Notebook for Real Analysis

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This document serves as a notebook for Tao's analysis book.

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Part I
Natural Numbers

1 The Peano Axioms

I have learned the Peano axioms. They are descriptive rather than constructive. That is, when we are using this axiom system, we assume that natural numbers do exist, and their properties are described by these axioms. The intuition of the Peano axioms might have been counting from 0 (or 1) by the successor function, but if we try to understand the Peano axioms as constructive, for example, giving 0 first and constructing other numbers via the successor function, things may look a little weird, and the axioms may seem incomplete.

There are some remarkable things regarding the axioms. For the first axiom, Peano originally used 1 instead of 0. This is merely a difference of symbols here, though 0 and 1 have unique meanings in other areas, so 0 is more widely used today than 1. Peano also gave four axioms about the equality relation, and the first three of them (i.e. except the one that says the equality relation is closed under natural numbers) are used as more generic assumptions for general mathematical objects.

Once we have described the basic properties of natural numbers and the successor function S, we can apply our common symbol system to it. We define $1 := S(0),\ 2 := S(1)$ and so on.

1.1 Addition of Natural Numbers

Then we define operations, such as addition, on natural numbers.

Addition The intuition is, the successor function acts like a $+1$ function. That is,

$$n + 1 := S(n) \tag{1}$$

And to add 2 to a number is to merely apply S two times to it. So from the informal equation 1, we can furthermore define

$$n + 2 := S(S(n)) = S(n + 1) \tag{2}$$

Note that by definition, $2 = S(1)$. Apply the substitution to equation 2, we can see that

$$n + S(1) := S(n + 1)$$

We may notice that if we define $n+0 := n$, then equation 1 can be rewritten as

$$n + S(0) := S(n + 0)$$

So now we could try to assume two rules here:

Definition 1. *1.* $0 + n := n$,

2. $S(m) + n := S(m + n)$

and see if it is a good definition of addition.

For every natural number n, we first have $0 + n = n$. Then if we want to know what $1 + n$ is, we have

$$\begin{aligned} 1 + n &= S(0) + n && \text{(By Def. of 1)} \\ &= S(0 + n) && \text{(By the second rule)} \\ &= S(n) && \text{(By the first rule)} \end{aligned}$$

Repeat the process to gain more results:

$$\begin{aligned} 2 + n &= S(1) + n && \text{(By Def. of 2)} \\ &= S(1 + n) && \text{(By the second rule)} \\ &= S(S(n)) && \text{(By the result of } 1 + n\text{)} \end{aligned}$$

Use induction (Suppose we have known $m + n = \underbrace{S(S(\ldots(}_{m \text{ times}} n \underbrace{)\ldots))}_{m \text{ times}}$):

$$\begin{aligned} S(m) + n &= S(m + n) && \text{(By the second rule)} \\ &= \underbrace{S(S(\ldots(}_{m+1 \text{ times}} n \underbrace{)\ldots))}_{m+1 \text{ times}} && \text{(By the result of } m + n\text{)} \end{aligned}$$

And then the add operation is defined for every natural number.

Afterward we will turn to some properties of the newly defined operation – addition. We are going to prove the commutativity and associativity of addition.

Lemma 1. *For any natural number n, $n + 0 = n$*

Proof. Firstly, by definition, $0 + 0 = 0$.

Secondly, if for natural number n, $n + 0 = n$ is true, then $S(n) + 0 = S(n + 0) = S(n)$. This closes the induction, so the proposition is right. □

Lemma 2. *For any natural number m, n, $n + S(m) = S(n + m)$*

Proof. For any fixed natural number m:

1. $0 + S(m) = S(m) + 0 = S(m + 0)$
2. Suppose that $n + S(m) = S(n + m)$, then

$$\begin{aligned} S(n) + S(m) &= S(n + S(m)) && \text{(By Def.)} \\ &= S(S(n + m)) && \text{(By assumption)} \\ &= S(S(n) + m) && \text{(By Def.)} \end{aligned}$$

This closes the induction and the proof is over. □

Proposition 1. *The addition of natural numbers is commutative. That is,*

$$m + n = n + m$$

Proof. First of all, $0 + n = n + 0$.

Then, assume that $m + n = n + m$. Thus:

$$\begin{aligned} S(m) + n &= S(m + n) && \text{(By Def.)} \\ &= S(n + m) && \text{(By Assumption)} \\ &= n + S(m) && \text{(By Lemma 2)} \end{aligned}$$

, which closes the induction. □

Proposition 2. *(Exercise 2.2.1) The addition of natural numbers is associative. That is, $(a + b) + c = a + (b + c)$.*

Proof. Use induction: First, $(0 + b) + c = b + c = 0 + (b + c) = b + c$.

Then, assume that $(n + b) + c = n + (b + c)$, thus

$$\begin{aligned} (S(n) + b) + (c) &= S(n + b) + c \\ &= S(n + b + c) \\ &= S(n + (b + c)) && \text{(By assumption)} \\ &= S(n) + (b + c) \end{aligned}$$

, which closes the induction. □

How fascinating! We have proven the basic properties of addition with only the definition of addition and the axioms. It seemed that we have to define these properties, but we did prove them!

Now we are about to prove some useful propositions about addition.

Proposition 3. *The cancellation law: If $a + b = a + c$, then $b = c$.*

Proof. Use induction: $0 + b = 0 + c \Longrightarrow b = c$.

Assume that $a + b = a + c \Longrightarrow b = c$, thus

$$\begin{aligned} S(a) + b = S(a) + c &\Longrightarrow \\ S(a + b) = S(a + c) &\Longrightarrow \\ S(b) = S(c) &\Longrightarrow \\ b = c& \end{aligned}$$

□

Then we describe natural numbers that are not equal to 0 as **positive**.

Proposition 4. *If a is positive, then for any natural number b, $a + b$ is positive.*

Proof. Use induction: $a + 0 = a$ is positive.

Assume that $a + b$ is positive, then

$$a + S(b) = S(a + b)$$

can not be 0, for 0 is not a successor of any natural number. This closes the induction. □

Corollary 1. *If for natural number a, b, $a + b = 0$, then $a = 0 \wedge b = 0$*

Proof. Presume the contradiction, that there exist $a \neq 0, b \neq 0$, $a + b = 0$.

$$a \neq 0 \Longrightarrow a \text{ is positive}$$

Then according to proposition 4, $a + b$ is also positive, which can not be 0. □

We may wonder something like is it true for every natural number $n \neq 0$, n is always the successor of some other natural number. That is, 0 is the only natural number that is not the successor of any natural number. Or we can convey it in such a way as following:

Proposition 5. *(Exercise 2.2.2) For any positive natural number n, there is exactly one natural number m that $S(m) = n$.*

Proof. Use induction: When $n = 0$, the statement is vacuously right.

Assume that the statement is true for a natural number n, thus

Existence

$$S(S(m)) = S(n)$$

Uniqueness It is obvious according to axiom 3.

□

Proposition 6. *For any natural number n, $S(n) \neq n$*

Proof. Use induction: $S(0) \neq 0$ for 0 is not the successor of any natural number.

Assume that $S(n) \neq n$. Suppose that $S(S(n)) = S(n)$, then by axiom 3, $n = S(n)$, then we have a contradiction. This closes the induction. □

1.2 Order of Natural Numbers

Then I learned the order of natural numbers.

Here I introduce one my own lemma:

Lemma 3. $a = a + n \iff n = 0$

Proof. On one hand, suppose that $a = a + n$ but $n \neq 0$. Try to prove the contradiction. Use induction: First, n is positive, so $0 + n \neq 0$.

Assume that $n \neq 0 \implies a \neq a + n$, thus

$$S(a) + n = S(a + n) \neq S(a) \text{ by assumption}$$

, which closes the induction. Then by the axiom of induction, we have a contradiction, so $a = a + n \implies n = 0$.

On the other hand, $n = 0 \implies a + n = a$. □

Hereby proposition 2.2.12 of Tao's book is proven.

Proposition 7. *(Exercise 2.2.3)*

1. *Order is reflective* $a \geq a$
2. *Order is transitive* $a \geq b \wedge b \geq c \implies a \geq c$
3. *Order is anti-symmetric* $a \geq b \wedge b \geq a \implies a = b$
4. *Addition preserves order* $a \geq b \implies a + c \geq b + c$
5. $a < b \iff S(a) \leq b$
6. $a < b$ *Iff for positive natural number* c, $b = a + c$

Proof. (1) It is immediately proven by $a = a + 0$.

(2)

$$\begin{aligned} a \geq b \wedge b \geq c &\implies \\ a = b + m \wedge b = c + n &\implies \\ a = c + n + m = c + (n + m) &\implies \\ a \geq c & \end{aligned}$$

(3) By definition of order,

$$\begin{aligned} a \geq b \wedge b \geq a &\implies \\ a = b + m \wedge b = a + n &\implies \\ a = a + (n + m) &\implies \\ n + m = 0 &\implies && \text{(By Lemma 3)} \\ n = m = 0 &\implies && \text{(By Corollary 1)} \\ b = a + 0 = a & \end{aligned}$$

(4)

$$\begin{aligned} a \geq b &\implies \\ a = b + n &\implies \\ a + c = (b + n) + c = b + (n + c) = b + (c + n) = (b + c) + n &\implies \\ a + c \geq b + c & \end{aligned}$$

(5)

$$\begin{aligned} a < b &\iff \\ b = a + p &\iff \\ b + 1 = a + p + 1 = a + 1 + p = S(a) + p & \\ = S(a) + S(n) & \\ & \text{(By proposition 5, } p \text{ is always some natural number } n\text{'s successor)} \\ = S(a) + n + 1 &\iff \\ b = S(a) + n &\iff && \text{(By cancellation law)} \\ b \geq S(a) & \end{aligned}$$

(6) On one hand, $b = a + c$ immediately gives $a < b$.

On the other hand, according to (5), $a < b$ gives

$$\begin{aligned} S(a) \leq b \implies \\ b = S(a) + n \\ = a + 1 + n = a + (n+1) \end{aligned}$$

, where $n + 1$ is positive. □

Proposition 8. *(Exercise 2.2.4) For two natural number m, n, m either $>$, or $=$, or $<$ n.*

Proof. Tao's book has proven that at most one statement can be true at a time.

Now we are proving the remnant. Use induction: When $m = 0$, for any natural number n, $0 = n$, or $0 \neq n$. Under the latter case:

Lemma 4. *n is positive* $\iff n > 0$

Proof. On one hand, $n > 0$ immediately gives n is positive.

On the other hand, $n = 0 + n$ gives $n \geq 0$. And n being positive implies that $n \neq 0$. So $n > 0$. □

According to the lemma, in this situation, $0 < n$. So 0 either $<$ or $=$ n.

Assume that we have proven the statement for a natural number m, thus when $m < n$, according to Proposition 1.2, $S(m) \leq n$, so $S(m)$ either $<$ or $=$ n. When $m = n$, $S(m) = n + 1 \implies S(m) > n$ by Proposition 1.2. When $m > n$, according to Proposition 1.2,

$$m = n + p \implies S(m) = n + (p+1) \implies S(m) > n$$

. This closes the induction, implying that at least one of the three statements is true. □

Exercise 2.2.5

Proof. Let $Q(n)$ be a property of a natural number n such that $Q(n)$ is true iff for all $m_0 \leq m' < n$, $P(m')$ is always true. Use induction: $Q(0)$ is vacuously true.

Assume that $Q(n)$ is true. Here we will be using the proposition we just proved, for because we have known that there will and only will be one true statement, we can classify the conditions as following: When $S(n) < m_0$, $Q(S(n))$ is also vacuously true. When $S(n) = m_0$, $Q(S(n))$ is true because $P(m_0)$ is true. And when $S(n) > m_0$:

First we need to prove that n is the only natural number $\geq m_0$ which satisfies $m_0 \leq n < S(n)$ but doesn't satisfy $m_0 \leq n < n$, so that we only need to prove $P(n)$ is true in the induction, which is obvious.

Lemma 5. *There is no natural number between n and $S(n)$. That is, there is no such natural number m that $n < m < S(n)$.*

Proof. Presume the contradiction. Thus, $m = n + p \wedge S(n) = m + q$, where p, q are positive. Substituting m with $n + p$ we have $S(n) = n + p + q$. Let $p = S(a) = a + 1$, which is always possible according to Proposition 5. Thus $n + 1 = n + 1 + a + q \implies n = n + a + q$, which means $a + q$ has to be 0, and which is impossible. □

Given a natural number a, it either $\geq$ or $<$ $S(n)$, and also either $\geq$ or $<$ n. Should it satisfy $m_0 \leq a < S(n)$ but doesn't satisfy $m_0 \leq a < n$, it then must satisfy $n \leq a < S(n)$, that is, either $a = n$ or $n < a < S(n)$. The latter, according to the lemma, is impossible. So n is the only natural number $\geq m_0$ which satisfies $m_0 \leq n < S(n)$ but doesn't satisfy $m_0 \leq n < n$.

Then $Q(S(n)) \iff Q(n) \wedge P(n)$, which is true. This closes the induction. So $Q(n)$ is true for all natural number $n \geq m_0$. And this implies that $P(n)$ is true. □

Exercise 2.2.6

Proof. Use induction: When $n = 0$, for all natural number $m \leq 0$, $P(m)$ is true.

Assume that we have proven for a natural number n that if $P(n)$ is true, then for all natural number $m \leq n$, $P(m)$ is also true. Thus, $P(S(n)) \implies P(n) \implies \forall m \leq n, P(m)$ is true. According to Lemma 5, $(\forall m \leq n, P(m)) \wedge P(S(n)) \iff \forall m \leq S(n), P(m)$. This closes the induction. □

1.3 Multiplication of Natural Numbers

Lemma 6. *(Exercise 2.3.1) Multiplication is commutative. That is, $a \times b = b \times a$.*

Proof. I

Try to imitate the way we prove the commutativity of addition.

Lemma 7.

$$0 \times a = a \times 0$$

Proof. Use induction: $0 \times 0 = 0$. Assume that $n \times 0 = 0$ is true. Thus, $S(n) \times 0 = (n \times 0) + 0 = 0$, which closes the induction. □

Lemma 8.

$$a \times S(b) = a \times b + a$$

Proof. Use induction: $0 \times S(b) = 0 = 0 \times b + 0$.

Assume that $a \times S(b) = ab + a$ is true. Thus,

$$\begin{aligned} S(a)S(b) &= aS(b) + S(b) && \text{(By Def.)} \\ &= ab + a + S(b) && \text{(By assumption)} \\ &= ab + S(a) + b && \text{(By addition's properties)} \\ &= (ab + b) + S(a) && \text{(By addition's properties)} \\ &= S(a)b + S(a) && \text{(By Def.)} \end{aligned}$$

, which closes the induction. □

Now use induction on a. First, when $a = 0$, by Lemma 7 we have $ab = ba$. Assume that $ab = ba$ is true. Thus,

$$\begin{aligned} S(a)b &= ab + b \\ &= ba + b \\ &= bS(a) && \text{(Lemma 8)} \end{aligned}$$

, which close the induction.

□

Proof. II In this proof we will use the distribution law of multiplication.

First, we have Lemma 7

Before we prove the remnant, we need to prove the distribution law. That is, $a \times (b + c) = ab + ac$

Proof. Use induction: $0 \times (b + c) = 0 \times b + 0 \times c = 0$.

Assume that $a \times (b + c) = ab + ac$ is ture. Thus,

$$\begin{aligned} S(a) \times (b + c) &= (a(b + c)) + (b + c) \\ &= (ab + ac) + (b + c) && \text{(By assumption)} \\ &= (ab) + b + (ac) + c \\ &= S(a)b + S(a)c \end{aligned}$$

, which closes the induction. □

We still have to prove $n \times 1 = n$ before proceeding. Use induction: $0 \times 1 = 0$. Assume that $n \times 1 = n$. Thus, $S(n) \times 1 = (n \times 1) + 1 = n + 1 = S(n)$.

Now we can proceed the proof. Assume that $a \times b = b \times a$. Thus,

$$\begin{aligned} S(a)b &= (ab) + b \\ &= (ba) + b && \text{(By assumption)} \\ &= b(a+1) && \text{(By } b \times 1 = b \text{ and the distribution law)} \\ &= b \times S(a) \end{aligned}$$

. This closes the induction. ☐

Lemma 9. *(Exercise 2.3.2)*

$$mn \neq 0 \Longleftrightarrow m \neq 0 \wedge n \neq 0$$

Proof. On one hand, let $m = S(a),\ n = S(b)$, where a, b are natural numbers.

$$\begin{aligned} mn &= S(a)S(b) \\ &= aS(b) + S(b) \end{aligned}$$

which, if $a \neq 0$, is the sum of two positive numbers, and is thus positive, and which, if $a = 0$, is a positive number $S(b)$.

On the other hand, if either of m, n is 0, then mn must be zero. So $mn \neq 0 \Longrightarrow m \neq 0 \wedge n \neq 0$. ☐

Distribution law has been proved here.

Proposition 9. *(Exercise 2.3.3)*

$$(ab)c = a(bc)$$

Proof. Use induction on a. First, $(0b)c = 0c = 0 = 0(bc)$.

Assume that $(ab)c = a(bc)$ is true. Thus,

$$\begin{aligned} (S(a)b)c &= (ab + b)c \\ &= c(ab + b) && \text{(Commutativity)} \\ &= c(ab) + cb && \text{(Distribution law)} \\ &= (ab)c + bc && \text{(Commutativity)} \\ &= a(bc) + bc && \text{(The induction hypothesis)} \\ &= S(a)(bc) \end{aligned}$$

. And now we can close the induction. ☐

Proposition 10. *Multiplication preserves order. That is, if $a > b \wedge c > 0$, then $ac > bc$.*

Proof.

$$\begin{aligned} a > b &\Longrightarrow \\ a = b + p &\Longrightarrow \\ ac = bc + pc \end{aligned}$$

According to Lemma 9, pc is positive. Therefore, $ac > bc$. □

Corollary 2. *Cancellation law.*

$$ac = bc \wedge c \neq 0 \Longrightarrow a = b$$

Proof. Either $a = b$, or $a < b$, or $a > b$. Suppose that $a \neq b$. Therefore, ac either $<$ or $>$ bc, which, according to Proposition 10, gives a contradiction. So $a = b$. □

Proposition 11. *(Exercise 2.3.4)*

$$(a + b)^2 = a^2 + 2ab + b^2$$

(Suppose that we have known $n^2 = n \times n$)

Proof.

$$\begin{aligned} (a + b)(a + b) &= (a + b)a + (a + b)b && \text{(Distribution law)} \\ &= a(a + b) + b(a + b) && \text{(Commutativity)} \\ &= a^2 + ab + ba + b^2 && \text{(Distribution law)} \\ &= a^2 + ab + ab + b^2 && \text{(Commutativity)} \end{aligned}$$

Now we prove that $2ab = ab + ab$.

$$\begin{aligned} 2ab &= S(1)ab \\ &= (1ab) + ab \\ &= (S(0)ab) + ab \\ &= (0ab + ab) + ab \\ &= ab + ab \end{aligned}$$

The proof is over. □

Proposition 12. *(Exercise 2.3.5) Euclidean algorithm. For any natural number n, positive number p, there exist natural numbers m, r such that $n = mp + r$.*

Proof. For any natural number p, we induct on n. Firstly, $0 = 0p + 0$.

Assume that the statement for n is true. We know that $r < p$. Then $S(r)$ either $=$ or $<$ p (Proposition 1.2). On the latter case, simply let $r' = S(r),\ m' = m$, which satisfies the restriction $0 \leq r' < p$

On the former case, let $m' = S(m),\ r' = 0$, and we have

$$\begin{aligned} m'p + r' &= S(m)p + 0 \\ &= mp + p \\ &= mp + S(r) && (p = S(r)) \\ &= S(mp + r) \\ &= S(n) \end{aligned}$$

. And now we can close the induction. ∎

Part II
Set Theory

2 Fundamentals

Exercise 3.1.1

Proof. Reflexive: $\forall x \in S, x \in S$.

Symmetric:

$$\begin{aligned} X = Y &\iff \\ \forall x \in X, x \in Y \land \forall x \in Y, x \in X &\iff \\ Y = X \end{aligned}$$

Transitive: $X = Y \implies \forall x \in X, x \in Y$. Because $x \in Y$ and $Y = Z$, we can conclude that $\forall x \in X, x \in Z$. Conduct the process from inversely, we can get $\forall x \in Z, x \in X$. Therefore, $X = Z$. □

The reason for the content beneath Axiom 3.2 is clearly demonstrated in the proof of Lemma 3.1.6.

In Remarks 3.1.9, there are three "Why"s. The reason can be concluded as: Because of the "if and only if" in Axiom 3.3, or more precisely, "only if", if x is a element in one of such sets, x must $= a$ or b. And because of the "if", x is thus in another set. So the two sets are equal according to Definition 3.1.4.

Exercise 3.1.2

Proof. According to Axiom 3.2, $\varnothing$ exists, and is thus an object as stated by Axiom 3.1. Therefore, by Axiom 3.3, $\{\varnothing\}$ also exists. $\varnothing$ is an element of $\{\varnothing\}$, but it is not an element of $\varnothing$ because any object $\notin \varnothing$.

For the same reason, any set that contains element(s) is not the same set as $\varnothing$. Furthermore, there exists an object $\{\varnothing\}$ (Axiom 3.3 and 3.1), which is an element of $\{\varnothing, \{\varnothing\}\}$, but which is not an element of $\{\varnothing\}$. So the two sets are not equal. □

Remarks 3.1.12

Proof. Let $x \in A' \cup B$. $x \in A' \implies x \in A$ And if $x \notin A'$, $x \in B$. So either way $x \in A \cup B$ and vice versa. □

Exercise 3.1.3

Proof. (1)

$$x \in A \cup B \equiv (x \in A \lor x \in B)$$

$$x \in A \Longrightarrow x \in B \cup A$$

$$x \in B \Longrightarrow x \in B \cup A$$

So $x \in A \cup B \Longrightarrow x \in B \cup A$. And vice versa.

(2) $x \in A \Rightarrow x \in A \cup A$ and $x \in A \cup A \Rightarrow x \in A$.

(3)

$$\begin{aligned} x \in A \cup \varnothing &\Longrightarrow \\ x \in A \lor x \in \varnothing &\Longrightarrow \\ x \in A & \qquad (\forall a, a \notin \varnothing) \end{aligned}$$

And obviously $x \in A \Rightarrow x \in A \cup \varnothing$. So $A \cup \varnothing = A$.

By transitivity of equality, and commutativity of pairwise union, we can conclude the others. □

Examples 3.1.17

Proof.

$$\forall x(x \in A \Longrightarrow x \in A)$$

And

$$\forall x(x \in \varnothing \Longrightarrow x \in A)$$

is vacuously true. □

Exercise 3.1.4

Proof. (1) On one hand,

$$A \subseteq B \equiv \forall x(x \in A \Longrightarrow x \in B)$$

. On the other hand,

$$B \subseteq A \equiv \forall x(x \in B \Longrightarrow x \in A)$$

. Thus $A = B$.

(2) First, we prove that $A \subsetneq B \Longrightarrow \exists x(x \in B \land x \notin A)$. Suppose the contradiction, that is, $\forall x(x \in B \Longrightarrow x \in A)$, which is impossible since $(A \subseteq B \equiv \forall x(x \in A \Longrightarrow x \in B)) \land A \neq B$.

According to what's proven in the book, $A \subsetneq B \wedge B \subsetneq C \Longrightarrow A \subseteq C$.

Now we prove that $\exists x(x \in C \wedge x \notin A)$. Since $x \in A \Longrightarrow x \in B$, $x \notin B \Longrightarrow x \notin A$. Because $B \subsetneq C$, $\exists x(x \in C \wedge x \notin B)$, and thus for such x, $x \notin A$. Then $A \neq C$.

So $A \subsetneq C$. □

Axiom 3.5 (1) Because $x \in \{x \in A : P(x)\} \Rightarrow x \in A$.

(2) Because both $\in$ and $P(x)$ obey the axiom of substitution.

Exercise 3.1.5

Proof. First we prove that $A \subseteq B \equiv A \cup B = B$. On one hand,

$$\begin{aligned} A \subseteq B \equiv \\ \forall x(x \in A \Longrightarrow x \in B) \Longrightarrow \\ \forall x((x \in A \vee x \in B) \Longrightarrow x \in B) \equiv \\ A \cup B = B \end{aligned}$$

.

On the other hand,

$$\forall x((x \in A \vee x \in B) \Longrightarrow x \in B) \Longrightarrow \forall x(x \in A \Longrightarrow x \in B)$$

. The statement is therefore proven.

Then we prove that $A \subseteq B \equiv A \cap B = A$. On one hand,

$$\begin{aligned} (A \cap B = A \equiv \forall x(x \in A \wedge x \in B \equiv x \in A)) \Longrightarrow \\ (\forall x(x \in A \Rightarrow x \in B) \equiv (A \subseteq B)) \end{aligned}$$

.

On the other hand,

$$\forall x(x \in A \wedge x \in B \Longrightarrow x \in A)$$

is always true (Vacuously true if $x \notin B$).

Logical equality is transitive, and thus all of the three statements are equal. □

Proposition 3.1.28 (Exercise 3.1.6)

Proof. (a) The two are identical to

$$\forall x(x \in A \vee x \in \varnothing \equiv x \in A)$$

, and

$$\nexists x(x \in A \wedge x \in \varnothing)$$

, which are all true since $\forall x(x \notin \varnothing)$.

(b) We have $A \subseteq X$. According to what we have proven in Exercise 3.1.5, the two statements are all true.

(c) Obvious since

$$\forall x(x \in A \vee x \in A \equiv x \in A)$$

and

$$\forall x(x \in A \wedge x \in A \equiv x \in A)$$

(d) All true since *logical or* and *logical and* are commutative.

(e) See Lemma 3.1.13. I believe that this can be concluded by the fact that *logical or* and *logical and* are also associative.

(f) First we prove the latter. On one hand, suppose

$$x \in A \cup (B \cap C)$$

is ture.

If $x \in A$, then $x \in$ both $A \cup B$ and $A \cup C$, and thus $\in (A \cup B) \cap (A \cup C)$.

If $x \notin A$, then $x \in B \cap C$, then $x \in$ both $A \cup B$ and $A \cup C$, and thus $\in (A \cup B) \cap (A \cup C)$.

On the other hand, suppose

$$x \in (A \cup B) \cap (A \cup C)$$

is true.

If $x \in A$, obviously $x \in A \cup (B \cap C)$.

If $x \notin A$, then x must $\in B \cap C$, and thus also $\in A \cup (B \cap C)$.

Now we prove the former. On one hand, suppose

$$x \in A \cap (B \cup C)$$

is true.

If $x \in A \wedge x \in B$, then $x \in A \cap B$, and thus $\in (A \cap B) \cup (A \cap C)$.

If $x \notin A \vee x \notin B$, then

1. if $x \notin A$, this is impossible.

2. if $x \in A$, then $x \notin B$. But $x \in B \cup C$, so $x \in C$. And thus $x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C)$.

On the other hand, suppose that

$$x \in (A \cap B) \cup (A \cap C)$$

is true.

First we can see that $x \in A$.

If $x \in B$, then $x \in B \cup C$, and thus $\in A \cap (B \cup C)$.

If $x \notin B$, then $x \in C$. So $x \in B \cup C$, and thus $\in A \cap (B \cup C)$.

(g) Now we prove the former: On one hand, suppose that

$$x \in A \cup (X - A)$$

.

If $x \in A$, then $x \in X$ since $A \subseteq X$.

If $x \notin A$, then $x \in X - A$, and thus also $\in X$.

On the other hand, suppose that

$$x \in X$$

If $x \in A$, then $x \in A \cup (X - A)$.

If $x \notin A$, then $x \in X - A$, and thus $\in A \cup (X - A)$.

(h) $x \in X - A$ requires $x \notin A$. So $\forall x(x \in A \cap (X - A))$ is always false. Thus

$$\forall x(x \in A \cap (X - A) \iff x \in \varnothing)$$

(vacuously true). □

Exercise 3.1.7

Proof. (1) $\forall x(x \in A \cap B \implies x \in A)$. Similarly, we can prove that $A \cap B \subseteq B$. (This can also be achieved via the commutativity).

(2) On one hand, suppose that

$$C \subseteq A \wedge C \subseteq B$$

is true. Then,

$$\forall x(x \in C \implies x \in A \wedge x \in B \implies x \in A \cap B)$$

.

On the other hand, suppose that

$$C \subseteq A \cap B$$

is true. Then,

$$\forall x(x \in C \implies x \in A \land x \in B)$$

. That is, $C \subseteq A \land C \subseteq B$.

(3) It is immediately given by

$$\forall x(x \in A \implies x \in A \cup B)$$

. Since $\cup$ is commutative, the latter case is proven.

(4) On one hand, suppose that $A \subseteq C \land B \subseteq C$ and let $x \in A \cup B$.

If $x \in A$, then $x \in C$.

If $x \notin A$, then $x \in B$, and thus $x \in C$.

On the other hand, suppose that $A \cup B \subseteq C$. Then,

$$\forall x(x \in A \implies x \in A \cup B \implies x \in C)$$

$$\forall x(x \in B \implies x \in A \cup B \implies x \in C) \qquad \square$$

.

Exercise 3.1.8

Proof. The former: On one hand, Suppose that

$$x \in A \cap (A \cup B)$$

.

If $x \in A$, then $x \in A$.

If $x \notin A$, this is impossible.

On the other hand, suppose that $x \in A$. Then $x \in A \land x \in (A \cup B)$, so $x \in A \cap (A \cup B)$.

The latter: On one hand, suppose that $x \in A \cup (A \cap B)$.

$$x \in A \implies x \in A.$$

$$x \notin A \implies x \in (A \cap B) \implies x \in A$$

.

On the other hand, Suppose that $x \in A$, then $x \in A \cup (A \cap B)$. $\square$

Exercise 3.1.9

Proof.

Lemma 10.

$$\nexists x \forall B \forall A (x \in A \land x \in B \land A \cap B = \varnothing)$$

Proof. Suppose the contradiction: $x \in A \land x \in B \land A \cap B = \varnothing$, then $x \in A \cap B$, and thus $\in \varnothing$, which is impossible. □

The former: On one hand, suppose that $x \in A$. Then $x \notin B$ by Lemma 10. And

$$x \in A \Longrightarrow x \in A \cup B \Longrightarrow x \in X$$

. So $x \in (X - B)$.

On the other hand, suppose that $x \in (X - B)$, then $x \in A \cup B$. But $x \notin B$, so $x \in A$ by Lemma 10.

The latter is immediately proven since $\cap, \cup$ are commutative. □

Exercise 3.1.10

Proof. Firstly we prove that $(A - B) \cap (A \cap B) = \varnothing$.

$x \in (A \cap B)$ gives $x \in B$, but $x \in (A - B)$ gives $x \notin B$. So the two statements can not be true simultaneously. Which means

$$x \in (A - B) \cap (A \cap B) \Longrightarrow x \in \varnothing$$

And obviously

$$x \in (A - B) \cap (A \cap B) \Longleftarrow x \in \varnothing$$

.

Similarly we can conclude all of the three sets are disjoint by the fact that $\nexists x \in$ either two of the three sets.

Now we are showing that their union is $A \cup B$.

On one hand, suppose that

$$x \in (A - B) \cap (A \cap B) \cap (B - A)$$

. x can at most be in one of these sets since they are disjoint. If $x \in A$, then $x \in A \cup B$.

If $x \notin A$, then $x \in (B - A)$, and thus $x \in B$. So $x \in A \cup B$.

On the other hand, suppose that $x \in A \cup B$. Then x either

1. $\in A$, but $\notin B$, or
2. $\in B$, but $\notin A$, or
3. $\in$ both A, B.

If (1), then $x \in (A - B)$.
If (2), then $x \in (B - A)$.
If (3), then $x \in A \cap B$.
In conclusion, we can see that $x \in (A - B) \cap (A \cap B) \cap (B - A)$. □

Exercise 3.1.11

Proof. Let $P(x, y)$ be a property pertaining to any object x, y, and is true iff $Q(x)$.

According to Axiom 3.6, for any set S, there exists a set Z, such that $x \in Z \equiv x \in S \wedge P(x, y)$, which means $x \in Z \equiv x \in S \wedge Q(x)$. So is the axiom of specification proven. □

Part III
Mathematical Logic

3 Mathematical Statements

Exercise A.1.1 It is ((both X, Y are false) or (both X, Y are true)).

Exercise A.1.2 It is ((Y can be true even if X is false) or (Y can be false even if X is true)).

Exercise A.1.3 Yes. That's the definition of logical equivalent.

Exercise A.1.4 No. It is still possible that (even if X is false, Y is still true).

Consider a statement Y that satisfies:

1. If X, then Y.
2. If X is false, then Y or (exclusively) Y is false.

X, Y satisfy the description in the exercise, but they are not logical equivalent.

Exercise A.1.5 Yes. (Now I'm using the symbols defined in the A.2 for the sake of simplification) $X \iff Y$ means $X \implies Y \wedge \neg X \implies \neg Y$. So does Y and Z. So

$$(X \implies Y \implies Z \wedge \neg X \implies \neg Y \implies \neg Z) \implies$$
$$(X \implies Z \wedge \neg X \implies \neg Z)$$

, which means X and Z are logical equivalent.

(Note that $A \implies B$ can also be interpreted as a statement, meaning "If A is true, then B is true", just like we did in this example.)

Exercise A.1.6 Yes. $(X \implies Y \implies Z) \implies (X \implies Z)$.

Now we are proving that $Z \implies X \equiv \neg X \implies \neg Z$. Assume that $\neg X \wedge Z$. Since $Z \implies X$, we have a contradiction: $X \wedge \neg X$.

So $X \implies Z \wedge \neg X \implies \neg Z$. Therefore, X, Z are logical equivalent. Besides, we can conclude that $Y \implies X$. Thus X, Y are also logical equivalent.

4 Implication

Why did Tao say

> If X, then Y can also be written as "X can only be true when Y is true"

?

Assume the $X \wedge \neg Y$, thus $X \implies Y$. So we have a contradiction $Y \wedge \neg Y$.

Define "when $x \neq 2$, $X : x = 2 \implies x^2 = 4$ is vacuously true" to ensure that X is always true regardless of the value of x.

5 Nested Quantifiers

Exercise A.5.1 **(a)** Let P be $y^2 = x$ is true for each positive number y. And this statement means P is true for each positive number x.

Gaming metaphor: Me and my friend each randomly pick up a positive, say x and y, and check if $y^2 = x$.

The statement is false.

(b) There is at least one positive number x such that for every positive number y, $y^2 = x$.

Gaming metaphor: I have to pick up a positive number x such that whatever positive number y my friend picks up, $y^2 = x$ is always true.

The statement is false.

(c) There is at least two positive numbers x, y such that $y^2 = x$.

Gaming metaphor: Me and my friend each have to pick up a positive number, say x and y, such that $y^2 = x$.

The statement is true. For example, $1^2 = 1$.

(d) The statement $\exists x > 0, y^2 = x$ is true for every $y > 0$.

Gaming metaphor: For each positive number y my friend picks up, I have to pick up a positive number x such that $y^2 = x$.

The statement is true, because y^2 is also positive.

(e) There is at least one positive number y such that for every positive number x, $y^2 = x$ is always true.

Gaming metaphor: I have to find a number $y > 0$ such that regardless of what number x my friend picks up, $y^2 = x$ is always true.

The statement is false.

6 Equality

Exercise A.7.1

Proof. Let $F(x) := x + c$. By axiom 4, $F(a) = F(b)$. That is, $a + c = b + c$. Similarly, by letting $G(x) := a + x$, we have $a + c = a + d$, which, according to axiom 2, becomes $a + d = a + c$. Now we have $a + d = a + c, a + c = b + c$. According to axiom 3, we can conclude that $a + d = b + c$. $\square$

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