An Improved Bound for a Signed Sum Problem

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January 4th, 2026

Abstract

This paper studies a signing problem from the 2021 China High School Mathematics League (Add-on Round). The main result of this work is a refined threshold: we define an explicit sequence c_n satisfying $c_n = \frac{n}{2} + O(1)$, depending on $n \bmod 8$, and prove that there exists an absolute constant $N \leq 151$ such that for all $n \geq N$, any subset $A \subseteq [n]$ with $|A| > c_n$ admits a signing with sum in $\{-1, 0, 1\}$ and c_n is optimal.

Our proof improves upon a known result that established the same conclusion with $N = 556$. The improvement is achieved through a detailed structural analysis of dense subsets, a classification of exceptional configurations, and a novel "gap-shifting" technique that leverages auxiliary notions of *s-excellence* and *strong expressiveness*. We also prove an intermediate theorem: if $|A| > \frac{n-7}{2}$, then A is 2-excellent for all sufficiently large n.

1 Original Problem and Solution

For readers to get familiar with the problems and solving techniques, we first introduce the original problem in mathematical competition and its solution. This provide original inspiration for our research.

Original Problem: Find the smallest positive constant c such that for any integer $n \geq 4$, and any subset $A \subseteq \{1, 2, 3, \ldots, n\}$, if $|A| > cn$, then there exists a function $f : A \to \{1, -1\}$ satisfying

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq 1.$$

On one hand, we can show that $c \geq \frac{2}{3}$. Conversely, suppose $c < \frac{2}{3}$. Take $n = 6$ and $A = \{1, 4, 5, 6\}$. Then $|A| > cn$. Note that

$$\sum_{a \in A} f(a) \cdot a \equiv \sum_{a \in A} a \equiv 0 \pmod 2,$$

and the condition $\left|\sum_{a \in A} f(a) \cdot a\right| \leq 1$ implies $\sum_{a \in A} f(a) \cdot a = 0$. However, this contradicts the fact that A contains no subset whose elements sum to $\frac{16}{2} = 8$.

We now prove that $c = \frac{2}{3}$ is sufficient. First, we establish the following lemma:

Lemma 1. *Let $a_1, a_2, \ldots, a_n$ be nonzero integers with $|a_i| \le i$ for $i = 1, 2, \ldots, n$. Then there exist signs $\varepsilon_i \in \{-1, 1\}$ such that*

$$\left| \sum_{i=1}^{n} \varepsilon_i a_i \right| \le 1.$$

Proof. We proceed by induction on n. The claim is trivial for $n = 1, 2$. Assume it holds for all $n \le k$, and consider $n = k + 1$. Without loss of generality, assume all a_i are positive (otherwise replace a_i by $-a_i$, which does not affect the conclusion). Since $1 \le a_{k+1} \le k + 1$ and $1 \le a_k \le k$, we have $|a_{k+1} - a_k| \le k$.

If $|a_{k+1} - a_k| = 0$, apply the induction hypothesis to $a_1, a_2, \ldots, a_{k-1}$. Otherwise, if $|a_{k+1} - a_k| \le k$, apply the induction hypothesis to $a_1, a_2, \ldots, a_{k-1}, |a_{k+1} - a_k|$. The lemma follows. $\square$

Now return to the original problem. Let $A = \{a_1, a_2, \ldots, a_t\} \subseteq \{1, 2, \ldots, n\}$ with $a_1 < a_2 < \cdots < a_t$, and $t > \frac{2n}{3}$.

(1) Suppose t is even. Consider the $\frac{t}{2}$ positive integers:

$$a_t - a_{t-1},\ a_{t-2} - a_{t-3},\ \ldots,\ a_2 - a_1.$$

We claim that for every $m \in \{0, 1, 2, \ldots, \frac{t}{2} - 1\}$, at most m of these $\frac{t}{2}$ numbers are $\ge \frac{t}{2} - m + 1$. (*)

Otherwise, suppose there exists $m \in \{0, 1, 2, \ldots, \frac{t}{2} - 1\}$ such that at least $m + 1$ of them are $\ge \frac{t}{2} - m + 1$. Then

$$\begin{aligned} a_t &= (a_t - a_{t-1}) + (a_{t-1} - a_{t-2}) + \cdots + (a_3 - a_2) + (a_2 - a_1) + a_1 \\ &\ge (m+1)\left(\frac{t}{2} - m + 1\right) + (t - (m+1)) \cdot 1 \\ &= t + (m+1)\left(\frac{t}{2} - m\right). \end{aligned}$$

Using the monotonicity of the quadratic function in m, we obtain

$$t + (m+1)\left(\frac{t}{2} - m\right) \ge \frac{3}{2}t,$$

with equality attained when $m = 0$ or $m = \frac{t}{2} - 1$. Thus $a_t \ge \frac{3}{2}t > n$, a contradiction.

From (*) it follows that for every $m \in \{0, 1, 2, \ldots, \frac{t}{2} - 1\}$, at least $\frac{t}{2} - m$ of the differences are $\le \frac{t}{2} - m$. Let $i = \frac{t}{2} - m$. Then for every $i \in \{1, 2, \ldots, \frac{t}{2}\}$, at least i of the $\frac{t}{2}$ numbers are $\le i$.

Rearranging $a_t - a_{t-1}, a_{t-2} - a_{t-3}, \ldots, a_2 - a_1$ in increasing order as $b_1 \le b_2 \le \cdots \le b_{\frac{t}{2}}$, we have $0 < b_i \le i$. By the lemma, there exist signs $\varepsilon_i \in \{-1, 1\}$ such that

$$\left| \sum_{i=1}^{t/2} \varepsilon_i b_i \right| \le 1.$$

Expressing each b_i as a difference of elements from A, we obtain the desired result.

(2) Suppose t is odd. Consider the $\frac{t+1}{2}$ positive integers:

$$a_t - a_{t-1},\ a_{t-2} - a_{t-3},\ \dots,\ a_3 - a_2,\ a_1.$$

A similar argument applies. We claim that for every $m \in \left\{0, 1, 2, \dots, \frac{t-1}{2}\right\}$, at most m of these $\frac{t+1}{2}$ numbers are $\geq \frac{t+1}{2} - m + 1$. $\quad(\triangle)$

Otherwise, suppose there exists $m \in \left\{0, 1, 2, \dots, \frac{t-1}{2}\right\}$ such that at least $m+1$ of them are $\geq \frac{t+1}{2} - m + 1$. Then

$$\begin{aligned} a_t &\geq (m+1)\left(\frac{t+1}{2} - m + 1\right) + (t - (m+1)) \cdot 1 \\ &= t + (m+1)\left(\frac{t+1}{2} - m\right). \end{aligned}$$

Using the monotonicity of the quadratic function, we get

$$t + (m+1)\left(\frac{t+1}{2} - m\right) \geq \frac{3t+1}{2},$$

so $a_t \geq \frac{3t+1}{2} > n$, again a contradiction.

From $(\triangle)$, for every $m \in \left\{0, 1, 2, \dots, \frac{t-1}{2}\right\}$, at least $\frac{t+1}{2} - m$ of the numbers are $\leq \frac{t+1}{2} - m$. Let $i = \frac{t+1}{2} - m$. Then for every $i \in \left\{1, 2, \dots, \frac{t+1}{2}\right\}$, at least i of the $\frac{t+1}{2}$ numbers are $\leq i$.

Rearranging them as $b_1 \leq b_2 \leq \cdots \leq b_{\frac{t+1}{2}}$, we have $0 < b_i \leq i$. Again, by the lemma, there exist signs $\varepsilon_i \in \{-1, 1\}$ such that

$$\left| \sum_{i=1}^{(t+1)/2} \varepsilon_i b_i \right| \leq 1.$$

Expressing each b_i as a difference of elements from A, the proof is complete.

2 Optimal Bound for Large n

In fact, for sufficiently large n, the constant $\frac{2}{3}$ is not optimal. For example, taking $A = \{2, 4, \dots, n\}$ when n is even shows that $c \geq \frac{1}{2}$. We will show that $c = \frac{1}{2}$ is sufficient—and hence optimal—for large n.

Define c_n as follows:

$$c_n = \begin{cases} 4k+1 & \text{if } n = 8k+2 \text{ or } 8k+3, \\ 4k+2 & \text{if } n = 8k+4, \dots, 8k+7, \\ 4k+3 & \text{if } n = 8k+8 \text{ or } 8k+9. \end{cases}$$

Our strengthened result is stated below:

Theorem 1. *There exists an absolute constant $N \leq 151$ such that for any integer $n \geq N$, and any subset $A \subseteq \{1, 2, 3, \ldots, n\}$, if $|A| > c_n$, then there exists a function $f : A \to \{1, -1\}$ satisfying*

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq 1.$$

It is straightforward to verify that c_n is optimal for each fixed n. A known result establishes Theorem 1 with $N = 556$. Our goal is to provide an alternative proof and improve the bound on N. We believe following the framework in this paper, with more exhaustive discussion, the bound can be improved to $N \approx 110$.

We first handle the case where there exists a subset $B \subset A$ such that either $B = \{1\}$ or $B = \{x, x+1\}$. In such cases, it suffices to consider $A \setminus B$. The remaining exceptional configurations can be dealt with manually.

Theorem 2. *There exists an absolute constant $N \leq 151$ such that for any integer $n \geq N$, and any subset $A \subseteq \{1, 2, 3, \ldots, n\}$, if $|A| > \frac{n-7}{2}$, then there exists a function $f : A \to \{1, -1\}$ satisfying*

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq 2.$$

2.1 Notation and Definitions

We begin by introducing some standard notation. Uppercase letters such as $A, B, C, \ldots$, along with their primed variants $A', B', C', \ldots$ and double-primed versions $A'', B'', C'', \ldots$, denote sets. Barred or tilded letters $\bar{A}, \bar{B}, \bar{C}, \tilde{A}, \tilde{B}, \tilde{C}, \ldots$ denote multisets.

For sets A and B, we write $A - B$ to mean $A \setminus (A \cap B)$. We use the standard notation $[n] = \{1, 2, \ldots, n\}$. For any set X or multiset $\bar{X}, \tilde{X}$, we adopt the following conventions:

1. $|X|$, $|\bar{X}|$, or $|\tilde{X}|$ denotes its cardinality;
2. $A^{\uparrow} = \{a_i\}_{i=1}^{|A|}$ or $\bar{A}^{\uparrow} = \{a_i\}_{i=1}^{|\bar{A}|}$ denotes the sequence of elements of A or $\bar{A}$ sorted in non-decreasing order;
3. $s(A) = \sum_{x \in A} x$, $s(\bar{A}) = \sum_{x \in \bar{A}} x$ denotes the sum of elements, with multiplicities counted for multisets.

Throughout this work, all sets are assumed to be subsets of $\mathbb{N}$. For a set A (or multiset $\bar{A}, \tilde{A}$) and an integer m, define

$$\mathcal{L}(A, m) = |A \cap [m, +\infty)|, \quad \mathcal{L}(\bar{A}, m) = |\bar{A} \cap [m, +\infty)|, \quad \mathcal{L}(\tilde{A}, m) = |\tilde{A} \cap [m, +\infty)|.$$

Denote $d(A) = \max_{x,y \in A} |x - y|$. If $|A|$ is even, define

$$\bar{A} = \{a_2 - a_1, a_4 - a_3, \ldots, a_{|A|} - a_{|A|-1}\}, \quad \tilde{A} = \{a_3 - a_2, \ldots, a_{|A|-1} - a_{|A|-2}\};$$

otherwise, define

$$\bar{A} = \{a_1, a_3 - a_2, \ldots, a_{|A|} - a_{|A|-1}\}, \quad \tilde{A} = \{a_2 - a_1, \ldots, a_{|A|-1} - a_{|A|-2}\}.$$

For any sequence $\{a_i\}$, we set $a_l = 0$ if $l \leq 0$.

Next, we introduce key definitions relevant to our theorems and proofs.

Definition 1. *We say that A (or $\bar{A}$) is s-**excellent** if there exists a function $f : A \to \{1, -1\}$ (or $f : \bar{A} \to \{1, -1\}$) such that*

$$\left| \sum_{a \in A} f(a) \cdot a \right| \leq s \quad \text{(or } \left| \sum_{a \in \bar{A}} f(a) \cdot a \right| \leq s\text{)}.$$

*If equality holds, we say that A (or $\bar{A}$) is **exactly** s-**excellent**.*

Definition 2. *We say that A (or $\bar{A}$) is **strongly expressive** if for every integer $x \leq s(A)$ with $x \equiv s(A) \pmod 2$, there exists a function $f : A \to \{1, -1\}$ (or $f : \bar{A} \to \{1, -1\}$) such that*

$$\sum_{a \in A} f(a) \cdot a = x \quad \text{(or } \sum_{a \in \bar{A}} f(a) \cdot a = x\text{)}.$$

By definition, if A or $\bar{A}$ is strongly expressive, then it is 1-excellent. The following proposition is straightforward to verify:

Proposition 1. *The following statements hold:*

1. *If $A, B \subset \mathbb{N}$ are disjoint, A is s-excellent and B is exactly t-excellent, then $A \cup B$ is $|s - t|$-excellent.*
2. *If $\bar{A}$ is s-excellent, then A is s-excellent.*
3. *If $\bar{A}$ is strongly expressive and $b \leq s(\bar{A})$, then $\bar{A} \cup \{b\}$ is strongly expressive.*

Proof. The second statement follows directly from the definition. Let f and g be the corresponding sign functions for A and B in Definition 1. Defining $h : A \cup B \to \{1, -1\}$ by $h|_A = f$ and $h|_B = \pm g$ proves the first statement.

For the third statement, note that $\bar{A}$ can represent all values $s(\bar{A}), s(\bar{A}) - 2, \ldots, \varepsilon$, where $\varepsilon \in \{0, 1\}$. Adding b allows us to represent $b + \varepsilon, \ldots, b + s(\bar{A})$, while adding $-b$ yields $b - \varepsilon, \ldots$. Since $b \leq s(\bar{A})$, this sequence covers all residues modulo 2 down to 0 or 1, depending on $b + s(\bar{A}) \pmod 2$. □

2.2 Useful Lemmas

Certain multisets are $(r + 1)$-excellent, as shown in the following lemmas.

Lemma 2. *Let $r \in \mathbb{N}$, and let $a_1, a_2, \ldots, a_n$ be nonzero integers with $|a_i| \leq i + r$ for $i = 1, 2, \ldots, n$. Then there exist signs $\varepsilon_i \in \{-1, 1\}$ such that*

$$\left| \sum_{i=1}^{n} \varepsilon_i a_i \right| \leq r + 1.$$

Proof. We proceed by induction on n. The claim is trivial for $n = 1, 2$. Assume it holds for all $n \leq k$, and consider $n = k + 1$. Without loss of generality, assume all $a_i > 0$. Since $1 \leq a_{k+1} \leq k + 1 + r$ and $1 \leq a_k \leq k + r$, we have $|a_{k+1} - a_k| \leq k + r$.

If $|a_{k+1} - a_k| = 0$, apply the induction hypothesis to $a_1, \ldots, a_{k-1}$. Otherwise, apply it to $a_1, \ldots, a_{k-1}, |a_{k+1} - a_k|$. The lemma follows. □

We now present a useful criterion for excellence.

Lemma 3 (Sufficient condition for an excellent set)**.** *Let $A \subseteq \mathbb{N}^*$ and $r \in \mathbb{N}$. If for all $m \in [0, |A|) \cap \mathbb{N}$, we have*

$$\mathcal{L}(\bar{A}, |\bar{A}| + r + 1 - m) \leq m,$$

then A is $(r+1)$-excellent.

Proof. Let $\bar{A}^{\uparrow} = \{a_1, \ldots, a_{|\bar{A}|}\}$. If $a_i \geq i + r + 1$, then $\mathcal{L}(\bar{A}, i + r + 1) > |\bar{A}| - i$, a contradiction. Hence $a_i \leq i + r$ for all i, and Lemma 2 implies $\bar{A}$ is $(r+1)$-excellent. Therefore, A is also $(r+1)$-excellent. □

The next two lemmas show that if $|A| > 0.4n + 1.2$, then the condition in Lemma 3 nearly holds for $r = 1, 2$, suggesting that A is likely 2- or 3-excellent.

Lemma 4. *Let $A \subseteq [n]$ and $|A| > \frac{2(n+3)}{5}$. Then for all $m \in \{2, 3, \ldots, |\bar{A}| - 2\}$, we have*

$$\mathcal{L}(\bar{A}, |\bar{A}| + 2 - m) \leq m. \tag{1}$$

Proof. Let $t = |A|$ and $A^{\uparrow} = \{a_i\}_{i=1}^{t}$.

If $t = 2p$ is even, consider the p positive integers:

$$\bar{A} = \{a_t - a_{t-1}, a_{t-2} - a_{t-3}, \ldots, a_2 - a_1\}.$$

Suppose there exists $m \in \{2, \ldots, p-2\}$ such that $\mathcal{L}(\bar{A}, p + 2 - m) \geq m + 1$. Then

$$\begin{aligned} a_t &\geq (m+1)(p - m + 2) + (t - (m+1)) \cdot 1 \\ &= t + (m+1)(p - m + 1). \end{aligned} \tag{2}$$

Using the monotonicity of the quadratic in m, we find

$$t + (m+1)(p - m + 1) \geq \frac{5}{2}t - 3 > n,$$

with minimum at $m = 2$ or $m = p - 2$. Thus $a_t > n$, a contradiction.

If $t = 2p + 1$ is odd, consider

$$\bar{A} = \{a_t - a_{t-1}, \ldots, a_2 - a_1, a_1\}.$$

Suppose there exists $m \in \{2, \ldots, p-1\}$ such that $\mathcal{L}(\bar{A}, p + 3 - m) \geq m + 1$. Then

$$\begin{aligned} a_t &\geq (m+1)(p + 3 - m) + (t - (m+1)) \cdot 1 \\ &= t + (m+1)(p + 2 - m). \end{aligned} \tag{3}$$

Again, the quadratic in m gives

$$t + (m+1)(p + 2 - m) \geq \frac{5}{2}t - \frac{3}{2} > n,$$

so $a_t > n$, a contradiction. □

Lemma 5. *Let $A \subseteq [n]$ and $|A| > \frac{2(n+3)}{5}$. Then for all $m \in \{3, \ldots, |\bar{A}|-1\}$, we have*

$$\mathcal{L}(\bar{A}, |\bar{A}|+3-m) \leq m-1. \tag{4}$$

Proof. Let $t = |A|$ and $A^{\uparrow} = \{a_i\}_{i=1}^{t}$.

If $t = 2p$, suppose $\mathcal{L}(\bar{A}, p+3-m) \geq m$ for some $m \in \{3, \ldots, p-1\}$. Then

$$a_t \geq m(p-m+3) + (t-m) \cdot 1 = t + m(p-m+2) \geq \frac{5}{2}t - 3 > n, \tag{5}$$

a contradiction.

If $t = 2p+1$, a similar computation yields $a_t > n$, again a contradiction. □

2.3 Proof of Theorem 2

We argue by contradiction. Suppose A is not 2-excellent.

2.3.1 Case 1: $\mathcal{L}(\bar{A}, 3) < |\bar{A}|$.

Let $t = |A|$ and $s = \lceil t/2 \rceil = |\bar{A}|$. Since $t > \frac{2n+6}{5}$ for $n \geq 61$, Lemmas 2 and 4 imply that $\mathcal{L}(\bar{A}, s+1) \geq 1$. Consequently, there exists an index $j \equiv t \pmod 2$ such that

$$a_j - a_{j-1} \geq s+1.$$

We claim that there exist two consecutive integers $v, v+1 \in A$ with $v, v+1 \notin \{a_j, a_{j-1}-1\}$. Otherwise, we would have

$$a_t = (a_t - a_j) + (a_j - a_{j-1}) + a_{j-1} \geq [2(t-j)-1] + (s+1) + [2(j-1)-1] = \frac{5t}{2} - 3 > n,$$

a contradiction.

Hence, the set $B = A \setminus \{v, v+1\}$ cannot be 3-excellent. Note that $|\bar{B}| = s-1$. By our choice of v, the difference $d = a_j - a_{j-1}$ must appear in $\bar{B}$ as an element generated by the pair (a_{j-1}, a_j). Applying Lemma 5, we obtain

$$\mathcal{L}(\bar{B}, s+2-m) \leq m-1, \quad \forall m \geq 3. \tag{6}$$

Moreover, we have $\mathcal{L}(\bar{B}, s) + \mathcal{L}(\tilde{B}, s) \leq 2$ and $\mathcal{L}(\tilde{B}, 2s) = 0$; otherwise,

$$a_t \geq 3s + 1 + (t-5) \geq \frac{5t}{2} - 4 > n.$$

Since $d = a_j - a_{j-1} \geq s+1$, inequality (6) implies that for any subset $\bar{X} \subseteq \bar{B} \setminus \{d\}$ and any $y \in \mathbb{N}^*$,

$$\mathcal{L}(\bar{X} \cup \{y\}, s+1-m) \leq m, \quad \forall m \geq 2. \tag{7}$$

We now establish several auxiliary lemmas by contradiction.

Lemma 6. $\mathcal{L}(\bar{B}, s) = 1$.

Proof. Suppose $\mathcal{L}(\bar{B}, s) = 2$. Choose $u \in B \setminus \{d\}$ with $u \geq s$. If $|u-d| > s$, then $u+d \geq 3s+1$, and hence

$$a_t \geq 3s + 1 + (t-2) \geq \frac{5t}{2} - 1 > n,$$

a contradiction. Thus $|u - d| \leq s$.

Now let $\bar{X} = \bar{B} \setminus \{d, u\}$ and $y = |u - d|$. One verifies that inequality (7) holds for all $m \geq 0$, which yields

$$\mathcal{L}(\bar{X} \cup \{y\}, |\bar{X} \cup \{y\}| + 3 - m) \leq m, \quad \forall\, m \geq 0. \tag{8}$$

By Lemma 2, the set $\bar{X} \cup \{y\}$ is 3-excellent, implying that B is 3-excellent—contradicting our earlier conclusion. $\square$

Lemma 7. *Let $B_1 = B \cap (0, a_{j-1})$ and $B_2 = B \cap (a_j, n]$. Then there exist $x, y \in B_1$ or $x, y \in B_2$ such that*

$$d - s \leq |x \pm y| \leq d + s.$$

Proof. Assume the contrary. Write $B_1^{\uparrow} = \{x_1 < \cdots < x_p\}$ and $B_2^{\uparrow} = \{y_1 < \cdots < y_q\}$. Then $x_p - x_1 \leq d - s - 1$; otherwise $x_p - x_1 \geq d + s + 1$. But since $\mathcal{L}(\bar{B}, 2s) = 0$ and $\mathcal{L}(\bar{B}, s) = 1$, we have $x_{i+1} - x_i \leq 2s$ for all i, and the intermediate value principle yields a contradiction. Similarly, $y_q - y_1 \leq d - s - 1$.

Therefore,

$$t - 6 = p + q - 2 \leq (x_p - x_1) + (y_q - y_1) \leq 2(d - s - 1),$$

which implies $d \geq t - 2$. On the other hand, $d \leq n + 1 - t$.

If $x_1 + x_2 \geq d + s + 1$, then $x_2 > \frac{d+1+s}{2}$, and consequently

$$y_q > a_2 + d + (t-6) \geq n \quad (\text{for } n \geq 28),$$

a contradiction. Hence $x_1 + x_2 \leq d - s - 1$, and by the same reasoning, $x_p + x_{p-1} \leq d - s - 1$, which gives $p \leq \frac{n+1-s-t}{2}$.

Similarly, $q - 1 \leq y_q - y_1 \leq d - s - 1 \leq n - t - s$. Combining these,

$$t - 4 = p + q \leq \frac{3}{2}(n + 1 - s - t) \leq \frac{3n+3}{2} - \frac{9}{4}t,$$

which contradicts the assumption $n \geq 122$. $\square$

Lemma 8. *Write $B^{\uparrow} = \{b_i\}_{i=1}^{t-2}$. Then there exists an index l such that $b_l = a_j$ and $b_{l-1} = a_{j-1}$, and there exist indices k, r with $1 \leq r \leq 4$ such that either $k \leq l - 1$ or $k \geq l + r$, and*

$$b_k - b_{k-r} \geq s - 1.$$

Proof. By Lemma 7, choose $x, y \in B_1$ or B_2 such that $z = |d - |x \pm y|| \leq s$. Define a set Y as follows:

$$Y = \begin{cases} B \setminus \{x, y, a_j, a_{j-1}\}, & \text{if } z = 0, \\ B \setminus \{x, y, a_j, a_{j-1}, z\}, & \text{if } z > 0 \text{ and } z \in B \setminus \{x, y, a_j, a_{j-1}\}, \\ (B \setminus \{x, y, a_j, a_{j-1}\}) \cup \{z\}, & \text{if } z > 0 \text{ and } z \notin B \setminus \{x, y, a_j, a_{j-1}\}. \end{cases}$$

Then $|Y| \geq t-7 > \frac{2n+6}{5}$ for $n \geq 113$, so by Lemma 2,

$$\mathcal{L}(\bar{Y}, |\bar{Y}|+3-m) \leq m-1, \quad \forall\, m \geq 3. \tag{9}$$

Note that $|\bar{Y}| \geq s-4$, and if $z \notin B \setminus \{x, y, a_j, a_{j-1}\}$, then $|\bar{Y}| \geq s-3$.
We claim $\mathcal{L}(\bar{Y}, |\bar{Y}|+2) \leq 1$; otherwise,

$$\max \bar{Y} \geq 2(s-2) + d + (t-10) > n \quad (\text{for } n \geq 88),$$

a contradiction. Similarly, $\mathcal{L}(\bar{Y}, |\bar{Y}|+1) \leq 2$.

Since Y cannot be 3-excellent, Lemma 2 forces $\mathcal{L}(\bar{Y}, |\bar{Y}|+3) = 1$. If $z \in (a_{j-1}, a_j)$, then $z \notin B \setminus \{x, y, a_j, a_{j-1}\}$, and $\mathcal{L}(\bar{Y}, s) = 1$. But $z \leq s$, so the only way z contributes to $\mathcal{L}(\bar{Y}, s)$ is if $z = s$ and $a_{j-1} = 0$. Even then, $|\bar{Y}| \geq s-2$ and $\mathcal{L}(\bar{Y}, s+1) = 1$, yet z cannot contribute to $\mathcal{L}(\bar{Y}, s+1)$; thus all contributions come from elements of B. If $z \notin (a_{j-1}, a_j)$, then contributions to $\mathcal{L}(\bar{Y}, s-1)$ also lie in B. Restoring the removed elements to Y completes the proof. □

Now define the interval

$$I = [b_l - 1, b_l] \cup [b_{k-r}, b_k].$$

Lemma 9. *There are at least $d-18$ elements in $\bar{B}$ equal to 1 that are not generated by elements within $[b_{k-r}, b_k]$.*

Proof. Let g denote the number of such elements. The gap $b_k - b_{k-r} \geq s-1$ involves at most two elements from $\bar{B}$ and two from $\tilde{B}$. Hence,

$$n \geq b_{t-2} \geq g + 2(s-4-g) + d + s - 1 + (t-2-s+1-2),$$

which simplifies to $g \geq d + t + 2s - 12 - n \geq d - 18$. □

Lemma 10. *For every $0 \leq i \leq t-4, p \geq 0$ such that $(b_i, b_{i+p}) \cap I = \varnothing$, we have*

$$b_{i+p} - b_i \leq 13 + p \quad \textit{and} \quad d \leq s + 14.$$

More precisely, if several such (b_i, b_{i+p}) are disjoint, then

$$\sum_i (b_{i+p} - b_i - p) + (d - (s+1)) \leq 13.$$

Proof. If some i violates the first inequality, then

$$n \geq b_{t-2} \geq d + s - 1 + (t-4-p) + 13 + p > n,$$

a contradiction. If $d \geq s+15$, then

$$n \geq b_{t-2} \geq (s+15) + s - 1 + (t-7) > n,$$

again a contradiction. The refined bound follows by a similar estimation. □

We now conclude this case. Let $\Delta = d - s - 1$. Our M_1, M_2 below are chosen according to different $0 \leq \Delta \leq 13$, as listed in Table 1.

By Lemmas 9, and since

$$d - 18 \geq 2M_1 - 1, \tag{10}$$

there are at least M_1 copies of the value 1 in $\bar{B}$ that are "free" from the interval I and they are **disjoint**. (For example, if $b_{i+1} - b_i = b_{i+3} - b_{i+2} = 1$, you can't take both of them)

Write $\bar{B} = \{x_1, x_2, \ldots, x_{s-1}\}$, where $x_i = b_{t-2i} - b_{t-2i-1}$. Select such $M_1 + 1$ of the free unit elements (including possibly $v + 1 - v = 1$), and remove any elements associated with the gap $b_l - b_{l-1}$ or the interval $[b_{k-r}, b_k]$. Denote the remaining multiset by $\bar{C}$. Then

$$|\bar{C}| \geq s - 4 - 2M_1.$$

The indices of the b_i involved in $\bar{C}$ lie in at most three disjoint subintervals of $(0, n] \setminus I$. Within each such block, we can pair adjacent differences $(x_w, x_{w+1}) = (b_{i+1} - b_i, b_{i+3} - b_{i+2})$ or $(b_{i+1} - b_i, b_{i+5} - b_{i+4})$ (depending on whether a copy between them had been removed) and replace them with

$$(y_w, y_{w+1}) = (b_{i+2} - b_i, b_{i+3} - b_{i+1}),$$

or

$$(y_w, y_{w+1}) = (b_{i+4} - b_i, b_{i+5} - b_{i+1}),$$

effectively "shifting" the pairing. Let $\bar{D}$ be the resulting multiset consisting of all such y_w, y_{w+1} and any unpaired x_i.

By Lemma 10, the sum of extra gap beyond 1 of $\bar{D}$ is at most $13 - \Delta$, every element in $\bar{D}$ is at most $17 - \Delta$, and the minimal M_2 number of y_w are no more than

$$4 + \left\lfloor (13 - \Delta) / \left(\left\lceil \frac{|\bar{C}| - 3}{2} \right\rceil - M_2 + 1 \right) \right\rfloor \leq M_1 + 1. \tag{11}$$

Together with $M_1 + 1$ copies of 1, the multiset $\bar{D}$ is strongly expressive (first select 1 then y_w from small to large) since

$$M_1 + 1 + 2M_2 \geq 17 - \Delta, \tag{12}$$

and

$$s(\bar{D}) \geq s(\bar{C}) + 2 \cdot \frac{|\bar{C}| - 3}{2} \geq 2s - 11 - 4M_1 \geq s + 1 + \Delta. \tag{13}$$

It is straightforward to check that our choice would let all inequalities hold. Thus, $\bar{D}$ can absorb all remaining elements of $\bar{B}$, implying that $\bar{A}$ is 1-excellent—a contradiction.

2.3.2 Case 2: $\mathcal{L}(\bar{A}, 3) \geq |\bar{A}|$.

Here, $a_{i+1} - a_i \geq 3$ for all $i \equiv t \pmod 2$. Let h be the number of 1's in $\tilde{A}$. Then

$$n \geq a_t \geq 3s + h + 2(t - s - h) = 2t + s - h,$$

so $h \geq \frac{t}{2} - 6 \geq 11$. Also, $a_{i+1} - a_i \leq 10$, otherwise $a_t > n$.

Consider $B = A \setminus \{a_1\}$. Then $\bar{B}$ contains at least 10 ones and no element exceeds 10. By induction, $\bar{B}$ can represent all $M \leq 10$ with $M \equiv s(\bar{B}) \pmod 2$. Since $a_1 \leq 10$, it follows that A is 2-excellent—a contradiction.

Δ	M_1	M_2	N
13	3	0	151
12	3	1	147
11	3	1	143
10	3	2	139
9	3	2	135
8	3	3	131
7	3	3	127
6	4	3	139
5	4	3	135
4	4	4	131
3	5	4	143
2	5	5	139
1	5	5	135
0	6	5	147

Table 1: Value for Δ, M_1, M_2, N, where N means when $n \geq N$, such $M_1.M_2$ would let all inequalities from hold.

2.4 Proof of Theorem 1

If $1 \in A$ or there exist $x, y \in A$ such that $|x - y| \leq 1$, then removing $\{x, y\}$ (or $\{1\}$) leaves a set whose size satisfies

$$|A \setminus \{x, y\}|,\ |A \setminus \{1\}| > c_n - 2 \geq \frac{n-7}{2}.$$

By the induction hypothesis, this remaining set is 2-excellent, and hence A itself is 1-excellent.

It remains to consider the case where all consecutive elements of A are at least distance 2 apart; that is, $a_i - a_{i-1} \geq 2$ for all i. In this situation, $1 \notin A$. As observed in Bingyuan Wang's solution, such a configuration is impossible when $n = 8k+1, 8k+2, \ldots, 8k+5$. Thus, we only need to analyze the cases $n = 8k+6, 8k+7, 8k+8, 8k+9$, under the assumption (by induction) that $n \in A$ and $|A| = \lfloor \frac{n}{2} \rfloor$ or $\lfloor \frac{n+1}{2} \rfloor$.

- **Case $n = 8k+6$:** Then $|A| = 4k+3$, and since $A \subseteq \{2, 3, \ldots, 8k+6\}$ with gaps of at least 2, the only possible choice is the even numbers:

  $$A = \{2, 4, 6, \ldots, 8k+6\},$$

  which is 0-excellent.

- **Case $n = 8k+7$:** The only admissible sets have the form

  $$A = \{2, 4, \ldots, 2m, 2m+3, 2m+5, \ldots, 8k+7\}$$

  for some m.

- If $m \geq 8$, we can perform the following reductions:

$$(2m, 2m-2) \to 2, \quad (2m-2, 2m-4) \to 2, \quad (2, 2m, 2m+3) \to 1.$$

  This yields a new set containing 1 along with smaller even numbers and the tail starting from $2m+5$. By the induction hypothesis, this set can generate all integers up to n.

- If $1 \leq m \leq 7$, we instead combine large elements:

$$(8k+7, 8k+5) \to 2, \quad (8k+3, 8k+1) \to 2, \quad (8k-1, 8k-3) \to 2,$$

  eventually reducing to a set of the form

$$\{3, 2, \ldots, 2, 2m-2, 2m+5, \ldots, 8k-5\},$$

  which again, by induction, generates all numbers $\leq n$.

- **Case** $n = 8k+8$**:** Here $|A| = 4k+4$, and the only feasible set with minimal spacing is

$$A = \{2, 4, 6, \ldots, 8k+8\},$$

  which is 0-excellent.

- **Case** $n = 8k+9$**:** Then $|A| = 4k+4$, and the only possible structure is

$$A = \{2, 4, \ldots, 2m, 2m+3, \ldots, 8k+9\}.$$

  The construction proceeds analogously to the $n = 8k+7$ case, using similar reduction steps to eventually produce a 1-excellent set.

Combining all these cases completes the proof.