

# Study on Problem 4 of the 2021 China High School Mathematics League (Add-on Round)

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## 1. Original Problem and solution

**Original Problem:** Find the smallest positive constant  $c$  such that for any integer  $n \geq 4$ , and any subset  $A \subseteq \{1, 2, 3, \dots, n\}$ , if  $|A| > cn$ , then there exists a function  $f : A \rightarrow \{1, -1\}$  satisfying

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq 1.$$

**Solution:**  $c_{\min} = \frac{2}{3}$ .

On one hand, we can prove  $c \geq \frac{2}{3}$ . Conversely, suppose  $c < \frac{2}{3}$ . Take  $n = 6$ ,  $A = \{1, 4, 5, 6\}$ . Then  $|A| > cn$ . Note that

$$\sum_{a \in A} f(a) \cdot a \equiv \sum_{a \in A} a \equiv 0 \pmod{2},$$

and  $|\sum_{a \in A} f(a) \cdot a| \leq 1$  implies  $\sum_{a \in A} f(a) \cdot a = 0$ . But this contradicts the fact that  $A$  has no subset summing to  $\frac{16}{2} = 8$ .

We now prove that  $c = \frac{2}{3}$  is feasible. First, we prove the following lemma:

**Lemma 1.** Let  $a_1, a_2, \dots, a_n$  be nonzero integers with  $|a_i| \leq i$  for  $i = 1, 2, \dots, n$ . Then there exist  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{i=1}^n \varepsilon_i a_i \right| \leq 1.$$

*Proof.* We proceed by induction on  $n$ . The result is trivial for  $n = 1, 2$ . Assume it holds for all  $n \leq k$ , and consider  $n = k + 1$ . Without loss of generality, assume all  $a_i$  are positive (otherwise replace  $a_i$  with  $-a_i$ , which does not affect the conclusion). Since  $1 \leq a_{k+1} \leq k + 1$  and  $1 \leq a_k \leq k$ , we have  $|a_{k+1} - a_k| \leq k$ .

If  $|a_{k+1} - a_k| = 0$ , apply the induction hypothesis to  $a_1, a_2, \dots, a_{k-1}$ . Otherwise, if  $|a_{k+1} - a_k| \leq k$ , apply the induction hypothesis to  $a_1, a_2, \dots, a_{k-1}, |a_{k+1} - a_k|$ . The lemma follows.  $\square$

Now return to the original problem. Let  $A = \{a_1, a_2, \dots, a_t\} \subseteq \{1, 2, \dots, n\}$  with  $a_1 < a_2 < \dots < a_t$ , and  $t > \frac{2n}{3}$ .

(1) Suppose  $t$  is even. Consider the  $\frac{t}{2}$  positive integers:

$$a_t - a_{t-1}, a_{t-2} - a_{t-3}, \dots, a_2 - a_1.$$

We claim that for every  $m \in \left\{0, 1, 2, \dots, \frac{t}{2} - 1\right\}$ , at most  $m$  of these  $\frac{t}{2}$  numbers are  $\geq \frac{t}{2} - m + 1$ . (\*)

Otherwise, suppose there exists  $m \in \left\{0, 1, 2, \dots, \frac{t}{2} - 1\right\}$  such that at least  $m + 1$  of them are  $\geq \frac{t}{2} - m + 1$ . Then

$$\begin{aligned} a_t &= (a_t - a_{t-1}) + (a_{t-1} - a_{t-2}) + \dots + (a_3 - a_2) + (a_2 - a_1) + a_1 \\ &\geq (m+1) \left( \frac{t}{2} - m + 1 \right) + (t - (m+1)) \cdot 1 \\ &= t + (m+1) \left( \frac{t}{2} - m \right). \end{aligned}$$

Using the monotonicity of the quadratic function in  $m$ , we know

$$t + (m+1) \left( \frac{t}{2} - m \right) \geq \frac{3}{2}t,$$

with equality minimized at  $m = 0$  or  $m = \frac{t}{2} - 1$ . Thus  $a_t \geq \frac{3}{2}t > n$ , a contradiction.

From (\*) it follows that for every  $m \in \left\{0, 1, 2, \dots, \frac{t}{2} - 1\right\}$ , at least  $\frac{t}{2} - m$  of the differences are  $\leq \frac{t}{2} - m$ . Let  $i = \frac{t}{2} - m$ . Then for every  $i \in \left\{1, 2, \dots, \frac{t}{2}\right\}$ , at least  $i$  of the  $\frac{t}{2}$  numbers are  $\leq i$ .

Rearranging  $a_t - a_{t-1}, a_{t-2} - a_{t-3}, \dots, a_2 - a_1$  in increasing order as  $b_1 \leq b_2 \leq \dots \leq b_{\frac{t}{2}}$ , we have  $0 < b_i \leq i$ . By the lemma, there exist  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{i=1}^{\frac{t}{2}} \varepsilon_i b_i \right| \leq 1.$$

Expressing each  $b_i$  as a difference of elements from  $A$ , we conclude the desired result.

(2) Suppose  $t$  is odd. Consider the  $\frac{t+1}{2}$  positive integers:

$$a_t - a_{t-1}, a_{t-2} - a_{t-3}, \dots, a_3 - a_2, a_1.$$

A similar argument applies. We claim that for every  $m \in \left\{0, 1, 2, \dots, \frac{t-1}{2}\right\}$ , at most  $m$  of these  $\frac{t+1}{2}$  numbers are  $\geq \frac{t+1}{2} - m + 1$ .  $(\Delta)$

Otherwise, suppose there exists  $m \in \left\{0, 1, 2, \dots, \frac{t-1}{2}\right\}$  such that at least  $m+1$  of them are  $\geq \frac{t+1}{2} - m + 1$ . Then

$$\begin{aligned} a_t &\geq (m+1) \left( \frac{t+1}{2} - m + 1 \right) + (t - (m+1)) \cdot 1 \\ &= t + (m+1) \left( \frac{t+1}{2} - m \right). \end{aligned}$$

Using the monotonicity of the quadratic function, we get

$$t + (m+1) \left( \frac{t+1}{2} - m \right) \geq \frac{3t+1}{2},$$

so  $a_t \geq \frac{3t+1}{2} > n$ , again a contradiction.

From  $(\Delta)$ , for every  $m \in \left\{0, 1, 2, \dots, \frac{t-1}{2}\right\}$ , at least  $\frac{t+1}{2} - m$  of the numbers are  $\leq \frac{t+1}{2} - m$ . Let  $i = \frac{t+1}{2} - m$ . Then for every  $i \in \left\{1, 2, \dots, \frac{t+1}{2}\right\}$ , at least  $i$  of the  $\frac{t+1}{2}$  numbers are  $\leq i$ .

Rearranging them as  $b_1 \leq b_2 \leq \dots \leq b_{\frac{t+1}{2}}$ , we have  $0 < b_i \leq i$ . Again, by the lemma, there exist  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{i=1}^{(t+1)/2} \varepsilon_i b_i \right| \leq 1.$$

Expressing each  $b_i$  as a difference of elements from  $A$ , the proof is complete.

## 2. Improvement to the conclusion

The lowerbound  $|A| > \frac{2n}{3}$  can be improved to  $|A| > \frac{1}{2}n + \log_2 n + c_1$  where  $c_1$  is an absolute constant. We study it right now.

**Lemma 2.** *Let  $n \in \mathbb{N}^*$ , subset  $A = \{a_1, \dots, a_t\} \subseteq [n]$  such that  $a_1 < a_2 \dots < a_t$ , the following facts hold:*

1. For all  $i \in [t-1]$ ,  $a_{i+1} - a_i \leq n + 1 - t$ .
2. Denote  $P_1(A) \triangleq \{i \in [t-1] : a_{i+1} - a_i = 1\}$ , then  $|P_1(A)| \geq 2t - n - 1$ .
3. For any  $x \in [n]$ , denote

$$d_k(x) \triangleq \left\{ \min \left| \sum_{i=1}^k \alpha_i b_i - x \right| : \alpha_i \in \{-1, 0, 1\}, \{b_i\}_{i=1}^k \subseteq A \right\}.$$

For all  $k \geq 2$ , if  $t - 2k > \frac{n+5}{2}$ ,  $n \geq 19$ ,  $2 \leq d_{2k-1}(x) \leq t - 2k - 2$ , then

$$d_{2k+1}(x) \leq \frac{1}{2}d_{2k-1}(x).$$

*Proof.* If 1 does not hold, then  $a_t - a_1 > n + 1 - t + t - 2 > n - 1$ , a contradiction. If 2 does not hold, then  $a_t - a_1 > 2t - n - 1 + 2(t - 1 - (2t - n - 1)) > n - 1$  (obviously we only need to discuss the case when  $2t - n - 1 > 0$ ) a contradiction.

Now we denote  $d_k$  as  $d_k(x)$  for simplicity. For  $d_{2k-1}$ , there exist  $x_1, \dots, x_{2k-1} \in A$  such that  $\left| \sum_{i=1}^{2k-1} \alpha_i x_i - x \right| = d_{2k-1}$ . Consider the set  $A_k = A - \{x_1, \dots, x_{2k-1}\}$ . If there exists  $u, v \in A_k$  such that  $\frac{3}{2}d_{2k-1} \geq |u - v| \geq \frac{1}{2}d_{2k-1}$ , then we can choose proper  $\alpha_{2k}, \alpha_{2k+1}$  such that  $|\alpha_1 x_1 - x + \alpha_{2k} u + \alpha_{2k+1} v| \leq \frac{1}{2}d_{2k-1}$ . Otherwise for all  $u, v \in A_k$ , either  $|u - v| < \frac{1}{2}d_{2k-1}$  or  $|u - v| > \frac{3}{2}d_{2k-1}$ . Let  $c_1 < \dots < c_{t-2k+1}$  be all elements of  $A_k$ , then it is easy to see that

$$c_{s+\lfloor d_{2k-1}/2 \rfloor + 1} - c_s \geq \left\lfloor \frac{d_{2k-1}}{2} \right\rfloor + \left\lceil \frac{3d_{2k-1}}{2} \right\rceil, \quad \forall s \in [t-2 - \lfloor \frac{d_{2k-1}}{2} \rfloor].$$

As a result,

$$c_{t-2k+1} - c_1 \geq 2d_{2k-1} \left\lfloor \frac{t-2k}{1 + \lfloor \frac{d_{2k-1}}{2} \rfloor} \right\rfloor \geq 2d_{2k-1} \left( \frac{t-2k - \lfloor \frac{d_{2k-1}}{2} \rfloor}{1 + \lfloor \frac{d_{2k-1}}{2} \rfloor} \right).$$

If  $d_{2k-1} = 2$  then  $c_{t-2k+1} - c_1 \geq 2t - 6 - 4k > n - 1$ ; if  $d_{2k-1} = 3$ , then  $c_{t-2k+1} - c_1 \geq 3t - 12 - 6k > n - 1$ ; if  $d_{2k-1} = 4$ , then  $c_{t-2k+1} - c_1 \geq \frac{8}{3}(t-2k) - 14 > n - 1$ ; if  $d_{2k-1} = 5$ , then  $c_{t-2k+1} - c_1 \geq \frac{10}{3}(t-2k) - 17 > n - 1$ ; if  $\frac{t}{2} \geq d_{2k-1} \geq 6$ , then  $c_{t-2k+1} - c_1 \geq 3t - 6 - 6k - 2d_1 > n - 1$ ; if  $d_{2k-1} > \frac{t}{2}$ , then  $c_{t-1} - c_1 \geq t \lfloor \frac{t-2k}{1+t/2-k-1} \rfloor = 2t > n - 1$ . Hence, a contradiction. This implies  $d_{2k+1} \leq \frac{1}{2}d_{2k-1}$ .  $\square$

**Lemma 3.** Let  $n \geq 19 \in \mathbb{N}^*$ , subset  $A = \{a_1, \dots, a_t\} \subseteq [n]$  such that  $a_1 < a_2 \dots < a_t$ . Denote  $k_n = \lceil \log_2 n \rceil$ , if  $t \geq \frac{n+5+4k_n}{2}$ , then for every  $0 < x \leq n/2$  there exist a set  $B(x) \subseteq A$ ,  $|B(x)| \leq k_n$  and  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{x_i \in B(x)} \varepsilon_i x_i - x \right| \leq 1.$$

*Proof.* For  $A$ , by the definition of  $d_k(x)$  we know that  $d_1(x) \leq x$ . For all  $k \leq k_n$  and  $d_{2k-1}(x) \geq 2$ , it is easy to check that all conditions in the 3th fact of Lemma 2 holds. Therefore, either for some  $k \leq k_n$  we have  $d_{2k-1}(x) = 1$  or  $d_{2k+1}(x) \leq \frac{1}{2}d_{2k-1}(x)$  holds for all  $k \leq k_n$ . The latter suggests  $d_{2k_n-1}(x) \leq 2^{-k_n}d_1(x) \leq 1$ . Then the proof is finished.  $\square$

**Lemma 4.** Let  $a_1, a_2, \dots, a_n$  be nonzero integers with  $|a_i| \leq i + 1$  for  $i = 1, 2, \dots, n$ . Then there exist  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{i=1}^n \varepsilon_i a_i \right| \leq 2.$$

*Proof.* We proceed by induction on  $n$ . The result is trivial for  $n = 1, 2$ . Assume it holds for all  $n \leq k$ , and consider  $n = k + 1$ . Without loss of generality, assume all  $a_i$  are positive (otherwise replace  $a_i$  with  $-a_i$ , which does not affect the conclusion). Since  $1 \leq a_{k+1} \leq k + 2$  and  $1 \leq a_k \leq k + 1$ , we have  $|a_{k+1} - a_k| \leq k + 1$ .

If  $|a_{k+1} - a_k| = 0$ , apply the induction hypothesis to  $a_1, a_2, \dots, a_{k-1}$ . Otherwise, if  $|a_{k+1} - a_k| \leq k + 1$ , apply the induction hypothesis to  $a_1, a_2, \dots, a_{k-1}, |a_{k+1} - a_k|$ . The lemma follows.  $\square$

**Lemma 5.** Let  $n \in \mathbb{N}^*$  and subset  $A = \{a_1, \dots, a_t\} \subseteq [n]$  such that  $a_1 < a_2 \dots < a_t$  and  $t > \frac{n}{2}$ . If  $a_1, a_{i+1} - a_i \leq \frac{n}{4} + 1$ , then there exist  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{i=1}^n \varepsilon_i a_i \right| \leq 2,$$

and there are exactly  $\frac{|A|}{2}$  number of  $\varepsilon_i$  taking value 1,  $\frac{|A|}{2}$  number of  $\varepsilon_i$  taking value  $-1$ .

*Proof.* If  $t = 2p$  is even, consider the  $p$  positive integers:

$$a_t - a_{t-1}, a_{t-2} - a_{t-3}, \dots, a_2 - a_1.$$

We claim that for every  $m \in \left\{-1, 0, 1, \dots, \frac{t}{2} - 2\right\}$ , at most  $m + 1$  of these  $\frac{t}{2}$  numbers are  $\geq \frac{t}{2} - m + 1$ . (\*)

When  $m = -1$ , the claim holds by our assumption  $a_{i+1} - a_i \leq \frac{n}{4} + 1$ . Otherwise, suppose there exists  $m \in \left\{0, 1, 2, \dots, \frac{t}{2} - 2\right\}$  such that at least  $m + 2$  of them are  $\geq \frac{t}{2} - m + 1$ . Then

$$\begin{aligned} a_t &= (a_t - a_{t-1}) + (a_{t-1} - a_{t-2}) + \dots + (a_3 - a_2) + (a_2 - a_1) + a_1 \\ &\geq (m+2) \left( \frac{t}{2} - m + 1 \right) + (t - (m+2)) \cdot 1 \\ &= t + (m+2) \left( \frac{t}{2} - m \right). \end{aligned}$$

Using the monotonicity of the quadratic function in  $m$ , we know

$$t + (m+2) \left( \frac{t}{2} - m \right) \geq 2t,$$

with equality minimized at  $m = 0$  or  $m = \frac{t}{2} - 2$ . Thus  $a_t \geq 2t > n$ , a contradiction.

If  $t = 2p + 1$  is odd, consider the  $p + 1$  positive integers:

$$a_t - a_{t-1}, a_{t-2} - a_{t-3}, \dots, a_2 - a_1, a_1.$$

A similar argument applies. We claim that for every  $m \in \left\{-1, 0, \dots, \frac{t-3}{2}\right\}$ , at most  $m+1$  of these  $\frac{t+1}{2}$  numbers are  $\geq \frac{t+1}{2} - m + 1$ .  $(\triangle)$

When  $m = 0$ , it contradicts to our assumption. Otherwise, suppose there exists  $m \in \left\{0, 1, 2, \dots, \frac{t-3}{2}\right\}$  such that at least  $m+2$  of them are  $\geq \frac{t+1}{2} - m + 1$ . Then

$$\begin{aligned} a_t &\geq (m+2) \left( \frac{t+1}{2} - m + 1 \right) + (t - (m+2)) \cdot 1 \\ &= t + (m+2) \left( \frac{t+1}{2} - m \right). \end{aligned}$$

Using the monotonicity of the quadratic function, we get

$$t + (m+2) \left( \frac{t+1}{2} - m \right) \geq 2t + 1,$$

so  $a_t \geq 2t + 1 > n$ , again a contradiction.

Therefore, in both cases, rearranging them as  $b_1 \leq b_2 \leq \dots \leq b_h$ , we have  $0 < b_i \leq i + 1$ . By Lemma 4, there exist  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{i=1}^h \varepsilon_i b_i \right| \leq 2.$$

Expressing each  $b_i$  as a difference of elements from  $A$ , the proof is complete.  $\square$

**Theorem 1.** Let  $n \geq 19$  be a positive integer, and subset  $A \subseteq [n]$ . If  $|A| \geq \frac{n}{2} + 2 \log_2 n + 6$ , then there exist  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{i=1}^n \varepsilon_i a_i \right| \leq 1.$$

*Proof.* Without the loss of generality, assume  $n \in A$ , otherwise we use induction. First, by Lemma 2, there exists  $u, v \in A - \{n\}$  such that  $|u - v| = 1$ . Let  $A' = A - \{u, v\}$  and  $A' = \{a_1, \dots, a_{t'}\}$  such that  $a_1 < a_2 < \dots < a_{t'}, t' = |A| - 2$ .

Denote  $a_0 = 0$  in the following proof. Let  $D = \max_{0 \leq i \leq t'-1} \{a_{i+1} - a_i\}$ . If  $D \leq \frac{n+1}{4}$ , then by Lemma 5 and  $u, v$  we finish the proof. Otherwise, there exists a unique  $j$  such that  $a_{j+1} - a_j \geq \frac{n+1}{4}$ . Let  $x = \min\{n - \frac{a_j + a_{j+1}}{2}, \frac{a_j + a_{j+1}}{2}\}$ , and  $A'' = A' - \{a_j, a_{j+1}\}$  if  $x = \frac{a_j + a_{j+1}}{2}$  and  $A'' = A' - \{a_j, a_{j+1}, n\}$  otherwise. For  $A''$ ,  $|A''| \geq t - 5 \geq \frac{n+5+4k_n}{2}$ ,  $d_1(x) \leq \frac{n}{2}$ . By Lemma 3, there exists  $B(x) \subseteq A'', |B(x)| \leq k_n$ , such that

$$\left| \sum_{x_i \in B(x)} \varepsilon_i x_i - x \right| \leq 1, \quad \varepsilon_i \in \{1, -1\}.$$

Denote  $y = \sum_{x_i \in B(x)} \varepsilon_i x_i$ , consider  $A''' = A'' - B(x) + \{y\}$ . Then  $A''' = \{a_{i_1}, \dots, a_j, y, a_{j+1}, \dots, a_s\}$

with  $1 \leq i_1 < \dots < i_r = j < j+1 = i_{r+2} < \dots < i_s$  and denote  $y = a_{i_{r+1}}$  (Some indexes are deleted by  $B(x)$ ) and  $|A'''| \geq t - 5 - 2k_n > \frac{n}{2} + 1$ . To use Lemma 5, we need to check that  $a_{i_{k+1}} - a_{i_k} \leq \frac{n}{4} + 1$ . This is indeed true. First,  $|y - a_j|, |y - a_{j+1}| \leq 1 + \frac{D}{2}$  and  $D < n + 1 - t < n/2$ . Next, if there is other  $i_k$  such that  $a_{i_{k+1}} - a_{i_k} > \frac{n}{4} + 1$ , then

$$n \geq a_{is} \geq a_{i_{k+1}} - a_{i_k} + D + \sum_{l=1, l \neq r, r+1, k}^s (a_{i_{l+1}} - a_{i_l}) \geq \frac{n}{2} + 2 + |A'''| - 3 > n.$$

A contradiction. Hence we can apply Lemma 5 to  $A'''$  and then finish the proof.  $\square$

### 3. Further Improvement- Using Pigeonhole Principle Trick

The lowerbound  $|A| > \frac{n}{2} + 2 \log_2 n + c$  can be improved to  $|A| > \frac{1}{2}n + 5$ . The further improvement is motivated from the observation that  $d_k(x) = 0$  for  $k \sim O(1)$  if  $2x\{\frac{n}{2x}\} \sim O(1)$ . The thought is that we first construct for  $x = n/2, n/4, 3n/4$  then plug it into the interval  $[a_j, a_{j+1}]$ .

**Lemma 6.** Let  $n, r, x \in \mathbb{N}^*$  and subset  $A \subseteq [n]$  such that  $|A| > \frac{n+r}{2} + 1$ . If  $2x\{\frac{n}{2x}\} \leq \min\{r, x-1\}$ , then there exist  $u, v \in A$  such that  $|u - v| = x$ .

*Proof.* Let  $n = 2kx + s$ , where  $k, s \in \mathbb{N}^*, 0 \leq s \leq 2x-1$ . Then the condition implies  $s \leq r$ . Construct congruence classes  $A_g = \{g, x+g, \dots, 2kx+g\}$  for  $g \leq s$  and  $A_g = \{g, x+g, \dots, (2k-1)x+g\}$  for  $x > g > s$ . If the lemma does not hold, then  $|A \cap A_g| \leq k+1$  for  $g \leq s$  and  $|A \cap A_g| \leq k$  for  $g > s$ . Hence,  $|A| \leq (k+1)(s+1) + k(x-s-1) \leq xk+s+1 \leq \frac{n+r+2}{2}$ , a contradiction.  $\square$

**Lemma 7.** Let  $n \in \mathbb{N}^*, n \geq 16$  and subset  $A \subseteq [n]$ . If  $|A| > \frac{n}{2} + 9$ . Then there exist disjoint  $A_1, A_2, A_3 \subseteq A$ , such that  $|A_1| = |A_2| = 2, |A_3| \leq 6$  and

$$\sum_{x \in A_1} \varepsilon_x x = \lfloor \frac{n}{4} \rfloor;$$

$$\sum_{x \in A_2} \varepsilon_x x = \lfloor \frac{n}{2} \rfloor;$$

$$\sum_{x \in A_3} \varepsilon_x x = \lfloor \frac{3n}{4} \rfloor,$$

where  $\varepsilon_x \in \{1, -1\}$  are chosen properly.

*Proof.* Take  $x_1 = \lfloor \frac{n}{4} \rfloor, r_1 = 3$  in Lemma 6, we know that there exist  $u_1, v_1 \in A$  such that  $|u_1 - v_1| = x_1$ . Then consider  $A' = A - \{u_1, v_1\}$ , there exist  $u_2, v_2 \in A'$  such that  $|u_2 - v_2| = x_1$ . Take  $x_2 = \lfloor \frac{n}{4} \rfloor, r_2 = 2$  in Lemma 6, we know that there exist  $u_3, v_3 \in A'' = A' - \{u_2, v_2\}$  such that  $|u_3 - v_3| = x_2$ . Again we know that exist  $u_4, v_4 \in A''' = A'' - \{u_3, v_3\}$  such that  $|u_4 - v_4| = x_2$ . By Lemma 2, we know that there exist  $u_5, v_5 \in A''' - \{u_4, v_4\}$  such that  $|u_5 - v_5| = 1$ . Note that  $|\lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{2} \rfloor - \lfloor \frac{3n}{4} \rfloor| \leq 1$ . Hence, we take  $A_1 = \{u_1, v_1\}, A_2 = \{u_3, v_3\}, A_3 = \{u_2, v_2, u_4, v_4\}$  or  $\{u_2, v_2, u_4, v_4, u_5, v_5\}$  and finish the proof.  $\square$

**Theorem 2.** Let  $n$  be a positive integer, and subset  $A \subseteq [n]$ . If  $|A| > \frac{n}{2} + 9$ , then there exist  $\varepsilon_i \in \{-1, 1\}$  such that

$$\left| \sum_{i=1}^n \varepsilon_i a_i \right| \leq 1.$$

*Proof.* First, since  $\frac{2n}{3} \leq \frac{n}{2} + 9$  when  $n \leq 16$ , we can assume  $n > 16$ . By Lemma 2, there exists  $u, v \in A$  such that  $|u - v| = 1$ . Let  $A' = A - \{u, v\}$  and  $A' = \{a_1, \dots, a_{t'}\}$  such that  $a_1 < a_2 < \dots < a_{t'}, t' = |A| - 2$ .

Denote  $a_0 = 0$  in the following proof. Let  $D = \max_{0 \leq i \leq t'-1} \{a_{i+1} - a_i\}$ . If  $D \leq \frac{n+1}{4}$ , then by Lemma 5 and  $u, v$  we finish the proof. Otherwise, there exists a unique  $j$  such that  $a_{j+1} - a_j \geq \frac{n+1}{4}$ . Let  $C = \{\lfloor \frac{n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor, \lfloor \frac{3n}{4} \rfloor\} \cap (a_j, a_{j+1})$ . Note that  $D \leq n + 1 - t' < \frac{n}{2} - 6$ , so  $|C| \leq 2$ . By Lemma 7, there exists  $A_0 \subseteq A'$ ,  $|A_0| \leq 8$  such that using  $A_0$ 's elements' signed sum, we can construct all elements in  $C$ . Then consider  $A'' = A' - A_0 + C$ , similar to the proof of Theorem 1, we can show that  $|A''| > \frac{n}{2} + 1$  and no consecutive  $a, b \in A''$  such that  $b - a \geq \frac{n}{4} + 1$ . Hence we apply Lemma 5 to  $A''$  and then finish the proof.  $\square$

**Remark 1.** A more careful analysis can improve the constant 9 to 7 by allowing choosing the number by  $\lfloor \frac{n+1}{2} \rfloor$  for example (You need to discuss the module of  $n$  to 4). Then  $|A_0| \leq 6$ . We leave details to the readers.

**Remark 2.** A more careful analysis can improve the constant 7 to 5. This is because if large  $D$  exists, then even if  $|A''|$  is slightly smaller than  $\frac{n}{2}$ , the key inequality in Lemma 5 still holds for all  $m$ , since in this case there can't be too many small 2-gaps.

**Remark 3.** We conjecture that the strongest lowerbound for  $|A|$  is  $\frac{n+1}{2}$  when  $n$  is larger than a certain constant. If  $D \leq \frac{n+1}{4}$ , then we can prove this bound. What is left to do is dealing with large gap's existence. We handle it by interpolate into the gap. We believe the next step to improve bound is to analyze the case when  $(a_j, a_{j+1})$  contains  $\lfloor \frac{3n}{4} \rfloor$  or its nearby, which is costly constructed. If it can be constructed by only 2 elements, then the constant can be improved to 3. The final improvement requires to discuss the case when  $(a_j, a_{j+1})$  contains  $\lfloor \frac{3n}{4} \rfloor, \lfloor \frac{n}{2} \rfloor$ .