CSM 165: Discrete Mathematics for Computer Science

Chapter 2: Set Theory

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Introduction

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Sets

Definition 1

*A set a **well-defined** collection of distinct objects, called **elements** or **members** of the set.*

*A set is said to **contain** its elements.*

Set are usually denoted by capital letters.

Sets

Elements of the set are represented by small letters

We write $\mathbf{a} \in \mathbf{A}$ to denote that a is an element of the set A.

The notation $\mathbf{a} \notin \mathbf{A}$ denotes that a is not an element of the set A.

Representation of Sets

1. **Roster Form**: By listing elements in braces {}, separated by comma.
   Example: $A = \{a, e, i, o, u\}$, $B = \{0, 2, 4, 6, 8, 10\}$

2. **Set-Builder Form:** By using statements depicting properties or relations among members.

   Example:
   $A = \{x \mid x \text{ is a vowel in English alphetic series}\}$
   $B = \{x : x \text{ is an even number less than } 12\}$

Sets

Example 1

1. $\mathbb{N} = \{1, 2, 3, \ldots\}$, the set of natural numbers
2. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of integers
3. $\mathbb{Z}^+ = \{1, 2, 3, ...\}$, the set of positive integers
4. $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z}, \text{and } q \neq 0\}$, the set of rational numbers
5. $R = \{x : x \text{ is a real number}\}$, the set of real numbers

Sets

Definition 2

*The **empty set** or the **null set** is a set that has no element, denoted by $\emptyset$ or { }*

Definition 3

*A set with only one element is called a **singleton set**.*

Example 2

1. $A = \{1\}$

2. $B = \{\emptyset\}$

Sets

Definition 4 (Subsets)

The set A is a subset of B if and only if every element of A is also an element of B.

We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

OR

$B \supseteq A$ (B is a superset of A)

Sets

NB: $A \subseteq B$ if and only if the quantification $\forall x(x \in A \to x \in B)$ is true

Properties

1. **Reflexivity:** $A \subseteq A$
2. **Transitivity:** $(A \subseteq B) \land (B \subseteq C) \to (A \subseteq C)$

Sets

Definition 5

Two sets are equal if and only if they have the same elements.

If A and B are sets, then A and B are equal if and only $\forall x(x \in A \leftrightarrow x \in B)$.

OR

Two sets A and B are said to be equal if $A \subseteq B$ *and* $B \subseteq A$

We write $A = B$ *if A and B are equal sets.*

Equal Sets

Properties:

1. **Reflexive:** $A = A$
2. **Symmetric:** $A = B \rightarrow B = A$
3. **Transitive:** $(A = B) \wedge (B = C) \rightarrow (A = C)$

Sets

Example 3

1. Sets $\{1,3,5\}$ and $\{3,5,1\}$ are equal.

2. If $A = \{a, e, i, o, u\}$ and $B = \{u, e, i, o, a\}$ then $A = B$.

Exercise A:

Let $A = \{\{1,2\},3,4,5,6\}$, $B = \{1,2,3,4,5\}$. Determine whether each of the following is true/false

1. $\{1,2\} \in B$
2. $A = B$
3. $A \subseteq B$
4. $B \subseteq A$

Sets

Definition 6

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the ***cardinality*** *of S.*

The cardinality of S is denoted by $|S|$

A set is said to be ***infinite*** *if it is not finite.*

Sets

Definition 7

Given a set S, the power set of S is the set of all subsets of the set S.

The power set of S is denoted by $\mathcal{P}(S)$.

NB: *the empty set and the set itself are members of this set of subsets.*

If a set has n elements, then its power set has 2^n *elements.*

Example 4

What is the power set of each of the following sets?

(a) $A=\{0, 1, 2\}$

(b) $B=\{\emptyset\}$

(c) $C=\{\emptyset, \{\emptyset\}\}$

Cartesian Products

Definition 8

The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ *is the ordered collection that has* a_1 *as its first element,* a_2 *as its second element, . . . , and* a_n *as its nth element.*

The ordered n-tuples $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$ if and only if $a_i = b_i \; \forall \, i = 1, 2, \ldots, n$

Definition 9

Let A and B be sets. The Cartesian product of A and B, denoted b y $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Cartesian Products

Example 5

(i) What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
Solution:
$A \times B = (1, a), (1, b), (1, c), (2, a), (2, b), (2, c)$.

(ii)

Cartesian Product

Definition 10

The Cartesian product of the sets $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for $i = 1, 2, \ldots, n$:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \textit{ for } i = 1, 2, \ldots, n$$

Example 6

What is the Cartesian product $A \times B \times C$, where $A = \{0,1\}$, $B = \{1,2\}$, and $C = \{0,1,2\}$?

Solution

$$\begin{aligned} A \times B \times C = \{&(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),\\ &(0,2,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),\\ &(1,2,1),(1,2,2)\}. \end{aligned}$$

Operations on Sets

Definition 11

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B.

$$A \cup B = \{x | x \in A \vee x \in B\}$$

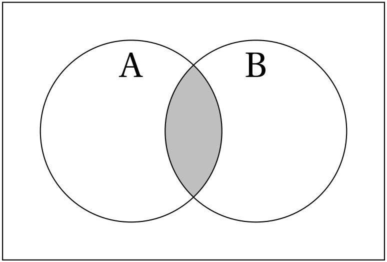
Operations on Sets

Definition 12

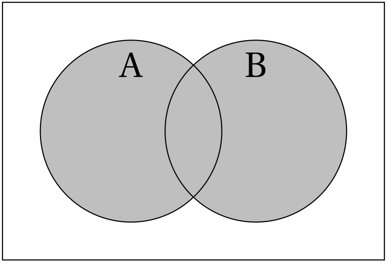
Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

$$A \cap B = \{x | x \in A \wedge x \in B\}$$

Operations on Sets



$A \cap B$



$A \cup B$

If $A_1, A_2, \ldots, A_n$ are subsets of the universal sets then,

$$\bigcup_{k=1}^{n} A_k = A_1 \cup A_2 \cup \cdots \cup A_n \quad \text{and} \quad \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap \cdots \cap A_n$$

Example 7

Let $A = \{1, 2, 3, 4, 5, 6, 7\}$, $B = \{1, 3, 5, 6, 8\}$, $C = \{2, 4, 5, 6, 9\}$, then

$A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A \cap B \cap C = \{5, 6\}$

Difference of two Sets

Definition 13

Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B.

$$A - B = \{x | x \in A \wedge x \notin B\}$$

Example 8

The difference of {1,3,5} and {1,2,3} is the set:
$\{1, 3, 5\} - \{1, 2, 3\} = \{5\}$

Principle of Inclusion & Exclusion

Let A and B be two finite sets in a universal set U. If A and B are disjoint, then

$$|A \cup B| = |A| + |B| \tag{1}$$

Theorem 1

If A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Similarly for three finite sets:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Principle of Inclusion & Exclusion

Theorem 2

If $A_1, A_2, \ldots, A_n$ are finite sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| +$$

$$\sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| - \cdots +$$

$$(-1)^{n+1} |A_1 \cap A_2 \cap \cdots \cap A_n| \qquad (2)$$

Computer Representation of Sets

Let $U = \{1,2,3,4,5,6,7,8,9,10\}$ $A = \{2,4,5,7,9\}$ and $B = \{1,3,4,6,7\}$, then $A \cap B = \{4,7\}$ and $A \cup B = \{1,2,4,5,6,7,9$

U	1	1	1	1	1	1	1	1	1	1
A	0	1	0	1	1	0	1	0	1	0
B	1	0	1	1	0	1	1	0	0	0
$A \cup B$	1	1	0	1	1	1	1	0	1	0
$A \cap B$	0	0	0	1	0	0	1	0	0	0

Set Identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

Set Identities

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

End of Lecture

Questions…???

Thanks

Reference Books

1. Kenneth H. Rosen, “Discrete Mathematics and Its Applications”, Tata Mcgraw Hill, New Delhi, India, seventh Edition, 2012.

2. J. P. Tremblay, R. Manohar, “Discrete Mathematical Structures with Applications to Computer Science”, Tata Mc Graw Hill, India, 1st Edition, 1997.