HOMEWORK A
PROPOSITIONAL LOGIC, PROOF TECHNIQUES[1]

CMU 10-607: COMPUTATIONAL FUNDAMENTALS FOR MACHINE LEARNING (FALL 2018)

https://piazza.com/cmu/fall2018/10606607

OUT: Tue, Oct 30, 2018

DUE: Wed, Nov 7, 2018, 11:59pm

START HERE: Instructions

- **Collaboration policy:** Collaboration on solving the homework is allowed, after you have thought about the problems on your own. It is also OK to get clarification (but not solutions) from books or online resources, again after you have thought about the problems on your own. There are two requirements: first, cite your collaborators fully and completely (e.g., "Jane explained to me what is asked in Question 2.1"). Second, write your solution *independently*: close the book and all of your notes, and send collaborators out of the room, so that the solution comes from you only. See the Academic Integrity Section on the course site for more information: http://www.cs.cmu.edu/~mgormley/courses/606-607-f18/about.html#7-academic-integrity-policies
- **Late Submission Policy:** See the late submission policy here: http://www.cs.cmu.edu/~mgormley/courses/606-607-f18/about.html#6-general-policies
- **Submitting your work to Gradescope:** For written problems such as short answer, multiple choice, derivations, proofs, or plots, we will be using Gradescope (https://gradescope.com/). Please use the provided template. Submissions can be handwritten onto the template, but should be labeled and clearly legible. If your writing is not legible, you will not be awarded marks. Alternatively, submissions can be written in LaTeX. Regrade requests can be made, however this gives the TA the opportunity to regrade your entire paper, meaning if additional mistakes are found then points will be deducted. For short answer questions you **should not** include your work in your solution. If you include your work in your solutions, your assignment may not be graded correctly by our AI assisted grader.

For multiple choice or select all that apply questions, shade in the box or circle in the template document corresponding to the correct answer(s) for each of the questions. For LaTeX users, use ■ and ● for shaded boxes and circles, and don't change anything else.

[1] Compiled on Wednesday 31st October, 2018 at 13:23

Instructions for Specific Problem Types

For "Select One" questions, please fill in the appropriate bubble completely:

Select One: Who taught this course?

- ● Matt Gormley
- ○ Marie Curie
- ○ Noam Chomsky

If you need to change your answer, you may cross out the previous answer and bubble in the new answer:

Select One: Who taught this course?

- ● Matt Gormley
- ○ Marie Curie
- ⊗ Noam Chomsky

For "Select all that apply" questions, please fill in all appropriate squares completely:

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- □ I don't know

Again, if you need to change your answer, you may cross out the previous answer(s) and bubble in the new answer(s):

Select all that apply: Which are scientists?

- ■ Stephen Hawking
- ■ Albert Einstein
- ■ Isaac Newton
- ⊠ I don't know

For questions where you must fill in a blank, please make sure your final answer is fully included in the given space. You may cross out answers or parts of answers, but the final answer must still be within the given space.

Fill in the blank: What is the course number?

10-606	10-6067

1 Proof Techniques [29 pts]

1. **[5 pts]** For any two finite sets A and B, prove that $\neg(A \cup B) = \neg A \cap \neg B$

① Let $P = \neg(A \cup B)$ and $Q = \neg A \cap \neg B$ [Assumption]

② Let x be an arbitrary element in P, then $x \in P \equiv x \in \neg(A \cup B)$ [Assumption]

③ $x \notin (A \cup B)$ [From ②]

④ $x \notin A$ and $x \notin B$ [From ③]

⑤ $x \in \neg A$ and $x \in \neg B$ [from ④]

⑥ $x \in \neg A \cap \neg B$ [from ⑤]

⑦ $x \in Q$ [from ⑥]

⑧ $P \subset Q$ [From ① ⑥]

⑨ Let y be an arbitrary element in Q, then $y \in Q \equiv y \in \neg A \cap \neg B$ [Assumption]

⑩ $y \notin A$ and $y \notin B$ [From ⑨]

⑪ $y \notin A \cup B$ [From ⑩]

⑫ $y \in \neg(A \cup B)$ [From ⑪]

⑬ $y \in P$ [From ⑫]

⑭ $Q \subset P$ [From ⑬]

⑮ $Q = P$, which is $\neg(A \cup B) = \neg A \cap \neg B$ [From ⑧ ⑬]

2. **[5 pts]** Prove there is no least positive real number.

① Assume there is a smallest positive real number, such that $x \in \mathbb{R}^+$ and for $\forall n \in \mathbb{R}^+$, $n \geq x$. [Assumption]

② $x = p/q$, $p, q \in \mathbb{Z}^+$ [from ①]

③ $\frac{x}{2} = p/2q$, $p, 2q \in \mathbb{Z}^+$ [from ②]

④ $\frac{x}{2} < n$ [from ③]

⑤ $\exists n \in \mathbb{R}^+, n \leq x$ [from ④]

⑥ ⑤ and ① contradict. So ① assumption must be wrong, so there is no least positive real number.

3. **[5 pts]** Let $n \in Z$. Prove that n^2 leaves a remainder of 0 or 1 when divided by 3. (Hint : Proof by cases. There are three possibilities for n. n = 3k, n = 3k +1, n= 3k +2)

① Let $n \in \mathbb{Z}$ [assumption]

② $\exists k \in \mathbb{Z}$, for $\forall n \in \mathbb{Z}$, $n=3k$ or $n=3k+1$ or $n=3k+2$ [from ①]

③ If $n=3k$, $n^2=9k^2$, $n^2 \bmod 3 = 0$ [from ②]

④ If $n=3k+1$, $n^2=9k^2+6k+1$, $n^2 \bmod 3 = 1$ [from ②]

⑤ If $n=3k+2$, $n^2=9k^2+12k+4$, $n^2 \bmod 4 = 1$ [from ②]

⑥ $n^2 \bmod 3 = 0$ or $n^2 \bmod 3 = 1$ [from ③④⑤]

4. **[5 pts]** Let r be a rational number and let a be an irrational number. Please provide a 2-column proof showing $r + a$ is irrational. (Hint : Try to prove by contradiction. To prove by contradiction, you start by assuming $r + a$ is rational and show why that cannot be true.)

① Assume r is rational, a is irrational
$r = \frac{p}{q}$; $p, q \in \mathbb{Z}$ a cannot be represented as ratio of integers
Then, $r+a$ is rational
$r + a = \frac{m}{n}$, where $m, n \in \mathbb{Z}$ [Assumption]

② $r + a = \frac{p}{q} + a = \frac{m}{n}$ [From ①]

③ $a = \frac{m}{n} - \frac{p}{q} = \frac{pn - qm}{qn}$ [From ②]

④ $(pn - qm) \in \mathbb{Z}$, $qn \in \mathbb{Z}$ [From ①]

⑤ $a \in \mathbb{R}$ [From ④]

→ Contradiction between ① ⑤
So, $r+a$ is irrational.

5. **[5 pts]** Prove by contraposition: Let $x \in Z$. If $x^2 - 6x + 5$ is even, then x is odd. (Hint : To prove by contraposition, assume x is even. For propositions p and q, if $\neg q \implies \neg p, p \implies q$.)

① Assume x is even, which is $x = 2k$, $k \in \mathbb{Z}$ [Assumption]

② $x^2 - 6x + 5 = 4k^2 - 12k + 5$, which is odd [From ①]

③ If $x^2 - 6x + 5$ is even, then x is odd [From ① → ②]

6. **[4 pts]** Suppose we have a binary classifier which predicts $y = 1$ if $f(x) > \theta$ and $y = 0$ if $f(x) <= \theta$, where $x \in \mathbf{X}^{n \times d}$ is a sample from your data and f is a function in $R^{1 \times d} \rightarrow R$. Prove that recall increases monotonically as we decrease θ. (Remember: recall is the proportion of truly positive ($y' = 1$) examples you also predicted as positive).

① $\exists x_1, x_1 \in A = \{x_1 \mid f(x_1) > \theta \wedge y_1' = 1\}$ [Assumption]

② Let $\theta' = \theta - \epsilon$, $\epsilon > 0$ [Assumption]

③ $\exists x_2, x_2 \in B = \{x_2 \mid f(x_2) > \theta' \wedge y_2' = 1\}$ [Assumption②]

④ $B = \{x_2 \mid f(x_2) > (\theta - \epsilon) \wedge y_2' = 1\}$ [From ③]

⑤ $\forall x_1 \in A \rightarrow x_1 \in B$ [From ①③④]

⑥ $A \subset B$ [From ⑤]

⑦ recall $= \frac{TP}{TP+FN}$ [Assumption]

⑧ $TP_\theta \leq TP_{\theta-\epsilon}$, $(TP+FN)$ does not change according to θ [From ⑥⑦]

⑨ $recall_\theta \leq recall_{\theta-\epsilon}$ [From ⑦⑧]

⑩ recall increases monotonically as we decrease θ [From ⑨]

2 Propositional Logic [16 pts]

1. **[6 pts]** The following questions describe two supporting statements (1. and 2.), and a conclusion (3.). Please indicate whether the conclusion can be reasonably inferred to be true from the supporting statements, using the rules of propositional logic.

   (a) **[1 pts]** 1. If pigs fly, then hell has frozen over. .
   2. Pigs fly.
   3. Therefore, hell has frozen over.

   ● Can be Inferred.

   ○ Cannot be Inferred.

   (b) **[1 pts]** 1. You took 10-606, or 10-607.
   2. It is not the case that you took 10-606.
   3. Therefore, you took 10-607.

   ● Can be Inferred.

   ○ Cannot be Inferred.

   (c) **[1 pts]** 1. If Elle Woods goes to Harvard, then Elle Woods passes the bar exam.
   2. It is not the case that Elle Woods passes the bar exam.
   3. Therefore, it is not the case that Elle Woods went to Harvard

   ● Can be Inferred.

   ○ Cannot be Inferred.

   (d) **[1 pts]** 1. All smurfs are snorks.
   2. All ewoks are snorks.
   3. Therefore, all smurfs are ewoks

   ○ Can be Inferred.

   ● Cannot be Inferred.

   (e) **[1 pts]** 1. If you are a deep learning researcher, you are a machine learning researcher.
   2. You are a machine learning researcher.
   3. Therefore, you are a deep learning researcher.

   ○ Can be Inferred.

   ● Cannot be Inferred.

   (f) **[1 pts]** 1. If you are a machine learning researcher, you know statistics, and if you are a deep learning researcher, you know statistics.
   2. You are a deep learning researcher or a machine learning researcher.
   3. Therefore, you know statistics.

   ● Can be Inferred.

   ○ Cannot be Inferred.

2. **[5 pts]** Select which conditional statements are true.

   - [x] If 1 + 1 = 3, then unicorns exist.
   - [x] If 1 + 1 = 3, then dogs can fly.
   - [ ] If 1 + 1 = 2, then dogs can fly.
   - [x] If 2 + 2 = 4, then 1 + 2 = 3.
   - [x] If 2 + 1 = 4, then 1 + 2 = 3.

3. **[5 pts]** Select which conditional statements are always true.

   - [x] $[(p \vee q) \wedge (p \implies r) \wedge (q \implies r)] \implies r$
   - [ ] $p \wedge \neg p$
   - [x] $p \implies q \Leftrightarrow \neg p \vee q$
   - [x] $\neg(p \implies q) \Leftrightarrow p \wedge \neg q$.
   - [x] $[(p \implies q) \wedge \neg q] \implies \neg p$

3 Collaboration Policy

After you have completed all other components of this assignment, report your answers to the collaboration policy questions detailed in the Academic Integrity Policies found here.

1. Did you receive any help whatsoever from anyone in solving this assignment? If so, include full details including names of people who helped you and the exact nature of help you received.

Receive help from Zixuan on question 1.6

2. Did you give any help whatsoever to anyone in solving this assignment? If so, include full details including names of people you helped and the exact nature of help you offered.

3. Did you find or come across code that implements any part of this assignment? If so, include full details including the source of the code and how you used it in the assignment.