Convex Sets

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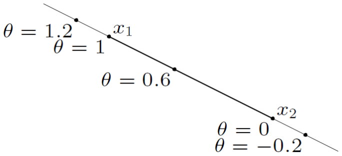
Outline

1. Affine and Convex Sets
2. Some Important Examples
3. Operations that Preserve Convexity
4. Generalized Inequalities
5. Separating and Supporting Hyperplanes

Definition of Affine Set

- **Line:** through $\boldsymbol{x}_1, \boldsymbol{x}_2$: all points

$$\boldsymbol{x} = \theta \boldsymbol{x}_1 + (1 - \theta)\boldsymbol{x}_2 \quad (\theta \in \mathbb{R})$$



- **Affine set:** contains the line through any two distinct points in the set
- **Example:** solution set of linear equations $\{\boldsymbol{x} | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$
  (conversely, every affine set can be expressed as solution set of system of linear equations)

$x_1 \in S$, $x_2 \in S$, $\theta \in R$

$x = \theta x_1 + (1-\theta) x_2 \in S$

$Ax = \theta \underbrace{A x_1}_{b} + (1-\theta) \underbrace{A x_2}_{b} = b$

Definition of Convex Set

- **Line segment:** between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$: all points

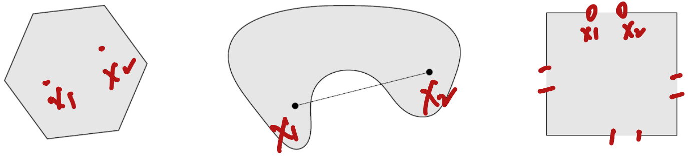
$$\boldsymbol{x} = \theta \boldsymbol{x}_1 + (1 - \theta)\boldsymbol{x}_2$$

with $0 \leq \theta \leq 1$

- **Convex set:** contains line segment between any two points in the set C

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta \boldsymbol{x}_1 + (1 - \theta)\boldsymbol{x}_2 \in C$$

- **Examples** (one convex, two nonconvex sets)



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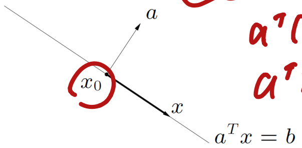
Examples: Hyperplanes and Halfspaces

- **Hyperplane:** set of the form $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} = b\} (\boldsymbol{a} \neq \mathbf{0})$

$a^T x_0 = b$

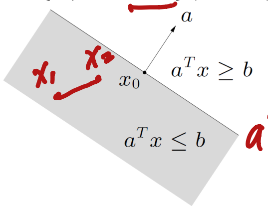
$a \perp (x - x_0)$



$a^T(x - x_0) = 0$

$a^T x = a^T x_0 = b$

- **Halfspace:** set of the form $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} \leq b\} (\boldsymbol{a} \neq \mathbf{0})$



$a^T x = b$

- $\boldsymbol{a}$ is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

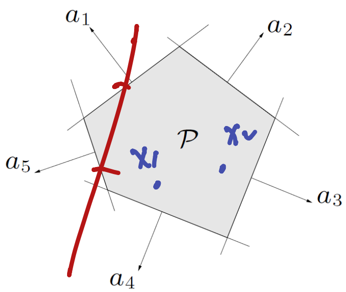
Example: Polyhedra

Solution set of finitely many linear inequalities and equalities

$$\boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b}, \qquad \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}$$

($\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $\preceq$ is componentwise inequality)

$$Ax = \begin{pmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_n^T x \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$



$a_i^T x \le b_i$

$i = 1, \dots, m$

polyhedron is intersection of finite number of halfspaces and hyperplanes

Examples: Euclidean Balls and Ellipsoids

$\Rightarrow \|ru\|_2 \le r \Rightarrow \|u\|_2 \le 1$

- **(Euclidean) Ball** with center $\boldsymbol{x}_c$ and radius r:

$$B(\boldsymbol{x}_c, r) = \{\boldsymbol{x} \,|\, \|\underbrace{\boldsymbol{x} - \boldsymbol{x}_c}_{=ru}\|_2 \le r\} = \{\boldsymbol{x}_c + r\boldsymbol{u} \,|\, \|\boldsymbol{u}\|_2 \le 1\}$$

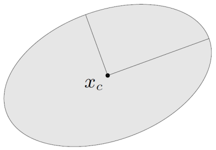
- **Ellipsoid:** set of the form

$$\begin{aligned} E(\boldsymbol{x}_c, \boldsymbol{P}) &= \{\boldsymbol{x} \,|\, (\boldsymbol{x} - \boldsymbol{x}_c)^T \boldsymbol{P}^{-1} (\boldsymbol{x} - \boldsymbol{x}_c) \le 1\} \\ &= \{\boldsymbol{x}_c + \boldsymbol{A}\boldsymbol{u} \,|\, \|\boldsymbol{u}\|_2 \le 1\} \end{aligned}$$

with $\boldsymbol{P} \in \mathbb{S}^n_{++}$ (i.e., $\boldsymbol{P}$ symmetric positive definite), $\boldsymbol{A}$ square and nonsingular

$$\frac{x_1^2}{r_1} + \frac{x_2^2}{r_2} = 1$$

$$P = \begin{pmatrix} r_1 & \\ & r_2 \end{pmatrix}$$



E

the length of the semi-axes of E

are given by $\sqrt{\lambda_i}$:

λ_i eigenvalues of φ .

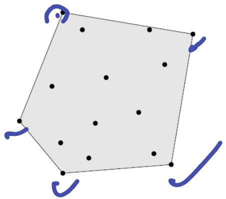
Convex Combination and Convex Hull

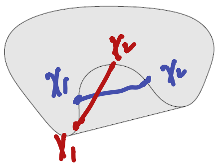
- **Convex combination** of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$: any point $\boldsymbol{x}$ of the form

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2 + \cdots + \theta_k \boldsymbol{x}_k$$

  with $\theta_1 + \cdots + \theta_k = 1, \theta_i \geq 0$

- **Convex hull** conv S: set of all convex combinations of points in S



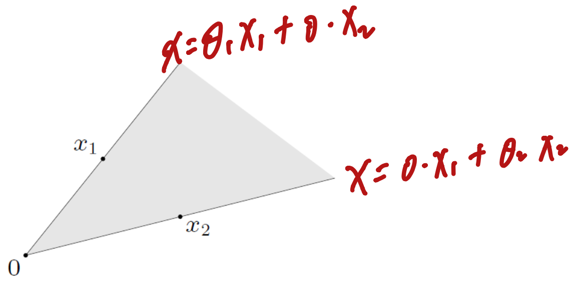


Conic Combination and Convex Cone

- **Conic (nonnegative) combination** of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$: any point of the form

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2$$

with $\theta_1 \geq 0, \theta_2 \geq 0$



- **Convex cone:** set that contains all conic combinations of points in the set

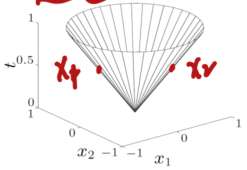
Convex Cones: Norm Balls and Norm Cones

- **Norm:** a function $\|\cdot\|$ that satisfies
  - $\|\boldsymbol{x}\| \geq 0$; $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \boldsymbol{0}$
  - $\|t\boldsymbol{x}\| = |t|\|\boldsymbol{x}\|$ for $t \in \mathbb{R}$
  - $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

  notation: $\|\cdot\|$ general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ a particular norm

- **Norm ball** with center $\boldsymbol{x}_c$ and radius r: $\{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_c\| \leq r\}$

- **Norm cone:** $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} \mid \|\boldsymbol{x}\| \leq t\}$



Euclidean norm cone or second-order cone (aka ice-cream cone)

Positive Semidefinite Cone

Notation

- $\mathbb{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{S}^n | \boldsymbol{X} \succeq 0\}$: positive semidefinite $n \times n$ matrices

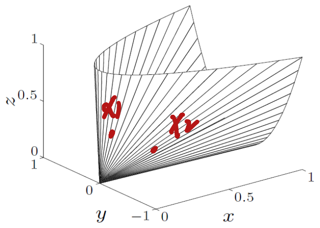
$$\boldsymbol{X} \in \mathbb{S}^n_+ \quad \Longleftrightarrow \quad \boldsymbol{z}^\top \boldsymbol{X} \boldsymbol{z} \geq 0 \text{ for all } \boldsymbol{z}$$

  $\mathbb{S}^n_+$ is a convex cone
- $\mathbb{S}^n_{++} = \{\boldsymbol{X} \in \mathbb{S}^n | \boldsymbol{X} \succ 0\}$: positive definite $n \times n$ matrices

Example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2_+$

$xz - y^2 \geq 0$

$x \geq 0, \ z \geq 0$



$M_1, \; M_2 \in S_+^n$

$M = \theta_1 M_1 + \theta_2 M_2 \in S_+^n, \quad \theta_1, \theta_2 \geq 0$

$$z^T M z = \theta_1 \underbrace{z^T M_1 z}_{\geq 0} + \theta_2 \underbrace{z^T M_2 z}_{\geq 0}$$

≥ 0

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Operations that Preserve Convexity

How to establish the convexity of a given set C

- Apply the definition (can be cumbersome)

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in C$$

- Show that C is obtained from simple convex sets(hyperplanes, halfspaces, norm balls,$\cdots$) by operations that preserve convexity
  - intersection
  - affine functions
  - perspective function
  - linear-fractional functions

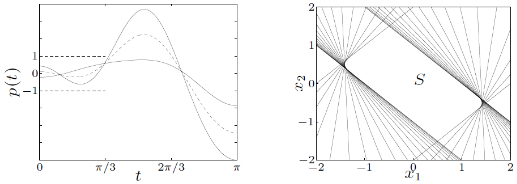
Intersection

$S :=$

- **Intersection:** if $S_1, S_2, \ldots, S_k$ are convex, then $S_1 \cap S_2 \cap \cdots \cap S_k$ is convex (k can be any positive integer)

- Example 1: a polyhedron is the intersection of halfspaces and hyperplanes

- Example 2:
$$S = \{\boldsymbol{x} \in \mathbb{R}^m \,|\, |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$
where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$



for $m = 2$

S_α is convex for any $\alpha \in A$

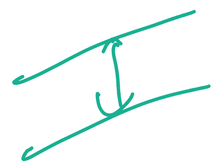
$\bigcap_{\alpha \in A} S_\alpha$ is convex

Examples: infinite number of $\underset{\text{convex}}{\underline{\text{slabs}}}$

given t,

$$S_t = \{ x \in \mathbb{R}^n \mid -1 \le \underbrace{(\cos t, \dots, \cos mt)}_{a(t)}^T x \le 1 \}$$

$$S = \bigcap_{|t| \le \frac{\pi}{3}} S_t \text{ . convex}$$



Affine Function

suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is affine ($f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{b} \in \mathbb{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(\boldsymbol{x}) | \boldsymbol{x} \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ a convex set under f is convex

$$C \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{\boldsymbol{x} \in \mathbb{R}^n | f(\boldsymbol{x}) \in C\} \text{ convex}$$

Examples

- scaling, translation, projection
- inverse solution set of linear matrix inequality $\{\boldsymbol{x} | x_1 \boldsymbol{A}_1 + \cdots + x_m \boldsymbol{A}_m \preceq \boldsymbol{B}\}$ (with $\boldsymbol{A}_i, \boldsymbol{B} \in \mathbb{S}^p$)

  inverse img.

  $A(x) :=$

  $f(x) = B - A(x) \succeq 0$

  PSD cone

- $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} | \|\boldsymbol{x}\| \le t\}$ is convex, so is

$$\{\boldsymbol{x} \in \mathbb{R}^n | \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\| \le \boldsymbol{c}^T\boldsymbol{x} + d\}$$

$f(x) = \begin{pmatrix} Ax+b \\ c^Tx+d \end{pmatrix} \begin{matrix} \to y \\ \to s \end{matrix}$

$\Rightarrow \|y\| \le s$, $f(x) \in$ second-order cone

Perspective and Linear-fractional Function I

(x_1, x_2, x_3)

$x_3 = 0$

- **Perspective function** $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$

$$P(\boldsymbol{x}, t) = \boldsymbol{x}/t, \quad \mathrm{dom}P = \{(\boldsymbol{x}, t) | t > 0\}$$

images and inverse images of convex sets under perspective are convex

$x_3 = -1$

$-\left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, 1\right)$

- **Linear-fractional function** $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(\boldsymbol{x}) = \frac{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^T\boldsymbol{x} + d}, \quad \mathrm{dom}f = \{\boldsymbol{x} | \boldsymbol{c}^T\boldsymbol{x} + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

$g(x) = \begin{pmatrix} A \\ c^T \end{pmatrix} x + \begin{pmatrix} b \\ d \end{pmatrix}$, affine function

$f(x) = P \circ g$

if C is convex, then

$P^{-1}(C) = \{(x,t) \mid \frac{x}{t} \in C,\ t>0\}$ is convex

proof: $(x,t) \in P^{-1}(C)$, $(y,s) \in P^{-1}(C)$, $0 \le \theta \le 1$

$\theta(x,t) + (1-\theta)(y,s) \in P^{-1}(C)$

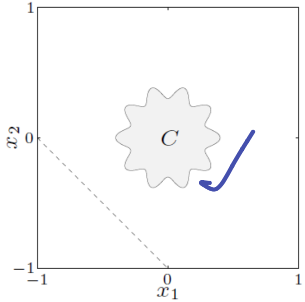
$\frac{\theta x + (1-\theta) y}{\theta t + (1-\theta) s} \in C \iff u\left(\frac{x}{t}\right) + (1-u)\left(\frac{y}{s}\right) \in C$

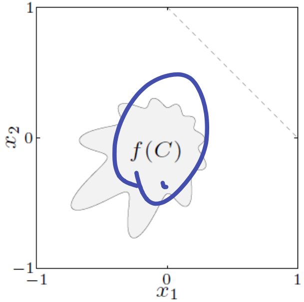
$u = \frac{\theta t}{\theta t + (1-\theta)s} \in [0,1]$

Perspective and Linear-fractional Function II

❧ **Examples** of a linear-fractional function

$$f(\boldsymbol{x}) = \frac{1}{x_1 + x_2 + 1}\boldsymbol{x}$$





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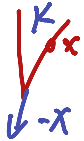
Generalized Inequalities I

- A convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if



  - K is closed (contains its boundary)
  - K is solid (has nonempty interior)
  - K is pointed (contains no line)

$x \in K,\ -x \in K \Rightarrow x = 0$

- **Examples**
  - nonnegative orthant

$$K = \mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n | x_i \geq 0, i = 1, \cdots, n\}$$

  - positive semidefinite cone

(PSD)

$$K = \mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{R}^{n\times n} | \boldsymbol{X} = \boldsymbol{X}^T \succeq \boldsymbol{0}\}$$

$\lambda_i \geq 0,\ i = 1, \cdots, n$

  - nonnegative polynomials on $[0, 1]$:

$$K = \{\boldsymbol{x} \in \mathbb{R}^n | x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

Generalized Inequalities II

Generalized inequality defined by a proper cone K:

$$\boldsymbol{y} \succeq_K \boldsymbol{x} \iff \boldsymbol{y} - \boldsymbol{x} \succeq_K \boldsymbol{0} \text{ or } \boldsymbol{y} - \boldsymbol{x} \in K$$

Examples

- Componentwise inequality ($K = \mathbb{R}^n_+$) $y_i - x_i \geq 0, \; i = 1, \ldots, n$

$$\boldsymbol{y} \succeq_{\mathbb{R}^n_+} \boldsymbol{x} \iff y_i \geq x_i, \quad i = 1, \cdots, n$$

- Matrix inequality ($K = \mathbb{S}^n_+$)

$$\boldsymbol{Y} \succeq_{\mathbb{S}^n_+} \boldsymbol{X} \iff \boldsymbol{Y} - \boldsymbol{X} \text{ positive semidefinite}$$

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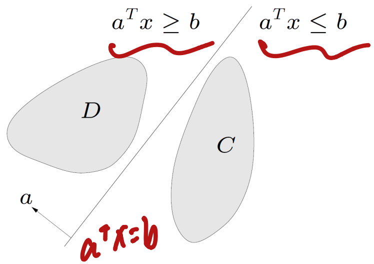
Separating Hyperplane Theorem

If C and D are nonempty disjoint convex sets, there exist $\boldsymbol{a} \neq \mathbf{0}$ and b, such that

$$\boldsymbol{a}^T\boldsymbol{x} \leq b \text{ for } \boldsymbol{x} \in C, \quad \boldsymbol{a}^T\boldsymbol{x} \geq b \text{ for } \boldsymbol{x} \in D$$



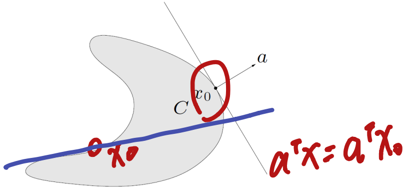
the hyperplane $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} = b\}$ separates C and D

Supporting Hyperplane Theorem

Supporting hyperplane to set C at boundary point $\boldsymbol{x}_0$:

$$\{\boldsymbol{x} | \boldsymbol{a}^T \boldsymbol{x} = \boldsymbol{a}^T \boldsymbol{x}_0\}$$

where $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{a}^T \boldsymbol{x} \leq \boldsymbol{a}^T \boldsymbol{x}_0$ for all $\boldsymbol{x} \in C$

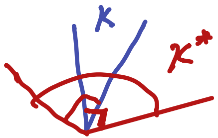


Supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Dual Cones and Generalized Inequalities

- **Dual cone** of a cone K:

$\langle y, x \rangle \geq$



$$K^* = \{\boldsymbol{y} | \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in K\}$$

- Examples
  - $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
  - $K = \mathbb{S}^n_+$: $K^* = \mathbb{S}^n_+$
  - $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$
  - $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_1 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_\infty \leq t\}$

  First three examples are **self-dual** cones

- Dual cones of proper cones are proper, hence define generalized inequalities:

$$\boldsymbol{y} \succeq_{K^*} \boldsymbol{0} \quad \Longleftrightarrow \quad \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \succeq_K \boldsymbol{0}$$

① nonnegative orthant

$y^T x \geq 0$, $\forall x \geq 0 \iff y \geq 0$

② positive semidefinite cone:

$\langle X, Y \rangle = \text{Trace}(XY^T) = \sum_{i,j=1}^{n} X_{ij} Y_{ij}$

The PSD cone is self-dual, i.e., for

$X, Y \in S^n$, $\underline{\text{Trace}(XY^T) \geq 0, \forall X \geq 0} \iff \underline{Y \geq 0}$

Proof: ① Suppose $\underline{Y \notin S_+^n}$, there exists $q \in R^n$

$q^T Y q = \langle q, Yq \rangle = \text{Trace}(qq^T Y^T) < 0$

$X := $ PSD matrix

Have the PSD matrix $X = qq^T$ satisfies

$\text{Trace}(XY) < 0$,

base on the definition of K^*

it follows that $Y \notin (S_+^n)^*$

② Now suppose $X, \underline{Y \in S_+^n}$

we can express X in terms of eigenvalue decomposition as

$X = \sum_{i=1}^{n} \lambda_i q_i q_i^T$, $\lambda_i \geq 0$, $i=1, \dots, n$

$\text{Trace}(YX) = \underline{\text{Trace}(Y \sum_{i=1}^{n} \lambda_i q_i q_i^T)}$

$= \underline{\sum_{i=1}^{n} \lambda_i q_i^T Y q_i} \geq 0$

this shows that $Y \in (S_+^n)^*$

(definition of K^*)

second-order cone is self-dual

$\underbrace{\text{second-order cone}}_{\mathcal{L}^d}$

proof: the dual cone of $\mathcal{L}^d$ is

$$C = \left\{ \begin{pmatrix} y \\ \delta \end{pmatrix} \;\middle|\; 0 \le \underbrace{y^T x + \delta t}_{\langle \begin{pmatrix} y \\ \delta \end{pmatrix}, \begin{pmatrix} x \\ t \end{pmatrix} \rangle \ge 0}, \ \forall \begin{pmatrix} x \\ t \end{pmatrix} \in \mathcal{L}^d \right\}$$

$= \mathcal{L}^d$? $\quad \|y\| \le t$?

inner product

$$= \left\{ \begin{pmatrix} y \\ \delta \end{pmatrix} \;\middle|\; 0 \le \inf_{t \ge 0} \inf_{\|x\| \le t} (y^T x + \delta t) \right\}$$

$$= \left\{ \begin{pmatrix} y \\ \delta \end{pmatrix} \;\middle|\; 0 \le \inf_{t \ge 0} \inf_{\|x\| \le t} (-\|y\|\|x\| + \delta t) \right\}$$

$$= \left\{ \begin{pmatrix} y \\ \delta \end{pmatrix} \;\middle|\; 0 \le \inf_{t \ge 0} (\delta - \|y\|) t \right\}$$

since $t \ge 0$, one has

$\begin{pmatrix} y \\ \delta \end{pmatrix} \in C$ iff $\|y\| \le \delta \Rightarrow C = \mathcal{L}^d$

second-order cone

Dual norm:

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The associated dual norm, $\|\cdot\|_*$, is defined as

$$\|z\|_* = \sup \{ z^T x \mid \|x\| \le 1 \}$$

fact: $\|x\|_{**} = \|x\|$, $\forall x$

Example 1: The dual of Euclidean norm is the Euclidean norm, i.e.,

$$\sup\{z^T x \mid \|x\|_2 \le 1\} = \|z\|_2$$

This follows from Cauchy-Schwarz inequality

for nonzero z, the value of x that (given z)

$$\left.\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{maximize}} & z^T x \\ \text{subject to} & \|x\|_2 \le 1 \end{array}\right\} \Rightarrow x^* = \frac{z}{\|z\|_2}$$

optimal objective value: $z^T x^* = \|z\|_2$

Example 2: The dual of the l_∞-norm is the l_1-norm

$$\sup\{z^T x \mid \underbrace{\|x\|_\infty}_{\max_i |x_i|} \le 1\} = \sum_{i=1}^n |z_i| = \|z\|_1$$

$$\left.\begin{array}{ll} \text{maximize} & z^T x \\ \text{subject to} & \|x\|_\infty \le 1 \end{array}\right\} \Rightarrow x_i^* = \frac{z_i}{|z_i|}$$

General results: the dual of the l_p-norm is the l_q-norm, q satisfies

$$\frac{1}{p} + \frac{1}{q} = 1$$

Reference

Chapter 2 of:

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.