《计算方法》之

插值

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Newton插值

- Lagrange插值的缺点：无承袭性。增加一个节点，所有的基函数都要重新计算
- 承袭性：$N_{n+1}(x) = N_n(x) + q_{n+1}(x)$
  - $N_n(x)$是利用$\{x_0, x_1, \cdots, x_n\}$插值得到的$n$阶多项式
  - $N_{n+1}(x)$是利用$\{x_0, x_1, \cdots, x_{n+1}\}$插值得到的$n+1$阶多项式
  - 增加一个节点，仅需在原有n个节点的多项式基础上添加多项式$q_{n+1}(x)$

如何构造

- 由$N_{n+1}(x_i) = N_n(x_i) = f(x_i), i = 0, \cdots, n$ 可知，$q_{n+1}(x)$有$\{x_0, x_1, \cdots, x_n\}$这$n+1$个零点
  则有$q_{n+1}(x) = a_{n+1}(x - x_0)(x - x_1)\cdots(x - x_n)$,其中$a_{n+1}$为实数
- $N_n(x) = N_{n-1}(x) + q_n(x)$
  则有$q_n(x) = a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$,其中$a_n$为实数
- $N_1(x) = N_0(x) + q_1(x)$
  则有$q_1(x) = a_1(x - x_0)$,其中a_1为实数

Newton插值多项式

$$N_n(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)\cdots(x - x_{n-1})$$

确定系数a_n

$$
\begin{aligned}
N_n(x_0) &= a_0 = f(x_0), \\
N_n(x_1) &= a_0 + a_1(x_1 - x_0) = f(x_1), \\
N_n(x_2) &= a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2), \\
&\cdots \\
N_n(x_n) &= a_0 + a_1(x_n - x_0) + \cdots + a_n(x_n - x_0)\cdots(x_n - x_{n-1}) = f(x_n)
\end{aligned}
$$

由此可得

$$
\begin{aligned}
a_0 &= f(x_0), \\
a_1 &= \frac{f(x_1) - f(x_0)}{x_1 - x_0} \\
a_2 &= \frac{1}{x_2 - x_1}\left(\frac{f(x_2) - f(x_0)}{x_2 - x_0} - a_1\right) \\
a_3 &= \frac{1}{x_3 - x_2}\left(\frac{f(x_3) - f(x_0)}{x_3 - x_0}\frac{1}{x_3 - x_1} - a_2\right)
\end{aligned}
$$

差商定义

定义

- 一阶差商

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

- k阶差商
设$\{x_0, x_1, \cdots, x_k\}$互不相同，$f(x)$关于$\{x_0, x_1, \cdots, x_k\}$的$k$阶插商为

$$f[x_0, x_1 \cdots, x_k] = \frac{f[x_1, \cdots, x_k] - f[x_0, \cdots, x_{k-1}]}{x_k - x_0}$$

差商算法

$$
\begin{array}{ll|llll}
x_0 & f(x_0) & & & & \\
x_1 & f(x_1) & f[x_0, x_1] & & & \\
x_2 & f(x_2) & f[x_1, x_2] & f[x_0, x_1, x_2] & & \\
\cdots & & & & & \\
x_n & f(x_n) & f[x_{n-1}, x_n] & f[x_{n-2}, x_{n-1}, x_n] & f[x_0, x_1, \cdots, x_n] &
\end{array}
$$

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Newton插值多项式表示为

$$N_n(x) = f(x_0)+f[x_0,x_1](x-x_0)+\cdots+f[x_0,x_1,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})$$

$$
\begin{aligned}
a_0 &= f(x_0),\\
a_1 &= \frac{f(x_1)-f(x_0)}{x_1-x_0} = f[x_0,x_1]\\
a_2 &= \frac{1}{x_2-x_1}\left(\frac{f(x_2)-f(x_0)}{x_2-x_0}-a_1\right)\\
&= \frac{1}{x_2-x_1}(f[x_2,x_0]-f[x_1,x_0]) = f[x_0,x_1,x_2]\\
&\cdots\\
a_n &= f[x_0,x_1,\cdots,x_n]
\end{aligned}
$$

算法

- 计算Newton多项式的值

```
for(i=1;i<=n;i++) !计算差商表
{
    for(j=n;j>=i;j--)
        y[j]=(y[j]-y[j-1])/(x[j]-x[j-i]);
}

fx=y[n]; !求Newton多项式的值
for(i=n;i<=1;i--)
{
    fx=y[i-1]+(x-x[i-1])y[i-1];
}
```

差商性质

- k阶差商$f[x_0, x_1, \cdots, x_k]$可由$f(x_0), f(x_1), \cdots, f(x_k)$的线性表示
  - 由多项式插值的唯一性，知$N_k(x) = L_k(x)$.
  - x^k的系数相同
  - $f[x_0, x_1, \cdots, x_k] = \sum\limits_{i=0}^{n} \frac{f(x_i)}{(x_i - x_0)\cdots(x - x_{i-1})(x_i - x_{i+1}\cdots(x_i - x_n)}$
- 对称性：若$i_0, i_1, \cdots, i_k$为$0, 1, \cdots, k$的任意排列，则有

$$f[x_0, x_1, \cdots, x_k] = f[x_{i_0}, x_{i_1}, \cdots, x_{i_k}]$$

- 若$f(x)$为m次多项式，则$f[x_0, x_1, \cdots, x_{k-1}, x]$为$m - k$次多项式。
- 函数差商与函数导数的关系

$$f[x, x_0, x_1, \cdots, x_n] = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)$$

多项式插值误差定理对于Newton插值多项式同样成立，故有对$[a, b]$中每个x，都有$\xi_x \in (a, b)$使得

$$f(x) - N_n(x) = R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x) \prod_{i=0}^{n} (x - x_i)$$

而

$$R_n(x) = f[x, x_0, x_1 \cdots, x_n] \prod_{i=0}^{n} (x - x_i)$$

故有

$$f[x, x_0, x_1 \cdots, x_n] = \frac{1}{(n+1)!} f^{(n+1)}(\xi_x)$$

上机作业3

对函数

$$f(x) = \frac{1}{1+x^2}, x \in [-5,5]$$

构造牛顿插值多项式$p_N(x)$，插值节点取为:

1. $x_i = 5 - \frac{10}{N} i, \ \ i = 0, 1, \cdots, N$
2. $x_i = -5\cos(\frac{2i+1}{2N+2}\pi), \ \ i = 0, 1, \cdots, N \quad$ (Chebyshev point)

并计算如下误差

$$\max_i \{|f(y_i) - p(y_i)|, \ y_i = \frac{i}{10} - 5, \ i = 0, 1, \cdots, 100\}$$

对$N = 5, 10, 20, 40$比较以上两组节点的结果。

输出形式如下:

N=5

Max Error of grid (1) : XXXXXXXXXXXXXXXXX

Max Error of grid (2) : XXXXXXXXXXXXXXXXX

算法

- 计算Newton多项式的值

```
for(i=1;i<=n;i++) !计算差商表
{
    for(j=n;j>=i;j--)
        y[j]=(y[j]-y[j-1])/(x[j]-x[j-i]);
}

fx=y[n]; !求Newton多项式的值
for(i=n;i<=1;i--)
{
    fx=y[i-1]+(x-x[i-1])y[i-1];
}
```

Hermite插值指的是对一个函数在一组结点上的函数值和导数值进行插值。
给定函数f以及结点$x_0, \ldots, x_n$，求多项式p:

$$p^{(j)}(x_i) = f^{(j)}(x_i), \quad j = 0, 1, i = 0, \ldots, n$$

- 多项式插值空间的维数，
- 共有2(n+1)个条件
- 多项式最高次数为2n+1

$$H(x) = \sum_{i=0}^{n} h_i(x) f(x_i) + \sum_{i=0}^{n} g_i(x) f'(x_i)$$

问题变为求解插值基函数$\{h_i(x)\}_i^n$, $\{g_i(x)\}_i^n \in P^{2n+1}(x)$,满足

$$\begin{cases} h_i(x_j) = \delta_{ij} \\ h_i'(x_j) = 0 \end{cases}, \quad \begin{cases} g_i(x_j) = 0 \\ g_i'(x_j) = \delta_{ij} \end{cases},$$

	h_0	$\cdots$	h_n	g_0	$\cdots$	g_n
x_0	1	$\cdots$	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_n	0	$\cdots$	1	0	$\cdots$	0
x_0'	0	$\cdots$	0	1	$\cdots$	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
x_n'	0	$\cdots$	0	0	$\cdots$	1

Hermite 插值基函数

$$h_i(x) = \left(1 - 2(x - x_i)\sum_{i \neq j} \frac{1}{x_i - x_j}\right) \ell_i^2(x)$$

$$g_i(x) = (x - x_i)\ell_i^2(x)$$

当取2个节点时的Hermite插值多项式基函数为

$$h_0(x) = \left(1 - 2\frac{x - x_0}{x_0 - x_1}\right)\left(\frac{x - x_1}{x_0 - x_1}\right)^2$$

$$h_1(x) = \left(1 - 2\frac{x - x_1}{x_1 - x_0}\right)\left(\frac{x - x_0}{x_1 - x_0}\right)^2$$

$$g_0(x) = (x - x_0)\left(\frac{x - x_1}{x_0 - x_1}\right)^2$$

$$g_1(x) = (x - x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)^2$$

例 给定$f(-1)=0,\ f(1)=4,\ f'(-1)=2,\ f'(1)=0$,
求Hermite插值多项式，并计算$f(0.5)$
解：

$$H_3(x)=h_0(x)\cdot 0+h_1(x)\cdot 4+g_0(x)\cdot 2+g_1(x)\cdot 0$$

只需计算$h_1(x)$和$g_0(x)$

$$h_1(x)=\left(1-2\frac{x-1}{1+1}\right)\left(\frac{x+1}{1+1}\right)^2=\frac{1}{4}(2-x)(x+1)^2$$

$$g_0(x)=(x+1)\left(\frac{x-1}{-1-1}\right)^2=\frac{1}{4}(x+1)(x-1)^2$$

$$H_3(x)=(2-x)(x+1)^2+\frac{1}{2}(x+1)(x-1)^2$$

$$H_3(0.5)=3.5625$$

误差估计

Theorem (Hermite插值误差估计定理)

若$f \in C^{2n+2}[a,b]$，$[a,b]$内的插值结点为$x_0,\ldots,x_n$，$p(x)$为相应的Hermite插值多项式，$\deg p \leqslant 2n+1$，则对于任意$x \in [a,b]$，存在$\xi_x \in (a,b)$使得

$$f(x) - p(x) = \frac{f^{(2n+2)}(\xi_x)}{(2n+2)!} \prod_{i=0}^{n} (x - x_i)^2$$

证明方法与无重结点的多项式插值误差估计定理完全类似。