# CSM 166: Discrete Mathematics for Computer Science

FINITE DIFFERENCE EQUATIONS - RECURRENCE RELATIONS

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# Content

# Introduction

## Definition 1 (Recurrence relation)

*A recurrence relation for a sequence $a_0, a_1, , \ldots$ is a relation that defines $a_n$ in terms of $a_0, a_1, , \ldots a_{n-1}$.*

The formula relating $a_n$ to earlier values in the sequence is called the **generating rule**.

The assignment of a value to one of the $a$'s is called an **initial condition**.

# Introduction

## Example 1

The Fibonacci Sequence $1, 1, 2, 3, 5, \ldots$ is a sequence in which every number after the first two is the sum of the preceding two numbers.

The initial conditions are $a_0 = a_1 = 1$ and the generating rule is

$$a_n = a_{n-1} + a_{n-2}; \quad n \geq 2$$

# Solution to a Recurrence Relation

A **solution** to a recurrence relation is an explicit formula for $a_n$ in terms of $n$ .

The fundamental method for finding the solution of a sequence defined recursively is by using **iteration**.

This involves starting with the initial values of the sequence and then calculates successive terms of the sequence until a pattern is observed.

An explicit formula for the sequence and then uses mathematical induction to prove its validity.

# Solution to a Recurrence Relation

## Example 2

Find a solution for the recurrence relation

$$\begin{cases} a_0 = 1 \\ a_n = a_{n-1} + 2, \quad n \geq 1 \end{cases}$$

# Solution to Example 2

Listing out some terms of the sequence:

$$a_0 = 1$$
$$a_1 = 1 + 2$$
$$a_2 = 1 + 4$$
$$a_3 = 1 + 6$$
$$a_4 = 1 + 8$$
$$a_1 = 1 + 10$$

A guessed formular is $a_n = 2n + 1$ , $n \geq 0$ and thus needs to be proven using mathematical induction.

# Solution to a Recurrence Relation

## Example 3

Consider the arithmetic sequence

$$a_n = a_{n-1} + d, \quad n \geq 1$$

where $a_0$ is the initial value. Find an explicit formula for $a_n$ .

# Solution to Example 3

Listing out the first 4 terms of the sequence:

$$a_1 = a_0 + d$$
$$a_2 = a_0 + 2d$$
$$a_3 = a_0 + 3d$$
$$a_4 = a_0 + 4d$$
$$a_5 = a_0 + 5d$$

A guessed formular is $a_n = a_0 + nd$ and thus

needs to be proven using mathematical induction.

# Solution to a Recurrence Relation

**Exercise A:**

1. Consider the geometric sequence

$$a_n = ra_{n-1}, \quad n \geq 1$$

   Where $a_0$ is the initial value. Find an explicit formula for $a_n$.

2. Find a solution to the recurrence relation

$$\begin{cases} a_0 = 0 \\ a_n = a_{n-1} + (n+1), \quad n \geq 1 \end{cases}$$

**Exercise B:** A function $y_n$ is defined recursively as follows:

$$\begin{cases} y_1 = 3 \\ y_2 = 7 \\ y_n = 3y_{n-1} - 2y_{n-2} \quad \text{for } n \geq 3 \end{cases}$$

Find an explicit formula or solution for $y_n$ in terms of $n$.

# Classification of Recurrence Relations

A recurrence relation is of first order if $y_n$ is defined only in terms of $y_{n-1}$.

It is of second order if $y_n$ is defined in terms of $y_{n-1}$ and $y_{n-2}$ , and so on.

A recurrence relation of the form $y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_k y_{n-k}$ is called a linear homogenous recurrence relation of order $k$

If the $a_i$ are **constants**, then the above equation is said to have constant coefficients.

# Classification of Recurrence Relations

Linear recurrence relations have the following important properties:

1. multiplying any solution by a constant gives another solution,
2. adding two or more solutions give another solution.

# First-Order Recurrence Relations

First-order recurrence relations are of the form:

$$\begin{cases} y_n = ay_{n-1} \\ y_0 = c \end{cases}$$

where $a$ and $c$ are constants.

First-order recurrence relations are solved by iteration:

$$\begin{aligned} y_n &= ay_{n-1} \\ &= a(ay_{n-2}) \\ &= a^2(ay_{n-3}) \\ &= \ldots\ldots \\ &= a^{n-1}y_1 \\ &= a^n y_0 \end{aligned}$$

Using the initial condition, we have $y_n = ca^n$, $n \in \mathbb{Z}^+$

# Second-Order Recurrence Relations

Second-order recurrence relations are of the form:

$$\begin{cases} y_n = ay_{n-1} + by_{n-2} \quad \text{for } n \geq 2 \\ y_0 = c_1 \\ y_0 = c_0 \end{cases} \tag{1}$$

Assuming $a$, $b$, $c_0$ and $c_1$ are constants and a trial function $y_n = ct^n$ to solve the relation above.

Using this assumption, $y_{n-1} = ct^{n-1}$ and $y_{n-2} = ct^{n-2}$

# Second-Order Recurrence Relations

Subtituting these into the recurrence relation (2:

$$ct^n = act^{n-1} + bct^{n-2}$$

Dividing through by $ct^{n-2}$:

$$t^2 = at + b \Rightarrow t^2 - at - b = 0 \qquad (2)$$

(2) is called the **auxiliary** or **characteristic**

equation of the recurrence relation.

# Roots of the Characteristic Equation

The characteristic equation is a quadratic equation whose roots may be:

I. two distinct real roots $t = t_1$ and $t = t_2$

II. repeated real roots $t = t_0$

III. two complex roots $t = t_1$ and $t_2 = \overline{t_1}$

# Solution to the Recurrence Relation

**CASE I**

Since $y_n = t_1^n$ and $y_n = t_2^n$ are solutions of the linear recurrence relation then another solution (the general solution) is

$$y_n = At_1^n + Bt_2^n \tag{3}$$

where $A$ and $B$ are arbitrary constants.

$A$ and $B$ are determined using the initial values $y_0 = c_0$ and $y_1 = c_1$.

# Solution to the Recurrence Relation

**CASE II**

Since $t = t_0$ is the repeated root of the characteristic equation, $y_n = t_0^n$ is a solution of the recurrence relation as well as the linearly independent solution $y_n = nt_0^n$.

Thus a general solution:

$$y_n = At^n + Bnt^n \tag{4}$$

where $A$ and $B$ are arbitrary constants.

$A$ and $B$ are determined using the initial values $y_0 = c_0$ and $y_1 = c_1$.

# Solution to the Recurrence Relation I

**CASE III**

The complex roots of the auxiliary equation with real coefficients occur in conjugate pair. i.e if $t_1 = u + iv$ then $t_2 = u - iv$ with $v \neq 0$

By the general rule, the solution

$$\begin{aligned} y_n &= At_1^n + Bt_2^n \\ &= A(u+iv)^n + B(u-iv)^n \end{aligned} \tag{5}$$

# Solution to the Recurrence Relation II

In polar form

$$u + iv = r(\cos\theta + i\sin\theta)$$
$$u + iv = r(\cos\theta - i\sin\theta)$$

And by DeMoivre's Theorem

$$[\rho(\cos\theta \pm i\sin\theta]^n = \rho^n(\cos n\theta \pm i\sin n\theta)$$

Thus

$$\begin{aligned} y_n &= A\rho^n(\cos n\theta + i\sin n\theta) + B\rho^n(\cos n\theta - i\sin n\theta) \\ &= (A+B)\rho^2(\cos n\theta) + i(A-B)\rho^2 \sin n\theta \end{aligned}$$

# Solution to the Recurrence Relation III

If we substitute $A = B = \frac{1}{2}$ , then $y_n = \rho^n \cos n\theta$ is a particular solution.

Similarly taking $A = -\frac{1}{2}i$ and $B = \frac{1}{2}i$, then $y_n = \rho^n \sin n\theta$ is also a particular solution. Thus the general solution is

$$y_n = A\rho^n \sin n\theta + B\rho^n \cos n\theta \tag{6}$$

where $\rho = \sqrt{u^2 + v^2}$ and $\theta = \tan^{-1} \frac{1}{2}$.

## Example 4

Solve the following

i $\begin{cases} y_n = 3y_{n-1} - 2y_{n-2} \text{ for } n \geq 2 \\ y_2 = 7 \\ y_1 = 3 \end{cases}$

ii $\begin{cases} y_n = 6y_{n-1} - 9y_{n-2} \text{ for } n \geq 1 \\ y_1 = 3 \\ y_0 = 5 \end{cases}$

iii $y_n + 2y_{n-1} + 2y_{n-2} = 0$

**Solution:**

The Characteristic equation is $t^2 - 6t + 2 = 0$

**Exercise C:** Solve the following difference equations:

1. $y_{n+1} - ay_n = 0,\ y_0 = 1$
2. $y_n = -3y_{n-1}$
3. $y_{n+2} + 2y_n = 0$
4. $y_n - 2y_{n-1} + 2y_{n-2} = 0$
5. $y_n + 4y_{n-1} + 8y_{n-2} = 0;\ y_1 = -1$
6. $y_n - 4y_{n-1} + 8y_{n-2} = 0;\ y_2 = 1, y_3 = -2$
7. $y_{n+2} + 2y_{n+1} + 4y_n = 0$

End of Lecture

Questions...???

Thanks