American Digital Put Option

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1 Analytical price & delta formula

Because for American digital option, we will surely exercise the option once the stock reaches the strike price, we can use this strong constraint to price it.

Proof:

Assuming we have a stopping time (exercising criterion) τ and $r \geq 0$, there's always

$$sup\left\{\tilde{E}\left[e^{-r(\tau-t)}c \cdot \mathbb{1}_{\{T\geq\tau>t\}}\middle|\mathcal{F}(t)\right]\right\} = c$$

Therefore, no matter how we define τ, we always have

$$\left\{c \cdot \mathbb{1}_{\{T\geq\tau\}}\middle|t = \tau\right\} = c \geq \tilde{E}\left[e^{-r(\tau-t)}c \cdot \mathbb{1}_{\{T\geq\tau>t\}}\middle|\mathcal{F}(t)\right]$$

That means if we have reached the stopping time now and the payoff is a constant number, exercising immediately is always better than waiting.

1.1 Risk-neutral approach

Note that a martingale stopped at a stopping time is still a martingale, so we can use the risk-neutral pricing formula:

$$e^{-rt}V(t) = \tilde{E}[e^{-rT}V(T)|\mathcal{F}(t)]$$

with GBM assumption

$$\frac{dS(t)}{S(t)} = rdt + \sigma d\widetilde{W}(t)$$

Let's start from call option.

Concretely, if we denote $v(t, S)$ as the payoff function, then

$$\tau_K = \min\{u \geq t|S(u) \geq K\}$$

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$$v\big(t,S(t)\big)=\mathbb{I}_{\{S(t)\geq K\}}, V(t)=e^{r(t-t\wedge\tau_K)}v\big(t\wedge\tau_K,S(t\wedge\tau_K)\big)$$

The above equation means $V(t)$ equals to \$1 by early exercising plus corresponding interest.

$$\therefore V(t)=e^{r(t-\tau_K)}\mathbb{I}_{\{t\geq\tau_K\}}$$

$$\therefore V(T)=e^{r(T-\tau_K)}\mathbb{I}_{\{t\geq\tau_K\}}\to e^{-rT}V(T)=e^{-r\tau_K}\mathbb{I}_{\{T\geq\tau_K\}}$$

Now, we know the only random variable in the pricing formula is τ_K, so we need to identify its distribution. For simplicity, consider that there's no condition (i.e. $\mathcal{F}(0)$).

Note that Brownian motion is easier to handle than geometric Brownian motion, so we consider

$$Z(t)=\frac{\ln\frac{S(t)}{S_0}}{\sigma}, S_0=S(0)\to dZ(t)=\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)dt+d\widetilde{W}(t), Z(0)=0$$

Since $S(t)$ *and* $\ln S(t)$ are in monotonically increasing relationship, so the stopping time τ_K also applies to $Z(t)$. We can just change the definition to another expression:

$$\tau_K=\min\left\{u\geq t\,\middle|\,Z(u)\geq\frac{\ln\frac{K}{S_0}}{\sigma}\right\}$$

Recall the 1st passage time's pdf of a Brownian motion with drift term, for a standard form $X(t)=ct+W(t), a>0$:

$$P(\tau_a<t)=\Phi\left(\frac{-a+ct}{\sqrt{t}}\right)+e^{2ac}\Phi\left(\frac{-a-ct}{\sqrt{t}}\right)$$

$$f_{\tau_a}(t)=\frac{ae^{\frac{-(a-ct)^2}{2t}}}{\sqrt{2\pi t^3}}$$

Proof:

$$Define\ d\widetilde{W}(t)=cdt+dW(t), by\ Girsanov\ theorem, we\ have$$

$$Z(t)=\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}=e^{-cW(t)-\frac{1}{2}c^2t}$$

To make the notation easier to read, we use $P(\tau_a\in\Delta t)$ to represent $\tau_a's$ pdf $f_{\tau_a}(t)$, where τ_a is the 1st passage time to level a and Δt refers to interval $\lim_{dt\to0}[t,t+dt]$.

$$P(\tau_a\in\Delta t)=E\left[\mathbb{I}_{\{\tau_a\in\Delta t\}}\right]=\tilde{E}\left[Z(t)\mathbb{I}_{\{\tau_a\in\Delta t\}}\right]=\tilde{E}\left[e^{-cW(t)-\frac{1}{2}c^2t}\mathbb{I}_{\{\tau_a\in\Delta t\}}\right]$$

$$=\tilde{E}\left[e^{-cW(t)-\frac{1}{2}c^2t}\mathbb{I}_{\{\tau_a\in\Delta t\}}\middle|\tau_a\in\Delta t\right]\cdot\tilde{P}(\tau_a\in\Delta t)+\tilde{E}\left[e^{-cW(t)-\frac{1}{2}c^2t}\mathbb{I}_{\{\tau_a\in\Delta t\}}\middle|\tau_a\notin\Delta t\right]\cdot\tilde{P}(\tau_a\notin\Delta t)$$

$$=\tilde{E}\left[e^{-cW(t)-\frac{1}{2}c^2t}\middle|\tau_a\in\Delta t\right]\cdot\tilde{P}(\tau_a\in\Delta t)+0$$

$$= e^{-cW(t)-\frac{1}{2}c^2 t} \cdot \tilde{P}(\tau_a \in \Delta t)$$

Note that under $\widetilde{\mathbb{P}}$ measure, our process $\widetilde{W}(t)$ doesn't have drift term, so we can apply reflection principle here easily. Assuming $a > 0$,

$$\because \tilde{P}\big(\widetilde{W}(t) > a\big) = \tilde{P}\big(\widetilde{W}(t) > a \big| \tau_a < t\big) \cdot \tilde{P}(\tau_a < t) + \tilde{P}\big(\widetilde{W}(t) > a \big| \tau_a \geq t\big) \cdot \tilde{P}(\tau_a \geq t)$$

$$= \tilde{P}\big(\widetilde{W}(t) > a \big| \tau_a < t\big) \cdot \tilde{P}(\tau_a < t) = \frac{1}{2}\tilde{P}(\tau_a < t)$$

$$\therefore \tilde{P}(\tau_a < t) = 2\tilde{P}\big(\widetilde{W}(t) > a\big) = 2\left(1 - \Phi\left(\frac{a}{\sqrt{t}}\right)\right)$$

$$\therefore \tilde{P}(\tau_a \in \Delta t) = \frac{\partial \tilde{P}(\tau_a < t)}{\partial t} = \phi\left(\frac{a}{\sqrt{t}}\right)\frac{a}{\sqrt{t^3}} = \frac{a e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}}$$

(Similarly, when $a < 0$, we can derive $\tilde{P}(\tau_a \in \Delta t) = -\frac{a e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}}$.)

Therefore, we conclude

$$f_{\tau_a}(t) = P(\tau_a \in \Delta t) = e^{-cW(t)-\frac{1}{2}c^2 t} \cdot \frac{a e^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}} = \frac{a e^{\frac{-(a-ct)^2}{2t}}}{\sqrt{2\pi t^3}}$$

In our case, we can write

$$P(\tau_K < t) = \Phi\left(\frac{-\frac{\ln\frac{K}{S_0}}{\sigma} + \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t}{\sqrt{t}}\right) + e^{2\frac{\ln\frac{K}{S_0}}{\sigma}\left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)}\Phi\left(\frac{-\frac{\ln\frac{K}{S_0}}{\sigma} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t}{\sqrt{t}}\right)$$

$$f_{\tau_K}(t) = \frac{\frac{\ln\frac{K}{S_0}}{\sigma}}{\sqrt{2\pi t^3}} \exp\left\{\frac{-\left[\frac{ln\frac{K}{S_0}}{\sigma} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)t\right]^2}{2t}\right\}$$

Next*,

$$V(0) = \tilde{E}[e^{-rT}V(T)|\mathcal{F}(0)] = \int_0^\infty e^{-ru}\mathbb{I}_{\{T \geq \tau_K\}}\, dP(\tau_K < u)$$

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* The following deduction is for $K > S_0$ case, otherwise $V(0) = 1$ trivially; similarly, for put, the formula is only for $K < S_0$ case, otherwise $V(0) = 1$.

$$= \int_0^T e^{-ru} f_{\tau_K}(u)du = \int_0^T e^{-ru} \cdot \frac{\frac{\ln \frac{K}{S_0}}{\sigma}}{\sqrt{2\pi u^3}} \exp\left\{ \frac{-\left[\frac{ln \frac{K}{S_0}}{\sigma} - \left(\frac{r}{\sigma} - \frac{\sigma}{2}\right) u \right]^2}{2u} \right\} du$$

Let $\boldsymbol{\alpha = r - \frac{\sigma^2}{2}}$, $\boldsymbol{\beta = \sqrt{\alpha^2 + 2\sigma^2 r}}$, we can write

$$\ldots = \int_0^T e^{-ru} \frac{\ln \frac{K}{S_0}}{\sigma\sqrt{2\pi u^3}} \exp\left\{ \frac{-\left(ln \frac{K}{S_0} - \alpha u\right)^2}{2\sigma^2 u} \right\} du$$

$$= \int_0^T e^{-ru} \frac{\ln \frac{K}{S_0}}{\sigma\sqrt{2\pi u^3}} \exp\left\{ -\frac{ln^2 \frac{K}{S_0} - 2\alpha u \, ln \frac{K}{S_0} + \alpha^2 u^2}{2\sigma^2 u} \right\} du$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha}{\sigma^2}} \int_0^T e^{-ru} \frac{\ln \frac{K}{S_0}}{\sigma\sqrt{2\pi u^3}} \exp\left\{ -\frac{ln^2 \frac{K}{S_0} + \alpha^2 u^2}{2\sigma^2 u} \right\} du$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha}{\sigma^2}} \int_0^T e^{-ru} \frac{\ln \frac{K}{S_0}}{\sigma\sqrt{2\pi u^3}} \exp\left\{ -\frac{ln^2 \frac{K}{S_0} + \beta^2 u^2 - 2\sigma^2 r u^2}{2\sigma^2 u} \right\} du$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha}{\sigma^2}} \int_0^T \frac{\ln \frac{K}{S_0}}{\sigma\sqrt{2\pi u^3}} \exp\left\{ -\frac{ln^2 \frac{K}{S_0} + \beta^2 u^2}{2\sigma^2 u} \right\} du$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha \pm \beta}{\sigma^2}} \int_0^T \frac{\ln \frac{K}{S_0}}{\sigma\sqrt{2\pi u^3}} \exp\left\{ -\frac{ln^2 \frac{K}{S_0} \pm 2\beta u \, ln \frac{K}{S_0} + \beta^2 u^2}{2\sigma^2 u} \right\} du$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha \pm \beta}{\sigma^2}} \int_0^T \frac{\ln \frac{K}{S_0}}{\sigma\sqrt{2\pi u^3}} \exp\left\{ -\frac{\left(ln \frac{K}{S_0} \pm \beta u\right)^2}{2\sigma^2 u} \right\} du$$

Note that $\frac{ln\frac{K}{S_0}}{\sigma\sqrt{u}} = \frac{ln\frac{K}{S_0} - \beta u}{2\sigma\sqrt{u}} + \frac{ln\frac{K}{S_0} + \beta u}{2\sigma\sqrt{u}}$, so

$$\ldots = \left(\frac{K}{S_0}\right)^{\frac{\alpha + \beta}{\sigma^2}} \int_0^T \frac{1}{2\sqrt{2\pi u}} \frac{\ln \frac{K}{S_0} - \beta u}{\sigma\sqrt{u}} \exp\left\{ -\frac{\left(ln \frac{K}{S_0} + \beta u\right)^2}{2\sigma^2 u} \right\} du$$

$$+\left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\int_0^T \frac{1}{2\sqrt{2\pi}u}\frac{\ln\frac{K}{S_0}+\beta u}{\sigma\sqrt{u}}\exp\left\{-\frac{\left(ln\frac{K}{S_0}-\beta u\right)^2}{2\sigma^2 u}\right\}du$$

Now we can perform variable transformation. Let $x_\pm = \frac{\ln\frac{K}{S_0}\pm\beta u}{\sigma\sqrt{u}}$, we will have

$$\ldots = \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}}\int_0^T \frac{1}{2\sqrt{2\pi}u}x_-\cdot e^{-\frac{x_+^2}{2}}du + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\int_0^T \frac{1}{2\sqrt{2\pi}u}x_+\cdot e^{-\frac{x_-^2}{2}}du$$

Also, because $dx_\pm = -\frac{1}{2}\frac{\ln\frac{K}{S_0}\mp\beta u}{\sigma u\sqrt{u}}du$, then $\frac{du}{u} = -\frac{2}{x_\mp}dx_\pm$, and $x_\pm(0)\to\infty$. Substitute in

$$\ldots = \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}}\int_\infty^{x_+(T)} -\frac{1}{2\sqrt{2\pi}}x_-\cdot e^{-\frac{x_+^2}{2}}\frac{2}{x_-}dx_+ + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\int_\infty^{x_-(T)} -\frac{1}{2\sqrt{2\pi}}x_+\cdot e^{-\frac{x_-^2}{2}}\frac{2}{x_+}dx_-$$

$$= -\left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}}\int_\infty^{x_+(T)}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x_+^2}{2}}dx_+ - \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\int_\infty^{x_-(T)}\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x_-^2}{2}}dx_-$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}}\int_{x_+(T)}^\infty\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x_+^2}{2}}dx_+ + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\int_{x_-(T)}^\infty\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{x_-^2}{2}}dx_-$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}}\Phi\big(-x_+(T)\big) + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\Phi\big(-x_-(T)\big)$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}}\Phi\left(-\frac{\ln\frac{K}{S_0}+\beta T}{\sigma\sqrt{T}}\right) + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\Phi\left(-\frac{\ln\frac{K}{S_0}-\beta T}{\sigma\sqrt{T}}\right)$$

Moreover,

$$\delta = \frac{\partial V(0)}{\partial S_0} = \frac{\partial}{\partial S_0}\left[\left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}}\Phi\left(-\frac{\ln\frac{K}{S_0}+\beta T}{\sigma\sqrt{T}}\right) + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\Phi\left(-\frac{\ln\frac{K}{S_0}-\beta T}{\sigma\sqrt{T}}\right)\right]$$

$$= -\frac{K}{S_0^2}\frac{\alpha+\beta}{\sigma^2}\left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}-1}\Phi\left(-\frac{\ln\frac{K}{S_0}+\beta T}{\sigma\sqrt{T}}\right) + \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}}\phi\left(-\frac{\ln\frac{K}{S_0}+\beta T}{\sigma\sqrt{T}}\right)\frac{1}{\sigma\sqrt{T}S_0}$$

$$-\frac{K}{S_0^2}\frac{\alpha-\beta}{\sigma^2}\left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}-1}\Phi\left(-\frac{\ln\frac{K}{S_0}-\beta T}{\sigma\sqrt{T}}\right) + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}}\phi\left(-\frac{\ln\frac{K}{S_0}-\beta T}{\sigma\sqrt{T}}\right)\frac{1}{\sigma\sqrt{T}S_0}$$

Similarly, for American digital put, we have

$\tau_K = \min\left\{u \geq t \middle| Z(u) \leq \frac{\ln\frac{K}{S_0}}{\sigma}\right\}$, the only change would be $f_{\tau_K}(t) \to -f_{\tau_K}(t)$ and $x_\pm(0) \to -\infty$.

$$\therefore V(0) = \tilde{E}\left[e^{-rT}\mathbb{1}_{\{T \geq \tau_K\}}\middle|\mathcal{F}(t)\right]$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \int_{-\infty}^{x_+(T)} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x_+^2}{2}} dx_+ + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \int_{-\infty}^{x_-(T)} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x_-^2}{2}} dx_-$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \Phi\big(x_+(T)\big) + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \Phi\big(x_-(T)\big)$$

$$= \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \Phi\left(\frac{\ln\frac{K}{S_0} + \beta T}{\sigma\sqrt{T}}\right) + \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \Phi\left(\frac{\ln\frac{K}{S_0} - \beta T}{\sigma\sqrt{T}}\right)$$

Furthermore,

$$\delta = -\frac{K}{S_0^2}\frac{\alpha+\beta}{\sigma^2}\left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}-1} \Phi\left(\frac{\ln\frac{K}{S_0} + \beta T}{\sigma\sqrt{T}}\right) - \left(\frac{K}{S_0}\right)^{\frac{\alpha+\beta}{\sigma^2}} \phi\left(\frac{\ln\frac{K}{S_0} + \beta T}{\sigma\sqrt{T}}\right)\frac{1}{\sigma\sqrt{T}S_0}$$

$$-\frac{K}{S_0^2}\frac{\alpha-\beta}{\sigma^2}\left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}-1} \Phi\left(\frac{\ln\frac{K}{S_0} - \beta T}{\sigma\sqrt{T}}\right) - \left(\frac{K}{S_0}\right)^{\frac{\alpha-\beta}{\sigma^2}} \phi\left(\frac{\ln\frac{K}{S_0} - \beta T}{\sigma\sqrt{T}}\right)\frac{1}{\sigma\sqrt{T}S_0}$$

1.2 PDE approach

1. First, we write the BS PDE and boundary conditions for the option

Write $P_d^{Am}(S,t)$ for the value of the American digital put. It satisfies the Black-Scholes equation

$$\frac{\partial P_d^{Am}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P_d^{Am}}{\partial S^2} + rS\frac{\partial P_d^{Am}}{\partial S} - rP_d^{Am} = 0 \quad \text{for } 0<E<S$$

On S=E, we receive the payoff:

$$P_d^{Am}(E,t) = 1$$

At expiry,

$$P_d^{Am}(S,T) = 0$$

Then we define new variables

$$S = Ee^x, t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, P_d^{Am}(S,t) = v(x,\tau), k = \frac{r}{\frac{1}{2}\sigma^2}, v = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau}u(x,\tau)$$

Finally, the problem becomes

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

$$\text{with } \begin{cases} u(0,\tau) = e^{\frac{1}{4}(k+1)^2\tau} \\ u(x,0) = 0 \end{cases}$$

2. Represent the solution as a sum of a time independent particular solution and a conventional barrier option

We can write the value of option as

$$P_d^{Am}(S,t) = (S/E) - P_1(S,t)$$

Where S/E is a time-independent solution and $P_1(S,t)$ is a barrier contract with payoff

$$\begin{cases} 0 & \text{for } 0<\text{S} \leq E \\ S/E & \text{for E}<\text{S}<\infty \end{cases}$$

Then we can write the boundary condition for this barrier option

$$\begin{cases} u_1(0,\tau) = 0 \\ u_1(x,0) = e^{\frac{1}{2}(k+1)x}, x \geq 0 \end{cases}$$

3. Solve these two assets and combine them

We use reflection to solve barrier option for all x that subject to

$$u_1(x,0) = \begin{cases} -e^{-\frac{1}{2}(k+1)x} & \text{for x}<0 \\ e^{\frac{1}{2}(k+1)x} & \text{for x}>0 \end{cases}$$

We make a change of variable$y = \frac{s-x}{\sqrt{2\tau}}$, then we can write the integral

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(y\sqrt{2\tau}+x)}\, e^{-\frac{y^2}{2}} dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}} e^{-\frac{1}{2}(k+1)(y\sqrt{2\tau}+x)}\, e^{-\frac{y^2}{2}} dy = I_1 - I_2$$

For the first integral of the right side

$$I_1 = e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(y - \frac{(k+1)\sqrt{2\tau}}{2}\right)^2}{2}} dy$$

$$= e^{\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau}\, \Phi\left(\frac{(k+1)\sqrt{2\tau}}{2} + \frac{x}{\sqrt{2\tau}}\right)$$

Similarly

$$I_2 = e^{-\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau} \int_{-\infty}^{-\frac{x}{\sqrt{2\tau}}} e^{-\frac{\left(y+\frac{(k+1)\sqrt{2\tau}}{2}\right)^2}{2}} dy$$

$$= e^{-\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau} \Phi\left(\frac{(k+1)\sqrt{2\tau}}{2} - \frac{x}{\sqrt{2\tau}}\right)$$

Recall the definition of all functions and variables to get to the original inputs

$$P_d^{Am}(S,t) = (S/E) - P_1(S,t)$$

$$= (S/E) - (S/E)\Phi(d_+) + (S/E)^{-2\frac{r}{\sigma^2}} \Phi(-d_-)$$

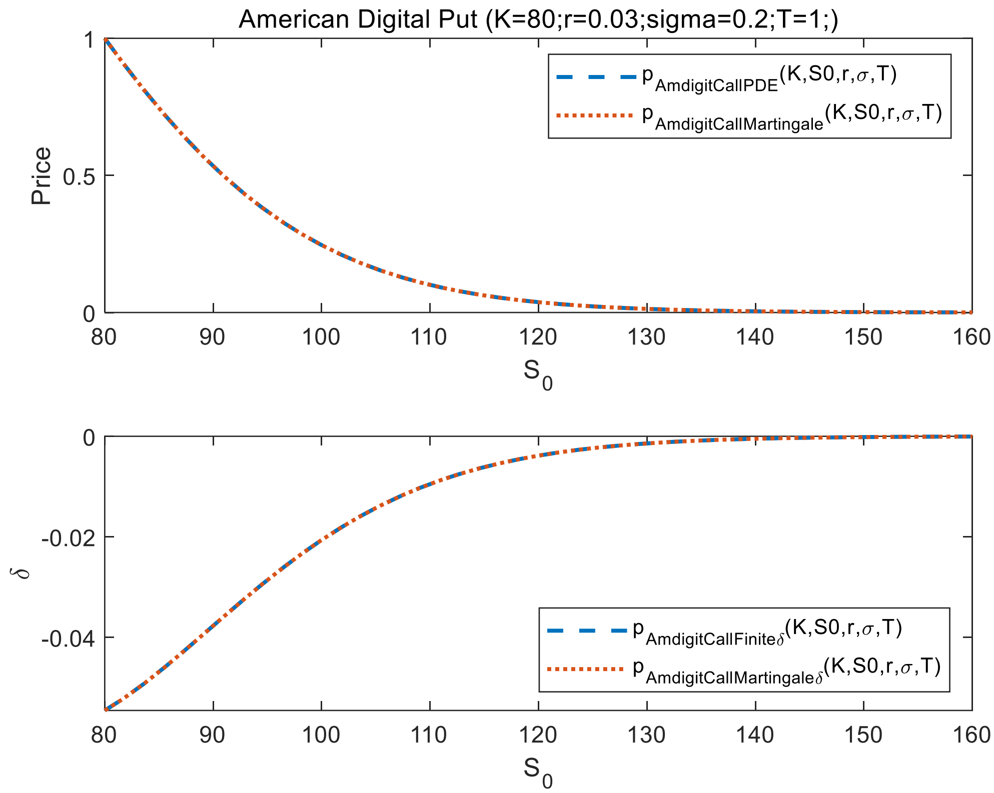
$$= (S/E)\Phi(-d_+) + (S/E)^{-2\frac{r}{\sigma^2}} \Phi(-d_-)$$

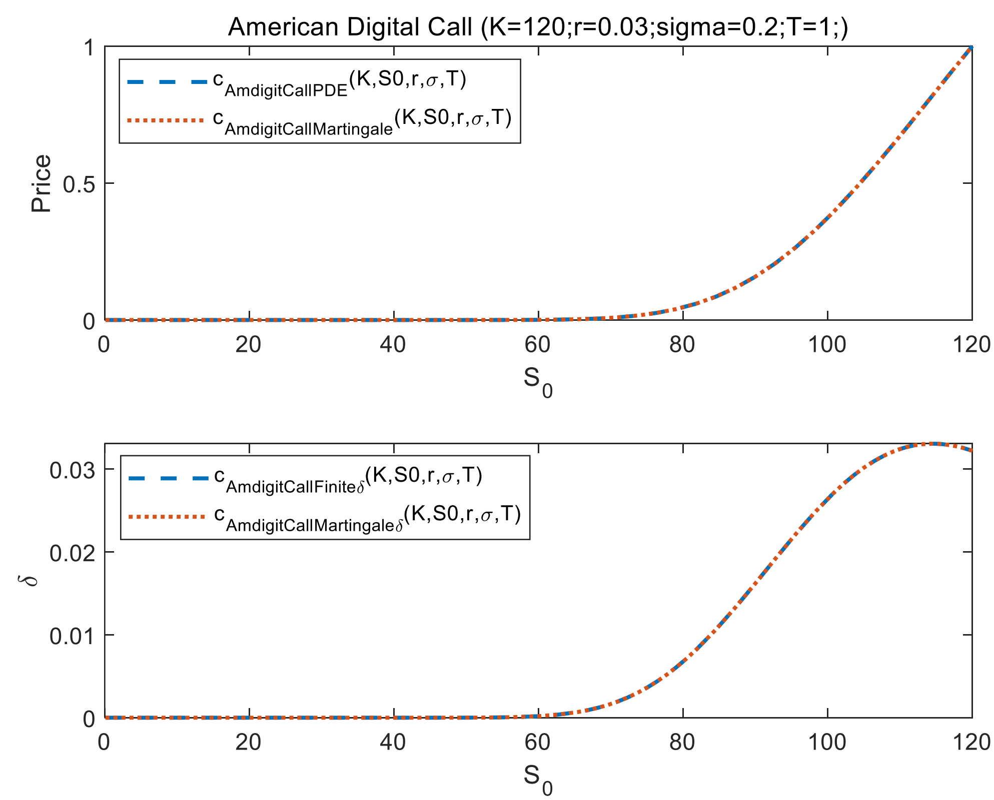
Where

$$d_+ = \frac{ln(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_- = \frac{ln(S/E) - \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

1.3 Numerical test between models





2 Model Validation

To validate our pricing model, the most popular and reliable ways are Monte Carlo and extreme case check. (Also, to validate our delta formula, the best way is finite differentiation and that has already been present in the previous paragraph)

2.1 Monte Carlo

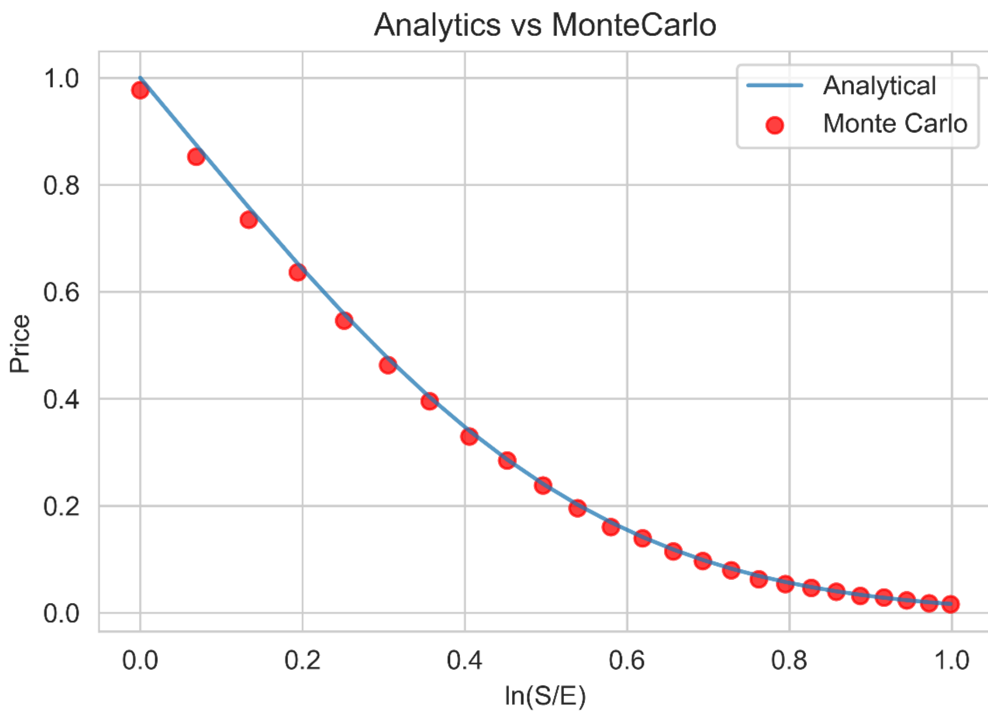


Figure. Analytical Vs Monte Carlo, Put price. T=2, r=0.02, σ=0.285, $\ln\frac{S}{E} \in [0,1]$.

From the above figure we can see that the price by Monte Carlo (1e4 paths, 252 nodes per year) is very close to the analytical price*. This demonstrated the correctness of our models.

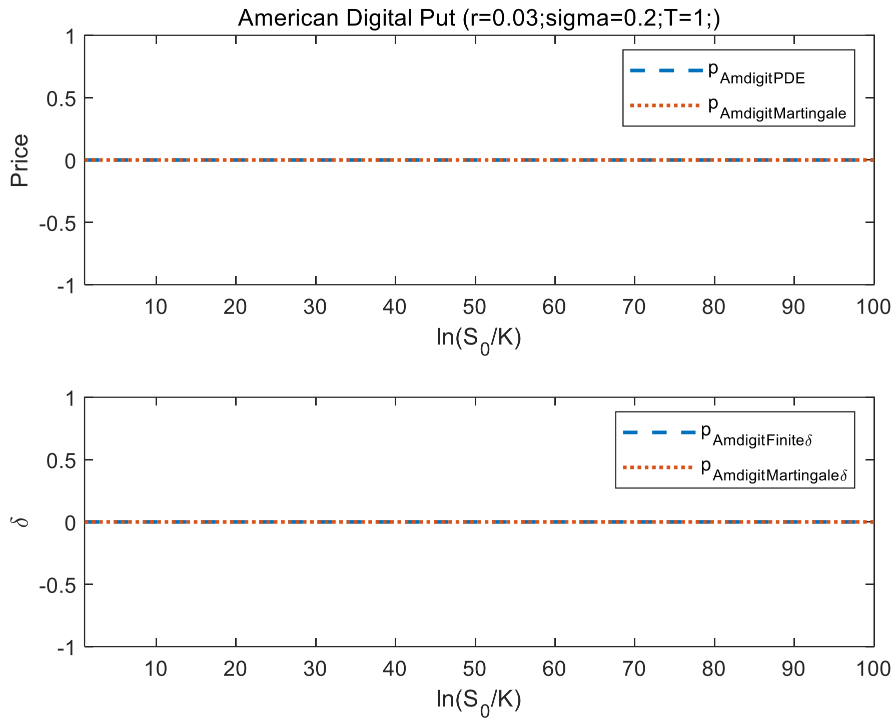
2.2 Extreme case

In some circumstances, American option will degrade towards European option or constants, so that provides us a chance to verify the model.

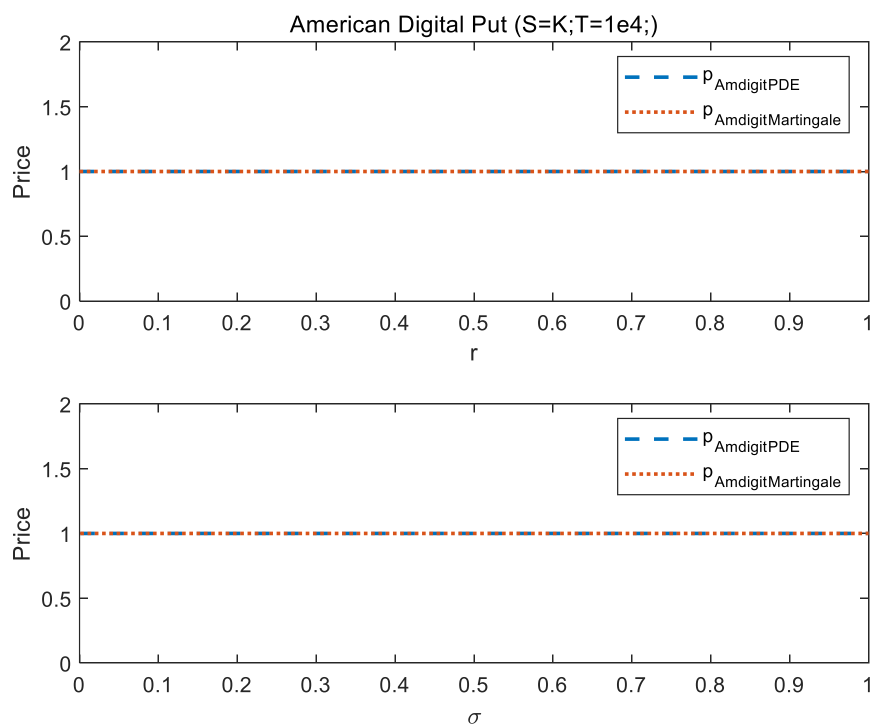
Particularly,

* Because the risk-neutral & PDE model have been verified to be the same, so here's only 1 analytical price line.

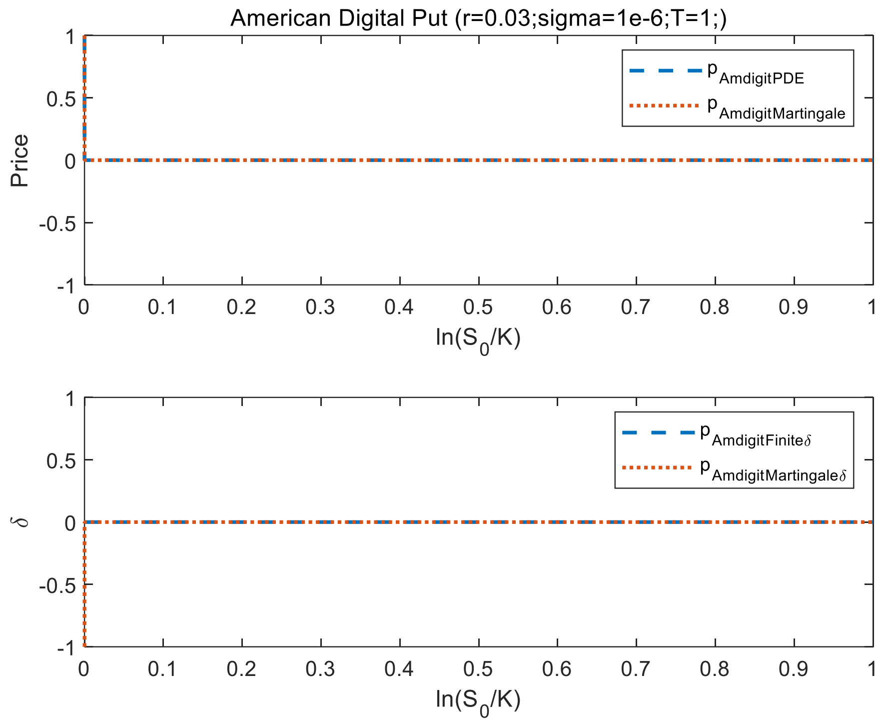
1) When $S \gg K$, American digital put nearly has no chance to be exercise, so its value should be 0;



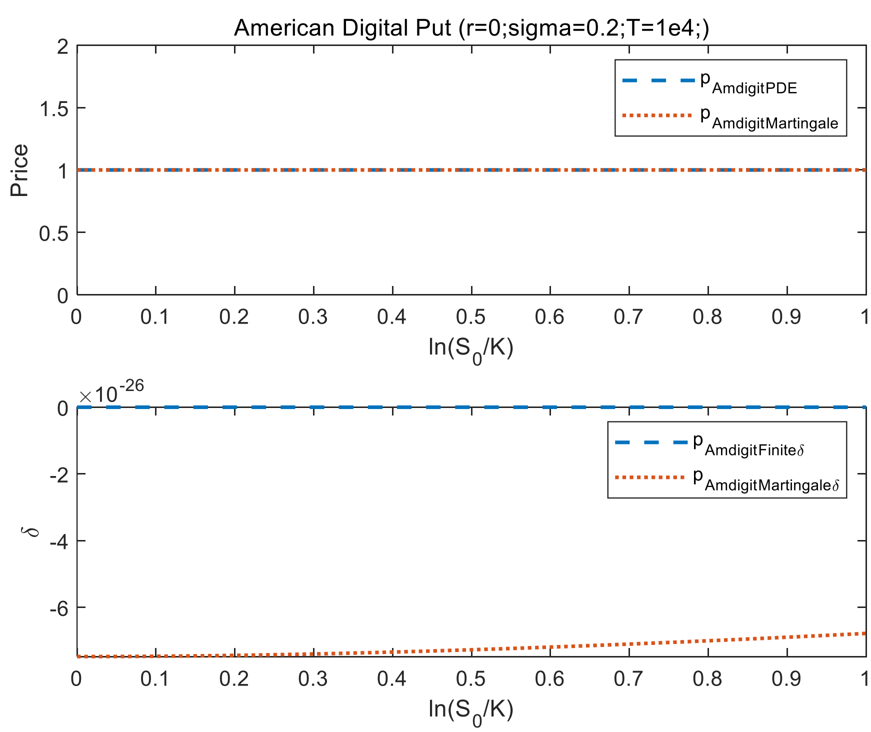
2) When $S = K$, American digital put should be exercise immediately, so its value should be 1;



3) When $\sigma \to 0$ & $r > 0$, American digital put has no chance to be exercise, so its value should be 0;



4) When $r = 0$ & $T \to \infty$, since $P(T \geq \tau_K) \to 1$, so its price should be 1;



3 Delta hedging backtest

In this part, we create a portfolio which contains 1000 contracts our American digital put option and short positions in 1000 share stocks with position equals to Delta of the option each share, each day we adjust positions of stocks according to Delta.

3.1 Parameters selection

Let the option to be set up on December 3 2018, expiration in 2 years. We use Apple Inc. stock (AAPL) as our underlying asset. Because the minimum stock price is slightly above 140 over the time period, we choose strike price equal to 140 so that the price never "touched" during the last year.

Then we calculate the historical volatility on log returns in the last year. To be specific, we have 252 working days in total, assuming stock price is $S(i)$at day$i, i = 1,2 \ldots 252$. We define our return as

$$R_i = ln\left(\frac{S(i+1)}{S(i)}\right), i = 1,2, \ldots ,251$$

To get annually volatility

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(R_i - R_{avg}\right)^2}{n-1}} \times \sqrt{252}$$

In our case, $\sigma \approx 0.2866$.

3.2 Comparison of American digital put and vanilla digital

To test our formula further, we compare the price with vanilla digital put with the same strike and expiration by plot. By definition, the price of vanilla digital put is

$$P = e^{-r(T-t)}N(-d_2)$$
$$d_2 = \frac{ln\left(\frac{S}{E}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

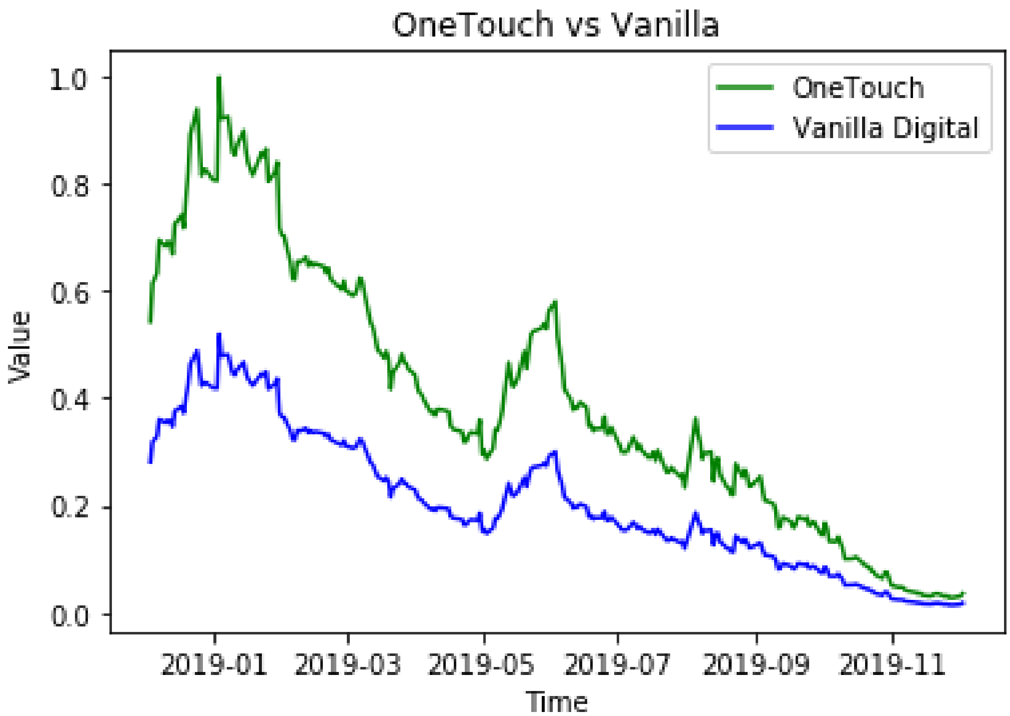


Figure. OneTouch Vs Vanilla, Put price. T=2, r=0.02, σ=0.285, $t \in [0,1]$

From the above figure we can see that the price of American digital put is approximately double of vanilla digital put, which meets the theoretical situation.

3.3 Calculation of PnL

We have a history of prices $S_{i,i=1,\dots 252}$ and dates D_i

- For each date D_i calculate the time t_i to expire T, the option value $V(S_i, t_i)$ and Delta.
- For each day, starting with day 2, calculate PnL:

$$PnL_i = 1000 * \left(V\left(t_{i,}S_i\right) - V\left(t_{i-1,}S_{i-1}\right) - \Delta_{i-1}(S_i - S_{i-1})\right)$$

- At last calculate cumulative PnL at day n:

$$CPnL_n = \sum_{i=1}^{n} PnL_i$$

We plot the PnL and cumulative PnL in same plot

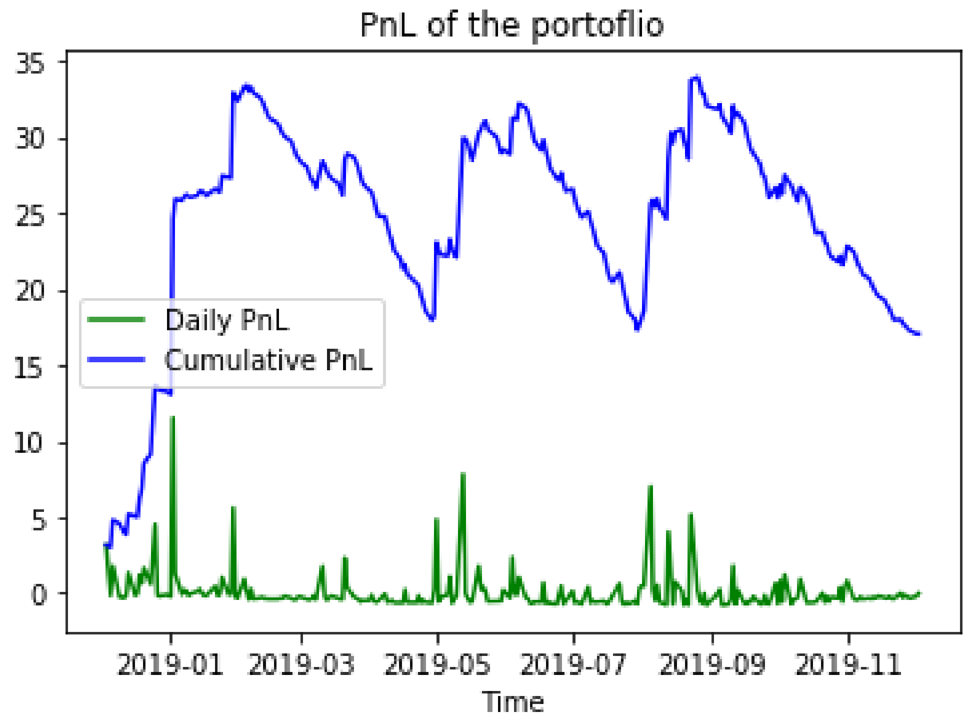


Figure. PnL of the portfolio, put option and stock

From the above figure we can see that PnL of the portfolio fluctuates around zero and is significantly greater than zero at certain points, so cumulative PnL is always positive as a whole, which indicates that our hedging is effective.