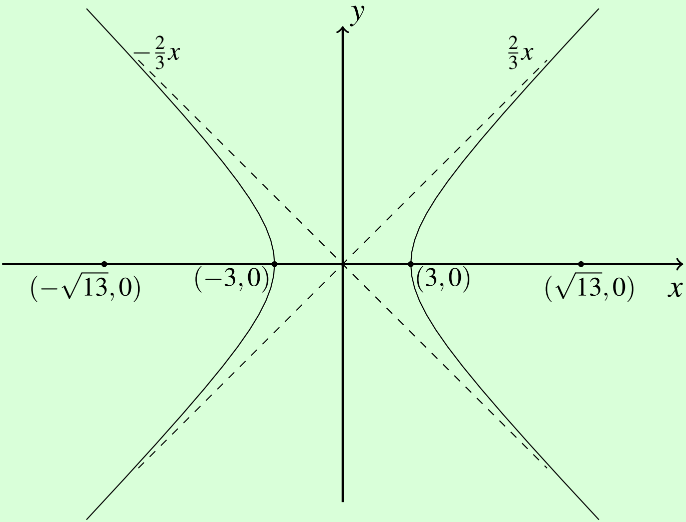
$$F(-ae,0) \text{ and } F'(ae,0)$$

$$\text{i.e. } \left(-\frac{\sqrt{13}}{3}(3),\ 0\right) \text{ and } \left(\frac{\sqrt{13}}{3}(3),\ 0\right)$$

$$= (-\sqrt{13},0) \text{ and } (\sqrt{13},0)$$

The asymptotes are $y = \pm\dfrac{b}{a}x = \pm\dfrac{2}{3}x$

Sketch of the Graph for (i)



Example (cont.)

Solution

(ii)

$$
\begin{aligned}
4x^2 - y^2 + 24x + 4y + 28 &= 0 \\
4x^2 + 24x - (y^2 - 4y) + 28 &= 0 \\
4(x^2 + 6x) - \left[(y^2 - 2) - 4\right] + 28 &= 0 \\
4\left[(x+3)^2 - 9\right] - (y-2)^2 + 4 + 28 &= 0 \\
4(x+3)^2 - (y-2)^2 - 36 + 4 + 28 &= 0 \\
4(x+3)^2 - (y-2)^2 &= 4 \\
\frac{(x+3)^2}{1} - \frac{(y-2)^2}{4} &= 1 \quad \Rightarrow a = 1 \text{ and } b = 2 \\
\implies \quad e = \sqrt{\frac{b^2}{a^2} + 1} &= \sqrt{\frac{4}{1} + 1} = \sqrt{5}
\end{aligned}
$$

Example (cont.)

(ii)

$$\text{The translation vector} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

$$\text{The center of the hyperbola} = (-3, 2)$$

The foci are

$$(-\sqrt{5} - 3, 2) \text{ and } (\sqrt{5} - 3, 2)$$

$$\text{i.e. } \left(-(3+\sqrt{5}),\ 0\right) \text{ and } \left(-3+\sqrt{5},\ 0\right)$$

The asymptotes are $y - 2 = \pm\frac{b}{a}(x+3) = \pm\frac{2}{1}(x+3)$. Thus, the asymptotes have equation

$$y = 2x + 8 \text{ and } y = -2x - 4$$

.

Example (cont.)

Solution

(iii)

$$a = b = 1$$
$$b^2 = a^2(e^2 - 1) \Rightarrow 1^2 = 1^2(e^2 - 1)$$
$$1 = e^2 - 1 \Rightarrow e^2 = 1 + 1 = 2$$
$$e = \sqrt{2}$$

The foci are $(-\sqrt{2}, 0)$ and $(\sqrt{2}, 0)$.
The center is $(0, 0)$
Asymptotes is $y = \pm x$

Example

2. The tangents at the points $T(a\sec t_1, b\tan t_1)$ and $S(a\sec t_2, b\tan t_2)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meet at the point N. Find the coordinates of N.

Example

2. The tangents at the points $T(a\sec t_1, b\tan t_1)$ and $S(a\sec t_2, b\tan t_2)$ on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ meet at the point N. Find the coordinates of N.

Solution

At $T(a\sec t_1, b\tan t_1)$

$$\frac{dy}{dx} = \frac{b^2(a\sec t_1)}{a^2 b\tan t_1} = \frac{b\sec t_1}{a\tan t_1} = \frac{b}{a}\cdot\frac{1}{\cos t_1}\cdot\frac{\cos t_1}{\sin t_1} = \frac{b}{a\sin t}$$

Example (cont.)

Equation of the tangent at T is given by

$$
\begin{aligned}
y - b\tan t_1 &= \frac{b}{a\sin t_1}(x - a\sec t_1) \\
ay\sin t_1 - ab\tan t_1 \sin t_1 &= bx - ab\sec t_1 \\
(a\sin t_1)y - bx &= ab(\tan t_1 \sin t_1 - \sec t_1) \\
&= ab(\sin^2 t_1 - 1) \\
&= -\frac{ab\cos^2 t_1}{\cos t_1} \\
&= -ab\cos t_1 \\
\implies \quad bx - a(\sin t_1)y &= ab\cos t_1
\end{aligned}
$$

Example (cont.)

The equation of tangent at T:

$$bx - (a\sin t_1)y = ab\cos t_1 \tag{1}$$

Also, the equation of tangent at S:

$$bx - (a\sin t_2)y = ab\cos t_2 \tag{2}$$

(2) – (1):

$$(-a\sin t_2 + a\sin t_1)y = ab(\cos t_2 - \cos t_1)$$

$$\implies \quad y = \frac{b(\cos t_2 - \cos t_1)}{\sin t_1 - \sin t_2}$$

Try

A hyperbola has foci $(2,5)$ and $(2,-3)$ and vertices at $(2,3)$and $(2,1)$. Find the equation of the hyperbola.