DECOMPOSITION METHOD

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Outline

Introduction

In the earlier lecture, we presented the process of solving a nonsingular linear system $Ax = b$ using Gaussian elimination. We formed the augmented matrix $A|b$ and applied the elementary row operations

1. Multiplying a row by a scalar.
2. Subtracting a multiple of one row from another
3. Exchanging two rows

to reduce A to upper-triangular form. Following this step, back substitution computed the solution.
We are moving a step forward on this algorithm.

Outline of Presentation

- Introduction

1. LU Decomposition with Gaussian Elimination
2. Gaussian Elimination with Partial Pivoting
3. LU decomposition using Doolittle's method
4. Crout Decomposition

LU Decomposition with Gaussian Elimination

This decomposes a given matrix A into a product of a lower-triangular matrix L and an upper-triangular matrix U .

$$A = LU$$

For a 4×4 matrix, the general form is

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For a 4×4 matrix, the general form is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & l_{43} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix} \tag{1}$$

1. The main idea of the LU decomposition is to record the steps used in Gaussian elimination with A in the places that would normally become zero. Consider the matrix:

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \tag{2}$$

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2. The first step of Gaussian elimination is to use $a_{11} = 1$ as the pivot and subtract 2 times the first row from the second and 3 times the first row from the third.

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3. Record these actions by placing the multipliers 2 and 3 into the entries they made zero. In order to make it clear that we are recording multipliers and not elements of A, put the entries in parentheses. This leads to:

$$\begin{bmatrix} 1 & -1 & 3 \\ (2) & -1 & -5 \\ (3) & 5 & -8 \end{bmatrix} \tag{3}$$

1. To zero-out the element in the third row, second column, the pivot is -1, and we need to subtract -5 times the second row from the third row. Record the -5 in the spot made zero.

$$\begin{bmatrix} 1 & -1 & 3 \\ (2) & -1 & -5 \\ (3) & (-5) & -33 \end{bmatrix} \tag{4}$$

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2. Then U is the upper-triangular matrix produced by Gaussian elimination and L be the lower-triangular matrix with the multipliers and ones on the diagonal, that is,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -33 \end{bmatrix} \tag{5}$$

1. Thus

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -33 \end{bmatrix} = A \tag{6}$$

2. We can see that A is the product of the lower triangular L and the upper triangular U.

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3. When a matrix can be written as a product of simpler matrices, we call that a decomposition and this one we call the LU decomposition.

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Note

As the elimination process continues, the pivots are on the diagonal of U.

Using LU to Solve Equations

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Example

Solve the following using the LU decomposition method.

$$\begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

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We have

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & -3 & 1 \\ 3 & 2 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \tag{12}$$

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A should be decomposed to L and U, from the previous illustration

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & -1 & 3 \\ 0 & -1 & -5 \\ 0 & 0 & -33 \end{bmatrix} \tag{13}$$

1. First solve $Ly = b$ to find y by forward substitution.

$$(1)y_1 = 1, \implies y_1 = 1, \tag{14}$$

$$2(1) + (1)y_2 = 3, \implies y_2 = 1, \tag{15}$$

$$(3)(1) - 5(1) + (1)y_3 = 1, \implies y_3 = 3. \tag{16}$$

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2. Then solve $Ux = y$ to find x by back substitution.

$$-33x_3 = 3, \implies x_3 = -1/11, \tag{17}$$

$$-x_2 - 5(-1/11) = 1, \implies x_2 = -6/11, \tag{18}$$

$$x_1 - (-6/11) + 3(-1/11) = 1, \implies x_1 = 8/11 \tag{19}$$

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Thus, the solution to the system is

$$x_1 = 8/11, \qquad x_2 = -6/11, \qquad x_3 = -1/11$$

Example

Solve

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 5 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

using the LU decomposition with Gaussian elimination method.

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Solve

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Perform Gaussian elimination to find U.

1st Iteration

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 2 \\ 5 & 1 & 4 \end{bmatrix} \begin{matrix} \longrightarrow \\ NR_2 = R_2 - (2)R_1 \\ NR_3 = R_3 - (5)R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -1 \\ (2) & -1 & 4 \\ (5) & -9 & 9 \end{bmatrix} \tag{20}$$

2nd Iteration

$$\begin{bmatrix} 1 & 2 & -1 \\ (2) & -1 & 4 \\ (5) & -9 & 9 \end{bmatrix} \overrightarrow{NR_3 = R_3 - (9)R_2} \begin{bmatrix} 1 & 2 & -1 \\ (2) & -1 & 4 \\ (5) & (9) & -27 \end{bmatrix} \tag{21}$$

2nd Iteration

$$\begin{bmatrix} 1 & 2 & -1 \\ (2) & -1 & 4 \\ (5) & -9 & 9 \end{bmatrix} NR_3 = \overrightarrow{R_3} - (9)R_2 \begin{bmatrix} 1 & 2 & -1 \\ (2) & -1 & 4 \\ (5) & (9) & -27 \end{bmatrix} \tag{21}$$

so

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 5 & 9 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & -27 \end{bmatrix} \tag{22}$$

1. Forward substitution:

$$y_1 = 1, \tag{23}$$

$$y_2 = -1 - 2(1) = -3, \tag{24}$$

$$y_3 = 2 - 5(1) - 9(-3) = 24. \tag{25}$$

1. Forward substitution:

$$y_1 = 1, \tag{23}$$
$$y_2 = -1 - 2(1) = -3, \tag{24}$$
$$y_3 = 2 - 5(1) - 9(-3) = 24. \tag{25}$$

2. Back substitution:

$$x_3 = -24/27 = -8/9, \tag{26}$$
$$x_2 = 3 - 4(8/9) = -5/9, \tag{27}$$
$$x_1 = 1 - 2(-5/9) - 8/9 = 11/9. \tag{28}$$

Outline of Presentation

Gaussian Elimination with Partial Pivoting (GEPP)

1. The Gaussian elimination method may fail when any one of the pivot points is zero or a very small number relative to the other values.
2. To overcome such computational difficulty, we use a procedure called **Partial Pivoting** to solve the given problem.
3. With this technique, first search through a given pivot column to find the largest number in magnitude. That number is used as the pivot by interchanging rows.
4. The procedure is continued until an upper triangular matrix is obtained.

Definition (Permuation Matrix)

Let P be a permutation matrix, also called the pivot matrix. Start with $P = I$(identity matrix), and swap rows i and j of the permutation matrix whenever rows i and j are swapped during GEPP. For instance

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{29}$$

would be the permutation matrix if the second and third rows of A are interchanged during pivoting.

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would be the permutation matrix if the second and third rows of A are interchanged during pivoting.

If we use GEPP, then an LU decomposition for A consists of three matrices $P, L,$ and U such that

$$PA = LU \tag{30}$$

Example

Compute the permutation matrix, the upper-triangular matrix and the lower-triangular matrix from $A = \begin{bmatrix} 3 & 8 & 1 \\ 5 & 2 & 0 \\ 6 & 1 & 12 \end{bmatrix}$

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Let

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 3 & 8 & 1 \\ 5 & 2 & 0 \\ 6 & 1 & 12 \end{bmatrix} \tag{31}$$

Iteration 1

Pivot row = 1. Swap rows 1 and 3, and permute P. Do not interchange rows of L until arriving at the pivot in row 2, column 2. This lead to

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 6 & 1 & 12 \\ 5 & 2 & 0 \\ 3 & 8 & 1 \end{bmatrix} \tag{32}$$

Iteration 2

Apply the pivot element, and add multipliers to L.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 5/6 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}, \qquad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 6 & 1 & 12 \\ 0 & 7/6 & -10 \\ 0 & 15/2 & -5 \end{bmatrix} \tag{33}$$

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$$L = \begin{bmatrix} 1 & 0 & 0 \\ 5/6 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 6 & 1 & 12 \\ 0 & 7/6 & -10 \\ 0 & 15/2 & -5 \end{bmatrix} \tag{33}$$

Iteration 3

Pivot row $= 2$. Swap rows 2 and 3. Permute P and L.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 5/6 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 6 & 1 & 12 \\ 0 & 15/2 & -5 \\ 0 & 7/6 & -10 \end{bmatrix} \tag{34}$$

Iteration 4

Apply the pivot element and update L.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 5/6 & 7/45 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 6 & 1 & 12 \\ 0 & 15/2 & -5 \\ 0 & 0 & -83/9 \end{bmatrix} \tag{35}$$

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Thus

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 5/6 & 7/45 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 6 & 1 & 12 \\ 0 & 15/2 & -5 \\ 0 & 0 & -83/9 \end{bmatrix} \tag{36}$$

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You can verify that

$$PA = LU$$

Outline of Presentation

- Introduction

LU decomposition using Doolittle's method

It is always possible to factor a square matrix into a lower triangular matrix and an upper triangular matrix.

$$A = LU$$

Doolittle's method provides an alternative way to factor A into an LU decomposition without going through the hassle of Gaussian Elimination.

For a 3×3 matrix, the general form is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \tag{37}$$

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Therefore

$$u_{11} = a_{11} \tag{39}$$

$$u_{12} = a_{12} \tag{40}$$

$$u_{13} = a_{13} \tag{41}$$

$$a_{21} = l_{21}u_{11} \implies l_{21} = \frac{a_{21}}{u_{11}} \tag{42}$$

$$a_{31} = l_{31}u_{11} \implies l_{31} = \frac{a_{31}}{u_{11}} \tag{43}$$

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$$a_{22} = l_{21}u_{12} + u_{22} \implies u_{22} = a_{22} - l_{21}u_{12} \tag{44}$$

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$$a_{33} = l_{31}u_{13} + l_{32}u_{23} + u_{33} \implies u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} \tag{47}$$

Using LU to Solve Equations

1. Factor A into the product of L and U: that is

$$Ax = b \tag{48}$$

$$LUx = b \tag{49}$$

$$L(Ux) = b \tag{50}$$

$$\text{Let } Ux = y \quad \text{where } y = n \times 1 \text{ vector} \tag{51}$$

Now equation (50) reduce to

$$Ly = b \tag{52}$$

2. First solve $Ly = b$ to find y by forward substitution.
3. Then solve $Ux = y$ to find x by back substitution.

Example

Solve the following using the LU with Doolittle's decomposition method.

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

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$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

$$A = LU \tag{53}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 1 & -5 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} \tag{54}$$

Example

Solve the following using the LU with Doolittle's decomposition method.

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 1 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix}$$

$$A = LU \tag{53}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 1 & -5 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} \tag{54}$$

$$u_{11} = a_{11} = 2 \qquad u_{12} = a_{12} = 3 \qquad u_{13} = a_{13} = -1 \tag{55}$$

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{3}{2} \tag{56}$$

$$l_{31} = \frac{a_{31}}{u_{11}} = \frac{1}{2} \tag{57}$$

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{3}{2} \tag{56}$$

$$l_{31} = \frac{a_{31}}{u_{11}} = \frac{1}{2} \tag{57}$$

$$u_{22} = a_{22} - l_{21}u_{12} = 2 - 3(3/2) = \frac{-5}{2} \tag{58}$$

$$u_{23} = a_{23} - l_{21}u_{13} = 1 - (-1)(3/2) = \frac{5}{2} \tag{59}$$

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$$l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}} = \frac{-5 - 3(1/2)}{-5/2} = \frac{13}{5} \tag{60}$$

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$$u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 3 - (-1)(1/2) - (13/5)(5/2) = -3 \tag{61}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 13/5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5/2 & 5/2 \\ 0 & 0 & -3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 13/5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5/2 & 5/2 \\ 0 & 0 & -3 \end{bmatrix}$$

First solve $Ly = b$ to find y by forward substitution.

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 13/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 0 \end{bmatrix} \tag{62}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 1/2 & 13/5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -5/2 & 5/2 \\ 0 & 0 & -3 \end{bmatrix}$$

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$$y_1 = 5 \tag{63}$$

$$\frac{15}{2} + y_2 = 10 \implies y_2 = \frac{5}{2} \tag{64}$$

$$(1/2)(5) + (13/5)(5/2) + y_3 = 0 \implies y_3 = -9 \tag{65}$$

Then solve $Ux = y$ to find x by back substitution.

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -5/2 & 5/2 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ \frac{5}{2} \\ -9 \end{bmatrix} \tag{66}$$

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$$-3x_3 = -9 \implies x_3 = 3 \tag{67}$$

$$\frac{-5}{2}x_2 + \frac{5}{2}(3) = \frac{5}{2} \implies x_2 = 2 \tag{68}$$

$$2x_1 + 3(2) - 3 = 5 \implies x_1 = 1 \tag{69}$$

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The solution by Doolittle's method is

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 3$$

Outline of Presentation

- Introduction

1. LU Decomposition with Gaussian Elimination
2. Gaussian Elimination with Partial Pivoting
3. LU decomposition using Doolittle's method
4. Crout Decomposition

Crout Decomposition

1. The Crout matrix decomposition is an LU decomposition which decomposes a matrix into a lower triangular matrix L, an upper triangular matrix U.
2. The Crout matrix decomposition algorithm differs slightly from the Doolittle method. Doolittle's method returns a unit lower triangular matrix and an upper triangular matrix, while the Crout method returns a lower triangular matrix and a unit upper triangular matrix.

For a $3\times$ matrix, the general form is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{70}$$

For a $3\times$ matrix, the general form is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} \tag{70}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} \tag{71}$$

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Therefore

$$l_{11} = a_{11} \tag{72}$$

$$l_{21} = a_{21} \tag{73}$$

$$l_{31} = a_{31} \tag{74}$$

$$a_{12} = l_{11}u_{12} \implies u_{12} = \frac{a_{12}}{l_{11}} \tag{75}$$

$$a_{13} = l_{11}u_{13} \implies u_{13} = \frac{a_{13}}{l_{11}} \tag{76}$$

$$a_{12} = l_{11}u_{12} \implies u_{12} = \frac{a_{12}}{l_{11}} \tag{75}$$

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$$a_{22} = l_{21}u_{12} + l_{22} \implies l_{22} = a_{22} - l_{21}u_{12} \tag{77}$$

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$$a_{33} = l_{31}u_{13} + l_{32}u_{23} + l_{33} \implies l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} \tag{80}$$

Using LU to Solve Equations

1. Factor A into the product of L and U: that is

$$Ax = b \tag{81}$$

$$LUx = b \tag{82}$$

$$L(Ux) = b \tag{83}$$

$$\text{Let } Ux = y \quad \text{where } y = n \times 1 \text{ vector} \tag{84}$$

Now equation (83) reduce to

$$Ly = b \tag{85}$$

2. First solve $Ly = b$ to find y by forward substitution.
3. Then solve $Ux = y$ to find x by back substitution.

Example

Solving the following system of equation with the Crout Method

$$\begin{bmatrix}1 & 1 & 1\\ 2 & -1 & -1\\ 1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}3\\ 3\\ 9\end{bmatrix}$$

Example

Solving the following system of equation with the Crout Method

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} \tag{86}$$

Example

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Therefore

$$l_{11} = a_{11} = 1 \qquad l_{21} = a_{21} = 2 \qquad l_{31} = a_{31} = 1 \tag{87}$$

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{1}{1} = 1 \tag{88}$$

$$u_{13} = \frac{a_{13}}{l_{11}} = \frac{1}{1} = 1 \tag{89}$$

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$$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23} = 1 - 1 + 2 = 2 \tag{93}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -2 & 2 \end{bmatrix} \tag{94}$$

and

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{95}$$

First solve $Ly = b$ to find y by forward substitution.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 9 \end{bmatrix} \tag{96}$$

First solve $Ly = b$ to find y by forward substitution.

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so

$$y_1 = 3 \tag{97}$$

$$2(3) - 3y_2 = 3 \implies y_2 = 1 \tag{98}$$

$$3 - 2(1) + 2y_3 = 9 \implies y_3 = 4 \tag{99}$$

Then solve $Ux = y$ to find x by back substitution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \tag{100}$$

Then solve $Ux = y$ to find x by back substitution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \tag{100}$$

Solving

$$x_3 = 4 \tag{101}$$

$$x_2 + 4 = 1 \implies x_2 = -3 \tag{102}$$

$$x_1 + (-3) + 4 = 3 \implies x_1 = 2 \tag{103}$$

Then solve $Ux = y$ to find x by back substitution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \tag{100}$$

Solving

$$x_3 = 4 \tag{101}$$

$$x_2 + 4 = 1 \implies x_2 = -3 \tag{102}$$

$$x_1 + (-3) + 4 = 3 \implies x_1 = 2 \tag{103}$$

The solution by the Crout's method is

$$x_1 = 2, \quad x_2 = -3, \quad x_3 = 4$$

Definition (Tridiagonal Systems)

When a matrix T is tridiagonal and nonsingular, its LU decomposition without pivoting yields bidiagonal matrices L and U. L has 1's on the main diagonal as usual, but the superdiagonal entries of U are the same as those of T.

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Example

If $T = \begin{bmatrix} 1 & 4 & 0 & 0 \\ -1 & 5 & 1 & 0 \\ 0 & 2 & -1 & -9 \\ 0 & 0 & 3 & 7 \end{bmatrix}$ then

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0.2 & 1 & 0 \\ 0 & 0 & -2.45 & 1 \end{bmatrix} \qquad (104)$$

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Example

If $T = \begin{bmatrix} 1 & 4 & 0 & 0 \\ -1 & 5 & 1 & 0 \\ 0 & 2 & -1 & -9 \\ 0 & 0 & 3 & 7 \end{bmatrix}$ then

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0.2 & 1 & 0 \\ 0 & 0 & -2.45 & 1 \end{bmatrix} \quad (104) \qquad U = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 9 & 1 & 0 \\ 0 & 0 & -1.2 & -9 \\ 0 & 0 & 0 & -15.09 \end{bmatrix} \quad (105)$$

Exercise

Solving the following system of equations using

1. LU decomposition method with Gaussian elimination
2. Doolittle's decomposition method
3. Crout decomposition method

$$
\begin{aligned}
10x + 4y - 2z &= 20 \\
3x + 12y - z &= 28 \\
x + 4y + 7z &= 2
\end{aligned}
$$

$$
\begin{aligned}
2a + b + c + d &= 2 \\
4a + 2c + d &= 3 \\
3a + 2b + 2c &= -1 \\
a + 3b + 2c + 6d &= 2
\end{aligned}
$$

END OF LECTURE

THANK YOU