CSM 165: Discrete Mathematics for Computer Science

Chapter 4: Relations and Functions

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Content

Relations

Representation of Relations

Relation

Definition 1

Let A and B be sets. A binary relation from A to B is a subset of $A \times B$.

Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B.

$\forall a \in A$ and $\forall b \in B$:

1. $(a, b) \in R$; a is related to b, written aRb.
2. $(a, b) \notin R$; a is not related to b, written $a \not R\, b$.

Relations

The **domain** of a relation R is the set of all first elements of the ordered pairs which belong to R

The **range** is the set of second elements.

Example 1

Given $A = (1, 2, 3)$ and $B = \{x, y, z\}$, and $R = \{(1, y), (1, z), (3, y)\}$. Then is a relation from A to B since $R \subseteq A \times B$:

$1Ry$, $1Rz$, $3Ry$, but $1 \not R\, x$, $2 \not R\, x$, $2 \not R\, y$, $2 \not R\, z$, $3 \not R\, x$, $3 \not R\, z$
The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.

Relations on a set

Definition 2

A relation on a set A is a relation from A to A.
i.e. a relation on a set A is a subset of $A \times A$.

Example 2

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) | \text{a divides } b\}$?

Solution:

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$

Relations

Exercise A: Consider the following relations on the set of integers:

$$R_1 = \{(a,b) | a \le b\}$$
$$R_2 = \{(a,b) | a > b\}$$
$$R_3 = \{(a,b) | a = b \text{ or } a = -b\}$$
$$R_4 = \{(a,b) | a = b\}$$
$$R_5 = \{(a,b) | a = b+1\}$$
$$R_6 = \{(a,b) | a+b \le 3\}$$

Which of these relations contain each of the pairs $(1,1), (1,2), (2,1), (1,-1), (2,2)$?

Properties of Relations

Definition 3 (Reflexive)

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Example 3

Which of these relations are reflexive on $\{1, 2, 3, 4\}$:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (3, 4), (4, 4)\}$

Properties of Relations

Definition 4 (Symmetric)

*A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.*

i.e. $\forall a \forall b((a, b) \in R \rightarrow (b, a)) \in R$

*A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called **antisymmetric**.*

i.e $\forall a \forall b((a, b) \in R \wedge (b, a) \in R) \rightarrow (a = b))$

Properties of Relations

Example 4

Which of the following relations on $\{1, 2, 3, 4\}$ are symmetric or antisymmetric :

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

$R_2 = \{(1,1),(1,2),(2,1)\}$,

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,

$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$,

$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(3,4),(4,4)\}$

Solution: R_2 and R_3 are symmetric and R_4, R_5 are antisymmetric.

Properties of Relations

Note: The terms **symmetric** and **antisymmetric** are not opposites, because a relation can have both of these properties or may lack both of them.

Exercise B:

1. Determine which of the relations from **Exercise A** are symmetric and which are antisymmetric?

2. Give an example of a relation on a set that is
   2.1 both symmetric and antisymmetric.
   2.2 neither symmetric nor antisymmetric.

Properties of Relations

Definition 5 (Transitive)

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Example 5

(i) Which of the relations in **Example 3** are transitive?

(ii) Is the "divides" relation on the set of positive integers transitive?

(iii) Is $R = \{4,3\}$ on the set $A = \{1,2,3,4\}$ transitive?

Solution:

(i) R_4, R_5 are transitive and R_1, R_2, R_3 are not transitive

(ii) The divides relation is transitive

(iii) R is transitive.

Properties of Relation

Definition 6 (Composite)

Let R be a relation from a set A to a set B and S a relation from B to a set C.

The composite of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$

The composite of R and S is denoted by $S \circ R$

Properties of Relation

Example 6

What is the composite of the relations R and S, where R is the relation from $\{1,2,3\}$ to $\{1,2,3,4\}$ with $R = \{(1,1),(1,4),(2,3),(3,1),(3,4)\}$ and S is the relation from $\{1,2,3,4\}$ to $\{0,1,2\}$ with S = {(1, 0), (2, 0), (3, 1), (3, 2), (4, 1)}?

Solution

$S \circ R = \{(1,0),(1,1),(2,1),(2,2),(3,0),(3,1)\}$

Properties of Relations

Definition 7

Let R be a relation on the set A. The powers R^n, $n = 1, 2, 3, \ldots$, are defined recursively by $R^1 = R$ and $R^{n+1} = R^n \circ R$

Properties of Relations

Example 7

Let $R = \{(1,1),(2,1),(3,2),(4,3)\}$. Find the powers $R^n, n = 2,3,4,....$

Solution:

$R^2 = R \circ R = \{(1,1),(2,1),(3,1),(4,2)\}$

$R^3 = R^2 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

$R^4 = R^3 \circ R = \{(1,1),(2,1),(3,1),(4,1)\}$

$\therefore R^n = R^3$ for $n = 4,5,7\ldots$

Properties of Relations

Theorem 1

The relation R on a set A is transitive if and only if $R^n \subseteq R$ for $n = 1, 2, 3, \ldots$.

Properties of Relations

Exercise C:

For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, symmetric, antisymmetric, and whether it is transitive

(a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

(b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

(c) $\{(2, 4), (4, 2)\}$

(d) $\{(1, 2), (2, 3), (3, 4)\}$

(e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

(f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

Matrix representation of relations

Suppose that R is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$. The relation R can be represented by the matrix $M_R = [m_{ij}]$ defined as

$$m_{ij} = \begin{cases} 1 \text{ if } (a_i, b_j) \in R, \\ 0 \text{ if } (a_i, b_j) \notin R. \end{cases}$$

Matrix representation of relations

Example 8

Suppose that $A = \{1, 2, 3\}$ and $B = \{1, 2\}$. Let R be the relation from A to B containing (a, b) if $a \in A$, $b \in B$, and $a > b$.

What is the matrix representing R?

Solution

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Matrix representation of relations

Example 9

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pairs are in the relation R represented by the matrix

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution:

$$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$$

Matrix representation of relations

Remarks:

1. If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.

2. R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$.

3. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$

Matrix representation of relations

Example 10

Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Is R reflexive, symmetric, and/or antisymmetric?

Matrix representation of relations

Example 11

Suppose that the relations R_1 and R_2 on a set A are represented by the matrices

$$M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } M_{R_1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing $R_1 \cup R_2$ and $R_1 \cap R_2$?

Matrix representation of relations

Solution:

$$M_{R_1 \cup R_2} = M_{R_1} \vee M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$M_{R_1 \cap R_2} = M_{R_1} \wedge M_{R_2} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix representation of relations

Exercise D:

Find the matrix representing the relations $S \circ R$, where the matrices representing R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } M_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Representing Relations Using Digraphs

Definition 8

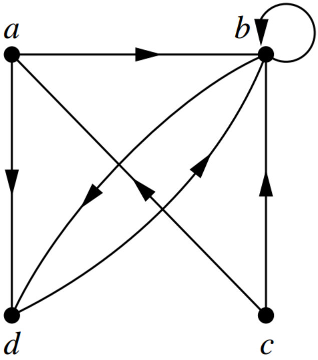
A directed graph, or digraph, consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges.

*The vertex a is called the **initial vertex** of the edge (a, b), and the vertex b is called the **terminal vertex** of this edge.*

Representing Relations Using Digraphs

Example 12

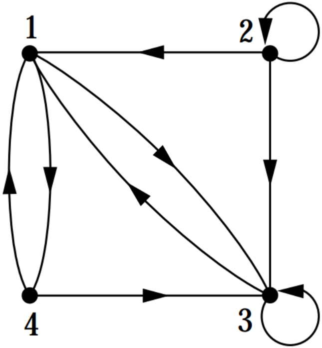
The directed graph with vertices a, b, c, and d, and edges $(a, b), (a, d), (b, b), (b, d), (c, a), (c, b)$, and (d, b) is given by

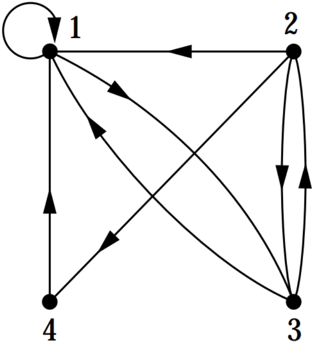


Representing Relations Using Digraphs

Example 13

What are the ordered pairs in the relation R represented by the directed graphs shown below?





Representing Relations Using Digraphs

Remarks:

1. **Reflexivity**: A loop must be present at all vertices in the graph.

2. **Symmetry**: If (a, b) is an edge,then so is (b, a).

3. **Antisymmetry**: If (a, b) with $a \neq b$ is an edge, then (a,b) is not an edge.

4. **Transitivity**: If (a, b) and (b, c) are edges, then so is (a, c).

Representing Relations Using Digraphs

Example 14

Determine whether the following relations for the directed graphs are reflexive, symmetric, antisymmetric, and/or transitive.

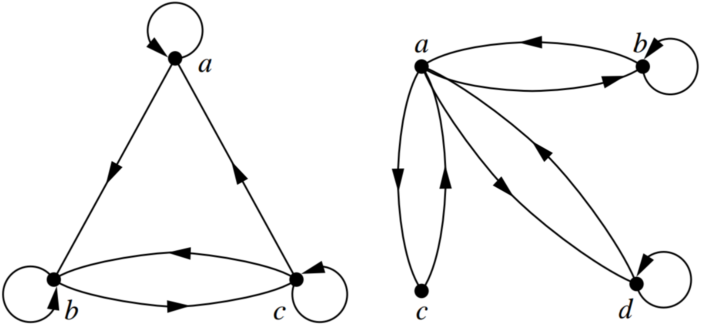
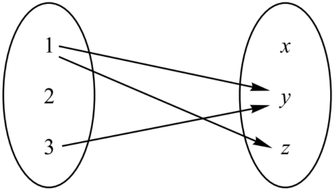


Table of Values & Arrow Diagram

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0



$$R = \{(1, y), (1, z), (3, y)\}$$

End of Lecture

Questions…???

Thanks

Reference Books

1. Kenneth H. Rosen, “Discrete Mathematics and Its Applications”, Tata Mcgraw Hill, New Delhi, India, seventh Edition, 2012.

2. J. P. Tremblay, R. Manohar, “Discrete Mathematical Structures with Applications to Computer Science”, Tata Mc Grforaw Hill, India, 1st Edition, 1997.