Chapter 18. State space models

18.1.

$$Q(\boldsymbol{\theta}, \boldsymbol{\theta}_{\text{old}}) = \mathbb{E}_{p(\mathbf{z}_{1:T}|\mathbf{y}_{1:T},\boldsymbol{\theta}_{\text{old}})}[\log p(\mathbf{z}_{1:T}, \mathbf{y}_{1:T}|\boldsymbol{\theta})]$$

$$= \mathbb{E}_{p(\mathbf{z}_{1:T}|\mathbf{y}_{1:T},\boldsymbol{\theta}_{\text{old}})}[\log p(\mathbf{z}_1) + \sum_{t=2}^{T} \log p(\mathbf{z}_t|\mathbf{z}_{t-1}) + \sum_{t=1}^{T} \log p(\mathbf{y}_t|\mathbf{z}_t)]$$

$$= \mathbb{E}_{p(\mathbf{z}_{1:T}|\mathbf{y}_{1:T},\boldsymbol{\theta}_{\text{old}})}[\log \mathcal{N}(\mathbf{z}_1|\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) + \sum_{t=2}^{T} \log \mathcal{N}(\mathbf{z}_t|\mathbf{A}_t\mathbf{z}_{t-1}, \mathbf{Q}_t) + \sum_{t=1}^{T} \log \mathcal{N}(\mathbf{y}_t|\mathbf{C}_t\mathbf{z}_t, \mathbf{R}_t)]$$

$$= \mathbb{E}_{p(\mathbf{z}_{1:T}|\mathbf{y}_{1:T},\boldsymbol{\theta}_{\text{old}})}[-\frac{1}{2}(\mathbf{z}_1 - \boldsymbol{\mu}_1)^T \boldsymbol{\Sigma}_1^{-1}(\mathbf{z}_1 - \boldsymbol{\mu}_1)$$

$$+ \sum_{t=2}^{T} -\frac{1}{2}(\mathbf{z}_t - \mathbf{A}_t\mathbf{z}_{t-1})^T \mathbf{Q}_t^{-1}(\mathbf{z}_t - \mathbf{A}_t\mathbf{z}_{t-1}) + \sum_{t=1}^{T} -\frac{1}{2}(\mathbf{y}_t - \mathbf{C}_t\mathbf{z}_t)^T \mathbf{R}_t^{-1}(\mathbf{y}_t - \mathbf{C}_t\mathbf{z}_t)$$

$$-\frac{1}{2}\log|\boldsymbol{\Sigma}_1| - \frac{1}{2}\sum_{t=2}^{T}\log|\mathbf{Q}_t| - \frac{1}{2}\sum_{t=1}^{T}\log|\mathbf{R}_t|] + \text{constant}$$

This quantity depends on three kinds of expectations, which will be denoted by the followings:

$$\hat{\mathbf{z}}_t = \mathbb{E}[\mathbf{z}_t|\mathbf{y}_{1:T}]$$

$$\mathbf{P}_t = \mathbb{E}[\mathbf{z}_t\mathbf{z}_t^T|\mathbf{y}_{1:T}]$$

$$\mathbf{S}_t = \mathbb{E}[\mathbf{z}_t\mathbf{z}_{t-1}^T|\mathbf{y}_{1:T}]$$

Note that the first state estimates differs from the one computed in a Kalman filter, because it depends on both past and future observations.

E-step:

We borrow auxillary statistics from the Section 18.3.2.1:

$$\boldsymbol{\mu}_{t|T} = \boldsymbol{\mu}_{t|t} + \mathbf{J}_t(\boldsymbol{\mu}_{t+1|T} - \boldsymbol{\mu}_{t+1|t})$$
$$\boldsymbol{\Sigma}_{t|T} = \boldsymbol{\Sigma}_{t|t} + \mathbf{J}_t(\boldsymbol{\Sigma}_{t+1|T} - \boldsymbol{\Sigma}_{t+1|t})\mathbf{J}_t^T$$
$$\mathbf{J}_t = \boldsymbol{\Sigma}_{t|t}\mathbf{A}_{t+1}^T\boldsymbol{\Sigma}_{t+1|t}^{-1}$$

Then the first two expectations can be computed as:

$$\hat{\mathbf{z}}_t = \boldsymbol{\mu}_{t|T}$$
$$\mathbf{P}_t = \boldsymbol{\Sigma}_{t|T} + \boldsymbol{\mu}_{t|T}\boldsymbol{\mu}_{t|T}^T$$

The third one can be computed as:

$$\mathbf{S}_t = \boldsymbol{\Sigma}_{t,t-1|T} + \boldsymbol{\mu}_{t|T}\boldsymbol{\mu}_{t-1|T}^T$$

where $\boldsymbol{\Sigma}_{t,t-1|T}$ can be obtained through the backward recursions:

$$\boldsymbol{\Sigma}_{t-1,t-2|T} = \boldsymbol{\Sigma}_{t-1|t-1}\mathbf{J}_{t-2}^T + \mathbf{J}_{t-1}(\boldsymbol{\Sigma}_{t,t-1|T} - \mathbf{A}_{t-1}\boldsymbol{\Sigma}_{t-1|t-1})\mathbf{J}_{t-2}^T$$

which is initialized by

$$\boldsymbol{\Sigma}_{T,T-1|T} = (\mathbf{I} - \boldsymbol{\Sigma}_{T|T-1}\mathbf{C}_T^T(\mathbf{C}_T\boldsymbol{\Sigma}_{T|T-1}\mathbf{C}_T^T + \mathbf{R}_T)^{-1}\mathbf{C}_T)\mathbf{A}_{T-1}\boldsymbol{\Sigma}_{T-1|T-1}.$$

M-step:
Output matrix:

$$\frac{\partial Q}{\partial \mathbf{C}_t} = -\mathbf{R}_t^{-1}\mathbf{y}_t\hat{\mathbf{z}}_t^T + \mathbf{R}_t^{-1}\mathbf{C}_t\mathbf{P}_t = 0$$
$$\Rightarrow \mathbf{C}_{t,\text{new}} = \mathbf{y}_t\hat{\mathbf{z}}_t^T\mathbf{P}_t^{-1}.$$

Output noise covariance:

$$\frac{\partial Q}{\partial \mathbf{R}_t^{-1}} = -\frac{\mathbf{R}_t}{2} - \frac{1}{2}\mathbf{y}_t\mathbf{y}_t^T + \mathbf{C}_t\hat{\mathbf{z}}_t\mathbf{y}_t^T - \frac{1}{2}\mathbf{C}_t\mathbf{P}_t\mathbf{C}_t^T = 0$$
$$\Rightarrow \mathbf{R}_{t,\text{new}} = \mathbf{y}_t\mathbf{y}_t^T - \mathbf{C}_{t,\text{new}}\hat{\mathbf{z}}_t\mathbf{y}_t^T.$$

State dynamics matrix:

$$\frac{\partial Q}{\partial \mathbf{A}_t} = -\mathbf{Q}_t^{-1}\mathbf{S}_t + \mathbf{Q}_t^{-1}\mathbf{A}_t\mathbf{P}_{t-1} = 0$$

$$\Rightarrow \mathbf{A}_{t,\text{new}} = \mathbf{S}_t \mathbf{P}_{t-1}^{-1}.$$

State noise covariance:

$$\frac{\partial Q}{\partial \mathbf{Q}_t^{-1}} = \frac{1}{2}\mathbf{Q}_t - \frac{1}{2}(\mathbf{P}_t - \mathbf{A}_{t,\text{new}}\mathbf{S}_t^{-1}) = 0$$

$$\Rightarrow \mathbf{Q}_{t,\text{new}} = \mathbf{P}_t - \mathbf{A}_{t,\text{new}}\mathbf{S}_t^{-1}.$$

Initial state mean:

$$\frac{\partial Q}{\partial \boldsymbol{\mu}_1} = (\hat{\mathbf{z}}_1 - \boldsymbol{\mu}_1)\boldsymbol{\Sigma}_1^{-1} = 0$$

$$\Rightarrow \boldsymbol{\mu}_{1,\text{new}} = \hat{\mathbf{z}}_1.$$

Initial state covariance:

$$\frac{\partial Q}{\partial \boldsymbol{\Sigma}_1^{-1}} = \frac{1}{2}\boldsymbol{\Sigma}_1 - \frac{1}{2}(\mathbf{P}_1 - \hat{\mathbf{z}}_1\boldsymbol{\mu}_1^T - \boldsymbol{\mu}_1\hat{\mathbf{z}}_1^T + \boldsymbol{\mu}_1\boldsymbol{\mu}_1^T) = 0$$

$$\Rightarrow \boldsymbol{\Sigma}_{1,\text{new}} = \mathbf{P}_1 - \hat{\mathbf{z}}_1\hat{\mathbf{z}}_1^T.$$

18.2.
$y_t = a_t + c_t + \epsilon_t^y$
$a_t = a_{t-1} + b_{t-1} + \epsilon_t^a$
$b_t = b_{t-1} + \epsilon_t^b$
$c_t = -c_{t-1} - c_{t-2} - c_{t-3} + \epsilon_t^c$
If we define $\mathbf{z}_t = (a_t, b_t, c_t, c_{t-1}, c_{t-2})$ for $t > 3$, then we have:

$$\begin{pmatrix} a_t \\ b_t \\ c_t \\ c_{t-1} \\ c_{t-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{t-1} \\ b_{t-1} \\ c_{t-1} \\ c_{t-2} \\ c_{t-3} \end{pmatrix} + \begin{pmatrix} \epsilon_t^a \\ \epsilon_t^b \\ \epsilon_t^c \\ 0 \\ 0 \end{pmatrix}$$

$$y_t = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_t \\ b_t \\ c_t \\ c_{t-1} \\ c_{t-2} \end{pmatrix} + \epsilon_t^y$$

Therefore,

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{Q} = \begin{pmatrix} Q_a & 0 & 0 & 0 & 0 \\ 0 & Q_b & 0 & 0 & 0 \\ 0 & 0 & Q_c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} R \end{pmatrix}.$$