VECTOR SPACES

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Introduction: Vectors in R^n

Euclidean 2-space, denoted by R^2, is the set of all vectors with two entries, that is,

$$R^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}; \quad x_1, x_2 \text{ are real numbers} \right\} \tag{1}$$

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Similarly, Euclidean 3-space, denoted by R^3, is the set of all vectors with three entries, that is,

$$R^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; \quad x_1, x_2, x_3 \text{ are real numbers} \right\} \tag{2}$$

Definition

Vectors in R^n Euclidean n-space, denoted by R^n, is the set of all vectors with n entries, that is,

$$R^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} ; \quad x_i, x_2, \cdots, x_n \text{ are real numbers} \right\} \tag{3}$$

Vector Spaces

In the previous lectures, we saw that algebra of vectors and matrices are similar in many respects. Vectors and matrices can be **added** and again be **multiplied** by scalars. In this section, we use these properties to define generalized *vectors* that arise in a wide variety of examples.

Definition (Vector Spaces)

1. Let V be a set on which two operations, called addition and scalar multiplication are defined.
2. If $\vec{u}$ and $\vec{v}$ are in V, then the sum of $\vec{u}$ and $\vec{v}$ is denoted by $\vec{u} + \vec{v}$, and if k is a scalar, then the scalar multiple of $\vec{u}$ by k is denoted by $k\vec{u}$.
3. If the following axioms hold for all $\vec{u}, \vec{v}, \vec{w} \in V$ and for all scalars k, k', then V is called a vector space and its elements are called vectors.

1. Closure:

$$\vec{u} + \vec{v} \in V; \qquad k\vec{u} \in V \tag{4}$$

2. Commutativity:

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \tag{5}$$

$$k\vec{v} = \vec{v}k \tag{6}$$

3. Associativity:

$$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \tag{7}$$

4. Compatibility

$$k(k'\vec{u}) = (kk')\vec{u} \tag{8}$$

5. Distributivity:

$$k(\vec{u} + \vec{v}) = k\vec{u} + k\vec{v} \tag{9}$$

$$(k + k')\vec{u} = k\vec{u} + k'\vec{u} \tag{10}$$

6. Identity element:

$$\vec{u} + \vec{0} = \vec{u}, \quad \text{and} \quad 1\vec{u} = \vec{u}, \quad 0\vec{u} = \vec{0} \tag{11}$$

7. Additive inverse:

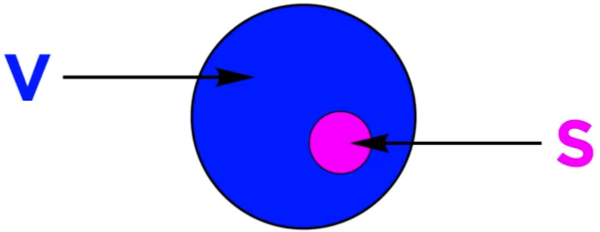
$$\vec{u} + (-\vec{u}) = \vec{0} \tag{12}$$

Subspaces of $\mathbb{R}^n$

Definition (Subspace)

A subset S of $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if

1. The zero vector belongs to S (i.e., $0 \in S$)
2. If $\vec{u} \in S$ and $\vec{v} \in S$, then $\vec{u} + \vec{v} \in S$ (S is said to be closed under vector addition);
3. If $\vec{u} \in S$ and $t \in R$, then $t\vec{u} \in S$ (S is said to be closed under scalar multiplication).



R^n is a subspace of itself, and we call R^n a vector space.

Example

Show that the set W of all vectors of the form $[a, b, -b, a]'$ is a subspace of $\mathbb{R}^n$

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2. $$\text{Let } \vec{u} = \begin{bmatrix} a \\ b \\ -b \\ a \end{bmatrix} \in W, \vec{v} = \begin{bmatrix} c \\ d \\ -d \\ c \end{bmatrix} \in W, \text{ then } \vec{u} + \vec{v} = \begin{bmatrix} a+c \\ b+d \\ -(b+d) \\ a+c \end{bmatrix} \tag{13}$$

   so $\vec{u} + \vec{v}$ is also in W because it has the right form.

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   so $\vec{u} + \vec{v}$ is also in W because it has the right form.
3. If k is a scalar then
   $$k\vec{u} = [ka, kb, -kb, ka]' \tag{14}$$
   so $k\vec{u}$ is in W

Thus , W is a nonempty subset of $\mathbb{R}^4$ that is closed under addition and scalar multiplication. Therefore W is a subspace of R^4.

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1. If $a = b = 0$, then W is nonempty because it contains the zero polynomial.
2. Let $p(x), q(x) \in W$ defined as

$$p(x) = a + bx - bx^2 + ax^3 \quad (15)$$

$$q(x) = c + dx - dx^2 + cx^3 \quad (16)$$

Then

$$p(x) + q(x) = (a + c) + (b + d)x - (b + d)x^2 + (a + c)x^3 \quad (17)$$

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$p(x) + q(x) \in W$ because it has the right form.

3. If k is a scalar then

$$kp(x) = ka + kbx - kbx^2 + kax^3, \qquad \text{so } kp(x) \in W \tag{18}$$

Thus , W is a nonempty subset of $\mathbb{P}^3$ that is closed under addition and scalar multiplication. Therefore, W is a subspace of P_3

Zero and Trivial Subspaces

Zero Subspaces

The set $\{0\}$ consisting of only the zero vector is also a subspace of V, called the **zero subspace**. That is, the two closure properties are satisfied

$$0 + 0 = 0 \quad \text{and} \quad k(0) = 0$$

If V is a vector space, then V is a subspace of itself.

Trivial Subspace

The subspaces $\{0\}$ and V are called the trivial subspaces of V.

Linear Combination

Definition (Linear Combination)

Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be a set of vectors in $\mathbb{R}^n$ and $k_1, k_2, \cdots, k_n$ be scalars. An expression of the form

$$k_1\vec{v_1} + k_2\vec{v_2} + \cdots + k_n\vec{v_n} = \sum_{i=1}^{k} k_i\vec{v_i} \tag{19}$$

is called a linear combination of the vectors of S.

Definition (Linear Combination)

A vector $\vec{v}$ is a linear combination of vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ if there are scalars $k_1, k_2, \cdots, k_n$ such that

$$\vec{v} = k_1\vec{v_1} + k_2\vec{v_2} + \cdots + k_n\vec{v_n} \tag{20}$$

The scalars $k_1, k_2, \cdots, k_n$ are called the **coefficients** of the linear combination.

Example

The vector $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix}$ Since

$$3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \tag{21}$$

This is a consistent system of the form $Ax = b$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & -3 & -4 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \tag{22}$$

Example

The vector $\begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}$ because there are no scalars k_1, k_2 and k_3 where

$$k_1 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} + k_3 \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix} \tag{23}$$

Note the reduced augmented form is inconsistent, that is

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 5 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Linear Combination and System of Equations

The general system of equation of the form $Ax = b$ can be recast into linear combination as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{24}$$

which is the same as

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \tag{25}$$

Thus, for any consistent system, the column vectors of A could be expressed as a linear combination of the right-hand side; b.

Theorem

A system of linear equations with augmented matrix $[A/b]$ is consistent if and only if b is a linear combination of the columns of A.

Spanning Set

If $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ is a set of vectors in R^n, then an expression of the form

$$k_1\vec{v_1} + k_2\vec{v_2} + \cdots + k_n\vec{v_n} \tag{26}$$

is said to be a linear combination of $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$.

Definition (Span)

If $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ is a set of vectors in a vector space V, then the set all linear combination of $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ is called the span of $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ and is denoted by span$(\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n})$ or span(S).

Example

Let S be the subset of the vector space R^3 defined by

$$S = \left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right)$$

Show that $\vec{v} = [-4, 4, -6]'$ is in the span(S).

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Show that $\vec{v} = [-4, 4, -6]'$ is in the span(S).

To determine if $\vec{v}$ is in the span of S, we consider the equation

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -6 \end{bmatrix} \tag{27}$$

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Solving this linear system, we obtain

$$c_1 = -2 \qquad c_2 = 1 \qquad c_3 = -1$$

This shows that $\vec{v}$ is a linear combination of the vectors in S and is thus in span(S).

Example

Consider the vector $[1,\ 2,\ 3]^T$ and determine if it can be written as a linear combination of $\begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{-5}{2} \\ 0 \\ 1 \end{bmatrix}$.

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There must be scalars c_1 and c_2 such that

$$c_1 \begin{bmatrix} \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \frac{-5}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \Longrightarrow \qquad c_1 = 2, \quad c_2 = 3$$

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Cross-checking with the first row:

$$\frac{3c_1}{2} - \frac{5c_2}{2} = 1 \Longrightarrow \frac{3(2)}{2} - \frac{5(3)}{2} = -4.5 \neq 1$$

Thus, the two vectors do not span R^3.

Null and Column Space

Definition (Null Space)

Let A be an $n \times n$ matrix. Then the set of vectors $x \in R^n$ satisfying $Ax = 0$ is a subspace of R^n called the null space of A and is denoted by $N(A)$.
To find the null space of A, we solve the homogeneous equation $Ax = 0$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{28}$$

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Definition (Column Space)

1. The column space of A, denoted by $col(A)$, is the set of all linear combinations of the column vectors of A.
2. If A is an $n \times n$ matrix, the subspace spanned by the columns of A is a subspace of R^n, called the column space of A.

Range and Row Space

Definition (Row Space)

Also, the subspace spanned by the rows of A is a subspace of R^n called the row space of A denoted by $row(A)$.
It is the set of all linear combinations of the row vectors of A.

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Definition (Range)

Let A be an $m \times n$ matrix, the range of A denoted as $(R(A))$ consists of the set of all vectors b in $\mathbb{R}^m$ such that the linear system

$$Ax = b$$

is consistent.

Theorem

Let A be an $m \times n$ matrix. The linear system $Ax = b$ is consistent if and only if b is in the column space of A.

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Example

Let

$$A = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \\ 2 & -2 & -2 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

Determine whether b is in $col(A)$.

By the Theorem above, the vector b is in $col(A)$ if and only if there is a vector x such that $Ax = b$.

Solving, the augmented matrix is

$$\begin{bmatrix} 1 & -1 & -2 & 3 \\ -1 & 2 & 3 & 1 \\ 2 & -2 & -2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & -4 \end{bmatrix} \tag{29}$$

Thus

$$x_1 = 3 \qquad x_2 = 8 \qquad x_3 = -4 \tag{30}$$

Hence, the linear system $Ax = b$ is consistent, and vector b is in $col(A)$. Specifically,

$$\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + 8\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - 4\begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix} \tag{31}$$

Linear Dependent

Definition

Vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ in R^n are said to be linearly dependent if there exist scalars $k_1, k_2, \cdots, k_n$ not all zero, such that

$$k_1\vec{v_1} + k_2\vec{v_2} + \cdots k_i\vec{v_i} \cdots + k_n\vec{v_n} = 0 \tag{32}$$

1. Suppose $k_i \neq 0$, then

$$\vec{v_i} = -\frac{k_1\vec{v_1} + k_2\vec{v_2} + \cdots + k_{i-1}\vec{v_{i-1}} + k_{i+1}\vec{v_{i+1}} + \cdots + k_n\vec{v_n}}{k_i} \tag{33}$$

2. Here, the vector $\vec{v_i}$ is written as a linear combination of the remaining vectors; in other words, it is dependent on them.

Linear Independent

The vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ are called linearly independent if they are not linearly dependent.

Definition

1. The vectors are linearly independent if the system

$$k_1\vec{v_1} + k_2\vec{v_2} + \cdots k_i\vec{v_i} \cdots + k_n\vec{v_n} = 0 \tag{34}$$

   has only the trivial solution

$$k_1 = 0,\ k_2 = 0, \cdots, k_n = 0 \tag{35}$$

   as its solution.
2. Conversely, if $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ are linearly independent, then the homogeneous system has only the trivial solution.

Example

Are the following three vectors in R^3 linearly independent or dependent?

$$x_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad x_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \qquad x_3 = \begin{bmatrix} -1 \\ 7 \\ 12 \end{bmatrix}$$

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$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This corresponds to the homogeneous system.

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 7 \\ 3 & 2 & 12 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$$

1st Iteration

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & 7 \\ 3 & 2 & 12 \end{bmatrix} \xrightarrow[NR_3 = R_3 - 3R_1]{NR_2 = R_2 - 2R_1} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 9 \\ 0 & 5 & 15 \end{bmatrix}$$

2nd Iteration

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 9 \\ 0 & 5 & 15 \end{bmatrix} \left[\xrightarrow[NR_3 = R_3 - (5/3)R_2]{}\right] \begin{bmatrix} 1 & -1 & -1 \\ 0 & 3 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

The row of zeros in the row-reduced matrix indicates that there are infinitely many solutions to the homogeneous system, so x_1, x_2, x_3 are linearly dependent.

Example

Are the vectors linearly dependent?

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$$c_1 x_1 + c_2 x_2 + c_3 x_3 = 0$$

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This corresponds to the homogeneous system.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$$

1st Iteration

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 3 & 3 \end{bmatrix} \begin{matrix} \longrightarrow \\ NR_2 = R_2 - 2R_1 \\ NR_3 = R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

2nd Iteration

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} \longrightarrow \\ NR_3 = R_3 + (4/3)R_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The final system has the trivial solution

$$c_1 = c_2 = c_3 = 0$$

so $x_1, x_2,$ and x_3 are linearly independent.

Alternative Approach

Linear Independence and Determinants

If A is a square matrix, then the column vectors of A are linearly independent if and only if

$$det(A) \neq 0 \tag{36}$$

Theorem

Let $Ax = b$ be a consistent $m \times n$ linear system. The solution is unique if and only if the column vectors of A are linearly independent.

Basis

Definition (Basis)

A subset B of a vector space V is a basis for V provided that

1. B is a linearly independent set of vectors in V
2. $span(B) = V$

Example

The standard basis vectors $e_1, \cdots, e_n$ form a basis for R^n. This is the reason for the term **standard basis**.

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If $\vec{x} = [x_1, \cdots, x_n]'$, then

$$\vec{x} = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \tag{37}$$

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They are linearly independent, since if

$$x_1 e_1 + x_2 e_2 + \cdots + x_n e_n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

then

$$x_1 = x_2 = \cdots = x_n = 0$$

Example

The following vectors are linearly independent. Show that the vectors are a basis for R^3

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It is necessary to show that the vectors span R^3. Let x be any vector in R^3. There must be a linear combination of $\vec{u}, \vec{v}$, and $\vec{w}$ that equals $\vec{x}$; in other words, there must be scalars c_1, c_2, c_3, such that $c_1\vec{u} + c_2\vec{v} + c_3\vec{w} = \vec{x}$. This is a system of linear equations

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 2 \\ -1 & 3 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Form the augmented matrix $\left.\begin{array}{ccc|c} 1 & 1 & 1 & x_1 \\ 2 & -1 & 2 & x_2 \\ -1 & 3 & 3 & x_3 \end{array}\right.$

In the previous illustration the row reduction gave

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

This shows that there is a unique solution for $[c_1,\ c_2,\ c_3]$ and so $\vec{u}, \vec{v}$, and $\vec{w}$ form a basis for R^3.

Finding a Basis

Given a set $S = \vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ the following can be used to find a basis for $span(S)$:

1. Form a matrix A whose column vectors are $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$.
2. Reduce A to row echelon form.
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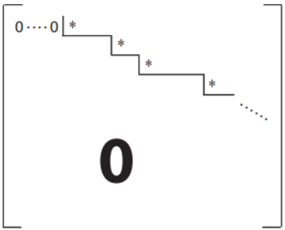
Definition (Row Echelon)

The matrix is in reduced row echelon form if,

1. Every row with all 0 entries is below every row with nonzero entries.
2. The first nonzero entry ((counting from left to right)) of each row is a 1.
3. Each column that contains a pivot has all other entries 0.



The height of each step is one row, and the first nonzero term in a row is denoted by $*$

Example

Find a basis for the span of S, if

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$$

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Start by constructing the matrix whose column vectors are the vectors in S. So we reduce the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 & -2 \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 0 & 1 & 0 & -2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Observe that the leading 1s in the reduced matrix are in columns $1, 2$, and 4. Therefore, a basis B for $span(S)$ is given by $\{\vec{v_1}, \vec{v_2}, \vec{v_4}\}$, that is

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$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

Dimension

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Example

The standard bases for R^n is $\{e_1, e_2, \cdots, e_n\}$ thus $dim(R^n) = n$. Similarly, the standard bases for are polynomial P_n is $\{1, x, x^2, \cdots, x^n\}$ thus $dim(P_n) = n + 1$

Example

The vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \right\}$$

has basis

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

So then

$$dim(S) = 3$$

Orthogonal Basis

Theorem

Let $S = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be a set of nonzero vectors in $\mathbb{R}^n$. If S is an orthogonal set of vectors, then S is a linearly independent set of vectors.

Definition (Orthogonal and Orthonormal Basis)

1. Let W be a subspace of $\mathbb{R}^n$, and let $B = \{\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}\}$ be a basis for W. If B is an orthogonal set of vectors, then B is called an orthogonal basis for W.
2. Furthermore, if $||v_i|| = 1$ for $1 \leq i \leq n$, then B is said to be an orthonormal basis for W.

Example

The basis vectors

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} \tag{38}$$

does not form an orthogonal basis,

Example

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while the basis vectors

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 7 \end{bmatrix} \right\} \tag{39}$$

forms an orthogonal basis for $\mathbb{R}^3$.
Note that these vectors are linearly independent.

Ordered Basis and Coordinates

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Definition (Coordinates)

Let $B = \vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ be an ordered basis for the vector space V. Let $\vec{v}$ be a vector in V, and let $k_1, k_2, \cdots, k_n$ be the unique scalars such that $\vec{v} = k_1\vec{v_1} + k_2\vec{v_2} + \cdots + k_n\vec{v_n}$. Then $k_1, k_2, \cdots, k_n$ are called the coordinates of $\vec{v}$ relative to B. In this case we write

$$[\vec{v}]_B = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} \tag{40}$$

and refer to the vector $[\vec{v}]_B$ as the coordinate vector of $\vec{v}$ relative to B.

Order Basis and Coordinates

Changing the order of the basis vectors will change the order of the scalars. For example, the sets

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad B' = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \tag{41}$$

are both bases for R^2. Then the list of scalars associated with the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\{1, 2\}$ relative to B, but is $\{2, 1\}$ relative to B'.

1. In R^n the **coordinates** of a vector relative to the standard basis $B = \{e_1, e_2, \cdots, e_n\}$ are simply the components of the vector.
2. Similarly, the **coordinates** of a polynomial $p(x) = a_0 + a_1x + a_2x_2 + \cdots + a_nx_n$ in P_n relative to the standard basis $\{1, x, x^2, \cdots, x^n\}$ are the coefficients of the polynomial.

Example

Let $V = R^2$ and B be the ordered basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Find the coordinates of the vector $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ relative to B.

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The coordinates k_1 and k_2 are found by writing $\vec{v}$ as a linear combination of the two vectors in B. That is, we solve the equation

$$k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{aligned} k_1 - k_2 &= 1 \\ k_1 + k_2 &= 5 \end{aligned} \tag{42}$$

Thus $k_1 = 3$ and $k_2 = 2.$. So the coordinate vector of $\vec{v}$ relative to B is

$$[\vec{v}]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \tag{43}$$

Example

Let $V = P_2$ and B be the ordered basis: $B = \{1, x-1, (x-1)^2\}$. Find the coordinates of $p(x) = 2x^2 - 2x + 1$ relative to B.

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We must find k_1, k_2, and k_3 such that

$$k_1(1) + k_2(x-1) + k_3(x-1)2 = 2x^2 - 2x + 1 \tag{44}$$

Expanding the left-hand side and collecting like terms give

$$k_3x^2 + (k_2 - 2k_3)x + (k_1 - k_2 + k_3) = 2x^2 - 2x + 1 \tag{45}$$

Equating the coefficients of like terms gives the linear system

$$k_1 - k_2 + k_3 = 1, \qquad k_2 - 2k_3 = -2, \qquad k_3 = 2 \tag{46}$$

The unique solution to this system is $k_1 = 1,\ k_2 = 2$, and $k_3 = 2$, so that $[\vec{v}]_B = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

Change of Basis

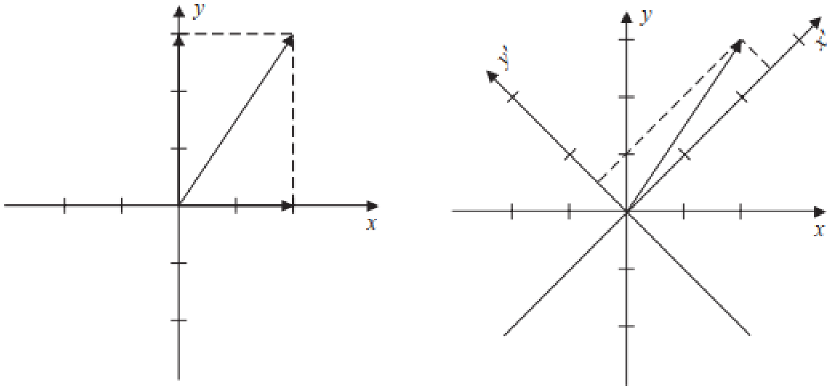
1. In the Euclidean space we have used rectangular coordinates, (or xy coordinates), to specify the location of a point in the plane.
2. Equipped with our knowledge of linear combinations, we now understand these xy coordinates to be the scalar multiples required to express the vector as a linear combination of the standard basis vectors e_1 and e_2. For example, the vector $\vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, with xy coordinates $(2, 3)$, can be written as

$$\vec{v} = 2e_1 + 3e_2 \tag{47}$$

3. This point (or vector) can also be specified relative to another pair of linearly independent vectors, describing an $x'y'$ coordinate system. For example, since

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{48}$$

the $x'y'$ coordinates of $\vec{v}$ are given by $\left(\frac{5}{2}, \frac{1}{2}\right)$



(a) (b)

Change of Basis

Definition

Let V be a vector space of dimension 2 and let

$$B = \{\vec{v_1}, \vec{v_2}\} \quad \text{and} \quad B' = \{\vec{v_1'}, \vec{v_2'}\} \tag{49}$$

be ordered bases for V. Now let $\vec{v}$ be a vector in V, and suppose that the coordinates of $\vec{v}$ relative to B are given by $[v]_B = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that is $\vec{v} = x_1\vec{v_1} + x_2\vec{v_2}$

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Then the coordinates of $\vec{v}$ relative to B' is given by,

$$[\vec{v}]_{B'} = [I]_B^{B'}[\vec{v}]_B \tag{50}$$

where $[I]_B^{B'} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is called the transition matrix from B to B'. The column vectors of the transition matrix are the coordinate vectors $[\vec{v_1}]_{B'}$ and $[\vec{v_2}]_{B'}$.

Example

Let $V = R^2$ with bases $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ and $B' = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$.

1. Find the transition matrix from B to B'.
2. Given that $[\vec{v}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, find $[\vec{v}]_{B'}$.

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1. Find the transition matrix from B to B'.
2. Given that $[\vec{v}]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, find $[\vec{v}]_{B'}$.

By denoting the vectors in B by $\vec{v_1}$ and $\vec{v_2}$ and those in B' by $\vec{v_1'}$ and $\vec{v_2'}$, then the column vectors of the transition matrix are $[\vec{v_1}]_{B'}$ and $[\vec{v_2}]_{B'}$ obtain by the equations:

$$c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad d_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + d_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \tag{51}$$

Solving these equations gives $c_1 = 2$ and $c_2 = 3$, and $d_1 = 0$ and $d_2 = -1$, so that

$$[\vec{v_1}]_{B'} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and} \quad [\vec{v_2}]_{B'} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

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$$[\vec{v}]_{B'} = [I]_B^{B'}[\vec{v}]_B \qquad \Longrightarrow [\vec{v}]_{B'} = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} \qquad \Longrightarrow [\vec{v}]_{B'} = \begin{bmatrix} 6 \\ 11 \end{bmatrix} \tag{52}$$

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Note

$$x_1\vec{v_1} + x_2\vec{v_2} = \vec{v} = x_1'\vec{v_1'} + x_2'\vec{v_2'} \tag{53}$$

$$3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 6\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 11\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{54}$$

Rank of a Matrix

Definition (Rank of a Matrix)

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The process we develop to find the rank of a matrix will involve row reductions, but we will go beyond just getting to upper-triangular form and will also **zero out** as many elements in the upper triangle as we can. The process is illustrated with examples.

Lemma

Subspaces spanning $\{x_1, x_2, \cdots, x_r\}$ and spanning $\{y_1, y_2, \cdots, y_s\}$ are equal if each of $x_1, x_2, \cdots, x_r$ is a linear combination of $y_1, y_2, \cdots, y_s$ and each of $y_1, y_2, \cdots, y_s$ is a linear combination of $x_1, x_2, \cdots, x_r$.

Theorem

The row space of a matrix is the same as the row space of any matrix derived from it using row reduction.

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

The upper-triangular form for A is

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

We cannot eliminate any more elements. The row space of A and B is the same and consists of all multiples of the vector $[1\ 2]$. Hence, $[1\ 2]$ is a basis for the row space of A and the rank of A is 1.

Example

Find the row space and rank of

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The row reduction is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left[\overrightarrow{NR_2 = R_2 - 3R_1} \right] \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

Continue on and use the -2 in row 2, column 2 to eliminate the element above it in row 1, column 2.

$$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \left[\overrightarrow{NR_1 = R_1 + R_2} \right] \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \left[\overrightarrow{NR_2 = -0.5R_2} \right] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The row space consists of all linear combinations of the vectors $[1\ 0]$ and $[0\ 1]$, and so the row space is R^2, and the rank of A is 2.

Example

Consider

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 4 & 5 \end{bmatrix}$$

Perform row reductions to determine a basis for the row space and the rank of A

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1st Iteration

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 4 & 5 \end{bmatrix} \left[NR_3 \overset{\longrightarrow}{=} R_3 - R_1 \right] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix}$$

2nd Iteration

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix} \left[NR_3 = \overrightarrow{R_3 - 4R_2} \right] \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{bmatrix}$$

This is upper-triangular form, but continue eliminating as many elements as we can. To make things easier, divide row 3 by -9.

3rd Iteration

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & -9 \end{bmatrix} \begin{bmatrix} \longrightarrow \\ NR_3 = R_3/(-9) \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

4th Iteration

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \longrightarrow \\ NR_2 = R_2 - 3R_3 \\ NR_1 = R_1 - 2R_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row space of A is R^3, and the rank of A is 3.

Definition

Given an $m \times n$ matrix A then

$$rank(A) + nullity(A) = n$$

where

1. The dimension of row space $row(A)$ or the column space $col(A)$ is the rank of A.
2. The dimension of null space $N(A)$ is called the nullity of A.

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Example

Find the rank, nullity, and dimension of the row space for the matrix A, where

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ -1 & 0 & 2 & -3 \\ 2 & 4 & 8 & 5 \end{bmatrix}$$

1. To find the dimension of the row space of A, observe that A is row equivalent to the matrix
$$A' = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
2. A' is in echelon form. Since the nonzero rows of A' form a basis for the row space of A, the row space of A has dimension 3.
3. Again from the reduced form, we can observe that the number of linear independent row of A' is 3 and hence the rank of the matrix A is 3

To find the nullity of A, we must determine the dimension of the null space. Since the homogeneous system $Ax = 0$ is equivalent to $A'x = 0$, the null space of A can be determined by solving $A'x = 0$. This gives

$$x_1 = 2x_3 \qquad x_2 = -3x_3 \qquad x_4 = 0$$

Thus

$$N(A) = \begin{bmatrix} 2x_3 \\ -3x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$$

It now follows that the nullity of A is 1

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Note

$$rank(A) + nullity(A) = n$$
$$3 + 1 = 4$$

Let A be an $n \times n$ matrix. Then the following statements are equivalent.

1. The matrix A is invertible.
2. The linear system $Ax = b$ has a unique solution for every vector b.
3. The homogeneous linear system $Ax = 0$ has only the trivial solution.
4. The matrix A is row equivalent to the identity matrix.
5. The determinant of the matrix A is nonzero.
6. The column vectors of A are linearly independent.
7. The column vectors of A span $\mathbb{R}^n$.
8. The column vectors of A are a basis for $\mathbb{R}^n$.
9. $rank(A) = n$
10. $R(A) = col(A) = \mathbb{R}^n$
11. $N(A) = \{0\}$
12. $row(A) = \mathbb{R}^n$
13. The number of pivot columns of the reduced row echelon form of A is n.

Exercises

1. Which of the following subsets of R^2 are subspaces?
   1. $[x\ y]^T$ satisfying $x = 2y$
   2. $[x\ y]^T$ satisfying $xy = 0$
2. Determine if the following vectors are linearly independent in R^4.

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} x_4 = \begin{bmatrix} 0 \\ 1 \\ 4 \\ 5 \end{bmatrix}$$

3. Determine for which values of k the vectors form a basis for R^4

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 3 \\ 4 \\ k \end{bmatrix}$$

Exercise

Find the transition matrix between the ordered bases B_1 and B_2; then given $[\vec{v}]_{B_1}$, find $[\vec{v}]_{B_2}$.

1. $B_1 = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}, \quad [\vec{v}]_{B_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
2. $B_1 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}, \quad B_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}, \quad [\vec{v}]_{B_1} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$
3. $B_1 = \{1,\ x,\ x^2\}, \quad B_2 = \{x^2,\ 1,\ x\}, \quad [\vec{v}]_{B_1} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

Exercise

1. Determine the rank of A, and find a basis for the column space of A.

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & -4 \\ 1 & 0 & 1 & -1 \\ 1 & 1 & 3 & -5 \end{bmatrix}$$

END OF LECTURE

THANK YOU