Study on Problem 4 of the 2021 China High School Mathematics League (Add-on Round)

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1. Original Problem and solution

Original Problem: Find the smallest positive constant c such that for any integer $n \geq 4$, and any subset $A \subseteq \{1, 2, 3, \ldots, n\}$, if $|A| > cn$, then there exists a function $f : A \to \{1, -1\}$ satisfying

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq 1.$$

On one hand, we can prove $c \geq \frac{2}{3}$. Conversely, suppose $c < \frac{2}{3}$. Take $n = 6$, $A = \{1, 4, 5, 6\}$. Then $|A| > cn$. Note that

$$\sum_{a \in A} f(a) \cdot a \equiv \sum_{a \in A} a \equiv 0 \pmod 2,$$

and $\left|\sum_{a \in A} f(a) \cdot a\right| \leq 1$ implies $\sum_{a \in A} f(a) \cdot a = 0$. But this contradicts the fact that A has no subset summing to $\frac{16}{2} = 8$.

We now prove that $c = \frac{2}{3}$ is feasible. First, we prove the following lemma:

Lemma 1. *Let $a_1, a_2, \ldots, a_n$ be nonzero integers with $|a_i| \leq i$ for $i = 1, 2, \ldots, n$. Then there exist $\varepsilon_i \in \{-1, 1\}$ such that*

$$\left| \sum_{i=1}^{n} \varepsilon_i a_i \right| \leq 1.$$

Proof. We proceed by induction on n. The result is trivial for $n = 1, 2$. Assume it holds for all $n \leq k$, and consider $n = k + 1$. Without loss of generality, assume all a_i are positive (otherwise replace a_i with $-a_i$, which does not affect the conclusion). Since $1 \leq a_{k+1} \leq k+1$ and $1 \leq a_k \leq k$, we have $|a_{k+1} - a_k| \leq k$.

If $|a_{k+1} - a_k| = 0$, apply the induction hypothesis to $a_1, a_2, \ldots, a_{k-1}$. Otherwise, if $|a_{k+1} - a_k| \leq k$, apply the induction hypothesis to $a_1, a_2, \ldots, a_{k-1}, |a_{k+1} - a_k|$. The lemma follows. $\square$

Now return to the original problem. Let $A = \{a_1, a_2, \ldots, a_t\} \subseteq \{1, 2, \ldots, n\}$ with $a_1 < a_2 < \cdots < a_t$, and $t > \frac{2n}{3}$.

(1) Suppose t is even. Consider the $\frac{t}{2}$ positive integers:

$$a_t - a_{t-1},\ a_{t-2} - a_{t-3},\ \ldots,\ a_2 - a_1.$$

We claim that for every $m \in \{0, 1, 2, \ldots, \frac{t}{2} - 1\}$, at most m of these $\frac{t}{2}$ numbers are $\geq \frac{t}{2} - m + 1$. (*)

Otherwise, suppose there exists $m \in \{0, 1, 2, \ldots, \frac{t}{2} - 1\}$ such that at least $m + 1$ of them are $\geq \frac{t}{2} - m + 1$. Then

$$\begin{aligned} a_t &= (a_t - a_{t-1}) + (a_{t-1} - a_{t-2}) + \cdots + (a_3 - a_2) + (a_2 - a_1) + a_1 \\ &\geq (m+1)\left(\frac{t}{2} - m + 1\right) + (t - (m+1)) \cdot 1 \\ &= t + (m+1)\left(\frac{t}{2} - m\right). \end{aligned}$$

Using the monotonicity of the quadratic function in m, we know

$$t + (m+1)\left(\frac{t}{2} - m\right) \geq \frac{3}{2}t,$$

with equality minimized at $m = 0$ or $m = \frac{t}{2} - 1$. Thus $a_t \geq \frac{3}{2}t > n$, a contradiction.

From (*) it follows that for every $m \in \{0, 1, 2, \ldots, \frac{t}{2} - 1\}$, at least $\frac{t}{2} - m$ of the differences are $\leq \frac{t}{2} - m$. Let $i = \frac{t}{2} - m$. Then for every $i \in \{1, 2, \ldots, \frac{t}{2}\}$, at least i of the $\frac{t}{2}$ numbers are $\leq i$.

Rearranging $a_t - a_{t-1}, a_{t-2} - a_{t-3}, \ldots, a_2 - a_1$ in increasing order as $b_1 \leq b_2 \leq \cdots \leq b_{\frac{t}{2}}$, we have $0 < b_i \leq i$. By the lemma, there exist $\varepsilon_i \in \{-1, 1\}$ such that

$$\left| \sum_{i=1}^{t/2} \varepsilon_i b_i \right| \leq 1.$$

Expressing each b_i as a difference of elements from A, we conclude the desired result.

(2) Suppose t is odd. Consider the $\frac{t+1}{2}$ positive integers:

$$a_t - a_{t-1},\ a_{t-2} - a_{t-3},\ \ldots,\ a_3 - a_2,\ a_1.$$

A similar argument applies. We claim that for every $m \in \left\{0, 1, 2, \ldots, \frac{t-1}{2}\right\}$, at most m of these $\frac{t+1}{2}$ numbers are $\geq \frac{t+1}{2} - m + 1$. ($\triangle$)

Otherwise, suppose there exists $m \in \left\{0, 1, 2, \ldots, \frac{t-1}{2}\right\}$ such that at least $m+1$ of them are $\geq \frac{t+1}{2} - m + 1$. Then

$$\begin{aligned} a_t &\geq (m+1)\left(\frac{t+1}{2} - m + 1\right) + (t - (m+1)) \cdot 1 \\ &= t + (m+1)\left(\frac{t+1}{2} - m\right). \end{aligned}$$

Using the monotonicity of the quadratic function, we get

$$t + (m+1)\left(\frac{t+1}{2} - m\right) \geq \frac{3t+1}{2},$$

so $a_t \geq \frac{3t+1}{2} > n$, again a contradiction.

From $(\triangle)$, for every $m \in \left\{0, 1, 2, \ldots, \frac{t-1}{2}\right\}$, at least $\frac{t+1}{2} - m$ of the numbers are $\leq \frac{t+1}{2} - m$. Let $i = \frac{t+1}{2} - m$. Then for every $i \in \left\{1, 2, \ldots, \frac{t+1}{2}\right\}$, at least i of the $\frac{t+1}{2}$ numbers are $\leq i$.

Rearranging them as $b_1 \leq b_2 \leq \cdots \leq b_{\frac{t+1}{2}}$, we have $0 < b_i \leq i$. Again, by the lemma, there exist $\varepsilon_i \in \{-1, 1\}$ such that

$$\left| \sum_{i=1}^{(t+1)/2} \varepsilon_i b_i \right| \leq 1.$$

Expressing each b_i as a difference of elements from A, the proof is complete.

2. Optimal bound for large n

In fact, for large n, $\frac{2}{3}$ is not the best factor. Take $A = \{2, 4, \cdots, n\}$ when n is even we know that $c \geq \frac{1}{2}$. We show that $c = \frac{1}{2}$ is sufficient and thus optimal when n is large.

First, we introduce some commonly used notation. We use uppercase letters such as $A, B, C, \ldots$ or $A', B', C', \ldots$ or $A'', B'', C'', \ldots$ and so on, to denote sets, and barred letters $\bar{A}, \bar{B}, \bar{C}, \ldots$ to denote multisets. By $A - B$ we means $A \backslash (A \cap B)$. We write $[n] = \{1, 2, \ldots, n\}$, and for any set X or multiset $\bar{X}$, we let $|X|$ or $|\bar{X}|$ denote its cardinality, counted with multiplicity in the case of a multiset. In our context, all sets are subsets of $\mathbb{N}$. Denote $\mathcal{L}(A, m) = |A \cap [m, +\infty)|$ and $\mathcal{L}(\bar{A}, m) = |\bar{A} \cap [m, +\infty)|$.

Let $A^{\uparrow} = \{a_i\}_{i=1}^{|A|}$ denote the sequence of elements of A sorted in increasing order , $D(A) = \max_{i=0}^{|A|-1}(a_{i+1} - a_i)$ $(a_0 = 0)$. If $|A|$ is even, we denote $\bar{A} = \{a_2 - a_1, a_4 - a_3, \cdots, a_{|A|} - a_{|A|-1}\}$; otherwise we denote $\bar{A} = \{a_1, a_3 - a_2, \cdots, a_{|A|} - a_{|A|-1}\}$.

Our strengthened theorem is stated as follows:

Theorem 1. *There exists an absolute constant N such that For any integer $n \geq N$, and any subset $A \subseteq \{1, 2, 3, \ldots, n\}$, if $|A| > \frac{n}{2}$, then there exists a function $f : A \to \{1, -1\}$ satisfying*

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq 1.$$

Next, we introduce key definitions related to our theorem and proof.

Definition 1. *We say that A (or $\bar{A}$) is s-**excellent** if there exists a function $f : A \to \{1, -1\}$ or $f : \bar{A} \to \{1, -1\}$ satisfying*

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq s \quad \text{or} \quad \left| \sum_{a \in \bar{A}} f(a) \cdot a \right| \leq s$$

*If equality holds, we say that A (or $\bar{A}$) is **exactly** s-**excellent**.*

By definition, it is easy to check the proposition below:

Proposition 1. *The following statements hold:*

1. *If $A, B \subset \mathbb{N}$ are disjoint, A is s-excellent and B is exactly t-excellent, then $A \cup B$ is $|s-t|$- excellent.*
2. *If $\bar{A}$ is s-excellent, then A is s-excellent.*
3. *Denote $P_1(A) \triangleq \{i \in [0, |A|-1] : a_{i+1} - a_i = 1\}$, then $|P_1(A)| \geq 2|A| - n$.*

Proof. The second statement is obvious by definition. Let f, g be the corresponding function for A, B in Definition 1. By taking $h : A \cup B \to \{1, -1\}$ as $h(A) = f(A), h(B) = \pm g(B)$ we can prove the first statement. If the third statement does not hold, let $t = |A|$, then $a_t > 2t-n+2(t-(2t-n)) = n$ (obviously we only need to discuss the case when $2t-n-1 > 0$) a contradiction. $\square$

A special kind of multi-sets are $(r+1)$-excellent, as the following lemmas stated:

Lemma 2. *Let $r \in \mathbb{N}$, $a_1, a_2, \ldots, a_n$ be nonzero integers with $|a_i| \leq i + r$ for $i = 1, 2, \ldots, n$. Then there exist $\varepsilon_i \in \{-1, 1\}$ such that*

$$\left| \sum_{i=1}^{n} \varepsilon_i a_i \right| \leq r + 1.$$

Proof. We proceed by induction on n. The result is trivial for $n = 1, 2$. Assume it holds for all $n \leq k$, and consider $n = k+1$. Without loss of generality, assume all a_i are positive (otherwise replace a_i with $-a_i$, which does not affect the conclusion). Since $1 \leq a_{k+1} \leq k + 1 + r$ and $1 \leq a_k \leq k + r$, we have $|a_{k+1} - a_k| \leq k + r$.

If $|a_{k+1} - a_k| = 0$, apply the induction hypothesis to $a_1, a_2, \ldots, a_{k-1}$. Otherwise, if $|a_{k+1} - a_k| \leq k + r$, apply the induction hypothesis to $a_1, a_2, \ldots, a_{k-1}, |a_{k+1} - a_k|$. The lemma follows. $\square$

We now present a useful lemma that characterizes when a set is excellent.

Lemma 3. *Let $A \subseteq \mathbb{N}^*$, $r \in \mathbb{N}$. If for all $m \in [0, |A|) \cap \mathbb{N}$, $\mathcal{L}(\bar{A}, |\bar{A}| + r + 1 - m) \leq m$, then A is $(r+1)$-excellent.*

Proof. Consider $\bar{A}^{\uparrow} = \{a_1, \cdots, a_{|\bar{A}|}\}$. If $a_i \geq i + r + 1$, then $\mathcal{L}(\bar{A}, i + r + 1) > |\bar{A}| - i$, a contradiction. Hence, by Lemma 2, $\bar{A}$ is $(r+1)$-excellent, so A is also $(r+1)$-excellent. $\square$

From now on, we assume n is sufficiently **large** ($n \geq 612$ actually) and prove Theorem 1 in this case. Then the rest could be verified by computer.

The next lemma shows that if $|A| > 0.4n + 1.2$, then Lemma 3's condition almost holds for $r = 1$. This suggests A is probably 2-excellent.

Lemma 4. *Let $A \subseteq [n]$ and $|A| > \frac{2(n+3)}{5}$, then for all $m \in \{2, 3, \cdots, |\bar{A}| - 2\}$, we have*

$$\mathcal{L}(\bar{A}, |\bar{A}| + 2 - m) \leq m. \tag{1}$$

Proof. Let $t = |A|$ and $A^{\uparrow} = \{a_i\}_{i=1}^{t}$. If $t = 2p$ is even, consider the p positive integers:

$$\bar{A} = \{a_t - a_{t-1},\ a_{t-2} - a_{t-3},\ \ldots,\ a_2 - a_1\}.$$

Suppose there exists $m \in \{2, 3, \ldots, p-2\}$ such that $\mathcal{L}(\bar{A}, p+2-m) \geq 1+m$. Then

$$\begin{aligned} a_t &= (a_t - a_{t-1}) + (a_{t-1} - a_{t-2}) + \cdots + (a_3 - a_2) + (a_2 - a_1) + a_1 \\ &\geq (m+1)(p-m+2) + (t-(m+1)) \cdot 1 \\ &= t + (m+1)(p-m+1). \end{aligned} \tag{2}$$

Using the monotonicity of the quadratic function in m, we know

$$t + (m+1)(p-m+1) \geq \frac{5}{2}t - 3 > n,$$

with equality minimized at $m = 2$ or $m = p-2$. Thus $a_t > n$, a contradiction.

If $t = 2p+1$ is odd, consider the $p+1$ positive integers:

$$\bar{A} = \{a_t - a_{t-1},\ a_{t-2} - a_{t-3},\ \ldots,\ a_2 - a_1, a_1\}.$$

A similar argument applies. Suppose there exists $m \in \{2, 3, \ldots, p-1\}$ such that $\mathcal{L}(\bar{A}, p+3-m) \geq 1+m$. Then

$$\begin{aligned} a_t &\geq (m+1)(p+3-m) + (t-(m+1)) \cdot 1 \\ &= t + (m+1)(p+2-m). \end{aligned} \tag{3}$$

Using the monotonicity of the quadratic function, we get

$$t + (m+1)(p+2-m) \geq \frac{5}{2}t - \frac{3}{2} > n,$$

so $a_t > n$, again a contradiction. □

Let us see what is remained to discuss. If (1) does not hold for $m = 0$ or $m = 1$, then $\mathcal{L}(\bar{A}, |\bar{A}|+1) \geq 1$, so there exists j such that $a_{j+1} - a_j \geq |\bar{A}| + 1$. That is to say, if $D(A) < \frac{|A|}{2} + 1$, then (1) holds for $m = 0, 1$. We can first deal with this case as follows:

Theorem 2. *Let $A \subseteq [n]$. Suppose that $|A| > \frac{n}{2}$, and for any $B = A - \{x, x+1\}$ such that $x, x+1 \in A$ or $B = A - \{1\}$ such that $1 \in A$, $D(B) < \frac{|A|}{2}$, then A is 1-excellent.*

Proof. First consider the case when $|A| > \frac{n+1}{2}$.

(1) $1 \in A$. Let $A' = A - \{1\}$. Then $D(A') < \frac{|A|}{2} = \frac{|A'|}{2} + 1$. Let $A'^{\uparrow} = \{a_i\}_{i=1}^{t}$. Consider (1) for $\bar{A}'$ and $m = |\bar{A}'| - 1$. If $t = 2p+1$ is odd, then (3) becomes

$$a_t \geq t + (m+1)(p+2-m) = 2t + 1 > 2 \cdot \frac{n-1}{2} + 1 = n,$$

a contradiction. If $t = 2p$ is even, then (2) becomes

$$a_t \geq t + (m+1)(p-m+1) = 2t > n-1.$$

This implies that all inequalities we use in (2) become equality. In this case, $a_1 = 1$, but this contradicts to $1 \notin A'$. Hence, A' satisfied the conditions in Lemma 4, so A' is 2-excellent. Since $\{1\}$ is exactly 1-excellent, A is 1-excellent by Proposition 1.

(2) $1 \notin A$. By Proposition 1, there exists $x, y \in A$ such that $x - y = 1$. Denote $x^* = \min\{x : x, x+1 \in A\}$ and $A' = A - \{x^*, x^*+1\}$. Then $D(A') < \frac{|A|}{2} \leq \frac{|A'|}{2} + 1$. Consider the case when (1) does not hold for $\bar{A}'$ and $m = |\bar{A}'| - 1$. If t is even, similar to **(1)** we have $a_t \geq 2t - 1 + a_1 > n - 4 + a_1$, so $a_1 \leq 3$. Since we have deleted x^* from A, and $a_1 \leq 3$, it suggests that $a_{i+1} - a_i \leq 3$ cannot hold for all i. So the bound improves to $a_1 \leq 2$, $a_1 = 2$. Then by Proposition 1, we only need $A'' = A - \{x^*, x^*+1, a_1\}$ to be 4-excellent. By Lemma 4, it only remains to show $\mathcal{L}(\bar{A}'', 5) \leq |\bar{A}''| - 1$, which is obvious since $|\bar{A}''| = \lceil \frac{t-3}{2} \rceil > \frac{n}{5} + 1$.

If t is odd and (1) does not hold for $m = |\bar{A}'| - 1$, then $a_1 \geq 3$. If $a_1 = 3$, then $a_2 - a_1 \geq 2$ by the definition of x^*. So a_2 is at least 5. $n - 5 \geq a_t - a_2 \geq 2t - 2$ implies all inequalities for $a_2, a_{i+1} - a_i$, $i \geq 2$ become equality. But in this case $a_{i+1} - a_i \leq 3$ for all i, a contradiction to a possible choice of x^*. Hence, A' satisfied the conditions in Lemma 4, so A' is 2-excellent. Since $\{x^*, x^*+1\}$ is exactly 1-excellent, we have A is 1-excellent by Proposition 1.

Second, consider the case when $n = 2t + 1$ is even and $|A| = t + 1$.

(1) $1 \in A$. Let $A' = A - \{1\}$. Then $D(A') < \frac{|A|}{2} = \frac{|A'|}{2} + 1$. Let $A'^{\uparrow} = \{a_i\}_{i=1}^t$. Consider the case when (1) does not hold for $\bar{A}'$ and $m = |\bar{A}'| - 1$. If $t = 2p + 1$ is odd, then (3) becomes

$$a_t \geq t + (m+1)(p + 2 - m) = 2t + 1 = n,$$

so all inequalities becomes equality. We have $A' = \{3, 4, 7, 8, 11, \cdots\}$. Note that both $\{3, 4, 7\}$ and $\{3, 4, 7, 8, 11, 12, 15, 16, 19\}$ are 2-excellent, so A' must be 2-excellent, then A is 1-excellent.

If $t = 2p$ is even, then (2) becomes

$$a_t - a_1 \geq t - 1 + (m+1)(p - m + 1) = 2t - 1.$$

Since $a_1 \geq 2$, all inequalities we use in (2) for $a_1, a_{i+1} - a_i$ become equality. Hence $a_1 = 2$ and $A' = \{2, 5, 6, 9, \cdots\}$. Both $\{2, 5, 6, 9\}$ and $\{2, 5, 6, 9, 10, 13\}$ are 2-excellent, so A' must be 2-excellent, then A is 1-excellent.

(2) $1 \notin A$. By Proposition 1, there exists $x, y \in A$ such that $x - y = 1$. Denote $x^* = \min\{x : x, x+1 \in A\}$ and $A' = A - \{x^*, x^*+1\}$. Then $|A'| = t - 1$, $D(A') < \frac{|A|}{2} \leq \frac{|A'|}{2} + 1$. Consider the case when (1) does not hold for $\bar{A}'$ and $m = |\bar{A}'| - 1$

If $t - 1$ is even, similar to **(1)** we have $a_{t-1} \geq 2t - 3 + a_1 \geq n - 4 + a_1$, so $a_1 \leq 4$.

- $a_1 = 3$. Then $3 \leq a_2 - a_1 \leq 4$. If $a_2 = 7$, then all inequalities for $a_{i+1} - a_i, i \geq 2$ we use must become equality. Since we have deleted x^* from A, and $a_1 \leq 3$, $a_{i+1} - a_i \leq 3, i \geq 3$, it suggests that $x^* = 5$, $A = \{3, 5, 6, 7, 8, 11, \cdots\}$. Since both $\{3, 5, 6, 7\}$ and $\{3, 5, 6, 7, 8, 11\}$ are 1-excellent. A is 1-excellent. If $a_2 = 6$, then we only need $A'' = A - \{x^*, x^*+1, a_1, a_2\}$ to be 5-excellent. Since $D(A'') = D(A') \leq 5$, by Lemma 4, it only remains to show $\mathcal{L}(\bar{A}'', 6) \leq |\bar{A}''| - 1$, which is obvious since $|\bar{A}''| = \lceil \frac{t-3}{2} \rceil > \frac{n}{6} + 1$.

- $a_1 = 4$. Then all inequalities for $a_{i+1} - a_i, i \geq 1$ must become equality. In this case, $a_{i+1} - a_i \leq 3$, so $x^* = 2$, $A = \{2, 3, 4, 7, 8, 11, \cdots\}$. It is easy to check that A is 1-excellent.

If $t-1$ is odd, then $n - 3 \geq a_{t-1} - a_2 \geq 2t - 5 = n - 6$, so $2 \leq a_1 < a_2 \leq 6$ and $a_{i+1} - a_i \leq 6$ for $i \geq 2$.

- $a_2 = 6$, then all inequalities for $a_{i+1} - a_i, i \geq 2$ we use must become equality. Hence, $2 \leq x^* < x^* + 1 < 6$. This implies $A = B \cup \{6, 9, 10, 13, \cdots\}$ where $B \subset [2, 5]$ and $|B| = 3$. It is easy to check that A is 1-excellent whatever B is.
- $a_2 \leq 5$, then $a_2 - a_1 = 1$ or 2 or 3. If $a_2 - a_1 = 2$ or 1, we only need $A'' = A - \{x^*, x^* + 1, a_2, a_1\}$ will be 3-excellent. Since $D(A'') = D(A') \leq 6$, by Lemma 4, it only remains to show $\mathcal{L}(\bar{A}'', 4) \leq |\bar{A}''| - 1$, which is obvious since $|\bar{A}''| = \lceil \frac{t-3}{2} \rceil$ and $n - 4 \geq a_t - a_3 \geq 4\mathcal{L}(\bar{A}'', 4) + |A''| - |\bar{A}''|$. The right side is about $\frac{5n}{4}$.
  Otherwise $a_2 - a_1 = 3$, only possible if $a_2 = 5, a_1 = 2$. In this case, $x^* > a_2$. so there is a unique j such that $x^* = a_{2j+1} - 2$, $a_{2j+1} - a_{2j} = 4$ and $a_{2i+1} - a_{2i} = 3$ for $i \neq j$, and $a_{2i+2} - a_{2i+1} = 1$ for $i > 0$. Let a_{2i+2}, a_{2i+1} be a pair for $1 \leq i \leq \frac{t-3}{2}$. Then A can transform to $\bar{X} = \{2, 5, 1, 1, \cdots\}$, which is 1-excellent as long as $t \geq 7$, i.e., $n \geq 15$.

In both cases, we have proved A is 1-excellent and the proof is complete. □

It remains to see what happens when $A - \{1\}$ or some $D(B) \geq \frac{n}{4}$. Note that there is at most 2 j such that $a_{j+1} - a_j \geq \frac{n}{4}$. Our thought is that we first construct for x which is close to n/k, $k \in \mathbb{N}^*$, then plug it into the interval $[a_j, a_{j+1}]$.

Lemma 5. *Suppose $A \subseteq [n]$ such that there exists $x \leq \frac{3n}{4}$ such that $A \cap [x, x + \frac{n}{4}] = \varnothing$. Then there are at least $|A| - \frac{5n}{12} - 11$ disjoint pairs of $\{x, y\} \subset A$ such that $|x - y| = \lfloor \frac{n}{12} \rfloor$.*

Proof. Denote $k = \lfloor \frac{n}{12} \rfloor$ and divided $[12k]$ into $[2ak + 1, 2(a + 1)k]$, $0 \leq a \leq 5$. Among these intervals, at least one is contained in $[x, x + \frac{n}{4}]$. So we have at least $|A| - 11$ numbers in the rest 5 intervals. We can pair like $(2ak + z, 2ak + k + z)$, $1 \leq z \leq k$ and there are $5k$ pairs in the rest 5 intervals. By pigeonhole principle we finish the proof. □

Theorem 3. *Let $A \subseteq [n]$. If $|A| > \frac{n}{2}$ and there exists $B = A - \{1\}$ (and $1 \in A$) or $B = A - \{x, x + 1\}$ (and $x, x + 1 \in A$) such that $D(B) \geq \frac{|A|}{2}$, then A is 1-excellent.*

Proof. Since $|A| > \frac{n}{2}$, we have $D(B) > \frac{n}{4}$. Let $B^{\uparrow} = \{a_i\}_{i=1}^{t}$. Denote j or j_1, j_2 as the index such that $a_{i+1} - a_i > \frac{n}{4}$ and J is the collection of such a_i, a_{i+1}. Let $C = \lfloor \frac{n}{12} \rfloor \cdot \{2, 4, 6, 8, 10\}$. Then $|B - C - J| \geq |A| - 10$. By Lemma 5, there exists at least $\frac{n}{12} - 21$ pairs $\{x, y\} \subset (B - C - J)$ such that $|x - y| = \lfloor \frac{n}{12} \rfloor$. Let $D = C - B$. Take at most 30 pairs from $B - C - J$ (denoted their collection as $E \subset B - C - J$), and transform them to each element of D. Note that D, E are disjoint. Then consider set $B' = (B \backslash E) \cup D$. Since $C \subset B'$ and $|B'| \geq \frac{n}{2} - 31$, we have $D(B') \leq \frac{n}{6} + 11 < \frac{|B'|}{2} + 1$.

Applying Lemma 4 to B', to prove B' is 2-excellent, we only need to check $\mathcal{L}(\bar{B}', 3) < |\bar{B}'|$. If not, suppose that elements of $J = \{x_1, \cdots, x_{2m}\}$ $(m \leq 2)$ divide $[n]$ into no more than 3 intervals $I_1, \cdots, I_p$ $(p \leq 3)$: $[1, x_1), (x_2, n]$ or $[1, x_1), (x_2, x_3), (x_4, n]$. Denote $J_k = B' \cap I_k$

and $B'^{\uparrow} = \{b_i\}_{i=1}^{|B'|}$. If $[b_{u_k}, b_{v_k}] \subset J_k$, then there are at least $\lfloor \frac{v_k - u_k}{2} \rfloor$ indexes l such that $b_{l+1} - b_l \in \bar{B}'$ and $b_l, b_{l+1} \in J_k$. Since $\bigcup_{k=1}^{p} J_k = B' - J$, there are at least

$$\sum_{k=1}^{p} \left(\frac{|J_k|}{2} - 1 \right) \geq \frac{|B' - J|}{2} - 3 \geq \frac{|A|}{2} - 38$$

indexes l such that $b_{l+1} - b_l \in \bar{B}'$ and $b_l, b_{l+1} \in J_k$. Counted the difference in $B' - J, J$ respectively, we know that

$$n - 1 \geq b_{|B'|} - b_1 \geq \frac{n}{4} + 2\left(\frac{|B'| - |J|}{2} - 3\right) + (|B'| - 1) \geq \frac{5n}{4} - 73,$$

a contradiction. Hence, B' is 2-excellent and then B is 2-excellent. By Proposition 1, $A = B \cup \{1\}$ or $B \cup \{x, x+1\}$ is 1-excellent. $\square$

Combining Theorem 2, 3, we have the following result:

Theorem 4. *For any integer $n \geq 612$, and any subset $A \subseteq \{1, 2, 3, \ldots, n\}$, if $|A| > \frac{n}{2}$, then there exists a function $f : A \to \{1, -1\}$ satisfying*

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq 1.$$

Proof. It is sufficient to check when $n \geq 612$, all inequalities in the proof of Theorem 2, 3 hold. We omit the details since it is just numerical calculation. $\square$

Remark 1. *When $n = 11$, the result does not hold because there is a counterexample $A = \{1, 3, 4, 7, 8, 11\}$. Hence, we have proved that $13 \leq N \leq 612$. We leave the work for finding the optimal N in the future or the reader could try to improve the bound or use computers to verify.*

Remark 2. *Our proof not only shows the existence of such f, but also gives an explicit algorithm to find such f when $N \geq 612$ and $|A| > \frac{n}{2}$.*

Remark 3. *A more careful analysis can improve the bound $n/2$ to $n/2 - c$ where $0 \leq c \leq 2$ according to the residue of n mod 8. From our proof, as long as $|A'| > n/2 - c$ (allowing remove a constant number of elements from A) and there is a large gap, then A' is 2-excellent. It remains to discuss when gaps are small. In this case, if there is 1-gap, then take it out and there is also another 1 or 2 gap, which allows the rest to be only 3-excellent (so the proof of Theorem 2 can be greatly simplified). If there is no 1-gap, then the situation is very restricted and can be solved by discussion.*