An elementary proof of QA-P2 and QA-DT's concyclicity

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1 Theorem Statement and its Proof

This is the most interesting plane geometry theorem that I found in high school:

Theorem 1. *Let $ABCD$ be a complete quadrilateral, and denote its three diagonal points by:*

$$E = AB \cap CD, \quad F = AC \cap BD, \quad G = AD \cap BC.$$

Let Q be the unique common point of the nine-point circles of the four triangles:

$$\triangle ABC, \quad \triangle BCD, \quad \triangle CDA, \quad \text{and} \quad \triangle DAB.$$

Then the four points E, F, G, and Q are concyclic; that is, they lie on a common circle.

The proof of Theorem 1 needs the following lemma:

Lemma 1. *Let A, B, C, P be four distinct points in the plane, and let S be a point not equal to A, B, C, P. For any point E, let E' denote its reflection across S. If the following conditions hold:*

1. *A, B, C', P' are concyclic,*
2. *A, B', C, P' are concyclic,*
3. *A', B, C, P' are concyclic,*

then the points A, B, C, P are on a circle.

Proof of Lemma 1. We denote the directed angle between lines XY and ZW as $\angle(XY, ZW)$, taken modulo $180°$. It suffices to prove that

$$\angle(AC, AB) = \angle(PC, PB). \tag{1}$$

Since A, B', C, P' are concyclic, we have

$$\angle(AP', AC) = \angle(B'P', B'C). \tag{2}$$

Add (1), (2) together we only need to prove

$$\angle(AP', AB) = \angle(PC, PB) + \angle(B'P', B'C). \tag{3}$$

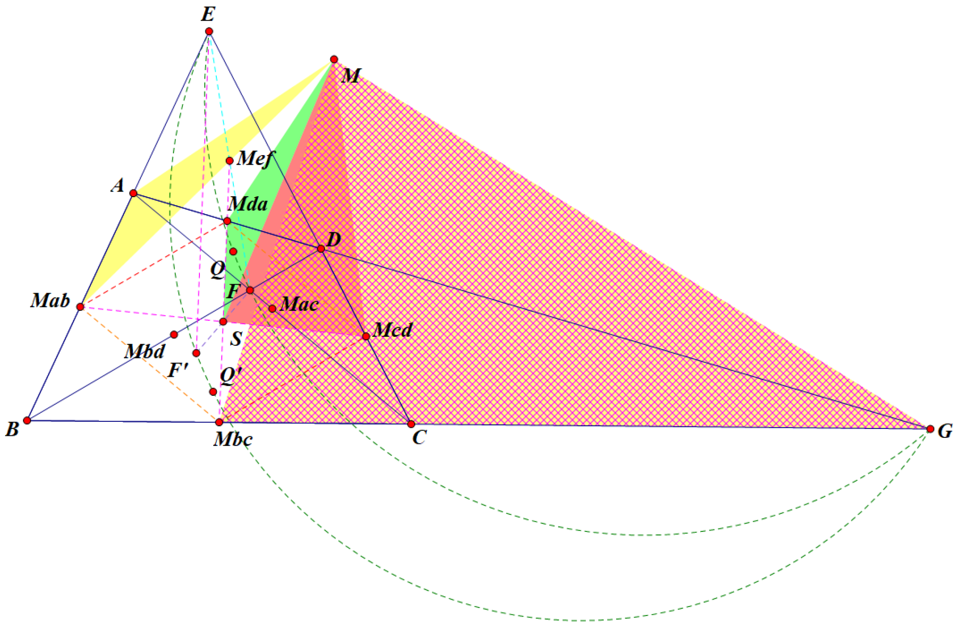

Figure 1: Illustration

Since A, B, C', P' are concyclic and $PC \parallel P'C', PB \parallel P'B'$, we have

$$\angle(AP', AB) = \angle(C'P', C'B), \quad \angle(PC, PB) = \angle(P'C', P'B'). \tag{4}$$

Take (4) into (3), it remains to prove

$$\angle(C'P', C'B) = \angle(P'C', P'B') + \angle(B'P', B'C), \tag{5}$$

which is obvious. Then we finish the proof. □

Proof of Theorem 1. Let $M_{ab}, M_{bc}, M_{cd}, M_{da}, M_{ac}, M_{bd}$ be the midpoint of line segment AB, BC, CD, DA, AC, BD respectively. It is easy to see that $M_{ab}M_{bc} \parallel M_{cd}M_{da}$ and $M_{ab}M_{bc} = M_{cd}M_{da}$. Let S be the center of parallelogram $M_{ab}M_{bc}M_{cd}M_{da}$. For any point P, let P' denote its reflection across S. Let M be another intersection of circumcircles of $\triangle EAD$ and $\triangle EBC$. Then M is the Miquel point.

We claim that Q', M, M_{ab}, M_{cd} are concyclic. This is because

$$\begin{aligned}
\angle(M_{ab}Q', M_{cd}Q') &= \angle(M_{cd}Q, M_{ab}Q) \\
&= \angle(M_{cd}Q, M_{ac}Q) + \angle(M_{ac}Q, M_{ab}Q) \\
&= \angle(M_{cd}M_{da}, M_{ac}M_{da}) + \angle(M_{ac}M_{bc}, M_{ab}M_{bc}) \\
&= \angle(AC, DC) + \angle(BA, CA) \\
&= \angle(BA, DC) = \angle(BE, CE) = \angle(M_{ab}M, M_{cd}M),
\end{aligned} \tag{6}$$

where the last equality uses the property of Miquel point.

By the same reason we have Q', M, M_{ad}, M_{bc} are concyclic. Hence, we have

$$\begin{aligned}
\angle(EQ', GQ') &= \angle(EQ', MQ') + \angle(MQ', GQ') \\
&= \angle(EM_{ab}, MM_{ab}) + \angle(MM_{bc}, GM_{bc}) \\
&= \angle(M_{da}S, MS) + \angle(MS, M_{cd}S) \\
&= \angle(M_{da}S, M_{cd}S),
\end{aligned} \tag{7}$$

where in the third equality, we use the property that $\triangle MAB \sim \triangle MM_{da}M_{bc}, \triangle MBC \sim \triangle MM_{ab}M_{cd}$ and S is the midpoint of both $M_{da}M_{bc}$ and $M_{ab}M_{cd}$.

Denote M_{ef} be the midpoint of EF, by the property of complete quadrilateral, we have M_{ef}, M_{ad}, M_{bc} are collinear, so SM_{da} is the midline of $\triangle FF'E$ and $SM_{da} \parallel EF'$. Similarly we have $SM_{bc} \parallel GF'$. Then (7) becomes

$$\angle(EQ', GQ') = \angle(EF', GF'). \tag{8}$$

Therefore, Q', F', E, G are concyclic. Similarly, Q', F, E', G are concyclic and Q', F, E, G' are concyclic. By Lemma 1, we have Q, E, F, G are cyclic. □

2 Literature Review (by LLM)

The geometry of the complete quadrilateral is a classical field of study that reveals deep connections between projective and metric properties. A central result in this configuration is the theorem stating that the **Euler-Poncelet point** (Q)—the common intersection of the four nine-point circles of the triangles formed by a quadrangle—is concyclic with the three diagonal points (E, F, G). This theorem serves as a bridge between the metric world of circles and the projective world of line intersections.

2.1 Historical Background: The Rectangular Hyperbola View

The discovery of Q dates back to **Charles Brianchon and Jean-Victor Poncelet (1821)**. Their proof was rooted in the theory of conics rather than synthetic circle geometry.

A unique rectangular hyperbola $\mathcal{H}$ passes through the four vertices A, B, C, D. Brianchon and Poncelet established two fundamental properties that prove the theorem:

1. **The Center Lemma:** The center of any rectangular hyperbola passing through the vertices of a triangle lies on the nine-point circle of that triangle. Thus, the center Q of $\mathcal{H}$ must be the concurrence point of the four nine-point circles.

2. **The Self-Conjugate Property:** The diagonal triangle EFG is *self-conjugate* with respect to any conic passing through A, B, C, D. A known property of the rectangular hyperbola is that its center Q lies on the circumcircle of any triangle self-conjugate to it.

This duality between the center of a conic and the circumcircle of its self-conjugate triangle provides the most direct classical explanation for the concyclicity.

2.2 The Complex Number Method

The complex coordinate approach provides a robust analytical confirmation of the theorem, effectively bridging the gap between specific configurations and the general case. The Cyclic Case and the Anticenter. The proof is most intuitively initiated by considering the case where the quadrilateral $ABCD$ is cyclic (inscribed in a circle). In this specific setting, the Euler-Poncelet point Q coincides with the "anticenter" of the quadrilateral. By representing the vertices as complex numbers on the unit circle, one can demonstrate that Q is the common intersection of the four nine-point circles of the triangles formed by the vertices. In this restricted case, the coordinates of Q and the diagonal points E, F, G exhibit a high degree of symmetry, and their concyclicity is confirmed by showing that their complex cross-ratio is a purely real number. Generalization via Rectangular Hyperbolas. The extension of this result to any general quadrilateral (non-cyclic) was addressed through the study of rectangular hyperbolas, a method notably advanced by the **Morley school** and analytical geometers like **Roger A. Johnson**. The key insight lies in the fact that any four points in a plane define a unique rectangular hyperbola. The Euler-Poncelet point Q is identified as the center of this hyperbola. The Projective Argument. To solve the general case without exhaustive coordinate expansion, geometers employ a "circular" transformation or projective mapping. By treating the rectangular hyperbola as a projection of a circle in a higher-dimensional or inversive space, the geometric properties are preserved. The diagonal triangle EFG is shown to be self-conjugate with respect to the hyperbola. Under this specific algebraic symmetry, the relationship between the points and their complex conjugates allows the complex components of the cross-ratio to cancel out entirely. This ensures that the cross-ratio remains real regardless of the quadrilateral's shape, proving that Q must always lie on the circumcircle of the diagonal triangle.

2.3 The Projective-Metric Duality

The significance of this theorem lies in its synthesis of two distinct geometric frameworks:

- **The Metric Framework:** The four nine-point circles are defined by midpoints and orthogonality, making them dependent on Euclidean distance and angles.
- **The Projective Framework:** The diagonal points E, F, and G are defined purely by the incidences of lines ($AB \cap CD$, etc.), remaining invariant under projective transformations.

The concyclicity of these four points demonstrates that the diagonal triangle is not merely an incidental construction but acts as a structural anchor for the quadrilateral's higher-order centers.