POLYNOMIAL APPROXIMATION AND INTERPOLATION II

(APPROXIMATION WITH EVENLY SPACED POINTS)

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Lecture Outline

1. Finite difference operators
2. Relations between differences and derivatives
3. Relation between Divided, Forward, and Backward differences
4. Newton's Forward Difference Interpolation Formula
5. Newton's Backward Difference Interpolation Formula

Introduction

1. Let the data $(x_i, f(x_i))$ be given with uniform spacing, that is, the nodal points are given by $x_i = x_0 + ih;\ i = 0, 1, 2, \cdots, n.$
2. In this case, Lagrange and Newton's divided difference interpolation polynomials can also be used for interpolation.
3. However, we can derive simpler interpolation formulas for the uniform mesh case.
4. We define finite difference operators

Some Operators

We begin by defining the following five difference operators

1. Shift operator E
2. Forward difference operator Δ
3. Backward difference operator ∇
4. Central difference operator δ
5. Mean operator μ

Shift operator E

Definition (Shift operator (E))

The shift operator is denoted as E and when applied on $f(x)$ shifts it to the value at the next nodal point. When the operator E is applied on $f(x_i)$, we obtain

$$Ef(x_i) = f(x_i + h) = f(x_{i+1}) \tag{1}$$

Example

$$\begin{aligned} Ef(x_0) &= f(x_0 + h) = f(x_1), && (2)\\ Ef(x_1) &= f(x_1 + h) = f(x_2) && (3)\\ \vdots \quad &= \quad \vdots \qquad\qquad \vdots \end{aligned}$$

Shift operator E

2nd Order

$$E^2 f(x_i) = E[Ef(x_i)] = E[f(x_i + h)] \tag{4}$$
$$= f(x_i + 2h) = f(x_{i+2}) \tag{5}$$

k_{th} Order

$$E^k f(x_i) = f(x_i + kh) = f(x_{i+k}); \quad k \in \mathbb{R} \tag{6}$$

when k = 1/2

$$E^{1/2} f(x_i) = f\left(x_i + \frac{h}{2}\right) = f_{i+1/2} \tag{7}$$

Forward difference operator Δ

Definition

When the operator Δ is applied on $f(x_i)$, we obtain

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = f_{i+1} - f_i \tag{8}$$

These differences are called the first forward differences.

Example

$$\Delta f(x_0) = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0) \tag{9}$$

$$\Delta f(x_1) = f(x_1 + h) - f(x_1) = f(x_2) - f(x_1) \tag{10}$$

$$\vdots \quad = \quad \vdots \qquad \vdots$$

Forward difference operator Δ

The second forward difference is defined by

$$
\begin{aligned}
\Delta^2 = \Delta[\Delta f(x_i)] = \Delta[f(x_i+h) - f(x_i)] = \Delta f(x_i+h) - \Delta f(x_i) \quad (11)\\
= [f(x_i+2h) - f(x_i+h)] - [f(x_i+h) - f(x_i)] \quad (12)\\
= f(x_i+2h) - 2f(x_i+h) + f(xi) \quad (13)\\
= f_{i+2} - 2f_{i+1} + f_i. \quad (14)
\end{aligned}
$$

The third forward difference is defined by

$$
\begin{aligned}
\Delta^3 f(x_i) = \Delta[\Delta^2 f(x_i)] = \Delta f(x_i+2h) - 2\Delta f(x_i+h) + \Delta f(x_i) \quad (15)\\
= f_{i+3} - 3f_{i+2} + 3f_{i+1} - fi \quad (16)
\end{aligned}
$$

The Relation between the Shift Operator E and The Forward difference operator Δ

From the definitions above

$$\Delta f(x_i) = f(x_i + h) - f(x_i) = Ef_i - f_i = (E-1)f_i \tag{17}$$

Comparing, we obtain the operator relation

$$\Delta = E - 1, \quad \text{or} \quad E = 1 + \Delta \tag{18}$$

Moreover, we can write the n_{th} forward difference of $f(x_i)$ as

$$\Delta^n f(x_i) = (E-1)^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+n-k} \tag{19}$$

Forward Difference Table

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
x_0	$f(x_0)$			
		$\Delta f_0 = f_1 - f_0$		
x_1	$f(x_1)$		$\Delta^2 f_0 = \Delta f_1 - \Delta f_0$	
		$\Delta f_1 = f_2 - f_1$		$\Delta^3 f_0 = \Delta^2 f_1 - \Delta^2 f_0$
x_2	$f(x_2)$		$\Delta^2 f_1 = \Delta f_2 - \Delta f_1$	
		$\Delta f_2 = f_3 - f_2$		
x_3	$f(x_3)$			

Backward difference operator (∇)

Definition

When the operator ∇ is applied on $f(x_i)$, we obtain

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f_i - f_{i-1} \tag{20}$$

These differences are called the first backward differences.

Example

$$\nabla f(x_1) = f(x_1) - f(x_0) \tag{21}$$

$$\nabla f(x_2) = f(x_2) - f(x_1) \tag{22}$$

$$\vdots \quad = \quad \vdots$$

Backward difference operator ∇

The second backward difference is defined by

$$\nabla^2 f(x_i) = \nabla[\nabla f(x_i)] = \nabla[f(x_i) - f(x_i - h)] = \nabla f(x_i) - \nabla f(x_i - h) \tag{23}$$
$$= [f(x_i) - f(x_i - h)] - [f(x_i - h) - f(x_i - 2h)] \tag{24}$$
$$= f(x_i) - 2f(x_i - h) + f(x_i - 2h) \tag{25}$$
$$= f_i - 2f_{i-1} + f_{i-2} \tag{26}$$

The third backward difference is defined by

$$\nabla^3 f(x_i) = \nabla[\nabla^2 f(x_i)] = \nabla f(x_i) - 2\nabla f(x_i - h) + \nabla f(x_i - 2h) \tag{27}$$
$$= f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3} \tag{28}$$

The Relation between the Shift Operator E and The Backward difference operator ∇

From the above equations

$$\nabla f(x_i) = f(x_i) - f(x_i - h) = f_i - E^{-1} f_i = (1 - E^{-1}) f_i \tag{29}$$

Comparing, we obtain the operator relation

$$\nabla = 1 - E^{-1}, \quad \text{or} \quad E^{-1} = 1 - \nabla, \quad \text{or} \quad E = (1 - \nabla)^{-1} \tag{30}$$

Moreover, we can write the n_{th} backward difference of $f(x_i)$ as

$$\nabla^n f(x_i) = (1 - E^{-1})^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i-k} \tag{31}$$

Backward Difference Table

x	$f(x)$	∇f	$\nabla^2 f$	$\nabla^3 f$
x_0	$f(x_0)$			
		$\nabla f_1 = f_1 - f_0$		
x_1	$f(x_1)$		$\nabla^2 f_2 = \nabla f_2 - \nabla f_1$	
		$\nabla f_2 = f_2 - f_1$		$\nabla^3 f_3 = \nabla^2 f_3 - \nabla^2 f_2$
x_2	$f(x_2)$		$\nabla^2 f_3 = \nabla f_3 - \nabla f_2$	
		$\nabla f_3 = f_3 - f_2$		
x_3	$f(x_3)$			

Remarks

From the two difference Tables, we note that the numbers (values of differences) in all the columns in the two tables are same. Some examples are

$$\Delta f_0 = \nabla f_1, \quad \Delta f_1 = \nabla f_2, \quad \Delta f_2 = \nabla f_3, \cdots, \Delta^3 f_0 = \nabla^3 f_3 \tag{32}$$

Central difference operator (δ)

Definition

When the operator δ is applied on $f(x_i)$, we obtain

$$\delta f(x_i) = f\left(x_i + \frac{h}{2}\right) - f\left(x_i - \frac{h}{2}\right) \tag{33}$$

$$= f_{i+1/2} - f_{i-1/2} \tag{34}$$

These differences are called the first central differences.

Note

$$\delta f\left(x_i + \frac{h}{2}\right) = \delta f_{i+1/2} = f(x_i + h) - f(x_i) = f_{i+1} - f_i \tag{35}$$

That is, $\delta f_{1/2} = f_1 - f_0$, $\delta f_{3/2} = f_2 - f_1$ etc.

Central difference operator δ

The second central difference is defined by

$$\delta^2 f(x_i) = \delta[\delta f(x_i)] = \delta[f_{i+1/2} - f_{i-1/2}] = \delta f_{i+1/2} - \delta f_{i-1/2} \quad (36)$$

$$= [f_{i+1} - f_i] - [f_i - f_{i-1}] \quad (37)$$

$$= f_{i+1} - 2f_i + f_{i-1} \quad (38)$$

The third central difference is defined by

$$\delta^3 f(x_i) = \delta[\delta^2 f(x_i)] = \delta f_{i+1} - 2\delta f_i + \delta f_{i-1} \quad (39)$$

$$= (f_{i+3/2} - f_{i+1/2}) - 2(f_{i+1/2} - f_{i-1/2}) + (f_{i-1/2} - f_{i-3/2}) \quad (40)$$

$$= f_{i+3/2} - 3f_{i+1/2} + 3f_{i-1/2} - f_{i-3/2} \quad (41)$$

All the odd central differences contain non-nodal values and the even central differences contain nodal values.

The Shift Operator E and The Central difference operator δ

From the equations above

$$\delta f(x_i) = f_{i+1/2} - f_{i-1/2} = E^{1/2} f_i - E^{-1/2} f_i = (E^{1/2} - E^{-1/2}) f_i \tag{42}$$

Comparing, we obtain the operator relation

$$\delta = (E^{1/2} - E^{-1/2}) \tag{43}$$

Using this relation, we can write the n_{th} central difference of $f(x_i)$ as

$$\delta^n f(x_i) = (E^{1/2} - E^{-1/2})^n f(x_i) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f_{i+(n/2)-k} \tag{44}$$

Central Difference Table

x	$f(x)$	δf	$\delta^2 f$	$\delta^3 f$
x_0	$f(x_0)$			
		$\delta f_{1/2} = f_1 - f_0$		
x_1	$f(x_1)$		$\delta^2 f_1 = \delta f_{3/2} - \delta f_{1/2}$	
		$\delta f_{3/2} = f_2 - f_1$		$\delta^3 f_{3/2} = \delta^2 f_2 - \delta^2 f_1$
x_2	$f(x_2)$		$\delta^2 f_2 = \delta f_{5/2} - \delta f_{3/2}$	
		$\delta f_{5/2} = f_3 - f_2$		
x_3	$f(x_3)$			

Mean operator μ

Definition

When the operator μ is applied on $f(x_i)$, we obtain

$$\mu f(x_i) = \frac{1}{2}\left[f\left(x_i + \frac{h}{2}\right) + f\left(x_i - \frac{h}{2}\right)\right] \tag{45}$$

$$= \frac{1}{2}\left[f_{i+1/2} + f_{i-1/2}\right] \tag{46}$$

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f_i \tag{47}$$

Deductively, the mean operator and the shift operator are related by the equation

$$\mu = \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] \tag{48}$$

Relation between The Shift Operator and derivatives

Recall from lecture 1:

$f(x+h) = \sum_{k=0}^{n} \dfrac{f^{(k)}(x)}{k!} h^k + E_{n+1}$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \ldots$

$$\begin{aligned}
Ef(x) &= f(x+h) && (49)\\
&= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots && (50)\\
&= \left[1 + h\frac{f'(x)}{f(x)} + \frac{h^2}{2!}\frac{f''(x)}{f(x)} + \cdots\right] f(x) && (51)\\
&= \left[1 + hD + \frac{h^2}{2!} D^2 + \cdots\right] f(x); \quad \frac{f'(x)}{f(x)} = D,\ \frac{f''(x)}{f(x)} = D^2,\ \frac{f^k(x)}{f(x)} = D^k && (52)\\
&= e^{hD} f(x) && (53)
\end{aligned}$$

Relation between Forward difference, & derivatives (1st)

The forward difference of $f(x)$ is defined as:

$$\Delta f(x) = f(x+h) - f(x) \tag{54}$$

Using the Taylor series expansions, we get

$$\begin{align}\Delta f(x) &= \left[f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots\right] - f(x) \tag{55}\\ &= hf'(x) + \frac{h^2}{2!}f''(x) + \cdots \tag{56}\end{align}$$

Neglecting the higher order terms, we get the approximation

$$\Delta f(x) \approx hf'(x) \implies f'(x) \approx \frac{1}{h}\Delta f(x) \tag{57}$$

The error term is given by

$$f'(x) - \frac{1}{h}\Delta f(x) = -\frac{h}{2} f''(x) + \cdots \tag{58}$$

So, we call the approximation eq. (57) as a first order approximation or of order $\mathcal{O}(h)$.

Relation between Forward difference, & derivatives (2nd)

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x) \tag{59}$$

$$= \left[f(x) + 2hf'(x) + \frac{4h^2}{2} f''(x) + \frac{8h^3}{6} f'''(x) + \cdots \right] - \tag{60}$$

$$2\left[f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \cdots \right] + f(x)$$

$$= h^2 f''(x) + h^3 f'''(x) + \cdots \tag{61}$$

Neglecting the higher order terms, we get the approximation

$$\Delta^2 f(x) \approx h^2 f''(x) \implies f''(x) \approx \frac{1}{h^2} \Delta^2 f(x) \tag{62}$$

The error term is given by

$$f''(x) - \frac{1}{h^2}\Delta^2 f(x) = -hf'''(x) + \cdots \tag{63}$$

We call the approximation eq. (62) as a first order approximation or of order $\mathcal{O}(h)$.

Relation between Backward differences and 1st, 2nd derivatives

Similarly, we have the following results for backward differences.

$$\nabla f(x) \approx h f'(x) \implies f'(x) \approx \frac{1}{h} \nabla f(x) \tag{64}$$

$$\nabla^2 f(x) \approx h^2 f''(x) \implies f''(x) \approx \frac{1}{h^2} \nabla^2 f(x) \tag{65}$$

These are also first order approximation or of order $\mathcal{O}(h)$.

Relation between Central difference, & derivatives (1st)

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \tag{66}$$

$$= \left[f(x) + \frac{h}{2}f'(x) + \frac{h^8}{2}f''(x) + \frac{h^3}{48}f'''(x) + \cdots\right] - \tag{67}$$

$$\left[f(x) - \frac{h}{2}f'(x) + \frac{h^2}{8}f''(x) - \frac{h^3}{48}f'''(x) + \cdots\right]$$

$$= hf'(x) + \frac{h^3}{24}f'''(x) + \cdots \tag{68}$$

Neglecting the higher order terms, we get the approximation

$$\delta f(x) \approx hf'(x) \implies f'(x) \approx \frac{1}{h}\delta f(x) \tag{69}$$

Relation between Central difference, & derivatives (1st)

The error term is given by

$$f'(x) - \frac{1}{h}\delta f(x) = -\frac{h^2}{24} f'''(x) + \cdots \tag{70}$$

So, we call the approximation eq. (69) as a second order approximation or of order $\mathcal{O}(h^2)$.

Relation between Central difference, & derivatives (2nd)

$$\delta^2 f(x) = f(x+h) - 2f(x) + f(x-h) \tag{71}$$

$$= \left[f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots \right] - 2f(x) + \tag{72}$$

$$\left[f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \cdots \right]$$

Neglecting the higher order terms, we get the approximation

$$\delta^2 f(x) \approx h^2 f''(x) \implies f''(x) \approx \frac{1}{h^2} \delta^2 f(x) \tag{73}$$

The error term is given by

$$f''(x) - \frac{1}{h^2} \delta^2 f(x) = -\frac{h^2}{12} f''''(x) + \cdots \tag{74}$$

So, we call the approximation eq. (73) as a second order approximation or of order $\mathcal{O}(h^2)$.

Relation between Divided, Forward, and Backward differences

1st divided difference

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{1}{h}\Delta f_i = \frac{1}{h}\nabla f_{i+1} \tag{75}$$

Similarly, the 2nd divided difference is

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i} = \frac{(1/h)\Delta f_{i+1} - (1/h)\Delta f_i}{2h} \tag{76}$$

$$= \frac{1}{2!h^2}\Delta^2 f_i = \frac{1}{2!h^2}\nabla^2 f_{i+2} \tag{77}$$

Generally, the nth divided difference is

$$f[x_0, x_1, \cdots, x_n] = \frac{1}{n!h^n}\Delta^n f_0 = \frac{1}{n!h^n}\nabla^n f_n \tag{78}$$

Newton's Forward Difference Interpolation Formula

Let h be the step length in the given data. In terms of the divided differences, we have the interpolation formula as

$$f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots \tag{79}$$

But

$$f[x_0, x_1, \cdots, x_n] = \frac{1}{n!h^n}\Delta^n f_0 = \frac{1}{n!h^n}\nabla^n f_n \tag{80}$$

Substituting eq. (80) into eq. (79), we have

$$\begin{aligned} f(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!h^2} + \cdots \\ + (x - x_0)(x - x_1)\cdots(x - x_{n-1})\frac{\Delta^n f_0}{n!h^n} \end{aligned} \tag{81}$$

This relation (81) is called the Newton's forward difference interpolation formula.

1. The Newton's forward difference formula has the permanence property.
2. Suppose we add a new data value $(x_{n+1}, f(x_{n+1}))$ at the end of the given table of values.
3. Then, the $(n+1)_{th}$ column of the forward difference table has the $(n+1)_{th}$ forward difference.
4. Then, the Newton's forward difference formula becomes

$$
\begin{aligned}
f(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!h^2} + \cdots \\
+ (x - x_0)(x - x_1)\cdots(x - x_n)\frac{\Delta^{n+1} f_0}{(n+1)!h^{n+1}} \quad (82)
\end{aligned}
$$

Example

For the data construct the forward difference formula. Hence, find $f(0.5)$.

x	-2	-1	0	1	2	3
$f(x)$	15	5	1	3	11	25

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
-2	15			
		-10		
-1	5		6	
		-4		0
0	1		6	
		2		0
1	3		6	
		8		0
2	11		6	
		14		
3	25			

From the table, we conclude that the data represents a quadratic polynomial. We have $h = 1$. The Newton's forward difference formula is given by

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0)\frac{\Delta f_0}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!h^2} && (83)\\ &= 15 + (x+2)(-10) + (x+2)(x+1)\frac{6}{2} && (84)\\ &= 15 - 10x - 20 + 3x^2 + 9x + 6 && (85)\\ &= 3x^2 - x + 1. && (86) \end{aligned}$$

Therefore,

$$f(0.5) = 3(0.5)2 - 0.5 + 1 = 0.75 - 0.5 + 1 = 1.25. \quad (87)$$

Newton's Backward Difference Interpolation Formula

Similarly, the Newton's backward difference interpolation formula is given as

$$f(x) = f(x_n) + (x - x_n)\frac{\nabla f(x_n)}{1!h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 f(x_n)}{2!h^2} + \cdots + (x - x_n)(x - x_{n-1})\cdots(x - x_1)\frac{\nabla^n f(x_n)}{n!h^n} \quad (88)$$

Note

We use the forward difference interpolation when we want to interpolate near the top of the table and backward difference interpolation when we want to interpolate near the bottom of the table.

Remark

1. As in divided differences, given a table of values, we can determine the degree of the forward/ backward difference polynomial using the difference table.
2. The k_{th} column of the difference table contains the k_{th} forward/ backward differences.
3. If the values of these differences are same, then the $(k+1)_{th}$ and higher order differences are zero.
4. Hence, the given data represents a k_{th} degree polynomial.

Example

For the following data, calculate the differences and obtain the Newton's forward and backward difference interpolation polynomials.

1. Are these polynomials different?
2. Interpolate at $x = 0.25$ and $x = 0.35$

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

x	$f(x)$	∇f	$\nabla^2 f$
0.1	1.40		
		0.16	
0.2	1.56		0.04
		0.20	
0.3	1.76		0.04
		0.24	
0.4	2.00		0.04
		0.28	
0.5	2.28		

The step length is $h = 0.1$. Since, the third and higher order differences are zero, the data represents a quadratic polynomial. The third column represents the first forward/ backward differences and the fourth column represents the second forward/ backward differences.

The forward difference polynomial is given by

$$f(x) = f(x_0) + (x - x_0)\frac{\Delta f_0}{1!h} + (x - x_0)(x - x_1)\frac{\Delta^2 f_0}{2!h^2} \tag{89}$$
$$= 1.4 + (x - 0.1)\frac{0.16}{0.1} + (x - 0.1)(x - 0.2)\frac{0.04}{0.02} \tag{90}$$
$$= 2x^2 + x + 1.28 \tag{91}$$

The backward difference polynomial is given by

$$f(x) = f(x_n) + (x - x_n)\frac{\nabla f_n}{1!h} + (x - x_n)(x - x_{n-1})\frac{\nabla^2 f_n}{2!h^2} \tag{92}$$
$$= 2.28 + (x - 0.5)\frac{0.28}{0.1} + (x - 0.5)(x - 0.4)\frac{0.04}{0.02} \tag{93}$$
$$= 2x^2 + x + 1.28 \tag{94}$$

Both the polynomials are identical, since the interpolation polynomial is unique.

$$f(0.25) = 2(0.25)^2 + 0.25 + 1.28 = 1.655 \tag{95}$$

$$f(0.35) = 2(0.35)^2 + (0.35) + 1.28 = 1.875 \tag{96}$$

Exercise

1. Using the Newton's forward difference formula, find the polynomial $f(x)$ satisfying the following data. Hence, evaluate y at $x = 5$.

x	4	6	8	10
y	1	3	8	10

2. A third degree polynomial passes through the points $(0, -1), (1, 1), (2, 1)$ and $(3, -2)$. Determine this polynomial using Newton's forward interpolation formula. Hence, find the value at 1.5.

3. Using the Newton's backward interpolation formula, find the cubic polynomial which takes the following values. Hence, find $y(4)$.

x	0	1	2	3
y	1	2	1	10

END OF LECTURE

THANK YOU