## 内容简介

一、分别编写用复化 Simpson 积分公式和复化梯形积分公式计算积分的通用程序。
二、用如上程序计算积分

$$I(f) = \int_0^4 \sin(x)dx \tag{1}$$

取节点 $x_i$，$i = 0, \cdots, N$，$N$ 为 $2^k$，$k = 1, \cdots, 12$，并分析误差。
利用公式计算算法的收敛阶：

$$Ord = \frac{\ln(Error_{old}/Error_{now})}{ln(N_{now}/N_{old})} \tag{2}$$

## 输出结果

```
Integrate[sin(x), 0, 4] = 1.653643620864

Composite Trapezoid Method:
I(k = 1) = 1.061792358343   Error = 0.591851262520   Order = 0.0000
I(k = 2) = 1.513487172039   Error = 0.140156448824   Order = 2.0782
I(k = 3) = 1.619048306831   Error = 0.034595314033   Order = 2.0184
I(k = 4) = 1.645021908709   Error = 0.008621712154   Order = 2.0045
I(k = 5) = 1.651489878133   Error = 0.002153742731   Order = 2.0011
I(k = 6) = 1.653105290366   Error = 0.000538330498   Order = 2.0003
I(k = 7) = 1.653509044811   Error = 0.000134576053   Order = 2.0001
I(k = 8) = 1.653609977261   Error = 0.000033643602   Order = 2.0000
I(k = 9) = 1.653635209989   Error = 0.000008410875   Order = 2.0000
I(k = 10) = 1.653641518146   Error = 0.000002102717   Order = 2.0000
I(k = 11) = 1.653643095184   Error = 0.000000525679   Order = 2.0000
I(k = 12) = 1.653643489444   Error = 0.000000131420   Order = 2.0000
Composite Simpson Method:
I(k = 1) = 1.920258141330   Error = 0.266614520466   Order = 0.0000
I(k = 2) = 1.664052109938   Error = 0.010408489075   Order = 4.6789
I(k = 3) = 1.654235351762   Error = 0.000591730898   Order = 4.1367
I(k = 4) = 1.653679776002   Error = 0.000036155138   Order = 4.0327
I(k = 5) = 1.653645867940   Error = 0.000002247077   Order = 4.0081
I(k = 6) = 1.653643761110   Error = 0.000000140246   Order = 4.0020
I(k = 7) = 1.653643629626   Error = 0.000000008762   Order = 4.0005
I(k = 8) = 1.653643621411   Error = 0.000000000548   Order = 4.0001
I(k = 9) = 1.653643620898   Error = 0.000000000034   Order = 4.0000
I(k = 10) = 1.653643620866   Error = 0.000000000002   Order = 3.9997
I(k = 11) = 1.653643620864   Error = 0.000000000000   Order = 4.0053
I(k = 12) = 1.653643620864   Error = 0.000000000000   Order = 3.6742
```

| n | Composite Trapezoid Error | order | Composite Simpson Error | order |
|---|---|---|---|---|
| 1 | 5.9185E-01 | – | 2.6661E-01 | – |
| 2 | 1.4015E-01 | 2.0782 | 1.0408E-02 | 4.6789 |
| 3 | 3.4595E-02 | 2.0184 | 5.9173E-04 | 4.1367 |
| 4 | 8.6217E-03 | 2.0045 | 3.6155E-05 | 4.0327 |
| 5 | 2.1537E-03 | 2.0011 | 2.2470E-06 | 4.0081 |
| 6 | 5.3833E-04 | 2.0003 | 1.4024E-07 | 4.0020 |
| 7 | 1.3457E-04 | 2.0001 | 8.7623E-09 | 4.0005 |
| 8 | 3.3643E-05 | 2.0000 | 5.4759E-10 | 4.0001 |
| 9 | 8.4108E-06 | 2.0000 | 3.4224E-11 | 4.0000 |
| 10 | 2.1027E-06 | 2.0000 | 2.1394E-12 | 3.9997 |
| 11 | 5.2567E-07 | 2.0000 | 1.3322E-13 | 4.0053 |
| 12 | 1.3141E-07 | 2.0000 | 1.0436E-14 | 3.6742 |

表 1: $L_\infty$ 范数意义下的精度检验

## 误差分析

一、本实验中使用的复化梯形公式为：

$$\int_a^b f(x)dx \approx \frac{h}{2}\left(f(a)+2\sum_{i=1}^{2^k-1} f(a+ih)+f(b)\right) \tag{3}$$

其中 $h=\dfrac{b-a}{2^k}$，记结点为 $x_0,\cdots,x_{2^k}$。下面对其作简单推导并分析该公式的误差。对子区间 $[x_i,\,x_{i+1}]=[a+ih,\,a+(i+1)h]$，用 Lagrange 插值法逼近 $f$ 得：

$$f(x)=p(x)+\frac{1}{2!}f''(\xi_i)(x-x_i)(x-x_{i+1}) \tag{4}$$

其中

$$\begin{aligned} p(x) &= f(x_i)\frac{x_{i+1}-x}{x_{i+1}-x_i}+f(x_{i+1})\frac{x-x_i}{x_{i+1}-x_i} \qquad (5)\\ &= \frac{1}{h}\left(f(x_i)(x_{i+1}-x)+f(x_{i+1})(x-x_i)\right) \qquad (6)\end{aligned}$$

故

$$\begin{aligned}\int_{x_i}^{x_{i+1}} f(x)dx &= \int_{x_i}^{x_{i+1}} p(x)dx+\frac{1}{2}\int_{x_i}^{x_{i+1}} f''(\xi_i)(x-x_i)(x-x_{i+1})dx\\ &= \frac{h}{2}\Big(f(x_i)+f(x_{i+1})\Big)+\frac{1}{12}f''(\xi_i)h^3\end{aligned}$$

对等号两端求和：

$$\begin{aligned}\int_a^b f(x)dx &= \sum_{i=0}^{2^k-1}\int_{x_i}^{x_{i+1}} f(x)dx\\ &= \frac{h}{2}\sum_{i=0}^{2^k-1}\left(f(x_i)+f(x_{i+1})+\frac{1}{12}f''(\xi_i)h^3\right)\\ &= \frac{h}{2}\left(f(a)+2\sum_{i=1}^{2^k-1}f(a+ih)+f(b)\right)+\frac{h^3}{12}\sum_{i=0}^{2^k-1}f''(\xi)\end{aligned}$$

设 $f''(x)$ 连续。那么在 $(a,\, b)$ 中存在一点 $\xi$ 使得

$$f''(\xi) = \frac{1}{n}\sum_{i=0}^{n-1} f''(\xi_i) \tag{7}$$

其中 $\xi_i \in (x_i, x_{i+1})$，$n = \dfrac{b-a}{h} = 2^k$。那么：

$$\int_a^b f(x)dx = \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{2^k-1} f(a+ih) + f(b)\right) + \frac{h^2}{12}(b-a)f''(\xi) \tag{8}$$

误差项为 $O(h^2)$，与实验所得结果是一致的。

二、本实验使用的复化 Simpson 公式为：

$$\int_a^b f(x)dx \approx \frac{h}{6}\left(f(a) + 2\sum_{i=1}^{2^{k-1}-1} f(a+ih) + 4\sum_{i=0}^{2^{k-1}-1} f(a+(i+\frac{1}{2})h) + f(b)\right) \tag{9}$$

其中 $h = \dfrac{b-a}{2^{k-1}}$。这里不再列出误差项推导，而直接由 [1] 得该公式的误差项为 $-\dfrac{1}{180}(2h)^4 f^{(4)}(\xi) = -\dfrac{8}{45}h^4 f^{(4)}(\xi) = O(h^4)$，$\xi \in (a,\, b)$。

$O(h^4)$ 的误差项与实验结果基本一致。但注意到实验中，在 $k = 12$ 处收敛阶突然减小。为进一步观察此现象，修改源代码，增加 $k$，得如下结果：

```
Integrate[sin(x), 0, 4] = 1.65364362086361182946
...
Composite Simpson Method:
...
I(k = 11) = 1.65364362086374505623   Error = 1.332268e-13   Order = 4.0053
I(k = 12) = 1.65364362086362226556   Error = 1.043610e-14   Order = 3.6742
I(k = 13) = 1.65364362086361538218   Error = 3.552714e-15   Order = 1.5546
I(k = 14) = 1.65364362086361005311   Error = 1.776357e-15   Order = 1.0000
I(k = 15) = 1.65364362086362137738   Error = 9.547918e-15   Order = -2.4263
I(k = 16) = 1.65364362086360472404   Error = 7.105427e-15   Order = 0.4263
I(k = 17) = 1.65364362086360938697   Error = 2.442491e-15   Order = 1.5406
I(k = 18) = 1.65364362086363958504   Error = 2.775558e-14   Order = -3.5064
I(k = 19) = 1.65364362086361693649   Error = 5.107026e-15   Order = 2.4422
I(k = 20) = 1.65364362086360716653   Error = 4.662937e-15   Order = 0.1312
```

注意到 $k = 15$ 比 $k = 14$ 误差反而更大，说明数值积分中的舍入误差影响此时不可忽视。

另外，python3.7 的 numpy.float64 类型的精度为 52 位，而 $2^{52}$ 的十进制数量级为 $10^{16}$。本实验中，由 scipy 包的 integrate 函数计算出的积分值量级为 $10^0$，将其视为精确值。$k > 11$ 后，数值积分值与精确值的偏差的量级在 $10^{-15}$ 左右，接近 float64 的精度极限。计算误差 Error 中浮点数减法导致的有效位数字损失的影响在 $k > 11$ 后将不可忽略。

# 工作环境

主要程序语言：**python**
软件：**JupyterLab**
使用的包：**numpy, scipy**

# 参考资料

[1] David R. Kincaid & E. Ward Cheney. *Numerical Analysis: Mathematics of Scientific of Computing Third Edition*, Brooks/Cole, 2002.