Institute of Distance Learning, KNUST

Template
For
Course Material

A 3-credit course outlay

2011

Updated

KWAME NKRUMAH UNIVERSITY OF SCIENCE AND TECHNOLOGY, KUMASI

INSTITUTE OF DISTANCE LEARNING

(BSC COMPUTER SCIENCE, FIRST YEAR)

CSM 166: DISCRETE MATHEMATICS FOR COMPUTER SCIENCE II

[Credit: 3]

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ISBN:

Editors:

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			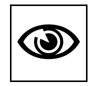	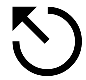
Learning Objective	Learning Activity	Unit Assignments	Review	Summary
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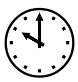		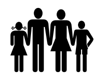		
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			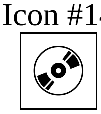	
Answer Tips	Note/Learning Tip	Pause	Interactive CD	Online

2. Guidelines for making use of learning support (virtual classroom, etc.)

This course material is also available online at the virtual classroom (v-classroom) Learning Management System. You may access it at www.kvcit.org

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Acknowledgement

I want to thank the almighty God for the knowledge and power granted me to produce this piece.

I am also grateful to the following persons without whose effort this piece might not have been a reality.

1. Mr. Robert Acquah, demonstrator at the mathematics department, KNUST.
2. Mr. Moses Obiri Yeboah, a teaching assistant at the mathematics department,KNUST.

Course Introduction

Discrete Mathematics is essentially that branch of mathematics that does not depend on limits; in this sense, it is the anti-thesis of Calculus. As computers are discrete object operating one jumpy, discontinuous step at a time, Discrete Mathematics is the right framework for describing precisely Computer Science concepts. Elucidate

COURSE OVERVIEW

The course starts with combinatorial analysis. Combinatorial analysis deals with counting techniques, an important concept in programming design. This is to establish several techniques for counting large finite sets without actually listing their elements. We expatiate on the techniques of counting as already treated under permutations and combinations and binomial expansion which is very much linked with combinations. From binomial, we extend to multinomial expansion and coefficients.

Often it is difficult to express the members of an object or numerical sequence explicitly. In other to generate infinite set of numbers, we look at recurrence relation, or simply put algorithm. We then add the study of the discrete analogue of differential equations. We will also look into the basic mathematical logic behind language of computer usage. This will help ease up complex structures in the complex language of the computer.

Since most students are familiar with mathematical relations such as “is a subset of”,” is less than” and so on, one frequently want to compare or contrast various members of a set by trying to arrange them in some appropriate order, or perhaps, to group together those with similar properties. The mathematical framework to describe this kind of organization of sets is the theory of relations. There are two kinds of relations, equivalence relations and order relations which we will discuss.

We end the course with basic basic concepts related to graphs and trees which has use in car navigation system, efficient database, build a boot to retrieve info off WWW, representing computational models and many other applications.

COURSE OBJECTIVES

1. Count large finite sets without listing them.
2. Have a recurrent relation that can generate infinite sets
3. Understand the mathematical logic behind the use of computer.
4. Use a mathematical framework to organise structures in computer.

COURSE OUTLINE

- Unit 1: FUNDAMENTALS OF COUNTING
- Unit 2: MULTINOMIAL EXPANSION AND COEFFICIENT
- Unit 3: RECURRENCE RELATION AND LINEAR DIFFERENCE METHOD
- Unit 4: BOOLEAN ALGEBRA AND LOGIC GATE
- Unit 5: RELATION AND ORDER IN SET
- Unit 6: ELEMENTS OF GRAPH THEORY

COURSE STUDY GUIDE

This provides a monthly/weekly schedule of progress of your learning.

Week #	Unit/Session	FFFS/Practical/Exam/Quiz
1	1/1.1,1.2	FFFS
2	2/2.1,2.2	FFFS
3	3/3.1,3.2	FFFS/Quiz
4	4/4.1,4.2	FFFS
5	5/5.1,5.3	FFFS
6	6/6.1,6.2	FFFS/Quiz

GRADING

Continuous assessment: 30%

End of semester examination: 70%

RESOURCES

Choose an item.

You will require Click here to enter text.for this course.

To complete this course you would need to accomplish [theory], [number of laboratory/workshop/ practical/ tutorial] and 3 credits

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2. Seymour Lipschutz and Marc Lars Lipson.(1992), Discrete Mathematics. Second Edition,McGraw – Hill Inc.

3. Marcel B. Finan, Arkansas Tech University

4. W W L CHEN(1982,2008), Imperial College, University of London.

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Unit 1

FUNDAMENTALS OF COUNTING

Introduction

Counting of large finite sets has been on the major hurdles in mathematics. The major goal of this unit is to establish several techniques for counting large finite sets without actually listing their elements. There are two basic counting principles used throughout. One involves addition and the other multiplication.

Learning Objectives

After reading this unit you should be able to:

1. Count large finite sets without listing them
2. Arrange objects with or without replacement
3. Arrange objects in an orderly manner
4. Difference between the permutation and combination
5. Enumerate all the logical possibilities of a sequence of events.

Unit content

Session 1-1: Elements of Counting

1-1.1 Sum Rule Principle

1-1.2 Product Rule Principle

Session 2-1: Permutation and Combination

2-1.1 Permutation

2-1.2 Ordered Sample

2-1.3 Combination

2-1.4 Tree Diagram

SESSION 1-1: **ELEMENTS OF COUNTING**

1-1.1 Sum Rule Principle

For a set X , $|X|$ denotes the number of elements in X . It is easy to see that for any two sets A and B we have the following result known as the

Inclusion- Exclusion Principle

$$|A \cup B| = |A| + |B| - |A \cap B| .$$

Indeed, $|A|$ gives the number of elements in A including those that are common to A and B. The same holds for $|B|$. Hence, $|A| + |B|$ includes twice the number of common elements. Hence, to get an accurate count of the elements of $A \cup B$, it is necessary to subtract $|A \cap B|$ from $|A| + |B|$.

Note that if A and B are disjoint then $|A \cap B| = 0$ and consequently $|A \cup B| = |A| + |B|$.

The first counting principle (***Sum Rule Principle***): Suppose some event A can occur in m ways and a second event B can occur in n ways, and suppose both events cannot occur simultaneously. Then A or B can occur in $m + n$ ways.

In other words (***Sum Rule Principle***): Suppose A and B are disjoints sets.

Then $|A \cup B| = |A| + |B|$.

Clearly, the principle can be extended to three or more events. Specifically, suppose an event A_1 can occur in n_1 ways, an event A_2 can occur in n_2 ways, an event A_3 can occur in n_3 ways, and so on, and suppose no two of the events can occur at the same time. Then one of the events can occur in $n_1 + n_2 + n_3 + \cdots$ ways.

Example 1.1 (The Addition Rule)

Show by induction on n, that if $\{A_1, A_2, ..., A_n\}$ is a collection of pairwise disjoint sets then

$$|A_1 \cup A_2 \cup ... \cup A_n| = |A_1| + |A_2| + ... + |A_n|$$

Solution:

Basis of induction: For $n = 2$ the result holds by the Inclusion-Exclusion Principle.

Induction hypothesis: Suppose that for any collection $\{A_1, A_2, ..., A_n\}$ of pairwise disjoint sets we have

$$|A_1 \cup A_2 \cup ... \cup A_n| = |A_1| + |A_2| + ... + |A_n|.$$

Induction step: Let $\{A_1, A_2, ..., A_n, A_{n+1}\}$ be a collection of pairwise disjoint sets.

Since

$$(A_1 \cup A_2 \cup ... \cup A_n) \cap A_{n+1} = (A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \cdots (A_n \cap A_{n+1}) = \varnothing$$

then by the Inclusion-Exclusion Principle and the induction hypothesis we have

$$|A_1 \cup A_2 \cup ... \cup A_n \cup A_{n+1}| = |A_1 \cup A_2 \cup ... \cup A_n| + |A_{n+1}|$$

$$= |A_1| + |A_2| + \cdots + |A_n| + |A_{n+1}|$$

Example 1.2

A total of 35 programmers interviewed for a job; 25 knew FORTRAN, 28 knew PASCAL, and 2 knew neither language. How many knew both languages?

Solution:

Let A be the group of programmers that knew FORTRAN, B those

Who knew PASCAL. Then $A \cap B$ is the group of programmers who knew both languages. By the Inclusion-Exclusion Principle we have:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

That is,

$$33 = 25 + 28 - |A \cap B|.$$

Solving for $|A \cap B|$ we find $|A \cap B| = 20$ |.

1-1.2 Product Rule Principle

Another important rule of counting is the multiplication rule.

The second Rule Principle (***Product Rule Principle***): Suppose there is an event A which can occur in m ways and, independent of this event, there is a second event B which can occur in n ways. Then combinations of A and B can occur in mn ways.

In other words (***Product Rule Principle***): Suppose A and B are finite sets.

Then $|A \times B| = |A| \cdot |B|$

Clearly, the principle can be stated as: If a decision consists of k steps, where the first step can be made in n_1 different ways, the second step in n_2 ways,..., the k th step in n_k ways, then the decision itself can be made in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways.

For example,

. Tossing a coin has two possible outcomes and tossing a die has six possible outcomes. Then the combined experiment, tossing the coin and die together results in 2 $\times$ 6 = 12 possible outcomes: $H1, H2, H3, H4, H5, H6,\ T1, T2, T3, T4,\ T5, T6$

. The number of different ways for a man to get dressed if he has 8 different shirts and 6 different pairs of trousers is 8 $\times$ 6 = 48

. The number of ways a three-figure integer be formed from the numbers, 4, 3, 5, 6 and 7 if no number is used twice or more is 5 $\times$ 4 $\times$ 3 = 60.

ACTIVITY **Example 1.3**

1. Suppose a college has 3 different computer science courses, 4 different mathematics courses, and 2 different actuarial courses (with no prerequisites)

   a) There are $n = 3 + 4 + 2 = 9$ ways to choose 1 of the courses.

   b) There are $n = (3)(4)(2) = 24$ ways to choose one of each of the courses.

2. Suppose Airline **A** has three daily flights between Kumasi and Accra, and Airline **B** has two daily flights between Kumasi and Accra

   a) There are $n = 3 + 2 = 5$ ways to fly from Kumasi to Accra.

   b) There are $n = (3)(2) = 6$ ways to fly Airline **A** from Kumasi to Accra, and then Airline **B** from Accra back to Kumasi.

   c) There are $n = (5)(5) = 25$ ways to fly from Kumasi and Accra , and then back again.

3. How many possible outcomes are there if 2 distinguishable dice are rolled?

4. Suppose that a state's license plates consist of 3 letters followed by four digits. How many different plates can be manufactured?(No repetitions)

Solution:

3. By the multiplication rule there are $6 \times 6 = 36$ possible outcomes.

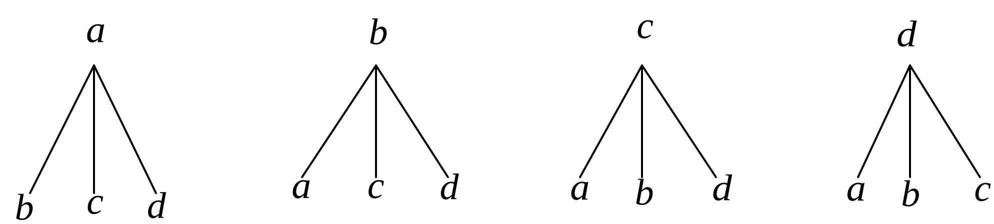
4. By the multiplication rule there are $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$ possible license plates.

Example 1.3

Let $\sum = \{a, b, c, d\}$ be an alphabet with 4 letters. Let $\sum^2$ be the set of all words of length 2 with letters from $\sum$. Find the number of all words of length 2 where the letters are not repeated. First use the **product rule**. List the words by means of a **tree diagram**.

Solution:

By the multiplication rule there are $4 \times 3 = 12$ different words. Constructing a tree diagram:



we find that the words are:

$$\{ab,\ ac,\ ad,\ ba,\ bc,\ bd, ca,\ cb,\ cd,\ da,\ db,\ dc\}$$

Applying the multiplication principle, results in the other two counting techniques, namely *Permutation and Combination*, used to find the number of possible ways when a fixed number of items are to be picked from a lot without replacement

SESSION 2-1: PERMUTATION AND COMBINATION

2-1.1 Permutation

Any arrangement of a set of n object in a given order is called a *permutation* of the objects (taken all at a time). Any arrangement of any $r \le n$ of those objects in a given order is called an r-*permutation* of n objects or a *permutation of the n objects taken r at a time*. In symbol $P(n,\ r)$, is an ordered selection of r objects from a given n objects.

Consider, for example, the set of letters $a, b, c,$ and d. Then:

a) $abcd, bcda, acdb,$ and $dcba$ are permutations of the four letters (taken all at a time);

b) $bad, adb, cbd,$ and bca are permutations of the four letters taken three at a time;

c) $ad, cb, da,$ and bd are permutations of the four letters taken two at a time.

Example 1.4

a) Use the product rule to show that $P(n,\ r) = \dfrac{n!}{(n-r)!}$

b) Find all possible 2 - permutations of the set $\{1,\ 2,\ 3\}$.

Solution:

a) We can treat a permutation as a decision with r steps. The first step can be made in n different ways, the second in $n-1$ different ways, ..., the rth in $n-(r-1) = n - r + 1$ different ways. Thus, by the multiplication rule there are $n(n-1)\cdots(n-r+1)$ r - permutations of n objects.

That is $P(n,\ r)\ = n(n-1)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$

b) $\quad P(3,\ 2)\ = \dfrac{3!}{(3-2)!} = 6$

Example 1.5

How many license plates are there that start with three letters followed by 4 digits (no repetitions)?

Solution:

$$P(26,\ 3)\ \cdot P(10,\ 4)\ =\ 78,\ 624,\ 000$$

An ordered arrangement of objects is called a *permutation*. The number of permutations of

(i) n distinct objects, taken all together is $n!$ = $n(n-1)(n-2)\times...\times 3\times 2\times 1$

(ii) n distinct objects taken r at a time is ${}^{n}P_{r}$ or $P(n,r) = \dfrac{n!}{(n-r)!}$, *where* $r \le n.$

(iii) n objects consisting of groups of which n_1 of the first group are alike, n_2 of the second group are alike and so on for the k^{th} group with n_k objects which are alike is $\dfrac{n!}{n_1!.n_2!.n_3!\ .\ .\ .n_k\,!}$, where $n = n_1 + n_2 + \ .\ .\ .+ n_k$

(iv) n distinct objects arranged in a circle, called *circular permutations* is given by

$$\frac{n!}{n} = (n-1)!.$$

For example,

1. The number of possible permutations of the letters, *A*, *B* and *C* is $3! = 6$. The required permutations are *ABC, BAC, ACB, BCA, CAB* and *CBA.*
2. The number of permutations of 10 distinct digits taken two at a time

$$= {}^{10}P_2 = \frac{10!}{(10-2)!} = 10 \times 9 = 90.$$

3. The number of permutations of the letters forming the following 14-letter word, *S C I E N T I F I C A L L Y*, which contains 2*C's*, 3I's, *2L's*, and 1's of the rest of letters $= \dfrac{14!}{2!.3!.2!}$ = 3,632,428,800
4. The number of circular permutations of 6 persons sitting around a circular table

= 5! = 120

2-1.2 Ordered Samples

When we choose one element after another from the set S containing n elements, say r times, we call the choice an ordered sample of size r. We consider two cases:

I. *Sampling with replacement*

Here the element is replaced in the set S before the next element is chosen. Since there are n different ways to choose each element (repetitions are allowed), the product rule principle tells us that there are

$$\overbrace{n \cdot n \cdot n \cdots n}^{r \text{ times}} = n^r$$

different ordered samples with replacement of size r.

II. Sampling without replacement

Here the element is not replaced in the set S before the next element is chosen. Thus there are no repetitions in the ordered sample. According, an ordered sample of size r without replacement is simply an $r-$ permutation of the elements in the set S with n elements. Thus there are

$$P(n,r)=n(n-1)(n-2)\cdots(n-r+1)=\frac{n!}{(n-r)!}$$

different ordered samples without replacement of size r from a population (set) with n elements. In other words, by the product rule, the first element can be chosen in n ways, the second in $n-1$ ways, and so on.

Example 1.6

Three cards are chosen in succession from a deck with 52 cards. Find the number of ways this can be done (a) with replacement (b) without replacement.

Solution:

a) Since each card is replaced before the next card is chosen, each card can be chosen in 52 ways. Thus there are

   $$52(52)(52)=52^3=140,608$$

   different ordered samples of size $r=3$ with replacement.

b) Since there is no replacement, the first card can be chosen in 52 ways, the second card in 51 ways, and the last card 50 ways. Thus there are

$$
\begin{aligned}
P(52,3) &= 52(52-1)(52-(3-1)) \\
&= 52(52-1)(52-3+1) \\
&= 52(51)(50) = 132,600
\end{aligned}
$$

different ordered samples of size $r = 3$ without replacement.

2-1.3 Combination

Suppose we have a collection of n objects. A *combination* of these n objects taken r at a time is any selection of r of the objects without taking order in account. An r - combination of n objects, in symbol $C(n,\ r)$, is an unordered selection of r of the n objects. In other words, an $r-$*combination* of a set of n objects is any subset of r elements. But the number of different ways that r objects can be ordered is $r!$. Since there are $C(n,\ r)$ groups of r objects from a given n objects then the number of ordered selection of r objects from n given objects is $r!C(n,\ r) = P(n,\ r)$.Thus

$$
C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.
$$

For example, the combinations of the letters a,b,c,d taken three at a time are:

$\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$ or simply abc, abd, acd, bcd

Observe that the following combinations are equal:

$abc, acb, bac, bca, cab, cba$

That is, each denotes the same set $\{a,b,c\}$

Example 1.7

Find the number of combinations of four objects, a,b,c,d taken three at a time.

Solution:

Each combination consisting of three objects determines $3!=6$ permutations of the objects in the combination. Thus the number of combination multiplied by $3!$ equals the number of permutations. That is:

$$C(4,3)\cdot 3!=P(4,3) \text{ or } C(4,3)=\frac{P(4,3)}{3!}$$

But $P(4,3)=4\cdot 3\cdot 2=24$ and $3!=6$. Thus $C(4,3)=4$, which is shown in the table below:

Combinations	Permutations
abc	$abc, acb, bac, bca, cab, cba$
abd	$abd, adb, bad, bda, dab, dba$
acd	$acd, adc, cad, cda, dac, dca$
bcd	$bcd, bdc, cbd, cdb, dbc, dcb$

Example 1.8

In how many different ways can a hand of 5 cards be selected from a deck of 52 cards? (no repetition)

Solution:

$$C(52,5) = 2,\ 598,\ 960$$

Example 1.9

Prove the following identities:

a) $C(n,0) = C(n,n) = 1$ and $C(n,1) = C(n,n-1) = n$.

b) Symmetry property: $C(n,\ r) = C(n,\ n-r),\ r \le n$.

c) Pascal's identity: $C(n+1,k) = C(n,k-1) + C(n,\ k),\ n \le k$.

Solution:

a) Follows immediately from the definition of $C(n,\ r)$. Check yourself.

b) Indeed, we have

$$C(n,n-r) = \frac{n!}{(n-r)!(n-n+r)!} = \frac{n!}{r!(n-r)!} = C(n,\ r)$$

c) $$C(n,k-1) + C(n,k) = \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!}$$

$$= \frac{n!k}{k!(n-k+1)!} + \frac{n!(n-k+1)}{k!(n-k)!}$$

$$= \frac{n!}{k!(n-k+1)!}(k+n-k+1) = \frac{(n+1)!}{(n+1-k)!} = C(n+1,k)$$

Example 1.10

1. Find the number m of committees of three that can be formed from eight people. Each committee is, essentially, a combination of the eight people take three at a time.

   Solution:

$$m = C(8,3) = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

2. A farmer buys three cows, two pigs, and four hens from a man who has six cows, five pigs, and eight hens. How many choices does the farmer have?

   Solution:

   The farmer can choose the cows in $\binom{6}{3}$ ways, the pigs in $\binom{5}{2}$ ways, and the hens in $\binom{8}{4}$ ways. Hence altogether he can choose the animals in

$$\binom{6}{3} \cdot \binom{5}{2} \cdot \binom{8}{4} = 20 \cdot 10 \cdot 70 = 14,000 \text{ ways}$$

3. Find the number m of ways that 9 toys can be divided between 4 children if the youngest is to receive 3 toys and each of the others 2 toys.

   Solution:

   There are $C(9,3) = 84$ ways to first choose 3 toys for the youngest. Then there are $C(6,2) = 15$ ways to choose 2 of the remaining 6 toys for the oldest. Next, there are $C(4,2) = 6$ ways to choose 2 of the remaining 4 toys for the second oldest. The third oldest receives the remaining 2 toys. Thus, by the product rule:

$$m = 84(15)(6)(1) = 7560$$

Example 1.11

1.11(a) (i) In how many ways can a three-figure integer is formed from the numbers: 4, 3, 5, 6 and 7 if any number can be used more than once?

(ii) In a certain examination paper, students are required to answer 5 out of 10 questions from *Section A* another 3 out of 5 questions from *Section B* and 2 out of 5 questions from *Section C*. In how many ways can the students answer the examination paper?

Solution:

(i) The first, second and third numbers, each can be chosen in 5 ways. The total number of ways = $5 \times 5 \times 5 = 125$

(ii) The number of ways of answering the questions in *Section A*

$= 10 \times 9 \times 8 \times 7 \times 6 = 30{,}240$

The number of ways of answering the questions in section B

$= 5 \times 4 \times 3 = 60$

The number of ways of answering the questions in section C

$= 5 \times 4 = 20$

Hence the students can answer the questions in the three sections in

$= 30{,}240 \times 60 \times 20 = 36{,}288{,}000$

1.11(b) A company codes its customers by giving each customer an eight character code. The first 3 characters are the letter *A, B* and *C* in any order and the remaining 5 are the digits 1, 2, 3, 4 and 5 also in any order. If each letter and digit can appear only once then number of customers the company can code is obtained as follows:

The first 3 letters can be filled in 3!

The next 5 digits can be filled in 5!

Then the required number = $3! \times 5! = 720$

1.11(c) In many ways can 4 boys and 2 girls seat themselves in a row if :

(i) The 2 girls are to sit next to each other?

(ii) The 2 girls are not to sit next to each other?

Solution:

(i) If we regard the 2 girls as a separate persons ($\underline{B_1}\ \underline{B_2}\ \underline{B_3}\ \underline{B_4}\ \underline{G_1 G_2}$), then the number of arrangements of 5 different persons, taken all at a time = 5!

The 2 girls can exchange places and so the required number of ways they can seat themselves = $5! \times 2! = 240$

(ii) The number of ways the boys can arrange themselves = 4!

The number of ways the 2 girls can occupy the arrowed places:

$$\uparrow B_1 \uparrow B_2 \uparrow B_3 \uparrow B_4 \uparrow \quad = \quad {}^5P_2 = 5 \times 4$$

The required number of permutations (with the 2 girls not sitting next to each other) = $4! \times 5 \times 4 = 480$

1.11(d) Find the number of ways in which a committee of 4 can be chosen from 6 boys and 5 girls if it must

(i) Consist of 2 boys and 2 girls.

(ii) Consist of at least 1 boy and 1 girl.

Solution:

(i) The number of ways of choosing 2 bys from 6 and 2 girls from 5

$$= \binom{6}{2}.\binom{5}{2} = 15 \times 10 = 150$$

(ii) For the committee to contain at least 1 boy and 1 girl we have

1B3G, 2B2G or 3B1G

The required number of ways

$$= \binom{6}{1}.\binom{5}{3} + \binom{6}{2}.\binom{5}{2} + \binom{6}{3}.\binom{5}{1}$$

$$= 6(10) + 15(10) + 20(5) = 130$$

1.11(e) (i) A school Parent-Teacher committee of 5 members is to be formed from 6 parents, 2 teachers and the principal. In how many ways can the committee be formed in order to include

(α) The principal? (β) Exactly four parents?

(γ) Not more than four parents?

(ii) Four balls are drawn from a bag of 12 balls of which 7 are blue and 5 are red. In how many of the possible combinations of 4 balls is at least a red?

Solution:

(i) (α) If the principal is to be included then we select 4 people from the remaining 8. Hence required number of ways the committee is formed

$$= \binom{1}{1}.\binom{8}{4} = 70$$

(β) The number of ways of selecting 4 parents out of 6 = $\binom{6}{4}$. The number of ways of selecting the remaining number from the 3 (2 teachers and the principal) = $\binom{3}{1}$

Therefore the number of ways of selecting exactly 4 parents

$$= \binom{6}{4} . \binom{3}{1} = 15 \times 3 = 45$$

(γ) The number of ways of forming a 5-member committee = $\binom{12}{5}$

The number of ways of selecting 5 parents from 6 = $\binom{6}{5}$

Therefore the required number of ways of selecting a committee with not more than 4 parents = $\binom{12}{5} - \binom{6}{5}$ = 126 – 6 = 120

(ii) If at least one red is to be included then the combinations include

1R 3B, with number of combinations = $\binom{5}{1}\binom{7}{3}$ = 175

2R 2B, with number of combinations = $\binom{5}{2}\binom{7}{2}$ = 210

3R 1B, with number of combinations = $\binom{5}{3}\binom{7}{1}$ = 70

4R, with number of combinations = $\binom{5}{4}\binom{7}{0}$ = 5

2-1.4 Tree Diagram

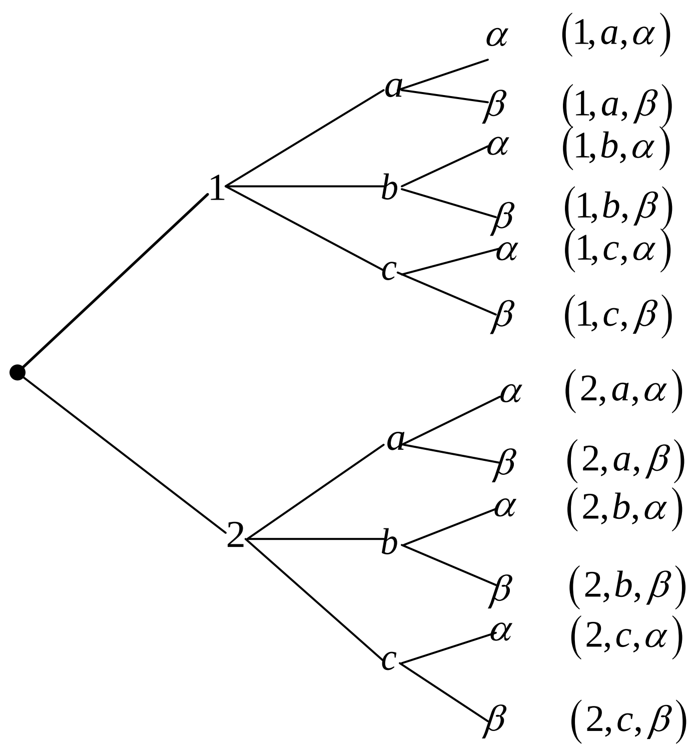
A (rooted) tree diagram is a useful to enumerate all the logical possibilities of a sequence of events where each event can occur in a finite number of ways.

A tree diagram is constructed from left to right and that the number of branches at each point corresponds to the number of ways the next event can occur.

Example 1.12

1. Find the product set $A \times B \times C$ where $A = \{1,2\}$, $B = \{a,b,c\}$, $C = \{\alpha, \beta\}$.

Solution:



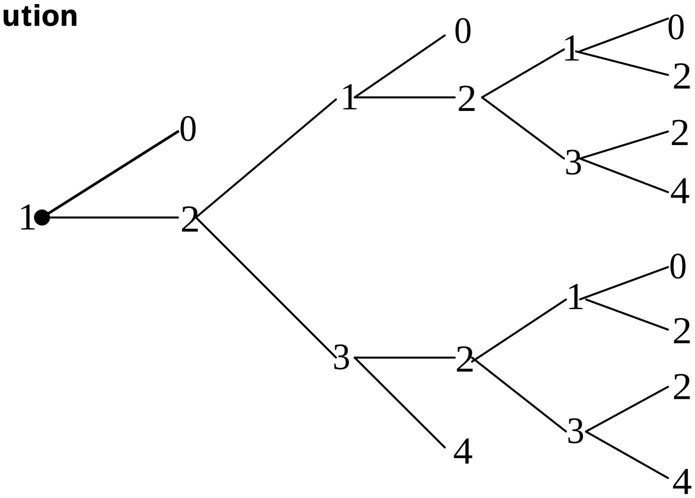
There are 12 endpoints corresponding to the 12 elements in $A \times B \times C$. Specifically, each path from the beginning of the tree to the endpoints designates an element of $A \times B \times C$ which is listed to the right of the tree.

That is,

$$A \times B \times C = \{(1,a,\alpha),(1,a,\beta),(1,b,\alpha),(1,b,\beta),(1,c,\alpha),(1,c,\beta),$$
$$(2,a,\alpha),(2,a,\beta),(2,b,\alpha),(2,b,\beta),(2,c,\alpha),(2,c,\beta),\}$$

2. A man has time to play roulette at most five times. At each play he wins or loses a cedi. The man begins with one cedi and will stop playing before the five times if he loses all his money or if he wins three cedis, i.e., if he has four cedis.
   a) Find the number of possible ways that the playing can occur.
   b) Determine the number of times the betting will stop before he has played five times.

Solution



a) There are 11 endpoints; hence the betting can occur in 11different ways.
b) The betting will stop before the five times are up in only three of the ways.

Review Probl

ems

Problem 1.1

a) How many ways can we get a sum of 4 or a sum of 8 when two distinguishable dice are rolled?

b) Suppose a bookcase shelf has 6 mathematics texts, 3 physics texts, 4 chemistry texts, and 5 computer science texts. Find the number n of ways a student can choose: (a) one of the texts $(\text{Ans}:18)$, (b) one of each type of text $(\text{Ans}:360)$

c) How many ways can we get a sum of 8 when two undistinguishable dice are rolled?

Problem 1.2

a) How many 4 - digit numbers can be formed using the digits, $1,2,\cdots,9$ (with repetitions)? How many can be formed if no digit can be repeated?

b) How many different license plates are there that involve $1,2,$ or 3 letters followed by 4 - digits (with repetitions)?

Problem 1.3

a) In how many ways can 4 cards be drawn, with replacement, from a deck of 52 cards?

b) In how many ways can 4 cards be drawn, without replacement, from a deck of 52 cards?

Problem 1.4

In how many ways can 7 women and 3 men be arranged in a row if the three men must always stand next to each other.

Problem 1.5

A menu in a Chinese restaurant allows you to order exactly two of

eight main dishes as part of the dinner special. How many different combinations of main dishes could you order?

Problem 1.6

There are 12 students in a class. Find the number n of ways that 12 students can take three different tests if four students are to take each test. $\left(\text{Ans}: 34,650\right)$

· ANS. 12C4*8C4*4C4

Problem 1.7

Teams A and B play in a basketball tournament. The team that first wins three games wins the tournament. Find the number of possible ways in which the tournament can occur.

Problem 1.8

A woman is at the origin on the x axis and takes a one unit step either to the left or to the right. She stops if she reaches 3 or -3, or if she occupies any position, other than the origin, more than once. Find the number of different paths the woman can travel.

Unit 2

MULTINOMIAL EXPANSION AND COEFFICIENT

Introduction

This unit extends the technique of counting as already treated under permutations and combinations. It may be recalled that binomial expansion is very much linked with combinations. We are going to have a short but concise study of multinomial expansion, which is naturally linked with multinomial coefficients.

Learning Objectives

After reading this unit you should be able to:

1. Determine the coefficient of a term in binomial expansion without necessary listing every term
2. Determine the coefficient of a term in multinomial expansion without necessary listing every term

Unit content

Session 1-2: Binomial Coefficients

Session 2-2: Multinomial Coefficients

SESSION 1-2: **BINOMIAL COEFFICIENTS**

Choosing a subset of size r out of a set of size n is logically equivalent to partitioning the set of size n into two subsets, one of size r and the other of size $(n-r)$. The number of such partitions if by definition ${}^nC_r = \frac{n!}{r!(n-r)!}$. Suppose x, y are variable and $n \in Z^+$ the positive integers, then

$$(x+y)^n = \sum_{r=o}^{n} {}^nC_r x^{n-r} y^r = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_n x^0 y^n$$

The coefficients ${}^nC_r,\ n \geq r,\ r \in \{0,1,\ldots,n\}$ are called the binomial coefficients of the binomial expression $(x+y)^n,\ n \in Z^+$.

Definition 2.1

$${}^nC_r = \frac{n!}{r!(n-r)!} \qquad (2)$$

Where $0! = 1 \text{ and } n! = n(n-1)(n-2)\ldots1$

Pascal's identity allows one to construct the following triangle known as

Pascal's triangle (for n = 5) as follows

1

$1 \to 1$

$1 \to 2 \to 1$

$1 \to 3 \to 3 \to 1$

$1 \to 4 \to 6 \to 4 \to 1$

The following theorem provides an expansion of $(x + y)^n$ where n is a nonnegative integer.

Theorem 2.1 (Binomial Theorem)

Let x and y be variables, and let n be a positive integer. Then

$$(x+y)^n = \sum_{r=0}^{n} C(n,r) x^{n-r} y^r$$

where $C(n, r)$ is called the **binomial coefficient**.

Proof.

The proof is by induction.

Basis of induction: For $n=1$ we have

$$(x+y)^1 = \sum_{r=0}^{1} C(n,r) x^{n-r} y^r = \binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1 = x+y$$

Induction hypothesis: Suppose that the theorem is true for n.

Induction step: Let us show that it is still true for $n+1$. That is

$$(x+y)^{n+1} = \sum_{r=0}^{n+1} C(n+1,r) x^{n-r+1} y^r .$$

Indeed, we have:

$$(x+y)^{n+1} = (x+y)(x+y)^n = x(x+y)^n + y(x + y)^n$$

$$= x\sum_{r=0}^{n} C(n,r) x^{n-r} y^r + y\sum_{r=0}^{n} C(n,r) x^{n-r} y^r$$

$$= \sum_{r=0}^{n} C(n,r) x^{n-r+1} y^{r} + y \sum_{r=0}^{n} C(n,r) x^{n-r} y^{r+1}$$

$$= C(n,0)x^{n+1} + C(n,1)x^{n}y + C(n,2)x^{n-1}y^{2} + \cdots + C(n,n)xy^{n}$$
$$+ C(n,0)x^{n}y + C(n,1)x^{n-1}y^{2} + \cdots + C(n,n-1)xy^{n} + C(n,n)y^{n+1}$$

$$= C(n+1,0)x^{n+1} + C(n+1,1)x^{n}y + C(n+1,2)x^{n-1}y^{2} +$$
$$+ \cdots + C(n+1,n)x^{n}y + C(n+1,n+1)y^{n+1}$$

$$= \sum_{r=0}^{n+1} C(n+1,r) x^{n-r+1} y^{r}$$

Example 2.1

Expand $(x+y)^6$ using the binomial theorem.

Solution:

By the Binomial Theorem and Pascal's triangle we have

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Example 2.2

a) Show that $\sum_{r=0}^{n} C(n,r) = 2^n$

b) Show that $\sum_{r=0}^{n} (-1)^r C(n,r) = 0$

Solution:

a) Letting $x = y = 1$ in the binomial theorem we find

$$2^n = (1+1)^n = \sum_{r=0}^{n} C(n,r)$$

b) This follows from the binomial theorem by letting $x = 1$ and $y = -1$

Example 2.3

Expand $(2x-3)^5$

Solution:

$$(2x-3)^5 = \sum_{r=0}^{5} {}^5C_r (2x)^{n-r} (-3)^r$$
$$= {}^5C_0 (2x)^5 (-3)^0 + {}^5C_1 (2x)^4 (-3)^1 + {}^5C_2 (2x)^3 (-3)^2$$
$$+ {}^5C_3 (2x)^2 (-3)^3 + {}^5C_4 (2x)(-3)^4 + {}^5C_5 (2x)^0 (-3)^5$$

$$= \frac{5!}{5!0!} 2^5 x^5 + \frac{5!}{4!1!} 2^4 x^4 (-3) + \frac{5!}{3!2!} 2^3 x^3 (9) + \frac{5!}{2!3!} 2^2 x^2 (-27)$$
$$+ \frac{5!}{1!4!} 2x(81) + \frac{5!}{0!5!}(-243)$$
$$= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 81x - 243$$

Example 2.4

Obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$

Solution:

To obtain the coefficient of the term a^5b^2, we notice that in (2), we set $r=2$ to obtain

${}^7C_2(2)^5(-3)^2$ as the coefficient of a^5b^2

But

$$\begin{aligned}{}^7C_2(2)^5(-3)^2 &= \frac{7!}{5!2!}(2)^5(-3)^2 \\ &= 6048\end{aligned}$$

Therefore, the coefficient of a^5b^2 of the binomial expression is 6048

Self Assessment 1-2

1. Expand the following binomial expressions:

(i) $(2x-4y)^5$ (ii) $(3x+2y^2)$ (iii) $(2-5xy)^4$

(iv) $[3z+(1-b)]^3$ (v) $[3x-(7+2y)]^3$

2. Find the *coefficients* of the *indicated terms* in the given *binomial expressions*:

(i) $x^{12}y^{13}$ in $(x+y)^{25}$ (ii) $x^{12}y^{13}$ in $(2x-3y)^{25}$

(ii) x^{9} in $(2-x)^{19}$ (iv) y^{6} in $\left[3-5(1-y)\right]^{8}$

(v) x^{19} in $\left[(3x+1)-2\right]^{20}$

3. Prove that for any $n \in Z^{+}$

(i) $\sum_{r=0}^{n} {}^{n}C_{r} = 2^{n}$ (ii) $\sum_{r=0}^{n} (-1)^{r}\, {}^{n}C_{r} = 0$.

4. Using the binomial expression $(1+x)^{n}$ to find the approximate values of the following to 5 decimal places:

(i) 0.9^5 (ii) 1.99^4

5. Expand $(1-2x)^{6}$ and hence evaluate 0.9^{6} to six decimal places.

6. Show that ${}^{k}C_{k} + {}^{k+1}C_{k} + \ldots + {}^{k+r}C_{k} = {}^{k+r+1}C_{k+1}$

7. Show that $\ln(1+x) = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} - \frac{x^{4}}{4} + \ldots$

Hence show that $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^{3}}{3} + \frac{x^{5}}{5} + \ldots$

8. Solve the following

a) Find the coefficient of $a^{5}b^{7}$ in the binomial expansion of $(1-2b)^{12}$

b) Use the binomial theorem to prove that

$$3^{n} = \sum_{r=0}^{n} 2^{r} C(n,r)$$

SESSION 2-2: MULTINOMIAL COEFFICIENTS

Let $k_1,\ k_2,\ ...,\ k_r$ be integers satisfying the relation $k_1 + k_2 + ... + k_r = n$. Then

The number of ways a set of n elements can be partitioned into classes of

sizes $k_1,\ k_2,\ ...,\ k_r$ equal

$$^{n}C_{k_1,k_2,...,k_r} = \frac{n!}{k_1!k_2!...k_r!}$$

Proof

We obtain the partitioning in steps: First, we choose k_1 out of n elements to form the first partition. Next, we choose k_2 elements out of the remaining $(n-k_1)$ elements, and so on until we have $n-k_1-k_2-...-k_{r-2} = k_{r-1}+k_r$ elements from which we choose k_{r-1} to form the next-to-last class. The remaining k_r elements forms the last class. This has been accomplished in

$\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}...\binom{n-k_1-k_2-...-k_{r-2}}{k_{r-1}}\binom{n-k_1-k_2-...-k_{r-1}}{k_r}$ ways.

Simple algebra shows that

$$\binom{n}{k_1}\binom{n-k_1}{k_2}...\binom{n-k_1-k_2-k_3}{k_{r-1}}\binom{n-k_1-k_2......k_{r-1}}{k_r} = \frac{n!}{k_1!k_2!...k_r!}$$

Suppose $x_1, x_2, x_3, ..., x_r$ are variables and $n \in Z^+$, then

$$(x_1 + x_2 + ... + x_r)^n = \sum_{k_1,k_2...,k_r} {}^{n}C_{k_1,k_2...,k_r} x_1^{k_1} x_2^{k_2} ... x_r^{k_r}$$

where $k_1 + k_2 ... + k_r = n$

Example 2.5

Expand $(x+y+z)^3$

Solution:

$$\begin{aligned}
(x+y+z)^3 &= {}^3C_{3,0,0}x^3y^0z^0 + {}^3C_{0,3,0}x^0y^3z^0 + {}^3C_{0,0,3}x^0y^0z^3 + {}^3C_{2,1,0}x^2y^1z^0 + {}^3C_{2,0,1}x^2y^0z^1 \\
&\quad + {}^3C_{1,2,0}x^1y^2z^0 + {}^3C_{0,2,1}x^0y^2z^1 + {}^3C_{1,0,2}x^1y^0z^2 + {}^3C_{0,1,2}x^0y^1z^2 + {}^3C_{1,1,1}x^1y^1z^1 \\
&= \frac{3!}{3!0!0!}x^3 + \frac{3!}{0!3!0!}y^3 + \frac{3!}{0!0!3!}z^3 + \frac{3!}{2!1!0!}x^2y + \frac{3!}{2!0!1!}x^2z \\
&\quad + \frac{3!}{1!2!0!}xy^2 + \frac{3!}{0!2!1!}y^2z + \frac{3!}{1!0!2!}xz^2 + \frac{3!}{0!1!2!}yz^2 + \frac{3!}{1!1!1!}xyz \\
&= x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3xy^2 + 3y^2z + 3xz^2 + 3yz^2 + 6xyz
\end{aligned}$$

Example 2.6

Evaluate the following (a) $\begin{pmatrix} 6 \\ 4,2,0 \end{pmatrix}$ (b) $\begin{pmatrix} 5 \\ 3,2 \end{pmatrix}$ (c) $\begin{pmatrix} 10 \\ 5,3,0,2 \end{pmatrix}$

Solution: (a) $\begin{pmatrix} 6 \\ 4,2,0 \end{pmatrix} = \frac{6!}{4!2!0!} = \frac{6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times2\times1} = 15$

(b) $\begin{pmatrix} 5 \\ 3,2 \end{pmatrix} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{3\times2\times1\times2\times1} = 10$

(c) $\begin{pmatrix} 10 \\ 5,3,0,2 \end{pmatrix} = \frac{10!}{5!3!0!2!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1\times3\times2\times1\times2\times1} = 2520$

Example 2.7 (Previous example)

Find the number m of ways that 9 toys can be divided between 4 children if the youngest is to receive 3 toys and each of the others 2 toys.

$$m = \frac{9!}{3!2!2!2!} = 7560$$

Self Assessment 2-2

1. Show that $\binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2}$

2. Show that if $i \geq 1,\ j \geq 1,\ k \geq 1$ and $i + j + k = n + 1$, then

$$\left(\frac{(n+1)!}{i!\,j!\,k!}\right) = \frac{n!}{(i-1)!\,j!\,k!} + \frac{n!}{i!(j-1)!\,k!} + \frac{n!}{i!\,j!(k-1)!}$$

3. Prove that for any $n \in Z^+$, $\sum_{k_1, k_2, \ldots, k_r}^{n} {}^{n}C_{k_1, k_2, \ldots, k_r} = r^n$

4. In the expansion of $(x + y + z)^7$ find the coefficient of the terms:

(i) xyz^5 (ii) $x^2y^2z^3$ (iii) x^3z^4

5. Determine the coefficient of the following terms in the indicated multinomial expressions.

(i) xyz^2 in $(2x - y - z)^4$

(ii) xyz^{-2} in $\left(x - 2y + 3z^{-1}\right)^4$

(iii) $w^3x^2yz^2$ in $(2w - x + 3y - 2z)^8$

(iv) $x^{11}y^4z^2$ in $\left(2x^3 - 3xy^2 + z^2\right)^6$

(v) $x^3y^4z^5$ in $(x - 2y + 3z)^{12}$

6. The letters B,C,E,E,N,R,S,S,Y,Z,Z,Z,Z are arranged at random.

Determine the probability that these letters will spell the word

SZCZEBRZESZYN

7. Show that if $a \le b \le c \le n$, then

$$\binom{n}{c}\binom{c}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}\binom{n-b}{c-b}$$

(i) Use the definition of binomial coefficients as ratios of factorials.

(ii) Use the interpretation of the binomial coefficients directly as the number of subsets of a given size.

(iii) Generalize the above formula to answer the question: In how many ways can one choose an a-element subset from a b-element subset from a c-element subset from a d-element subset from an n-element set, where $a \le b \le c \le d \le n$?

8. Expand $(x+y+z)^6$. Hence with $x = y = z = 0.3$ evaluate 0.9^6 to six decimal places. (Compare this approximation with that of Q5 of the previous exercise).

Unit 3

RECURRENCE RELATION AND LINEAR DIFFERENCE METHOD

Introduction

Welcome to this unit. Any equation involving several elements of a sequence is called a recurrence relation. Here, we shall learn more about recurrence relations, or simply put, algorithms. This would entail one, two or three expressions that would generate an infinite set of numbers. On the other hand, given a set of recurrence relations, a solution would be found for such a relation.

Learning Objectives

After reading this unit you should be able to:

1. Use the difference operator method analogous to differentiation of a real function
2. Find a generic form of a recurrence relation

Unit content

Session 1-3: The Difference Operator
Session 2-3: Recurrence Relations -Finite Difference Equations

2-3.1Recursion
2-3.2 Classification of Recurrence Relations
2-3.3The Homogeneous Case
2-3.4The Non-Homogeneous Case

SESSION 1-3: THE DIFFERENCE OPERATOR

The difference operator Δ is defined by $(\Delta y)(n) = y(n+1) - y(n)$. This is called the **forward** difference operator. The difference operator Δ is the analogous of the differential operator in calculus. Indeed, $y'(x) = \lim_{h\to 0} \frac{y(x+h)-y(x)}{h}$ is the definition of the derivative $f'(x)$ of $f(x)$. Similarly $(\Delta y)(x) = \frac{y(x+h)-y(x)}{h}$, when $h = 1$.

Familiar properties of $\frac{d}{dx}$ carry over as follows:

Linearity

$\Delta(y+v) = \Delta y + \Delta v;\ \ \Delta(\alpha y) = \alpha\Delta(y),\ \alpha$ scalar

Product Rule:

$$\begin{aligned}(\Delta(y\cdot v))(n) &= y(n+1)v(n+1) - y(n)v(n) \\ &= [y(n+1) - y(n)]v(n+1) + y(n)[v(n+1) - v(n)] \\ &= (\Delta y)(n)v(n+1) + y(n)(\Delta v)(n)\end{aligned}$$

Exponential:

$$\begin{aligned}\text{If } y(n) = 2^n,\ (\Delta y)(n) &= y(n+1) - y(n) \\ &= 2^{n+1} - 2^n \\ &= 2^n = f(n) \\ \text{that is} \qquad \Delta y &= y\end{aligned}$$

Thus 2^n for difference equations plays the same role e^x does for differential equations.

Constants:

$\Delta \cdot c = 0,\ c$ constant

Conversely, suppose that $\Delta y = 0$, that is $(\Delta y)(n) = y(n+1) - y(n) = 0$ for all n, then y is a constant.

The operation analogous to integration is that of taking finite sums. The following result is a discrete version of the fundamental theorem of calculus, which essentially says that differentiation and integration are inverse operations.

Theorem

If $\Delta y = v$, then $\sum_{r=m}^{n} v(r) = \sum_{r=m}^{n} (\Delta y)(r) = \left[y(r) \right]_m^{n+1} = y(n+1) - y(m)$.

Proof

We have

$$
\begin{aligned}
v(n) &= y(n+1) - y(n) \\
v(n-1) &= y(n) - y(n-1) \\
v(n-2) &= y(n-1) - y(n-2) \\
&\vdots \\
v(m+1) &= y(m+2) - y(m+1) \\
v(m) &= y(m+1) - y(m) \\
\hline
\sum_{r=m}^{n} v(r) &= y(n+1) - y(m)
\end{aligned}
$$

Example 3.1

Exponentials:

$$\Delta a^n = a^{n+1} - a^n = a^n\left(a-1\right),\ a \text{ constant}$$

Hence

$$a^n = \frac{\Delta a^n}{a-1} = \Delta\left(\frac{a^n}{a-1}\right)\ \left(\text{by linearity}\right)$$

$$\therefore \sum_{r=m}^{n} a^r = \frac{1}{a-1}\sum_{r=m}^{n}\Delta a^r = \frac{a^{n+1}-a^m}{a-1}$$

(This is just the usual method for summing a geometric progression).

Example 3.2

Polynomials:

We consider easier polynomials of the form

$$f_r\left(n\right) = n\left(n-1\right)\left(n-2\right)...\left(n-r+1\right)$$

Then, $\Delta f_r(n) = y_r(n+1) - y_r(n)$

$$\begin{aligned}\therefore \Delta f_r\left(n\right) &= \left(n+1\right)n\left(n-1\right)\left(n-2\right)...\left(n-r+2\right) - n\left(n-1\right)\left(n-2\right)...\left(n-r+1\right)\\ &= n\left(n-1\right)...\left(n-r+2\right)\left[\left(n+1\right)-\left(n-r+1\right)\right]\\ &= rn\left(n-1\right)\left(n-2\right)...\left(n-r+2\right)\\ &= rf_{r-1}\left(n\right)\end{aligned}$$

This is analogous to $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

We may easily deduce that $\sum\limits_{s=m}^{n} f_{r-1}(s) = \frac{1}{r}\left[f_r(n+1) - f_r(m) \right]$.

So in particular $\sum\limits_{s=0}^{n} y_r(s) = \frac{1}{r+1} y_{r+1}(n+1)$, if $m = 0$.

With this, we have a method of summation for any polynomial.

For example,

$$\sum_{s=1}^{n} s^3 = \sum_{s=0}^{n} s^3$$

$$= \sum_{s=0}^{n} \left[s(s-1)(s-2) + 3s(s-1) + s \right]$$

$$= \sum_{s=0}^{n} \left[y_3(s) + 3y_2(s) + y_1(s) \right] = \frac{1}{4}\left[y_4(n+1) + y_3(n+1) + \frac{1}{2} y_2(n+1) \right]$$

$$= \frac{1}{4}(n+1)n(n-1)(n-2) + (n+1)n(n-1) + \frac{1}{2}(n+1)n$$

$$= \frac{(n+1)n}{4}\left[(n-1)(n-2) + 4(n-1) + 2 \right]$$

$$= \frac{1}{4} n^2 (n+1)^2$$

ACTIVITY **Example 3.3**

Let $y(n) = na^n$. [a , constant]

Using example 1, we study Δy.

$$(\Delta y)(n) = (n+1)a^{n+1} - na^{n}$$
$$= a^{n+1} + na^{n}(a-1) = a^{n+1} + (a-1)y(n)$$

Hence $y(n) = \dfrac{1}{a-1}(\Delta y)(n) - \dfrac{a^{n+1}}{a-1}$

$$= \frac{1}{a-1}(\Delta y)(n) - \frac{a}{(a-1)^2}\Delta a^{n}$$
$$= \Delta\left[\frac{1}{a-1}y(n) - \frac{a}{(a-1)^2}a^{n}\right]$$

Then by the theorem,

$$\sum_{r=m}^{n} y(r) = \frac{1}{a-1}\left[y(n+1) - y(m)\right] - \frac{a}{(a-1)^2}\left[a^{n+1} - a^{m}\right]$$
$$= \frac{(n+1)a^{n+1} - ma^{m}}{a-1} - \frac{a^{n+2} - a^{m+1}}{(a-1)^2}$$

ACTIVITY **Example 3.4**

Trigonometric Functions:

We study this specialized branch of mathematics using forward differences:

$$\Delta \sin(wx+\alpha) = \sin(w(x+1)+\alpha) - \sin(wx+\alpha)$$
$$= 2\sin\left(\tfrac{w}{2}\right)\cos\left[wx+\alpha+\tfrac{w}{2}\right]$$

Then

$$\sum_{r=0}^{n} 2\sin\left(\tfrac{w}{2}\right)\cos\left[wr+\alpha+\tfrac{w}{2}\right] = \sin\left[w(n+1)+\alpha\right] - \sin\alpha$$

so that

$$\sum_{r=0}^{n} \cos\left[wr+\alpha+\tfrac{w}{2}\right] = \frac{\sin\left[w(n+1)+\alpha\right] - \sin\alpha}{2\sin\tfrac{w}{2}}$$

Setting $\alpha = -\dfrac{w}{2}$, we have

$$\sum_{r=0}^{n} \cos wr = \frac{\sin\left[w\left(n+\frac{1}{2}\right)\right] + \sin\frac{w}{2}}{2\sin\frac{w}{2}}$$

A similar formula may be derived for $\sum_{r=0}^{n} \sin wr$.

Self Assessment 1-3

1. Verify the following differences

   (a) $\Delta n^2 4^n = (3n+2)(n+2)4^n;$

   (b) $\Delta\left[3n(n+1)(n+2)(n+3) - 4(n+1)(n+2)(n+3)\right]$
   $= 12n(n+2)(n+3)$

   (c) $\Delta(-1)^n \binom{N}{n} = (-1)^{n+1}\binom{N+1}{n+1}$

   (d) $\Delta \dfrac{2n+1}{n(n+1)} = -\dfrac{2}{n(n+2)}$

2. From (1) deduce the value of

   (a) $\sum_{r=0}^{n} (3r+2)(r+2)4^r$ (b) $\sum_{r=1}^{n} r(r+2)(r+3)$

   (c) $\sum_{n=3}^{10} (-1)^n \binom{15}{n}$ (d) $\sum_{n=0}^{N} \dfrac{1}{n(n+2)}$

   From your answer to (d), deduce that $\sum_{n=1}^{\infty} \dfrac{1}{n(n+2)}$ exists and find its value.

3. Show that $\sum_{r=1}^{n} r^4 = \dfrac{n}{30}(n+1)\left(6n^3 + 9n^2 + n - 1\right)$.

SESSION 2-1: FINITE DIFFERENCE EQUATIONS - RECURRENCE RELATIONS

What we are about to study is the discrete analogue of differential equations. Let a_n be a function defined on the set of positive integers. An inductive or recurrence definition of the function y specifies the starting value, such as a_0 (or several starting values of the inductive or recursive definition, such as, $a_0,\ a_1,\ a_2,\ldots$) and then a formula specified as to how to generate subsequent function values from the starting value (or values).

We shall think of the integer n as the independent variable, and restrict our attention to real sequences, so that the sequence a_n is considered as a function of the type

$$f : \mathbb{N} \cup \{0\} \to \mathbb{R};\ n \mapsto a_n$$

2-3.1 Recursion

A **recurrence relation** for a sequence $a_0, a_1, \ldots$ is a relation that defines a_n in terms of $a_0, a_1, \ldots, a_{n-1}$. The formula relating a_n to earlier values in the sequence is called the **generating rule**. The assignment of a value to one of the a's is called an **initial condition**.

ACTIVITY **Example 3.5**

The Fibonacci sequence

$$1, 1, 2, 3, 5, \ldots$$

is a sequence in which every number after the first two is the sum of the preceding two numbers. Find the generating rule and the initial conditions.

Solution:

The initial conditions are $a_0 = a_1 = 1$ and the generating rule is $a_n = a_{n-1} + a_{n-2}$; $n \geq 2$

Example 3.6

Let $n \geq 0$ and find the number s_n of words from the alphabet $\sum = \{0,1\}$ of length n not containing the pattern 11 as a subword.

Solution:

Clearly, $s_0 = 1$ (empty word) and $s_1 = 2$. We will find a recurrence relation for s_n, $n \geq 2$. Any word of length n with letters from $\sum$ begins with either 0 or 1. If the word begins with 0, then the remaining $n-1$ letters can be any sequence of 0's or 1's except that 11 cannot happen. If the word begins with 1 then the next letter must be 0 since 11 cannot happen; the remaining $n-2$ letters can be any sequence of 0's and 1's with the exception that 11 is not allowed. Thus the above two categories form a partition of the set of all words of length n with letters from $\sum$ and that do not contain 11. This implies the recurrence relation $s_n = s_{n-1} + s_{n-2}$, $n \geq 2$

A **solution** to a recurrence relation is an explicit formula for a_n in terms of n.

The most basic method for finding the solution of a sequence defined recursively is by using **iteration**. The iteration method consists of starting with the initial values of the sequence and then calculates successive terms of the sequence until a pattern is observed. At that point one guesses an explicit formula for the sequence and then uses mathematical induction to prove its validity.

Example 3.7

Find a solution for the recurrence relation

$$\begin{cases} a_0 & = 1 \\ a_n = a_{n-1} + 2, n \geq 1 \end{cases}$$

Solution:

Listing the first five terms of the sequence one finds

$$a_0 = 1$$

$$a_1 = 1 + 2$$

$$a_2 = 1 + 4$$

$$a_3 = 1 + 4$$

$$a_4 = 1 + 8$$

Hence, a guess is $a_n = 2n + 1$, $n \geq 0$. It remains to show that this formula is valid by using mathematical induction.

Basis of induction: For $n = 0, a_0 = 1 = 2(0) + 1$.

Induction hypothesis: Suppose that $a_n = 2n + 1$.

Induction step: We must show that $a_{n+1} = 2(n+1) + 1$. By the definition of a_{n+1} we have $a_{n+1} = a_n + 2 = 2n + 1 + 2 = 2(n+1) + 1$.

Example 3.8

Consider the arithmetic sequence

$$a_n = a_{n-1} + d\ ,\ n \geq 1$$

where a_0 is the initial value. Find an explicit formula for a_n.

Solution:

Listing the first four terms of the sequence after a_0 we find

$$a_1 = a_0 + d$$

$$a_2 = a_0 + 2d$$

$$a_3 = a_0 + 3d$$

$$a_4 = a_0 + 4d$$

Hence, a guess is $a_n = a_0 + nd$. Next, we prove the validity of this formula by induction.

Basis of induction: For $n = 0$ $a_0 = a_0 + (0)d$.

Induction hypothesis: Suppose that $a_n = a_0 + nd$.

Induction step: We must show that $a_{n+1} = a_0 + (n+1)d$. By the definition of a_{n+1} we have $a_{n+1} = a_n + d = a_0 + nd + d = a_0 + (n+1)d$.

Example 3.9

Consider the geometric sequence

$$a_n = ra_{n-1},\ n \geq 1$$

Where a_0 is the initial value. Find an explicit formula for a_n.

Solution:

Listing the first four terms of the sequence after a_0 we find

$$a_1 = ra_0$$

$$a_2 = r^2 a_0$$

$$a_3 = r^3 a_0$$

$$a_4 = r^4 a_0$$

Hence, a guess is $a_n = r^n a_0$. Next, we prove the validity of this formula by induction.

Basis of induction: For $n = 0$, $a_0 = r^0 a_0$.

Induction hypothesis: Suppose that $a_n = r^n a_0$.

Induction step: We must show that $a_{n+1} = r^{n+1} a_0$. By the definition of a_{n+1} we have $a_{n+1} = ra_n = r\left(ra_0\right) = r^{n+1} a_0$.

Example 3.10

Find a solution to the recurrence relation

$$\begin{cases} a_0 & 0 \\ a_n = a_{n+1} + (n+1), & n \geq 1 \end{cases}$$

Solution:

Writing the first five terms of the sequence we find

$$a_0 = \quad 0$$

$$a_1 = \quad 0$$

$$a_2 = \quad 0+1$$

$$a_3 = \quad 0+1+2$$

$$a_4 = \quad 0+1+2+3$$

A guessing formula is that

$$a_n = 0+1+2+\cdots+(n-1) = \frac{n(n-1)}{2}$$

We next show that the formula is valid by using induction on $n \geq 0$.

Basis of induction: $a_0 = 0 = \dfrac{0(0-1)}{2} = 0$.

Induction hypothesis: Suppose that $a_n = \dfrac{n(n-1)}{2}$.

Induction step: We must show that $a_{n+1} = \frac{n(n+1)}{2}$. Indeed,

$$a_{n+1} = a_n + n$$

$$= \frac{n(n-1)}{2} + n$$

$$= \frac{n(n+1)}{2}$$

Example 3.11

Consider the recurrence relation

$$\begin{cases} a_0 & = 1 \\ a_n = 2a_{n-1} + n & n \geq 1 \end{cases}$$

Is it true that $a_n = 2^n + n$ is a solution to the given recurrence relation?

Solution:

If so then we must be able to prove its validity by mathematical induction.

Basis of induction: $a_0 = 2^0 + 1$.

Induction hypothesis: Suppose that $a_n = 2^n + n$.

Induction step: We must show that $a_{n+1} = 2^{n+1} + (n+1)$. If this is so then we will have $2^{n+1} + (n+1) = 2a_n + n = 2^{n+1} + 2n + n + 1$. But this would imply that $n = 0$ which contradicts the fact that n is any nonnegative integer.

Example 3.12

Define a sequence, $a_1, a_2, \ldots$, recursively as follows:

$$\begin{aligned} a_1 &= 1 \\ a_n &= 2 \cdot a_{\lfloor \frac{n}{2} \rfloor}, \ n \geq 2 \end{aligned}$$

a. Use iteration to guess an explicit formula for this sequence.

b. Use induction to prove the validity of the formula found in a.

Solution:

Computing the first few terms of the sequence we find

$a_1 =$	1	$a_2 =$	2
$a_3 =$	2	$a_4 =$	4
$a_5 =$	4	$a_6 =$	4
$a_7 =$	4	$a_8 =$	$\cdots = a_{15} = 8$

Hence, for $2^i \leq n < 2^{i+1}$, $a_n = 2^i$. Moreover, $i \leq \log_2 n < i + 1$ so that $i = \lfloor \log_2 n \rfloor$ and a formula for a_n is

$$a_n = 2^{\lfloor \log_2 n \rfloor}, n \geq 1$$

b. We prove the above formula by mathematical induction.

Basis of induction: For $n=1$, $a_1 = 1 = 2^{\lfloor \log_2 1 \rfloor}$.

Induction hypothesis: Suppose that $a_n = 2^{\lfloor \log_2 n \rfloor}$.

Induction step: We must show that $a_{n+1} = 2^{\lfloor \log_2 n+1 \rfloor}$.

Indeed, for n odd (i.e. $n+1$ even) we have:

$$a_{n+1} = 2 \cdot a_{\lfloor \frac{n+1}{2} \rfloor}$$

$$= 2 \cdot a_{\frac{n+1}{2}} \quad = 2 \cdot 2^{\lfloor \log_2 \frac{n+1}{2} \rfloor} = 2^{\lfloor \log_2 (n+1) - 1 \rfloor + 1} \quad = 2^{\lfloor \log_2 (n+1) \rfloor - 1 + 1}$$

$$= 2^{\lfloor \log_2 (n+1) \rfloor}$$

A similar argument holds when n is even.

NOTATION:

We write $a(n) = a_n$, or $y(n) = y_n$, $n \in \mathbb{Z}^{+}$

Example 3.13

Let n be the number of memory locations referenced by a certain computer program. Suppose that the algorithm implemented by the program requires y_n bytes of the memory, where y_n depends on n. Let y_n be defined inductively by

$$\begin{cases} y_1 = 3 \\ y_n = 4y_{n-1} \text{ if } n > 1 \end{cases}$$

By this inductive definition, we generate the following table of Fibonacci sequence:

n	1	2	3	4	5	...
y_n	3	12	48	192	768	...

Sometimes it is more convenient to start at $n=0$ instead of $n=1$. In fact, any integer could be used as the starting value in the inductive definition or recursive relation.

Example 3.14

The factorial function $n!$ is defined as follows:

$$\begin{cases} 0! = 1 \\ n! = n(n-1)! \text{ if } n \geq 1 \end{cases}$$

It is desirable to find an explicit expression of y_n as a function of n. To find such an expression is called the solution process of the inductive definition or recurrence relation.

A general solution for the recurrence relation is a solution process involving some arbitrary constants. The initial conditions enable us to find the values of those arbitrary constants. Sometimes by studying the Fibonacci sequence, an intuition gives the solution of y_n in terms of n.

Example 3.15

A function a_n is defined recursively as follows:

$$\begin{cases} a_1 = 3 \\ a_2 = 7 \\ a_n = 3a_{n-1} - 2a_{n-2} \text{ for } n \geq 3 \end{cases}$$

We want to find an explicit formula or solution for a_n in terms of n. We display some few initial terms of the Fibonacci sequence:

n	1	2	3	4	5	6	...
a_n	3	7	15	31	63	127	...

By intuition, $a_n = 2^{n+1} - 1$. This formula clearly satisfies the initial conditions a_1 and a_2. We verify if it satisfies the recurrence relation

$$
\begin{aligned}
&a_n = 3a_{n-1} - 2a_{n-2} \\
&\text{But} \\
&R.H.S = 3a_{n-1} - 2a_{n-2} \\
&\qquad = 3\left(2^n - 1\right) - 2\left(2^{n-1} - 1\right) \\
&\qquad = 3 \cdot 2^n - 3 - 2^n + 2 \\
&\qquad = 2^n\left(3 - 1\right) - 1 \\
&\qquad = 2^{n+1} - 1 \\
&\qquad = a_n \\
&\qquad = L.H.S
\end{aligned}
$$

When iteration does not apply, other methods are available for finding explicit formulas for special classes of recursively defined.

Exercise 3.1

1. Use iteration to guess a formula for the following recursively defined sequence and then use mathematical induction to prove the validity of your formula:

   $a_n = 1$, $a_n = 3a_{n-1} + 1$, for all $n \geq 2$

2. Use iteration to guess a formula for the following recursively defined sequence and then use mathematical induction to prove the validity of your formula:

   $a_0 = 1, a_n = 2^n - a_{n-1}$, for all $n \geq 2$.

3. Determine whether the recursively defined sequence: $a_1 = 0$ and $a_n = 2a_{n-1} + n - 1$ satisfies the explicit formula $a_n = \left(n-1\right)^2, n \geq 1$.

2-3.2 Classification of Recurrence Relation

A recurrence relation is of first order if a_n is defined only in terms of a_{n-1}. It is of second order if a_n is defined in terms of a_{n-1} and a_{n-2}, and so on.

A recurrence relation is then an equation of the type

$$F\left(n, a_n, a_{n+1}, \ldots, a_{n+k}\right) = 0,$$

where $k \in \mathbb{N}$ is fixed.

ACTIVITY **Example 3.16**. $a_{n+1} = 5a_n$ is a recurrence relation of order 1.

ACTIVITY **Example 3.17** $a_{n+1}^4 + a_n^5 = n$ is a recurrence relation of order 1.

Example 3.18 $a_{n+3}+5a_{n+2}+4a_{n+1}+a_n=\cos n$ is a recurrence relation of order 3.

Example 3.19 $a_{n+2}+5\left(a_{n+1}^2+an\right)^{1/3}=0$ is a recurrence relation of order 2.

We now define the order of a recurrence relation.

Definition 3.1: The order of a recurrence relation is the difference between the greatest and lowest sub – scripts of the terms of the sequence in the equation.

Hence $F\left(n,a_n,a_{n+1},...,a_{n+k}\right)=0$ is of order k.

Definition 3.2: A recurrence relation of order k is said to be linear if it is linear in $a_n,a_{n+1},...,a_{n+k}$. Otherwise, the recurrence relation is said to be non-linear.

Example 3.20. The recurrence relations in **Examples 3.16** and **3.18** are linear, while those in **Examples 3.17** and **3.19** are non-linear.

Example 3.21. $a_{n+1}a_{n+2}=5a_n$ is a non-linear recurrence relation of order 2.

Remark. The recurrence relation $a_{n+3} + 5a_{n+2} + 4a_{n+1} + a_n = \cos n$ can also be written in the form $a_{n+2} + 5a_{n+1} + 4a_n + a_{n-1} = \cos(n-1)$. There is no reason why the term of the sequence in the equation with the lowest subscript should always have subscript n.

For the sake of uniformity and convenience, we shall in this unit always follow the convention that the element of the sequence in the equation with the lowest subscript has subscript n.

Elimination of Arbitrary constants

We shall first of all consider a few examples. Do not worry about the details.

Example 3.22. Consider the equation $a_n = A(n!)$, where A is an arbitrary constant. Replacing n by $(n+1)$ in the equation, we obtain $a_{n+1} = A((n+1)!)$. Combining the two equations and eliminating A, we obtain the first-order recurrence relation $a_{n+1} = (n+1)a_n$.

Example 3.23. Consider the equation

$$a_n = (A + Bn)3^n \qquad (1)$$

where A and B are arbitrary constants. Replacing n by $(n+1)$ and (n+2) in the equation, we obtain respectively

$$a_{n+1} = \left(A + B\left(n \ +1\right)\right)3^{n+1} = \left(3A + 3B\right)3^{n} + 3Bn3^{n} \qquad (2)$$

and

$$a_{n+2} = \left(A + B\left(n \ +2\right)\right)3^{n+2} = \left(3A + 3B\right)3^{n+1} + 3B\left(n+1\right)3^{n+1}$$

which simplifies

$$a_{n+2} = \left(A + B\left(n \ +2\right)\right)3^{n+2} = \left(9A + 18B\right)3^{n} + 9Bn3^{n} \qquad (3)$$

Combining $(1)-(3)$ and eliminating A and B, we obtain the second-order recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

Example 3.24 Consider the equation

$$a_n = A(-1)^n + B(-2)^n + C3^n \qquad (4)$$

where A, B and C are constants. Replacing n by $(n+1)$, $(n+2)$ and $(n+3)$ in the equation, we obtain respectively

$$a_{n+1} = A(-1)^{n+1} + B(-2)^{n+1} + C3^{n+1} = -A(-1)^n - B(-2)^n + 3C3^n \qquad (5)$$

$$a_{n+2} = A(-1)^{n+2} + B(-2)^{n+2} + C3^{n+2} = A(-1)^n + 4B(-2)^n + 9C3^n \qquad (6)$$

and

$$a_{n+3} = A(-1)^{n+3} + B(-2)^{n+3} + C3^{n+3} = -A(-1)^n - 8B(-2)^n + 27C3^n \quad (7)$$

Combining $(4)-(7)$ and eliminating $A, B,$ and C, we obtain the third-order recurrence relation

$$a_{n+3} - 7a_{n+1} - 6a_n = 0$$

Note that in these three examples, the expression of a_n as a function of n contains respectively one, two and three constants. By writing down one, two and three extra expressions respectively, using subsequent terms of the sequence, we are in a position to eliminate these constants.

In general, the expression of a_n as a function of n may contain k arbitrary constants. By writing down k further equations, using subsequent terms of the sequence, we expect to be able to eliminate these constants. After eliminating these constants, we expect to end up with a recurrence relation of order k.

If we reverse the argument, it is reasonable to define the general solution of a recurrence relation of order k as that solution containing k arbitrary constants. This is, however, not very satisfactory. Instead, the following is true: Any solution of a recurrence relation of order k containing fewer than k arbitrary constants cannot be the general solution.

In many situations, the solution of a recurrence relation has to satisfy certain specified conditions. These are called initial conditions, and determine the values of the arbitrary constants in the solution.

ACTIVITY **Example 3.25**. The recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

has general solution $a_n = (A + Bn)3^n$, where A and B are arbitrary constants. Suppose that we have the initial conditions $a_0 = 1$ and $a_1 = 15$. Then we must have an

$$a_n = (1 + 4n)3^n.$$

Linear Recurrence Relations

Non-linear recurrence relations are usually very difficult, with standard techniques only for very few cases. We shall therefore concentrate on linear recurrence relations.

The general linear recurrence relation of order k is the equation

$$s_0(n)a_{n+k} + s_1(n)a_{n+k-1} + \cdots + s_k(n)a_n = f(n) \qquad (3.1)$$

where $s_0(n), s_1(n), ..., s_k(n)$ and $f(n)$ are given functions. Here we are primarily concerned with (3.1) only when the coefficients $s_0(n), s_1(n), ..., s_k(n)$ are constants and hence independent of n. We therefore study equations of the type

$$s_0 a_{n+k} + s_1 a_{n+k-1} + \cdots + s_k a_n = f(n) \qquad (3.2)$$

where $s_0, s_1, ..., s_k$ are constants, and where $f(n)$ is a given function.

Linear recurrence relations have the following important properties:

- multiplying any solution by a constant gives another solution,
- adding two or more solutions give another solution.

First-Order Recurrence Relations

First-order recurrence relations are of the form:

$$\begin{cases} a_n = sa_{n-1} \\ a_0 = t \end{cases}$$

We assume s and t are constants. We solve first-order recurrence relations by iteration; a process of repetitive procedure, as displayed:

$$\begin{aligned} a_n &= sa_{n-1} \\ &= s\left(sa_{n-2}\right) \\ &= s^2\left(sa_{n-3}\right) \\ &= \ldots\ldots\ldots\ldots \\ &= s^{n-1}a_1 \\ &= s^n a_0 \end{aligned}$$

Using the initial condition, we have $a_n = ts^n$, $n \in \square^{+}$. By this procedure, **Example 3.13** above, with $s = 4$ and $t = 3 = a_1$, we have $a_n = s^{n-1}a_1 = 3 \cdot 4^{n-1}$.

The Second-Order Recurrence Relation

Second-order recurrence relation has the form:

$$\begin{cases} a_n = s_1 y_{n-1} + s_2 a_{n-2} \text{ for } n \geq 2 \\ a_1 = t_1 \\ a_0 = t_0 \end{cases}$$

2-3.3 The Homogeneous Case

If the function $f(n)$ on the right-hand side of (3.1) is identically zero, then we say that the recurrence relation (3.1) is **homogeneous**. If the function $f(n)$ on the right-hand side of (3.1) is not identically zero, then we say that the recurrence relation

$$s_0(n) a_{n+k} + s_1 a_{n+k-1} + \ldots + s_k a_n = 0 \quad \text{.............................(3.3)}$$

is the reduced recurrence relation of (3.2).

In this subunit, we study the problem of finding the general solution of a homogeneous recurrence relation of the type (3.3)(called the linear homogeneous recurrence relation)

Suppose that $a_n^{(1)}, \ldots, a_n^{(k)}$ are k independent solutions of the recurrence relation (3.3), so that no linear combination of them with constant coefficient is identically zero, and

$$s_0 a^{(1)}{}_{n+k} + s_1 a^{(1)}{}_{n+k-1} + \ldots + s_k a_n^{(1)} = 0, \qquad \ldots, \qquad s_0 a^{(k)}{}_{n+k} + s_1 a^{(k)}{}_{n+k-1} + \ldots s_k a_n^{(k)} = 0.$$

We consider the linear combination

$$a_n = c_1 a_n^{(1)} + \ldots + c_k a_n^{(k)} \ldots\ldots\ldots\ldots\ldots\ldots(3.4)$$

where $c_1, \ldots, c_k$ are arbitrary constants. Then a_n is clearly also a solution of (3.3), for

$$s_0 a_{n+k} + s_1 a_{n+k-1} + \ldots + s_k a_n = s_0 \left(c_1 a^{(1)}{}_{n+k} + \ldots + c_k a^{(k)}{}_{n+k} \right)$$
$$+ s_1 \left(c_1 a^{(1)}{}_{n+k-1} + \ldots + c_k a^{(k)}{}_{n+k-1} \right) + \ldots + s_k \left(c_1 a_n^{(1)} + \ldots + c_k a_n^{(k)} \right)$$
$$= c_1 \left(s_0 a_{n+k}^{(1)} + s_1 a_{n+k-1}^{(1)} + \ldots + s_k a_n^{(1)} \right) + \ldots + c_k \left(s_0 a_{n+k}^{(k)} + s_1 a_{n+k-1}^{(k)} + \ldots s_k a_n^{(k)} \right) = 0.$$

Since (3.4) contains k constants, it is reasonable to take this as the general solution of (3.3). It remains to find k independent solutions $a_n^{(1)}, \ldots, a_n^{(k)}$.

Consider first of all the case $k = 2$. We are therefore interested in the homogeneous recurrence relation

$$s_0 a_{n+2} + s_1 a_{n+1} + s_2 a_n = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.5)$$

where s_0, s_1, s_2 are constants, with $s_0 \neq 0$ and $s_2 \neq 0$. Let us try a solution of the form

$$a_n = \lambda^n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.6)$$

where $\lambda \neq 0$. Then clearly $a_{n+1} = \lambda^{n+1}$ and $a_{n+2} = \lambda^{n+2}$, so that

$$\left(s_0 \lambda^2 + s_1 \lambda + s_2 \right) \lambda^n = 0$$

Since $\lambda \neq 0$, we must have

$$s_0 \lambda^2 + s_1 \lambda + s_2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.7)$$

This is called the **characteristic polynomial** (or **auxiliary equation**) of the recurrence relation (3.5).

It follows that (3.6) is a solution of the recurrence relation (3.5) whenever λ satisfies the characteristic polynomial (3.7). Suppose that λ_1 and λ_2 are the two roots of (3.7). Then

$$a_n^{(1)} = \lambda_1^n \text{ and } a_n^{(2)} = \lambda_2^n$$

are both solutions of the recurrence relation (3.5). It follows that the general solution of the recurrence relation (3.5) is

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n \text{ ....(3.8)}$$

Characteristic Polynomial

The **auxiliary or characteristic equation** , $s_0\lambda^2 + s_1\lambda + s_2 = 0$, is a quadratic equation and to solve this, we have three possibilities

I. two distinct real roots $\lambda = \lambda_1 \text{ and } \lambda = \lambda_2$

II. repeated real root $\lambda(\text{twice}) = \lambda_1 = \lambda_2$

III. two complex roots $\lambda = \lambda_1 \text{ and } \lambda_2 = \overline{\lambda_1}$

Case I

Since $a_n = \lambda_1^n \text{ and } a_n = \lambda_2^n$ are solutions of the linear recurrence relation, then another solution (the general solution) is $a_n = c_1\lambda_1^n + c_2\lambda_2^n$. Where $c_1 \text{ and } c_2$ are arbitrary constants. Using the initial values $a_0 = t_0 \text{ and } a_1 = t_1, c_1 \text{ and } c_2$ are easily determined.

Case II

However, if $\lambda_1 = \lambda_2$, then (3.8) does not qualify as the general solution of the recurrence relation (3.5), as it contains only one arbitrary constant. We therefore try for a solution of the form

$$a_n = u_n \lambda^n \text{ ....(3.9)}$$

where u_n is a function of n, and where λ is the repeated root of the characteristic polynomial (3.7). Then

$$a_{n+1} = u_{n+1}\lambda^{n+1} \text{ And } a_{n+2} = u_{n+2}\lambda^{n+2} \text{ ....(3.10)}$$

Substituting (3.9) and (3.10) into (3.5), we obtain

$$s_0 u_{n+2}\lambda^{n+2} + s_1 u_{n+1}\lambda^{n+1} + s_2 u_n \lambda^n = 0$$

Note that the left-hand side is equal to

$$\begin{aligned} & s_0\left(u_{n+2} - u_n\right)\lambda^{n+2} + s_1\left(u_{n+1} - u_n\right)\lambda^{n+1} + u_n\left(s_0\lambda^2 + s_1\lambda + s_2\right)\lambda^n \\ & = s_0\left(u_{n+2} - u_n\right)\lambda^{n+2} + s_1\left(u_{n+1} - u_n\right)\lambda^{n+1} \end{aligned}$$

It follows that

$$s_0\lambda\left(u_{n+2} - u_n\right) + s_1\left(u_{n+1} - u_n\right) = 0$$

Note now that since $s_0\lambda^2 + s_1\lambda + s_2 = 0$ and that λ is a repeated root, we must have $2\lambda = -s_1/s_0$. It follows that we must have $\left(u_{n+2} - u_n\right) - 2\left(u_{n+1} - u_n\right) = 0$, so that

$$u_{n+2} - 2u_{n+1} + u_n = 0$$

This implies that the sequence u_n is an arithmetic progression, so that $u_n = c_1 + c_2 n$, where c_1 and c_2 are constants. It follows that the general solution of the recurrence relation (3.5) in this case is given by

$$a_n = \left(c_1 + c_2 n\right)\lambda^n$$

where λ is the repeated root of the characteristic polynomial (3.7).

Case III

Since the characteristic equation has real coefficients, the complex roots occur in conjugate pairs. In other words, if $\lambda = \lambda_1 = u + iv$ is a root of the characteristic equation with real coefficients, then its complex conjugate $\lambda_2 = \overline{\lambda}_1 = u - iv$ is also a root with $v \neq 0$.

By the general rule, the solution

$$a_n = A\lambda_1^n + B\lambda_2^n = A\left(u+iv\right)^n + B\left(u-iv\right)^n$$

Converting $u+iv$ and $u-iv$ into polar coordinates,

$$u+iv = \rho\left(\cos\theta + i\sin\theta\right), \qquad u-iv = \rho\left(\cos\theta - i\sin\theta\right)$$

And by DeMoivre's Theorem,

$$\left[\rho\left(\cos\theta \pm i\sin\theta\right)\right]^n = \rho^n\left(\cos n\theta \pm i\sin n\theta\right)$$

Then

$$\begin{aligned} a_n &= A\rho^n\left(\cos n\theta + i\sin n\theta\right) + B\rho^n\left(\cos n\theta - i\sin n\theta\right) \\ &= \left(A+B\right)\rho^n(\cos n\theta) + i(A-B)\rho^n \sin n\theta \end{aligned}$$

If we substitute $A = B = \frac{1}{2}$, then $a_n = \rho^n \cos n\theta$ is a particular solution.

Similarly if we substitute $A = -\frac{1}{2}i$ and $B = \frac{1}{2}i$, then $a_n = \rho^n \sin n\theta$ is also a particular solution. Therefore the general solution is

$$a_n = c_1\rho^n \sin n\theta + c_2\rho^n \cos n\theta = \rho^n \left(c_1 \sin n\theta + c_2 \cos n\theta\right).$$

where $\rho = \sqrt{u^2 + v^2}$ and $\theta = \tan^{-1}\frac{v}{u}$.

ACTIVITY **Example 3.26**. The recurrence relation

$$a_{n+2} + 4a_{n+1} + 3a_n = 0$$

has characteristic polynomial $\lambda^2 + 4\lambda + 3 = 0$, with roots $\lambda_1 = -3$ and $\lambda_2 = -1$. It follows that the general solution of the recurrence relation is given by

$$a_n = c_1\left(-3\right)^n + c_2\left(-1\right)^n$$

ACTIVITY **Example 3.27**

Solve

$$\begin{cases} y_n = 3y_{n-1} - 2y_{n-2} \text{ for } n \geq 2 \\ y_2 = 7 \\ y_1 = 3 \end{cases}$$

Solution:

The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$. Factorizing the left-hand side gives

$(\lambda - 1)(\lambda - 2) = 0$ so that $\lambda = 1$ or $\lambda = 2$. The general solution is $y_n = c_1 \cdot 1^n + c_2 \cdot 2^n$.

Using the initial conditions specified, $c_1 = -1$ and $c_2 = 2$, therefore the solution is

$$\begin{aligned} y_n &= -1 + 2 \cdot 2^n \\ &= 2^{n+1} - 1 \end{aligned}$$

Example 3.28. The recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 0$$

has characteristic polynomial $\lambda^2 - 6\lambda + 9 = 0$, with repeated roots $\lambda = 3$. It follows that the general solution of the recurrence relation is given by

$$a_n = (c_1 + c_2 n)3^n$$

Example 3.29

Solve

$$\begin{cases} y_n = 6y_{n-1} - 9y_{n-2} \text{ for } n \geq 1 \\ y_1 = 3 \\ y_0 = 5 \end{cases}$$

Solution:

The characteristic equation is $\lambda^2 - 6\lambda + 9 = 0$. Factorizing the left-hand side gives $(\lambda - 3)^2 = 0 \text{ so that } \lambda = 3(\text{repeated})$. The general solution is

$y_n = c_1 \cdot 3^n + c_2 n \cdot 3^n$.

Using the initial condition specified, $c_1 = 5 \text{ and } c_2 = -4$, therefore the solution is

$$y_n = 5 \cdot 3^n - 4n \cdot 3^n$$
$$= 3^n (5 - 4n)$$

Example 3.30. The recurrence relation

$$a_{n+2} + 4a_n = 0$$

has characteristic polynomial $\lambda^2 + 4 = 0$, with roots $\lambda_1 = 2i$ and $\lambda_2 = -2i$. It follows that the general solution of the recurrence relation is given by

$$a_n = b_1 (2i)^n + b_2 (-2i)^n = 2^n \left(b_1 i^n + b_2 (-i)^n \right)$$

$$= 2^n \left(b_1 \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2} \right)^n + b_2 \left(\cos\frac{\pi}{2} - i \sin\frac{\pi}{2} \right)^n \right)$$

$$= 2^n \left(b_1 \left(\cos\frac{n\pi}{2} + i \sin\frac{n\pi}{2} \right) + b_2 \left(\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2} \right) \right)$$

$$= 2^n \left((b_1 + b_2) \cos\frac{n\pi}{2} + i (b_1 - b_2) \sin\frac{n\pi}{2} \right)$$

$$= 2^n \left(c_1 \cos\frac{n\pi}{2} + c_2 \sin\frac{n\pi}{2} \right)$$

Example 3.31. The recurrence relation

$$a_{n+2} + 4a_{n+1} + 16a_n = 0$$

has characteristic polynomial $\lambda^2 + 4\lambda + 16 = 0$, with roots $\lambda_1 = -2 + 2\sqrt{3}i$ and $\lambda_2 = -2 - 2\sqrt{3}i$. It follows that the general solution of the recurrence relation is given by

$$a_n = b_1 \left(-2 + 2\sqrt{3}i \right)^n + b_2 \left(-2 - 2\sqrt{3}i \right)^n$$

$$= 4^n \left(b_1 \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i \right)^n + b_2 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i \right)^n \right)$$

$$= 4^n \left(b_1 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)^n + b_2 \left(\cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3} \right)^n \right)$$

$$= 4^n \left(b_1 \left(\cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3} \right) + b_2 \left(\cos \frac{2n\pi}{3} - i \sin \frac{2n\pi}{3} \right) \right)$$

$$= 4^n \left(\left(b_1 + b_2 \right) \cos \frac{2n\pi}{3} + i \left(b_1 - b_2 \right) \sin \frac{2n\pi}{3} \right)$$

$$= 4^n \left(c_1 \cos \frac{2n\pi}{3} + c_2 \sin \frac{2n\pi}{3} \right)$$

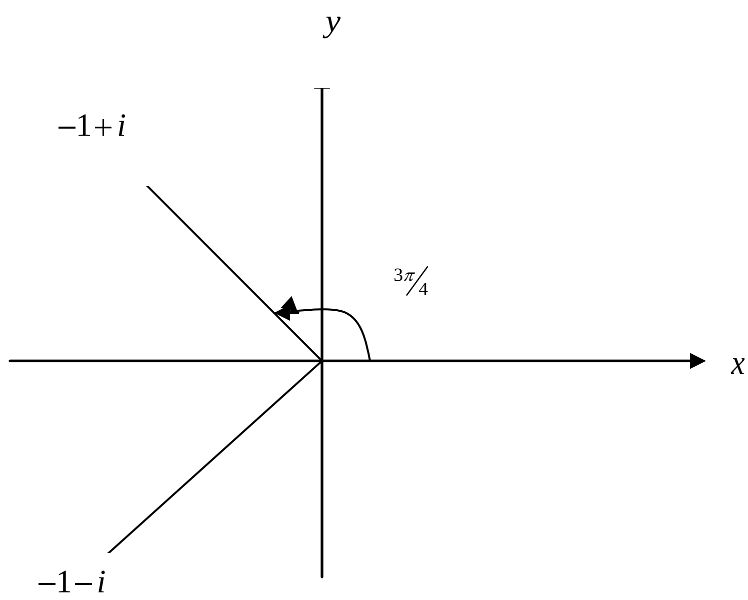
Example 3.32

Solve $y_n + 2y_{n-1} + 2y_{n-2} = 0$

Solution:

The characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$. Using the quadratic formula, we have the roots: $\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. Thus $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$.

Using the diagram,



$\rho = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$.

Hence, $y_n = A\left(\sqrt{2}\right)^n \sin\frac{3n\pi}{4} + B\left(\sqrt{2}\right)^n \cos\frac{3n\pi}{4}$ is the general solution.

We note that a characteristic equation is equally used for higher-order recurrence relations.We now consider the general case. We are therefore interested in the homogeneous recurrence relation

$$s_0 a_{n+k} + s_1 a_{n+k-1} + \ldots + s_k a_n = 0 \text{......................(3.11)}$$

where $s_0, s_1, \ldots, s_k$ are constants, with $s_0 \neq 0$. If we try a solution of the form $a_n = \lambda^n$ as before, where $\lambda \neq 0$, then it can easily be shown that we must have

$$s_0 \lambda^k + s_1 \lambda^{k-1} + \ldots + s_k = 0 \text{..............................(3.12)}$$

This is called the characteristic polynomial of the recurrence relation (3.11).

We shall state the following theorem without proof.

Proposition 3.1 *Suppose that the characteristic polynomial (3.12) of the homogeneous recurrence relation (3.11) has distinct roots* $\lambda_1,...,\lambda_s$*, with multiplicities* $m_1,...,m_s$ *respectively (where, of course,* $k = m_1 + ... + m_s$ *). Then the general solution of the recurrence relation (3.11) is given by*

$$a_n = \sum_{j=1}^{s}\left(b_{j,1} + b_{j,2}n + ... + b_{j,m_j}n^{m_j-1}\right)\lambda_j^n,$$

where, for every $j = 1,...,s,$ the coefficients $b_{j,1},...,b_{j,m_j}$ are constants.

Example 3.33. The recurrence relation

$$a_{n+5} + 7a_{n+4} + 19a_{n+3} + 25a_{n+2} + 16a_{n+1} + 4a_n = 0$$

has characteristic polynomial

$\lambda^5 + 7\lambda^4 + 19\lambda^3 + 25\lambda^2 + 16\lambda + 4 = (\lambda+1)^3(\lambda+2)^2 = 0$ with roots $\lambda_1 = -1$ and $\lambda_2 = -2$ with multiplicities $m_1 = 3$ and $m_2 = 2$ respectively. It follows that the general solution of the recurrence relation is given by

$$a_n = (c_1 + c_2 n + c_3 n^2)(-1)^n + (c_4 + c_5 n)(-2)^n$$

Example 3.34

Solve

$$\begin{cases} y_n = 7y_{n-2} - 6y_{n-3} \\ y_0 = 1,\ y_1 = -2 \text{ and } y_2 = 3 \end{cases}$$

Solution:

The characteristic equation is $\lambda^3 - 7\lambda + 6 = 0$. Factorizing the right-hand side gives

$$(\lambda - 1)(\lambda - 2)(\lambda + 3) = 0 \text{ so that } \lambda = 1,\ 2 \text{ and } -3.$$

The general solution is $y_n = c_1 \cdot 1^n + c_2 \cdot 2^n + c_3(-3)^n$.

Using the initial conditions, $c_1 = \frac{5}{4},\ c_2 = -\frac{4}{5}$ and $c_3 = \frac{11}{20}$.

So the solution is $y_n = \frac{5}{4} - \left(\frac{4}{5}\right)2^n + \left(\frac{11}{20}\right)(-3)^n$.

Sometimes a recurrence relation can be converted into linear, constant-coefficient form although it might not originally be in that form.

Example 3.35

Consider the recurrence relation:

$$\begin{cases} \sqrt{y_n} = -\sqrt{y_{n-1}} + 6\sqrt{y_{n-2}} \\ y_1 = 1 \\ y_0 = 9 \end{cases}$$

Solution:

Using the substitution $g_n = \sqrt{y_n}$ transforms the recurrence relation into

$$\begin{cases} g_n = -g_{n-1} + 6g_{n-2} \\ g_1 = 1 \\ g_0 = 3 \end{cases}$$

Using the appropriate standard method

$g_n = 1 \cdot (-3)^n + 2 \cdot 2^n$ and therefore $y_n = g_n^2 = \left[1(-3)^n + 2 \cdot 2^n\right]^2$

2-3.4 The Non – Homogeneous Case

We study equations of the type

$$s_0 a_{n+k} + s_1 a_{n+k-1} + \ldots + s_k a_n = f(n) \text{..}(3.13)$$

where $s_0, s_1, \ldots, s_k$ are constants, and where $f(n)$ is a given function.

Suppose that $a_n^{(c)}$ is the general solution of the reduced recurrence relation

$$s_0 a_{n+k} + s_1 a_{n+k-1} + \ldots + s_k a_n = 0 \text{......................................}(3.14)$$

so that the expression of $a_n^{(c)}$ involves k arbitrary constants. Suppose further that $a_n^{(p)}$ is any solution of the non-homogeneous recurrence relation (3.13). Then

$$s_0 a_{n+k}^{(c)} + s_1 a_{n+k-1}^{(c)} + \ldots + s_k a_n^{(c)} = 0 \text{ and } s_0 a_{n+k}^{(p)} + s_1 a_{n+k-1}^{(p)} + \ldots + s_k a_n^{(p)} = f(n)$$

Let

$$a_n = a_n^{(c)} + a_n^{(p)} \ldots\ldots\ldots\ldots\ldots\ldots(3.15)$$

Then

$$\begin{aligned} s_0 a_{n+k} + s_1 a_{n+k-1} + \ldots + s_k a_n &= s_0\left(a_{n+k}^{(c)} + a_{n+k}^{(p)}\right) + \\ &\qquad s_1\left(a_{n+k-1}^{(c)} + a_{n+k-1}^{(p)}\right) + \ldots + s_k\left(a_n^{(c)} + a_n^{(p)}\right) \\ &= \left(s_0 a_{n+k}^{(c)} + s_1 a_{n+k-1}^{(c)} + \ldots + s_k a_n^{(c)}\right) + \left(s_0 a_{n+k}^{(p)} + s_1 a_{n+k-1}^{(p)} + \ldots + s_k a_n^{(p)}\right) \\ &= 0 + f(n) = f(n). \end{aligned}$$

It is therefore reasonable to say that (3.15) is the general solution of the non-homogeneous recurrence relation (3.13).

The term $a_n^{(c)}$ is usually known as the complementary function of the recurrence relation (3.13), while the term $a_n^{(p)}$ is usually known as a particular solution of the recurrence relation (3.13). Note that $a_n^{(p)}$ is in general not unique.

To solve the recurrence relation (3.13), it remains to find a particular solution $a_n^{(p)}$.

The Method of Undetermined Coefficients

In this section, we are concerned with the question of finding particular solutions of recurrence relations of the type

$$s_0 a_{n+k} + s_1 a_{n+k-1} + \ldots + s_k a_n = f(n) \quad \text{..............................(3.16)}$$

where $s_0, s_1, \ldots s_k$ are constants, and where $f(n)$ is a given function.

$f(n)$	trial $a_n^{(p)}$	$f(n)$	trial $a_n^{(p)}$

The method of undetermined coefficients is based on assuming a trial form for the particular solution $a_n^{(p)}$ of (3.16) which depends on the form of the function $f(n)$ and which contains a number of arbitrary constants. This trial function is then substituted into the recurrence relation (3.16) and the constants are chosen to make this a solution.

The basic trial forms are given in the table below (c denotes a constant in the expression of $f(n)$ and A (with or without subscripts) denotes a constant to be determined):

c	A	$c\sin\alpha n$	$A_1\cos\alpha n + A_2\sin\alpha n$
cn	$A_0 + A_1 n$	$c\cos\alpha n$	$A_1\cos\alpha n + A_2\sin\alpha n$
cn^2	$A_0 + A_1 n + A_2 n^2$	$cr^n\sin\alpha n$	$A_1 r^n\cos\alpha n + A_2 r^n\sin\alpha n$
$cn^m\left(m\in\mathbb{N}\right)$	$A_0 + A_1 n + A_2 n^2 + \cdots A_m n^m$	$cr^n\cos\alpha n$	$A_1 r^n\cos\alpha n + A_2 r^n\sin\alpha n$
$cr^n\left(r\in\mathbb{R}\right)$	Ar^n	$cn^m r^n$	$r^n\left(A_0 + A_1 n + \cdots A_m r^m\right)$

Example 3.36 Consider the recurrence relation

$$a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)n$$

It has been shown in **Example 3.26** that the reduced recurrence relation has complementary function

$$a_n^{(c)} = c_1(-3)^n + c_2(-1)^n$$

For a particular solution, we try

$$a_n^{(p)} = A(-2)^n$$

Then

$$a_{n+1}^{p} = A(-2)^{n+1} = -2A(-2)^n$$

and

$$a_{n+2}^{(p)} = A(-2)^{n+2} = 4A(-2)^n$$

It follows that

$$a_{n+2}^{(p)}+4a_{n+1}^{(p)}+3a_{n}^{(p)}=(4A-8A+3A)(-2)^{n}=-A(-2)^{n}=5(-2)^{n}$$

if $A=-5$.

Hence

$$a_n=a_n^{(c)}+a_n^{(p)}=c_1(-3)^n+c_2(-1)^n-5(-2)^n$$

Example 3.37. Consider the recurrence relation

$$a_{n+2}+4a_n=6\cos\frac{n\pi}{2}+3\sin\frac{n\pi}{2}$$

It has been shown in **Example 3.30** that the reduced recurrence relation has complementary function

$$a_n^{(c)}=2^n\left(c_1\cos\frac{n\pi}{2}+c_2\sin\frac{n\pi}{2}\right)$$

For a particular solution, we try

$$a_n^{(p)}=A_1\cos\frac{n\pi}{2}+A_2\sin\frac{n\pi}{2}$$

Then

$$a_{n+1}^{(p)}=A_1\cos\frac{(n+1)\pi}{2}+A_2\sin\frac{(n+1)\pi}{2}$$

$$=A_1\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{2}-\sin\frac{n\pi}{2}\sin\frac{\pi}{2}\right)+A_2\left(\sin\frac{n\pi}{2}\cos\frac{\pi}{2}+\cos\frac{n\pi}{2}\sin\frac{\pi}{2}\right)$$

$$=A_2\cos\frac{n\pi}{2}-A_1\sin\frac{n\pi}{2}$$

and

$$a_{n+2}^{(p)} = A_1 \cos\frac{(n+2)\pi}{2} + A_2 \sin\frac{(n+2)\pi}{2}$$

$$= A_1\left(\cos\frac{n\pi}{2}\cos\pi - \sin\frac{n\pi}{2}\sin\pi\right) + A_2\left(\sin\frac{n\pi}{2}\cos\pi + \cos\frac{n\pi}{2}\sin\pi\right)$$

$$= -A_1\cos\frac{n\pi}{2} - A_2\sin\frac{n\pi}{2}$$

It follows that

$$a_{n+2}^{(p)} + 4a_n^{(p)} = 3A_1\cos\frac{n\pi}{2} + 3A_2\sin\frac{n\pi}{2} = 6\cos\frac{n\pi}{2} + 3\sin\frac{n\pi}{2}$$

if $A_1 = 2$ and $A_2 = 1$.

Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = 2^n\left(c_1\cos\frac{n\pi}{2} + c_2\sin\frac{n\pi}{2}\right) + 2\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2}$$

Example 3.38. Consider the recurrence relation

$$a_{n+2} + 4a_{n+1} + 16a_n = 4^{n+2}\cos\frac{n\pi}{2} - 4^{n+3}\sin\frac{n\pi}{2}$$

It has been shown in **Example 3.31** that the reduced recurrence relation has complementary function

$$a_n^{(c)} = 4^n\left(c_1\cos\frac{2n\pi}{3} + c_2\sin\frac{2n\pi}{3}\right)$$

For a particular solution, we try

$$a_n^{(p)} = 4^n\left(A_1\cos\frac{n\pi}{2} + A_2\sin\frac{n\pi}{2}\right)$$

Then

$$a_{n+1}^{(p)} = 4^{n+1}\left(A_1 \cos\frac{(n+1)\pi}{2} + A_2 \sin\frac{(n+1)\pi}{2} \right) = 4^n \left(4A_2 \cos\frac{n\pi}{2} - 4A_1 \sin\frac{n\pi}{2} \right)$$

and

$$\begin{aligned} a_{n+2}^{(p)} &= 4^{n+2}\left(A_1 \cos\frac{(n+2)\pi}{2} + A_2 \sin\frac{(n+2)\pi}{2} \right) \\ &= 4^n \left(-16A_2 \cos\frac{n\pi}{2} - 16A_1 \sin\frac{n\pi}{2} \right) \end{aligned}$$

It follows that

$$\begin{aligned} a_{n+2}^{(p)} + 4a_{n+1}^{(p)} + 16a_n^{(p)} &= 16A_2 4^n \cos\frac{n\pi}{2} - 16A_1 4^n \sin\frac{n\pi}{2} \\ &= 4^{n+2} \cos\frac{n\pi}{2} - 4^{n+3} \sin\frac{n\pi}{2} \end{aligned}$$

If $A_1 = 4$ and $A_2 = 1$. Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = 4^n \left(c_1 \cos\frac{2n\pi}{3} + c_2 \sin\frac{2n\pi}{3} + 4\cos\frac{n\pi}{2} + \sin\frac{n\pi}{2} \right)$$

For $f(n) = cr^n$ c, r are constants, if the characteristic polynomial (3.14) has $\lambda = r$ as a root of multiplicity k, then the trial function is of the form $a_n^{(p)} = An^k r^n$, where A constant to be determined. If $\lambda = r$ is **Not** a root, take $k = 0$. $k = 1$ if $\lambda = r$, $k = 2$ if $\lambda_1 = \lambda_2 = r$, and so on.

ACTIVITY **Example 3.39**.

Solve the recurrence relation $a_{n+2} + 4a_{n+1} + 3a_n = 12(-3)^n$.

Solution:

$$\lambda^2 + 4\lambda + 3 = 0 \qquad \Rightarrow \lambda = -3, -1$$

$$a_n^{(c)} = c_1(-3)n + c_2(-1)n$$

For a particular solution, the trial solution

$$a_n^{(p)} = An(-3)^n$$

Then

$$a_{n+1}^{(p)} = A(n+1)(-3)^{n+1} = -3An(-3)^n - 3A(-3)^n$$

and

$$a_{n+2}^{(p)} = A(n+2)(-3)^{n+2} = 9An(-3)^n + 18A(-3)^n$$

It follows that

$$a_{n+2}^{(p)} + 4a_{n+1}^{(p)} + 3a_n^{(p)} = (9A - 12 + 3A)n(-3)^n + (18A - 12A)(-3)^n = 12(-3)^n$$

$$\Rightarrow 6A(-3)^n = 12(-3)^n$$

$$\Rightarrow A = 2$$

Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = c_1(-3)^n + c_2(-1)^n + 2n(-3)^n$$

Example 3.40.

Consider the recurrence relation

$$a_{n+2} - 6a_{n+1} + 9a_n = 3^n$$

The characteristic polynomial of the homogeneous part is:

$$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow \lambda = \lambda_1 = \lambda_2 = 3$$

The complementary solution is: $a_n^{(c)} = (c_1 + c_2 n)3^n$

For the roots of the polynomial , then $k = 2$, so we have the particular solution as:

$$a_n^{(p)} = An^2 3^n$$

Then

$$a_{n+1}^{(p)} = A(n+1)^2 3^{n+1} = A(3n^2 + 6n + 3)3^n$$

and $$a_{n+2}^{(p)} = A(n+2)^2 3^{n+2} = A(9n^2 + 36n + 36)3^n$$

It follows that

$$a_{n+2}^{(p)} - 6a_{n+1}^{(p)} + 9a_n^{(p)} = 18A3^n = 3^n$$

If $A = 1/18$.

Hence

$$a_n = a_n^{(c)} + a_n^{(p)} = \left(c_1 + c_2 n + \frac{n^2}{18}\right)3^n$$

For $f(n) = cr^n \sin\alpha n$ or $cr^n \cos\alpha n$, or both, a,b constants, if $\lambda^2 + r^2 = 0$ or $\lambda^2 + r^2$ is a factor of the auxiliary equation of multiplicity k, then the trial function would be $a_n^{(p)} = n^k \left(A_1 \sin \alpha n + A_2 \cos \alpha n \right)$, where A, B are constant to be determined. If $\lambda^2 + r^2$ is **Not** a factor, then $k = 0$. $k = 1$ if $\lambda^2 + r^2$ is the only factor.

Example 3.41.

Consider the recurrence relation

$$a_{n+2} + 4a_n = 2^n \cos \frac{n\pi}{2}$$

It has been shown in **Example 3.30** that the reduced recurrence relation has complementary function

$$a_n^{(c)} = 2^n \left(c_1 \cos \frac{n\pi}{2} + c_2 \sin \frac{n\pi}{2} \right)$$

For a particular solution, we will then have:

$$a_n^{(p)} = n2^n \left(A_1 \cos \frac{n\pi}{2} + A_2 \sin \frac{n\pi}{2} \right)$$

Then

$$a_{n+1}^{(p)} = (n+1)2^{n+1} \left(A_1 \cos \frac{(n+1)\pi}{2} + A_2 \sin \frac{(n+1)\pi}{2} \right)$$

$$\Rightarrow a_{n+1}^{(p)} = (n+1)2^{n+1} \left(A_2 \cos \frac{n\pi}{2} - A_1 \sin \frac{n\pi}{2} \right)$$

and

$$a_{n+2}^{(p)}=(n+2)2^{n+2}\left(A_1\cos\frac{(n+2)\pi}{2}+A_2\sin\frac{(n+2)\pi}{2}\right)$$

$$\Rightarrow a_{n+2}^{(p)}=(n+2)2^{n+2}\left(-A_1\cos\frac{n\pi}{2}-A_2\sin\frac{n\pi}{2}\right)$$

It follows that

$$a_{n+2}^{(p)}+4a_n^{(p)}=(n+2)2^{n+2}\left(-A_1\cos\frac{n\pi}{2}-A_2\sin\frac{n\pi}{2}\right)+4\left[n2^n\left(A_1\cos\frac{n\pi}{2}+A_2\sin\frac{n\pi}{2}\right)\right]$$

$$\Rightarrow a_{n+2}^{(p)}+4a_n^{(p)}=\left[-(n+2)2^{n+2}+4n2^n\right]A_1\cos\frac{n\pi}{2}+\left[-(n+2)2^{n+2}+4n2^n\right]A_2\sin\frac{n\pi}{2}$$

$$=-2^{n+3}A_1\cos\frac{n\pi}{2}-2^{n+3}A_2\sin\frac{n\pi}{2}=2^n\cos\frac{n\pi}{2}$$

If $A_1=-1/8$ and $A_2=0$. Hence

$$a_n=a_n^{(c)}+a_n^{(p)}=2^n\left(c_1\cos\frac{n\pi}{2}+c_2\frac{n\pi}{2}-\frac{n}{8}\cos\frac{n\pi}{2}\right)$$

Self Assessment 2-3

1. Solve each of the following homogeneous linear recurrences:

a) $a_{n+2}-6a_{n+1}-7a_n=0$ b) $a_{n+2}+10a_{n+1}+25a_n=0$ e) $a_{n+3}+5a_{n+2}+12a_{n+1}-18a_n=0$

c) $a_{n+3}-6a_{n+2}+9a_{n+1}-4a_n=0$ d) $a_{n+2}-4a_{n+1}+8a_n=0$

2. For each of the following linear recurrences, write down its characteristic polynomial, the general solution of the reduced recurrence, and the form of a particular solution to the recurrence:

a) $a_{n+2}+4a_{n+1}-5a_n=4$ b) $a_{n+2}+4a_{n+1}-5a_n=n^2+n+1$

c) $a_{n+2}-2a_{n+1}-5a_n=\cos\ n\pi$ d) $a_{n+2}+4a_{n+1}+8a_n=2^n\sin\left(n\pi/4\right)$

e) $a_{n+2}-9a_n=3^n$ f) $a_{n+2}-9a_n=n3^n$

g) $a_{n+2}-9a_n=n^2 3^n$ h) $a_{n+2}-6a_{n+1}+9a_n=3^n$

i) $a_{n+2}-6a_{n+1}+9a_n=3^n+7^n$ j) $a_{n+2}+4a_n=2^n\cos\left(n\pi/2\right)$

k) $a_{n+2}+4a_n=2^n\cos n\pi$ l) $a_{n+2}+4a_n=n2^n\sin\ n\pi$

3. For each of the following functions $f\left(n\right)$ use the method of undetermined coefficients to find a particular solution of the non-homogeneous linear recurrence $a_{n+2}-6a_{n+1}-7a_n=f\left(n\right)$.

Write down the general solution of the recurrence, and then find the solution that satisfies the given initial conditions:

a) $f\left(n\right)=24\left(-5\right)^n;a_0=3,a_1=-1$ b) $f\left(n\right)=16\left(-1\right)^n;a_0=4,a_1=2$

c) $f\left(n\right)=8\left(\left(-1\right)^n+7^n\right);\ a_0=5,\ a_1=11$ d) $f\left(n\right)=12n^2-4n+10;\ a_0=0,\ a_1=-10$

e) $f\left(n\right)=2\ \cos\ \dfrac{n\pi}{2}+36\sin\dfrac{n\pi}{2};a_0=20,a_1=3$

Unit 4

BOOLEAN ALGEBRA, LOGIC GATES

Introduction

We shall have an insight into the basic mathematical logic behind language of computer usage. An understanding of this unit will ease up complex structures in the complex language of the computer. You are encouraged to put in some effort to follow this unit

Learning Objectives

After reading this unit you should be able to:

1. Use Boolean symbols as analogous to logical symbols
2. Use Boolean symbols for electrical circuits

Unit content

Session 1-4: Boolean Algebra and Function

1-4.1Basic Definitions and Theorems
1-4.2Boolean Expressions and Functions
1-4.3Duality
1-4.4Representation of Boolean Functions

Session 2-4: Logic Gates

2-4.1Logic Gates Forms
2-4.2Combinations of Gates
2-4.3Examples of circuit

SESSION 1-4: BOOLEAN ALGEBRA AND FUNCTIONS

1-4.1 Basic Definitions and Theorems

Definition (Boolean Algebra)

Let B be a set on which are defined two binary defined operators, $+$ and $*$, and a unary operation, denoted by $'$; let 0 and 1 denote two distinct elements of B. Then the sextuplet

$$\langle B,\ +,\ *,\ ',\ 0,\ 1\rangle$$

is called a **Boolean algebra** if the following axioms hold for any elements: $a,b,c \in B$;

$[\mathbf{B}_1]$ Commutative laws:

$$(1\text{a})\ a+b=b+a \qquad (1\text{b})\ a*b=b*a$$

$[\mathbf{B}_2]$ Distributive laws:

$$(2\text{a})\ a+(b*c)=(a+b)*(a+c) \qquad (2\text{b})\ a*(b+c)=(a*b)+(a*c)$$

$[\mathbf{B}_3]$ Identity Laws:

$$(3\text{a})\ a+0=a \qquad (3\text{b})\ a*1=a$$

$[\mathbf{B}_4]$ Complement Laws:

$$(4\text{a})\ a+a'=1 \qquad (4\text{b})\ a*a'=0$$

The sextuplet is denoted by B when the operations are understood. The element 0 is called the **zero** element, the element 1 is called the **unit** element, and a' is called the **complement** of a. The results of the operations $+$ and $*$ are called the **sum** and **product,** respectively. We will frequently drop the symbol $*$ and use juxtaposition instead. Then (2b) and (2a) are written

$$(2\text{b})\ a(b+c)=ab+ac \quad (2\text{a})\ a+bc=(a+b)(a+c)$$

The following convention, unless guided by parenthesis, is that $'$ has precedence over $*$, and $*$ has precedence over $+$.

For example

$$a+b*c \text{ means } a+(b+c) \text{ not } (a+b)*c$$
$$a*b' \text{ means } a*(b)' \text{ and not } (a*b)'$$

The Boolean Algebra B With Two Elements 0 And 1 (Called Bits)

Let $+$ and $*$ be the binary operations in B. Let $0'=1$ and $1'=0$.

•	'
1 0	0 1

Then we have the following two fundamental tables:

+	1	0
1 0	1 1	1 0

*	1	0
1 0	1 0	0 0

Example 4.1

Find the value of $1*0+(0+1)'$

Solution:

From the tables:

$$
\begin{aligned}
&(0+1)=1 \\
&\text{and } (0+1)'=1'=0 \\
&1*0+(0+1)'=1*0+0 \\
&\qquad\qquad\quad =0+0 \\
&\qquad\qquad\quad =0
\end{aligned}
$$

The complement, Boolean sum and Boolean product correspond to the logical operators $\neg$, $\vee$ and $\wedge$ respectively, where 0 corresponds to F (false) and 1 to T (true).

1-4.2 Boolean Expressions and Boolean Functions

Let $B=\{0,1\}$. The variable x is called a **Boolean variable** if it takes values only from B. A function $F: B^n = \{(x_1, x_2, ..., x_n) | x_i \in B,\ 1 \le i \le n\} \to B$ is called a **Boolean function of degree n.** The values of a Boolean function are displayed in tables.

Example 4.2

The Boolean function $F(x,y)$ with the value 1 when $x=1 \text{ and } y=0$ and the value 0 for all other choice of $x \text{ and } y$ is represented by the table

x	y	$F(x,y)$
1	1	0
1	0	1
0	1	0
0	0	0

Boolean functions are represented by Boolean expressions made up of the variables and the Boolean operations. Boolean algebra $(B,\ +,\ *,\ ',\ 0,\ 1)$ are defined recursively as

For each $s \in B$, s is a Boolean expression. $x_1, x_2, ..., x_n$ are Boolean expressions.

If $x_1 \text{ and } x_2$ are Boolean expressions, so are $x_1', x_2', x_1 + x_2 \text{ and } x_1 * x_2$.

Each Boolean expression represents a Boolean function. The values of this function are obtained by substituting 0 and 1 for the variables in the expression.

Example 4.3

Find the values of the Boolean function represented by $F(x, y, z) = xy + z'$

Solution:

x	y	z	$xy = x * y$	z'	$F(x, y, z)$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

The Boolean functions *F* and *G* of *n* variables are equal if and only if $F(b_1, b_2, ..., b_n) = G(b_1, b_2, ..., b_n)$ whenever $b_1, b_2, ..., b_n \in B$.

Two different expressions that represent the same function are called **equivalent**. For example, the Boolean expressions: $xy,\ xy + 0,\ xy * 1$ are equivalent.

The **complement** of the Boolean function F is the function F'.

Let F and G be Boolean functions of degree n. The **Boolean sum** $F+G$ and the **Boolean product** $F*G$ are defined by

$$(F+G)(x_1,x_2,...,x_n)=F(x_1,x_2,...,x_n)+G(x_1,x_2,...,x_n)$$
$$(F*G)(x_1,x_2,...,x_n)=F(x_1,x_2,...,x_n)*G(x_1,x_2,...,x_n)$$

Recalling the definition of a function or a given domain into a co-domain using exponentiation, we have $b(\{f:f:B\rightarrow A\})=b(A)^{b(B)}$. Where b denotes the cardinality or distinct number of elements in a given set.

Therefore a Boolean function of degree 2, by definition, is a function from a set with four elements, namely, pairs of elements from $B=\{0,1\}$, to B, a set of two elements. Hence, there are $2^4=16$ different Boolean functions of degree 2. That is, we want to find $b(F:B^2=B*B=\{x_1,x_2\}:x_1\in B=\{0,1\}\rightarrow B)$.

In a similar analysis, the different Boolean functions of degree n is 2^{2n}.

Identities of Boolean Algebra

There are many identities of Boolean algebra but we provide the most important of them as displayed in the following table:

Boolean Identities

Identity	Name
$(x')' = x$	Law of double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 0 = 0$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Dominance laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + (yz) = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$(xy)' = x' + y'$ $(x + y)' = x' \cdot y'$	De Morgan's laws

Example 4.4

Show that the distributive law $x(y+z)=xy+xz$ is valid.

Solution:

The verification is shown in the following table

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

The identity holds because the last two columns of the table agree. The basic important identities summarized in the previous table can be used to prove further identities.

Example 4.5

Prove the absorption law $x(x+y)=x$, verify the identities of Boolean algebra.

Solution:

The steps used to derive this identity and the law used in each step follows:

$x(x+y)=(x+0)(x+y)$ Identity law for the Boolean sum

$=x+(0\cdot y)$ Distributive law of the Boolean sum over the Boolean product

$=x+y\cdot 0$ Commutative law for the Boolean product

$=x+0$ Dominance law for the Boolean product

$=x$ Identity law for Boolean sum

1-4.3 Duality

Observe that in the table of Boolean Identities, the identities come in pairs (except for the double complement). To fully explain the relationship between the two identities in each pair, we use the concept of a **dual**. The **dual** of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0's and 1's.

ACTIVITY **Example 6**

Find the duals of $x(y+0)$ and $x'\cdot 1+(y'+z)$.

Solution:

Interchanging * signs and + signs and interchanging 0's and 1's in these expressions produces their duals. The duals are $x+(y\cdot 1)$ and $(x'+0)(y'z)$ respectively.

The dual of a Boolean function F represented by a Boolean expression is the function represented by the dual of this expression. This dual function, denoted by F^d, does not depend on the particular Boolean expression used to represent F.

An identity between functions represented by Boolean expressions remains valid when the duals of both sides of the identity are taken. That is, if F and G are Boolean functions represented by Boolean expressions in n variables and $F = G$, then $F^d = G^d$, where F^d and G^d are the Boolean functions represented by the duals of the Boolean expressions representing F and G, respectively. This result, called the **duality principle** is useful for obtaining new identities.

Example 4.7

Construct an identity from the absorption law: $x(x+y)=x$ given in example 5 by taking duals.

Solution:

Taking duals of both sides of this identity produces the identity $x+(xy)=x$, which is also called an absorption law.

1. Find the values of the following expressions

   (a) $1\cdot 0'$ (b) $1+1'$ (c) $0'\cdot 0$ (d) $(1+0)'$

2. Find the values, if any, of the Boolean variable x that satisfy the following equations:

   (a) $x\cdot 1=0$ (b) $x+x=0$ (c) $x\cdot 1=x$ (d) $x\cdot x'=1$

   $\left[\text{Hint: use tables for } x\in\{0,1\}\right]$

3. What values of the Boolean variables x and y satisfy $xy=x+y$? [Hint: use table]
4. How many different Boolean functions are there of degree 7?
5. Prove the absorption law $x+xy=x$ using the laws in table 5.
6. Show that $F(x,y,z)=xy+xz+yz$ has the value 1 if and only if at least two of the variables x, y and z have 1. (use tables)
7. Show that $xy'+yz'+x'z=x'y+y'z+xz'$. (use tables)

Exercise 8-15 deal with the Boolean algebra defined by the Boolean sum and Boolean product on $\{0,1\}$

8. Verify the law of double complement
9. Verify the idempotent laws
10. Verify the identity laws
11. Verify the dominance laws
12. Verify the commutative laws
13. Verify the associative laws
14. Verify the first-distributive law in Table 5
15. Verify De Morgan's laws.

The operator $\oplus$, called the xOR operator, is defined by

(a) $1\oplus 1=0,\qquad 1\oplus 0=1\qquad 1\oplus 0=1\qquad \text{and}\qquad 0\oplus 0=0$

16. Simplify the following expressions

    (a) $x\oplus 0$ $\quad$ (b) $x\oplus 1$ $\quad$ (c) $x\oplus x$ $\quad$ (d) $x\oplus x'$

    (use tables)

17. Show that the following identities hold:

    (a) $x\oplus y=(x+y)(xy)'$ $\qquad$ (b) $x\oplus y=(xy')+(x'y)$

18. Show that $x\oplus y=y\oplus x$

19. Prove or disprove the following equalities

    (a) $x\oplus(y\oplus z)=(x\oplus y)\oplus z$

    (b) $x+(y\oplus z)=(x+y)\oplus(x+z)$

    (c) $x\oplus(y+z)=(x\oplus y)+(x\oplus z)$

1-4.4 Representation Boolean Functions

Two important problems of Boolean algebra will be examined.

The first problem is: Given the values of a Boolean function, can the function be found? This problem will be solved by showing that any Boolean function may be represented by a Boolean sum or Boolean product of the variables and their complements, that is, every Boolean function can be represented using the three Boolean operators:

$$\cdot,\ +\ \text{and}\ '.$$

The second problem is: Is there a smaller set of operators that can be used to represent all Boolean functions? This will be answered by showing that all Boolean functions can be represented using only one operator.

The foregoing problems have practical importance in circuit design.

Sum of Products Expansions

We use examples to illustrate one important way to find a Boolean expression that represents a Boolean function.

Example 4.8

Find Boolean expressions that represent the functions $F(x, y, z)$ and $G(x, y, z)$, which are given in Table 1.

Table 1

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Solution:

F has value 1 when $x = z = 1 \text{ and } y = 0$ and θ otherwise. Such an expression is formed by taking the Boolean product of $x,\ y' \text{ and } z$. This product $xy'z$ has value 1 if and only if $x = y' = z = 1$, which holds if and only if $x = z = 1 \text{ and } y = 0$.

To represent G, we need an expression that equals 1 when $x = y = 1 \text{ and } z = 0$, **or** when $x = z = 0 \text{ and } y = 1$. We can form an expression with these values by taking the Boolean sum of two different Boolean products. The Boolean product xyz' has the value 1 if and only if $x = z = 0 \text{ and } y = 1$. The Boolean sum of these two products $xyz' + x'yz'$ represent *G*, since it has the value 1 if and only if $x = y = 1 \text{ and } z = 0 \text{ or } x = z = 0 \text{ and } y = 1$.

Definition

A **literal** is a Boolean variable or its complement. A **minterm** of the Boolean variables $x_1, x_2, ..., x_n$ is a Boolean product $y_1, y_2, ..., y_n$ where $y_i = x_i$ or $y_i = x_i^{'}$. Hence, a minterm is a product of n literals with one literal for each variable.

Example 4.9

Find a minterm that equals 1 if $x_1 = x_3 = 0$ and $x_2 = x_4 = x_5 = 1$, and equals 0 otherwise.

Solution:

The minterm $x_1^{'} x_2 x_3^{'} x_4 x_5$ has the correct set of values.

By taking Boolean sums of distinct minterms we can build up a Boolean expression with a specified set of values. In particular, a Boolean sum of minterms has the value 1 when exactly one of the minterms in the sum has the value 1. Consequently, given a Boolean function, a Boolean sum of minterms can be formed that has the value 1 when this Boolean function has the value 1. The minterms in this Boolean sum correspond to those combinations of values for which the function has the value 1. The sum of minterms that represents the function is called the **sum of products expansion** or the **disjunction normal forms** of the Boolean function.

Example 4.10

Find the sum of products expansion for the function $F(x, y, z) = (x + y)z'$

Solution:

The first step is to find the values of F. These are found in table 2. The sum-of-products expansion of F is the Boolean sum of three minterms corresponding to the three rows of this table that give the value 1 for the function.

This gives $F(x, y, z) = xyz' + xy'z' + x'yz'$.

Table 2

x	y	z	$x+y$	z'	$(x+y)z' = F$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

It is also possible to find a Boolean expression that represents a Boolean function by taking a Boolean product of Boolean sums. The resulting expansion is called the

conjugate normal form or **product-of-sums expansion** of the function. These expansions can be found from the sum-of-products expansions by taking duals.

Functional Completeness

Every Boolean function can be expressed as a Boolean sum of minterms, which are Boolean products of Boolean variables or their complements. This shows that every Boolean function can be represented using the Boolean operations $\cdot,\ +$ and $'$. We therefore say that the set $\{\cdot,\ +,\ '\}$ is **functionally complete**.

Can we find a smaller set of functionally complete operators? This is achievable if one of the three operators of this set can be expressed in terms of the other two.

There are two such procedures, in a De Morgan's laws:

I. Eliminate all Boolean sums using the identity

$$x + y = (x'y')'$$

Thus the set $\{\cdot,\ '\}$ is functionally complete.

II. Eliminate all Boolean products using the identity

$$xy = (x' + y')'$$

Thus the set $\{+,\ '\}$ is functionally complete.

However, the set $\{\cdot,\ +\}$ is not functionally complete.

Can we find a smaller set of functionally complete operators, namely, a set containing just one operator? Such sets exist.

1	1	0
1 0	0 1	1 1

$\downarrow$	1	0
1 0	0 0	0 1

Define two operator | or NAND (not AND) operator, defined by

$1|1=0$ and $1|0=0|1=0|0=0$; and $\downarrow$ or NOR $\left(\text{not OR}\right)$

defined by $1\downarrow 1=1\downarrow 0=0\downarrow 1=0$ and $0\downarrow 0=1$.

Both of the sets {|} is functionally complete. Since $\{\cdot,\ '\}$ is functionally complete, all we need to do is show that both operators $\cdot$ and $'$ can be expressed using just | operator. This is done as follows:

$$x'=x|x$$
$$xy=(x|y)|(x|y)$$

Exercise

1. Find a Boolean product of the Boolean variable $x,\ y \text{ and } z$ or their complements, that has the value 1 if and only if

   $(\text{a})\ \ x = y = 0,\ z = 1$

   $(\text{b})\ \ x = 0,\ \ y = 1\ \ z = 0$

   $(c)\ \ x = 0,\ \ y = z = 1$

   $(d)\ x = y = z = 0$

2. Find the sum-of-products expansions of the following Boolean functions

   $(a)\ \ F(x, y) = x' + y$

   $(b)\ \ F(x, y) = xy'$

   $(c)\ \ F(x, y) = 1$

   $(d)\ \ F(x, y) = y'$

3. Find the sum-of-products expansions of the following Boolean functions

   $(a)\ \ F(x, y, z) = x + y + z$

   $(b)\ \ F(x, y, z) = (x + z)y$

   $(c)\ \ F(x, y, z) = x$

   $(d)\ \ F(x, y, z) = xy'$

4. Find the sum-of-products expansions of the Boolean function $F(x, y, z)$ that equals 1 if and only if

   $(a)\ \ x = 0$ $\qquad$ $(\text{b})\ \ xy = 0$

   $(c)\ \ x + y = 0$ $\qquad$ $(\text{d})\ \ xyz = 0$

5. Find the sum-of-products of the Boolean function $F(w, x, y, z)$ that has the value 1 if and only if are odd number of $w, x, y \text{ and } z$ have the value 1

6. Find the sum-of-products expansions of the Boolean function $F\left(x_1, x_2, x_3, x_4, x_5\right)$ that has the value 1 if and only if three or more of the variables x_1, x_2, x_3, x_4, x_5 have the value 1.

In question numbers 7-11, find a Boolean expression that represents a Boolean function formed from a Boolean product of Boolean sums of literals.

7. Find a Boolean sum containing either x or x', either y or y' and either z or z' that have the value 0 if and only if

   (a) $x = y = 1,\ z = 0$

   (b) $x = y = z = 0$

   (c) $x = z = 0, y = 1$

8. Find a Boolean product of Boolean sums of literals that has the value 0 if and only if either $x = y = 1$ and $z = 0$, $x = z = 0$ and $y = 1$ or $x = y = z = 0$ (Hint: take the Boolean product of Boolean sums found in parts (a), (b) and (c) in7).
9. Show that the Boolean sum $y_1 + y_2 + ... + y_n$, where $y_i = x_i$ and $x_i = 1$ if $y_i = x_i^{'}$. This Boolean sum is called a **maxterm**.

10. Show that a Boolean function can be represented as a Boolean product of maxterms. This representation is called the **product-of-sums expansion or conjugate normal form** of the function. (Hint: include one maxterm in this product for each confirmation of the variables where the function has the value 0).
11. Find the product-of-sums expansion of each of the Boolean functions in (3).
12. Express each of the following Boolean functions using the operators $\cdot$ and $'$.

    (a) $x + y + z$

    (b) $x + y'\left(x' + z\right)$

    (c) $\left(x + y'\right)'$

    (d) $x'\left(x + y' + z'\right)$

13. Express each of the Boolean functions in (12) using the operators $\cdot$ and $'$.

14. Show that

$(a)\ \ x' = x|x$ $\qquad$ $(\mathrm{b})\ \ xy = (x|y)(x|y)$

$(c)\ \ x + y = (x|x)(y|y)$

15. Show that

$(a)\ \ x' = x \downarrow x$ $\qquad$ $(\mathrm{b})\ \ xy = (x \downarrow x) \downarrow (y \downarrow y)$

$(c)\ \ x + y = (x \downarrow y) \downarrow (x \downarrow y)$

16. Show that $\{\downarrow\}$ is functionally complete using (15)

17. Express each of the Boolean functions in (3) using the operator $|$.

18. Express each of the Boolean functions in (3) using the operator $\downarrow$.

SESSION 2-4: LOGIC GATES

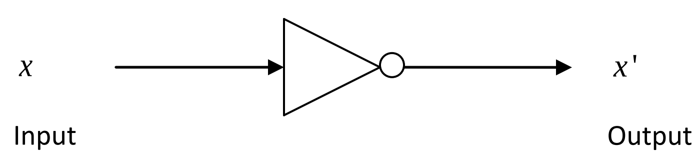
2-4.1 Logic Gates Forms

We use Boolean algebra to model the circuiting of electronic devices. We take each input and each output of such a device as a member of the set $\{0,1\}$.

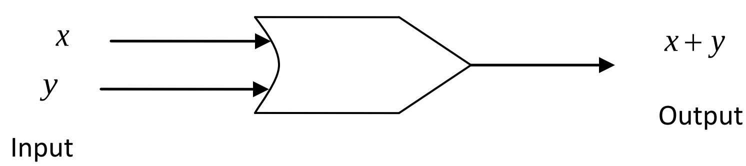
A computer, or any electronic device, is made up of circuits. Each circuit is designed using the rules of Boolean algebra so far studied. The basic elements of circuits are called **gates**, so that each type of gate implements a Boolean operation. With these gates, we apply the rules of Boolean algebra to design circuits to perform variety of tasks. Such circuits give **output**, that depends only on the **inputs**, and not on memory capacities. Such circuits are called **combinatorial circuits**.

Combinatorial circuits are constructed using three basic types of elements which we describe as follows:

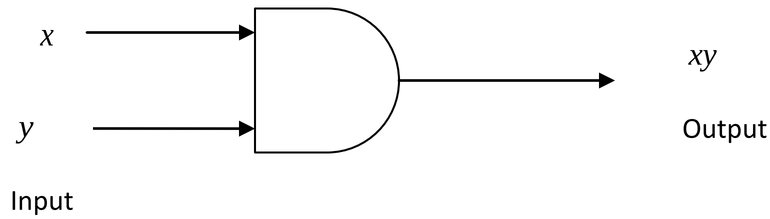
I. **An inverter**: This accepts the value of a Boolean variable as an input and produces its complement as its output. This is represented by



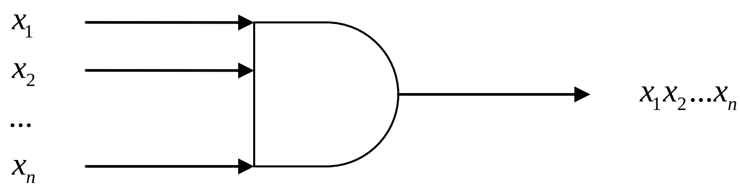
II. **OR gate:** The inputs of this gate are two or more Boolean variables. The output is the Boolean **sum** of their values. This is represented by

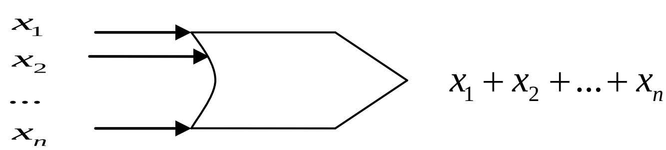


III. **AND gate:** The inputs to this gate are of two or more Boolean variables. The output is the Boolean **product** of their values. This is represented by



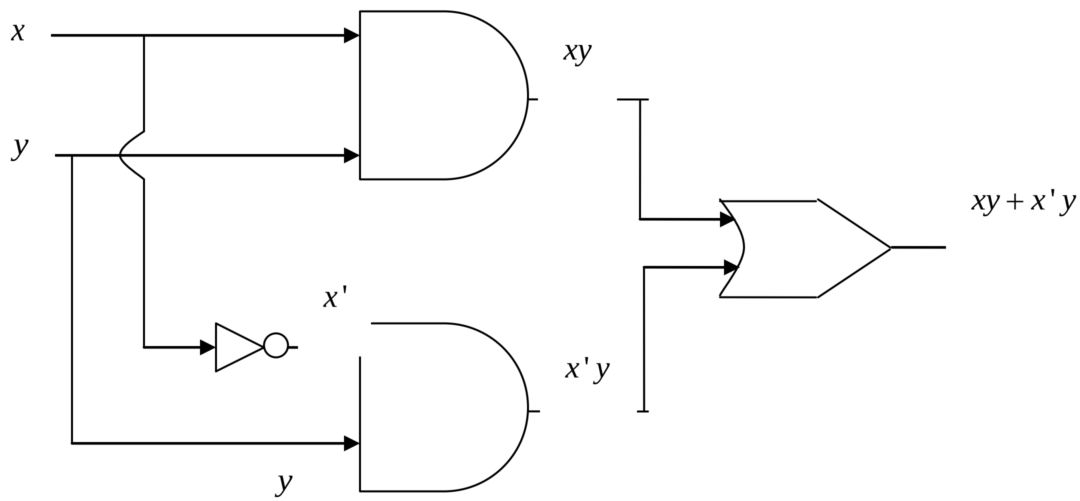
More inputs are permitted to AND and OR gates. Such situations are shown below



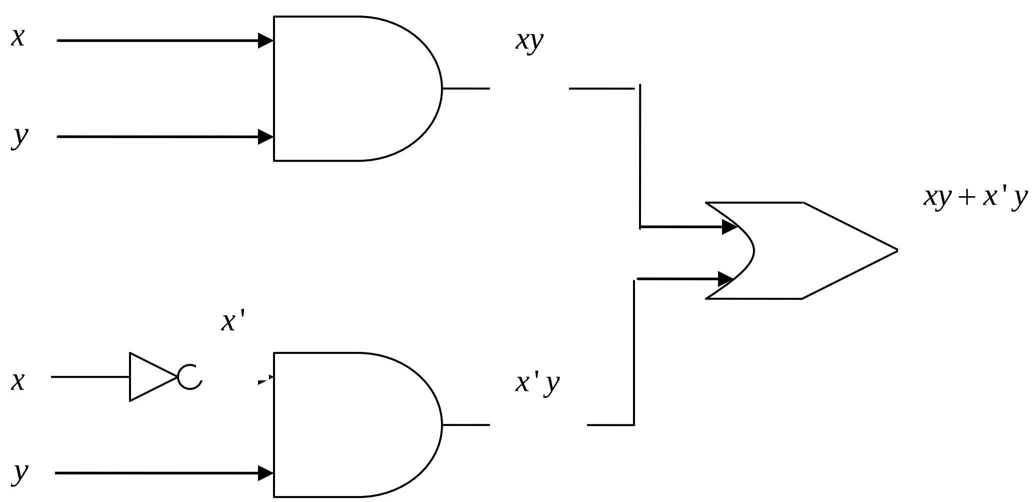


2-4.2 Combinations of Gates

Combination circuits can be constructed using a combination of inverters, OR gates, and AND gates. When combinations of circuits are formed, some gates may share inputs. One method is the use of **branching** to indicate all the gates that use a given input.



The other method is to indicate this input separately for each gate.



Note also that the output from a gate may be used as input by one or more elements as shown in the above two diagrams. Note also that the two diagrams represent the **same** input and output circuiting.

EXAMPLE 4.11

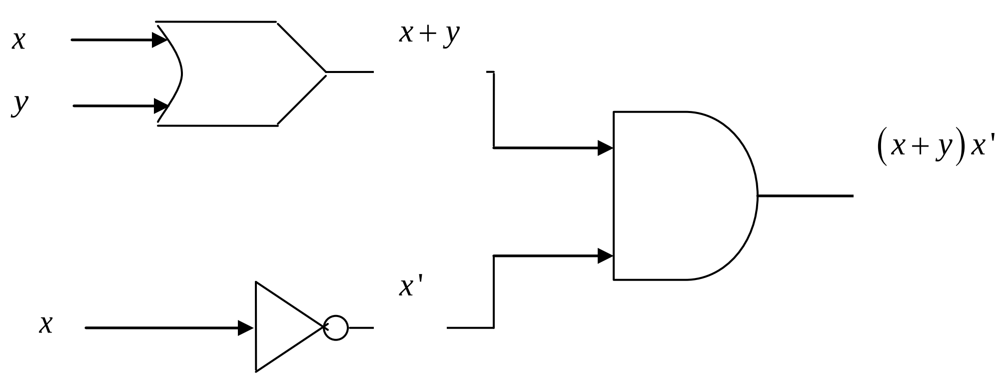
Construct circuits that produce the following outputs:

$(a)\ \ (x+y)x'$; $\qquad$ (b) $\ x'(y+z')'$;

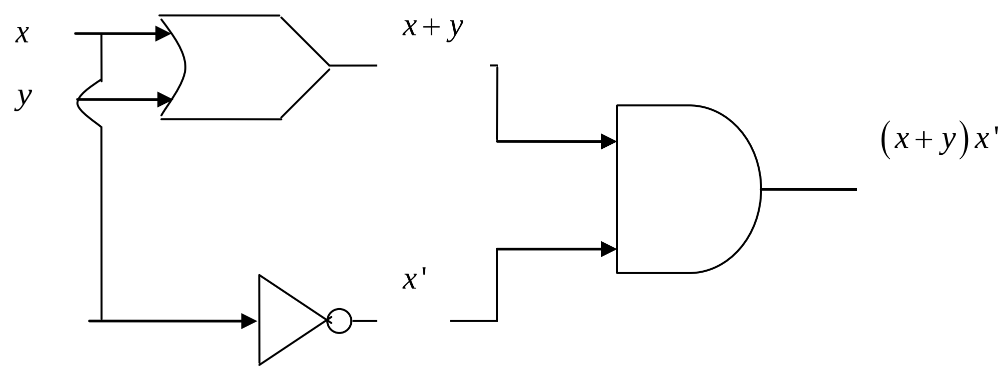
$(c)\ \ (x+y+z)(x'y'z')$

Solution:

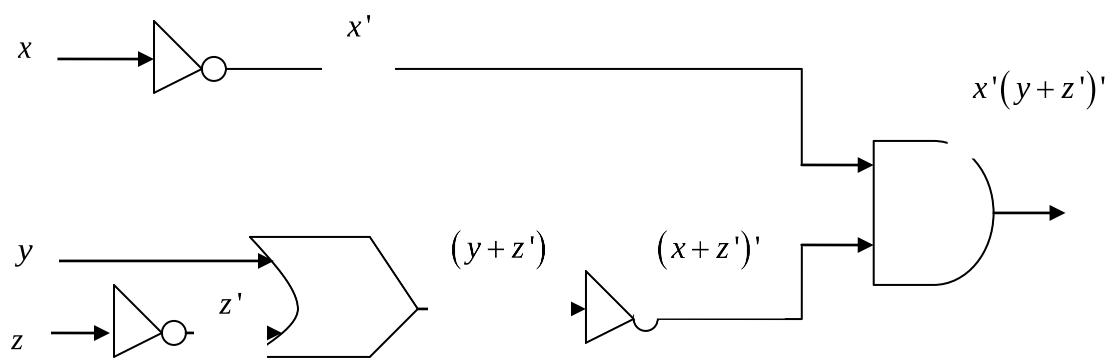
(a)

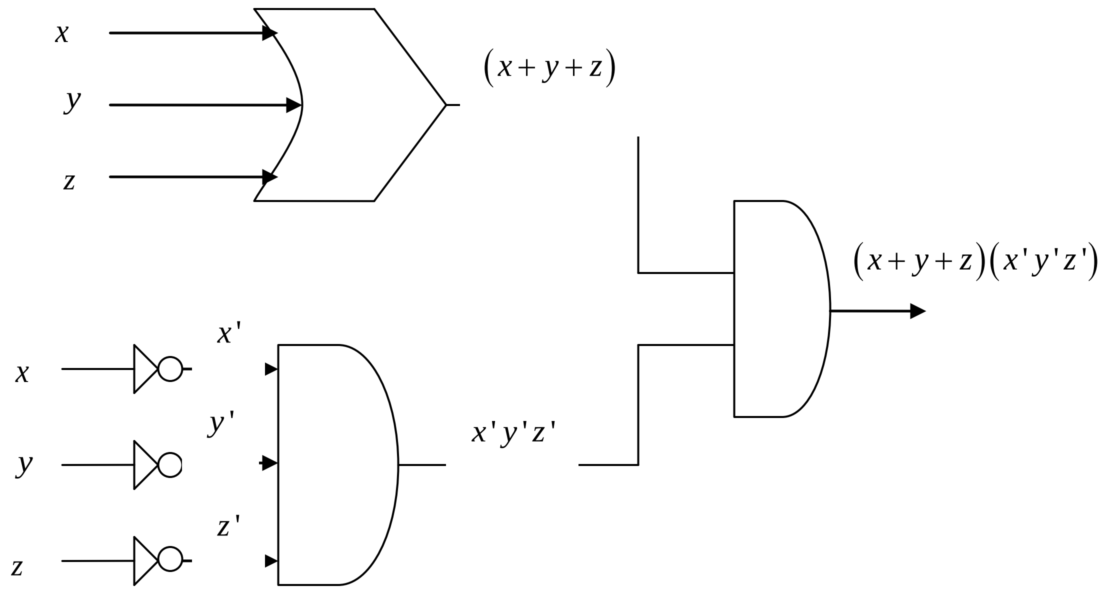


OR



(b)





x
y
z
$(x+y+z)$
$(x+y+z)(x'y'z')$
x
y
z
x'
y'
z'
$x'y'z'$

2-4.3 Examples of Circuit

We give some examples of circuits that perform some useful functions.

Example 4.12

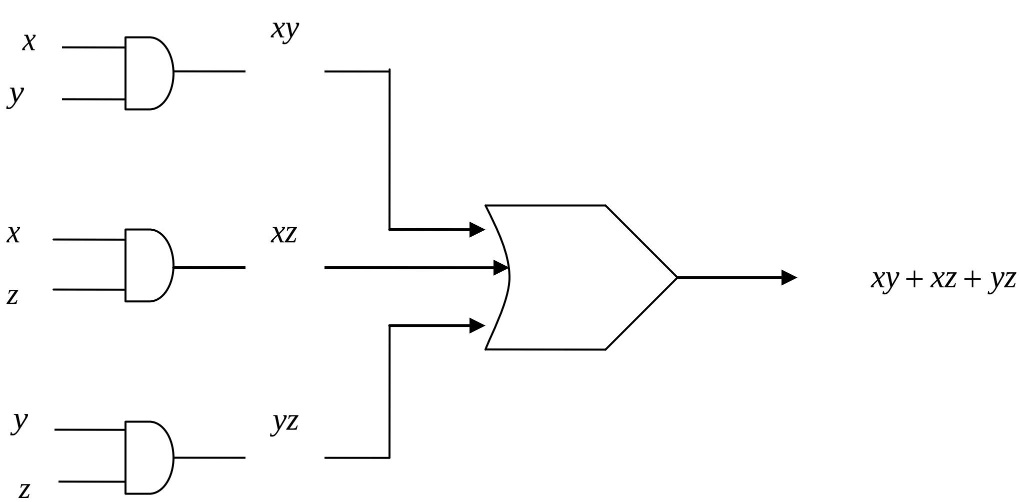
A committee of three individuals decides issues for an organization. Each individual votes either yes or no for each proposal that arises. A proposal is passed if it received **at least** two yes votes. Design a circuit that determines whether a proposal passes.

Solution:

Let $x=1$ if the first individual votes yes, and $x=0$ if he votes no; let $y=1$ if the second votes yes and $y=0$ if he votes no; let $z=1$ if the third individual votes yes, and $z=0$ if this individual votes no. Then a circuit must be designed that produces the output 1 from the inputs $x,\ y$ and z when **two or more** of $x,\ y$ and z are 1. The representations of the Boolean function that have these output values are

$F(x,y,z)=xy+xz+yz$ or $F(x,y,z)=xy+xz+xyz+yz$.

We draw the circuit of $F(x,y,z)$.



Draw the circuit of $F(x,y,z)=xy+xz+xyz+yz$.

Example 3

Sometimes light fixtures are controlled by more than one switch. Circuits need to be designed so that flipping any one of the switches turns the light on when it is off and turns the light off when it is on. Design circuits that accomplish this when there are two switches and when there are three switches.

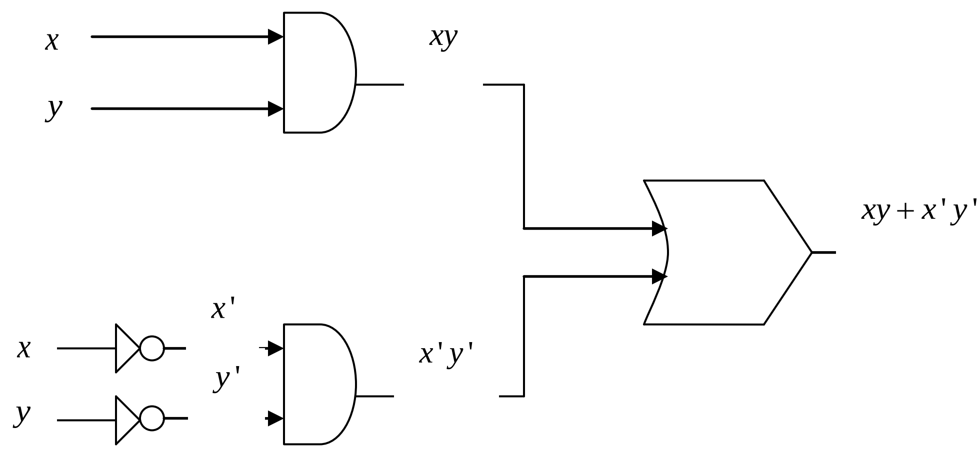
Solution:

I. **For two switches**

Let $x=1$ when the switch is closed and $x=0$ when it is opened and let $y=1$ when the second switch is closed and $y=0$ when it is opened. Let $F(x,y)=1$ when the light is on and $F(x,y)=0$ when it is off. We **arbitrary** decide that the light is on when both switches are closed, so that $F(1,1)=1$. **This determines all the other values of F**. When one of the two switches is opened, the light goes off, so $F(1,0)=F(0,1)=0$. When the other switch is opened, the light goes on, so that $F(0,0)=1$. The following table displays these values.

x	y	$F(x,y)$
1	1	1
1	0	0
0	1	0
0	0	1

Then we see that $F(x, y) = xy + x'y'$ with the following circuit:



II. **For three switches**

Let x, y and z be the Boolean variables that indicate whether each of the three switches are closed.

Let $x = 1$ when first switch is closed and $x = 0$ when it is opened;

Let $y = 1$when the second switch is closed and $y = 0$ when it is opened;

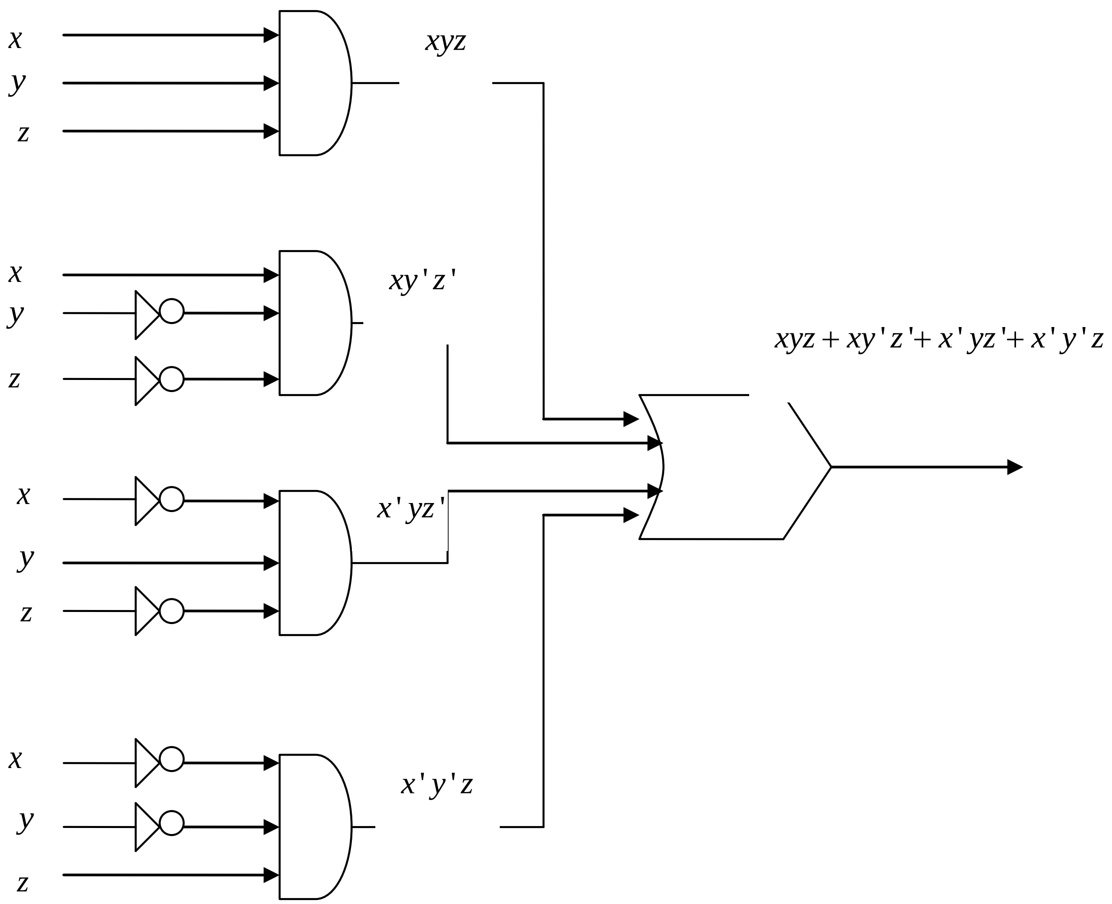
Let $z = 1$ when the third switch is closed and $z = 0$ when it is opened.

Let $F(x, y, z) = 1$ when light is on and $F(x, y, z) = 0$ when the light is off.

We **arbitrary** specify that the light be on when all three switches are closed, so that $F(1,1,1) = 1$. **This determines all the values of** F **.** When one switch is open the light goes off, so that $F(1,1,0) = F(1,0,1) = F(0,1,1) = 0$.

When a second switch is opened, the light goes on, so that
$F(1,0,0) = F(0,0,1) = F(0,1,0) = 1$. Finally, when the third switch is opened, the light goes off again, so that $F(0,0,0) = 0$. The following table and circuit display the foregoing analysis.

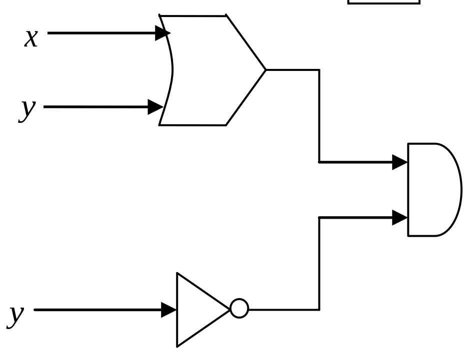
x	y	z	$F(x,y,z)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0



Where $F = xyz + xy'z' + x'yz' + x'y'z$ is the sum-of-products expansion.

In (1) to (5), find the output of the given circuit.

1



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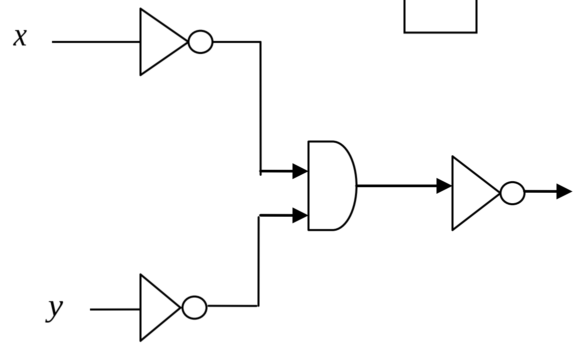

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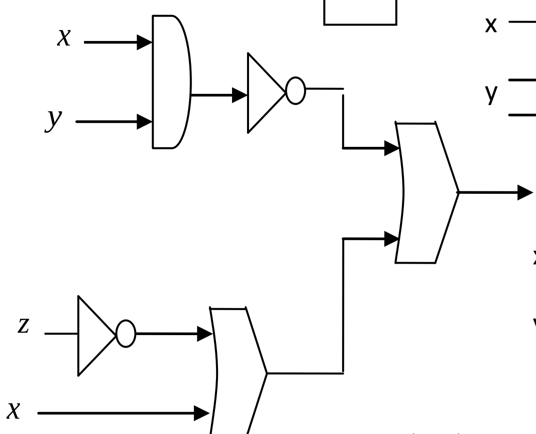

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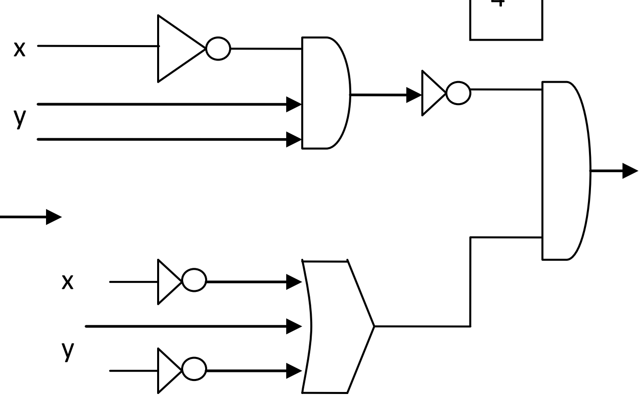

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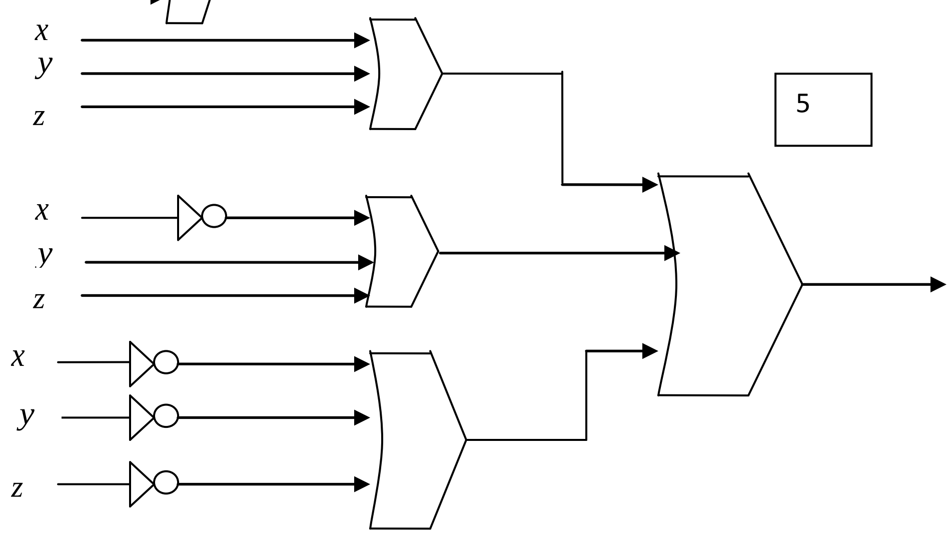


6. Construct circuits from inverters, AND gates, and OR gates to produce the following inputs: $(a)\ x'+y$; $(\mathrm{b})\ (x+y)'x$; $(\mathrm{c})\ xyz+x'y'z'$; (d) $\left[(x'+z)(y+z')\right]'$

7. Design a circuit that implements majority voting for five individuals. A proposal is passed if it receives at least three yes votes.

8. Design a circuit for a light fixture controlled by four switches where flipping one of the switches turns the light on when it is off and turns it off when it is on.

Unit 5

RELATIONS AND ORDER IN A SET

Introduction

The reader is familiar with many relations which are used in mathematics and computer science, i.e. ”is a subset of”, ” is less than” and so on. One frequently wants to compare or contrast various members of a set, perhaps to arrange them in some appropriate order or to group together those with similar properties. The mathematical framework to describe this kind of organization of sets is the theory of relations.

Learning Objectives

After reading this unit you should be able to:

1. Have ordered arrangement of members of set
2. Find minimal and maximal elements

Unit content

Session 1-5: Relations

1-5.1Relations in a set

1-5.2Equivalence Relations

Session 2-5: Ordering of a set

2-5.1Partial Ordering

2-5.2Subsets of Ordered Sets

2-5.3First and Last Elements

2-5.4Maximal and Minimal Elements

SESSION 1-5: RELATIONS

1-5.1 Relations in a Set

Definitions

A binary relation (or relation) $\mathcal{R}$ from a set of A to a set B assigns to each pair $\langle a,b\rangle$ in $A\times B$ exactly one of the following statements:

(i) "a is related to b", written $a\mathcal{R}b$

(ii) "a is not related to b", written $a\not\mathcal{R}b$.

A relation from a set A to the same set A is called a **relation in A.**

Observe that any relation $\mathcal{R}$ from a set A to a set B uniquely defines a subset $\mathcal{R}^*$ of $A\times B$ as follows: $\mathcal{R}^* = \{\langle a,b\rangle : a\mathcal{R}b\}$

On the other hand, any subset $\mathcal{R}^*$ of $A\times B$ defines a relation $\mathcal{R}$ from A to B as:

$a\mathcal{R}b$ iff $\langle a,b\rangle \in \mathcal{R}^*$

Domain of $\mathcal{R} = \{a : \langle a,b\rangle \in \mathcal{R}\}$

Range of $\mathcal{R} = \{b : \langle a,b\rangle \in \mathcal{R}\}$

1-5.2 Equivalence Relation

Definition

A relation $\square$ in a set A, that is, a subset $\square$ of $A \times A$ is called an equivalence relation if and only if it satisfies the following axioms:

$$[E_1] \text{ For every } a \in A,\ \langle a,a \rangle \in \square,\ \text{reflexive property}$$
$$[\mathrm{E}_2] \text{ If } \langle a,b \rangle \in \square^*,\ \text{then } \langle b,a \rangle \in \square,\ \text{symmetric property}$$
$$[\mathrm{E}_3] \text{ If } \langle a,b \rangle \in \square^*,\ \text{and } \langle b,c \rangle \in \square,\ \text{then } \langle a,c \rangle \in \square^*,\ \text{transition property}$$

Accordingly, a relation $\square$ is an equivalence relation if and only if it is reflexive, symmetric and transitive.

If $\square$ is an equivalence relation in A, then the equivalence class of any element $a \in A$, denoted by $[a]$, is the set of elements to which a is related. $[a] = \{x : \langle a, x \rangle \in \square\}$

The collection of equivalence classes of A denoted by $A/\square$, is called the quotient of A by $\square$. $A/\square = \{[a] : a \in A\}$

The quotient set $A/\square$ possesses the following properties:

Theorem

Let $\square$ be an equivalence relation in A and let $[a]$ be the equivalence class of $a \in A$. Then:

$$(\mathrm{i}) \text{ For every } a \in A, a \in [a]$$
$$(\mathrm{ii})\ [a] = [b], \text{ iff } \langle a,b \rangle \in \square$$
$$(\mathrm{iii}) \text{ If } [a] \neq [b], \text{ then } [a] \text{ and } [b] \text{ are disjoint}$$

A class *a* of non-empty subsets of *A* is called a partition of A if and only if

1. each $a \in A$ belongs to some member of a and
2. the members of a are pairwise disjoint.

Therefore, the theorem above implies the following fundamental theorem of equivalence relations.

Theorem

Let □ be an equivalence relation in A. Then the quotient set $A/□$ is a partition of A.

Example 5.1

Let $□_5$ be the relation in □, the set of integers, defined by $x \equiv y(\mathrm{mod}\,5)$, which reads " x is congruent to y modulo 5" and which means " $x - y$ is divisible by 5". Then $□_5$ is an equivalence relation in □. There are exactly five distinct equivalence classes in $□/□_5$:

$$
\begin{aligned}
E_0 &= \{\ldots,-10,-5,0,5,10,\ldots\} = [0]\\
E_1 &= \{\ldots,-9,-4,1,6,11,\ldots\} \;\; = [1]\\
E_2 &= \{\ldots,-8,-3,2,7,12,\ldots\} = [2]\\
E_3 &= \{\ldots,-7,-2,3,8,13,\ldots\} = [3]\\
E_4 &= \{\ldots,-6,-1,4,9,14\} \quad = [4]
\end{aligned}
$$

We observe for each integer x, $x = 5p + r$ where $0 \le r < 5$, is a member of the equivalence class E_r, where r is the remainder when x is divided by 5.

Note that the equivalence classes are pairwise disjoint and that

$Z = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4$

Example 5.2

Consider the relation $\square = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$ in $A = \{1,2,3\}$. Then

$$\text{Domain of } \square = \{1,2\},$$
$$\text{Range of } \square = \{2,3\}$$

Let $\square^{-1}$ denote the relation that reverses the pairs in $\square$. Then $\square^{-1} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$.

We observe that $\square$ and $\square^{-1}$ are identical respectively, to the mathematical relations < and > in the set A. Thus $\langle a,b\rangle \in \square$ iff $a < b$, and $\langle a,b\rangle \in \square^{-1}$ iff $a < b$.

The identity relation in any set A, denoted by Δ or Δ_A, is the set of pairs in $A \times A$ with equal coordinates, that is, $\Delta_A = \{\langle a,a\rangle : a \in A\}$

1. Prove: let $\square$ be a relation in A, that is $\square \subset A \times A$. Then
   (i) $\square$ is reflexive if and only if $\Delta_A \subset \square$
   (ii) $\square$ is symmetric if and only $\square = \square^{-1}$
2. Consider the relation $\square = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ in $X = \{1,2,3\}$. Determine whether or not $\square$ is (i) reflexive, (ii) symmetric, (iii) transitive
3. Consider the set $\square \times \square$, that is, the set of ordered pairs of positive integers. Let $\square$ be the relation $\square$ in $\square \times \square$ which is defined by $\langle a,b\rangle \square \langle c,d\rangle$ iff $ad = bc$.

Prove that $\sim$ is an equivalence relation

4. Consider $\mathbb{N} \times \mathbb{N}$, the set of ordered pairs of positive integers. Let $\sim$ be the relation in $\mathbb{N} \times \mathbb{N}$ defined by $\langle a,b \rangle \sim \langle c,d \rangle$ iff $a+b=b+c$

   (i) Prove $\sim$ is an equivalence relation

   (ii) Find the equivalence class of $\langle 2,5 \rangle$ that is $[\langle 2,5 \rangle]$

SESSION 2-5: ORDERING OF A SET

2-5.1 Partial Ordering

A relation $\preceq$ in a set A is called a **partial order** (or order) if and only if, for every $a, b, c \in A$ such that

(i) $a \preceq a$, reflexive property

(ii) $a \preceq b$ and $b \preceq a$ implies $a = b$, anti-symmetric property

(iii) $a \preceq b$ and $b \preceq a$ implies $a \preceq c$, transitive property

The set A together with the partial order, that is, the pair $(A, \preceq)$ is called a **partially ordered set**. A partial order is a reflexive, anti-symmetric and transitive relation.

ACTIVITY **Example 5.3**

Set inclusion is a partial order in any class of sets since

(i) $A \subset A$ is true for any set A

(ii) $A \subset B$ and $B \subset A$ implies $A = B$ for any sets $A,\ B$

(iii) $A \subset B$ and $B \subset C$ implies $A \subset C$ for any sets $, A,\ B$ and C

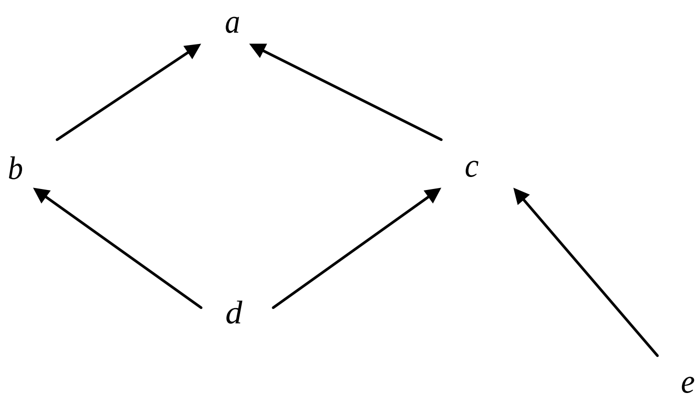
ACTIVITY **Example 5.4**

Let A be any set of real numbers. Then the relation in A defined by $x \preceq y$ is a partial order and is called the natural order in A.

Example 5.5

Let $X = \{a, b, c, d, e\}$. Then the diagram below defines a partial order in X as follows:

$x \preceq y$ iff $x = y$ or if one can go from x to y in the diagram, always moving in the indicated direction that is upward.



Definition

If $a \preceq b$ in an ordered set, we say, a precedes or is smaller than b and that b follows or dominates or is larger than a. We write $a \prec b$ if $a \preceq b$ but $a \neq b$.

A particular ordered set A is said to be **totally (or linearly) ordered** if, **for every** $a, b \in A$, either $a \preceq b$ or $b \preceq a$.

The set of real numbers, $\mathbb{R}$, with the natural order defined by $x \preceq y$ is an example of a totally ordered set.

Example 5.6

Let A and B be totally ordered. Then the product set $A \times B$ can be totally ordered as follows: $\langle a,b \rangle \prec \langle a',b' \rangle$ if $a \prec a'$ or $a = a'$, and $b \prec b'$.

This order is called **lexicographical** of $A \times B$ since it is similar to the way words are arranged in a dictionary.

REMARKS

If a relation □ in a set A defines a partial order, that is, is reflexive, anti-symmetric and transitive, then the inverse relation $□^{-1}$ is also a partial order, it is called the inverse order.

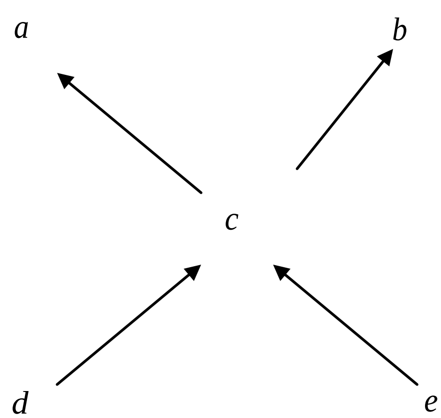
2-5.2 Subsets of Ordered Sets

Let A be a subset of a partially ordered set X. Clearly the order in X induces an order in A in a very natural way. If $a,b \in A$, then $a \preceq b$ as elements in A if and only if $a \preceq b$ as elements in X. More precisely, if $\preceq$ is a partial order in X, then the relation

$\preceq_A = \preceq \cap (A \times A)$, called the restriction of $\preceq$ to A, is a partial order in A. The ordered set $(A, \preceq_A)$ is called **partially ordered subset** of the ordered set $(X, \preceq)$. Some subsets of a partially ordered set X may, in fact, be totally ordered. Clearly, if X itself is totally ordered, every subset of X will also be ordered.

Example 5.7

Consider the partial order in $W = \{a,b,c,d,e\}$ defined by the diagram



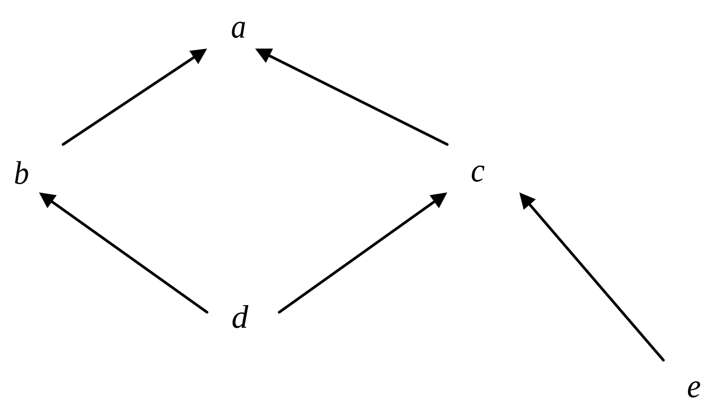
The sets $\{a,c,d\}$ and $\{b,e\}$ are totally ordered subsets, the sets $\{a,b,c\}$ and $\{d,e\}$ are **not** totally ordered subsets.

2-5.3 First and Last Elements

Let X be an ordered set. An element $a_0 \in X$ is a first or smallest element of X if and only if $a_0 \preceq x$ for all $x \in X$. Similarly, an element $b_0 \in X$ is a last or largest element of X if and only if $x \preceq b_0$ for all $x \in X$.

Example 5.8

Let $X = \{a, b, c, d, e\}$ be ordered by the diagram,



Then a is a last element of X since a follows every element. We note that X has no first element, since d is not a first element because d does not proceed e.

Example 5.9

The positive integers $\mathbb{N}$ with the natural order have 1 as a first element; there is no last element. Similarly, the set of integers $\mathbb{Z}$ with the natural order has no first element and no last element.

2-5.4 Maximal and Minimal Elements

Let X be an ordered set. An element $a_0 \in X$ is maximal if and only if $a_0 \preceq x$ implies $x = a_0$, that is, if no element follows a_0 except itself. Similarly, an element $b_0 \in X$ is minimal if and only if $x \preceq b_0$ implies $x = b_0$, that is, if no element precedes b_0 except itself.

Example 5.9

Let $X = \{a, b, c, d, e\}$ be ordered by the **diagram of Example 5.7**. Then both d and e are minimal elements. The element a is a maximal element.

Example 5.10

Although □ with the natural order is totally ordered it has no minimal and no maximal elements.

Example 5.11

Let $A = \{a_1, a_2, ..., a_m\}$ be a finite totally ordered set. Then A contains precisely one minimal element and precisely one maximal element, denoted respectively by

$$\min\{a_1, ..., a_m\} \text{ and } \max\{a_1, ..., a_m\}$$

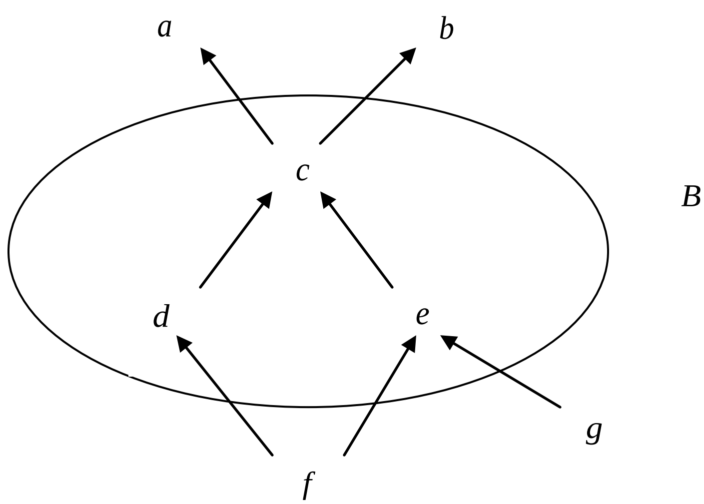
2-5.5 Upper and Lower Bounds

Let A be a subset of a partially ordered set X . An element $m \in X$ is a lower bound of A if and only if $m \preceq x$ for all $x \in A$, that is, if m precedes every element in A. If some lower bound of A follows every other lower bound of A, then it is called the greatest lower bound (G.L.B) or *infimum* of A and is denoted by $\inf(A)$. Similarly, an element $\mu \in X$ is an upper bound of A if and only if $x \preceq \mu$ for all $x \in A$, that is, if μ follows every element in A. If some upper bound of A precedes every other upper bound of A, then it is called the least upper bound (L.U.B) or *supremum* of A and is denoted by $\sup(A)$.

A is said to be bounded above if it has an upper bound, and bounded below if it has a lower bound. If A has both an upper and lower bounds, then it is said to be bounded.

ACTIVITY Example 5.12

Let $X = \{a, b, c, d, e, f, g\}$ be ordered as shown in the following diagram:



Let $B=\{c,d,e\}$. Then a, b and c are upper bound of B, and f is the only lower bound of B. We note that g is not a lower bound of B since g does not precede d. Furthermore, $c=\sup(B)$ belongs to B, while $f=\inf(B)$ does not belong to B.

Example 5.13

Let A be a bounded set of real numbers. Then, a fundamental theorem about real numbers states that, under the natural order, $\inf(A)$ and $\sup(A)$ exist.

Example 5.14

Let $\mathbb{Q}$ be the set of rational numbers. Let $B=\{x : x\in\mathbb{Q}, x>0, 2<x^2<3\}$. Then B has an infinite number of upper and lower bounds, but $\inf(B)$ and $\sup(B)$ do not exist, because the real numbers $\sqrt{2}$ and $\sqrt{3}$ do not belong to $\mathbb{Q}$ and therefore cannot be considered as upper and lower bounds of B.

EXERCISE

1. Let $\mathbb{N}$, the positive integers, be ordered as follows: each pair of elements $a, a'\in\mathbb{N}$ can be written uniquely in the form $a=2^r(2s+1)$, $a'=a^{r'}(as'+1)$

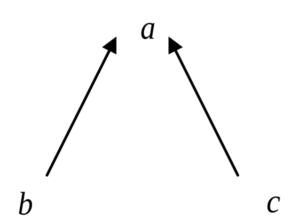
   Where $r, r', s, s'\in\{0,1,2,3,...\}$. Let $a\prec a'$ if $r<r$ or if $'r=r'$ but $s<s'$.

Insert the correct symbol, $\prec$ or $\succ$, between each of the following pairs of numbers. (Note: $x \succ y$ iff $y \prec x$).

(i) 5......14 (ii) 6...........9

(iii) 3........20 (iv) 14.........21

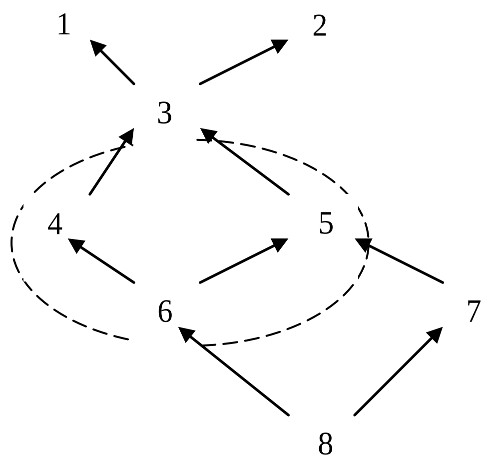
2. Let $A = \{a, b, c\}$ be ordered as in the diagram below. Let $\mathcal{A}$ be the collection of all non-empty totally ordered subsets of A, and let $\mathcal{A}$ be partially ordered by set inclusion. Construct a diagram of the order of $\mathcal{A}$.



3. Let $A = \{2, 3, 4, ...\} = \mathbb{N} \setminus \{1\}$ and let A be ordered by "x divides y".
   (i) Determine the minimal elements of A
   (ii) Determine the maximal elements of A.

4. Let $B = \{2, 3, 4, 5, 6, 8, 9, 10\}$ be ordered by "x is a multiply of y".
   (i) Find all maximal elements of B
   (ii) Find all minimal elements of B.

5. Let $W = \{1, 2, ..., 7, 8\}$ be ordered as follows:



Consider $V = \{4,5,6\}$, a subset of W

(i) Find the set of upper bounds of V

(ii) Find the set of lower bounds of V

(iii) Does $\sup(V)$ exist?

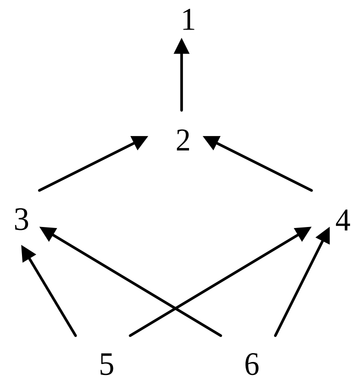
(iv) Does $\inf(V)$ exist?

6. Let $A = (\mathbb{N}, \leq)$, the positive integers with the natural order; and let $B = (\mathbb{N}, \geq)$, the positive integer with the inverse order. Furthermore, let $A \times B$ denote the lexicographical ordering of $\mathbb{N} \times \mathbb{N}$ according to the order of A and then B. Insert the correct symbol, < or >, between each pair of elements of $\mathbb{N} \times \mathbb{N}$.

(i) $\langle 3,8 \rangle$..........$\langle 1,1 \rangle$ (ii) $\langle 2,1 \rangle$.............$\langle 2,8 \rangle$

(iii) $\langle 3,3 \rangle$..........$\langle 3,1 \rangle$ (iv) $\langle 4,9 \rangle$............$\langle 7,15 \rangle$

7. Let $X = \{1,2,3,4,5,6\}$ be ordered as in the diagram below. Consider the subset $A = \{2,3,4\}$ of X.



(i) Find the maximal elements of X

(ii) Find the minimal elements of X

(iii) Does X have a first element?

(iv) Does X have a last element?

(v) Find the set of upper bounds of A

(vi) Find the set of lower bounds

(vii) Does sup (A) exist?

(viii) Does inf (A) exist?

8. Consider $\mathbb{Q}$, the set of rational numbers, with the natural order, and its subset $A = \{x : x \in \mathbb{Q}, x^3 < 3\}$

(i) Is A bounded above?

(ii) Is A bounded below?

(iii) Does $\sup(A)$ exist?

(iv) Does $\inf(A)$ exist?

9. Let $\mathbb{N}$, the positive integers, be ordered by "x divides y" and let $A \subset \mathbb{N}$

(i) Does $\inf(A)$ exist?

(ii) Does $\sup(A)$ exists?

ZORN'S LEMMA

Zorn's lemma: Let X be a non-empty partially ordered set in which every totally ordered subset has an upper bound, then X contains at least one maximal element.

[based on respective learning points in learning objective add interactive subject matter; with examples, exercises with answers, insert captioned illustrations, diagrams, tables, figures, etc.]

Unit 6

ELEMENTS OF GRAPH THEORY

Introduction

In this unit we present the basic concepts related to graphs and trees such as the degree of a vertex, connectedness, Euler and Hamiltonians circuits.

Learning Objectives

After reading this unit you should be able to:

1. Graphs are binary – relations on their vertex set(except for multigraphs)
2. Different purposes require different types of graphs

Unit content

Session 1-6: Basics of Graphs

1-6.1Definitions
1-6.2Complete Graph
1-6.3Degree

Session 2-6: Paths and Circuits

2-6.1Paths
2-6.2Connectivity
2-6.3Euler Path and Circuit
2-6.4Hamiltonian Path and Circuit

SESSION 1-6: BASICS OF GRAPHS

1-6.1 Definitions

An **undirected graph** G consists of a set V_G of **vertices**, **points**, or **nodes** and a set E_G of **edges** such that each edge $e \in E_G$ is associated with an unordered pair of vertices, called its **endpoints**.

A **directed graph** or **digraph** G consists of a set V_G of vertices, points or nodes and a set E_G of ordered pairs of vertices called ***arcs*** or ***directed edges*** or, simply, ***edges*** .

We denote a graph by $G = (V_G, E_G)$.

Two vertices are said to be **adjacent** if there is an edge connecting the two vertices. Two edges associated to the same vertices are called **parallel**. An edge incident to a single vertex is called a **loop**. A vertex that is not incident on any edge is called an **isolated** vertex. A graph with neither loops nor parallel edges is called **simple** graph.

Example 6.1

Describe formally the graph shown in fig. 6-1.

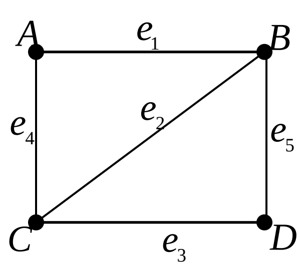
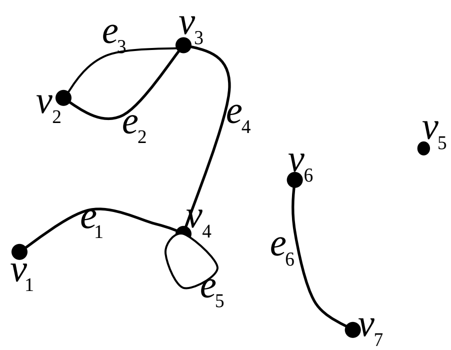


Figure 6-1 shows the graph $G = G(V, E)$ where: (i) V consists of the vertices A, B, C, D; and (ii) E consists of the five edges $e_1 = \{A, B\}$, $e_2 = \{B, C\}$, $e_3 = \{C, D\}$, $e_4 = \{A, C\}$, $e_5 = \{B, D\}$.

ACTIVITY **Example 6.2**

Consider the following graph G



a. Find E_G and V_G.

b. List the isolated vertices.

c. List the loops.

d. List the parallel edges.

e. List the vertices adjacent to v_3..

f. Find all edges incident on v_4.

Solution.

a. $E_G = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ and $V_G = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

b. There is only one isolated vertex, v_5.

c. There is only one loop, e_5.

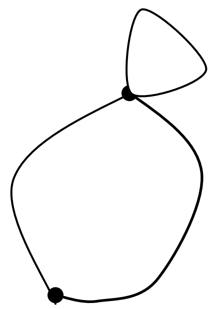
d. $\{e_2, e_3\}$.

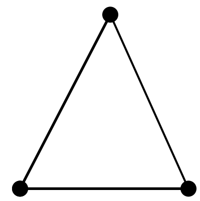
e. $\{v_2, v_4\}$.

f. $\{e_1, e_4, e_5\}$

Example 6.3

Which one of the following graphs is simple.





Solution.

a. G is not simple since it has a loop and parallel edges.

b. G is simple.

1-6.2 Complete Graph

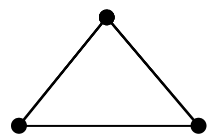
A **complete graph** on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

Example 6.4

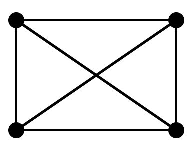
Draw $K_2, K_3, K_4,$ and K_5.

Solution.

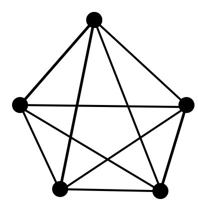
K_2



K_3



K_4

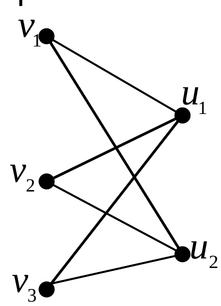


K_5

A graph in which the vertices can be partitioned into two disjoint sets V_1 and V_2 with every edge incident on one vertex in V_1 and one vertex of V_2 is called **bipartite graph**.

Example 6.5

a. Show that the graph G is bipartite.



b. Show that K_3 is not bipartite.

Solution:

a. Clear from the definition and the graph.

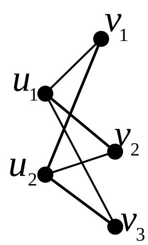
b. Any two sets of vertices of K_3 will have opposite parity. Thus, according to the definition of bipartite graph, K_3 is not bipartite.

A complete bipartite graph $K_{m,n,}$ is the graph that has its vertex set partitioned into two disjoint subsets of m and n vertices, respectively. More – over, there is an edge between two vertices if and only if one vertex is in the first set and the other vertex is in the second set.

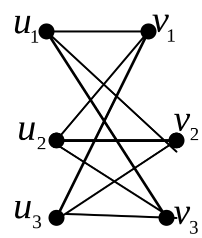
Example 6.6

Draw $K_{2,3}, K_{3,3}$.

Solution:



$K_{2,3}$



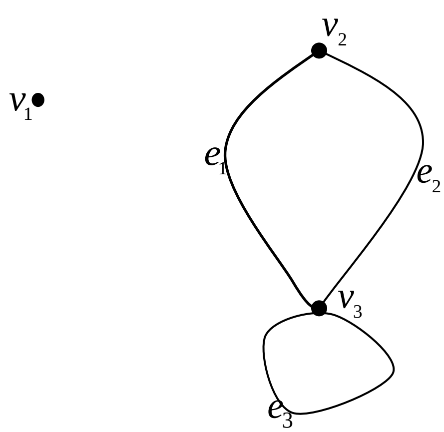
$K_{3,3}$

1-6.3 Degree

The **degree** of a vertex v in an undirected graph, in symbol $\deg(v)$, is the number of edges incident on it. By definition, a loop at a vertex contributes twice to the degree of that vertex. **The total degree of** G is the sum of the degrees of all the vertices of G.

ACTIVITY **Example 6.7**

What are the degrees of the vertices in the following graph



Solution:

$\deg(v_1) = 0, \deg(v_2) = 2, \deg(v_3) = 4$.

Theorem 6.1

For any graph $G=\left(V_G,E_G\right)$ we have

$$2\,|\,E_G\,|\;=\sum_{v\in V(G)}\deg\left(v\right)$$

[The sum of the degrees of the vertices of a graph is equal to twice the number of edges]

Proof:

Suppose that $V_G=\left\{v_1,v_2,...,v_n\right\}$ and $|\,E_G\,|\;=m$. Let $e\in E_G$. If e is a loop then it contributes 2 to the total degree of G. If e is not a loop then let v_i and v_j denote the endpoints of e. Then e contributes 1 to $\deg\left(v_i\right)$ and contributes 1 to the $\deg(v_j)$.Therefore, e contributes 2 to the total degree of G. Since e was arbitrarily, this shows that each edge of G contributes 2 to the total degree of G. Thus,

$$2|\,E_G\,|\;=\sum_{v\in V(G)}\deg(v)$$

The following is easily deduced from the previous theorem.

Theorem 6.2

In any graph there are an even number of vertices of odd degree.

Proof.

Let $G=\left(V_G,E_G\right)$ be a graph. By the previous theorem, the sum of all the degrees of the vertices is $T\ =2\,|\,E_G\,|$, an even number. Let E be the sum of the numbers $\deg(v)$, each which is even and O the sum of numbers $\deg(v)$ each which is odd. Then $T=E+O$. That is, $O=T-E$. Since both T and E are even then T is also even. This implies that there must be an even number of the odd degrees. Hence, there must be an even number of vertices with odd degree.

ACTIVITY **Example 6.8**

Find a formula for the number of edges in K_n.

Solution:

Since G is complete, each vertex is adjacent to the remaining vertices. Thus, the degree of each of the n vertices is $n-1$, and we have the sum of the degrees of all of the vertices being $n\left(n-1\right)$.

By Theorem 6.1 $n\left(n-1\right)=2\,|\,EG\,|$.

This completes a proof of the theorem.

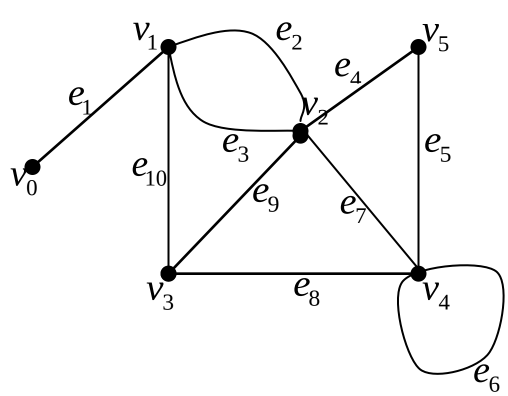
SESSION 2-6: PATHS AND CIRCUITS

2-6.1 Paths

In an undirected graph G a sequence P of the form $v_0 e_1 e_2 \ldots v_{n-1} e_n v_n$ is called a **path of length** n or a path connecting v_0 to v_n. If P is a path such that $v_0 = v_n$ then it is called a **circuit** or a **cycle**. A path or circuit is **simple** if it does not contain the same edge more than once. A graph that does not contain any circuit is called **acyclic**.

ACTIVITY **Example 6.9**

In the graph below, determine whether the following sequences are paths, simple paths, circuits, or simple circuits.



a. $v_0 e_1 v_1 e_{10} v_5 e_9 v_2 e_2 v_1$.

b. $v_3 e_5 v_4 e_8 v_5 e_{10} v_1 e_3 v_2$.

c. $v_1 e_2 v_2 e_3 v_1$.

d. $v_5 e_9 v_2 e_4 v_3 e_5 v_4 e_6 v_4 e_8 v_5$.

Solution:

a. a path (no repeated edge), not a simple path (repeated vertex v_1), not a circuit

b. a simple path

c. a simple circuit

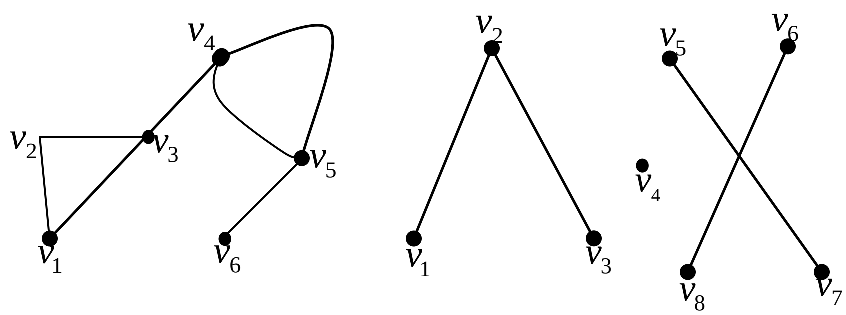
d. a circuit, not a simple circuit (vertex v_4 is repeated)

2-6.2 Connectivity

An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph. A graph that is not connected is said to be **disconnected**.

Example 6.10

Determine which graph is connected and which one is disconnected.



a *b*

Solution:

a. Connected.

b. Disconnected since there is no path connecting the vertices v_1 and v_4.

2-6.3 Euler Path and Circuit

A simple path that contains all edges of a graph G is called an **Euler path**.

If this path is also a circuit, it is called an **Euler circuit**.

Theorem 6.3

If a graph G has an Euler circuit then every vertex of the graph has even degree.

Proof:

Let G be a graph with an Euler circuit. Start at some vertex on the circuit and follow the circuit from vertex to vertex, erasing each edge as you go along it. When you go through a vertex you erase one edge going in and one edge going out, or else you erase a loop. Either way, the erasure reduces the degree of the vertex by 2. Eventually every edge gets erased and all the vertices have degree 0. So all vertices must have had even degree to begin with.

It follows from the above theorem that if a graph has a vertex with odd

degree then the graph cannot have an Euler circuit.

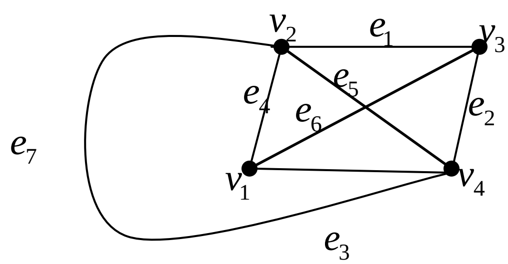
The following provide a converse to the above theorem.

Theorem 6.4 (Euler Theorem)

If all the vertices of a connected graph have even degree, then the graph has an Euler circuit.

Example 6.11

Show that the following graph has no Euler circuit.



Solution:

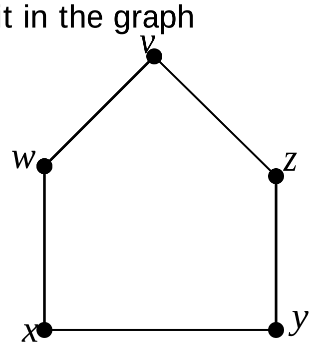
Vertices v_1 and v_3 both have degree 3, which is odd. Hence, by the remark following the previous theorem, this graph does not have an Euler circuit.

2-6.4 Hamiltonian Path and Circuit

A path is called a **Hamiltonian path** if it visits every vertex of the graph exactly once. A circuit that visits every vertex exactly once except for the last vertex which duplicates the first one is called a **Hamiltonian circuit**.

Example 6.12

Find a Hamiltonian circuit in the graph

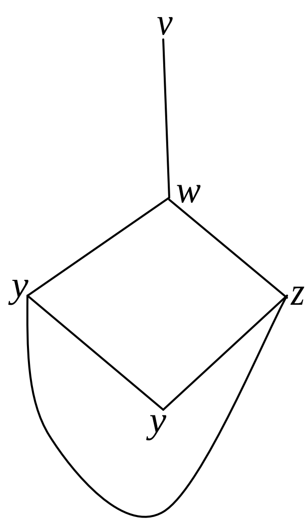


Solution:

$vwxyzv$.

Example 6.13

Show that the following graph has a Hamiltonian path but no Hamiltonian circuit.



Solution:

$vwxyz$ is a Hamiltonian path. There is no Hamiltonian circuit since no cycle goes through v .

MY PAGE

Name: ____ Learning Centre: _________________

Contact: Tel. ____________ Email: ________________ Emergency Name/Phone: __________

Important numbers: Student number ______________ Examination number ________________

Program: ___________ Year: _______ Course code/title: _______________________________

Course objectives: ____

____

Course dates/Semester No (   ): Starts_____________________ Ends ____________________

FFFS schedule/Dates: ____

Quiz dates: ____

Assignments hand in dates: ____

Revision dates: ____

Group discussion/work members (names and contacts): __________________________________

____

____

End of course Self Evaluation:
I have completed all Units & interactive sessions☐, mastered all learning objectives, completed all self Assessments, unit summary, key words and terms, discussion questions, review questions, reading activity, web activity, unit assignments, and submitted all CA scoring assignments, learner feedback on this course and submitted my comments and course focus contributory questions to facilitator for discussion.

Self-grading: self assessment questions score ______ %   Unit Assignments scored ______ %

My course conclusion remarks: ____

____

____

____ (may continue on reverse side) detach and return to IDL, KNUST

Learner Feedback Form/[insert course code]

Dear Learner,
While studying the units in the course, you may have found certain portions of the text difficult to comprehend. We wish to know your difficulties and suggestions, in order to improve the course. Therefore, we request you to fill out and send the following questionnaire, which pertains to this course. If you find the space provided insufficient, kindly use a separate sheet.

1. How many hours did you need for studying the units

Unit no.	1	2	3	4	5	6
No. of hours						

2. Please give your reactions to the following items based on your reading of the course

Items	**Excellent**	**Very good**	**Good**	**Poor**	*Give specific examples, if poor*
Presentation quality	☐	☐	☐	☐	
Language and style	☐	☐	☐	☐	
Illustrations used (diagrams, tables, etc.)	☐	☐	☐	☐	
Conceptual clarity	☐	☐	☐	☐	
Self assessment	☐	☐	☐	☐	
Feedback to SA	☐	☐	☐	☐	

3. Any other comments (may continue on reverse side)

Unit 1: ____________________

Unit 2: ____________________

Unit 3: ____________________

Unit 4: ____________________

Unit 5: ____________________

Unit 6: ____________________