Constructive Convex Analysis and Disciplined Convex Programming

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Outline

Convex Optimization

Constructive Convex Analysis

Disciplined Convex Programming

Modeling Frameworks

Conclusions

Outline

Convex optimization problem — standard form

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

with variable $x \in \mathbf{R}^n$

- objective and inequality constraints $f_0, \ldots, f_m$ are convex

  for all x, y, $\theta \in [0, 1]$,

  $$f_i(\theta x + (1 - \theta)y) \leq \theta f_i(x) + (1 - \theta) f_i(y)$$

  i.e., graphs of f_i curve upward
- equality constraints are linear

Convex optimization problem — conic form

cone program:

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b, \quad x \in \mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_n
\end{array}
$$

with variable $x \in \mathbf{R}^n$

- linear objective, equality constraints; $\mathcal{K}$ is convex cone
- special cases:
  - linear program (LP)
  - semidefinite program (SDP)

- the modern canonical form
- *there are well developed solvers for cone programs*

Why convex optimization?

- beautiful, fairly complete, and useful theory
- solution algorithms that work well in theory and practice
  - convex optimization is **actionable**
- **many applications** in
  - control
  - combinatorial optimization
  - signal and image processing
  - communications, networks
  - circuit design
  - machine learning, statistics
  - finance

. . . and many more

How do you solve a convex problem?

- use an existing custom solver for your specific problem
- develop a new solver for your problem using a currently fashionable method
  - requires work
  - but (with luck) will scale to large problems

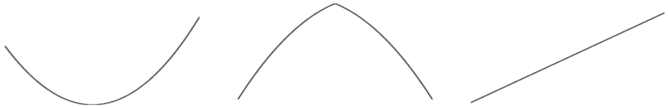
step 1: CVX

- transform your problem into a cone program, and use a standard cone program solver
  - can be *automated* using *domain specific languages*

step 2: SDPT3, MOSEK, Sedum

Outline

Curvature: Convex, concave, and affine functions



- f is *concave* if $-f$ is convex, *i.e.*, for any x, y, $\theta \in [0,1]$,

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y)$$

- f is *affine* if it is convex and concave, *i.e.*,

$$f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta)f(y)$$

  for any x, y, $\theta \in [0,1]$

- f is affine $\iff$ it has form $f(x) = a^T x + b$

Verifying a function is convex or concave

(verifying affine is easy)

approaches:

- via basic definition (inequality)
- via first or second order conditions, *e.g.*, $\nabla^2 f(x) \succeq 0$
- via convex calculus: construct f using
  - library of basic functions that are convex or concave
  - calculus rules or transformations that preserve convexity

Convex functions: Basic examples

- x^p ($p \geq 1$ or $p \leq 0$), *e.g.*, x^2, $1/x$ ($x > 0$)
- e^x
- $x \log x$
- $a^T x + b$
- $x^T P x$ ($P \succeq 0$)
- $\|x\|$ (any norm)
- $\max(x_1, \ldots, x_n)$

Concave functions: Basic examples

- x^p $(0 \leq p \leq 1)$, *e.g.*, $\sqrt{x}$
- $\log x$
- $\sqrt{xy}$
- $x^T P x$ $(P \preceq 0)$
- $\min(x_1, \ldots, x_n)$

Convex functions: Less basic examples

- x^2/y $(y > 0)$, $x^T x/y$ $(y > 0)$, $x^T Y^{-1} x$ $(Y \succ 0)$
- $\log(e^{x_1} + \cdots + e^{x_n})$
- $f(x) = x_{[1]} + \cdots + x_{[k]}$ (sum of largest k entries)
- $f(x, y) = x \log(x/y)$ $(x, y > 0)$
- $\lambda_{\max}(X)$ $(X = X^T)$

Concave functions: Less basic examples

- $\log\det X$, $(\det X)^{1/n}$ $(X \succ 0)$
- $\log \Phi(x)$ (Φ is Gaussian CDF)
- $\lambda_{\min}(X)$ $(X = X^T)$

Calculus rules

- **nonnegative scaling**: f convex, $\alpha \geq 0 \implies \alpha f$ convex
- **sum**: f, g convex $\implies f + g$ convex
- **affine composition**: f convex $\implies f(Ax + b)$ convex
- **pointwise maximum**: $f_1, \ldots, f_m$ convex $\implies \max_i f_i(x)$ convex
- **composition**: h convex increasing, f convex $\implies h(f(x))$ convex

...and similar rules for concave functions

(there are other more advanced rules)

Examples

from basic functions and calculus rules, we can show convexity of . . .

- piecewise-linear function: $\max_{i=1,\ldots,k}(a_i^T x + b_i)$ Lasso
- ℓ_1-regularized least-squares cost: $\|Ax - b\|_2^2 + \lambda\|x\|_1$, with $\lambda \geq 0$
- sum of largest k elements of x: $x_{[1]} + \cdots + x_{[k]}$
- log-barrier: $-\sum_{i=1}^m \log(-f_i(x))$ (on $\{x \mid f_i(x) < 0\}$, f_i convex)
- KL divergence: $D(u, v) = \sum_i (u_i \log(u_i/v_i) - u_i + v_i)$ $(u, v > 0)$

A general composition rule

$h(f_1(x), \ldots, f_k(x))$ is convex when h is convex and for each i

- h is increasing in argument i, and f_i is convex, or
- h is decreasing in argument i, and f_i is concave, or
- f_i is affine

- there's a similar rule for concave compositions (just swap convex and concave above)
- this one rule subsumes all of the others
- *this is pretty much the only rule you need to know*

Example

let's show that

$$f(u, v) = (u + 1) \log((u + 1)/ \min(u, v))$$

is convex

- u, v are variables with $u, v > 0$
- $u + 1$ is affine; $\min(u, v)$ is concave
- since $x \log(x/y)$ is convex in (x, y), decreasing in y,

$$f(u, v) = (u + 1) \log((u + 1)/ \min(u, v))$$

  is convex

Example

- $\log(e^{u_1} + \cdots + e^{u_k})$ is convex, increasing
- so if $f(x,\omega)$ is convex in x for each ω and $\gamma > 0$,

$$\log\left(\left(e^{\gamma f(x,\omega_1)} + \cdots + e^{\gamma f(x,\omega_k)}\right)/k\right)$$

  is convex
- this is $\log \mathbf{E}\, e^{\gamma f(x,\omega)}$, where $\omega \sim \mathcal{U}(\{\omega_1,\ldots,\omega_k\})$
- arises in stochastic optimization via bound

$$\log \mathbf{Prob}(f(x,\omega) \geq 0) \;\leq\; \log \mathbf{E}\, e^{\gamma f(x,\omega)}$$

Constructive convexity verification

- start with function given as **expression**
- build parse tree for expression
  - leaves are variables or constants/parameters
  - nodes are functions of children, following general rule
- tag each subexpression as convex, concave, affine, constant
  - variation: tag subexpression signs, use for monotonicity
    e.g., $(\cdot)^2$ is increasing if its argument is nonnegative
- sufficient (but not necessary) for convexity

Example

for $x < 1$, $y < 1$

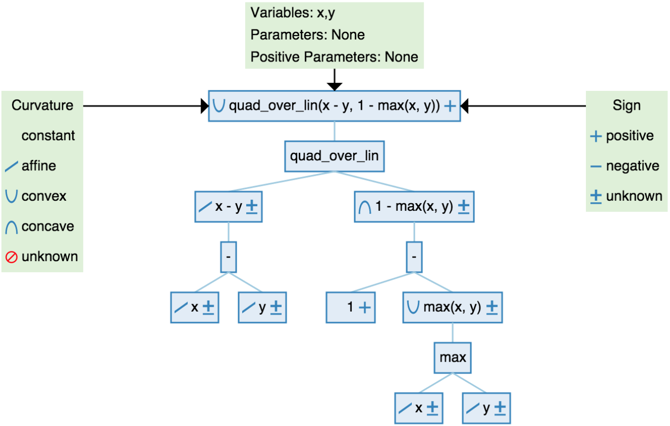
$$\frac{(x-y)^2}{1-\max(x,y)}$$

is convex

- (leaves) x, y, and 1 are affine expressions
- $\max(x,y)$ is convex; $x-y$ is affine
- $1-\max(x,y)$ is concave
- function u^2/v is convex, monotone decreasing in v for $v > 0$
  hence, convex with $u = x-y$, $v = 1-\max(x,y)$

Example

analyzed by `dcp.stanford.edu` (*Diamond 2014*)



Example

- $f(x) = \sqrt{1 + x^2}$ is convex
- but cannot show this using constructive convex analysis
  - (leaves) 1 is constant, x is affine
  - x^2 is convex
  - $1 + x^2$ is convex
  - but $\sqrt{1 + x^2}$ **doesn't match general rule**
- writing $f(x) = \|(1, x)\|_2$, however, works
  - $(1, x)$ is affine
  - $\|(1, x)\|_2$ is convex

Outline

Disciplined convex programming (DCP)

(*Grant, Boyd, Ye, 2006*)

- framework for describing convex optimization problems
- based on constructive convex analysis
- sufficient but not necessary for convexity
- basis for several domain specific languages and tools for convex optimization

Disciplined convex program: Structure

a DCP has

- zero or one **objective**, with form
  - minimize {scalar convex expression} or
  - maximize {scalar concave expression}
- zero or more **constraints**, with form
  - {convex expression} <= {concave expression} or
  - {concave expression} >= {convex expression} or
  - {affine expression} == {affine expression}

Disciplined convex program: Expressions

- expressions formed from
  - **variables**,
  - **constants/parameters**,
  - and **functions** from a library
- library functions have known convexity, monotonicity, and sign properties
- all subexpressions match general composition rule

Disciplined convex program

- a valid DCP is
  - convex-by-construction (cf. posterior convexity analysis)
  - ‘syntactically’ convex (can be checked ‘locally’)

- convexity depends only on *attributes* of library functions, and not their meanings
  - *e.g.*, could swap $\sqrt{\cdot}$ and $\sqrt[4]{\cdot}$, or $\exp \cdot$ and $(\cdot)_+$, since their attributes match

Canonicalization

- easy to build a DCP parser/analyzer
- not much harder to implement a *canonicalizer*, which transforms DCP to equivalent cone program
- then solve the cone program using a generic solver

- yields a *modeling framework* for convex optimization

Outline

Optimization modeling languages

- domain specific language (DSL) for optimization
- express optimization problem in high level language
  - declare variables
  - form constraints and objective
  - solve
- long history: AMPL, GAMS, ...
  - no special support for convex problems
  - very limited syntax
  - callable from, but not embedded in other languages

Modeling languages for convex optimization

all based on DCP

YALMIP	Matlab	Löfberg	2004
CVX	Matlab	Grant, Boyd	2005
CVXPY	Python	Diamond, Boyd	2013
Convex.jl	Julia	Udell et al.	2014

some precursors

- SDPSOL (*Wu, Boyd, 2000*)
- LMITOOL (*El Ghaoui et al., 1995*)

CVX

minimize $\|Ax-b\|^2 + \lambda \|x\|_1$

subject to $\|x\|_\infty \le 1$

```
cvx_begin
    variable x(n)    % declare vector variable
    minimize sum(square(A*x-b)) + gamma*norm(x,1)
    subject to norm(x,inf) <= 1
cvx_end
```

solver SDPT3, mosek

- A, b, gamma are constants (gamma nonnegative)
- variables, expressions, constraints exist inside problem
- after cvx_end
  - problem is canonicalized to cone program
  - then solved

Some functions in the CVX library

function	meaning	attributes
`norm(x, p)`	$\|x\|_p$, $p \geq 1$	cvx
`square(x)`	x^2	cvx
`pos(x)`	x_+	cvx, nondecr
`sum_largest(x,k)`	$x_{[1]} + \cdots + x_{[k]}$	cvx, nondecr
`sqrt(x)`	$\sqrt{x}$, $x \geq 0$	ccv, nondecr
`inv_pos(x)`	$1/x$, $x > 0$	cvx, nonincr
`max(x)`	$\max\{x_1, \ldots, x_n\}$	cvx, nondecr
`quad_over_lin(x,y)`	x^2/y, $y > 0$	cvx, nonincr in y
`lambda_max(X)`	$\lambda_{\max}(X)$, $X = X^T$	cvx

DCP analysis in CVX

```
cvx_begin
    variables x y
    square(x+1) <= sqrt(y) % accepted
    max(x,y) == 1 % not DCP
    ...

Disciplined convex programming error:
   Invalid constraint: {convex} == {real constant}
```

CVXPY

```
from cvxpy import *
x = Variable(n)
cost = sum_squares(A*x-b) + gamma*norm(x,1)
prob = Problem(Minimize(cost),
               [norm(x,"inf") <= 1])
opt_val = prob.solve()
solution = x.value
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- variables, expressions, constraints exist outside of problem
- `solve` method canonicalizes, solves, assigns `value` attributes

Signed DCP in CVXPY

function	meaning	attributes
`abs(x)`	$\lvert x\rvert$	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$
`huber(x)`	$\begin{cases} x^2, & \lvert x\rvert \leq 1 \\ 2\lvert x\rvert - 1, & \lvert x\rvert > 1 \end{cases}$	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$
`norm(x, p)`	$\lVert x\rVert_p$, $p \geq 1$	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$
`square(x)`	x^2	cvx, nondecr for $x \geq 0$, nonincr for $x \leq 0$

DCP analysis in CVXPY

$$\text{expr} = \frac{(x-y)^2}{1-\max(x,y)}$$

```
x = Variable()
y = Variable()
expr = quad_over_lin(x - y, 1 - max_elemwise(x,y))
expr.curvature # CONVEX
expr.sign # POSITIVE
expr.is_dcp() # True
```

Parameters in CVXPY

- symbolic representations of constants
- can specify sign
- change value of constant without re-parsing problem

- for-loop style trade-off curve:

```
x_values = []
for val in numpy.logspace(-4, 2, 100):
    gamma.value = val
    prob.solve()
    x_values.append(x.value)
```

Parallel style trade-off curve

```
# Use tools for parallelism in standard library.
from multiprocessing import Pool

# Function maps gamma value to optimal x.
def get_x(gamma_value):
    gamma.value = gamma_value
    result = prob.solve()
    return x.value

# Parallel computation with N processes.
pool = Pool(processes = N)
x_values = pool.map(get_x, numpy.logspace(-4, 2, 100))
```

Convex.jl

```
using Convex
x = Variable(n);
cost = sum_squares(A*x-b) + gamma*norm(x,1);
prob = minimize(cost, [norm(x,Inf) <= 1]);
opt_val = solve!(prob);
solution = x.value;
```

- `A`, `b`, `gamma` are constants (`gamma` nonnegative)
- similar structure to CVXPY
- `solve!` method canonicalizes, solves, assigns `value` attributes

Outline

Conclusions

- DCP is a formalization of constructive convex analysis
  - simple method to certify problem as convex (sufficient, but not necessary)
  - basis of several DSLs/modeling frameworks for convex optimization

- modeling frameworks make rapid prototyping of convex optimization based methods easy

References

- *Disciplined Convex Programming* (Grant, Boyd, Ye)
- *Graph Implementations for Nonsmooth Convex Programs* (Grant, Boyd)
- *Matrix-Free Convex Optimization Modeling* (Diamond, Boyd)

- CVX: `http://cvxr.com/`
- CVXPY: `http://www.cvxpy.org/`
- Convex.jl: `http://convexjl.readthedocs.org/`