CSM 166: DISCRETE MATHEMATICS FOR COMPUTER SCIENCEII

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Unit 1

Fundamentals of Counting

The major goal of this unit is to establish several techniques for counting large finite sets without actually listing their elements. There are two basic counting principles used throughout. One involves addition and the other multiplication.

Elements of Counting

Sum Rule Principle

For a set X, $|X|$ denotes the number of elements of X. It is easy to see that for any two sets A and B we have the following result known as the **Inclusion- Exclusion Principle**

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Indeed, $|A|$ gives the number of elements in A including those that are common to A and B. The same holds for $|B|$. Hence, $|A| + |B|$ includes twice the number of common elements. Hence, to get an accurate count of the elements of $A \cup B$, it is necessary to subtract $|A \cap B|$ from $|A| + |B|$.

Note that if A and B are disjoint then $|A \cap B| = 0$ and consequently $|A \cup B| = |A| + |B|$.

The first counting principle (Sum Rule Principle): Suppose some event A can occur in m ways and a second event B can occur in n ways, and suppose both events cannot occur simultaneously. Then A or B can occur in $m+n$ ways.

In other words (Sum Rule Principle): Suppose A and B are disjoints sets.

Then $|A \cup B| = |A| + |B|$.

Clearly, the principle can be extended to three or more events. Specifically, suppose an event A_1 can occur in n_1 ways, an event A_2 can occur in n_2 ways, an event A_3 can occur in n_3 ways, and so on, and suppose no two of the events can occur at the same time. Then one of the events can occur in $n_1 + n_2 + n_3 + \cdots$ ways.

Example 1.1 (The Addition Rule)

Show by induction on n, that if $\{A_1, A_2, ..., A_n\}$ is a collection of pairwise disjoint sets then

$$|A_1 \cup A_2 \cup ... \cup A_n| = |A_1| + |A_2| + ... + |A_n|$$

Solution:

Basis of induction: For $n=2$ the result holds by the Inclusion-Exclusion Principle.

Induction hypothesis: Suppose that for any collection $\{A_1, A_2, ..., A_n\}$ of pairwise disjoint sets we have

$$|A_1 \cup A_2 \cup ... \cup A_n| = |A_1| + |A_2| + ... + |A_n|.$$

Induction step: Let $\{A_1, A_2, ..., A_n, A_{n+1}\}$ be a collection of pairwise disjoint sets.

Since

$$(A_1 \cup A_2 \cup ... \cup A_n) \cap A_{n+1} = (A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \cdots (A_n \cap A_{n+1}) = \varnothing$$

then by the Inclusion-Exclusion Principle and the induction hypothesis we have

$$\begin{aligned} |A_1 \cup A_2 \cup ... \cup A_n \cup A_{n+1}| &= |A_1 \cup A_2 \cup ... \cup A_n| + |A_{n+1}| \\ &= |A_1| + |A_2| + \cdots + |A_n| + |A_{n+1}| \end{aligned}$$

Example 1.2

A total of 35 programmers interviewed for a job; 25 knew FORTRAN, 28 knew PASCAL, and 2 knew neither language. How many knew both languages?

Solution:

Let A be the group of programmers that knew FORTRAN, B those Who knew PASCAL. Then $A \cap B$ is the group of programmers who knew both languages. By the Inclusion-Exclusion Principle we have:

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

That is,

$$33 = 25 + 28 - |A \cap B|.$$

Solving for $|A \cap B|$ we find $|A \cap B| = 20$|.

Product Rule Principle

Another important rule of counting is the multiplication rule.

The second Rule Principle (Product Rule Principle): Suppose there is an event A which can occur in m ways and, independent of this event, there is a second event B which can occur in n ways. Then combinations of A and B can occur in mn ways.

In other words (Product Rule Principle): Suppose A and B are finite sets.

Then $|A \times B| = |A| \cdot |B|$

Clearly, the principle can be stated as: If a decision consists of k steps, where the first step can be made in n_1 different ways, the second step in n_2 ways,…, the k th step in n_k ways, then the decision itself can be made in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ ways.

For example,

. Tossing a coin has two possible outcomes and tossing a die has six possible outcomes. Then the combined experiment, tossing the coin and die together results in $2 \times 6 = 12$ possible outcomes: $H1, H2, H3, H4, H5, H6,\ T1, T2, T3, T4,\ T5, T6$

. The number of different ways for a man to get dressed if he has 8 different shirts and 6 different pairs of trousers is $8 \times 6 = 48$

. The number of ways a three-figure integer be formed from the numbers, 4, 3, 5, 6 and 7 if no number is used twice or more is $5 \times 4 \times 3 = 60$.

Example 1.3

1. Suppose a college has 3 different computer science courses, 4 different mathematics courses, and 2 different actuarial courses (with no prerequisites)

   a) There are $n = 3 + 4 + 2 = 9$ ways to choose 1 of the courses.

   b) There are $n = (3)(4)(2) = 24$ ways to choose one of each of the courses.

2. Suppose Airline **A** has three daily flights between Kumasi and Accra, and Airline **B** has two daily flights between Kumasi and Accra

   a) There are $n = 3 + 2 = 5$ ways to fly from Kumasi to Accra.

   b) There are $n = (3)(2) = 6$ ways to fly Airline **A** from Kumasi to Accra, and then Airline **B** from Accra back to Kumasi.

   c) There are $n = (5)(5) = 25$ ways to fly from Kumasi and Accra , and then back again.

3. How many possible outcomes are there if 2 distinguishable dice are rolled?

4. Suppose that a state's license plates consist of 3 letters followed by four digits. How many different plates can be manufactured?(No repetitions)

Solution:

3. By the multiplication rule there are $6 \times 6 = 36$ possible outcomes.

4. By the multiplication rule there are $26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000$ possible license plates.

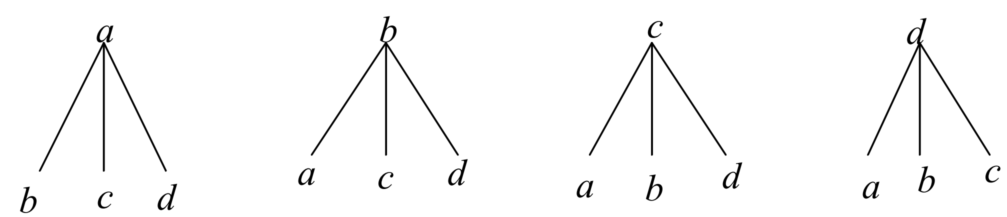
Example 1.3

Let $\Sigma = \{a, b, c, d\}$ be an alphabet with 4 letters. Let Σ^2 be the set of all words of length 2 with letters from Σ. Find the number of all words of length 2 where the letters are not repeated. First use the **product rule**. List the words by means of a **tree diagram**.

Solution:

By the multiplication rule there are $4 \times 3 = 12$ different words.

Constructing a tree diagram:



we find that the words are:

$$\{ab,\ ac,\ ad,\ ba,\ bc,\ bd, ca,\ cb,\ cd,\ da,\ db,\ dc\}$$

Applying the multiplication principle, results in the other two counting techniques, namely *Permutation and Combination*, used to find the number of possible ways when a fixed number of items are to be picked from a lot without replacement

PERMUTATIONS

Any arrangement of a set of n object in a given order is called a *permutation* of the objects (taken all at a time). Any arrangement of any $r \leq n$ of those objects in a given order is called an r-*permutation* of n objects or a *permutation of the* n *objects taken* r *at a time*. In symbol $P(n,\ r)$, is an ordered selection of r objects from a given n objects.

Consider, for example, the set of letters $a, b, c,$ and d. Then:

a) $abcd, bcda, acdb,$ and $dcba$ are permutations of the four letters (taken all at a time);

b) $bad, adb, cbd,$ and bca are permutations of the four letters taken three at a time;

c) $ad, cb, da,$ and bd are permutations of the four letters taken two at a time.

Example 1.4

a) Use the product rule to show that $P(n,\ r) = \dfrac{n!}{(n-r)!}$

b) Find all possible 2 - permutations of the set $\{1,\ 2,\ 3\}$.

Solution:

a) We can treat a permutation as a decision with r steps. The first step can be made in n different ways, the second in $n-1$ different ways, ..., the rth in $n-(r-1) = n - r + 1$ different ways. Thus, by the multiplication rule there are $n(n-1)\cdots(n-r+1)$ r-permutations of n objects.

That is $P(n, r) = n(n-1)\cdots(n-r+1) = \dfrac{n!}{(n-r)!}$

b) $P(3, 2) = \dfrac{3!}{(3-2)!} = 6$

Example 1.5

How many license plates are there that start with three letters followed by 4 digits (no repetitions)?

Solution:

$$P(26, 3)\cdot P(10, 4) = 78,\ 624,\ 000$$

An ordered arrangement of objects is called a *permutation*. The number of permutations of

(i) n distinct objects, taken all together is $n! = n(n-1)(n-2)\times\ldots\times 3\times 2\times 1$

(ii) n distinct objects taken r at a time is ${}^nP_r$ or $P(n,r) = \dfrac{n!}{(n-r)!}$, *where* $r \le n$.

(iii) n objects consisting of groups of which n_1 of the first group are alike, n_2 of the second group are alike and so on for the k^{th} group with n_k objects which are alike is $\dfrac{n!}{n_1!.n_2!.n_3!.\,.\,.n_k!}$, where $n = n_1 + n_2 + \ .\ .\ . + n_k$

(iv) n distinct objects arranged in a circle, called *circular permutations* is given by

$$\frac{n!}{n} = (n-1)!.$$

For example,

1. The number of possible permutations of the letters, *A*, *B* and *C* is $3! = 6.$ The required permutations are *ABC, BAC, ACB, BCA, CAB* and *CBA*.
2. The number of permutations of 10 distinct digits taken two at a time

$$= {}^{10}P_2 = \frac{10!}{(10-2)!} = 10 \times 9 = 90.$$

3. The number of permutations of the letters forming the following 14- letter word, *S C I E N T I F I C A L L Y*, which contains 2*C's*, 3I's, *2L's*, and 1's of the rest of letters

$$= \frac{14!}{2!.3!.2!} = 3,632,428,800$$

4. The number of circular permutations of 6 persons sitting around a circular table

= 5! = 120

Ordered Samples

When we choose one element after another from the set S containing n elements, say r times, we call the choice an ordered sample of size r. We consider two cases:

I. *Sampling with replacement*

Here the element is replaced in the set S before the next element is chosen. Since there are n different ways to choose each element (repetitions are allowed), the product rule principle tells us that there are

$$\overbrace{n \cdot n \cdot n \cdots n}^{r \text{ times}} = n^r$$

different ordered samples with replacement of size r.

II. *Sampling without replacement*

Here the element is not replaced in the set S before the next element is chosen. Thus there are no repetitions in the ordered sample. According, an ordered sample of size r without replacement is simply an $r-$permutation of the elements in the set S with n elements. Thus there are

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

different ordered samples without replacement of size r from a population (set) with n elements. In other words, by the product rule, the first element can be chosen in n ways, the second in $n-1$ ways, and so on.

Example 1.6

Three cards are chosen in succession form a deck with 52 cards. Find the number of ways this can be done (a) with replacement (b) without replacement.

Solution:

a) Since each card is replaced before the next card is chosen, each card can be chosen in 52 ways. Thus there are

$$52(52)(52) = 52^3 = 140{,}608$$

different ordered samples of size $r = 3$ with replacement.

b) Since there is no replacement, the first card can be chosen in 52 ways, the second card in 51 ways, and the last card 50 ways. Thus there are

$$\begin{aligned} P(52,3) &= 52(52-1)(52-(3-1)) \\ &= 52(52-1)(52-3+1) \\ &= 52(51)(50) = 132{,}600 \end{aligned}$$

different ordered samples of size $r = 3$ without replacement.

COMBINATIONS

Suppose we have a collection of n objects. A *combination* of these n objects taken r at a time is any selection of r of the objects without taking order in account. An r - combination of n objects, in symbol $C(n, r)$, is an unordered selection of r of the n objects. In other words, an $r-combination$ of a set of n objects is any subset of r elements. But the number of different ways that r objects can be ordered is $r!$. Since there are $C(n, r)$ groups of r objects from a given n objects then the number of ordered selection of r objects from n given objects is $r!C(n, r) = P(n, r)$.Thus

$$C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}.$$

For example, the combinations of the letters a,b,c,d taken three at a time are:

$\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\}$ or simply abc,abd,acd,bcd

Observe that the following combinations are equal:

abc,acb,bac,bca,cab,cba

That is, each denotes the same set $\{a,b,c\}$

Example 1.7

Find the number of combinations of four objects, a,b,c,d taken three at a time.

Solution:

Each combination consisting of three objects determines $3!=6$ permutations of the objects in the combination. Thus the number of combination multiplied by $3!$ equals the number of permutations. That is:

$$C(4,3)\cdot 3! = P(4,3) \text{ or } C(4,3) = \frac{P(4,3)}{3!}$$

But $P(4,3)=4\cdot 3\cdot 2=24$ and $3!=6$. Thus $C(4,3)=4$, which is shown in the table below:

Combinations	Permutations
abc	$abc, acb, bac, bca, cab, cba$
abd	$abd, adb, bad, bda, dab, dba$
acd	$acd, adc, cad, cda, dac, dca$
bcd	$bcd, bdc, cbd, cdb, dbc, dcb$

Example 1.8

In how many different ways can a hand of 5 cards be selected from a deck of 52 cards? (no repetition)

Solution:

$$C(52,5) = 2,\ 598,\ 960$$

Example 1.9

Prove the following identities:

a) $C(n,0)=C(n,n)=1$ and $C(n,1)=C(n,n-1)=n$.

b) Symmetry property: $C(n,\ r)\ =\ C(n,\ n\ -\ r),\ r\ \leq\ n$.

c) Pascal's identity: $C(n+1,k)=C(n,k-1)+C(n,\ k),\ n\leq k$.

Solution:

a) Follows immediately from the definition of $C(n,\ r)$. Check yourself.

b) Indeed, we have

$$
\begin{aligned}
C(n,n-r) &= \frac{n!}{(n-r)!(n-n+r)!} \\
&= \frac{n!}{r!(n-r)!} \\
&= C(n,\ r)
\end{aligned}
$$

c)
$$
\begin{aligned}
C(n,k-1)+C(n,k) &= \frac{n!}{(k-1)!(n-k+1)!}+\frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{k!(n-k+1)!}+\frac{n!(n-k+1)}{k!(n-k)!} \\
&= \frac{n!}{k!(n-k+1)!}(k+n-k+1) \\
&= \frac{(n+1)!}{(n+1-k)!}=C(n+1,k)
\end{aligned}
$$

Example 1.10

1. Find the number m of committees of three that can be formed from eight people. Each committee is, essentially, a combination of the eight people take three at a time.

   Solution:

   $$m = C(8,3) = \binom{8}{3} = \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} = 56$$

2. A farmer buys three cows, two pigs, and four hens from a man who has six cows, five pigs, and eight hens. How many choices does the farmer have?

   Solution:

   The farmer can choose the cows in $\binom{6}{3}$ ways, the pigs in $\binom{5}{2}$ ways, and the hens in $\binom{8}{4}$ ways. Hence altogether he can choose the animals in

   $$\binom{6}{3} \cdot \binom{5}{2} \cdot \binom{8}{4} = 20 \cdot 10 \cdot 70 = 14{,}000 \text{ ways}$$

3. Find the number m of ways that 9 toys can be divided between 4 children if the youngest is to receive 3 toys and each of the others 2 toys.

Solution:

There are $C(9,3)=84$ ways to first choose 3 toys for the youngest. Then there are $C(6,2)=15$ ways to choose 2 of the remaining 6 toys for the oldest. Next, there are $C(4,2)=6$ ways to choose 2 of the remaining 4 toys for the second oldest. The third oldest receives the remaining 2 toys. Thus, by the product rule:

$$m=84(15)(6)(1)=7560$$

Example 1.11

1.11(a) (i) In how many ways can a three-figure integer is formed from the numbers: 4, 3, 5, 6 and 7 if any number can be used more than once?

(ii) In a certain examination paper, students are required to answer 5 out of 10 questions from *Section A* another 3 out of 5 questions from *Section B* and 2 out of 5 questions from *Section C*. In how many ways can the students answer the examination paper?

Solution:

(i) The first, second and third numbers, each can be chosen in 5 ways. The total number of ways = $5 \times 5 \times 5 = 125$

(ii) The number of ways of answering the questions in *Section A*

$= 10 \times 9 \times 8 \times 7 \times 6 = 30{,}240$

The number of ways of answering the questions in section B

$$= 5 \times 4 \times 3 = 60$$

The number of ways of answering the questions in section C

$$= 5 \times 4 = 20$$

Hence the students can answer the questions in the three sections in

$$= 30{,}240 \times 60 \times 20 = 36{,}288{,}000$$

1.11(b) A company codes its customers by giving each customer an eight character code. The first 3 characters are the letter *A, B* and *C* in any order and the remaining 5 are the digits 1, 2, 3, 4 and 5 also in any order. If each letter and digit can appear only once then number of customers the company can code is obtained as follows:

The first 3 letters can be filled in 3!

The next 5 digits can be filled in 5!

Then the required number = 3! × 5! = 720

1.11(c) In many ways can 4 boys and 2 girls seat themselves in a row if :

(i) The 2 girls are to sit next to each other?

(ii) The 2 girls are not to sit next to each other?

Solution:

(i) If we regard the 2 girls as a separate persons ($\underline{B_1}$ $\underline{B_2}$ $\underline{B_3}$ $\underline{B_4}$ $\underline{G_1 G_2}$), then the number of arrangements of 5 different persons, taken all at a time = 5!

The 2 girls can exchange places and so the required number of ways they can seat themselves = 5! × 2! = 240

(ii) The number of ways the boys can arrange themselves = 4!

The number of ways the 2 girls can occupy the arrowed places:

$$\uparrow B_1 \uparrow B_2 \uparrow B_3 \uparrow B_4 \uparrow = {}^5P_2 = 5 \times 4$$

The required number of permutations (with the 2 girls not sitting next to each other) = 4! × 5 × 4 = 480

1.11(d) Find the number of ways in which a committee of 4 can be chosen from 6 boys and 5 girls if it must

(i) Consist of 2 boys and 2 girls.

(ii) Consist of at least 1 boy and 1 girl.

Solution:

(i) The number of ways of choosing 2 bys from 6 and 2 girls from 5

$$= \binom{6}{2} . \binom{5}{2} = 15 \times 10 = 150$$

(ii) For the committee to contain at least 1 boy and 1 girl we have

1B3G, 2B2G or 3B1G

The required number of ways

$$= \binom{6}{1} . \binom{5}{3} + \binom{6}{2} . \binom{5}{2} + \binom{6}{3} . \binom{5}{1}$$

$$= 6(10) + 15(10) + 20(5) = 130$$

1.11(e) (i) A school Parent-Teacher committee of 5 members is to be formed from 6 parents, 2 teachers and the principal. In how many ways can the committee be formed in order to include

(α) The principal? (β) Exactly four parents?

(γ) Not more than four parents?

(ii) Four balls are drawn from a bag of 12 balls of which 7 are blue and 5 are red. In how many of the possible combinations of 4 balls is at least a red?

Solution:

(i) (α) If the principal is to be included then we select 4 people from the remaining 8. Hence required number of ways the committee is formed

$$= \binom{1}{1}.\binom{8}{4} = 70$$

(β) The number of ways of selecting 4 parents out of 6 = $\binom{6}{4}$. The number of

ways of selecting the remaining number from the 3 (2 teachers and the principal) = $\binom{3}{1}$

Therefore the number of ways of selecting exactly 4 parents

$$= \binom{6}{4}.\binom{3}{1} = 15 \times 3 = 45$$

(γ) The number of ways of forming a 5-member committee = $\binom{12}{5}$

The number of ways of selecting 5 parents from 6 $= \binom{6}{5}$

Therefore the required number of ways of selecting a committee with not more than 4 parents $= \binom{12}{5} - \binom{6}{5} = 126 - 6 = 120$

(ii) If at least one red is to be included then the combinations include

1R 3B, with number of combinations $= \binom{5}{1}\binom{7}{3} = 175$

2R 2B, with number of combinations $= \binom{5}{2}\binom{7}{2} = 210$

3R 1B, with number of combinations $= \binom{5}{3}\binom{7}{1} = 70$

4R, with number of combinations $= \binom{5}{4}\binom{7}{0} = 5$

Review Problems

<u>Problem 1.1</u>

a) How many ways can we get a sum of 4 or a sum of 8 when two distinguishable dice are rolled?

b) Suppose a bookcase shelf has 6 mathematics texts, 3 physics texts, 4 chemistry texts, and 5 computer science texts. Find the number n of ways a student can choose: (a) one of the texts $(\text{Ans}:18)$, (b) one of each type of text $(\text{Ans}:360)$

c) How many ways can we get a sum of 8 when two undistinguishable dice are rolled?

Problem 1.2

a) How many 4 - digit numbers can be formed using the digits, $1, 2, \cdots, 9$ (with repetitions)? How many can be formed if no digit can be repeated?

b) How many different license plates are there that involve $1, 2,$ or 3 letters followed by 4 - digits (with repetitions)?

Problem 1.3

a) In how many ways can 4 cards be drawn, with replacement, from a deck of 52 cards?

b) In how many ways can 4 cards be drawn, without replacement, from a deck of 52 cards?

Problem 1.4

In how many ways can 7 women and 3 men be arranged in a row if the three men must always stand next to each other.

Problem 1.5

A menu in a Chinese restaurant allows you to order exactly two of eight main dishes as part of the dinner special. How many different combinations of main dishes could you order?

Problem 1.6

There are 12 students in a class. Find the number n of ways that 12 students can take three different tests if four students are to take each test. $(\text{Ans}: 34,650)$

Unit 2

MULTINOMIAL COEFFICIENTS

INTRODUCTION

This unit extends the technique of counting as already treated under permutations and combinations. It may be recalled that binomial expansion is very much linked with combinations. We are going to have a short but concise study of multinomial expansion, which is naturally linked with multinomial coefficients.

Objectives

On completion of this unit, you should be able to:

- Upgrade your knowledge on advanced techniques on counting process
- Extend knowledge of binomial expansion to multinomials
- Solve more practical problems

BINOMIAL COEFFICIENTS

Choosing a subset of size r out of a set of size n is logically equivalent to partitioning the set of size n into two subsets, one of size r and the other of size $(n-k)$. The number of such partitions if by definition ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$.

Suppose x, y are variable and $n \in Z^+$ the positive integers, then

$$\left(x+y\right)^n = \sum_{r=o}^{n} {}^nC_r x^{n-r} y^r = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + \ldots + {}^nC_n x^0 y^n$$

The coefficients ${}^nC_r,\ n \geq r,\ r \in \{0,1,\ldots,n\}$ are called the binomial coefficients of the binomial expression $\left(x+y\right)^n,\ n \in Z^+$.

DEFINITION

$${}^nC_r = \frac{n!}{r!\left(n-r\right)!} \qquad (2)$$

Where $0! = 1$ and $n! = n\left(n-1\right)\left(n-2\right)\ldots 1$

Pascal's identity allows one to construct the following triangle known as

Pascal's triangle (for n = 5) as follows

1

$1 \rightarrow 1$

$1 \rightarrow 2 \rightarrow 1$

$1 \rightarrow 3 \rightarrow 3 \rightarrow 1$

$1 \rightarrow 4 \rightarrow 6 \rightarrow 4 \rightarrow 1$

The following theorem provides an expansion of $\left(x + y\right)^n$ where n is a nonnegative integer.

Theorem 2.1 (Binomial Theorem)

Let x and y be variables, and let n be a positive integer. Then

$$\left(x+y\right)^n=\sum_{r=0}^{n}C\left(n,r\right)x^{n-r}y^r$$

where $C\left(n,\ r\right)$ is called the **binomial coefficient**.

Proof.

The proof is by induction.

Basis of induction: For $n=1$ we have

$$\left(x+y\right)^1 \ =\sum_{r=0}^{1}C\left(n,r\right)x^{n-r}y^r=\begin{pmatrix}1\\0\end{pmatrix}x^{1-0}y^0+\begin{pmatrix}1\\1\end{pmatrix}x^{1-1}y^1=x+y$$

Induction hypothesis: Suppose that the theorem is true for n.

Induction step: Let us show that it is still true for $n+1$. That is

$$\left(x+y\right)^{n+1} \ =\sum_{r=0}^{n+1}C\left(n+1,r\right)x^{n-r+1}y^r.$$

Indeed, we have:

$$\left(x+y\right)^{n+1}=\left(x+y\right)\left(x+y\right)^n=x\left(x+y\right)^n+y\left(x\ +\ y\right)^n$$

$$=x\sum_{r=0}^{n}C\left(n,r\right)x^{n-r}y^r+y\sum_{r=0}^{n}C\left(n,r\right)x^{n-r}y^r$$

$$=\sum_{r=0}^{n}C\left(n,r\right)x^{n-r+1}y^r+y\sum_{r=0}^{n}C\left(n,r\right)x^{n-r}y^{r+1}$$

$$=C\left(n,0\right)x^{n+1}+C\left(n,1\right)x^ny+C\left(n,2\right)x^{n-1}y^2+\cdots+C\left(n,n\right)xy^n$$
$$+C\left(n,0\right)x^ny+C\left(n,1\right)x^{n-1}y^2+\cdots+C\left(n,n-1\right)xy^n+C\left(n,n\right)y^{n+1}$$

$$=C\left(n+1,0\right)x^{n+1}+C\left(n+1,1\right)x^ny+C\left(n+1,2\right)x^{n-1}y^2+$$
$$+\cdots+C\left(n+1,n\right)x^ny+C\left(n+1,n+1\right)y^{n+1}$$

$$=\sum_{r=0}^{n+1}C\left(n+1,r\right)x^{n-r+1}y^r$$

Example

Expand $\left(x+y\right)^{6}$ using the binomial theorem.

Solution:

By the Binomial Theorem and Pascal's triangle we have

$$\left(x+y\right)^{6}=x^{6}+6x^{5}y+15x^{4}y^{2}+20x^{3}y^{3}+15x^{2}y^{4}+6xy^{5}+y^{6}$$

Example

a) Show that $\sum_{r=0}^{n}C\left(n,r\right)=2^{n}$

b) Show that $\sum_{r=0}^{n}\left(-1\right)^{r}C\left(n,r\right)=0$

Solution:

a) Letting $x=y=1$ in the binomial theorem we find

$$2^{n}=\left(1+1\right)^{n}=\sum_{r=0}^{n}C\left(n,r\right)$$

b) This follows from the binomial theorem by letting $x=1$ and $y=-1$

Example

Expand $(2x-3)^{5}$

Solution:

$$\begin{aligned}\left(2x-3\right)^{5}&=\sum_{r=0}^{5}{}^{5}C_{r}\left(2x\right)^{n-r}\left(-3\right)^{r}\\&={}^{5}C_{0}\left(2x\right)^{5}\left(-3\right)^{0}+{}^{5}C_{1}\left(2x\right)^{4}\left(-3\right)^{1}+{}^{5}C_{2}\left(2x\right)^{3}\left(-3\right)^{2}\\&\quad+{}^{5}C_{3}\left(2x\right)^{2}\left(-3\right)^{3}+{}^{5}C_{4}\left(2x\right)\left(-3\right)^{4}+{}^{5}C_{5}\left(2x\right)^{0}\left(-3\right)^{5}\end{aligned}$$

$$= \frac{5!}{5!0!}2^5x^5 + \frac{5!}{4!1!}2^4x^4(-3) + \frac{5!}{3!2!}2^3x^3(9) + \frac{5!}{2!3!}2^2x^2(-27)$$
$$+ \frac{5!}{1!4!}2x(81) + \frac{5!}{0!5!}(-243)$$
$$= 32x^5 - 240x^4 + 720x^3 - 1080x^2 + 81x - 243$$

Example 2

Obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$

Solution:

To obtain the coefficient of the term a^5b^2, we notice that in (2), we set $r = 2$ to obtain

${}^7C_2(2)^5(\text{-}3)^2$ as the coefficient of a^5b^2

But

$$ {}^7C_2(2)^5(-3)^2 = \frac{7!}{5!2!}(2)^5(-3)^2$$
$$= 6048$$

Therefore, the coefficient of a^5b^2 of the binomial expression is 6048

1. Expand the following binomial expressions:

(i) $(2x-4y)^5$ (ii) $(3x+2y^2)$ (iii) $(2-5xy)^4$

(iv) $[3z+(1-b)]^3$ (v) $[3x-(7+2y)]^3$

2. Find the *coefficients* of the *indicated terms* in the given *binomial expressions*:

(i) $x^{12}y^{13}$ in $(x+y)^{25}$ (ii) $x^{12}y^{13}$ in $(2x-3y)^{25}$

(ii) x^{9} in $(2-x)^{19}$ (iv) y^{6} in $\left[3-5(1-y)\right]^{8}$

(v) x^{19} in $\left[(3x+1)-2\right]^{20}$

3. Prove that for any $n \in Z^{+}$

(i) $\sum_{r=0}^{n} {}^{n}C_{r} = 2^{n}$ (ii) $\sum_{r=0}^{n} (-1)^{r}\, {}^{n}C_{r} = 0$.

4. Using the binomial expression $(1+x)^{n}$ to find the approximate values of the following to 5 decimal places:

(i) 0.9^{5} (ii) 1.99^{4}

5. Expand $(1-2x)^{6}$ and hence evaluate 0.9^{6} to six decimal places.

6. Show that ${}^{k}C_{k} + {}^{k+1}C_{k} + \ldots + {}^{k+r}C_{k} = {}^{k+r+1}C_{k+1}$

7. Show that $\ln(1+x) = x - \frac{x^{2}}{2} + \frac{x^{3}}{3} - \frac{x^{4}}{4} + \ldots$

Hence show that $\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = x + \frac{x^{3}}{3} + \frac{x^{5}}{5} + \ldots$

8. Solve the following

a) Find the coefficient of $a^{5}b^{7}$ in the binomial expansion of $(1-2b)^{12}$

b) Use the binomial theorem to prove that

$$3^{n} = \sum_{r=0}^{n} 2^{r} C(n,r)$$

MULTINOMIAL COEFFICIENTS

Let $k_1,\ k_2,\ ...,\ k_r$ be integers satisfying the relation $k_1+k_2+...+k_r=n$. Then The number of ways a set of n elements can be partitioned into classes of Sizes $k_1,\ k_2,\ ...,\ k_r$ equal

$$\frac{n!}{k_1!k_2!...k_r!}$$

Proof

We obtain the partitioning in steps: First, we choose k_1 out of n elements to form the first partition. Next, we choose k_2 elements out of the remaining $(n-k_1)$ elements, and so on until we have $n-k_1-k_2-...-k_{r-2}=k_{r-1}+k_r$ elements from which we choose k_{r-1} to form the next-to-last class. The remaining k_r elements forms the last class. This has been accomplished in

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}...\binom{n-k_1-k_2-...-k_{r-2}}{k_{r-1}}\binom{n-k_1-k_2-...-k_{r-1}}{k_r} \text{ ways.}$$

Simple algebra shows that

$$\binom{n}{k_1}\binom{n-k_1}{k_2}...\binom{n-k_1-k_2-k_3}{k_{r-1}}\binom{n-k_1-k_2......k_{r-1}}{k_r}=\frac{n!}{k_1!k_2!...k_r!}$$

Suppose $x_1,x_2,x_3,...,x_r$ are variables and $n\in Z^+$, then

$$(x_1+x_2+...+x_r)^n=\sum_{k_1,k_2...,k_r} {}^nC_{k_1,k_2...,k_r}\,x_1^{k_1}x_2^{k_2}...x_r^{k_r}$$

where $k_1+k_2...+k_r=n$

Example 1

Expand $(x+y+z)^3$

Solution:

$$\begin{aligned}(x+y+z)^3 &= {}^3C_{3,0,0}x^3y^0z^0 + {}^3C_{0,3,0}x^0y^3z^0 + {}^3C_{0,0,3}x^0y^0z^3 + {}^3C_{2,1,0}x^2y^1z^0 + {}^3C_{2,0,1}x^2y^0z^1 \\ &\quad + {}^3C_{1,2,0}x^1y^2z^0 + {}^3C_{0,2,1}x^0y^2z^1 + {}^3C_{1,0,2}x^1y^0z^2 + {}^3C_{0,1,2}x^0y^1z^2 + {}^3C_{1,1,1}x^1y^1z^1 \\ &= \frac{3!}{3!0!0!}x^3 + \frac{3!}{0!3!0!}y^3 + \frac{3!}{0!0!3!}z^3 + \frac{3!}{2!1!0!}x^2y + \frac{3!}{2!0!1!}x^2z \\ &\quad + \frac{3!}{1!2!0!}xy^2 + \frac{3!}{0!2!1!}y^2z + \frac{3!}{1!0!2!}xz^2 + \frac{3!}{0!1!2!}yz^2 + \frac{3!}{1!1!1!}xyz \\ &= x^3 + y^3 + z^3 + 3x^2y + 3x^2z + 3xy^2 + 3y^2z + 3xz^2 + 3yz^2 + 6xyz\end{aligned}$$

Example 2

Evaluate the following (a) $\binom{6}{4,2,0}$ (b) $\binom{5}{3,2}$ (c) $\binom{10}{5,3,0,2}$

Solution: (a) $\binom{6}{4,2,0} = \frac{6!}{4!2!0!} = \frac{6\times5\times4\times3\times2\times1}{4\times3\times2\times1\times2\times1} = 15$

(b) $\binom{5}{3,2} = \frac{5!}{3!2!} = \frac{5\times4\times3\times2\times1}{3\times2\times1\times2\times1} = 10$

(c) $\binom{10}{5,3,0,2} = \frac{10!}{5!3!0!2!} = \frac{10\times9\times8\times7\times6\times5\times4\times3\times2\times1}{5\times4\times3\times2\times1\times3\times2\times1\times2\times1} = 2520$

Example (Previous example)

Find the number m of ways that 9 toys can be divided between 4 children if the youngest is to receive 3 toys and each of the others 2 toys.

$$m = \frac{9!}{3!2!2!2!} = 7560$$

ACTIVITY

1. Show that $\binom{n}{n_1, n_2} = \binom{n}{n_1} = \binom{n}{n_2}$

2. Show that if $i \geq 1,\ j \geq 1,\ k \geq 1$ and $i + j + k = n + 1$, then

$$\left(\frac{(n+1)!}{i!\,j!\,k!}\right) = \frac{n!}{(i-1)!\,j!\,k!} + \frac{n!}{i!\,(j-1)!\,k!} + \frac{n!}{i!\,j!\,(k-1)!}$$

3. Prove that for any $n \in Z^+$, $\sum\limits_{k_1, k_2, \ldots, k_r}^{n} {}^{n}C_{k_1, k_2, \ldots, k_r} = r^n$

4. In the expansion of $(x + y + z)^7$ find the coefficient of the terms:

   (i) xyz^5 (ii) $x^2y^2z^3$ (iii) x^3z^4

5. Determine the coefficient of the following terms in the indicated multinomial expressions.

   (i) xyz^2 in $(2x - y - z)^4$

   (ii) xyz^{-2} in $\left(x - 2y + 3z^{-1}\right)^4$

   (iii) $w^3x^2yz^2$ in $(2w - x + 3y - 2z)^8$

   (iv) $x^{11}y^4z^2$ in $\left(2x^3 - 3xy^2 + z^2\right)^6$

   (v) $x^3y^4z^5$ in $(x - 2y + 3z)^{12}$

6. The letters B,C,E,E,N,R,S,S,Y,Z,Z,Z,Z are arranged at random. Determine the probability that these letters will spell the word SZCZEBRZESZYN

7. Show that if $a \leq b \leq c \leq n$, then

$$\binom{n}{c}\binom{c}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}\binom{n-b}{c-b}$$

(i) Use the definition of binomial coefficients as ratios of factorials.

(ii) Use the interpretation of the binomial coefficients directly as the number of subsets of a given size.

(iii) Generalize the above formula to answer the question: In how many ways can one choose an a-element subset from a b-element subset from a c-element subset from a d-element subset from an n-element set, where $a \le b \le c \le d \le n$?

8. Expand $(x+y+z)^6$. Hence with $x = y = z = 0.3$ evaluate 0.9^6 to six decimal places. (Compare this approximation with that of Q5 of the previous exercise).

Unit 3

THEORY OF DIFFERENCE EQUATIONS/RECURRENCE RELATIONS

INTRODUCTION

Welcome to this unit. Here, we shall learn more about recurrence relations, or simply put, algorithms. This would entail one, two or three expressions that would generate an infinite set of numbers. On the other hand, given a set of recurrence relations, a solution would be found for such a relation.

THE DIFFERENCE OPERATOR

The difference operator Δ is defined by $(\Delta y)(n) = y(n+1) - y(n)$. This is called the **forward** difference operator. The difference operator Δ is the analogous of the differential operator in calculus. Indeed, $y'(x) = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}$ is the definition of the derivative $f'(x)$ of $f(x)$. Similarly $(\Delta y)(x) = \frac{y(x+h) - y(x)}{h}$, when $h = 1$.

Familiar properties of $\frac{d}{dx}$ carry over as follows:

Linearity

$\Delta(y+v) = \Delta y + \Delta v;\ \Delta(\alpha y) = \alpha \Delta(y),\ \alpha$ scalar

Product Rule:

$$\begin{aligned}(\Delta(y\cdot v))(n) &= y(n+1)v(n+1)-y(n)v(n)\\ &=\left[y(n+1)-y(n)\right]v(n+1)+y(n)\left[v(n+1)-v(n)\right]\\ &=(\Delta y)(n)v(n+1)+y(n)(\Delta v)(n)\end{aligned}$$

Exponential:

If $y(n)=2^n$, $(\Delta y)(n)=y(n+1)-y(n)$

$$\begin{aligned}&=2^{n+1}-2^n\\ &=2^n=f(n)\end{aligned}$$

that is $\qquad \Delta y = y$

Thus 2^n for difference equations plays the same role e^x does for differential equations.

Constants:

$\Delta\cdot c=0,\ c$ constant

Conversely, suppose that $\Delta y=0$, that is $(\Delta y)(n)=y(n+1)-y(n)=0$ for all n, then y is a constant.

The operation analogous to integration is that of taking finite sums. The following result is a discrete version of the fundamental theorem of calculus, which essentially says that differentiation and integration are inverse operations.

Theorem

If $\Delta y = v$, then $\sum_{r=m}^{n} v(r) = \sum_{r=m}^{n} (\Delta y)(r) = \left[y(r) \right]_{m}^{n+1} = y(n+1) - y(m)$.

Proof

We have

$$\begin{aligned}
v(n) &= y(n+1) - y(n) \\
v(n-1) &= y(n) - y(n-1) \\
v(n-2) &= y(n-1) - y(n-2) \\
&\vdots \\
v(m+1) &= y(m+2) - y(m+1) \\
v(m) &= y(m+1) - y(m)
\end{aligned}$$

....

$$\sum_{r=m}^{n} v(r) = y(n+1) - y(m)$$

Example 1

Exponentials:

$\Delta a^n = a^{n+1} - a^n = a^n(a-1)$, a constant

Hence

$$a^n = \frac{\Delta a^n}{a-1} = \Delta\left(\frac{a^n}{a-1}\right) \text{ (by linearity)}$$

$$\therefore \sum_{r=m}^{n} a^r = \frac{1}{a-1} \sum_{r=m}^{n} \Delta a^r = \frac{a^{n+1} - a^m}{a-1}$$

(This is just the usual method for summing a geometric progression).

Example 2

Polynomials:

We consider easier polynomials of the form

$f_r(n) = n(n-1)(n-2)\ldots(n-r+1)$

Then, $\Delta f_r(n) = y_r(n+1) - y_r(n)$

$$\begin{aligned}\therefore \Delta f_r(n) &= (n+1)n(n-1)(n-2)\ldots(n-r+2) - n(n-1)(n-2)\ldots(n-r+1) \\ &= n(n-1)\ldots(n-r+2)\big[(n+1)-(n-r+1)\big] \\ &= rn(n-1)(n-2)\ldots(n-r+2) \\ &= rf_{r-1}(n)\end{aligned}$$

This is analogous to $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.

We may easily deduce that $\sum_{s=m}^{n} f_{r-1}(s) = \frac{1}{r}\left[f_r(n+1) - f_r(m)\right]$.

So in particular $\sum_{s=0}^{n} y_r(s) = \frac{1}{r+1} y_{r+1}(n+1)$, if $m = 0$.

With this, we have a method of summation for any polynomial.

For example,

$$\begin{aligned}\sum_{s=1}^{n} s^3 &= \sum_{s=0}^{n} s^3 \\ &= \sum_{s=0}^{n}\left[s(s-1)(s-2) + 3s(s-1) + s\right]\end{aligned}$$

$$= \sum_{s=0}^{n}\left[y_3(s) + 3y_2(s) + y_1(s)\right] = \frac{1}{4}\left[y_4(n+1) + y_3(n+1) + \frac{1}{2}y_2(n+1)\right]$$

$$=\frac{1}{4}(n+1)n(n-1)(n-2)+(n+1)n(n-1)+\frac{1}{2}(n+1)n$$

$$=\frac{(n+1)n}{4}\left[(n-1)(n-2)+4(n-1)+2\right]$$

$$=\frac{1}{4}n^2(n+1)^2$$

Example 3

Let $y(n)=na^n$. [a, constant]

Using example 1, we study Δy.

$$(\Delta y)(n)=(n+1)a^{n+1}-na^n$$

$$=a^{n+1}+na^n(a-1)=a^{n+1}+(a-1)y(n)$$

Hence $y(n)=\dfrac{1}{a-1}(\Delta y)(n)-\dfrac{a^{n+1}}{a-1}$

$$=\frac{1}{a-1}(\Delta y)(n)-\frac{a}{(a-1)^2}\Delta a^n$$

$$=\Delta\left[\frac{1}{a-1}y(n)-\frac{a}{(a-1)^2}a^n\right]$$

Then by the theorem,

$$\sum_{r=m}^{n}y(r)=\frac{1}{a-1}\left[y(n+1)-y(m)\right]-\frac{a}{(a-1)^2}\left[a^{n+1}-a^m\right]$$

$$=\frac{(n+1)a^{n+1}-ma^m}{a-1}-\frac{a^{n+2}-a^{m+1}}{(a-1)^2}$$

Example 4

Trigonometric Functions:

We study this specialized branch of mathematics using forward differences:

$$\Delta \sin\left(wx+\alpha\right) = \sin\left(w\left(x+1\right)+\alpha\right) - \sin\left(wx+\alpha\right)$$
$$= 2\sin\left(\tfrac{w}{2}\right)\cos\left[wx+\alpha+\tfrac{w}{2}\right]$$

Then

$$\sum_{r=0}^{n} 2\sin\left(\tfrac{w}{2}\right)\cos\left[wr+\alpha+\tfrac{w}{2}\right] = \sin\left[w\left(n+1\right)+\alpha\right] - \sin\alpha$$

so that

$$\sum_{r=0}^{n} \cos\left[wr+\alpha+\tfrac{w}{2}\right] = \frac{\sin\left[w\left(n+1\right)+\alpha\right]-\sin\alpha}{2\sin\frac{w}{2}}$$

Setting $\alpha = -\dfrac{w}{2}$, we have

$$\sum_{r=0}^{n} \cos wr = \frac{\sin\left[w\left(n+\tfrac{1}{2}\right)\right]+\sin\frac{w}{2}}{2\sin\frac{w}{2}}$$

A similar formula may be derived for $\sum_{r=0}^{n} \sin wr$.

ACTIVITY

1. Verify the following differences

   (a) $\Delta n^2 4^n = \left(3n+2\right)\left(n+2\right)4^n$;

   (b) $\Delta\left[3n\left(n+1\right)\left(n+2\right)\left(n+3\right)-4\left(n+1\right)\left(n+2\right)\left(n+3\right)\right]$
   $= 12n\left(n+2\right)\left(n+3\right)$

   (c) $\Delta\left(-1\right)^n \binom{N}{n} = \left(-1\right)^{n+1}\binom{N+1}{n+1}$

   (d) $\Delta \dfrac{2n+1}{n\left(n+1\right)} = -\dfrac{2}{n\left(n+2\right)}$

2. From (1) deduce the value of

(a) $\sum_{r=0}^{n}(3r+2)(r+2)4^{r}$ (b) $\sum_{r=1}^{n}r(r+2)(r+3)$

(c) $\sum_{n=3}^{10}(-1)^{n}\binom{15}{n}$ (d) $\sum_{n=0}^{N}\frac{1}{n(n+2)}$

From your answer to (d), deduce that $\sum_{n=1}^{\infty}\frac{1}{n(n+2)}$ exists and find its value.

3. Show that $\sum_{r=1}^{n}r^{4}=\frac{n}{30}(n+1)(6n^{3}+9n^{2}+n-1)$.

FINITE DIFFERENCE EQUATIONS - RECURRENCE RELATIONS

What we are about to study is the discrete analogue of differential equations. Let a_n be a function defined on the set of positive integers. An inductive or recurrence definition of the function y specifies the starting value, such as a_0 (or several starting values of the inductive or recursive definition, such as, $a_0,\ a_1,\ a_2,...$) and then a formula specified as to how to generate subsequent function values from the starting value (or values).

Recursion

A **recurrence relation** for a sequence $a_0, a_1, ...$ is a relation that defines a_n in terms of $a_0, a_1, ..., a_{n-1}$. The formula relating a_n to earlier values in the sequence is called the **generating rule**. The assignment of a value to one of the a's is called an **initial condition**.

Example 2.1

The Fibonacci sequence

$$1, 1, 2, 3, 5, ...$$

is a sequence in which every number after the first two is the sum of the preceding two numbers. Find the generating rule and the initial conditions.

Solution:

The initial conditions are $a_0 = a_1 = 1$ and the generating rule is

$a_n = a_{n-1} + a_{n-2};\ n \geq 2$

Example 2.2

Let $n \geq 0$ and find the number s_n of words from the alphabet $\sum = \{0,1\}$ of length n not containing the pattern 11 as a subword.

Solution:

Clearly, $s_0 = 1$ (empty word) and $s_1 = 2$. We will find a recurrence relation for $s_n, n \geq 2$. Any word of length n with letters from $\sum$ begins with either 0 or 1. If the word begins with 0, then the remaining $n-1$ letters can be any sequence of 0's or 1's except that 11 cannot happen. If the word begins with 1 then the next letter must be 0 since 11 cannot happen; the remaining $n-2$ letters can be any sequence of 0's and 1's with the exception that 11 is not allowed. Thus the above two categories form a partition of the set of all words of length n with letters from $\sum$ and that do not contain 11. This implies the recurrence relation

$$s_n = s_{n-1} + s_{n-2},\ n \geq 2$$

A **solution** to a recurrence relation is an explicit formula for a_n in terms of n.

The most basic method for finding the solution of a sequence defined recursively is by using **iteration**. The iteration method consists of starting with the initial values of the sequence and then calculates successive terms of the sequence until a pattern is observed. At that point one guesses an explicit formula for the sequence and then uses mathematical induction to prove its validity.

Example 2.3

Find a solution for the recurrence relation

$$\begin{cases} a_0 & = 1 \\ a_n = a_{n-1} + 2, n \geq 1 \end{cases}$$

Solution:

Listing the first five terms of the sequence one finds

$$a_0 = 1$$

$$a_1 = 1 + 2$$

$$a_2 = 1 + 4$$

$$a_3 = 1 + 4$$

$$a_4 = 1 + 8$$

Hence, a guess is $a_n = 2n + 1$, $n \geq 0$. It remains to show that this formula is valid by using mathematical induction.

Basis of induction: For $n = 0, a_0 = 1 = 2(0) + 1$.

Induction hypothesis: Suppose that $a_n = 2n + 1$.

Induction step: We must show that $a_{n+1} = 2(n+1) + 1$. By the definition of a_{n+1} we have $a_{n+1} = a_n + 2 = 2n + 1 + 2 = 2(n+1) + 1$.

Example 2.4

Consider the arithmetic sequence

$$a_n = a_{n-1} + d \, , \; n \geq 1$$

where a_0 is the initial value. Find an explicit formula for a_n.

Solution:

Listing the first four terms of the sequence after a_0 we find

$$a_1 = a_0 + d$$

$$a_2 = a_0 + 2d$$

$$a_3 = a_0 + 3d$$

$$a_4 = a_0 + 4d$$

Hence, a guess is $a_n = a_0 + nd$. Next, we prove the validity of this formula by induction.

Basis of induction: For $n = 0$ $a_0 = a_0 + (0)d$.

Induction hypothesis: Suppose that $a_n = a_0 + nd$.

Induction step: We must show that $a_{n+1} = a_0 + (n+1)d$. By the definition of a_{n+1} we have $a_{n+1} = a_n + d = a_0 + nd + d = a_0 + (n+1)d$.

Example 2.5

Consider the geometric sequence

$$a_n = ra_{n-1},\ n \geq 1$$

Where a_0 is the initial value. Find an explicit formula for a_n.

Solution:

Listing the first four terms of the sequence after a_0 we find

$$a_1 = ra_0$$

$$a_2 = r^2 a_0$$

$$a_3 = r^3 a_0$$

$$a_4 = r^4 a_0$$

Hence, a guess is $a_n = r^n a_0$. Next, we prove the validity of this formula by induction.

Basis of induction: For $n = 0$, $a_0 = r^0 a_0$.

Induction hypothesis: Suppose that $a_n = r^n a_0$.

Induction step: We must show that $a_{n+1} = r^{n+1} a_0$. By the definition of a_{n+1} we have $a_{n+1} = ra_n = r\left(ra_0\right) = r^{n+1} a_0$.

Example 2.6

Find a solution to the recurrence relation

$$\begin{cases} a_0 & 0 \\ a_n = a_{n+1} + \left(n+1\right), & n \geq 1 \end{cases}$$

Solution:

Writing the first five terms of the sequence we find

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 0 \\ a_2 &= 0+1 \\ a_3 &= 0+1+2 \\ a_4 &= 0+1+2+3 \end{aligned}$$

A guessing formula is that

$$a_n = 0+1+2+\cdots+\left(n-1\right) = \frac{n\left(n-1\right)}{2}$$

We next show that the formula is valid by using induction on $n \geq 0$.

Basis of induction: $a_0 = 0 = \frac{0(0-1)}{2} = 0$.

Induction hypothesis: Suppose that $a_n = \frac{n(n-1)}{2}$.

Induction step: We must show that $a_{n+1} = \frac{n(n+1)}{2}$. Indeed,

$$\begin{aligned} a_{n+1} &= a_n + n \\ &= \frac{n(n-1)}{2} + n \\ &= \frac{n(n+1)}{2} \end{aligned}$$

Example 2.7

Consider the recurrence relation

$$\begin{cases} a_0 & = 1 \\ a_n = 2a_{n-1} + n & n \geq 1 \end{cases}$$

Is it true that $a_n = 2^n + n$ is a solution to the given recurrence relation?

Solution:

If so then we must be able to prove its validity by mathematical induction.

Basis of induction: $a_0 = 2^0 + 1$.

Induction hypothesis: Suppose that $a_n = 2^n + n$.

Induction step: We must show that $a_{n+1} = 2^{n+1} + (n+1)$. If this is so then we will have $2^{n+1} + (n+1) = 2a_n + n = 2^{n+1} + 2n + n + 1$. But this would imply that $n = 0$ which contradicts the fact that n is any nonnegative integer.

Example 2.8

Define a sequence, $a_1, a_2, ...,$ recursively as follows:

$$\begin{aligned} a_1 &= 1 \\ a_n &= 2 \cdot a_{\lfloor \frac{n}{2} \rfloor}, \ n \geq 2 \end{aligned}$$

a. Use iteration to guess an explicit formula for this sequence.

b. Use induction to prove the validity of the formula found in a.

Solution.

Computing the first few terms of the sequence we find

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_3 &= 2 \\ a_4 &= 4 \\ a_5 &= 4 \\ a_6 &= 4 \\ a_7 &= 4 \\ a_8 &= \cdots = a_{15} = 8 \end{aligned}$$

Hence, for $2^i \leq n < 2^{i+1}$, $a_n = 2^i$. Moreover, $i \leq \log_2 n < i+1$ so that $i = \lfloor \log_2 n \rfloor$ and a formula for a_n is

$$a_n = 2^{\lfloor \log_2 n \rfloor}, n \geq 1$$

b. We prove the above formula by mathematical induction.

Basis of induction: For $n = 1$, $a_1 = 1 = 2^{\lfloor \log_2 1 \rfloor}$.

Induction hypothesis: Suppose that $a_n = 2^{\lfloor \log_2 n \rfloor}$.

Induction step: We must show that $a_{n+1} = 2^{\lfloor \log_2 n+1 \rfloor}$.

Indeed, for n odd (i.e. $n+1$ even) we have:

$$a_{n+1} = 2 \cdot a_{\lfloor \frac{n+1}{2} \rfloor}$$

$$= 2 \cdot a_{\frac{n+1}{2}} \quad = 2 \cdot 2^{\lfloor \log_2 \frac{n+1}{2} \rfloor} \quad = 2^{\lfloor \log_2(n+1) - 1 \rfloor + 1} \quad = 2^{\lfloor \log_2(n+1) \rfloor - 1 + 1}$$

$$= 2^{\lfloor \log_2(n+1) \rfloor}$$

A similar argument holds when n is even.

NOTATION:

We write $a(n) = a_n$, or $y(n) = y_n$, $n \in \mathbb{Z}^+$

Example 2.9

Let n be the number of memory locations referenced by a certain computer program. Suppose that the algorithm implemented by the program requires y_n bytes of the memory, where y_n depends on n. Let y_n be defined inductively by

$$\begin{cases} y_1 = 3 \\ y_n = 4y_{n-1} \text{ if } n > 1 \end{cases}$$

By this inductive definition, we generate the following table of Fibonacci sequence:

n	1	2	3	4	5	…
y_n	3	12	48	192	768	…

Sometimes it is more convenient to start at $n=0$ instead of $n=1$. In fact, any integer could be used as the starting value in the inductive definition or recursive relation.

Example 2.10

The factorial function $n!$ is defined as follows:

$$\begin{cases} 0!=1 \\ n!=n(n-1)! \text{ if } n\geq 1 \end{cases}$$

It is desirable to find an explicit expression of y_n as a function of n. To find such an expression is called the solution process of the inductive definition or recurrence relation.

A general solution for the recurrence relation is a solution process involving some arbitrary constants. The initial conditions enable us to find the values of those arbitrary constants. Sometimes by studying the Fibonacci sequence, an intuition gives the solution of y_n in terms of n.

Example 2.11

A function y_n is defined recursively as follows:

$$\begin{cases} y_1=3 \\ y_2=7 \\ y_n=3y_{n-1}-2y_{n-2} \text{ for } n\geq 3 \end{cases}$$

We want to find an explicit formula or solution for y_n in terms of n. We display some few initial terms of the Fibonacci sequence:

n	1	2	3	4	5	6	…
y_n	3	7	15	31	63	127	…

By intuition, $y_n = 2^{n+1} - 1$. This formula clearly satisfies the initial conditions y_1 and y_2. We verify if it satisfies the recurrence relation

$$y_n = 3y_{n-1} - 2y_{n-2}$$

But

$$\begin{aligned} R.H.S &= 3y_{n-1} - 2y_{n-2} \\ &= 3\left(2^n - 1\right) - 2\left(2^{n-1} - 1\right) \\ &= 3 \cdot 2^n - 3 - 2^n + 2 \\ &= 2^n\left(3 - 1\right) - 1 \\ &= 2^{n+1} - 1 \\ &= y_n \\ &= L.H.S \end{aligned}$$

When iteration does not apply, other methods are available for finding explicit formulas for special classes of recursively defined.

CLASSIFICATION OF RECURRENCE RELATIONS

A recurrence relation is of first order if y_n is defined only in terms of y_{n-1}. It is of second order if y_n is defined in terms of y_{n-1} and y_{n-2}, and so on.

A recurrence relation of the form $y_n = a_1 y_{n-1} + a_2 y_{n-2} + \ldots + a_k y_{n-k}$ is called a **linear homogenous** recurrence relation of order k. Assuming that the a_i are constants, then the above equation is said to have **constant** coefficients.

Linear recurrence relations have the following important properties:

- multiplying any solution by a constant gives another solution,
- adding two or more solutions give another solution.

FIRST-ORDER RECURRENCE RELATIONS

First-order recurrence relations are of the form:

$$\begin{cases} y_n = ay_{n-1} \\ y_0 = c \end{cases}$$

We assume a and c are constants. We solve first-order recurrence relations by iteration; a process of repetitive procedure, as displayed:

$$\begin{aligned} y_n &= ay_{n-1} \\ &= a\left(ay_{n-2}\right) \\ &= a^2\left(ay_{n-3}\right) \\ &= \ldots\ldots\ldots\ldots \\ &= a^{n-1}y_1 \\ &= a^n y_0 \end{aligned}$$

Using the initial condition, we have $y_n = ca^n$, $n \in \mathbb{Z}^+$. By this procedure, example 2.11 above, with $a = 4$ and $c = 3 = y_1$, we have $y_n = a^{n-1}y_1 = 3 \cdot 4^{n-1}$.

As an exercise, show that this solution satisfies the recursive definition of example 2, and generate the Fibonacci sequence as shown in example 2.

THE SECOND-ORDER RECURRENCE RELATION

Second-order recurrence relation has the form:

$$\begin{cases} y_n = ay_{n-1} + by_{n-2} \text{ for } n \geq 2 \\ y_1 = c_1 \\ y_0 = c_0 \end{cases}$$

We assume a, b, c_0 and c_1 are constants, and also a trial function $y_n = ct^n$ to solve the relation above. By this assumption, $y_{n-1} = ct^{n-1}$ and $y_{n-2} = ct^{n-2}$.

Substituting these into the recurrence relation $y_n = ay_{n-1} + by_{n-2}$, gives $ct^n = act^{n-1} + bct^{n-2}$.

Dividing through by ct^{n-2}, we have $t^2 = at + b$ OR $t^2 - at - b = 0$, which is called the **auxiliary or characteristic** equation of the recurrence relation. The auxiliary or characteristic equation is a quadratic equation and to solve this, we have three possibilities

I. two distinct real roots $t = t_1$ and $t = t_2$

II. repeated real root $t = t_0\,(\text{twice})$

III. two complex roots $t = t_1$ and $t_2 = \overline{t_1}$

CASE I

Since $y_n = t_1^n$ and $y_n = t_2^n$ are solutions of the linear recurrence relation, then another solution (the general solution) is $y_n = At_1^n + Bt_2^n$. Where A and B are arbitrary constants.

Using the initial values $y_0 = c_0$ and $y_1 = c_1$, A and B are easily determined.

CASE II

As $t = t_0$ is the repeated root of the characteristic equation, then $y_n = t_0^n$ is a solution of the recurrence relation. The other linearly independent solution is $y_n = nt_0^n$. So a general solution of the linear recurrence relation is $y_n = At_0^n + Bnt_0^n$, where A and B are arbitrary constants.

Once again, using the initial values $y_0 = c_0$ and $y_1 = c_1$, we can solve for A and B.

CASE III

Since the characteristic equation has real coefficients, the complex roots occur in conjugate pairs. In other words, if $u+iv$ is a root of the characteristic equation with real coefficients, then its complex conjugate $u-iv$ is also a root with $v \neq 0$.

By the general rule, the solution

$$\begin{aligned} y_n &= At_1^n + Bt_2^n \\ &= A(u+iv)^n + B(u-iv)^n \end{aligned}$$

Converting $u+iv$ and $u-iv$ into polar coordinates,

$$u+iv = \rho(\cos\theta + i\sin\theta)$$
$$u-iv = \rho(\cos\theta - i\sin\theta)$$

And by DeMoivre's Theorem,

$$\left[\rho(\cos\theta \pm i\sin\theta)\right]^n = \rho^n(\cos n\theta \pm i\sin n\theta)$$

Then

$$\begin{aligned} y_n &= A\rho^n(\cos n\theta + i\sin n\theta) + B\rho^n(\cos n\theta - i\sin n\theta) \\ &= (A+B)\rho^n(\cos n\theta) + i(A-B)\rho^n \sin n\theta \end{aligned}$$

If we substitute $A = B = \frac{1}{2}$, then $y_n = \rho^n \cos n\theta$ is a particular solution. Similarly if we substitute $A = -\frac{1}{2}i$ and $B = \frac{1}{2}i$, then $y_n = \rho^n \sin n\theta$ is also a particular solution. Therefore the general solution is

$$y_n = \tilde{A}\rho^n \sin n\theta + \tilde{B}\rho^n \cos n\theta.$$

Where $\rho = \sqrt{u^2+v^2}$ and $\theta = \tan^{-1}\frac{v}{u}$.

EXAMPLE 5

Solve

$$\begin{cases} y_n = 3y_{n-1} - 2y_{n-2} \text{ for } n \geq 2 \\ y_2 = 7 \\ y_1 = 3 \end{cases}$$

SOLUTION

The characteristic equation is $t^2 - 3t + 2 = 0$. Factorizing the left-hand side gives

$(t-1)(t-2) = 0$ so that $t = 1$ or $t = 2$. The general solution is $y_n = A \cdot 1^n + B \cdot 2^n$.

Using the initial conditions specified, $A = -1$ and $B = 2$, therefore the solution is

$$\begin{aligned} y_n &= -1 + 2 \cdot 2^n \\ &= 2^{n+1} - 1 \end{aligned}$$

EXAMPLE 6

Solve

$$\begin{cases} y_n = 6y_{n-1} - 9y_{n-2} \text{ for } n \geq 1 \\ y_1 = 3 \\ y_0 = 5 \end{cases}$$

SOLUTION

The characteristic equation is $t^2 - 6t + 9 = 0$. Factorizing the left-hand side gives $(t-3)^2 = 0$ so that $t = 3(\text{repeated})$. The general solution is $y_n = A \cdot 3^n + Bn \cdot 3^n$

.

Using the initial condition specified, $A = 5$ and $B = -4$, therefore the solution is

$$\begin{aligned} y_n &= 5\cdot 3^n - 4n\cdot 3^n \\ &= 3^n(5-4n) \end{aligned}$$

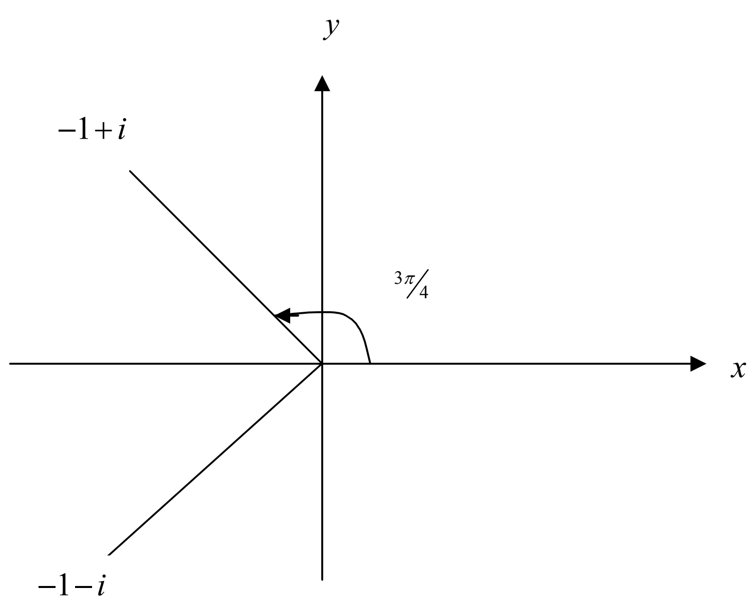
EXAMPLE 7

Solve $y_n + 2y_{n-1} + 2y_{n-2} = 0$

SOLUTION

The characteristic equation is $t^2 + 2t + 2 = 0$. Using the quadratic formula, we have the roots: $t = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. Thus $t_1 = -1 + i$ and $t_2 = -1 - i$.

Using the diagram,



$\rho = \sqrt{2}$ and $\theta = \frac{3\pi}{4}$. Hence, $y_n = A\left(\sqrt{2}\right)^n \sin\frac{3n\pi}{4} + B\left(\sqrt{2}\right)^n \cos\frac{3n\pi}{4}$ is the general solution.

We note that a characteristic equation is equally used for higher-order recurrence relations. Sometimes a recurrence relation can be converted into linear, constant-coefficient form although it might not originally be in that form.

EXAMPLE 8

Solve

$$\begin{cases} y_n = 7y_{n-2} - 6y_{n-3} \\ y_0 = 1,\ y_1 = -2 \text{ and } y_2 = 3 \end{cases}$$

SOLUTION

The characteristic equation is $t^3 - 7t + 6 = 0$. Factorizing the right-hand side gives

$(t-1)(t-2)(t+3) = 0$ so that $t = 1,\ 2$ and -3.

The general solution is $y_n = A \cdot 1^n + B \cdot 2^n + C(-3)^n$.

Using the initial conditions, $A = \frac{5}{4},\ B = -\frac{4}{5}$ and $C = \frac{11}{20}$.

So the solution is $y_n = \frac{5}{4} - \left(\frac{4}{5}\right)2^n + \left(\frac{11}{20}\right)(-3)^n$.

EXAMPLE 9

Consider the recurrence relation:

$$\begin{cases} \sqrt{y_n} = -\sqrt{y_{n-1}} + 6\sqrt{y_{n-2}} \\ y_1 = 1 \\ y_0 = 9 \end{cases}$$

SOLUTION

Using the substitution $g_n = \sqrt{y_n}$ transforms the recurrence relation into

$$
\begin{cases}
g_n = -g_{n-1} + 6g_{n-2} \\
g_1 = 1 \\
g_0 = 3
\end{cases}
$$

Using the appropriate standard method

$g_n = 1\cdot(-3)^n + 2\cdot 2^n$ and therefore $y_n = g_n^2 = \left[1(-3)^n + 2\cdot 2^n\right]^2$

Solve the following difference equations:

(a) $y_{n+1} - ay_n = 0,\ y_0 = 1$

(b) $y_n = -3y_{n-1}$

(c) $y_{n+2} + 2y_n = 0$

(d) $y_n - 2y_{n-1} + 2y_{n-2} = 0$

(e) $y_n + 4y_{n-1} + 8y_{n-2} = 0;\ \ y_0 = 2,\ y_1 = -1$

(f) $y_n - 4y_{n-1} + 8y_{n-2} = 0,\ \ y_2 = 1,\ y_3 = -2$

(g) $y_{n+2} + 2y_{n+1} + 4y_n = 0$

NON-HOMOGENEOUS DIFFERENCE EQUATIONS

So far we have considered difference equations of the form

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \ldots + a_0 y_n = \phi(n) \tag{1}$$

Where $\phi(n) = 0$ for all $n \in \mathbb{Z}^+$. For now, we consider cases where $\phi(n)$ is a non-zero function. In the case of a homogenous difference equations, that is, where $\phi(n) \neq 0$, the general solution of (1) is obtained by adding any particular solution of (1) to the general solution of the corresponding homogenous equation.

THEOREM

Let $\rho(n)$ be a particular solution to

$$a_k y_{n+k} + a_{k-1} y_{n+k-1} + \ldots + a_0 y_n = \phi(n) \tag{2}$$

and

$$\alpha_1 g_1(n) + \alpha_2 g_2(n) + \ldots + \alpha_k g_k(n)$$

Be the general solution to $a_k y_{n+k} + a_{k-1} y_{n+k-1} + \ldots + a_0 y_n = 0$.

Then

$$y(n) = \alpha_1 g_1(n) + \alpha_2 g_2(n) + \ldots + \alpha_k g_k(n) + \rho(n) \tag{3}$$

is the general solution to (2).

EXAMPLE 1

Use the theorem above to find the general solution to

$$y_{n+2} - 5y_{n+1} + 6y_n = 2^{n+2} \tag{4}$$

SOLUTION

We observe that $\rho(n) = -n2^{n+1}$ is a particular solution of (4) for if

$\rho(n) = -n2^{n+1}$ then $\rho(\text{n+1}) = -(n+1)2^{n+2}$ and $\rho(n+2) = -(n+2)2^{n+3}$
and therefore

$$\begin{aligned}\rho(\text{n+2}) - 5\rho(n+1) + 6\rho(n) &= -(n+2)2^{n+3} + 5(n+1)2^{n+2} - 6n2^{n+1} \\ &= 2^{n+1}\left[-2^2(n+2) + 10(n+1) - 6n\right] \\ &= 2^{n+1} \cdot 2 = 2^{n+2}\end{aligned}$$

As to how to have $\rho(n) = -n2^{n+1}$, will be studied later. We find the general solution of the associated homogenous equation

$$y_{n+2} - 5y_{n+1} + 6y_n = 0 \tag{5}$$

The characteristic equation of (5) is: $t^2 - 5t + 6 = 0$

With solution: $t = 2,\ 3$. Thus, the general solution to (5) is $\alpha_1 2^n + \alpha_2 3^n$, and by the Theorem, the general solution of the non-homogenous equation (4)

$y_n = \alpha_1 2^n + \alpha_2 3^n - n2^{n+1}$.

As an emphasis, we need to find only one particular solution of the non-homogenous equation (as well as the general solution to the associated homogenous equation), in order to completely solve the non-homogenous equation. The theorem is useful when $\phi(n)$ in (1) is the sum of various terms.

THEOREM:

If

$F_1(n)$ is a solution of

$$y_{n+k} + a_{n+k-1}y_{n+k-1} + \ldots + a_n y_n = \phi_1(n) \tag{6}$$

and

$F_2(n)$ is a solution of

$$y_{n+k} + a_{n+k-1}y_{n+k-1} + \ldots + a_n y_n = \phi_2(n) \tag{7}$$

then $F_1(n) + F_2(n)$ is a solution of

$$y_{n+k} + a_{n+k-1}y_{n+k-1} + \ldots + a_n y_n = \phi_1(n) + \phi_2(n)$$

Example

Find the general solution of

$$y_{n+2} - 3y_{n+1} + 2y_n = 4(2^n) + n + 2 \tag{8}$$

Solution

The characteristic equation of the associated homogenous equation is

$t^2 - 3t + 2 = (t-2)(t-1) = 0$.

Since the roots of this equation are $t = 1,\ 2$, the general solution of the homogenous equation is $\alpha_1 + \alpha_2 2^n$.

Direct substitution shows that $y_1(n) = n2^{n+1}$ is a solution of the non-homogenous difference equation

$$y_{n+2} - 3y_{n+1} + 2y_n = 4(2^n) \tag{9}$$

And that $F_2(n) = -\frac{1}{2}n^2 - \frac{5}{2}n$ is a solution of the non-homogenous difference equation

$$y_{n+2} - 3y_{n+1} + 2y_n = n + 2 \tag{10}$$

Thence from the theorem, $F_1(n)+F_2(n)=n2^{n+1}-\frac{1}{2}n^2-\frac{5}{2}n$ is a particular solution to (8). Adding this particular solution to the general solution of the homogenous equation gives $y_n=\alpha_1+\alpha_2 2^n+n2^{n+1}-\frac{1}{2}n^2-\frac{5}{2}n$ as the general solution of the non-homogenous difference equation (8).

ACTIVITY

In the problems below, show that $\rho(n)$ is a particular solution of the given difference equation, and find the general solution to the difference equation. If initial conditions are given, find also the particular solution that satisfies those conditions.

(1) $y_{n+2}-3y_{n+1}+2y_n=1;\ \rho(n)=-n$

(2) $y_{n+2}-2y_{n+1}+y_n=5+3n;\ \rho(n)=n^2+\dfrac{1}{2}n^3$

(3) $y_{n+2}-3y_{n+1}+2y_n=3^n;\ \rho(n)=\left(\dfrac{1}{2}\right)3^n,\ y_0=1,\ y_1=0$

(4) $y_{n+2}-4y_{n+1}+4y_n=2^n;\ \rho(n)=n(n-1)2^{n-3}$

(5) $y_{n+2}-4y_n=2^n;\ \rho(n)=\dfrac{1}{8}n2^n,\ y_1=3,\ y_2=1$

(6) $y_{n+2}-4y_n=n^2-1;\ \rho(n)=-\dfrac{n^2}{3}-\dfrac{4}{9}n-\dfrac{11}{27}$

(7) $y_n-10y_{n-1}+25y_{n-2}=2^n;\ \rho(n)=\left(\dfrac{1}{9}\right)2^{n+2}$

(8) $y_n-4y_{n-1}+4y_{n-2}=2^n;\ \rho(n)=n2^{n-3}$

(9) $y_n-3y_{n-1}+2y_{n-2}=2^n;\ \rho(n)=n^{2+1}$

(10) $y_{n+2}+5y_{n+1}-6y_n=n;\ \rho(n)=\dfrac{1}{14}\left(n-\dfrac{9}{14}\right)^2$

(11) $y_{n+2} + y_n = n + 1;\ \ \rho(n) = \dfrac{n}{2},\ y_0 = 1,\ y_1 = 0$

(12) $y_{n+2} + y_{n+1} - 3y_n = -2n + 1;\ \ \rho(n) = 2n + 3$

(13) $y_{n+1} = y_n + 1;\ \ \rho(n) = n,\ y_1 = 1$

(14) $y_{n+1} = 3y_n + 2;\ \ \rho(n) = 3^n - 1$

(15) $y_{n+2} + 5y_{n+1} + 6y_n = 4^{n+1};\ \ \rho(n) = \left(\dfrac{2}{21}\right)4^n$

(16) $y_n + 2y_{n-1} + y_{n-2} = 3^{n-2};\ \ \rho(n) = \dfrac{\left(3^{n-2}\right)}{16}$

THE METHOD OF UNDETERMINED COEFFICIENTS

The method of undetermined coefficients is introduced with an example. We consider the difference equation:

$$y_{n+2} + 2y_{n+1} - y_n = 6n^2 + 24n + 25 \qquad (1)$$

Clearly the sum $y_{n+2} + 2y_{n+1} - y_n$ will be equal to the polynomial $6n^2 + 24n + 25$ if and only if y is a second-degree polynomial. For a solution of (1), we consider

$y_n = y(n) = A_2n^2 + A_1n + A_0$, where A_2, A_1 and A_0 are constants to be determined. This is reasonably possible by substituting $y_n = A_2n^2 + A_1n + A_0$ into (1).

We however observe that if $y_n = A_2n^2 + A_1n + A_0$, then

$$\begin{aligned} y_{n+1} &= y(n+1) = A_2(n+1)^2 + A_1(n+1) + A_0 \\ &= A_2n^2 + (2A_2 + A_1)n + (A_2 + A_1 + A_0) \\ y_{n+2} &= A_2(n+2)^2 + A_1(n+2) + A_0 \\ &= A_2n^2 + (4A_2 + A_1)n + (4A_2 + 2A_1 + A_0) \end{aligned}$$

Consequently

$$\begin{aligned} y_{n+2} + 2y_{n+1} - y_n = {} & A_2n^2 + (4A_2 + A_1)n + (4A_2 + 2A_1 + A_0) \\ & + 2A_2n^2 + 2(2A_2 + A_1)n + 2(A_2 + A_1 + A_0) \\ & - A_2n^2 - A_1n - A_0 \\ = {} & 2A_2n^2 + (8A_2 + 2A_1)n + (6A_2 + 4A_1 + 2A_0) \end{aligned}$$

Thus for $y_n = A_2n^2 + A_1n + A_0$ to be a solution to (1), then

$$2A_2n^2 + (8A_2 + 2A_1)n + (6A_2 + 4A_1 + 2A_0) = 6n^2 + 24n + 25.$$

This equation is valid for all n if and only if the corresponding coefficients are equal. Thus

$$2A_2 = 6$$
$$8A_2 + 2A_1 = 24$$
$$6A_2 + 4A_1 + 2A_0 = 25$$

From these,

$$A_2 = 3, \quad A_1 = 0, \; A_0 = \frac{7}{2}$$

Substituting these values into $y_n = A_2 n^2 A_1 n + A_0$, we have $\rho(n) = 3n^2 + \frac{7}{2}$ as a particular solution to the non-homogenous equation (1).

To apply the method successfully, it is necessary to have the correct form for a trial solution. The following table gives the correct form of a trial solution to a difference equation: $y_{n+k} + a_{n+k-1} y_{n+k-1} + \ldots + a_n y_n = \phi(n)$

For various functions $\phi(n)$.

$\phi(n)$	Form of trial solution
$a_m n^m + a_{m-1} n^{m-1} + \ldots + a_1 n + a_0$ $k^n \left(a_m n^m + a_{m-1} n^{m-1} + \ldots + a_1 n + a_0\right)$	$A_m n^m + A_{m-1} n^{m-1} + \ldots + A_1 n + A_0$ $k^n \left(A_m n^m + A_{m-1} n^{m-1} + \ldots + A_1 n + A_0\right)$

Example 1

Find the general solution of

$$y_n - 3y_{n-1} - 10y_{n-2} = 3\left(2^n\right) \tag{2}$$

Solution:

The characteristic equation of the homogenous equation is

$t^2 - 3t - 10 = 0$ OR $(t-5)(t+2) = 0 \ \Rightarrow t = 5, \ -2$.

Therefore the general solution of the homogenous equation is: $\alpha_1 5^n + \alpha_2(-2)^n$, since $\phi(n) = 3(2^n)$ is of the form ck^n. By the table above, $c = 3$ and $k = 2$, we select a trial solution of (2) in the form $y_n = c2^n$. Substituting this into (2), we have

$$c2^n - 3c2^{n-1} - 10c2^{n-2} = 3(2^n) \tag{3}$$

Dividing through this by 2^{n-2}, we have

$$c2^2 - 3c2 - 10c = 3(2^2) \text{ or } -12c = 12$$

so that $c = -1$

Therefore $y_n = -2^n$ is a particular solution of (2).

Consequently by the above theorem, $y_n = \alpha_1 5^n + \alpha_2(-2)^n - 2^n$ is a general solution of (2).

Example 2

Find the general solution of

$$y_{n+2} - y_{n+1} - 6y_n = 6n^2 + 22n + 23 \tag{4}$$

Solution:

The characteristic solution of the associated homogenous equation is

$t^2 - t - 6 = 0$ or $(t-3)(t+2) = 0$ then $t = 3, \ -2$.

Therefore the general solution of the homogenous equation is $\alpha_1(-2)^n + \alpha_2 3^n$, since $\phi(n) = 6n^2 + 22n + 23$ is of the form $a_1 n^2 + a_1 n + a_0$.

By the table, we select the trial solution: $y_n = A_2 n^2 + A_1 n + A_0$.

Substituting $y_n = A_2 n^2 + A_1 n + A_0$ into (4) yields

$$\left[A_2(n+2)^2 + A_1(n+2) + A_0\right] - \left[A_2(n+1)^2 + A_1(n+1) + A_0\right] - 6\left[A_2 n^2 + A_1 n + A_0\right]$$

This reduces to $-6A_2 n^2 + (2A_2 - 6A_1)n + (3A_2 + A_1 - 6A_0) = 6n^2 + 22n + 23$.

In order for this equation to be valid for all n, the corresponding coefficients must be equal hence,

$-6A_2 = 6,\ 2A_2 - 6A_1 = 22,\ 3A_2 + A_1 - 6A_0 = 23.$

From these, $A_2 = -1,\ A_1 = -4$ and $A_0 = -5$ so that

$\rho(n) = -n^2 - 4n - 5$

is a particular solution to the non-homogenous equation (4).

The general solution, therefore, is $y_n = \alpha_1(-2)^n + \alpha_2 3^n - n^2 - 4n - 5$

A slight complication arises in the use of the table whenever a term of the selected trial solution is also a solution to the homogenous equation. If this occurs, we multiply every term of the trial solution by n. If this new trial solution still has a term that is a solution to the homogenous equation, we again multiply the trial solution by n. This process continues until no term of the trial solution is a solution of the homogenous equation.

For example, if $\phi(n) = n^2 + 1$, then the trial solution would ordinarily be of the form $A_2 n^2 + A_1 n + A_0$. Say, however, that the general solution of the homogenous equation is $\alpha_1 n + \alpha_2$. Then both $A_1 n$ and A_0 are solutions of the homogenous equation. Multiplying the original trial solution by n gives $A_2 n^3 + A_1 n^2 + A_0 n$ as a trial solution. However, since $A_0 n$ is still a solution to the homogenous equation, we again multiply by n to obtain $A_2 n^4 + A_1 n^3 + A_0 n^2$.

Since none of the terms $A_2 n^4$, $A_1 n^3$ or $A_0 n^2$ of the proposed solution is a solution to the homogenous equation, $A_2 n^4 + A_1 n^3 + A_0 n^2$ is the desired trial solution.

Example 3

Find the general solution of

$$y_n - 4y_{n-1} + 4y_{n-2} = 2^n \qquad (5)$$

SOLUTION

The characteristic equation of the associated homogenous equation is

$t^2 - 4t + 4 = 0$ or $(t-2)^2 = 0$, then $t = 2$ (repeated).

Therefore the general solution of the homogenous equation is $\alpha_1 2^n + \alpha_2 n 2^2$.

But $\phi(n) = 2^n$ so we are prompted to select our trial solution as $c2^n$, where c is a constant.

However $c2^n$ and $cn2^n$ are both solutions to the homogenous equation and, therefore, we multiply 2^n by n^2 to use $cn^2 2^n$ as the form of the trial solution. Substituting $y_n = cn^2 2^n$ into the non-homogenous equation, we have

$$cn^2 2^n - 4cn^2 2^{n-1} + 8cn2^{n-1} - 4c2^{n-1} + 4cn^2 2^{n-2} - 16cn2^{n-2} + 16c2^{n-2} = 2^n .$$

Dividing throughout by 2^{n-2}, we obtain

$4cn^2 - 8cn^2 + 16cn - 8c + 4cn^2 - 16cn + 16c = 4$ from which we have $8c = 4$ or $c = \frac{1}{2}$. Consequently, $\rho(n) = \frac{1}{2}n^2 2^n = n^2 2^{n-1}$ is a particular solution to the non-homogenous equation (5) and therefore the general solution of (5) is $y_n = \alpha_1 2^n + \alpha_2 n 2^2 + n^2 2^{n-1}$.

Example 4

Find the general solution of

$$y_{n+2} + w^2 y_n = 2^n; \;\; y_0 = \alpha, \; y_1 = \beta \tag{6}$$

Solution

The characteristic equation is $t^2 + w^2 = 0$ or $t = \pm\, iw$.

The general solution of the associated homogenous equation is

$$\begin{aligned}
\alpha_1 (iw)^n + \alpha_2 (-iw)^n &= w^n \left[\alpha_1 i^n + \alpha_2 (-i)^n \right] \\
&= w^n \left[\alpha_1 e^{n\pi i/2} + \alpha_2 e^{-n\pi i/2} \right] \text{ since } \left(i = e^{\pi i/2} \right) \\
&= w^n \left[\alpha_1 \left\{ \cos\tfrac{n\pi}{2} + i \sin\tfrac{n\pi}{2} \right\} + \alpha_2 \left\{ \cos\tfrac{n\pi}{2} - i \sin\tfrac{n\pi}{2} \right\} \right] \\
&= w^n \left[A \cos\tfrac{n\pi}{2} + B \sin\tfrac{n\pi}{2} \right]
\end{aligned}$$

To find a particular integral, our trial solution is $\rho(n) = c2^n$. Substituting this into the non-homogenous equation, we have

$c2^{n+2} + w^2 c 2^n = 2^n$ so that $c(4 + w^2) = 1$ or $c = \dfrac{1}{4 + w^2}$.

The general solution of the non-homogenous equation is

$$y_n = \frac{2^n}{4 + w^2} + w^n \left[A \cos\tfrac{n\pi}{2} + B \sin\tfrac{n\pi}{2} \right] \tag{7}$$

Finally we use the initial conditions $y_0 = \alpha$ and $y_1 = \beta$ to find the values of A and B in (6).

$$y_0 = \frac{1}{4+w^2} + A = \alpha, \quad y_1 = \frac{2}{4+w^2} + wB = \beta \text{ so that}$$

$$A = \alpha - \frac{1}{4+w^2} \qquad B = \frac{\beta}{w} - \frac{2}{w\left(4+w^2\right)}$$

Therefore the solution is

$$y_n = \frac{2^n}{4+w^2} + w^n\left[\left(\alpha - \frac{1}{4+w^2}\right)\cos\tfrac{n\pi}{2} + \left(\frac{\beta}{w} - \frac{2}{w\left(4+w^2\right)}\right)\sin\tfrac{n\pi}{2}\right]$$

Finally, we give an example of an important use of difference equations. Most especially, we use difference equations to evaluate certain sums.

Example 5

If $s_n = s(n) = \sum_{k=1}^{n} k^2$, find a simple expression for the summation.

Solution:

We write

$$s_n - s_{n-1} = n^2 \qquad (8)$$

This is a non-homogenous equation. We find its general solution. The associated homogenous equation is $s_n - s_{n-1} = 0$, whose general solution is α, since $s_n = s_{n-1} = s_{n-2} = \ldots = s_0 = \alpha$, a constant. Since, $\phi(n) = n^2$, we have a trial solution of the form $A_2 n^2 + A_1 n + A_0$.

However since A_0 is a constant and is also a solution of the homogenous equation, we multiply this trial solution through by n. The resulting trial solution is $A_2n^3 + A_1n^2 + A_0n$.

Substituting this into (8), we have $A_2n^3 + A_1n^2 + A_0n - A_2(n\text{-}1)^2 - A_0(n\text{-}1) = n^2$. Simplifying this, we select A_2, A_1 and A_0; $3A_2n^2 + (2A_1 - 3A_2)n + (A_0 - A_1 + A_2) = n^2$ for all integers n, from which it follows that

$3A_2 = 1, 2A_1 - 3A_2 = 0$ and $A_0 - A_1 + A_2 = 0$.

Therefore $A_2 = \frac{1}{3}$, $A_1 = \frac{1}{2}$ and $A_0 = \frac{1}{6}$, *so*

$\rho(n) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ is a particular solution of (8).

Adding this to the general solution of
the homogenous solution,

$s_n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + \alpha$. Since $s_1 = 1$, we have $1 = \frac{1}{3} + \frac{1}{2} + \frac{1}{6} + \alpha$,

which gives $\alpha = 0$, so $s_n = \sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$ (9)

ACTIVITY

a. In the following problems, find the general solution if the initial conditions are given, find the particular solution that satisfies those conditions.

(1) $y_{n+2} - y_{n+1} - 2y_n = 3^n$

(2) $y_n + 2y_{n-1} - 8y_{n-2} = 5^{n-2}$

(3) $y_n - 4y_{n-1} + 3y_{n-2} = -4$

(4) $y_{n+2} - y_{n+1} - 2y_n = 2^n;\ y_0 = 2,\ y_1 = 1$

(5) $y_{n+2} - 2y_{n+1} + y_n = -4;\ y_1 = 2,\ y_2 = -1$

(6) $y_{n+2} - 3y_{n+1} - 10y_n = 36n - 21$

(7) $y_n - 9y_{n-2} = n^2 - 4n - 1$

(8) $y_{n+2} - 9y_n = n^2 - 5n + 3^n$

(9) $y_{n+2} + 2y_{n+1} - 8y_n = -5n + 14;\ y_0 = 0,\ y_1 = -1$

(10) $2y_{n+1} - 3y_n - 5y_{n-1} = 5^{n-1} - 4$

(11) $y_n - 4y_{n-2} = -6n^2 + 32n - 23 - 2^{n-1}$

(12) $y_n + 6y_{n-1} + 12y_{n-2} + 8y_{n-3} = 3^n$

b. In the following problems, use the techniques of example 5 of this section to find simple expression for the given sum.

(13) $\sum_{k=1}^{n} k$

(14) $\sum_{k=1}^{n} r^k$

(15) $\sum_{k=1}^{n} (-1)^k k$

(16) $\sum_{k=1}^{n} 2^k$

(17) $\sum_{k=1}^{n} 2^k k$

(18) $\sum_{k=1}^{n} (-1)^k k^2$

(19) Use induction to show that for each positive integer n

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n .$$

Unit 4

BOOLEAN ALGEBRA AND BOOLEAN FUNCTIONS

INTRODUCTION

We shall have an insight into the basic mathematical logic behind language of computer usage. An understanding of this unit will ease up complex structures in the complex language of the computer. You are encouraged to put in some effort to follow this unit.

BASIC DEFINITIONS AND THEOREMS

DEFINITION (BOOLEAN ALGEBRA)

Let B be a set with two binary defined operators $+$ and $*$, and a unary operation, denoted by $'$; let 0 and 1 denote two distinct elements of B. Then the sextuplet

$(B,\ +,\ *,\ ',\ 0,\ 1)$ is called a **Boolean algebra** if the following axioms hold for any elements: $a, b, c \in B$;

B_1: **commutative laws:**

$$a + b = b + a$$
$$a * b = b * a$$

B_2: **Distributive laws:**

$$a + (b * c) = (a + b) * (a + c)$$
$$a * (b + c) = (a * b) + (a * c)$$

B_3: **Identity Laws**

$$a + 0 = a$$
$$a * 1 = a$$

B_4 : **Complement Laws**

$$a + a' = 1$$
$$a * a' = 0$$

The sextuplet is denoted by B when the operations are understood. The element 0 is called the **zero** element, the element 1 is called the **unit** element, and a' is called the **complement** of a. The results of the operations $+$ and $*$ are called the **sum** and **product** respectively. The following convention, unless guided by parenthesis, is that $'$ has precedence over $*$, and $*$ has precedence over +.

For example

$a + b * c$ means $a + (b + c)$ not $(a + b) * c$

$a * b'$ means $a * (b)'$ and not $(a * b)'$

THE BOOLEAN ALGEBRA B WITH TWO ELEMENTS 0 AND 1 (CALLED BITS)

Let $+$ and $*$ be the binary operations in B. Let $0' = 1$ and $1' = 0$.

•	'
1	0
0	1

Then we have the following two fundamental tables:

+	1	0
1	1	1
0	1	0

*	1	0
1	1	0
0	0	0

Example 1

Find the value of $1*0+(0+1)'$

Solution:

From the tables:

$(0+1)=1$

and $(0+1)'=1'=0$

$$\begin{aligned} 1*0+(0+1)' &= 1*0+0 \\ &= 0+0 \\ &= 0 \end{aligned}$$

The complement, Boolean sum and Boolean product correspond to the logical operators $\sim$, $\vee$ and $\wedge$ respectively, where 0 corresponds to F (false) and 1 to T (true).

BOOLEAN EXPRESSIONS AND BOOLEAN FUNCTIONS

Let $B = \{0,1\}$. The variable x is called a **Boolean variable** if it takes values only from B. A function $F : B^n = \{(x_1, x_2, ..., x_n) | x_i \in B,\ 1 \le i \le n\} \to B$ is called a **Boolean function of degree n.** The values of a Boolean function are displayed in tables.

Example 2

The Boolean function $F(x, y)$ with the value 1 when $x = 1$ and $y = 0$ and the value 0 for all other choice of x and y is represented by the table

x	y	$F(x,y)$
1	1	0
1	0	1
0	1	0
0	0	0

Boolean functions are represented by Boolean expressions made up of the variables and the Boolean operations. Boolean algebra $(B,\ +,\ *,\ ',\ 0,\ 1)$ are defined recursively as

For each $s \in B$, s is a Boolean expression. $x_1, x_2, ..., x_n$ are Boolean expressions.

If x_1 and x_2 are Boolean expressions, so are $x_1', x_2', x_1 + x_2$ and $x_1 * x_2$.

Each Boolean expression represents a Boolean function. The values of this function are obtained by substituting 0 and 1 for the variables in the expression.

Example 3

Find the values of the Boolean function represented by $F(x,y,z) = xy + z'$

Solution:

x	y	z	$xy = x * y$	z'	$F(x,y,z)$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

The Boolean functions F and G of n variables are equal if and only if $F(b_1, b_2, ..., b_n) = G(b_1, b_2, ..., b_n)$ whenever $b_1, b_2, ..., b_n \in B$.

Two different expressions that represent the same function are called **equivalent**. For example, the Boolean expressions: $xy,\ xy + 0,\ xy * 1$ are equivalent.

The **complement** of the Boolean function F is the function F'.

Let F and G be Boolean functions of degree n. The **Boolean sum** $F+G$ and the **Boolean product** $F*G$ are defined by

$$(F+G)(x_1,x_2,...,x_n)=F(x_1,x_2,...,x_n)+G(x_1,x_2,...,x_n)$$
$$(F*G)(x_1,x_2,...,x_n)=F(x_1,x_2,...,x_n)*G(x_1,x_2,...,x_n)$$

Recalling the definition of a function or a given domain into a co-domain using exponentiation, we have $b(\{f:f:B\to A\})=b(A)^{b(B)}$. Where b denotes the cardinality or distinct number of elements in a given set.

Therefore a Boolean function of degree 2, by definition, is a function from a set with four elements, namely, pairs of elements from $B=\{0,1\}$, to B, a set of two elements. Hence, there are $2^4=16$ different Boolean functions of degree 2. That is, we want to find $b\left(F:B^2=B*B=\{x_1,x_2\}:x_1\in B=\{0,1\}\to B\right)$.

In a similar analysis, the different Boolean functions of degree n is 2^{2n}.

IDENTITIES OF BOOLEAN ALGEBRA

There are many identities of Boolean algebra but we provide the most important of them as displayed in the following table:

BOOLEAN IDENTITES

Identity	Name
$(x')' = x$	Law of double complement
$x + x = x$ $x \cdot x = x$	Idempotent laws
$x + 0 = x$ $x \cdot 0 = 0$	Identity laws
$x + 1 = 1$ $x \cdot 0 = 0$	Dominance laws
$x + y = y + x$ $xy = yx$	Commutative laws
$x + (y + z) = (x + y) + z$ $x(yz) = (xy)z$	Associative laws
$x + (yz) = (x + y)(x + z)$ $x(y + z) = xy + xz$	Distributive laws
$(xy)' = x' + y'$ $(x + y)' = x' \cdot y'$	De Morgan’s laws

Example 4

Show that the distributive law $x(y+z)=xy+xz$ is valid.

Solution:

The verification is shown in the following table

x	y	z	$y+z$	xy	xz	$x(y+z)$	$xy+xz$
1	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1
1	0	1	1	0	1	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

The identity holds because the last two columns of the table agree. The basic important identities summarized in the previous table can be used to prove further identities.

Example 5

Prove the absorption law $x(x+y)=x$, verify the identities of Boolean algebra.

Solution:

The steps used to derive this identity and the law used in each step follows:

$x(x+y)=(x+0)(x+y)$ Identity law for the Boolean sum

$=x+(0\cdot y)$ Distributive law of the Boolean sum over the Boolean product

$=x+y\cdot 0$ Commutative law for the Boolean product

$=x+0$ Dominance law for the Boolean product

$=x$ Identity law for Boolean sum

DULITY

Observe that in the table of Boolean Identities, the identities come in pairs (except for the double complement). To fully explain the relationship between the two identities in each pair, we use the concept of a **dual**. The **dual** of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0's and 1's.

Example 6

Find the duals of $x(y+0)$ and $x'\cdot 1+(y'+z)$.

Solution

Interchanging * signs and + signs and interchanging 0's and 1's in these expressions produces their duals. The duals are $x+(y\cdot 1)$ and $(x'+0)(y'z)$ respectively.

The dual of a Boolean function F represented by a Boolean expression is the function represented by the dual of this expression. This dual function, denoted by F^d, does not depend on the particular Boolean expression used to represent F.

An identity between functions represented by Boolean expressions remains valid when the duals of both sides of the identity are taken. That is, if F and G are Boolean functions represented by Boolean expressions in n variables and $F = G$, then $F^d = G^d$, where F^d and G^d are the Boolean functions represented by the duals of the Boolean expressions representing F and G, respectively. This result, called the **duality principle** is useful for obtaining new identities.

Example 7

Construct an identity from the absorption law: $x(x+y)=x$ given in example 5 by taking duals.

Solution:

Taking duals of both sides of this identity produces the identity $x+(xy)=x$, which is also called an absorption law.

1. Find the values of the following expressions

   (a) $1\cdot 0'$ (b) $1+1'$ (c) $0'\cdot 0$ (d) $(1+0)'$

2. Find the values, if any, of the Boolean variable x that satisfy the following equations:

   (a) $x\cdot 1=0$ (b) $x+x=0$ (c) $x\cdot 1=x$ (d) $x\cdot x'=1$

   [Hint: use tables for $x\in\{0,1\}$]

3. What values of the Boolean variables x and y satisfy $xy=x+y$? [Hint: use table]

4. How many different Boolean functions are there of degree 7?

5. Prove the absorption law $x + xy = x$ using the laws in table 5.
6. Show that $F(x, y, z) = xy + xz + yz$ has the value 1 if and only if at least two of the variables x, y and z have 1. (use tables)
7. Show that $xy' + yz' + x'z = x'y + y'z + xz'$. (use tables)

Exercise 8-15 deal with the Boolean algebra defined by the Boolean sum and Boolean product on $\{0,1\}$

8. Verify the law of double complement
9. Verify the idempotent laws
10. Verify the identity laws
11. Verify the dominance laws
12. Verify the commutative laws
13. Verify the associative laws
14. Verify the first-distributive law in Table 5
15. Verify De Morgan's laws.

The operator $\oplus$, called the xOR operator, is defined by

(a) $1 \oplus 1 = 0,$ $1 \oplus 0 = 1$ $1 \oplus 0 = 1$ and $0 \oplus 0 = 0$

16. Simplify the following expressions

    (a) $x \oplus 0$ (b) $x \oplus 1$ (c) $x \oplus x$ (d) $x \oplus x'$

    (use tables)
17. Show that the following identities hold:

    (a) $x \oplus y = (x + y)(xy)'$ (b) $x \oplus y = (xy') + (x'y)$
18. Show that $x \oplus y = y \oplus x$

19. Prove or disprove the following equalities

(a) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

(b) $x + (y \oplus z) = (x + y) \oplus (x + z)$

(c) $x \oplus (y + z) = (x \oplus y) + (x \oplus z)$

REPRESENTATION BOOLEAN FUNCTIONS

Two important problems of Boolean algebra will be examined.

The first problem is: Given the values of a Boolean function, can the function be found? This problem will be solved by showing that any Boolean function may be represented by a Boolean sum or Boolean product of the variables and their complements, that is, every Boolean function can be represented using the three Boolean operators: $\cdot$, $+$ and $'$.

The second problem is: Is there a smaller set of operators that can be used to represent all Boolean functions? This will be answered by showing that all Boolean functions can be represented using only one operator.

The foregoing problems have practical importance in circuit design.

SUM OF PRODUCTS EXPANSIONS

We use examples to illustrate one important way to find a Boolean expression that represents a Boolean function.

Example 1

Find Boolean expressions that represent the functions $F(x, y, z)$ and $G(x, y, z)$, which are given in Table 1.

TABLE 1

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

Solution

F has value 1 when $x = z = 1$ and $y = 0$ and θ otherwise. Such an expression is formed by taking the Boolean product of x, y' and z. This product $xy'z$ has value 1 if and only if $x = y' = z = 1$, which holds if and only if $x = z = 1$ and $y = 0$.

To represent G, we need an expression that equals 1 when $x = y = 1$ and $z = 0$, **or** when $x = z = 0$ and $y = 1$. We can form an expression with these values by taking the Boolean sum of two different Boolean products. The Boolean product xyz' has the value 1 if and only if $x = z = 0$ and $y = 1$. The Boolean sum of these two products $xyz' + x'yz'$ represent G, since it has the value 1 if and only if $x = y = 1$ and $z = 0$ or $x = z = 0$ and $y = 1$.

DEFINITION

A **literal** is a Boolean variable or its complement. A **minterm** of the Boolean variables $x_1, x_2, ..., x_n$ is a Boolean product $y_1, y_2, ..., y_n$ where $y_i = x_i$ or $y_i = x_i^{'}$. Hence, a minterm is a product of n literals with one literal for each variable.

Example 2

Find a minterm that equals 1 if $x_1 = x_3 = 0$ and $x_2 = x_4 = x_5 = 1$, and equals 0 otherwise.

Solution

The minterm $x_1^{'} x_2 x_3^{'} x_4 x_5$ has the correct set of values.

By taking Boolean sums of distinct minterms we can build up a Boolean expression with a specified set of values. In particular, a Boolean sum of minterms has the value 1 when exactly one of the minterms in the sum has the value 1. Consequently, given a Boolean function, a Boolean sum of minterms can be formed that has the value 1 when this Boolean function has the value 1. The minterms in this Boolean sum correspond to those combinations of values for which the function has the value 1. The sum of minterms that represents the function is called the **sum of products expansion** or the **disjunction normal forms** of the Boolean function.

Example 3

Find the sum of products expansion for the function $F(x,y,z)=(x+y)z'$

Solution

The first step is to find the values of F. These are found in table 2. The sum-of-products expansion of F is the Boolean sum of three minterms corresponding to the three rows of this table that give the value 1 for the function.

This gives $F(x,y,z)=xyz'+xy'z'+x'yz'$.

Table 2

x	y	z	$x+y$	z'	$(x+y)z'=F$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

It is also possible to find a Boolean expression that represents a Boolean function by taking a Boolean product of Boolean sums. The resulting expansion is called the **conjugate normal form** or **product-of-sums expansion** of the function. These expansions can be found from the sum-of-products expansions by taking duals.

FUNCTIONAL COMPLETENESS

Every Boolean function can be expressed as a Boolean sum of minterms, which are Boolean products of Boolean variables or their complements. This shows that every Boolean function can be represented using the Boolean operations $\cdot$, $+$ and $'$. We therefore say that the set $\{\cdot,\ +,\ '\}$ is **functionally complete**.

Can we find a smaller set of functionally complete operators? This is achievable if one of the three operators of this set can be expressed in terms of the other two.

There are two such procedures, in a De Morgan's laws:

I. Eliminate all Boolean sums using the identity

   $x+y=(x'y')'$

   Thus the set $\{\cdot,\ '\}$ is functionally complete.

II. Eliminate all Boolean products using the identity

   $xy=(x'+y')'$

Thus the set $\{+,\ '\}$ is functionally complete.

However, the set $\{\cdot,\ +\}$ is not functionally complete.

Can we find a smaller set of functionally complete operators, namely, a set containing just one operator? Such sets exist.

1	1	0
1 0	0 1	1 1

$\downarrow$	1	0
1	0	0
0	0	1

Define two operator | or NAND (not AND) operator, defined by

$1|1=0$ and $1|0=0|1=0|0=0;$ and $\downarrow$ or NOR (not OR)

defined by $1\downarrow 1=1\downarrow 0=0\downarrow 1=0$ and $0\downarrow 0=1$.

Both of the sets {|} is functionally complete. Since $\{\cdot, '\}$ is functionally complete, all we need to do is show that both operators $\cdot$ and $'$ can be expressed using just | operator. This is done as follows:

$$x'=x|x$$

$$xy=(x|y)|(x|y)$$

Exercise

1. Find a Boolean product of the Boolean variable x, y and z or their complements, that has the value 1 if and only if

   (a) $x=y=0,\ z=1$

   (b) $x=0,\ \ y=1\ \ z=0$

   (c) $x=0,\ \ y=z=1$

   (d) $x=y=z=0$

2. Find the sum-of-products expansions of the following Boolean functions

(a) $F(x,y) = x' + y$
(b) $F(x,y) = xy'$
(c) $F(x,y) = 1$
(d) $F(x,y) = y'$

3. Find the sum-of-products expansions of the following Boolean functions
   (a) $F(x,y,z) = x + y + z$
   (b) $F(x,y,z) = (x+z)y$
   (c) $F(x,y,z) = x$
   (d) $F(x,y,z) = xy'$

4. Find the sum-of-products expansions of the Boolean function $F(x,y,z)$ that equals 1 if and only if
   (a) $x = 0$ (b) $xy = 0$
   (c) $x + y = 0$ (d) $xyz = 0$

5. Find the sum-of-products of the Boolean function $F(w,x,y,z)$ that has the value 1 if and only if are odd number of w, x, y and z have the value 1

6. Find the sum-of-products expansions of the Boolean function $F(x_1,x_2,x_3,x_4,x_5)$ that has the value 1 if and only if three or more of the variables x_1,x_2,x_3,x_4,x_5 have the value 1.

In question numbers 7-11, find a Boolean expression that represents a Boolean function formed from a Boolean product of Boolean sums of literals.

7. Find a Boolean sum containing either x or x', either y or y' and either z or z' that have the value 0 if and only if
   (a) $x = y = 1,\ z = 0$
   (b) $x = y = z = 0$
   (c) $x = z = 0, y = 1$

8. Find a Boolean product of Boolean sums of literals that has the value 0 if and only if either $x = y = 1$ and $z = 0, x = z = 0$ and $y = 1$ or $x = y = z = 0$ (Hint: take the Boolean product of Boolean sums found in parts (a), (b) and (c) in7).

9. Show that the Boolean sum $y_1 + y_2 + ... + y_n$, where $y_i = x_i$ and $x_i = 1$ if $y_i = x_i^{'}$. This Boolean sum is called a **maxterm**.

10. Show that a Boolean function can be represented as a Boolean product of maxterms. This representation is called the **product-of-sums expansion or conjugate normal form** of the function. (Hint: include one maxterm in this product for each confirmation of the variables where the function has the value 0).

11. Find the product-of-sums expansion of each of the Boolean functions in (3).

12. Express each of the following Boolean functions using the operators $\cdot$ and $'$.

(a) $x+y+z$

(b) $x+y'(x'+z)$

(c) $(x+y')'$

(d) $x'(x+y'+z')$

13. Express each of the Boolean functions in (12) using the operators $\cdot$ and $'$.

14. Show that

(a) $x' = x|x$ $\qquad$ (b) $xy = (x|y)|(x|y)$

(c) $x+y = (x|x)|(y|y)$

15. Show that

(a) $x' = x \downarrow x$ $\qquad$ (b) $xy = (x \downarrow x) \downarrow (y \downarrow y)$

(c) $x+y = (x \downarrow y) \downarrow (x \downarrow y)$

16. Show that $\{\downarrow\}$ is functionally complete using (15)

17. Express each of the Boolean functions in (3) using the operator $|$.

18. Express each of the Boolean functions in (3) using the operator $\downarrow$.

LOGIC GATES

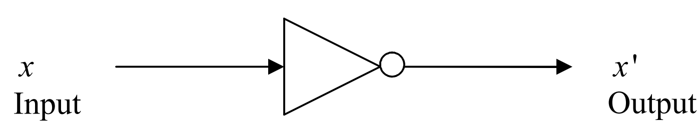
We use Boolean algebra to model the circuiting of electronic devices. We take each input and each output of such a device as a member of the set $\{0,1\}$

.

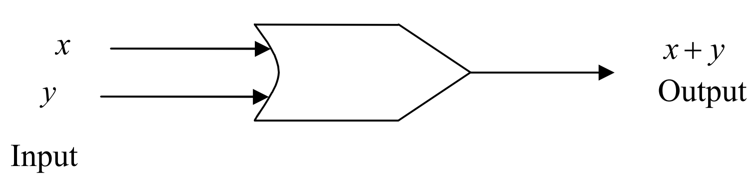
A computer, or any electronic device, is made up of circuits. Each circuit is designed using the rules of Boolean algebra so far studied. The basic elements of circuits are called **gates**, so that each type of gate implements a Boolean operation. With these gates, we apply the rules of Boolean algebra to design circuits to perform variety of tasks. Such circuits give **output**, that depends only on the **inputs**, and not on memory capacities. Such circuits are called **combinatorial circuits**.

Combinatorial circuits are constructed using three basic types of elements which we describe as follows:

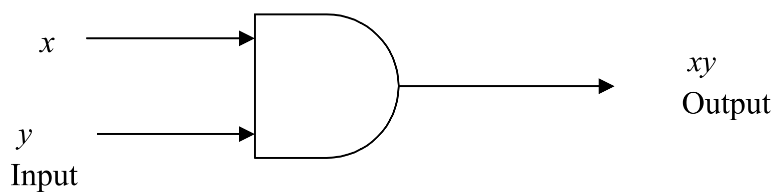
I. **An inverter**: This accepts the value of a Boolean variable as an input and produces its complement as its output. This is represented by



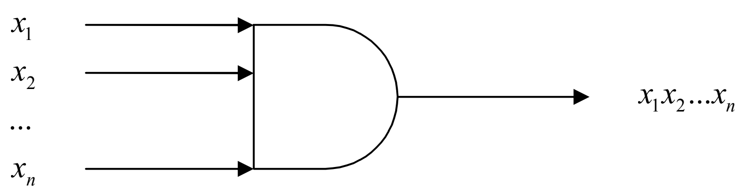
II. **OR gate:** The inputs of this gate are two or more Boolean variables. The output is the Boolean **sum** of their values. This is represented by

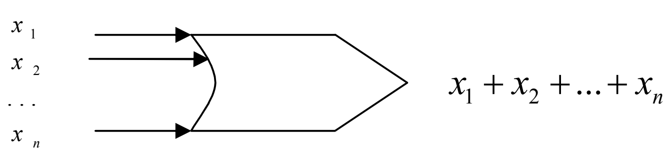


III. **AND gate:** The inputs to this gate are of two or more Boolean variables. The output is the Boolean **product** of their values. This is represented by



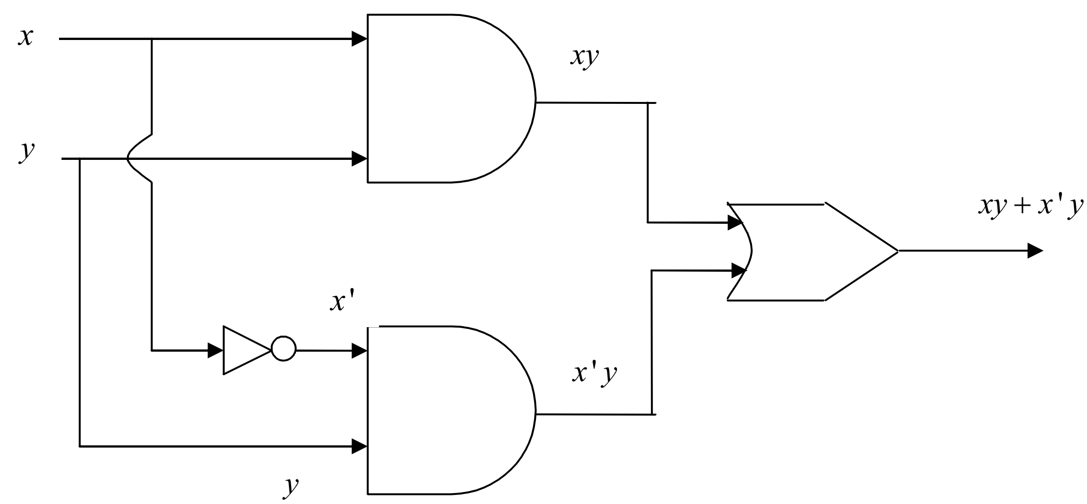
More inputs are permitted to AND and OR gates. Such situations are shown below



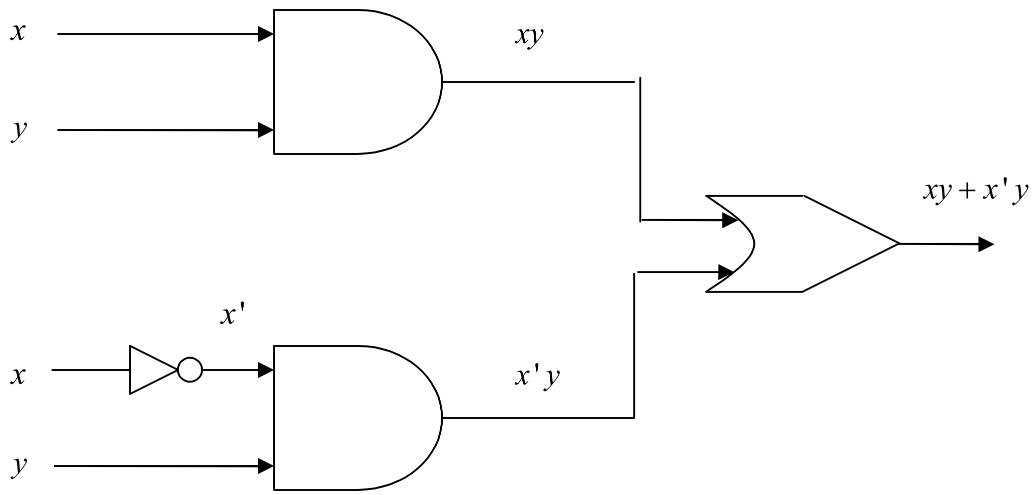


2-4.1 COMBINATIONS OF GATES

Combination circuits can be constructed using a combination of inverters, OR gates, and AND gates. When combinations of circuits are formed, some gates may share inputs. One method is the use of **branching** to indicate all the gates that use a given input.



The other method is to indicate this input separately for each gate.



Note also that the output from a gate may be used as input by one or more elements as shown in the above two diagrams. Note also that the two diagrams represent the **same** input and output circuiting.

EXAMPLE 1

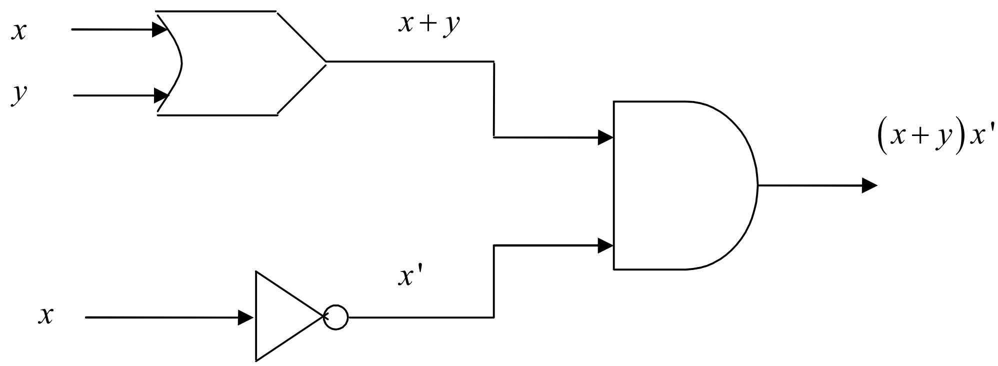
Construct circuits that produce the following outputs:

(a) $(x+y)x'$; (b) $x'(y+z')'$;

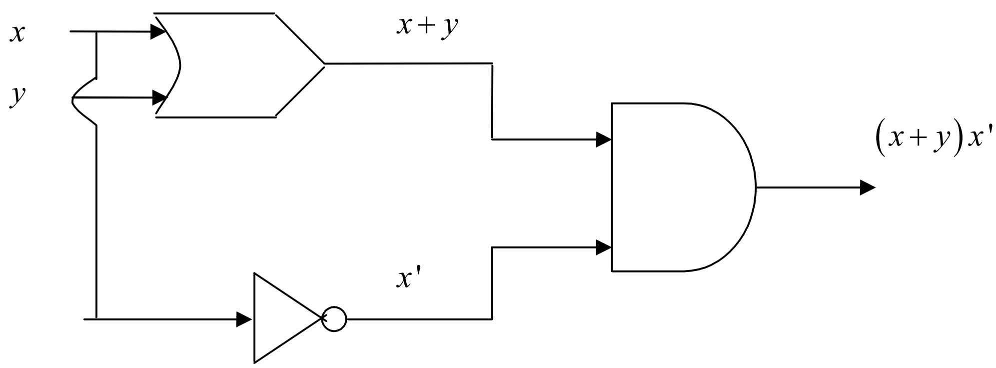
(c) $(x+y+z)(x'y'z')$

SOLUTION

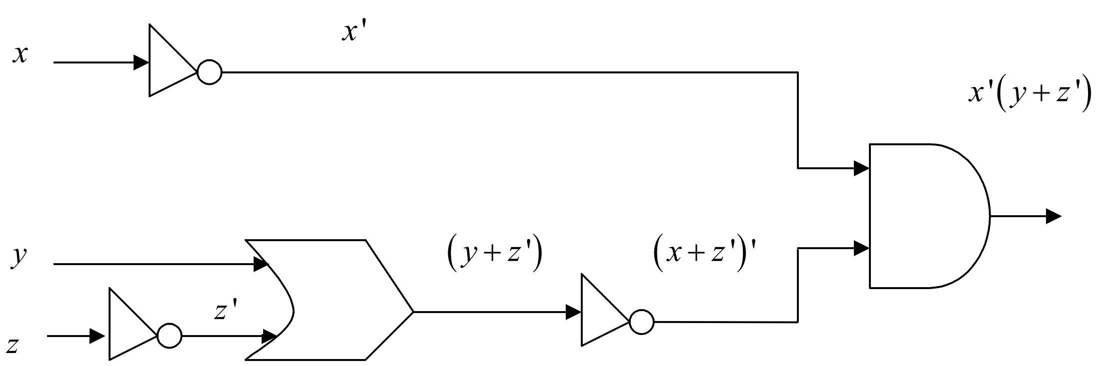
(a)

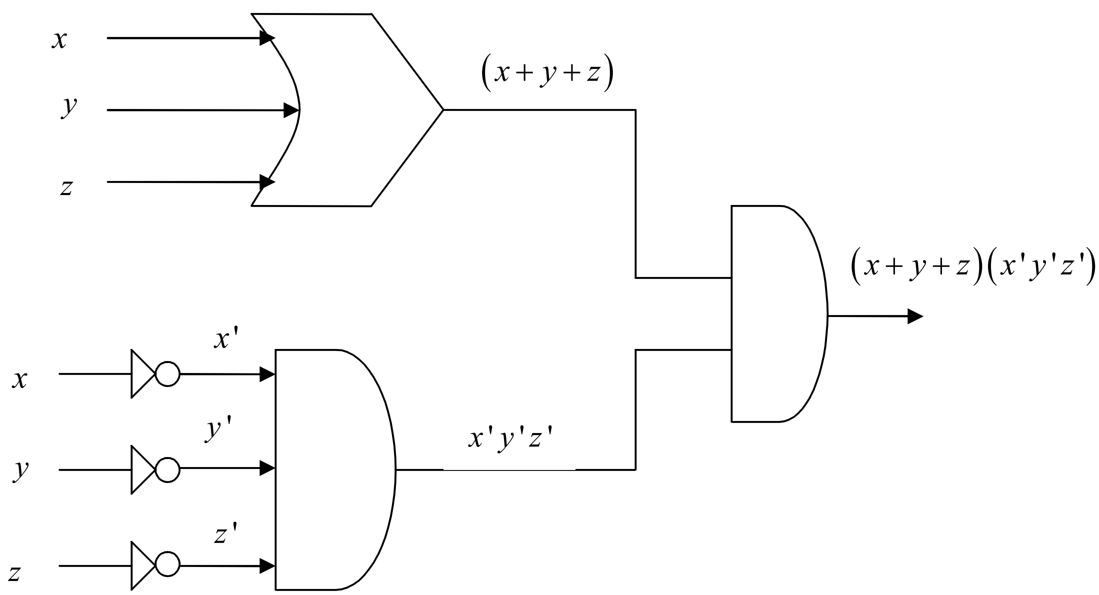


OR



(b)





EXAMPLES OF CIRCUIT

We give some examples of circuits that perform some useful functions.

Example 2

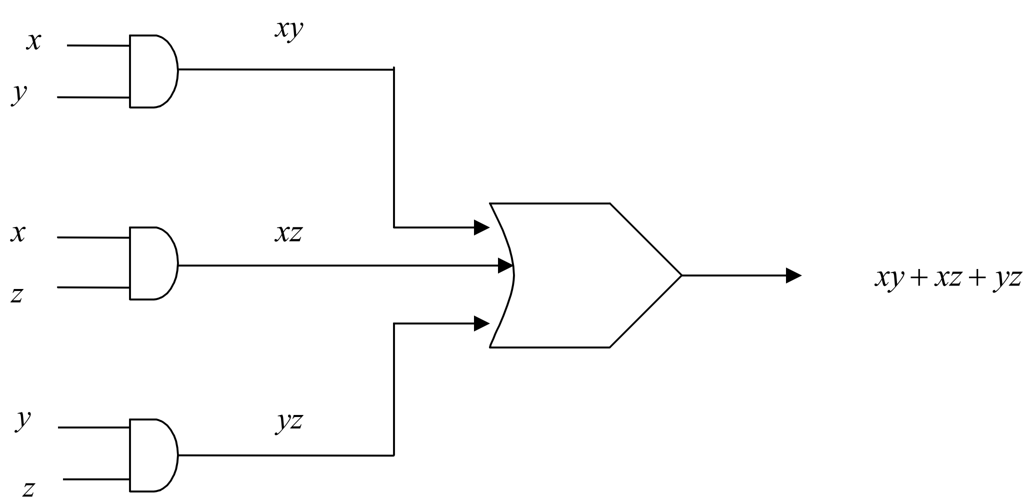
A committee of three individuals decides issues for an organization. Each individual votes either yes or no for each proposal that arises. A proposal is passed if it received **at least** two yes votes. Design a circuit that determines whether a proposal passes.

Solution:

Let $x=1$ if the first individual votes yes, and $x=0$ if he votes no; let $y=1$ if the second votes yes and $y=0$ if he votes no; let $z=1$ if the third individual votes yes, and $z=0$ if this individual votes no. Then a circuit must be designed that produces the output 1 from the inputs x, y and z when **two or more** of x, y and z are 1. The representations of the Boolean function that have these output values are

$F(x,y,z)=xy+xz+yz$ or $F(x,y,z)=xy+xz+xyz+yz$.

We draw the circuit of $F(x,y,z)$.



Draw the circuit of $F(x,y,z)=xy+xz+xyz+yz$.

Example 3

Sometimes light fixtures are controlled by more than one switch. Circuits need to be designed so that flipping any one of the switches turns the light on when it is off and turns the light off when it is on. Design circuits that accomplish this when there are two switches and when there are three switches.

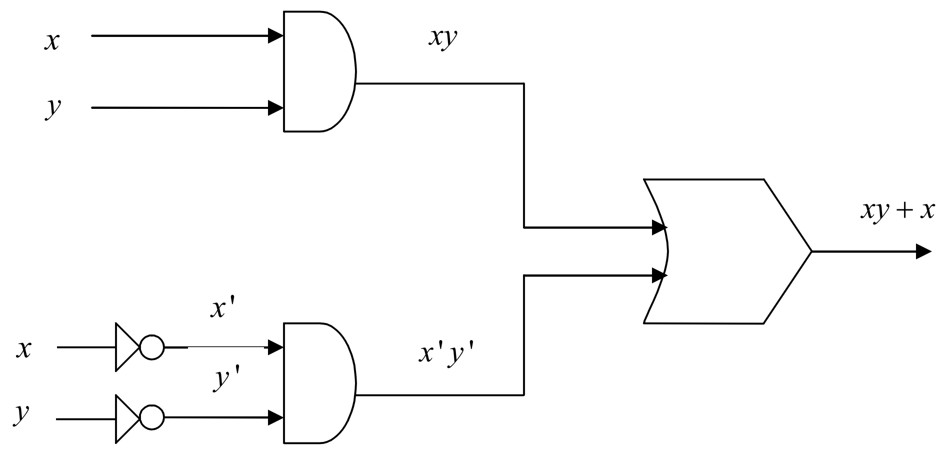
Solution:

I. **For two switches**

Let $x = 1$ when the switch is closed and $x = 0$ when it is opened and let $y = 1$ when the second switch is closed and $y = 0$ when it is opened. Let $F(x, y) = 1$ when the light is on and $F(x, y) = 0$ when it is off. We **arbitrary** decide that the light is on when both switches are closed, so that $F(1,1) = 1$. **This determines all the other values of F**. When one of the two switches is opened, the light goes off, so $F(1,0) = F(0,1) = 0$. When the other switch is opened, the light goes on, so that $F(0,0) = 1$. The following table displays these values.

x	y	$F(x, y)$
1	1	1
1	0	0
0	1	0
0	0	1

Then we see that $F(x,y) = xy + x'y'$ with the following circuit:



II. **For three switches**

Let x, y and z be the Boolean variables that indicate whether each of the three switches are closed.

Let $x = 1$ when first switch is closed and $x = 0$ when it is opened;

Let $y = 1$ when the second switch is closed and $y = 0$ when it is opened;

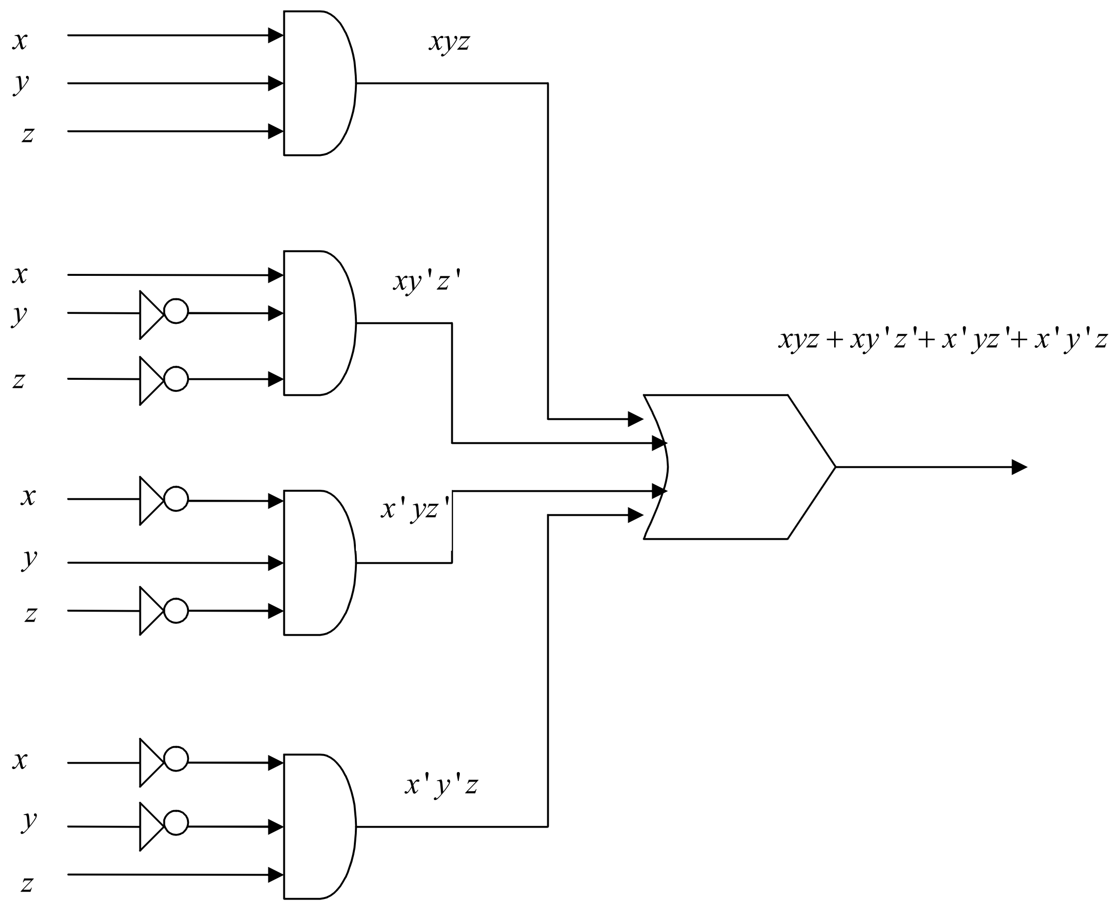
Let $z = 1$ when the third switch is closed and $z = 0$ when it is opened.

Let $F(x,y,z) = 1$ when light is on and $F(x,y,z) = 0$ when the light is off.

We **arbitrary** specify that the light be on when all three switches are closed, so that $F(1,1,1) = 1$. **This determines all the values of** F**.** When one switch is open the light goes off, so that $F(1,1,0) = F(1,0,1) = F(0,1,1) = 0$.

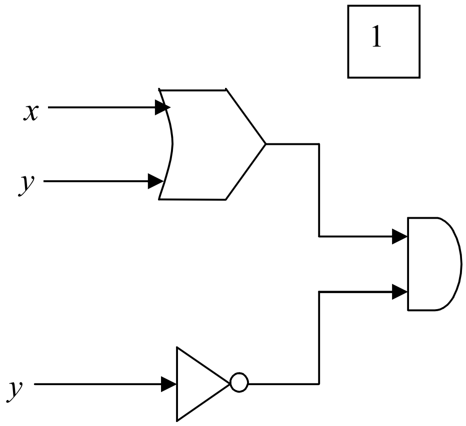
When a second switch is opened, the light goes on, so that $F(1,0,0) = F(0,0,1) = F(0,1,0) = 1$. Finally, when the third switch is opened, the light goes off again, so that $F(0,0,0) = 0$. The following table and circuit display the foregoing analysis.

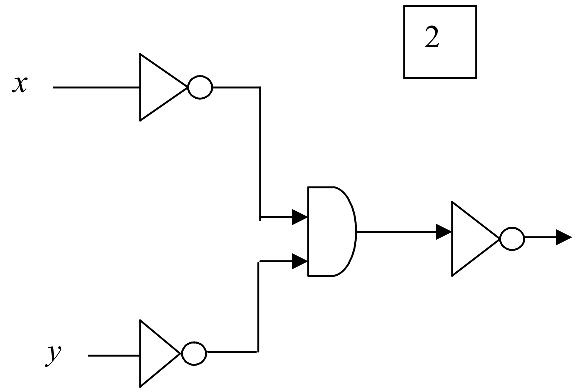
x	y	z	$F(x,y,z)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	1
0	0	0	0

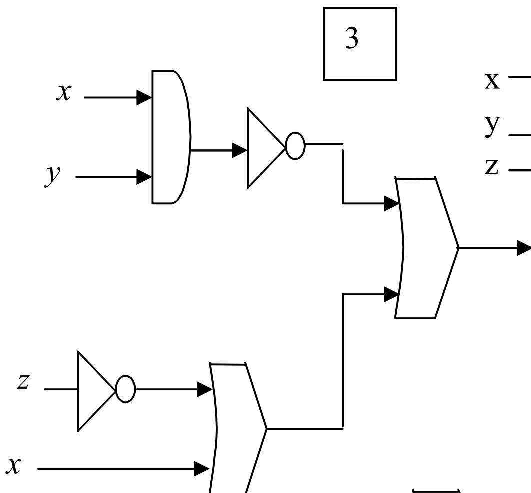


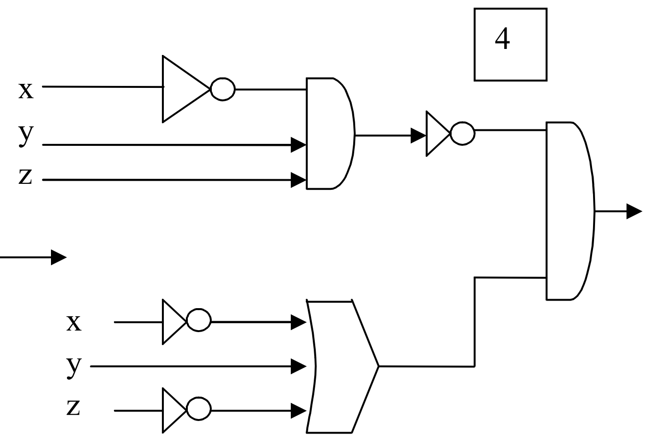
Where $F = xyz + xy'z' + x'yz' + x'y'z$ is the sum-of-products expansion.

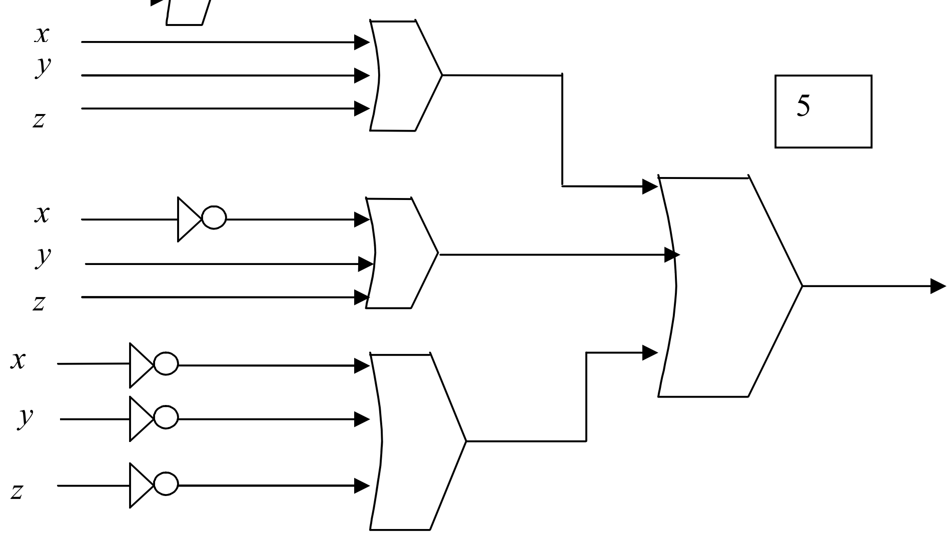
In (1) to (5), find the output of the given circuit.


1
x
y
y


2
x
y


3
x
y
z
x


4
x
y
z
x
y
z


x
y
z
5
x
y
z
x
y
z

6. Construct circuits from inverters, AND gates, and OR gates to produce the following inputs:

(a) $x'+y$; (b) $(x+y)'x$; (c) $xyz+x'y'z'$; (d) $\left[(x'+z)(y+z')\right]'$

7. Design a circuit that implements majority voting for five individuals. A proposal is passed if it receives at least three yes votes.

8. Design a circuit for a light fixture controlled by four switches where flipping one of the switches turns the light on when it is off and turns it off when it is on.

RELATIONS AND ORDER IN A SET

Introduction

There is the need to have a modicum knowledge of pure mathematics. This unit does exactly that. It ushers you into an interesting area of mathematics. This unit is a must-do unit.

Objectives

On completion of this unit, you would be able to:

- Identify some basic theories in pure mathematics
- Apply the knowledge gained to analytic courses in the years ahead

RELATIONS IN A SET

DEFINITIONS

A binary relation (or relation) $\mathbb{R}$ from a set of A to a set B assigns to each pair $\langle a,b \rangle$ in $A \times B$ exactly one of the following statements:

(i) "a is related to b", written $a\mathbb{R}b$

(ii) "a is not related to b", written $a \not\mathbb{R} b$.

A relation from a set A to the same set A is called a **relation in A.**

Observe that any relation $\mathbb{R}$ from a set A to a set B uniquely defines a subset $\mathbb{R}^*$ of $A \times B$ as follows: $\mathbb{R}^* = \{\langle a,b\rangle : a\mathbb{R}b\}$

On the other hand, any subset $\mathbb{R}^*$ of $A \times B$ defines a relation $\mathbb{R}$ from A to B as: $a\mathbb{R}b$ iff $\langle a,b\rangle \in \mathbb{R}^*$

Domain of $\mathbb{R} = \{a : \langle a,b\rangle \in \mathbb{R}\}$

Range of $\mathbb{R} = \{b : \langle a,b\rangle \in \mathbb{R}\}$

EQUIVALENCE RELATIONS

DEFINITION

A relation $\mathbb{R}$ in a set A, that is, a subset $\mathbb{R}$ of $A \times A$ is called an equivalence relation if and only if it satisfies the following axioms:

$[E_1]$ For every $a \in A$, $\langle a,a\rangle \in \mathbb{R}$, reflexive property

$[\mathrm{E}_2]$ If $\langle a,b\rangle \in \mathbb{R}^*$, then $\langle b,a\rangle \in \mathbb{R}$, symmetric property

$[\mathrm{E}_3]$ If $\langle a,b\rangle \in \mathbb{R}^*$, and $\langle b,c\rangle \in \mathbb{R}$, then $\langle a,c\rangle \in \mathbb{R}^*$, transition property

Accordingly, a relation $\mathbb{R}$ is an equivalence relation if and only if it is reflexive, symmetric and transitive.

If $\mathbb{R}$ is an equivalence relation in A, then the equivalence class of any element $a \in A$, denoted by $[a]$, is the set of elements to which a is related.

$[a] = \{x : \langle a,x\rangle \in \mathbb{R}\}$

The collection of equivalence classes of A denoted by $A/\mathbb{R}$, is called the quotient of A by $\mathbb{R}$. $A/\mathbb{R} = \{[a] : a \in A\}$

The quotient set $A/\mathbb{R}$ possesses the following properties:

THEOREM

Let $\mathbb{R}$ be an equivalence relation in A and let $[a]$ be the equivalence class of $a \in A$. Then:

(i) For every $a \in A, a \in [a]$
(ii) $[a] = [b]$, iff $\langle a,b \rangle \in \mathbb{R}$
(iii) If $[a] \neq [b]$, then $[a]$ and $[b]$ are disjoint

A class *a* of non-empty subsets of *A* is called a partition of A if and only if

1. each $a \in A$ belongs to some member of *a* and
2. the members of *a* are pairwise disjoint.

Therefore, the theorem above implies the following fundamental theorem of equivalence relations.

THEOREM

Let $\mathbb{R}$ be an equivalence relation in A. Then the quotient set $A/\mathbb{R}$ is a partition of A.

Example 1

Let $\mathbb{R}_5$ be the relation in $\mathbb{Z}$, the set of integers, defined by $x \equiv y(\text{mod}\,5)$, which reads "$x$ is congruent to y modulo 5" and which means "$x-y$ is divisible by 5". Then $\mathbb{R}_5$ is an equivalence relation in $\mathbb{Z}$. There are exactly five distinct equivalence classes in $\mathbb{Z}/\mathbb{R}_5$:

$$E_0 = \{\ldots,-10,-5,0,5,10,\ldots\} = [0]$$
$$E_1 = \{\ldots,-9,-4,1,6,11,\ldots\} \;\; = [1]$$
$$E_2 = \{\ldots,-8,-3,2,7,12,\ldots\} = [2]$$
$$E_3 = \{\ldots,-7,-2,3,8,13,\ldots\} = [3]$$
$$E_4 = \{\ldots,-6,-1,4,9,14\} \quad = [4]$$

We observe for each integer x, $x = 5p + r$ where $0 \le r < 5$, is a member of the equivalence class E_r, where r is the remainder when x is divided by 5.

Note that the equivalence classes are pairwise disjoint and that

$Z = E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4$

Example 2

Consider the relation $\mathbb{R} = \{\langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle\}$ in $A = \{1,2,3\}$. Then

Domain of $\mathbb{R} = \{1,2\}$,

Range of $\mathbb{R} = \{2,3\}$

Let $\mathbb{R}^{-1}$ denote the relation that reverses the pairs in $\mathbb{R}$. Then $\mathbb{R}^{-1} = \{\langle 2,1\rangle, \langle 3,1\rangle, \langle 3,2\rangle\}$.

We observe that $\mathbb{R}$ and $\mathbb{R}^{-1}$ are identical respectively, to the mathematical relations < and > in the set A. Thus $\langle a,b\rangle \in \mathbb{R}$ iff $a < b$, and $\langle a,b\rangle \in \mathbb{R}^{1}$ iff $a < b$.

The identity relation in any set A, denoted by Δ or Δ_A, is the set of pairs in $A \times A$ with equal coordinates, that is, $\Delta_A = \{\langle a,a\rangle : a \in A\}$

1. Prove: let $\mathbb{R}$ be a relation in A, that is $\mathbb{R} \subset A \times A$. Then
   (i) $\mathbb{R}$ is reflexive if and only if $\Delta_A \subset \mathbb{R}$
   (ii) $\mathbb{R}$ is symmetric if and only $\mathbb{R} = \mathbb{R}^{-1}$
2. Consider the relation $\mathbb{R} = \{\langle 1,1\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ in $\mathrm{X} = \{1,2,3\}$. Determine whether or not $\mathbb{R}$ is (i) reflexive, (ii) symmetric, (iii) transitive

3. Consider the set $\mathbb{N}\times\mathbb{N}$, that is, the set of ordered pairs of positive integers. Let $\mathbb{R}$ be the relation $\simeq$ in $\mathbb{N}\times\mathbb{N}$ which is defined by $\langle a,b\rangle \simeq \langle c,d\rangle$ iff $ad = bc$.

   Prove that $\mathbb{R}$ is an equivalence relation

4. Consider $\mathbb{N}\times\mathbb{N}$, the set of ordered pairs of positive integers. Let $\simeq$ be the relation in $\mathbb{N}\times\mathbb{N}$ defined by $\langle a,b\rangle \simeq \langle c,d\rangle$ iff $a+b=b+c$

   (i) Prove $\simeq$ is an equivalence relation

   (ii) Find the equivalence class of $\langle 2,5\rangle$ that is $[\langle 2,5\rangle]$

ORDERING OF A SET

PARTIAL ORDERING

A relation $\preceq$ in a set A is called a **partial order** (or order) if and only if, for every $a,b,c \in A$ such that

(i) $a \preceq a$, reflexive property

(ii) $a \preceq b$ and $b \preceq a$ implies $a = b$, anti-symmetric property

(iii) $a \preceq b$ and $b \preceq a$ implies $a \preceq c$, transitive property

The set A together with the partial order, that is, the pair $(A,\preceq)$ is called a **partially ordered set**. A partial order is a reflexive, anti-symmetric and transitive relation.

Example 1

Set inclusion is a partial order in any class of sets since

(i) $A \subset A$ is true for any set A

(ii) $A \subset B$ and $B \subset A$ implies $A = B$ for any sets $A,\ B$

(iii) $A \subset B$ and $B \subset C$ implies $A \subset C$ for any sets , $A,\ B$ and C

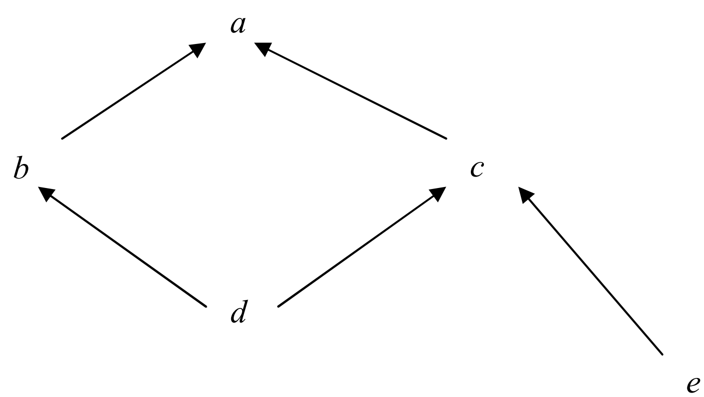
Example 2

Let A be any set of real numbers. Then the relation in A defined by $x \preceq y$ is a partial order and is called the natural order in A.

Example 3

Let $X = \{a, b, c, d, e\}$. Then the diagram below defines a partial order in X as follows:

$x \preceq y$ iff $x = y$ or if one can go from x to y in the diagram, always moving in the indicated direction that is upward.



DEFINITION

If $a \preceq b$ in an ordered set, we say, a precedes or is smaller than b and that b follows or dominates or is larger than a. We write $a \prec b$ if $a \preceq b$ but $a \neq b$.

A particular ordered set A is said to be **totally (or linearly) ordered** if, **for every** $a, b \in A$, either $a \preceq b$ or $b \preceq a$.

The set of real numbers, $\mathbb{R}$, with the natural order defined by $x \preceq y$ is an example of a totally ordered set.

Example 4

Let A and B be totally ordered. Then the product set $A \times B$ can be totally ordered as follows: $\langle a,b \rangle \prec \langle a',b' \rangle$ if $a \prec a'$ or $a = a'$, and $b \prec b'$.

This order is called **lexicographical** of $A \times B$ since it is similar to the way words are arranged in a dictionary.

REMARKS

If a relation $\mathbb{R}$ in a set A defines a partial order, that is, is reflexive, anti-symmetric and transitive, then the inverse relation $\mathbb{R}^{-1}$ is also a partial order, it is called the inverse order.

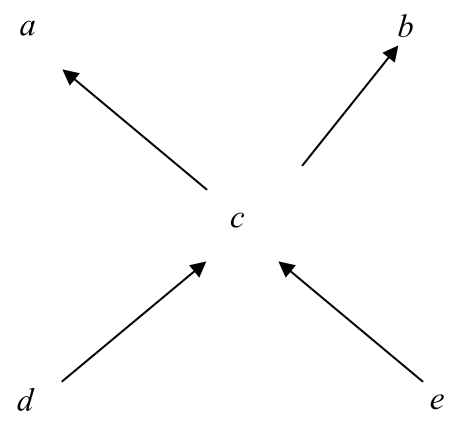
SUBSETS OF ORDERED SETS

Let A be a subset of a partially ordered set X. Clearly the order in X induces an order in A in a very natural way. If $a,b \in A$, then $a \preceq b$ as elements in A if and only if $a \preceq b$ as elements in X. More precisely, if $\mathbb{R}$ is a partial order in X, then the relation

$\mathbb{R}_A = \mathbb{R} \cap (A \times A)$, called the restriction of $\mathbb{R}$ to A, is a partial order in A. The ordered set $(A, \mathbb{R}_A)$ is called **partially ordered subset** of the ordered set $(X, \mathbb{R})$. Some subsets of a partially ordered set X may, in fact, be totally ordered. Clearly, if X itself is totally ordered, every subset of X will also be ordered.

Example 5

Consider the partial order in $W = \{a, b, c, d, e\}$ defined by the diagram



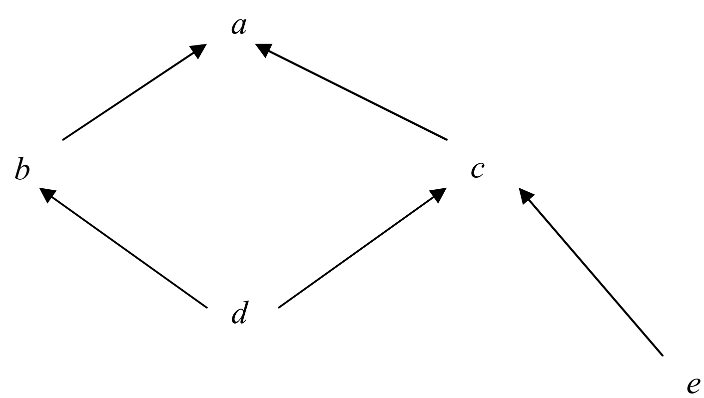
The sets $\{a, c, d\}$ and $\{b, e\}$ are totally ordered subsets, the sets $\{a, b, c\}$ and $\{d, e\}$ are **not** totally ordered subsets.

FIRST AND LAST ELEMENTS

Let X be an ordered set. An element $a_0 \in X$ is a first or smallest element of X if and only if $a_0 \preceq x$ for all $x \in X$. Similarly, an element $b_0 \in X$ is a last or largest element of X if and only if $x \preceq b_0$ for all $x \in X$.

Example 6

Let $X = \{a, b, c, d, e\}$ be ordered by the diagram,



Then a is a last element of X since a follows every element. We note that X has no first element, since d is not a first element because d does not proceed e.

Example 7

The positive integers $\mathbb{N}$ with the natural order have 1 as a first element; there is no last element. Similarly, the set of integers $\mathbb{Z}$ with the natural order has no first element and no last element.

MAXIMAL AND MINIMAL ELEMENTS

Let X be an ordered set. An element $a_0 \in X$ is maximal if and only if $a_0 \preceq x$ implies $x = a_0$, that is, if no element follows a_0 except itself. Similarly, an element $b_0 \in X$ is minimal if and only if $x \preceq b_0$ implies $x = b_0$, that is, if no element precedes b_0 except itself.

Example 8

Let $X = \{a, b, c, d, e\}$ be ordered by the diagram of Example 6. Then both d and e are minimal elements. The element a is a maximal element.

Example 9

Although $\mathbb{R}$ with the natural order is totally ordered it has no minimal and no maximal elements.

Example 10

Let $A = \{a_1, a_2, ..., a_m\}$ be a finite totally ordered set. Then A contains precisely one minimal element and precisely one maximal element, denoted respectively by

$\min\{a_1, ..., a_m\}$ and $\max\{a_1, ..., a_m\}$

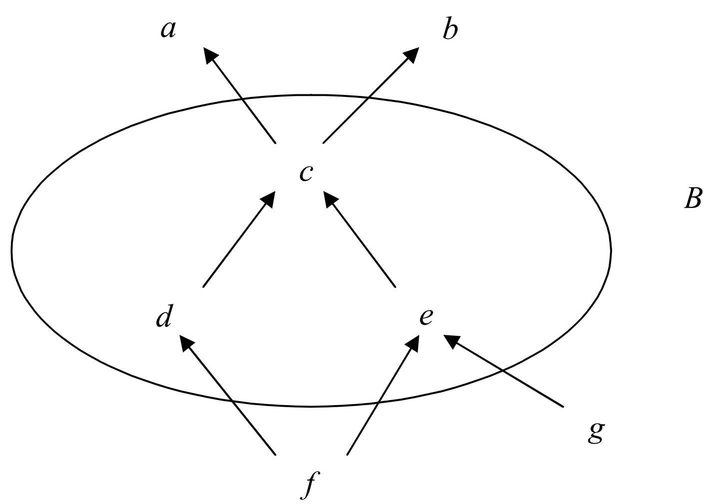
UPPER AND LOWER BOUNDS

Let A be a subset of a partially ordered set X. An element $m \in X$ is a lower bound of A if and only if $m \preceq x$ for all $x \in A$, that is, if m precedes every element in A. If some lower bound of A follows every other lower bound of A, then it is called the greatest lower bound (G.L.B) or *infimum* of A and is denoted by $\inf(A)$. Similarly, an element $\mu \in X$ is an upper bound of A if and only if $x \preceq \mu$ for all $x \in A$, that is, if μ follows every element in A. If some upper bound of A precedes every other upper bound of A, then it is called the least upper bound (L.U.B) or *supremum* of A and is denoted by $\sup(A)$.

A is said to be bounded above if it has an upper bound, and bounded below if it has a lower bound. If A has both an upper and lower bounds, then it is said to be bounded.

Example 11

Let $X = \{a, b, c, d, e, f, g\}$ be ordered as shown in the following diagram:



Let $B = \{c, d, e\}$. Then a, b and c are upper bound of B, and f is the only lower bound of B. We note that g is not a lower bound of B since g does not precede d. Furthermore, $c = \sup(B)$ belongs to B, while $f = \inf(B)$ does not belong to B.

Example 12

Let A be a bounded set of real numbers. Then, a fundamental theorem about real numbers states that, under the natural order, $\inf(A)$ and $\sup(A)$ exist.

Example 13

Let $\mathbb{Q}$ be the set of rational numbers. Let $B=\{x : x\in\mathbb{Q}, x>0, 2<x^2<3\}$. Then B has an infinite number of upper and lower bounds, but $\inf(B)$ and $\sup(B)$ do not exist, because the real numbers $\sqrt{2}$ and $\sqrt{3}$ do not belong to $\mathbb{Q}$ and therefore cannot be considered as upper and lower bounds of B.

EXERCISE

1. Let $\mathbb{N}$, the positive integers, be ordered as follows: each pair of elements $a, a' \in \mathbb{N}$ can be written uniquely in the form $a = 2^r(2s+1)$, $a' = a^{r'}(as'+1)$

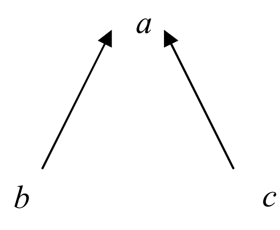
   Where $r, r', s, s' \in \{0,1,2,3,...\}$. Let $a \prec a'$ if $r < r$ or if $'r = r'$ but $s < s'$.

   Insert the correct symbol, $\prec$ or $\succ$, between each of the following pairs of numbers. (Note: $x \succ y$ iff $y \prec x$).

   (i) 5……14 (ii) 6………..9

   (iii) 3……..20 (iv) 14………21

2. Let $A=\{a,b,c\}$ be ordered as in the diagram below. Let $\mathcal{A}$ be the collection of all non-empty totally ordered subsets of A, and let $\mathcal{A}$ be partially ordered by set inclusion. Construct a diagram of the order of $\mathcal{A}$.



3. Let $A=\{2,3,4,...\}=\mathbb{N}\setminus\{1\}$ and let A be ordered by "x divides y".

   (i) Determine the minimal elements of A

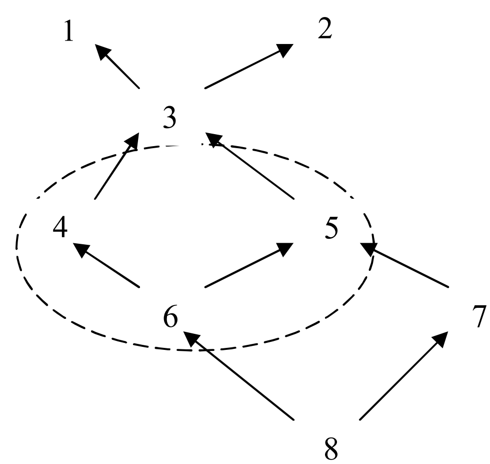
   (ii) Determine the maximal elements of A.

4. Let $B=\{2,3,4,5,6,8,9,10\}$ be ordered by "x is a multiply of y".

   (i) Find all maximal elements of B

   (ii) Find all minimal elements of B.

5. Let $W=\{1,2,...,7,8\}$ be ordered as follows:



   Consider $V=\{4,5,6\}$, a subset of W

   (i) Find the set of upper bounds of V

   (ii) Find the set of lower bounds of V

   (iii) Does $\sup(V)$ exist?

   (iv) Does $\inf(V)$ exist?

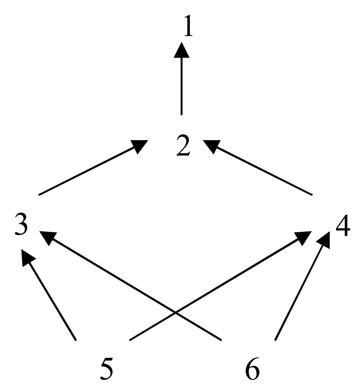
6. Let $A=(\mathbb{N},\leq)$, the positive integers with the natural order; and let $B=(\mathbb{N},\geq)$, the positive integer with the inverse order. Furthermore, let

$A\times B$ denote the lexicographical ordering of $\mathbb{N}\times\mathbb{N}$ according to the order of A and then B. Insert the correct symbol, < or >, between each pair of elements of $\mathbb{N}\times\mathbb{N}$.

(i) $\langle 3,8\rangle$..........$\langle 1,1\rangle$ (ii) $\langle 2,1\rangle$.............$\langle 2,8\rangle$

(iii) $\langle 3,3\rangle$..........$\langle 3,1\rangle$ (iv) $\langle 4,9\rangle$............$\langle 7,15\rangle$

7. Let $X=\{1,2,3,4,5,6\}$ be ordered as in the diagram below. Consider the subset $A=\{2,3,4\}$ of X.



(i) Find the maximal elements of X

(ii) Find the minimal elements of X

(iii) Does X have a first element?

(iv) Does X have a last element?

(v) Find the set of upper bounds of A

(vi) Find the set of lower bounds

(vii) Does sup (A) exist?

(viii) Does inf (A) exist?

8. Consider $\mathbb{Q}$, the set of rational numbers, with the natural order, and its subset $A = \{x : x \in \mathbb{Q}, x^3 < 3\}$
   (i) Is A bounded above?
   (ii) Is A bounded below?
   (iii) Does $\sup(A)$ exist?
   (iv) Does $\inf(A)$ exist?
9. Let $\mathbb{N}$, the positive integers, be ordered by $"x \text{ divides } y"$ and let $A \subset \mathbb{N}$
   (i) Does $\inf(A)$ exist?
   (ii) Does $\sup(A)$ exists?

ZORN'S LEMMA

Zorn's lemma: Let X be a non-empty partially ordered set in which every totally ordered subset has an upper bound, then X contains at least one maximal element.