CSM 166: Discrete Mathematics for Computer Science

THEORY OF DIFFERENCE EQUATIONS/RECURRENCE RELATIONS

Isaac Afari Addo <addoisaacafari@gmail.com>
National Institute for Mathematical Science (NIMS) - Ghana
Department of Mathematics, KNUST
Kumasi-Ghana.

19/07/2021

Content

Introduction

Classification of Recurrence Relations

Non-Homogeneous Difference Equations

The Difference Operator

The difference operator Δ is defined by

$$(\Delta y)(n) = y(n+1) - y(n) \tag{1}$$

This is called the **forward difference** operator.
The difference operator is analogous to the differential operator in calculus.
i.e.

$$y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h} = (\Delta y)(x) = \frac{y(x+h) - y(x)}{h}$$

for $h = 1$

The Difference Operator

The difference operator Δ is defined by

$$(\Delta y)(n) = y(n+1) - y(n) \tag{1}$$

This is called the **forward difference** operator.
The difference operator is analogous to the differential operator in calculus.
i.e.

$$y'(x) = \lim_{h\to 0} \frac{y(x+h) - y(x)}{h} = (\Delta y)(x) = \frac{y(x+h) - y(x)}{h}$$

for $h = 1$

Properties of Δ

Linearity:

$\Delta(y+v) = \Delta y + \Delta v;$

$\Delta(\alpha y) = \alpha\Delta(y)$, α a scalar

Product Rule:

$$
\begin{aligned}
(\Delta(y \cdot u))(n) &= y(n+1) - y(n)u(n) \\
&= [y(n+1) - y(n)]\,u(n+1) \\
&\quad + y(n)[u(n+1) - u(n)] \\
&= (\Delta y)(n)\,u(n+1) + y(n)(\Delta u)(n)
\end{aligned}
$$

Exponential:
If $y(n) = 2^n$,

$$\begin{aligned}(\Delta y)(n) &= y(n+1) - y(n) \\ &= 2^{n+1} - 2^n \\ &= 2^n \\ \Rightarrow \Delta y &= y\end{aligned}$$

Thus 2^n for difference equations plays same role as e^x for differential equations.

Constants:
$\Delta \cdot c = 0$, constant.

Exponential:

If $y(n) = 2^n$,

$$\begin{aligned}(\Delta y)(n) &= y(n+1) - y(n) \\ &= 2^{n+1} - 2^n \\ &= 2^n \\ \Rightarrow \Delta y &= y\end{aligned}$$

Thus 2^n for difference equations plays same role as e^x for differential equations.

Constants:

$\Delta \cdot c = 0$, constant.

The Difference Operator

Theorem 1

If $\Delta y = v$, *then*

$$\sum_{r=m}^{n} (\Delta y)(r) = [y(r)]_m^{n+1} = y(n+1) - y(m).$$

Proof.

Theorem 1

$$
\begin{aligned}
v(n) &= y(n+1) - y(n) \\
v(n-1) &= y(n) - y(n-1) \\
v(n-2) &= y(n-1) - y(n-2) \\
&\vdots \\
v(m+1) &= y(m+2) - y(m+1) \\
v(m) &= y(m+1) - y(m)
\end{aligned}
$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{r=m}^{n} v(r) = y(n+1) - y(m)$$

$\square$

The Difference Operator

Example 1

$\Delta a^n = a^{n+1} - a^n = a^n(a-1)$, a a constant

Thus

$$a^n = \frac{\Delta a^n}{a-1} = \Delta\left(\frac{a^n}{a-1}\right)$$

$$\therefore \sum_{r=m}^{n} a^r = \frac{1}{a-1}\sum_{r=m}^{n} \Delta a^r = \frac{a^{n+1}-a^m}{a-1}$$

Recurrence relation

Definition 1 (Recurrence relation)

A recurrence relation for a sequence $a_0, a_1, , \ldots$ is a relation that defines a_n in terms of $a_0, a_1, , \ldots a_{n-1}$.

The formula relating a_n to earlier values in the sequence is called the **generating rule**.

The assignment of a value to one of the a's is called an **initial condition**.

Recurrence relation

Definition 1 (Recurrence relation)

A recurrence relation for a sequence $a_0, a_1, , \ldots$ is a relation that defines a_n in terms of $a_0, a_1, , \ldots a_{n-1}$.

The formula relating a_n to earlier values in the sequence is called the **generating rule**.

The assignment of a value to one of the a's is called an **initial condition**.

Recurrence relation

Definition 1 (Recurrence relation)

A recurrence relation for a sequence $a_0, a_1, , \ldots$ *is a relation that defines* a_n *in terms of* $a_0, a_1, , \ldots a_{n-1}$.

The formula relating a_n to earlier values in the sequence is called the **generating rule**.

The assignment of a value to one of the a's is called an **initial condition**.

Introduction

Example 2

The Fibonacci Sequence $1, 1, 2, 3, 5, \ldots$ is a sequence in which every number after the first two is the sum of the preceding two numbers.

The initial conditions are $a_0 = a_1 = 1$ and the generating rule is

$$a_n = a_{n-1} + a_{n-2}; \quad n \geq 2$$

Solution to a Recurrence Relation

A **solution** to a recurrence relation is an explicit formula for a_n in terms of n .

The fundamental method for finding the solution of a sequence defined recursively is by using **iteration**.

This involves starting with the initial values of the sequence and then calculates successive terms of the sequence until a pattern is observed.

An explicit formula for the sequence and then uses mathematical induction to prove its validity.

Solution to a Recurrence Relation

A **solution** to a recurrence relation is an explicit formula for a_n in terms of n .

The fundamental method for finding the solution of a sequence defined recursively is by using **iteration**.

This involves starting with the initial values of the sequence and then calculates successive terms of the sequence until a pattern is observed.

An explicit formula for the sequence and then uses mathematical induction to prove its validity.

Solution to a Recurrence Relation

A **solution** to a recurrence relation is an explicit formula for a_n in terms of n .

The fundamental method for finding the solution of a sequence defined recursively is by using **iteration**.

This involves starting with the initial values of the sequence and then calculates successive terms of the sequence until a pattern is observed.

An explicit formula for the sequence and then uses mathematical induction to prove its validity.

Solution to a Recurrence Relation

A **solution** to a recurrence relation is an explicit formula for a_n in terms of n .

The fundamental method for finding the solution of a sequence defined recursively is by using **iteration**.

This involves starting with the initial values of the sequence and then calculates successive terms of the sequence until a pattern is observed.

An explicit formula for the sequence and then uses mathematical induction to prove its validity.

Solution to a Recurrence Relation

Example 3

Find a solution for the recurrence relation

$$\begin{cases} a_0 = 1 \\ a_n = a_{n-1} + 2, \quad n \geq 1 \end{cases}$$

Solution to Example 3

Listing out some terms of the sequence:

$$
\begin{aligned}
a_0 &= 1 \\
a_1 &= 1+2 \\
a_2 &= 1+4 \\
a_3 &= 1+6 \\
a_4 &= 1+8 \\
a_1 &= 1+10
\end{aligned}
$$

A guessed formular is $a_n = 2n+1$, $n \geq 0$ and thus needs to be proven using mathematical induction.

Solution to Example 3

Listing out some terms of the sequence:

$$a_0 = 1$$
$$a_1 = 1 + 2$$
$$a_2 = 1 + 4$$
$$a_3 = 1 + 6$$
$$a_4 = 1 + 8$$
$$a_1 = 1 + 10$$

A guessed formular is $a_n = 2n + 1$, $n \geq 0$ and thus needs to be proven using mathematical induction.

Solution to a Recurrence Relation

Example 4

Consider the arithmetic sequence

$$a_n = a_{n-1} + d, \quad n \geq 1$$

where a_0 is the initial value. Find an explicit formula for a_n .

Solution to Example 4

Listing out the first 4 terms of the sequence:

$$
\begin{aligned}
a_1 &= a_0 + d \\
a_2 &= a_0 + 2d \\
a_3 &= a_0 + 3d \\
a_4 &= a_0 + 4d \\
a_5 &= a_0 + 5d
\end{aligned}
$$

A guessed formular is $a_n = a_0 + nd$ and thus needs to be proven using mathematical induction.

Solution to Example 4

Listing out the first 4 terms of the sequence:

$$a_1 = a_0 + d$$
$$a_2 = a_0 + 2d$$
$$a_3 = a_0 + 3d$$
$$a_4 = a_0 + 4d$$
$$a_5 = a_0 + 5d$$

A guessed formular is $a_n = a_0 + nd$ and thus needs to be proven using mathematical induction.

Solution to a Recurrence Relation

Exercise A:

1. Consider the geometric sequence

$$a_n = ra_{n-1}, \quad n \geq 1$$

   Where a_0 is the initial value. Find an explicit formula for a_n.

2. Find a solution to the recurrence relation

$$\begin{cases} a_0 = 0 \\ a_n = a_{n-1} + (n+1), \quad n \geq 1 \end{cases}$$

Exercise B: A function y_n is defined recursively as follows:

$$\begin{cases} y_1 = 3 \\ y_2 = 7 \\ y_n = 3y_{n-1} - 2y_{n-2} \quad \text{for}\, n \geq 3 \end{cases}$$

Find an explicit formula or solution for y_n in terms of n.

Classification of Recurrence Relations

A recurrence relation is of first order if y_n is defined only in terms of y_{n-1}.

It is of second order if y_n is defined in terms of y_{n-1} and y_{n-2} , and so on.

A recurrence relation of the form $y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_k y_{n-k}$ is called a linear homogenous recurrence relation of order k

If the a_i are **constants**, then the above equation is said to have constant coefficients.

Classification of Recurrence Relations

A recurrence relation is of first order if y_n is defined only in terms of y_{n-1}.

It is of second order if y_n is defined in terms of y_{n-1} and y_{n-2} , and so on.

A recurrence relation of the form $y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_k y_{n-k}$ is called a linear homogenous recurrence relation of order k

If the a_i are **constants**, then the above equation is said to have constant coefficients.

Classification of Recurrence Relations

A recurrence relation is of first order if y_n is defined only in terms of y_{n-1}.

It is of second order if y_n is defined in terms of y_{n-1} and y_{n-2} , and so on.

A recurrence relation of the form $y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_k y_{n-k}$ is called a linear homogenous recurrence relation of order k

If the a_i are **constants**, then the above equation is said to have constant coefficients.

Classification of Recurrence Relations

A recurrence relation is of first order if y_n is defined only in terms of y_{n-1}.

It is of second order if y_n is defined in terms of y_{n-1} and y_{n-2} , and so on.

A recurrence relation of the form $y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_k y_{n-k}$ is called a linear homogenous recurrence relation of order k

If the a_i are **constants**, then the above equation is said to have constant coefficients.

Classification of Recurrence Relations

Linear recurrence relations have the following important properties:

1. multiplying any solution by a constant gives another solution,
2. adding two or more solutions give another solution.

First-Order Recurrence Relations

First-order recurrence relations are of the form:

$$\begin{cases} y_n = ay_{n-1} \\ y_0 = c \end{cases}$$

where a and c are constants.

First-order recurrence relations are solved by iteration:

$$
\begin{aligned}
y_n &= ay_{n-1} \\
&= a(ay_{n-2}) \\
&= a^2(ay_{n-3}) \\
&= \ldots\ldots \\
&= a^{n-1}y_1 \\
&= a^n y_0
\end{aligned}
$$

Using the initial condition, we have $y_n = ca^n$, $n \in \mathbb{Z}^+$

Second-Order Recurrence Relations

Second-order recurrence relations are of the form:

$$\begin{cases} y_n = ay_{n-1} + by_{n-2} & \text{for } n \geq 2 \\ y_0 = c_1 \\ y_0 = c_0 \end{cases} \tag{2}$$

Assuming a, b, c_0 and c_1 are constants and a trial function $y_n = ct^n$ to solve the relation above.

Using this assumption, $y_{n-1} = ct^{n-1}$ and $y_{n-2} = ct^{n-2}$

Second-Order Recurrence Relations

Second-order recurrence relations are of the form:

$$\begin{cases} y_n = ay_{n-1} + by_{n-2} \quad \text{for } n \geq 2 \\ y_0 = c_1 \\ y_0 = c_0 \end{cases} \tag{2}$$

Assuming a, b, c_0 and c_1 are constants and a trial function $y_n = ct^n$ to solve the relation above.

Using this assumption, $y_{n-1} = ct^{n-1}$ and $y_{n-2} = ct^{n-2}$

Second-Order Recurrence Relations

Subtituting these into the recurrence relation (3:

$$ct^n = act^{n-1} + bct^{n-2}$$

Dividing through by ct^{n-2}:

$$t^2 = at + b \Rightarrow t^2 - at - b = 0 \qquad (3)$$

(3) is called the **auxiliary** or **characteristic**

equation of the recurrence relation.

Roots of the Characteristic Equation

The characteristic equation is a quadratic equation whose roots may be:

I two distinct real roots $t = t_1$ and $t = t_2$

II repeated real roots $t = t_0$

III two complex roots $t = t_1$ and $t_2 = \overline{t_1}$

Roots of the Characteristic Equation

The characteristic equation is a quadratic equation whose roots may be:

I two distinct real roots $t = t_1$ and $t = t_2$

II repeated real roots $t = t_0$

III two complex roots $t = t_1$ and $t_2 = \overline{t_1}$

Roots of the Characteristic Equation

The characteristic equation is a quadratic equation whose roots may be:

I two distinct real roots $t = t_1$ and $t = t_2$

II repeated real roots $t = t_0$

III two complex roots $t = t_1$ and $t_2 = \overline{t_1}$

Solution to the Recurrence Relation

CASE I

Since $y_n = t_1^n$ and $y_n = t_2^n$ are solutions of the linear recurrence relation then another solution (the general solution) is

$$y_n = At_1^n + Bt_2^n \tag{4}$$

where A and B are arbitrary constants.

A and B are determined using the initial values $y_0 = c_0$ and $y_1 = c_1$.

Solution to the Recurrence Relation

CASE I

Since $y_n = t_1^n$ and $y_n = t_2^n$ are solutions of the linear recurrence relation then another solution (the general solution) is

$$y_n = At_1^n + Bt_2^n \tag{4}$$

where A and B are arbitrary constants.

A and B are determined using the initial values $y_0 = c_0$ and $y_1 = c_1$.

Solution to the Recurrence Relation

CASE II

Since $t = t_0$ is the repeated root of the characteristic equation, $y_n = t_0^n$ is a solution of the recurrence relation as well as the linearly independent solution $y_n = nt_0^n$.

Thus a general solution:

$$y_n = At^n + Bnt^n \qquad (5)$$

where A and B are arbitrary constants.

A and B are determined using the initial values $y_0 = c_0$ and $y_1 = c_1$.

Solution to the Recurrence Relation

CASE II

Since $t = t_0$ is the repeated root of the characteristic equation, $y_n = t_0^n$ is a solution of the recurrence relation as well as the linearly independent solution $y_n = nt_0^n$.

Thus a general solution:

$$y_n = At^n + Bnt^n \tag{5}$$

where A and B are arbitrary constants.

A and B are determined using the initial values $y_0 = c_0$ and $y_1 = c_1$.

Solution to the Recurrence Relation

CASE II

Since $t = t_0$ is the repeated root of the characteristic equation, $y_n = t_0^n$ is a solution of the recurrence relation as well as the linearly independent solution $y_n = nt_0^n$.

Thus a general solution:

$$y_n = At^n + Bnt^n \tag{5}$$

where A and B are arbitrary constants.

A and B are determined using the initial values $y_0 = c_0$ and $y_1 = c_1$.

Solution to the Recurrence Relation I

CASE III

The complex roots of the auxiliary equation with real coefficients occur in conjugate pair. i.e if $t_1 = u + iv$ then $t_2 = u - iv$ with $v \neq 0$

By the general rule, the solution

$$\begin{aligned} y_n &= At_1^n + Bt_2^n \\ &= A(u+iv)^n + B(u-iv)^n \end{aligned} \tag{6}$$

Solution to the Recurrence Relation II

In polar form

$$u + iv = r(\cos\theta + i\sin\theta)$$
$$u + iv = r(\cos\theta - i\sin\theta)$$

And by DeMoivre's Theorem

$$[\rho(\cos\theta \pm i\sin\theta]^n = \rho^n(\cos n\theta \pm i\sin n\theta)$$

Thus

$$\begin{aligned} y_n &= A\rho^n(\cos n\theta + i\sin n\theta) + B\rho^n(\cos n\theta - i\sin n\theta) \\ &= (A+B)\rho^n(\cos n\theta) + i(A-B)\rho^n \sin n\theta \end{aligned}$$

Solution to the Recurrence Relation III

If we substitute $A = B = \frac{1}{2}$, then $y_n = \rho^n \cos n\theta$ is a particular solution.

Similarly taking $A = -\frac{1}{2}i$ and $B = \frac{1}{2}i$, then $y_n = \rho^n \sin n\theta$ is also a particular solution.
Thus the general solution is

$$y_n = \tilde{A}\rho^n \sin n\theta + \tilde{B}\rho^n \cos n\theta \tag{7}$$

where $\rho = \sqrt{u^2 + v^2}$ and $\theta = \tan^{-1} \frac{v}{u}$.

Example 5

Solve the following

i $\begin{cases} y_n = 3y_{n-1} - 2y_{n-2} \text{ for } n \geq 2 \\ y_2 = 7 \\ y_1 = 3 \end{cases}$

ii $\begin{cases} y_n = 6y_{n-1} - 9y_{n-2} \text{ for } n \geq 1 \\ y_1 = 3 \\ y_0 = 5 \end{cases}$

iii $y_n + 2y_{n-1} + 2y_{n-2} = 0$

Solution:

(i) The Characteristic equation is $t^2 - 6t + 2 = 0$

Exercise C: Solve the following difference equations:

1. $y_{n+1} - ay_n = 0,\ y_0 = 1$
2. $y_n = -3y_{n-1}$
3. $y_{n+2} + 2y_n = 0$
4. $y_n - 2y_{n-1} + 2y_{n-2} = 0$
5. $y_n + 4y_{n-1} + 8y_{n-2} = 0;\ y_1 = -1$
6. $y_n - 4y_{n-1} + 8y_{n-2} = 0;\ y_2 = 1, y_3 = -2$
7. $y_{n+2} + 2y_{n+1} + 4y_n = 0$

Non-Homogeneous Difference Equations

Consider the difference equation of the form

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = \phi(n) \qquad (8)$$

Equation (8) is called **homogeneous** if $\phi(n) = 0$

and **non-homogeneous** when $\phi(n) \neq 0 \ \forall n \in \mathbb{Z}^+$

The general solution to the non-homogeneous is obtained by adding any particular solution of (8) to the general solution of the corresponding homogeneous equation.

Non-Homogeneous Difference Equations

Consider the difference equation of the form

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = \phi(n) \qquad (8)$$

Equation (8) is called **homogeneous** if $\phi(n) = 0$

and **non-homogeneous** when $\phi(n) \neq 0 \ \forall n \in \mathbb{Z}^+$

The general solution to the non-homogeneous is obtained by adding any particular solution of (8) to the general solution of the corresponding homogeneous equation.

Non-Homogeneous Difference Equations

Consider the difference equation of the form

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = \phi(n) \qquad (8)$$

Equation (8) is called **homogeneous** if $\phi(n) = 0$

and **non-homogeneous** when $\phi(n) \neq 0 \ \forall n \in \mathbb{Z}^+$

The general solution to the non-homogeneous is obtained by adding any particular solution of (8) to the general solution of the corresponding homogeneous equation.

Theorem 2

Let $p(n)$ be a particular solution to

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = \phi(n) \qquad (9)$$

and

$$a_1 g_1(n) + a_2 g_2(n) + \cdots + a_k g_k(n)$$

be the general solution to

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = 0 \qquad (10)$$

Then the general solution to (9) is

$$y(n) = a_1 g_1(n) + a_2 g_2(n) + \cdots + a_k g_k(n) + p(n) \qquad (11)$$

Theorem 2

Let $p(n)$ be a particular solution to

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = \phi(n) \qquad (9)$$

and

$$a_1 g_1(n) + a_2 g_2(n) + \cdots + a_k g_k(n)$$

be the general solution to

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \cdots + a_0 y_n = 0 \qquad (10)$$

Then the general solution to (9) is

$$y(n) = a_1 g_1(n) + a_2 g_2(n) + \cdots + a_k g_k(n) + p(n) \quad (11)$$

Example 6

Find the general solution to the difference equation $y_{n+1} - 2y_n = 2^n, \qquad n \geq 0 \;\; y_0 = 1$

Solution:

The homogeneous part

$r - 2 = 0 \Rightarrow r = 2$

$y_n^{(h)} = \alpha 2^n$

The non-homogeneous part

Guess of solution $y_n^{(p)} = A2^n n$

Substituting it into the relation

$A2^{n+1}(n+1) - 2A2^n n = 2^n$

$A2^n \cdot 2^1(n+1) - 2A \cdot 2^n n = 2^n$ dividing by 2^n

$$2A(n+1) - 2An = 1$$

$$2An + 2A - 2An = 1$$

$$2A = 1 \Rightarrow A = \tfrac{1}{2}$$

$$y_n^{(p)} = A2^n n = \tfrac{1}{2}2^n n = 2^{n-1} n$$

Now the general solution to the recurrence relation is

$$y(n) = y_n^{(h)} + y_n^{(p)}$$

$$y(n) = \alpha 2^n + 2^{n-1} n$$

Using the initial conditions

$$\alpha = 1$$

Thus

$$y_n = 2^n + 2^{n-1} n$$

$$\Rightarrow y_n = 2^{n-1}(2 + n)$$

First-Order Non-homogeneous Relation I

Consider the non-homogeneous first-order relation

$$y_n + a_1 y_{n-1} = kr^n \tag{12}$$

where k is a constant and $n \in \mathbf{Z}^+$. If r^n is not a solution of the associated homogeneous relation

$$y_n + a_1 y_{n-1} = 0, \tag{13}$$

then $y_n^{(p)} = Ar^n$ where A is a constant. When r^n is a solution of the associated homogeneous relation, then $y_n^{(p)} = Bnr^n$, for B a constant

2nd-Order Non-homogeneous Relation

Consider the non-homogeneous second-order relation

$$y_n + a_1 y_{n-1} a_2 y_{n-2} = kr^n \tag{14}$$

where k is a constant. We have that

(a) $a_n^{(p)} = Ar^n$, for A a constant, if r^n is not a solution of the associated homogeneous relation.

(b) $y_n^{(p)} = Bnr^n$, where B is a constant, if $y_n^{(h)} = a_1 r^n + a_2 r_1^n$, where $r_1 \neq r$; and

(c) $y_n^{(p)} = a_n^2 r^n$, for a a constant, when $y_n^{(h)} = (a_1 + a_2 n) r^n$.

2nd-Order Non-homogeneous Relation

Consider the non-homogeneous second-order relation

$$y_n + a_1 y_{n-1} a_2 y_{n-2} = kr^n \tag{14}$$

where k is a constant. We have that

(a) $a_n^{(p)} = Ar^n$, for A a constant, if r^n is not a solution of the associated homogeneous relation.

(b) $y_n^{(p)} = Bnr^n$, where B is a constant, if $y_n^{(h)} = a_1 r^n + a_2 r_1^n$, where $r_1 \neq r$; and

(c) $y_n^{(p)} = a_n^2 r^n$, for a a constant, when $y_n^{(h)} = (a_1 + a_2 n) r^n$.

2nd-Order Non-homogeneous Relation

Consider the non-homogeneous second-order relation

$$y_n + a_1 y_{n-1} a_2 y_{n-2} = kr^n \tag{14}$$

where k is a constant. We have that

(a) $a_n^{(p)} = Ar^n$, for A a constant, if r^n is not a solution of the associated homogeneous relation.

(b) $y_n^{(p)} = Bnr^n$, where B is a constant, if $y_n^{(h)} = a_1 r^n + a_2 r_1^n$, where $r_1 \neq r$; and

(c) $y_n^{(p)} = a_n^2 r^n$, for a a constant, when $y_n^{(h)} = (a_1 + a_2 n) r^n$.

Some functions $\phi(n)$ and the corresponding forms of paticula solutions $y_n^{(p)}$

$\phi(n)$	$y_n^{(p)}$
c, a constant	A, constant
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
$n^t, t \in \mathbf{Z}^+$ $r^n, r \in \mathbf{R}$	$A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0$ Ar^n
$\sin n\theta$	$A \sin n\theta + B \cos n\theta$
$\cos n\theta$	$A \sin n\theta + B \cos n\theta$
$n^t r^n$	$r^n(A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0)$
$r^n \sin n\theta$	$Ar^n \sin n\theta + Br^n \cos n\theta$
$r^n \cos n\theta$	$Ar^n \sin n\theta + Br^n \cos n\theta$

Example 7

i) Find the general solution of
$y_n - 3y_{n-1} - 10y_{n-2} = 3(2^n)$

ii) Solve the relation $y_{n+1} - 4y_{n+1} + 3y_n = -200$,
$n \geq 0$ $a_0 = 3000$, $a_1 = 3300$

iii) Find the solution of
$y_{n+2} - y_{n+1} - 6y_n = 6n^2 + 22n + 23$

Exercise D:

Find the general solution for the following.

1. $y_{n+2} - y_{n+1} - 2y_n = 3^n$

2. $y_n + 2y_{n-1} - 8y_{n-2} = 5^{n-2}$

3. $y_n - 9y_{n-2} = n^2 - 4n - 1$

4. $2y_{n+1} - 3y_n - 5y_{n-1} = 5^{n-1} - 4$

5. $y_{n+2} - y_{n+1} - 2y_n = 2^n;\ y_0 = 2,\ y_1 = 1$

6. $y_{n+2} + 2y_{n+1} - 8y_n = -5n + 14;\ y_0 = 0,\ y_1 = 1$

End of Lecture

Questions...???

Thanks