Notebook for Real Analysis

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This document serves as a notebook for Terence Tao's *Analysis, Third Edition*.

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You can get a copy of source code at https://github.com/Little-He-Guan/Notebook-for-Analysis-of-Tao.

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1 General Principles

This section describes the overall principles of the document. It illuminates how notations are explained, in what structure this document is written and so forth. This section should be read and understood comprehensively prior to reading the main content of the document.

1.1 Definitions

The document The phrase *the document* means this document (what you're reading) itself.

The book The phrase *the book* represents Tao's *Analysis* (both volume I and II).

1.2 Indices

The book has two volumes: *Analysis I* and *Analysis II*. We may notice that the indices of the two volumes both start from 1. It may lead to some confusions. So in the document, the indices are organized in such a way that: If the content comes from *Analysis I*, the corresponding index is the same as the book's. Otherwise, the corresponding index is prefixed with "2.".

For example, Exercise 3.1.3 in *Analysis I* is indexed as Exercise 3.1.3 in the document, but Exercise 3.1.3 in *Analysis II* is indexed as Exercise 2.3.1.3.

1.3 Notations

In the answers to some exercises, you may notice that the content are divided by numbers enclosed with parentheses (e.g. **(1)**, **(2)**). Tao often puts multiple questions into a single exercise, so these numbers indicates the number of the sub-questions.

For example, Exercise 3.5.4 is

> Exercise 3.5.4. Let A, B, C be sets. Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$, that $A \times (B \cap C) = (A \times B) \cap (A \times C)$, and that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

Then (1) indicates the question "Show that $A \times (B \cup C) = (A \times B) \cup (A \times C)$.", (2) indicates the question "Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$.", and (3) indicates the question "Show that $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$".

In logical contents,

$$\Longrightarrow, \Rightarrow, \longrightarrow, \rightarrow,$$

have the same meaning "implies". And

$$\Longleftarrow, \Leftarrow, \longleftarrow, \leftarrow,$$

also have the same meaning ($P \leftarrow Q$ means that Q implies P). Finally, these following symbols all indicate logical equality.

$$\leftrightarrow, \longleftrightarrow, \Leftrightarrow, \Longleftrightarrow, \equiv$$

.

For nested quantifiers, their order is "from left to right". For example, the following statement

$$\forall x \exists y (P(x, y))$$

means that for all object x, their exists a object y such that $P(x, y)$ is true. That is,

$$\forall x (\exists y (P(x, y)))$$

Tao uses ++ to denote the successor of a natural number. However, in the document, it is denoted by $S(n)$ most of the times.

$$\bigvee_{i=1}^{n} P(i), \bigwedge_{i=1}^{n} P(i)$$

mean that for $1 \leq i \leq n$, at least one $P(i)$ is true; and for all $1 \leq i \leq n$, $P(i)$ is true, respectively.

Some sets that have special meanings (e.g. the set of all natural numbers, the set of all real numbers) are denoted in whiteboard font (e.g. $\mathbb{N}, \mathbb{R}$).

Without special interpretation, the notation

$$(\forall x P(x))(Q(x))$$

is interpreted as

$$\forall x (P(x) \Longrightarrow Q(x))$$

For example,

$$(\forall x \in X)(Q(x)) \equiv \forall x (x \in X \Longrightarrow Q(x))$$

While

$$(\exists x P(x))(Q(x))$$

is interpreted as

$$\exists x (P(x) \wedge Q(x))$$

. For example,

$$(\exists x \in X)(Q(x)) \equiv \exists x (x \in X \wedge Q(x))$$

1.4 Abbreviations

We often leave off some descriptions for a number's properties. For example, we may refer a *positive natural number* n as a *positive number* when we haven't learned the rationals and the reals.

Why Axiomization

Before we enter the world of limits, we will axiomize the real numbers. We will start from as few as possible axioms to construct the natural numbers, integers, rational, and finally the real numbers. And along the way, basic operations and relations such as addition, multiplication, and order (i.e. $<, >, =$) will be defined, with their properties verified (e.g. The commutative law).

You might wonder the reason for all these long and tedious works. The laws of algebra and the existence of the reals have accompanied us for many years and we are very familiar with them now. Why should we verify all of them? Can't we just take them for granted and go ahead to limits? Well, here comes the very meaning of axiomization.

Now looking back, for these properties, we were just taught them. We have been using them without doubting their correctness. And to doubt it seems very funny. However, we must admit that we are merely thinking that they are right. Mathematics is a rigorous subject. It does not accept such things as "I believe that it is right." So to show that they are right, we need to *prove* them.

Nevertheless, we cannot prove something from absolutely nothing. We can only get nothing from the void. We should at least start from something. These beginnings are called the axioms. We admit their correctness without proof. Then one might ask a question: *How about treat the existence and the properties of the reals, the operations, and the relations as axioms?*. This is a good question. But have you ever thought about what if some of these properties are inconsistent, that is, they violate each other, for example, property A denies B? You can bet that they are consistent, but to show the consistence, we can do nothing but to prove. (However, Gödel's second incomplete theorem states that many axiom systems cannot demonstrate its own consistence. So we'd better say that for some axiom systems we can do nothing but to believe their consistence.)

So our idea is, to give as few axioms that are consistent with each other as possible (though most of the time we can only believe so), and then we start from this very fundamental. We will define and prove all other things along the way. Note that definitions can also be treated as some kind of axioms, after all. And when our journey is finished, all properties are verified. We can just proceed as we did before, but this time we are more certain and have a better understanding of what we are doing.

Part I
Natural Numbers

2 The Peano Axioms

I have learned the Peano axioms. They are descriptive rather than constructive. That is, when we are using this axiom system, we assume that natural numbers do exist, and their properties are described by these axioms. The intuition of the Peano axioms might have been counting from 0 (or 1) by the successor function, but if we try to understand the Peano axioms as constructive, for example, giving 0 first and constructing other numbers via the successor function, things may look a little weird, and the axioms may seem incomplete.

There are some remarkable things regarding the axioms. For the first axiom, Peano originally used 1 instead of 0. This is merely a difference of symbols here, though 0 and 1 have unique meanings in other areas, so 0 is more widely used today than 1. Peano also gave four axioms about the equality relation, and the first three of them (i.e. except the one that says the equality relation is closed under natural numbers) are used as more generic assumptions for general mathematical objects.

Once we have described the basic properties of natural numbers and the successor function S, we can apply our common symbol system to it. We define $1 := S(0),\ 2 := S(1)$ and so on.

There is something interesting about the mathematical induction. Consider the situation where we have a property $P(n)$ pertaining to all natural numbers n, and which is vacuously true if $n = 0$. Then do we need to check if it is true when $n = 1$, or can we just check if $P(n) \Longrightarrow P(S(n))$?

The answer is, we need to check if it is true when $n = 1$. We can just choose a property $P(n)$ such that: $\neg P(0) \wedge (P(n) \Longrightarrow P(S(n)))$. Then let $Q(n)$ be any property such that $Q(0)$ is vacuously true and $Q(n) \equiv P(n)$ when $n \neq 0$. We can see that $Q(n)$ may not be always true.

Another thing is, what if a property P does not pertain to 0? For example, if we let P be a property pertaining to all $n \in \mathbb{N} \wedge n \neq 0$, can we apply mathematical induction to it? Generally we can. We can define the property $Q(n)$ to be $P(S(n))$, then $Q(0) \equiv P(1)$ is the base case. We haven't learned addition and order here. But after we have, it is easy to extend this technique to: If P is a property pertaining to all $n \in \mathbb{N} \wedge n \geq m$, then we can define the property $Q(n)$ to be $P(n + m)$.

2.1 Addition of Natural Numbers

Then we define operations, such as addition, on natural numbers.

Addition The intuition is, the successor function acts like a $+1$ function. That is,

$$n + 1 := S(n) \tag{1}$$

And to add 2 to a number is to merely apply S two times to it. So from the informal equation 1, we can furthermore define

$$n + 2 := S(S(n)) = S(n + 1) \tag{2}$$

Note that by definition, $2 = S(1)$. Apply the substitution to equation 2, we can see that

$$n + S(1) := S(n + 1)$$

We may notice that if we define $n+0 := n$, then equation 1 can be rewritten as

$$n + S(0) := S(n + 0)$$

So now we could try to assume two rules here:

Definition 1. *1.* $0 + n := n$,

2. $S(m) + n := S(m + n)$

and see if it is a good definition of addition.

For every natural number n, we first have $0 + n = n$. Then if we want to know what $1 + n$ is, we have

$$\begin{aligned} 1 + n &= S(0) + n && \text{(By Def. of 1)} \\ &= S(0 + n) && \text{(By the second rule)} \\ &= S(n) && \text{(By the first rule)} \end{aligned}$$

Repeat the process to gain more results:

$$\begin{aligned} 2 + n &= S(1) + n && \text{(By Def. of 2)} \\ &= S(1 + n) && \text{(By the second rule)} \\ &= S(S(n)) && \text{(By the result of } 1 + n\text{)} \end{aligned}$$

Use induction (Suppose we have known $m+n=\underbrace{S(S(\ldots(}_{m\text{ times}}n\underbrace{)\ldots))}_{m\text{ times}}$):

$$\begin{aligned} S(m)+n &= S(m+n) && \text{(By the second rule)}\\ &= \underbrace{S(S(\ldots(}_{m+1\text{ times}}n\underbrace{)\ldots))}_{m+1\text{ times}} && \text{(By the result of } m+n\text{)} \end{aligned}$$

And then the add operation is defined for every natural number.

Afterward we will turn to some properties of the newly defined operation – addition. We are going to prove the commutativity and associativity of addition.

Lemma 1. *For any natural number n, $n+0=n$*

Proof. Firstly, by definition, $0+0=0$.

Secondly, if for natural number n, $n+0=n$ is true, then $S(n)+0=S(n+0)=S(n)$. This closes the induction, so the proposition is right. □

Lemma 2. *For any natural number m,n, $n+S(m)=S(n+m)$*

Proof. For any fixed natural number m:

1. $0+S(m)=S(m)+0=S(m+0)$
2. Suppose that $n+S(m)=S(n+m)$, then

$$\begin{aligned} S(n)+S(m) &= S(n+S(m)) && \text{(By Def.)}\\ &= S(S(n+m)) && \text{(By assumption)}\\ &= S(S(n)+m) && \text{(By Def.)} \end{aligned}$$

This closes the induction and the proof is over. □

Proposition 1. *The addition of natural numbers is commutative. That is,*

$$m+n=n+m$$

Proof. First of all, $0+n=n+0$.

Then, assume that $m+n=n+m$. Thus:

$$\begin{aligned} S(m)+n &= S(m+n) && \text{(By Def.)}\\ &= S(n+m) && \text{(By Assumption)}\\ &= n+S(m) && \text{(By Lemma 2)} \end{aligned}$$

, which closes the induction. □

Proposition 2. *(Exercise 2.2.1) The addition of natural numbers is associative. That is,* $(a + b) + c = a + (b + c)$.

Proof. Use induction: First, $(0 + b) + c = b + c = 0 + (b + c) = b + c$.

Then, assume that $(n + b) + c = n + (b + c)$, thus

$$\begin{aligned}(S(n) + b) + (c) &= S(n + b) + c \\ &= S(n + b + c) \\ &= S(n + (b + c)) \qquad \text{(By assumption)} \\ &= S(n) + (b + c)\end{aligned}$$

, which closes the induction. □

How fascinating! We have proven the basic properties of addition with only the definition of addition and the axioms. It seemed that we have to define these properties, but we did prove them!

Now we are about to prove some useful propositions about addition.

Proposition 3. *The cancellation law: If* $a + b = a + c$*, then* $b = c$.

Proof. Use induction: $0 + b = 0 + c \implies b = c$.

Assume that $a + b = a + c \implies b = c$, thus

$$\begin{aligned}S(a) + b = S(a) + c \implies \\ S(a + b) = S(a + c) \implies \\ S(b) = S(c) \implies \\ b = c\end{aligned}$$

□

Then we describe natural numbers that are not equal to 0 as **positive**.

Proposition 4. *If* a *is positive, then for any natural number* b*,* $a + b$ *is positive.*

Proof. Use induction: $a + 0 = a$ is positive.

Assume that $a + b$ is positive, then

$$a + S(b) = S(a + b)$$

can not be 0, for 0 is not a successor of any natural number. This closes the induction. □

Corollary 1. *If for natural number a, b, $a + b = 0$, then $a = 0 \wedge b = 0$*

Proof. Presume the contradiction, that there exist $a \neq 0, b \neq 0$, $a + b = 0$.

$$a \neq 0 \Longrightarrow a \text{ is positive}$$

Then according to proposition 4, $a + b$ is also positive, which can not be 0. □

We may wonder something like is it true for every natural number $n \neq 0$, n is always the successor of some other natural number. That is, 0 is the only natural number that is not the successor of any natural number. Or we can convey it in such a way as following:

Proposition 5. *(Exercise 2.2.2) For any positive natural number n, there is exactly one natural number m that $S(m) = n$.*

Proof. Use induction: When $n = 0$, the statement is vacuously right.

Assume that the statement is true for a natural number n, thus

Existence

$$S(S(m)) = S(n)$$

Uniqueness It is obvious according to axiom 3.

□

Proposition 6. *For any natural number n, $S(n) \neq n$*

Proof. Use induction: $S(0) \neq 0$ for 0 is not the successor of any natural number.

Assume that $S(n) \neq n$. Suppose that $S(S(n)) = S(n)$, then by axiom 3, $n = S(n)$, then we have a contradiction. This closes the induction. □

2.2 Order of Natural Numbers

Then I learned the order of natural numbers.

Here I introduce one my own lemma:

Lemma 3. $a = a + n \Longleftrightarrow n = 0$

Proof. On one hand, suppose that $a = a + n$ but $n \neq 0$. Try to prove the contradiction. Use induction: First, n is positive, so $0 + n \neq 0$.

Assume that $n \neq 0 \implies a \neq a + n$, thus

$$S(a) + n = S(a + n) \neq S(a) \text{ by assumption}$$

, which closes the induction. Then by the axiom of induction, we have a contradiction, so $a = a + n \implies n = 0$.

On the other hand, $n = 0 \implies a + n = a$. □

Hereby proposition 2.2.12 of Tao's book is proven.

Proposition 7. *(Exercise 2.2.3)*

1. *Order is reflective* $a \geq a$
2. *Order is transitive* $a \geq b \wedge b \geq c \implies a \geq c$
3. *Order is anti-symmetric* $a \geq b \wedge b \geq a \implies a = b$
4. *Addition preserves order* $a \geq b \implies a + c \geq b + c$
5. $a < b \iff S(a) \leq b$
6. $a < b$ *Iff for positive natural number* c, $b = a + c$

Proof. (1) It is immediately proven by $a = a + 0$.

(2)

$$\begin{aligned} a \geq b \wedge b \geq c \implies \\ a = b + m \wedge b = c + n \implies \\ a = c + n + m = c + (n + m) \implies \\ a \geq c \end{aligned}$$

(3) By definition of order,

$$\begin{aligned} a \geq b \wedge b \geq a \implies & \\ a = b + m \wedge b = a + n \implies & \\ a = a + (n + m) \implies & \\ n + m = 0 \implies & \quad \text{(By Lemma 3)} \\ n = m = 0 \implies & \quad \text{(By Corollary 1)} \\ b = a + 0 = a & \end{aligned}$$

(4)

$$\begin{aligned} a \geq b \equiv \\ a = b + n \equiv \\ a + c = (b + n) + c = b + (n + c) = b + (c + n) = (b + c) + n \equiv \\ a + c \geq b + c \end{aligned}$$

(5)

$$\begin{aligned} a < b \iff \\ b = a + p \iff \\ b + 1 = a + p + 1 = a + 1 + p = S(a) + p \\ = S(a) + S(n) \end{aligned}$$

(By proposition 5, p is always some natural number n's successor)

$$\begin{aligned} = S(a) + n + 1 \iff \\ b = S(a) + n \iff \quad \text{(By cancellation law)} \\ b \geq S(a) \end{aligned}$$

(6) On one hand, $b = a + c$ immediately gives $a < b$.
On the other hand, according to (5), $a < b$ gives

$$\begin{aligned} S(a) \leq b \implies \\ b = S(a) + n \\ = a + 1 + n = a + (n + 1) \end{aligned}$$

, where $n + 1$ is positive. □

Proposition 8. *(Exercise 2.2.4) For two natural number m, n, m either $>$, or $=$, or $< n$.*

Proof. Tao's book has proven that at most one statement can be true at a time.

Now we are proving the remnant. Use induction: When $m = 0$, for any natural number n, $0 = n$, or $0 \neq n$. Under the latter case:

Lemma 4. *n is positive $\iff n > 0$*

Proof. On one hand, $n > 0$ immediately gives n is positive.

On the other hand, $n = 0 + n$ gives $n \geq 0$. And n being positive implies that $n \neq 0$. So $n > 0$. □

According to the lemma, in this situation, $0 < n$. So 0 either $<$ or $= n$.

Assume that we have proven the statement for a natural number m, thus when $m < n$, according to Proposition 2.2, $S(m) \leq n$, so $S(m)$ either $<$ or $= n$. When $m = n$, $S(m) = n + 1 \implies S(m) > n$ by Proposition 2.2. When $m > n$, according to Proposition 2.2,

$$m = n + p \implies S(m) = n + (p + 1) \implies S(m) > n$$

. This closes the induction, implying that at least one of the three statements is true. □

Exercise 2.2.5

Proof. Let $Q(n)$ be a property of a natural number n such that $Q(n)$ is true iff for all $m_0 \leq m' < n$, $P(m')$ is always true. Use induction: $Q(0)$ is vacuously true.

Assume that $Q(n)$ is true. Here we will be using the proposition we just proved, for because we have known that there will and only will be one true statement, we can classify the conditions as following: When $S(n) < m_0$, $Q(S(n))$ is also vacuously true. When $S(n) = m_0$, $Q(S(n))$ is true because $P(m_0)$ is true. And when $S(n) > m_0$:

First we need to prove that n is the only natural number $\geq m_0$ which satisfies $m_0 \leq n < S(n)$ but doesn't satisfy $m_0 \leq n < n$, so that we only need to prove $P(n)$ is true in the induction, which is obvious.

Lemma 5. *There is no natural number between n and $S(n)$. That is, there is no such natural number m that $n < m < S(n)$.*

Proof. Presume the contradiction. Thus, $m = n + p \wedge S(n) = m + q$, where p, q are positive. Substituting m with $n + p$ we have $S(n) = n + p + q$. Let $p = S(a) = a + 1$, which is always possible according to Proposition 5. Thus $n + 1 = n + 1 + a + q \implies n = n + a + q$, which means $a + q$ has to be 0, and which is impossible. □

Given a natural number a, it either $\geq$ or $< S(n)$, and also either $\geq$ or $< n$. Should it satisfy $m_0 \leq a < S(n)$ but doesn't satisfy $m_0 \leq a < n$, it then must satisfy $n \leq a < S(n)$, that is, either $a = n$ or $n < a < S(n)$. The latter, according to the lemma, is impossible. So n is the only natural number $\geq m_0$ which satisfies $m_0 \leq n < S(n)$ but doesn't satisfy $m_0 \leq n < n$.

Then $Q(S(n)) \iff Q(n) \wedge P(n)$, which is true. This closes the induction. So $Q(n)$ is true for all natural number $n \geq m_0$. And this implies that $P(n)$ is true. □

Exercise 2.2.6

Proof. Use induction: When $n = 0$, for all natural number $m \leq 0$, $P(m)$ is true.

Assume that we have proven for a natural number n that if $P(n)$ is true, then for all natural number $m \leq n$, $P(m)$ is also true. Thus, $P(S(n)) \Longrightarrow P(n) \Longrightarrow \forall m \leq n, P(m)$ is true. According to Lemma 5, $(\forall m \leq n, P(m)) \wedge P(S(n)) \Longleftrightarrow \forall m \leq S(n), P(m)$. This closes the induction. □

2.3 Multiplication of Natural Numbers

Lemma 6. *(Exercise 2.3.1) Multiplication is commutative. That is,* $a \times b = b \times a$.

Proof. I

Try to imitate the way we prove the commutativity of addition.

Lemma 7.

$$0 \times a = a \times 0$$

Proof. Use induction: $0 \times 0 = 0$. Assume that $n \times 0 = 0$ is true. Thus, $S(n) \times 0 = (n \times 0) + 0 = 0$, which closes the induction. □

Lemma 8.

$$a \times S(b) = a \times b + a$$

Proof. Use induction: $0 \times S(b) = 0 = 0 \times b + 0$.

Assume that $a \times S(b) = ab + a$ is true. Thus,

$$\begin{aligned} S(a)S(b) &= aS(b) + S(b) && \text{(By Def.)} \\ &= ab + a + S(b) && \text{(By assumption)} \\ &= ab + S(a) + b && \text{(By addition's properties)} \\ &= (ab + b) + S(a) && \text{(By addition's properties)} \\ &= S(a)b + S(a) && \text{(By Def.)} \end{aligned}$$

, which closes the induction. □

Now use induction on a. First, when $a = 0$, by Lemma 7 we have $ab = ba$.

Assume that $ab = ba$ is true. Thus,

$$\begin{aligned} S(a)b &= ab + b \\ &= ba + b \\ &= bS(a) && \text{(Lemma 8)} \end{aligned}$$

, which close the induction.

□

Proof. II In this proof we will use the distribution law of multiplication.

First, we have Lemma 7

Before we prove the remnant, we need to prove the distribution law. That is, $a \times (b + c) = ab + ac$

Proof. Use induction: $0 \times (b + c) = 0 \times b + 0 \times c = 0$.

Assume that $a \times (b + c) = ab + ac$ is ture. Thus,

$$\begin{aligned} S(a) \times (b + c) &= (a(b + c)) + (b + c) \\ &= (ab + ac) + (b + c) && \text{(By assumption)} \\ &= (ab) + b + (ac) + c \\ &= S(a)b + S(a)c \end{aligned}$$

, which closes the induction. □

We still have to prove $n \times 1 = n$ before proceeding. Use induction: $0 \times 1 = 0$. Assume that $n \times 1 = n$. Thus, $S(n) \times 1 = (n \times 1) + 1 = n + 1 = S(n)$.

Now we can proceed the proof. Assume that $a \times b = b \times a$. Thus,

$$\begin{aligned} S(a)b &= (ab) + b \\ &= (ba) + b && \text{(By assumption)} \\ &= b(a + 1) && \text{(By } b \times 1 = b \text{ and the distribution law)} \\ &= b \times S(a) \end{aligned}$$

. This closes the induction. □

Lemma 9. *(Exercise 2.3.2)*

$$mn \neq 0 \iff m \neq 0 \wedge n \neq 0$$

Proof. On one hand, let $m = S(a),\ n = S(b)$, where a, b are natural numbers.

$$\begin{aligned} mn &= S(a)S(b) \\ &= aS(b) + S(b) \end{aligned}$$

which, if $a \neq 0$, is the sum of two positive numbers, and is thus positive, and which, if $a = 0$, is a positive number $S(b)$.

On the other hand, if either of m, n is 0, then mn must be zero. So $mn \neq 0 \Longrightarrow m \neq 0 \wedge n \neq 0$. □

Distribution law has been proved here.

Proposition 9. *(Exercise 2.3.3)*

$$(ab)c = a(bc)$$

Proof. Use induction on a. First, $(0b)c = 0c = 0 = 0(bc)$.

Assume that $(ab)c = a(bc)$ is true. Thus,

$$\begin{aligned}
(S(a)b)c &= (ab + b)c \\
&= c(ab + b) && \text{(Commutativity)} \\
&= c(ab) + cb && \text{(Distribution law)} \\
&= (ab)c + bc && \text{(Commutativity)} \\
&= a(bc) + bc && \text{(The induction hypothesis)} \\
&= S(a)(bc)
\end{aligned}$$

. And now we can close the induction. □

Proposition 10. *Multiplication preserves order. That is, if $a > b \wedge c > 0$, then $ac > bc$.*

Proof.

$$\begin{aligned}
a > b \implies \\
a = b + p \implies \\
ac = bc + pc
\end{aligned}$$

According to Lemma 9, pc is positive. Therefore, $ac > bc$. □

Corollary 2. *Cancellation law.*

$$ac = bc \wedge c \neq 0 \implies a = b$$

Proof. Either $a = b$, or $a < b$, or $a > b$. Suppose that $a \neq b$. Therefore, ac either $<$ or $>$ bc, which, according to Proposition 10, gives a contradiction. So $a = b$. □

Proposition 11. *(Exercise 2.3.4)*

$$(a + b)^2 = a^2 + 2ab + b^2$$

(Suppose that we have known $n^2 = n \times n$)

Proof.

$$\begin{aligned}
(a+b)(a+b) &= (a+b)a + (a+b)b && \text{(Distribution law)} \\
&= a(a+b) + b(a+b) && \text{(Commutativity)} \\
&= a^2 + ab + ba + b^2 && \text{(Distribution law)} \\
&= a^2 + ab + ab + b^2 && \text{(Commutativity)}
\end{aligned}$$

Now we prove that $2ab = ab + ab$.

$$\begin{aligned}
2ab &= S(1)ab \\
&= (1ab) + ab \\
&= (S(0)ab) + ab \\
&= (0ab + ab) + ab \\
&= ab + ab
\end{aligned}$$

The proof is over. □

Proposition 12. *(Exercise 2.3.5) Euclidean algorithm. For any natural number n, positive number p, there exist natural numbers m, r such that $n = mp + r$.*

Proof. For any natural number p, we induct on n. Firstly, $0 = 0p + 0$.

Assume that the statement for n is true. We know that $r < p$. Then $S(r)$ either $=$ or $< p$ (Proposition 2.2). On the latter case, simply let $r' = S(r),\ m' = m$, which satisfies the restriction $0 \leq r' < p$

On the former case, let $m' = S(m),\ r' = 0$, and we have

$$\begin{aligned}
m'p + r' &= S(m)p + 0 \\
&= mp + p \\
&= mp + S(r) && (p = S(r)) \\
&= S(mp + r) \\
&= S(n)
\end{aligned}$$

. And now we can close the induction. □

Part II
Set Theory

3 Fundamentals

Exercise 3.1.1

Proof. Reflexive: $\forall x \in S, x \in S$.

Symmetric:

$$\begin{aligned} X = Y \iff \\ \forall x \in X, x \in Y \land \forall x \in Y, x \in X \iff \\ Y = X \end{aligned}$$

Transitive: $X = Y \implies \forall x \in X, x \in Y$. Because $x \in Y$ and $Y = Z$, we can conclude that $\forall x \in X, x \in Z$. Conduct the process from inversely, we can get $\forall x \in Z, x \in X$. Therefore, $X = Z$. □

The reason for the content beneath Axiom 3.2 is clearly demonstrated in the proof of Lemma 3.1.6.

In Remarks 3.1.9, there are three "Why"s. The reason can be concluded as: Because of the "if and only if" in Axiom 3.3, or more precisely, "only if", if x is a element in one of such sets, x must $= a$ or b. And because of the "if", x is thus in another set. So the two sets are equal according to Definition 3.1.4.

Exercise 3.1.2

Proof. According to Axiom 3.2, $\varnothing$ exists, and is thus an object as stated by Axiom 3.1. Therefore, by Axiom 3.3, $\{\varnothing\}$ also exists. $\varnothing$ is an element of $\{\varnothing\}$, but it is not an element of $\varnothing$ because any object $\notin \varnothing$.

For the same reason, any set that contains element(s) is not the same set as $\varnothing$. Furthermore, there exists an object $\{\varnothing\}$ (Axiom 3.3 and 3.1), which is an element of $\{\varnothing, \{\varnothing\}\}$, but which is not an element of $\{\varnothing\}$. So the two sets are not equal. □

Remarks 3.1.12

Proof. Let $x \in A' \cup B$. $x \in A' \implies x \in A$ And if $x \notin A'$, $x \in B$. So either way $x \in A \cup B$ and vice versa. □

Exercise 3.1.3

Proof. (1)

$$x \in A \cup B \equiv (x \in A \vee x \in B)$$

$$x \in A \Longrightarrow x \in B \cup A$$

$$x \in B \Longrightarrow x \in B \cup A$$

So $x \in A \cup B \Longrightarrow x \in B \cup A$. And vice versa.

(2) $x \in A \Rightarrow x \in A \cup A$ and $x \in A \cup A \Rightarrow x \in A$.

(3)

$$\begin{aligned} x \in A \cup \varnothing &\Longrightarrow \\ x \in A \vee x \in \varnothing &\Longrightarrow \\ x \in A & \qquad (\forall a, a \notin \varnothing) \end{aligned}$$

And obviously $x \in A \Rightarrow x \in A \cup \varnothing$. So $A \cup \varnothing = A$.

By transitivity of equality, and commutativity of pairwise union, we can conclude the others. □

Examples 3.1.17

Proof.

$$\forall x(x \in A \Longrightarrow x \in A)$$

And

$$\forall x(x \in \varnothing \Longrightarrow x \in A)$$

is vacuously true. □

Exercise 3.1.4

Proof. (1) On one hand,

$$A \subseteq B \equiv \forall x(x \in A \Longrightarrow x \in B)$$

. On the other hand,

$$B \subseteq A \equiv \forall x(x \in B \Longrightarrow x \in A)$$

. Thus $A = B$.

(2) First, we prove that $A \subsetneq B \Longrightarrow \exists x(x \in B \wedge x \notin A)$. Suppose the contradiction, that is, $\forall x(x \in B \Longrightarrow x \in A)$, which is impossible since $(A \subseteq B \equiv \forall x(x \in A \Longrightarrow x \in B)) \wedge A \neq B$.

According to what's proven in the book, $A \subsetneq B \wedge B \subsetneq C \Longrightarrow A \subseteq C$.

Now we prove that $\exists x(x \in C \wedge x \notin A)$. Since $x \in A \Longrightarrow x \in B$, $x \notin B \Longrightarrow x \notin A$. Because $B \subsetneq C$, $\exists x(x \in C \wedge x \notin B)$, and thus for such x, $x \notin A$. Then $A \neq C$.

So $A \subsetneq C$. □

Axiom 3.5 (1) Because $x \in \{x \in A : P(x)\} \Rightarrow x \in A$.

(2) Because both $\in$ and $P(x)$ obey the axiom of substitution.

Exercise 3.1.5

Proof. First we prove that $A \subseteq B \equiv A \cup B = B$. On one hand,

$$\begin{aligned} A \subseteq B \equiv \\ \forall x(x \in A \Longrightarrow x \in B) \Longrightarrow \\ \forall x((x \in A \vee x \in B) \Longrightarrow x \in B) \equiv \\ A \cup B = B \end{aligned}$$

.

On the other hand,

$$\forall x((x \in A \vee x \in B) \Longrightarrow x \in B) \Longrightarrow \forall x(x \in A \Longrightarrow x \in B)$$

. The statement is therefore proven.

Then we prove that $A \subseteq B \equiv A \cap B = A$. On one hand,

$$\begin{aligned} (A \cap B = A \equiv \forall x(x \in A \wedge x \in B \equiv x \in A)) \Longrightarrow \\ (\forall x(x \in A \Rightarrow x \in B) \equiv (A \subseteq B)) \end{aligned}$$

.

On the other hand,

$$\forall x(x \in A \wedge x \in B \Longrightarrow x \in A)$$

is always true (Vacuously true if $x \notin B$).

Logical equality is transitive, and thus all of the three statements are equal. □

Proposition 3.1.28 (Exercise 3.1.6)

Proof. (a) The two are identical to

$$\forall x(x \in A \vee x \in \varnothing \equiv x \in A)$$

, and

$$\nexists x(x \in A \wedge x \in \varnothing)$$

, which are all true since $\forall x(x \notin \varnothing)$.

(b) We have $A \subseteq X$. According to what we have proven in Exercise 3.1.5, the two statements are all true.

(c) Obvious since

$$\forall x(x \in A \vee x \in A \equiv x \in A)$$

and

$$\forall x(x \in A \wedge x \in A \equiv x \in A)$$

(d) All true since *logical or* and *logical and* are commutative.

(e) See Lemma 3.1.13. I believe that this can be concluded by the fact that *logical or* and *logical and* are also associative.

(f) First we prove the latter. On one hand, suppose

$$x \in A \cup (B \cap C)$$

is ture.

If $x \in A$, then $x \in$ both $A \cup B$ and $A \cup C$, and thus $\in (A \cup B) \cap (A \cup C)$.

If $x \notin A$, then $x \in B \cap C$, then $x \in$ both $A \cup B$ and $A \cup C$, and thus $\in (A \cup B) \cap (A \cup C)$.

On the other hand, suppose

$$x \in (A \cup B) \cap (A \cup C)$$

is true.

If $x \in A$, obviously $x \in A \cup (B \cap C)$.

If $x \notin A$, then x must $\in B \cap C$, and thus also $\in A \cup (B \cap C)$.

Now we prove the former. On one hand, suppose

$$x \in A \cap (B \cup C)$$

is true.

If $x \in A \wedge x \in B$, then $x \in A \cap B$, and thus $\in (A \cap B) \cup (A \cap C)$.

If $x \notin A \vee x \notin B$, then

1. if $x \notin A$, this is impossible.
2. if $x \in A$, then $x \notin B$. But $x \in B \cup C$, so $x \in C$. And thus $x \in A \cap C \Rightarrow x \in (A \cap B) \cup (A \cap C)$.

On the other hand, suppose that

$$x \in (A \cap B) \cup (A \cap C)$$

is true.

First we can see that $x \in A$.

If $x \in B$, then $x \in B \cup C$, and thus $\in A \cap (B \cup C)$.

If $x \notin B$, then $x \in C$. So $x \in B \cup C$, and thus $\in A \cap (B \cup C)$.

(g) Now we prove the former: On one hand, suppose that

$$x \in A \cup (X - A)$$

.

If $x \in A$, then $x \in X$ since $A \subseteq X$.

If $x \notin A$, then $x \in X - A$, and thus also $\in X$.

On the other hand, suppose that

$$x \in X$$

If $x \in A$, then $x \in A \cup (X - A)$.

If $x \notin A$, then $x \in X - A$, and thus $\in A \cup (X - A)$.

(h) $x \in X - A$ requires $x \notin A$. So $\forall x(x \in A \cap (X - A))$ is always false. Thus

$$\forall x(x \in A \cap (X - A) \Longleftrightarrow x \in \varnothing)$$

(vacuously true). □

Exercise 3.1.7

Proof. (1) $\forall x(x \in A \cap B \Longrightarrow x \in A)$. Similarly, we can prove that $A \cap B \subseteq B$. (This can also be achieved via the commutativity).

(2) On one hand, suppose that

$$C \subseteq A \wedge C \subseteq B$$

is true. Then,

$$\forall x(x \in C \Longrightarrow x \in A \wedge x \in B \Longrightarrow x \in A \cap B)$$

.

On the other hand, suppose that

$$C \subseteq A \cap B$$

is true. Then,

$$\forall x(x \in C \Longrightarrow x \in A \wedge x \in B)$$

. That is, $C \subseteq A \wedge C \subseteq B$.

(3) It is immediately given by

$$\forall x(x \in A \Longrightarrow x \in A \cup B)$$

. Since $\cup$ is commutative, the latter case is proven.

(4) On one hand, suppose that $A \subseteq C \wedge B \subseteq C$ and let $x \in A \cup B$.

If $x \in A$, then $x \in C$.

If $x \notin A$, then $x \in B$, and thus $x \in C$.

On the other hand, suppose that $A \cup B \subseteq C$. Then,

$$\forall x(x \in A \Longrightarrow x \in A \cup B \Longrightarrow x \in C)$$

$$\forall x(x \in B \Longrightarrow x \in A \cup B \Longrightarrow x \in C) \qquad \square$$

.

Exercise 3.1.8

Proof. The former: On one hand, Suppose that

$$x \in A \cap (A \cup B)$$

.

If $x \in A$, then $x \in A$.

If $x \notin A$, this is impossible.

On the other hand, suppose that $x \in A$. Then $x \in A \wedge x \in (A \cup B)$, so $x \in A \cap (A \cup B)$.

The latter: On one hand, suppose that $x \in A \cup (A \cap B)$.

$$x \in A \Longrightarrow x \in A.$$

$$x \notin A \Longrightarrow x \in (A \cap B) \Longrightarrow x \in A$$

.

On the other hand, Suppose that $x \in A$, then $x \in A \cup (A \cap B)$. $\square$

Exercise 3.1.9

Proof.

Lemma 10.

$$\nexists x \forall B \forall A (x \in A \wedge x \in B \wedge A \cap B = \varnothing)$$

Proof. Suppose the contradiction: $x \in A \wedge x \in B \wedge A \cap B = \varnothing$, then $x \in A \cap B$, and thus $\in \varnothing$, which is impossible. □

The former: On one hand, suppose that $x \in A$. Then $x \notin B$ by Lemma 10. And

$$x \in A \Longrightarrow x \in A \cup B \Longrightarrow x \in X$$

. So $x \in (X - B)$.

On the other hand, suppose that $x \in (X - B)$, then $x \in A \cup B$. But $x \notin B$, so $x \in A$ by Lemma 10.

The latter is immediately proven since $\cap, \cup$ are commutative. □

Exercise 3.1.10

Proof. Firstly we prove that $(A - B) \cap (A \cap B) = \varnothing$.

$x \in (A \cap B)$ gives $x \in B$, but $x \in (A - B)$ gives $x \notin B$. So the two statements can not be true simultaneously. Which means

$$x \in (A - B) \cap (A \cap B) \Longrightarrow x \in \varnothing$$

And obviously

$$x \in (A - B) \cap (A \cap B) \Longleftarrow x \in \varnothing$$

.

Similarly we can conclude all of the three sets are disjoint by the fact that $\nexists x \in$ either two of the three sets.

Now we are showing that their union is $A \cup B$.

On one hand, suppose that

$$x \in (A - B) \cap (A \cap B) \cap (B - A)$$

. x can at most be in one of these sets since they are disjoint. If $x \in A$, then $x \in A \cup B$.

If $x \notin A$, then $x \in (B - A)$, and thus $x \in B$. So $x \in A \cup B$.

On the other hand, suppose that $x \in A \cup B$. Then x either

1. $\in A$, but $\notin B$, or
2. $\in B$, but $\notin A$, or
3. $\in$ both A, B.

If (1), then $x \in (A - B)$.
If (2), then $x \in (B - A)$.
If (3), then $x \in A \cap B$.
In conclusion, we can see that $x \in (A - B) \cap (A \cap B) \cap (B - A)$. □

Exercise 3.1.11

Proof. Let S be a set. Let $P(x, y)$ be a property pertaining to $x \in S$ and any object y, and is true iff $Q(x) \wedge y = x$, where $Q(x)$ is a property pertaining to $x \in S$.

According to Axiom 3.6, there exists a set Z, such that $y \in Z \equiv x \in S \wedge P(x, y)$, which means $y \in Z \equiv x \in S \wedge Q(x) \wedge x = y$. So is the axiom of specification proven. □

4 Russell's paradox

I think one major reason for building such a "cumbersome" axiom system is to restrict the way to construct sets. We can not construct just any set we want, there only exist certain kinds of sets.

Exercise 3.2.1

Proof. (Axiom 3.2) To prove the existence of the empty set, simply choose a property that is false for all objects.

(Axiom 3.3) To prove the existence of a *pair set*, say $\{a, b\}$, let $P(x)$ be a property pertaining to any object x, and is true iff $x = a \vee x = b$.

(Axiom 3.4) Let the property be $P(x) : x \in A \vee x \in B$.

(Axiom 3.5) Let the property be $Q(x) : x \in A \wedge P(x)$, where $P(x)$ is a property pertaining to elements of A.

(Axiom 3.6) Let the property be $Q(y) : P(x, y)$ is true for some $x \in A$. □

Exercise 3.2.2

Proof. (1) Suppose the contradiction: $\exists A(A \in A)$. Then by Axiom 3.3, construct a set $B := \{A\}$. A is the only element in B. A is a set. A is not disjoint from B, for $A \in A \wedge A \in B$.

(2) Suppose the contradiction: $A \in B \wedge B \in A$. Construct a set $S : \{A, B\}$. A is an element of S. A is a set. A is not disjoint from S, for $B \in A \wedge B \in S$. □

Exercise 3.2.3 On one hand, if Axiom 3.8 is true, we can choose a property $P(x)$ which is true for all objects. Thus we have Ω.

On the other hand, if there exists such a set as Ω, we can use Axiom 3.5 to construct any set we want from it. (e.g. If we want a set to have these elements: $a, b, \ldots$, we can let $P(x) := x = a \vee x = b, \vee \ldots.$)

5 Functions

In Example 3.3.3, Tao asked why $x' = x \Rightarrow f(x') = f(x)$. The reason is, the property $P(x, y)$ obeys the axiom of substitution. Thus, $P(x, y) \equiv P(x', y)$. According to definition, since $x' \in X$, y is unique.

In Example 3.3.9, Tao asked why all functions whose domain is $\varnothing$ and whose range is the same are equal. The reason is $x \in \varnothing \Longrightarrow f(x) = g(x)$ is vacuously true.

Exercise 3.3.1

Proof. The properties of equality are all true since in definition, we only use $f(x) = g(x)$, in which the $=$ obeys these rules, plus the fact that the output is unique.

Then the substitution:

$$\begin{aligned} f = \widetilde{f} &\Longrightarrow \\ f(x) = \widetilde{f}(x) &\Longrightarrow \\ g(f(x)) = g(\widetilde{f}(x)) & \end{aligned}$$

. And then $\widetilde{g}(\widetilde{f}(x)) = g(\widetilde{f}(x)) = g(x)$. □

Exercise 3.3.2

Proof. The former: Suppose the contradiction:

$$\exists x \exists x' (g(f(x)) = g(f(x')) \land x \neq x')$$

Then,

$$\begin{aligned} g(f(x)) = g(f(x')) &\implies \\ f(x) = f(x') &\implies && (g \text{ is injective}) \\ x = x' && && (f \text{ is injective}) \end{aligned}$$

, which is impossible.

The latter: Suppose the contradiction:

$$\exists z \forall x (z \in Z \land g \circ f(x) \neq z)$$

Then, we can conclude that $\exists y \forall x (y \in Y \land y \neq f(x))$, since g is surjective. This is impossible as f is surjective. □

Exercise 3.3.3

Proof. **Attention:** Different interpretations for injectivity may result in different conclusions. I have asked a question at Stack Exchange regarding this problem.

Let the range be Y, and the function be f. Injectivity:

$$\forall x' \forall x ((x \in \varnothing \land x' \in \varnothing) \implies (x \neq x' \implies f(x) \neq f(x')))$$

, which is always vacuously true.

Surjectivity:

$$\forall y (y \in Y \implies \exists x (x \in \varnothing \land f(x) = y))$$

, which is false if $Y \neq \varnothing$, and which is vacuously true if $Y = \varnothing$.

Bijective: True if $Y = \varnothing$. □

Exercise 3.3.4

Proof. The former: $f, \widetilde{f}$ have the same range and domain.

$$\forall x (g \circ f = g \circ \widetilde{f} \implies g(f(x)) = g(\widetilde{f}(x)))$$

We know that g is injective, so $\forall x \in X, f(x) = \widetilde{f}(x)$. Thus $f = \widetilde{f}$.

It is not true if g is not injective. Consider an extreme condition, where g is constant. So whatever $f, \widetilde{f}$ are, $g \circ f = g \circ \widetilde{f}$ are always equal.

The latter: Suppose the contradiction:$g \neq \widetilde{g}$. $g, \widetilde{g}$ have the same range and domain. But they are not equal, so $\exists y(y \in Y \wedge g(y) \neq \widetilde{g}(y))$. Because f is surjective, $\exists x(x \in X \wedge f(x) = y)$. However, $g \circ f(x) = \widetilde{g} \circ f(x)$, so this is impossible.

It is not true if f is not surjective. We can make $g(y) = \widetilde{g}(y)$ when $y = f(x)$, but as well make $g(y') \neq \widetilde{g}(y')$ if $\nexists x(y' = f(x))$. □

Exercise 3.3.5

Proof. Injectivity: Suppose the contradiction, that

$$\exists x \exists x'(x \neq x' \wedge f(x) = f(x'))$$

, which immediately gives

$$g(f(x)) = g(f(x'))$$

, and thus is impossible.

g has not to be also injective, because f being so ensures that an unique input x gives an unique input to g.

Surjectivity: If g is not surjective, then $\exists z \forall y(z \in Z \wedge y \in Y \wedge z \neq g(y))$ And whatever x is, $f(x) \in Y$, so $g(f(x)) \neq z$, which is a contradiction.

f has not to be surjective as long as its "real" domain is large enough to form the set Z through g. For example (Informal), let g be $z = |y|, \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$, and let f be $y = x, \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$. □

Exercise 3.3.6

Proof. The latter: By definition, $P(y, x)$ of $x = f^{-1}(y)$ is $f(x) = y$. Substitute x with $f^{-1}(y)$, and here we have $f(f^{-1}(y)) = y$, where $y \in Y$.

The former: Let $y = f(x)$. According to what we have proven, $f(f^{-1}(y)) = y$. Substitute y with $f(x)$, we have $f(f^{-1}(f(x))) = f(x)$. Since that $f(x)$ is injective, we have $f^{-1}(f(x)) = x$.

Now we need to show that f^{-1} is bijective. Assume that it is not injective, thus $\exists x \exists x'(x \in Y \wedge x' \in Y \implies (x \neq x' \implies f^{-1}(x) = f^{-1}(x')))$. However, according to the latter conclusion, $f^{-1}(x) = f^{-1}(x') \implies x = x'$, a contradiction, so f^{-1} must be injective.

And it is also surjective. $\forall x \in X$, $\exists y \in Y, f^{-1}(y) = x$. According to the former conclusion, y is $f(x)$.

So now f^{-1} is bijective, and thus has its inverse. By definition, $P(x, y)$ of $y = (f^{-1})^{-1}(x)$ is $f^{-1}(y) = x$, where $x \in X$. According to the former conclusion, $f^{-1}(f(x)) = x$. Thus

$$f^{-1}(y) = f^{-1}(f(x)) \Longrightarrow y = f(x) \Longrightarrow (f^{-1})^{-1}(x) = f(x)$$

, which is true $\forall x \in X$. And since they have the same domain and range, $(f^{-1})^{-1} = f$. □

Exercise 3.3.7

Proof. Injectivity:

$$g \circ f(x) = g \circ f(x') \Longrightarrow f(x) = f(x') \Longrightarrow x = x'$$

Surjectivity: For each $z \in Z$, we need to find $x \in X$ such that $g \circ f(x) = z$. By the surjectivity of g, we can find $y \in Y$ such that $g(y) = z$. We can also find $a \in X$ such that $f(a) = y$ as f is surjective. So a is our desired x.

The $P(z, x)$ of $x = (g \circ f)^{-1}(z)$ is $z = g \circ f(x)$. Consider the following expression:

$$\begin{aligned} f^{-1} \circ g^{-1}(z) &= f^{-1} \circ g^{-1}(g \circ f(x)) \\ &= f^{-1}(g^{-1}(g(f(x)))) \\ &= f^{-1}(f(x)) \\ &= x \end{aligned}$$

So $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Therefore, they are equal as they have the same domain and range. □

Exercise 3.3.8

Proof. (a) First they have the same domain and range. Finally,

$$\forall x(x \in X \Longrightarrow x = x \Longrightarrow \iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z})$$

(b) On one hand, they have the same domain and range.

On the other hand,

$$\begin{aligned} f \circ \iota_{A \to A}(x) &= f(\iota_{A \to A}(x)) \\ &= f(x) \\ &= \iota_{B \to B}(f(x)) \\ &= \iota_{B \to B} \circ f(x) \end{aligned}$$

(c) It is easy to see that they have the same domain and range.

$$f \circ f^{-1}(b) = b = \iota_{B\to B}$$
$$f \circ f^{-1}(a) = a = \iota_{A\to A}$$

(d) It is easy to see that they have the same domain and range.
Let h be $h(x) = f(x)$, if $x \in X$, $h(x) = g(x)$, if $x \in Y$.
For each $x \in X$, $\iota_{X\to X\cup Y}(x) = x$, so $h(\iota_{X\to X\cup Y}(x)) = f(x)$.
Similarly we can prove $h(\iota_{Y\to X\cup Y}(x)) = g(x)$ for each $x \in Y$. □

6 Images and inverse Images

Definition 3.4.1 To prove that $f(S)$ is well-defined by using the axiom of specification, we need to apply it to set Y, not X. Let $P(y)$ be a property pertaining to each $y \in Y$, which is true iff $\exists x(x \in S \wedge f(x) = y)$. According to the axiom of specification, there exists a set that contains every $y \in Y$ such that $P(y)$ is true.

In some places where Tao asked "(Why?)", the reason is obvious, so I don't write them here.

Example 3.4.6 This is because

$$f^{-1}(f(\{-1, 0, 1, 2\})) = f^{-1}(\{1, 0, 4\}) = \{-1, 1, 0, 2, -2\}$$

.

More generally, if f (whose domain is X, and whose range is Y) is not injective, then

$$\exists x \exists x'((x \in X \wedge x' \in X) \wedge (x \neq x' \wedge f(x) = f(x')))$$

. Let $D \subseteq X$ such that $x \in D \wedge x' \notin D$. Then $f(x) = f(x') \in f(D)$. And thus $x, x' \in f^{-1}(f(D)) \Longrightarrow f^{-1}(f(D)) \neq D$.

Exercise 3.4.1

Proof. $f^{-1}(V)$ may be interpreted in two different ways:
(1) Interpret $f^{-1}(V)$ as an inverse image, that is,

$$(\forall x \in X)(x \in f^{-1}(V) \equiv f(x) \in V)$$

$$(\forall x \notin X)(x \notin f^{-1}(V))$$

(2) Interpret $f^{-1}(V)$ as an image, where we regard f^{-1} as a function. So,

$$\forall x(\exists y(y \in V \wedge x = f^{-1}(y)) \equiv x \in f^{-1}(V))$$

We need to show that if the two statements are well-defined($x \in X$) , they are logically equivalent.

Let S_1 be the set defined in form (1), S_2 be the set defined in form (2). For every $x \in S_1$, $f(x) \in V$. Let $y' = f(x), x' = f^{-1}(y')$, then by definition (2) we have $x' \in S_2$. But $x' = x$, so $\forall x(x \in S_1 \Longrightarrow x \in S_2)$.

On the other hand, for every $a \in S_2$, $\exists b \in V$, such that $a = f^{-1}(b)$. Then $a \in X$. $f(a) = f(f^{-1}(b)) = b \in V$, so $a \in S_1$. Thus, $S_1 = S_2$. □

Exercise 3.4.2 (1) Generally we can say $S \subseteq f^{-1}(f(S))$ but we cannot say that they are equal; (2) we can say $f(f^{-1}(U)) \subseteq U$ but we cannot say that they are equal.

Proof. (1) $x \in S \Longrightarrow f(x) \in f(S) \Longrightarrow x \in f^{-1}(f(S))$. However, it is possible that $\exists x(x \in X \wedge x \notin S \wedge f(x) \in f(S))$

(2)

$$x \in f^{-1}(U) \Longrightarrow f(x) \in U \Longrightarrow (y \in f(f^{-1}(U)) \Longrightarrow y \in U)$$

However, it is still possible that

$$\exists y(y \in U \wedge \forall x(x \in X \Longrightarrow f(x) \neq y))$$

□

Exercise 3.4.3

Proof. (1)

$$x \in A \cap B \Longrightarrow f(x) \in f(A) \wedge f(x) \in f(B) \Longrightarrow f(x) \in f(A) \cap f(B)$$

$$y \in f(A \cap B) \equiv \exists x(x \in A \cap B \wedge f(x) = y)$$

So $y \in f(A) \cap f(B)$, thus $f(A \cap B) \subseteq f(A) \cap f(B)$.

(2)

$$x \in A \setminus B \Longrightarrow f(x) \in f(A \setminus B)$$

$$y \in f(A) \setminus f(B) \Longrightarrow \exists x(x \in A \wedge x \notin B \wedge f(x) = y)$$

So $y \in f(A \setminus B)$, thus $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

(3) On one hand,

$$y \in f(A \cup B) \equiv \exists x(x \in A \cup B \wedge f(x) = y)$$

$$x \in A \cup B \Longrightarrow x \in A \vee x \in B \Longrightarrow$$
$$f(x) \in f(A) \vee f(x) \in f(B) \Longrightarrow f(x) \in f(A) \cup f(B)$$

On the other hand,

$$y \in f(A) \cup f(B) \Longrightarrow \exists x((x \in A \vee x \in B) \wedge f(x) = y)$$

$$x \in A \vee x \in B \Longrightarrow x \in A \cup B \Longrightarrow f(x) \in f(A \cup B)$$

□

(1) $\subseteq$ can not be improved. Since it is possible that

$$\exists x \exists x'(x \in A \wedge x' \in B \wedge x \neq x' \wedge f(x) = f(x'))$$

(2) $\subseteq$ can not be improved. Since it is possible that

$$\exists x \exists x'(x \in A \setminus B \wedge x' \in B \wedge f(x) = f(x'))$$

Exercise 3.4.4

Proof. (1)

$$x \in f^{-1}(U \cup V) \equiv (x \in X \wedge f(x) \in U \cup V) \equiv$$
$$(x \in X \wedge (f(x) \in U \vee f(x) \in V))$$

$$x \in f^{-1}(U) \cup f^{-1}(V) \equiv (x \in X \wedge f(x) \in U) \vee (x \in X \wedge f(x) \in V)$$
$$\equiv (x \in X \wedge (f(x) \in U \vee f(x) \in V))$$

(2) and (3) can be proven in similar manners. □

Exercise 3.4.5

Proof. (1) On one hand, if $f(f^{-1}(S)) = S$ for every $S \subseteq Y$, then $f(f^{-1}(Y)) = Y$. That means, $y \in Y \Longrightarrow \exists x(x \in f^{-1}(Y) \wedge f(x) \in Y)$. So even $f^{-1}(Y)$ is enough for f to be surjective. And $f^{-1}(Y) \subseteq X$, so f is surjective.

On the other hand, if f is surjective, then for each $S \subseteq Y$,

$$y \in S \Longrightarrow \exists x(x \in X \wedge f(x) = y)$$

Such x are elements of $f^{-1}(S)$ of course, so $f(f^{-1}(S)) = S$.

(2) On one hand, we show that $\forall S(S \subseteq X \Longrightarrow f^{-1}(f(S)) = S)$ implies that f is injective. Suppose the contradiction, that when

$$\forall S(S \subseteq X \Longrightarrow f^{-1}(f(S)) = S)$$

, but f is not injective. Since f is not injective,

$$\exists x \exists x'(x \in X \wedge x' \in X \wedge x \neq x' \wedge f(x) = f(x'))$$

Let $S \subseteq X$ and $x \in S \wedge x' \notin S$. There is always such a set S. For example, we can let $S = X \setminus \{x'\}$. So we have $f^{-1}(f(S)) \neq S$ because $x' \in$ it.

On the other hand, if f is injective, then for every $S \subseteq X$, and for every $x \in S$, we have $f(x) \in f(S)$. And $f(x)$ are the only elements in $f(S)$, that is, $y \in f(S) \Longrightarrow y = f(x)$ for some $x \in S$. So now we know that $S \subseteq f^{-1}(f(S))$. Moreover, for every $x' \in f^{-1}(f(S))$, $f(x') \in f(S)$. We can let $f(x') = y = f(x)$. As f is injective, $x = x'$, so $x' \in S$. That means $f^{-1}(f(S)) \subseteq S$. So $f^{-1}(f(S)) = S$. □

Exercise 3.4.6

Proof. My own proof: According to Axiom 3.10, we can construct the set X^X. Apply the axiom of replacement to each element of X^X, we construct a set Z such that

$$\forall x(x \in Z \equiv \exists f(f \in X^X \wedge x = f(X)))$$

Let $Y = \{\varnothing\} \cup Z$.

Now we prove that Y is the set we want. On one hand, for any $S \subseteq X$, if $S = \varnothing$, then $S \in Y$, as $Y = \{\varnothing\} \cup Z$.

If $S \neq \varnothing$, there exists a surjective function $g : X \to S$. $g \in X^X$, and $g(X) = S$, so $S \in Z$, and thus $S \in Y$. (To show the existence of g, for

example, let $x \in X, g(x) = x$ if $x \in S$, and for $x \in X \wedge x \notin S$, $g(x)$ can be any element of X.)

On the other hand, for any $S' \nsubseteq X$, $\exists a(a \in S' \wedge a \notin X)$. To prove that $S' \notin Y$, we need to show that $\nexists f(f \in X^X \wedge S' = f(X))$. We know that $\nexists x(x \in X \wedge f(x) = a)$, so $a \notin f(X)$. Therefore $S' \neq f(X)$, so $S' \notin Y$.

Y is the set we want.

I posted a question here for verification for this proof. Thanks to answers of people at Stack Exchange so that my proof can be refined.

Proof By Tao's Hint: For each $S \subseteq X$, let a function f_S be $f_S(x) = 1$ if $x \in S$, and $f_S(x) = 0$ if $x \in X \wedge x \notin S$. Then $f_S^{-1}(\{1\})$ gives S.

Now we show that any element in $\{0,1\}^X$ is some f_S. Let $g \in \{0,1\}^X$. Then if $\forall x \in X, g(x) = 0$, then $g = f_\varnothing$. Otherwise, there exists a set that contains all x such that $g(x) = 0$ by axiom of specification, namely R. Then $g = f_R$.

On the other hand, each f_S is obviously an element of $\{0,1\}^X$.

Use the axiom of replacement, we construct a set Y such that

$$\forall x(x \in Y \equiv \exists f(f \in \{0,1\}^X \wedge x = f^{-1}(\{1\})))$$

According to what we have proven, Y is the set we want. □

Exercise 3.4.7

Proof. As stated by the previous exercise, there exists a set $\mathbb{X}$ whose elements are all subsets of X, and a set $\mathbb{Y}$ whose elements are all subsets of Y.

For every element $x \in \mathbb{X}$, apply the axiom of replacement to $\mathbb{Y}$, to obtain a set $S_x := \{y^x\}$ for every element $y \in \mathbb{Y}$.

According to the axiom of union, using $\mathbb{X}$ as the index set, we have the set

$$Z = \bigcup_{x \in \mathbb{X}} S_x$$

Apply again the axiom of union to Z to obtain R, which contains all elements of elements of Z. Now we show that R is the set we want.

On one hand, let f be an arbitrary function with the domain of $X' \subseteq X$, and the range of $Y' \subseteq Y$. We can see that $f \in {Y'}^{X'} \in S_{X'}$. ${Y'}^{X'}$ becomes an element of Z. And thus f becomes an element in R.

On the other hand, from the construction of R, we can see that R contains only these elements.

□

Exercise 3.4.8

Proof. Let A, B be two arbitrary sets. They are also objects as stated by Axiom 3.1. So according to Axiom 3.3, there exists a set $S = \{A, B\}$. By Axiom 3.11, we have a set Z such that

$$\forall x(x \in Z \equiv \exists X(X \in S \wedge x \in X))$$

Now we show that Z is the set we want. If $x \in A \vee x \in B$, then $\exists X(X \in S \wedge x \in X)$ is true. So $x \in Z$.

If $x \notin A \wedge x \notin B$, then $\forall X(X \in S \Longrightarrow x \notin X)$, that is, $\exists X(X \in S \wedge x \in X)$ is false. So $x \notin Z$.

Z is therefore the set we want. □

Example 3.4.11 In (3.3), why do Tao choose some element β of I? This is because we need to apply the axiom of specification to A_β with the restriction $x \in A_\alpha$ for all $\alpha \in I$.

Exercise 3.4.9

Proof. This is quiet easy to prove. Let the left-handed side set be S, the RHS set be S'. For any $x \in S$, $x \in A_\alpha$ for all $\alpha \in I$. So $x \in A_{\beta'}$. And $x \in A_\alpha$ for all $\alpha \in I$. Therefore $x \in S'$.

It is nearly the same the prove $x \in S' \Longrightarrow x \in S$. □

Exercise 3.4.10

Proof. For the sake of convenience, let $(\bigcup_{\alpha\in I} A_\alpha) \cup (\bigcup_{\alpha\in J} A_\alpha)$ be S, $\bigcup_{\alpha\in I\cup J} A_\alpha$ be S', $(\bigcap_{\alpha\in I} A_\alpha) \cap (\bigcap_{\alpha\in J} A_\alpha)$ be Z, $\bigcap_{\alpha\in I\cup J} A_\alpha$ be Z'.

(1) When $I, J \neq \varnothing$: On one hand,

$$x \in S \Longrightarrow (x \in \bigcup_{\alpha\in I} A_\alpha \vee x \in \bigcup_{\alpha\in J} A_\alpha)$$

If $x \in \bigcup_{\alpha\in I} A_\alpha$, then $x \in \bigcup_{\alpha\in I\cup J} A_\alpha$. If $x \in \bigcup_{\alpha\in J} A_\alpha$, then $x \in \bigcup_{\alpha\in I\cup J} A_\alpha$.

On the other hand, if $x \in S'$, then there exists an object $a \in I \cup J$ such that $x \in A_a$. If $a \in I$ then $x \in x \in \bigcup_{\alpha\in I} A_\alpha \Longrightarrow x \in S$. If $a \in J$ then $x \in x \in \bigcup_{\alpha\in J} A_\alpha \Longrightarrow x \in S$.

When I, J are both empty, S, S' are all empty.

When there is only one of I, J is empty, say it is I, then $S = \varnothing \cup \bigcup_{\alpha\in J} = \bigcup_{\alpha\in J}$. And $S' = \bigcup_{\alpha\in\varnothing\cup J} A_\alpha = \bigcup_{\alpha\in J}$.

(2)

$$x \in Z \equiv \forall a(a \in I \Longrightarrow x \in A_a) \wedge \forall b(b \in J \Longrightarrow x \in A_b)$$

, which is equal to $\forall a(a \in I \cup J \Longrightarrow x \in A_a) \equiv x \in Z'$. □

Exercise 3.4.11

Proof. (1) Let the LHS be S, the RHS be S'.

$$\begin{aligned} x \in S \equiv \\ x \in X \wedge x \notin \bigcup_{\alpha \in I} A_\alpha \equiv \\ x \in X \wedge \forall a(a \in I \Longrightarrow x \notin A_a) \end{aligned}$$

$$\begin{aligned} x \in S' \equiv \\ \forall a(a \in I \Longrightarrow x \in X \setminus A_a) \equiv \\ x \in X \wedge \forall a(a \in I \Longrightarrow x \notin A_a) \end{aligned}$$

.

So $S = S'$.

(2) Let the LHS be Z, the RHS be Z'.

$$\begin{aligned} x \in Z \equiv \\ x \in X \wedge x \notin \bigcap_{\alpha \in I} A_\alpha \equiv \\ x \in X \wedge \neg(\forall a(a \in I \Longrightarrow x \in A_a)) \equiv \\ x \in X \wedge \exists a(a \in I \Longrightarrow x \notin A_a) \end{aligned}$$

$$\begin{aligned} x \in Z' \equiv \\ x \in X \wedge \bigvee_{\alpha \in I} (x \notin A_\alpha) \equiv \\ x \in X \wedge \exists a(a \in I \Longrightarrow x \notin A_a) \end{aligned}$$

Thus, $Z = Z'$ □

7 Cartesian products

Exercise 3.5.1

Proof. First we show that $(x, y) = \{\{x\}, \{x, y\}\}$ is a good definition. Let S_1 denote $(x_1, y_1) = \{\{x_1\}, \{x_1, y_1\}\}$, S_2 denote $(x_2, y_2) = \{\{x_2\}, \{x_2, y_2\}\}$.

On one hand, if $x_1 = x_2 \wedge y_1 = y_2$, then obviously $S_1 = S_2$ for they have the same elements.

On the other hand, if $S_1 = S_2$, then

$$\{x_1\} \in S_2 \wedge \{x_1, y_1\} \in S_2 \wedge \{x_2\} \in S_1 \wedge \{x_2, y_2\} \in S_1$$

. We have that

$$\begin{aligned} \{x_1\} \in S_2 &\equiv \{x_1\} = \{x_2\} \vee \{x_1\} = \{x_2, y_2\} \\ &\equiv x_1 = x_2 \vee x_1 = x_2 = y_2 \\ &\Longrightarrow x_1 = x_2 \end{aligned}$$

$$\begin{aligned} \{x_1, y_1\} \in S_2 &\equiv \{x_1, y_1\} = \{x_2\} \vee \{x_1, y_1\} = \{x_2, y_2\} \\ &\equiv x_1 = x_2 = y_1 \\ &\vee ((x_1 = x_2 \wedge y_1 = y_2) \vee (x_1 = y_2 \wedge y_1 = x_2)) \end{aligned}$$

Similarly we have that

$$\begin{aligned} \{x_2, y_2\} \in S_1 &\equiv x_2 = x_1 = y_2 \\ &\vee ((x_2 = x_1 \wedge y_2 = y_1) \vee (x_2 = y_1 \wedge y_2 = x_1)) \end{aligned}$$

We may notice that the red-colored text are two same statements. Thus from $\{x_1, y_1\} \in S_2$ and $\{x_2, y_2\} \in S_1$ we can always conclude that $y_1 = y_2$. Therefore, $S_1 = S_2 \Longrightarrow x_1 = x_2 \wedge y_1 = y_2$.

Then we show that if X, Y are two sets, then $X \times Y$ is also a set. For each element $x \in X$, construct a set S_x, where we replace each element $y \in Y$ with (x, y). Then construct the set $\bigcup_{x \in X} S_x$. □

Exercise 3.5.2

Proof. Since x, y are two functions, they are equal means that $\forall 1 \leq i \leq n$, $x(i) = y(i)$. That is, $x_i = y_i, 1 \leq i \leq n$.

Now we show that $\prod_{1 \leq i \leq n} X_i$ is a set. Let set F be the set that contains all partial functions from $N = \{i \in \mathbb{N} : 1 \leq i \leq n\}$ to $X = \bigcup_{1 \leq i \leq n} X_i$ (Exercise 3.4.7). Use the axiom of specification, select such elements f from F that:

1. the element is surjective, and
2. its domain is N, and
3. $f(i) \in X_i$

, and use all of them to construct a set Z, which is the set we want. □

Exercise 3.5.3

Proof. The definition is entirely based on the equality of objects (e.g. $x = x'$). The proof is immediately done since this equality is reflective ($x = x$), symmetric ($x = x' \equiv x' = x$), and transitive ($x_0 = x_1 \wedge x_1 = x_2 \implies x_0 = x_2$). □

Exercise 3.5.4

Proof. (1)

$$\begin{aligned}(x,y) \in A \times (B \cup C) &\equiv x \in A \wedge y \in (B \cup C) \\ &\equiv x \in A \wedge (y \in B \vee y \in C) \\ &\equiv (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \\ &\equiv ((x,y) \in A \times B) \vee ((x,y) \in A \times C) \\ &\equiv (x,y) \in (A \times B) \cup (A \times C)\end{aligned}$$

(2)

$$\begin{aligned}(x,y) \in A \times (B \cap C) &\equiv x \in A \wedge y \in (B \cap C) \\ &\equiv x \in A \wedge (y \in B \wedge y \in C) \\ &\equiv (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C) \\ &\equiv ((x,y) \in A \times B) \wedge ((x,y) \in A \times C) \\ &\equiv (x,y) \in (A \times B) \cap (A \times C)\end{aligned}$$

(3)

$$\begin{aligned}(x,y) \in A \times (B \setminus C) &\equiv x \in A \wedge y \in (B \setminus C) \\ &\equiv x \in A \wedge (y \in B \wedge \neg(y \in C)) \\ &\equiv (x \in A \wedge y \in B) \wedge \neg(x \in A \wedge y \in C) \\ &\quad \text{(The statement } x \in A \text{ implies } \neg(x \in A \wedge y \in C) \implies \neg(y \in C)) \\ &\equiv ((x,y) \in A \times B) \wedge \neg((x,y) \in A \times C) \\ &\equiv (x,y) \in (A \times B) \setminus (A \times C)\end{aligned}$$

□

Exercise 3.5.5

Proof. (1)

$$\begin{aligned}(x,y) \in (A \times B) \cap (C \times D) &\equiv (x,y) \in (A \times B) \wedge (x,y) \in (C \times D) \\ &\equiv (x \in A \wedge y \in B) \wedge (x \in C \wedge y \in D) \\ &\equiv (x \in A \wedge x \in C) \wedge (y \in B \wedge y \in D) \\ &\equiv x \in A \cap C \wedge y \in B \cap D \\ &\equiv (x,y) \in (A \cap C) \times (B \cap D)\end{aligned}$$

(2) It is not true since

$$\begin{aligned}(x,y) \in (A \times B) \cup (C \times D) &\equiv (x,y) \in (A \times B) \vee (x,y) \in (C \times D) \\ &\equiv (x \in A \wedge y \in B) \vee (x \in C \wedge y \in D) \\ &\not\equiv (x \in A \vee x \in C) \wedge (y \in B \vee y \in D)\end{aligned}$$

Generally

$$(x \in A \wedge y \in B) \vee (x \in C \wedge y \in D) \Longrightarrow (x \in A \vee x \in C) \wedge (y \in B \vee y \in D)$$

, but

$$(x \in A \vee x \in C) \wedge (y \in B \vee y \in D) \not\Rightarrow (x \in A \wedge y \in B) \vee (x \in C \wedge y \in D)$$

.

(3) It is not true since

$$\begin{aligned}(x,y) \in (A \times B) \setminus (C \times D) &\equiv (x,y) \in (A \times B) \wedge (x,y) \notin (C \times D) \\ &\equiv (x \in A \wedge y \in B) \wedge (x \notin C \vee y \notin D) \\ &\not\equiv (x \in A \wedge x \notin C) \wedge (y \in B \wedge y \notin D)\end{aligned}$$

□

Exercise 3.5.6

Proof. (1) On one hand, if $A \subseteq C$ and $B \subseteq D$, then

$$\begin{aligned}(x,y) \in A \times B &\equiv x \in A \wedge y \in B \\ &\Longrightarrow x \in C \wedge y \in D \\ &\Longrightarrow (x,y) \in C \times D\end{aligned}$$

, which means $A \times B \subseteq C \times D$.

On the other hand, if $A \times B \subseteq C \times D$, but we suppose that

$$\neg(A \subseteq C \wedge B \subseteq D)$$

. We only consider that $A \nsubseteq C$, the other situations are similar. Then $\exists x(x \in A \wedge x \notin C)$. Let $p = (x, y)$, where $y \in B$, then $p \in A \times B$. But $x \notin C$, so $p \notin C \times D$, a contradiction. Therefore,

$$A \times B \subseteq C \times D \Longrightarrow A \subseteq C \wedge B \subseteq D$$

(2) On one hand, if $A = C \wedge B = D$, then

$$\begin{aligned}(x, y) \in A \times B &\equiv x \in A \wedge y \in B \\ &\equiv x \in C \wedge y \in D \\ &\equiv (x, y) \in C \times D\end{aligned}$$

.

On the other hand, if $A \times B = C \times D$, but we suppose that $\neg(A = C \wedge B = D)$. We only consider that $A \neq C$, the other situations are similar. Then we only consider $\exists x(x \in A \wedge x \notin C)$, for the other situations are similar.

(3) It is easy to prove that $X \times \varnothing = \varnothing$ and $\varnothing \times X = \varnothing$. Let $A = \varnothing$, we can see that even if $B \nsubseteq D$, $A \times B \subseteq C \times D$.

Let $A = D = \varnothing$, then even if $A \neq C$, $A \times B = C \times D$. □

Exercise 3.5.7

Proof. Existence: Let $h(t) := (f(t), y(t))$. It is easy to verify that $h(t) \in X \times Y$, and that given a $t \in Z$, $h(t)$ is unique. Therefore, h is a function. And it is obvious that $\pi_{X \times Y \to X} \circ h = f$ and that $\pi_{X \times Y \to Y} \circ h = g$.

Uniqueness: $\pi_{X \times Y \to X} \circ h = f$ and $\pi_{X \times Y \to Y} \circ h = g$ imply that if there is another function h' that satisfies the requirements, then $h'(t) = h(t)$. So h is unique. □

Exercise 3.5.8

Proof. On one hand, if for some $i, X_i = \varnothing$, then

$$\forall (x_i)_{1 \leq i \leq n} (\bigwedge_{i=1}^{n} x_i \in X_i \equiv (x_i)_{1 \leq i \leq n} \in \varnothing)$$

, which means that $\varnothing = \prod_{i=1}^{n} X_i$.

On the other hand, if $\prod_{i=1}^{n} X_i = \varnothing$ but we suppose that $X_i \neq \varnothing$. Then for each i, $\exists x_i \in X_i$. We thus have a tuple $(x_i)_{1 \leq i \leq n}$, which should be an element of $\prod_{i=1}^{n} X_i$. Therefore we have a contradiction. □

Exercise 3.5.9

Proof. On one hand, let $x \in (\bigcup_{\alpha\in I} A_\alpha) \cap (\bigcup_{\beta\in J} B_\beta)$. Then

$$\exists a(a \in I \wedge x \in A_a) \wedge \exists b(b \in J \wedge x \in B_b)$$

It is obvious that $x \in A_a \cap B_b$ and that $(a, b) \in I \times J$. Therefore

$$x \in \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta)$$

.

On the other hand, let $x \in \bigcup_{(\alpha,\beta)\in I\times J}(A_\alpha \cap B_\beta)$. Then

$$\begin{aligned}\exists(a, b) \in I \times J(x \in A_a \cap B_b) &\Longrightarrow x \in A_a \wedge x \in B_b \\ &\Longrightarrow x \in \bigcup_{\alpha\in I} A_\alpha \wedge x \in \bigcup_{\beta\in J} B_\beta \\ &\Longrightarrow x \in (\bigcup_{\alpha\in I} A_\alpha) \cap (\bigcup_{\beta\in J} B_\beta)\end{aligned}$$

□

Exercise 3.5.10

Proof. We denote $\widetilde{f}$ as f', the graph of f as G, and the graph of f' as G' for the sake of simplification.

(1) On one hand, if $f = f'$, then for every $(x, f(x)) \in G$, we can find $(x, f'(x)) \in G'$, and obviously $(x, f(x)) = (x, f'(x))$, and vice versa.

On the other hand, if $G = G'$, then for each $(x, f(x)) \in G$, $(x, f(x)) \in G'$. Note that each element of G' obeys the form $(x, f'(x))$, so $f(x) = f'(x)$ for every $x \in X$, that is, $f = f'$.

(2) Existence: Let $f(x)$ be such a value that $(x, f(x)) \in G$. Thus the value is unique, so f is a function. According to its definition, the graph of f is G.

Uniqueness: As proven in (1), if f, f' have the same graph, then they are equal. □

Exercise 3.5.11 I think this exercise is meaningless. Lemma 3.4.6 is proven by the fact that X^Y exists, which depends on Axiom 3.10. Then the exercise asks us to prove Axiom 3.10 using Lemma 3.4.6. So I looked up some books about set theory and found out that the power set axiom is essentially Lemma 3.4.6, not Axiom 3.10.

Nevertheless, here is the proof:

Proof. Let set Z contains all subsets of $X \times Y$. The specify such element in Z that obey the vertical line test, and let them form the set S. According to the previous exercise, for each element in S, there exists an unique function whose graph is the element. Then we replace all elements in S with these functions to construct the set F. Obviously, each element in F is a function with the domain X and the range Y.

Now we show that every function f from X to Y is in F. Denote the graph of f as G. We know that G obeys the vertical line test and $G \subseteq X \times Y$, so $G \in S$. Since G is the graph of f, $f \in F$. □

Exercise 3.5.12 I am confused by this exercise. It seems that simply applying induction to a can solve the problem, just like what we did in Proposition 2.1.16. What is wrong?

By the way, according to the corrections, edit the exercise as the following:

> Let X be an arbitrary set containing at least an element c and obeys the Peano axioms. Let f be a function from $N \times X$ to X. ...
>
> Show that there exists an unique function a from X to X such that
>
> $$a(0) = c$$
>
> and
>
> $$a(n++) = f(n, a(n)), \forall n \in X$$
>
> ...
>
> such that $a_N(0) = c$ and $a_N(n++) = f(n, a_N(n))$ ...

Note that all properties (e.g. orders, addition) in section 2 are deduced from the Peano axioms and their definitions. Since X obeys these rules, we use such properties on elements of X without proof.

The proof is now reserved for further research.

Proof. □

Exercise 3.5.13

Proof. Use induction.

Existence: We need to prove that for all $n \in \mathbb{N}$, $f(n)$ is defined. Use induction: $f(0) = 0'$ is define. And the definition is unique for 0 is not the successor of any natural number. Now suppose that $f(n) = n'$ is defined,

then $f(S(n)) = S'(f(n)) = S'(n')$ is also defined. The definition is also unique. So we know that f exists.

Injectivity: We need to prove that $f(m) = f(n) \Longrightarrow m = n$. If $f(m) = f(n)$, then $m' = n'$, and thus $m = n$.

Surjectivity: Use induction: The basic case is, for $0' \in \mathbb{N}'$, $f(0) = 0'$.

Now suppose that for $n' \in \mathbb{N}'$, we can find $n \in \mathbb{N}$ such that $f(n) = n'$, then for $S'(n')$, we have $f(S(n)) = S'(n')$. We can close the induction now. □

8 Cardinality of Sets

Exercise 3.6.1

Proof. Reflexivity: Let $f(x) := x, X \to X$. f is bijective since $f^{-1}(x) = x$ exists.

Symmetry: If X, Y have the same cardinality, then $\exists f : X \to Y$ which is bijective. So f^{-1} exists, and is also a bijection. Thus Y, X have the same cardinality. Since then, we can say that two sets have the same cardinality without caring about the order.

Transitivity: If X, Y have the same cardinality, and Y, Z also have the same cardinality, then there exist two bijections: $f : X \to Y$ and $g : Y \to Z$. It is easy to verify that $g \circ f$ is also a bijection and is from X to Z (See Exercise 3.3.7). □

Remark 3.6.6 It is $f(n) := S(n)$. We are now proving something stronger

Lemma 11. *For any natural number m, n, $\{i \in \mathbb{N} : 0 \leq i \leq n\}$ and $\{i \in \mathbb{N} : m \leq i \leq n + m\}$ have the same cardinality.*

Proof. Use induction on m. When $m = 0$, the statement is obviously true. Simply give the function $f(n) := n$.

Suppose that for some m, we have proven the statement. Then there exists a bijection:

$$f : \{i \in \mathbb{N} : 0 \leq i \leq n\} \to \{i \in \mathbb{N} : m \leq i \leq n + m\}$$

. Let g be a function from $\{i \in \mathbb{N} : 0 \leq i \leq n\}$ to $\mathbb{N}$ such that $g(x) = S(f(x))$. We prove that g is a bijection from $\{i \in \mathbb{N} : 0 \leq i \leq n\}$ to $\{i \in \mathbb{N} : S(m) \leq i \leq n + S(m)\}$.

First we prove that $g(n)$ always in $\{i \in \mathbb{N} : S(m) \leq i \leq n + S(m)\}$, which is immediately given by the fact that addition preserves order.

Surjectivity: For any $a \in \{i \in \mathbb{N} : S(m) \leq i \leq n + S(m)\}$, a is positive. Then a is always some number's successor, that is $a = S(b) = b+1$ for some natural number b. Since addition preserves order, $b \in \{i \in \mathbb{N} : m \leq i \leq n + m\}$. f being surjective implies that there is some x in the domain such that $f(x) = b$, and $g(x) = f(x) + 1 = a$.

Injectivity: By cancellation law, $f(x) + 1 \neq f(x') + 1 \equiv f(x) \neq f(x') \equiv x \neq x'$.

We can now close the induction. □

Lemma 3.6.9 Empty functions are not injective when the range is not empty (See Exercise 3.3.3).

Now we show that g is bijective:

Proof. Injectivity: f being injective implies that

$$\forall x \forall x'((x \in X \wedge x' \in X) \Longrightarrow (f(x) = f(x') \Rightarrow x = x'))$$

For $a, a' \in X - \{x\}$, they also $\in X$. If $g(a) = g(a')$, then either directly $f(a) = f(a')$ or $f(a) - 1 = f(a') - 1$, which gives $f(a) = f(a')$. Thus $a = a'$. (Note that subtraction is not defined yet, see the footnote about this in the book).

Surjectivity: The surjectivity of f gives

$$(\forall 1 \leq i \leq n)(\exists a(a \in X \wedge f(a) = i))$$

.

If $f(x) = n$, then $g(a) = f(a)$ for all meaningful a. Then for $1 \leq i \leq n-1$, we can find a such that $a \in X \wedge a \neq x$, that is, $x \in X - \{x\}$. So $g(a)$ is meaningful, then g is surjective.

If $f(x) \neq n$, then $f(x) < n$. For those $1 \leq i < f(x)$, g is obviously surjective. For $n - 1 \geq i \geq f(x)$, since $S(i) \leq n$, $\exists a(a \in X \wedge f(a) = S(i))$. And we know that $S(i) \neq f(x)$, then $a \in X - \{x\}$. So $g(a) = f(a) - 1 = i$. □

Exercise 3.6.2

Proof. On one hand, if X is empty, then we know that the empty function whose range is also empty is injective, (See Exercise 3.3.3) so its cardinality is 0.

On the other hand, if $\#X = 0$ but $X \neq \varnothing$, then there exists an bijection $f : X \to \varnothing$, which is impossible. □

Exercise 3.6.3

Proof. When $n = 0$, this is vacuously true. The base case then becomes $n = 1$. We simply let $M = f(1)$.

Suppose that the statement for n is true. And for $1 \leq i \leq n$ we have the number M. Then $f(S(n))$ either $\geq$ or $<$ M. On the former case, let $f(S(n))$ be M', and on the latter case, let $M' = M$. It is east to verify that M' is the number we want. □

From now on we will denote $\{i \in \mathbb{N} : 1 \leq i \leq n\}$ as $\mathbb{N}_n$

Exercise 3.6.4

Proof. (a) Let $n = \#X$. There is an injective f from X to $\{i \in \mathbb{N} : 1 \leq i \leq n\}$. Let g be a function from $X \cup \{x\}$ to $\{i \in \mathbb{N} : 1 \leq i \leq n+1\}$ such that $g(a) = f(a)$ if $a \neq x$, and $g(x) = n+1$. Now we show that g is bijective.

Injectivity: We know that $\forall x \in X$, g is already injective. Since that $g(x) = n+1 \neq g(a)$ for all $a \in X$, so g is injective on $X \cup \{x\}$.

Surjectivity: We know that $\forall i \in \{i \in \mathbb{N} : 1 \leq i \leq n\}$, we can find $a \in X \cup \{x\}$ such that $g(a) = i$. And we have $g(x) = n+1$, so $\forall a \in \{i \in \mathbb{N} : 1 \leq i \leq n+1\}$, we can find $a \in X \cup \{x\}$ such that $g(a) = i$.

(b) First we prove that if X, Y are disjoint, then $\#X + \#Y = \#(X \cup Y)$. Let f be a bijection from X to $\mathbb{N}_{\#X}$, and g be a bijection from Y to $\mathbb{N}_{\#Y}$. According to this Lemma, there exists a bijection h from $\mathbb{N}_{\#Y}$ to $\{i \in \mathbb{N} : \#X+1 \leq i \leq \#X+\#Y\}$. Thus $h \circ g$ is also a bijection. Let u be a function from $X \cup Y$ to $\mathbb{N}_{\#X} \cup \{i \in \mathbb{N} : \#X + 1 \leq i \leq \#X + \#Y\}$. Now we show that u is bijective.

Injectivity: For $x \neq x'$ in the domain. If x, x' are both in X or Y, then $f(x) \neq f(x')$ is immediately given by the injectivity of f and $h \circ g$. If one of them is in X, and the other is in Y, then they can also never be equal because the ranges of the two functions are disjoint.

Surjectivity: It is easy to verify that the range is equal to $\mathbb{N}_{\#X+\#Y}$. For any y in the range, if $y \in$ the range of f, then u is surjective since f is, and if $y \in$ the range of $h \circ g$, u is surjective for the same reason. The range consists of only this two sets, so u is surjective on the whole range.

The proof is over. This also implies that $X \cup Y$ is finite. Now we need only to show that $\#(X \cup Y) < \#X + \#Y$ when X, Y are not disjoint. It is

easy to see that

$$\begin{aligned}\#A + \#B &= \#(A - A \cap B) + \#(A \cap B) + \#(B - A \cap B) + \#(A \cap B) \\ &= (\#(A - A \cap B) + \#(A \cap B) + \#(B - A \cap B)) + \#(A \cap B) \\ &= \#(A \cup B) + \#(A \cap B) \\ &> \#(A \cup B)\end{aligned}$$

(c) If $X \subseteq Y \wedge X \neq Y$, then $\#(Y \setminus X) \neq 0$.

$$\#Y = \#X + \#(Y \setminus X) > \#X$$

.

If $X = Y$, then $\#(Y \setminus X) = 0$, and $\#Y$ becomes $\#X$.

(d) $f : X \to f(X)$ is always surjective. If f is also injective, then f is bijective. On this occasion, $\#f(X) = \#X$. If f is not injective, we can select a set $X' \subseteq X \wedge X' \neq X$, on which f is bijective. Then $\#X' = \#f(X') = \#f(X)$. According to (c), $\#X' < \#X$, so $\#f(X) < \#X$.

(e) Suppose that $\#Y = n$. Use induction on n.

When $n = 0$, Y is empty, then $\#(X \times Y) = 0 = \#X \times 0$. Here we additionally prove that when $n = 1$, this is also true for further usage. When $n = 1$, let $Y = \{a\}$. Then the bijection is $f(x) := (x, a), X \to X \times \{a\}$.

Suppose that we have proven for some n, $\#(X \times Y) = \#X \times \#Y$. Then when $\#Y = S(n)$, let $Y = Y \setminus \{x\} \cup \{x\}$, where $x \in Y$. Lemma 3.6.9 tells us that $\#(Y \setminus \{x\}) = S(n) - 1 = n$. And Exercise 3.5.4 tells us that $X \times Y = X \times (Y \setminus \{x\}) \cup X \times \{x\}$.

$$\begin{aligned}\#(X \times Y) &= \#(X \times (Y \setminus \{x\}) \cup X \times \{x\}) \\ &= \#(X \times (Y \setminus \{x\})) + \#(X \times \{x\}) \\ &= \#X \times n + \#X \\ &= \#X \times S(n)\end{aligned}$$

We can now close the induction.

(f) We should first define m^n for natural numbers m, n. It has not been done yet. Exponentiation is defined for rational numbers at Definition 4.3.9.

Definition 2.

- $m^0 = 1$,
- $m^{S(n)} = m^n \times m$

Suppose that $\#Y = m, \#X = n$. Use induction on n.

When $n = 0$, X is empty, then Y^X has one function $f : \varnothing \to Y$.

Suppose that we have proven the statement for some n. Before we proceed the proof, we need some lemmas.

Lemma 12. *If X is not empty,*

$$\#Y^{X\setminus\{x'\}\cup\{x'\}} = \#Y^{X\setminus\{x'\}} \times \#Y$$

, where x' is an element of X.

Proof. By (e) we know that

$$\#Y^{X\setminus\{x'\}} \times \#Y = \#(Y^{X\setminus\{x'\}} \times Y)$$

.

Try to build a bijection between $Y^{X\setminus\{x'\}} \times Y$ and Y^X. Let $f' \in Y^X$.

Let h be a function from Y^X to $Y^{X\setminus\{x'\}} \times Y$ such that

$$h(f') = (f, f'(x')),$$

where $f(x) := f'(x)$ when $x \neq x'$. Now we show that h is bijective.

Injectivity: If ${f_1}' \neq {f_2}'$, then

$${f_1}'(x') \neq {f_2}'(x') \vee \exists x(x \neq x' \wedge {f_1}'(x) \neq {f_2}'(x))$$

That is,

$${f_1}'(x') \neq {f_2}'(x') \vee f_1 \neq f_2,$$

which means

$$(f_1, {f_1}'(x')) \neq (f_2, {f_2}'(x')).$$

Surjectivity: For any $(f, a) \in Y^{X\setminus\{x'\}} \times Y$, let f' be f if $x \neq x'$, and $f'(x') = a$. Then $f' \in Y^X$ and $h(f') = (f, a)$.

So,

$$\#Y^X = \#(Y^{X\setminus\{x'\}} \times Y)$$

, which gives the lemma. □

Now we proceed the proof. Suppose that $\#X = n+1$, then $\#(X\setminus\{x'\}) = n$. By induction hypothesis, $\#(Y^{X\setminus\{x'\}}) = m^n$.

By the lemma,

$$\#Y^X = \#Y^{X\setminus\{x'\}\cup\{x'\}} = \#Y^{X\setminus\{x'\}} \times \#Y,$$

which equals to $m^n \times m$.

Now we can close the induction.

We have proven that the cardinality of power sets obeys the definition of power. This ensures the exercise. □

Exercise 3.6.5

Proof. Let $f((x,y)) := (y,x), A \times B \to B \times A$. The bijectivity is obvious.

Now we are using set theory to prove the commutativity of multiplication of natural number. For any natural number m, n, construct two sets: $M = \mathbb{N}_m, N = \mathbb{N}_n$. According to (e) in Proposition 3.6.14, we have that $\#(M \times N) = \#M \times \#N$. Then by what we have just proven,

$$\#(M \times N) = \#(N \times M) \Longrightarrow \#M \times \#N = \#N \times \#M \Longrightarrow mn = nm$$

□

Exercise 3.6.6

Proof. Let $c \in C$, $f \in (A^B)^C$. Then $f(c)$ is a function $B \to A$. Let $b \in B, h \in A^{B\times C}$. Let

$$g : A^{B\times C} \to (A^B)^C$$

be such a function that for all b, c,

$$g(h) = f \equiv h(b,c) = (f(c))(b)$$

. Now we show that g is bijective.

Injectivity: If $h \neq h'$, then $\exists b_0, c_0 (h(b_0, c_0) \neq h'(b_0, c_0))$. Let $g(h) = f, g(h') = f'$. Then we know that $(f(c_0))(b_0) \neq (f'(c_0))(b_0)$, so $f(c_0) \neq f'(c_0) \Longrightarrow f \neq f'$. That means, $g(h) \neq g(h')$.

Surjectivity: For any $f \in (A^B)^C$, let h be such a function $\in A^{B\times C}$ that for all $b \in B, c \in C$, $h(b,c) := (f(c))(b)$. It is easy to see that h is well-defined. So $g(h) = f$.

Note that by Proposition 3.6.14 we have $\#M^N = m^n$ and $\#(M \times N) = mn$, where $\#M = m, \#N = n$. Suppose that $\#A = a, \#B = b, \#C = c$, then

$$\#(A^B)^C = (\#A^B)^{\#C} = (a^b)^c$$
$$\#A^{B\times C} = \#A^{\#(B\times C)} = a^{bc}$$

So we have proven that $(a^b)^c = a^{bc}$.

Now we try to prove $a^b \times a^c = a^{b+c}$. Let B, C be disjoint sets with the cardinality b, c respectively. What we need to show is that

$$\#(A^B \times A^C) = \#(A^{B\cup C}).$$

Similarly, let

$$f : (A^{B\cup C}) \to (A^B \times A^C)$$

be such a function that

$$f(g) = (u, v) \equiv \forall x(x \in B \Rightarrow g(x) = u(x) \wedge x \in C \Rightarrow g(x) = v(x)),$$

where $g \in A^{B \cup C}, (u, v) \in A^B \times A^C$.

We can verify the bijectivity of f nearly in the same way as way did previously. So I won't write it down here.

Then, we know $B \cap C = \varnothing \Rightarrow \#(B \cup C) = \#B + \#C$. So we can conclude that

$$a^b \times a^c = a^{b+c}$$

□

Exercise 3.6.7

Proof. On one hand, if $\#A = a \leq \#B = b$, we show that A has lesser or equal cardinalty to B. Let f be a bijection from A to $\mathbb{N}_a$, g be a bijection from B to $\mathbb{N}_b$. Let $\iota(x) := x, \mathbb{N}_a \to \mathbb{N}_b$. Then $g^{-1} \circ \iota \circ f$ is an injection from A to B.

On the other hand, suppose that there is an injection f from A to B. We know that $f : A \to f(A)$ is bijective. So $\#A = \#f(A)$. Since $f(A) \subseteq B$, $\#f(A) \leq B$ (See (c) in Proposition 3.6.14). That is, $\#A \leq \#B$ □

Exercise 3.6.8

Proof. $f : A \to f(A)$ is bijective. So $f^{-1} : f(A) \to A$ is surjective. Let g be defined as:

- $b \in f(A) \Longrightarrow g(b) = f^{-1}(b)$
- $b \in B \setminus f(A) \Longrightarrow g(b)$ is any element of A.

Then g is surjective. □

Exercise 3.6.9

Proof.

$$\begin{aligned}
\#A + \#B &= \#(A - A \cap B) + \#(A \cap B) + \#(B - A \cap B) + \#(A \cap B) \\
&= (\#(A - A \cap B) + \#(A \cap B) + \#(B - A \cap B)) + \#(A \cap B) \\
&= \#(A \cup B) + \#(A \cap B)
\end{aligned}$$

□

Exercise 3.6.10

Proof. Presume the contradiction:

$$\forall i(i \in \{1, \ldots, n\} \Longrightarrow \#(A_i) < 2)$$

Use mathematical induction for (b) in Proposition 3.6.14, we can easily get:

$$\# \bigcup_{i \in \{1,\ldots,n\}} A_i \leq \sum_{i \in \{1,\ldots,n\}} \#A_i$$

We can also use mathematical induction to furthermore enhance what we proved while dealing with natural numbers to:

$$\bigwedge_i a_i \leq b_i \Longrightarrow \sum_i a_i \leq \sum_i b_i$$

.

Then because $\#A_i \leq 1$, so

$$\sum_{i \in \{1,\ldots,n\}} A_i \leq (\sum_{i \in \{1,\ldots,n\}} 1 = n)$$

, which is impossible. □

Part III
Integers and Rationals

Now we are going to extend natural numbers to integers and rationals.

9 The Integers

Exercise 4.1.1

Proof. It is immediately given by the fact that

$$a + b = a + b \equiv a - -b = a - -b$$

□

Lemma 4.1.3

$$(m - -0) + (n - -0) = (m + n) - -0$$

$$(m - -0) \times (n - -0) = (mn) - -0$$

ensures that the definition $m - -0 := m$ is consistent with addition and multiplication.

Exercise 4.1.2

Proof.

$$a - -b = a' - -b' \equiv a = b \wedge a' = b'$$

Then,

$$(b - -a) = (b' - -a') \equiv -(a - -b) = -(a' - -b')$$

□

Exercise 4.1.3

Proof.

$$\begin{aligned} -1 \times a &= (0 - -1) \times (a - -0) \\ &= (0 \times a + 1 \times 0) - -(0 \times 0 + 1 \times a) \\ &= 0 - -a \\ &= -a \end{aligned}$$

□

Exercise 4.1.4

Proof. Let $x = (a - -b), y = (c - -d), z = (e - -f)$.

(1)

$$\begin{aligned}(a - -b) + (c - -d) &= (a + c) - -(b + d)\\ &= (c + a) - -(d + b)\\ &= (c - -d) + (a - -b)\end{aligned}$$

(2)

$$\begin{aligned}((a - -b) + (c - -d)) + (e - -f) &= ((a + c) + e) - -((b + d) + f)\\ &= (a + (c + e)) - -(b + (d + f))\\ &= (a - -b) + ((c - -d) + (e - -f))\end{aligned}$$

(3) First ,

$$(a - -b) + (0 - -0) = (a - -b)$$

.

Second, by (1) we have $0 + x = x + 0$.

(4) First,

$$\begin{aligned}(a - -b) + (b - -a) &= (a + b) - -(a + b)\\ &= 0 - -0 \qquad\qquad (a + b + 0 = a + b + 0)\end{aligned}$$

Second, by (1) we have $x + (-x) = (-x) + x$.

(5)

$$\begin{aligned}(a - -b)(c - -d) &= (ac + bd) - -(ad + bc)\\ &= (ca + db) - -(cb + da)\\ &= (c - -d)(a - -b)\end{aligned}$$

(6) The book proved this.

(7) First,

$$(1 - -0)(a - -b) = (1a + 0b) - -(1b + 0a) = (a - -b)$$

Second, by (5) we have $1x = x1$.

(8)

$$\begin{aligned}
&(a--b)((c--d)+(e--f))\\
&=(a--b)((c+e)--(d+f))\\
&=(a(c+e)+b(d+f))--(a(d+f)+b(c+e))\\
&=((ac+bd)+(ae+bf))--((ad+bc)+(af+be))\\
&=(ac+bd)--(ad+bc)+(ae+bf)--(af+be)\\
&=(a--b)(c--d)+(a--b)(e--f)
\end{aligned}$$

(9) This can be easily concluded from (5) and (8). □

Exercise 4.1.5

Proof. We need to show that

$$a \neq 0 \wedge b \neq 0 \Longrightarrow ab \neq 0$$

Since a, b are not 0, they can be either positive or negative. If they are both positive, the case is already proven.

When at least one of them is negative, we can divide the -1 from the negative ones. That is, if $a = -m$, where m is positive, then we substitute a with $-1 \times m$. Then we may get ab in either the form $(-1)(-1)mn$ or $(-1)mn$, where the former is a positive number because $(-1)(-1) = 1$ and the latter is negative. □

Exercise 4.1.6

Proof. We check the value of $ac - bc$. We know that $ac = bc$, so $ac - bc = 0 - 0 = 0$. According to (9) in Proposition 4.1.6,

$$ac - bc = ac + (-b)c = (a + (-b))c = 0$$

As stated by Proposition 4.1.8, since that $c \neq 0$, $a + (-b) = 0$, which means $a - b = 0$. Then we have $a = b$. □

Exercise 4.1.7 In the following contents, p stands for a positive natural number, n stands for a natural number.

Proof. (a)

$$\begin{aligned} a > b &\equiv a = b + p \\ &\equiv a + (-b) = b + (-b) + p \qquad \text{(See the following explanation)} \\ &\equiv a - b = p \end{aligned}$$

We now explain why $a = b + p \equiv a + (-b) = b + (-b) + p$. Using the substitution law and the commutativity of addition, it is clear to see that $a = b + p \Longrightarrow a + (-b) = b + (-b) + p$. We now show the cancellation law of addition, that is,

Lemma 13.

$$a + c = b + c \Longrightarrow a = b$$

Proof.

$$\begin{aligned} a + c = b + c &\Longrightarrow a + c + (-c) = b + c + (-c) \\ &\Longrightarrow a + (c + (-c)) = b + (c + (-c)) \\ &\Longrightarrow a = b \end{aligned}$$

□

So we get the inverse result: $a = b + p \Longleftarrow a + (-b) = b + (-b) + p$.

Note that by the definition of integer and what we have know now, we can conclude that

Lemma 14. *For every integer $i = a - b, j = c - d$, there exists exactly one integer k such that $i = j + k$.*

(b)

$$\begin{aligned} a > b &\equiv a = b + p \\ &\Longrightarrow a + c = b + c + p \\ &\Longrightarrow a + c > b + c \end{aligned}$$

(c)

$$\begin{aligned} a > b &\equiv a = b + p \\ &\Longrightarrow ac = (b + p)c = bc + pc \\ &\Longrightarrow ac > bc \qquad (pc > 0 \text{ by Lemma 2.3.3}) \end{aligned}$$

(d)

$$a > b \equiv a = b + p$$

Then

$$-a = -(b+p) = (-1)(b+p) = -b - p$$

So

$$-a + p = -b - p + p$$

That is,

$$-b = -a + p \equiv -b > -a$$

(e) Let

$$a = b + p_1, b = c + p_2$$

Then $a = c + (p_1 + p_2)$. Obviously $p_1 + p_2$ is positive, so $a > c$.

Note that $-a, -b$ are also integers, and plus that $-(-a) = a$, so we can give a stronger conclusion:

$$a > b \equiv -a < -b$$

(f) If a, b are all natural numbers, the statement was proven before.

If one of them (say a) is negative, the other (b) is a natural number, then $a = -n$, and we know that $b > 0$ and $0 = a + n \implies 0 > -a$, so by (e) we have $b > a$.

If they are both negative, then their negations satisfy the statement. Then

$$-a < -b \equiv a > b, -a = -b \equiv a = b, -a > -b \equiv a < b$$

. □

Exercise 4.1.8 An example: $P(i) : i >= 0$. It is obvious that $P(0)$ and $P(i) \implies P(i+1)$ is true. But for any negative integer n, $P(n)$ is not true.

We additionally prove one more property:

Lemma 15. *For integers a, b,*

$$a - b = 0 \implies a = b$$

Proof. We can add b to both side to obtain $a = b$. □

10 The Rationals

Exercise 4.2.1

Proof. Reflectivity:

$$a//b = a//b \equiv ab = ab$$

Being Symmetric:

$$\begin{aligned} a//b = c//d &\equiv ad = bc \\ &\equiv cb = da \\ &\equiv c//d = a//d \end{aligned}$$

Transitivity:

$$a//b = c//d \equiv ad = bc$$

$$c//d = e//f \equiv cf = de$$

Thus,

$$(ad)(cf) = (bc)(de)$$

We then have

$$afcd = becd$$

We can cancel d since $d \neq 0$ to obtain $afc = bec$. If $a = 0$, we can conclude that c, e also must be 0. Under this occasion, $af = be$ is also true because they all equal to 0. □

Definition 4.2.2 It is useful to prove that

Lemma 16.

$$(-a)//b = a//(-b)$$

,

$$a//b = (-a)//(-b)$$

Proof. The first is immediately given since $(-a)(-b) = ab$. The latter is proven as $a(-b) = b(-a) = -ab$. □

We may notice that subtraction is not mentioned here. This is because that we can get $a - b$ by adding a and $-b$, where addition $+$ and negation $-$ are mentioned.

Exercise 4.2.2

Proof. (1) is deduced in the book.

(2) I don't quiet understand why Tao used this $*$ sign instead of $\times$. I know it is a new definition, but the $\times$ sign is undefined for rationals (except for integers, but for which we can verify that the two definitions are the same). We will use the $\times$ sign or just leave it off here.

$$(a'//b')(c//d) \equiv a'd = b'c \equiv ad = bc \equiv (a//b)(c//d)$$

Similarly we can verify this for $c'//d'$.

(3)

$$-ab' = -a'b \equiv (-a)//b = (-a')//b'$$

□

For the sake of simplification, we hereby introduce some useful lemmas:

Lemma 17.

$$b = d \neq 0 \Longrightarrow (a//b = c//d \equiv a = c)$$

Proof. Assume that $b = d \neq 0$.

On one hand, if $a//b = c//d$, then $ad = bc$. Since that $b = d \neq 0$, we can cancel them to obtain $a = c$.

On the other hand, if $a = c$, then if we multiply them by the same integer (namely $b = d$), and the results are still equal ($ad = bc$). So $a//b = c//d$. □

Lemma 18.

$$c \neq 0 \Longrightarrow a//b = ac//bc$$

Proof. Assume that $c \neq 0$.

First we know that $ab = ab$. Then we can further obtain $abc = abc$, which means $a//b = ac//bc$. □

Exercise 4.2.3

Proof. (1) We have

$$a//b + c//d = (ad + bc)//(bd)$$

$$c//d + a//b = (cb + da)//(db)$$

It is easy to see that they are equal.

(2) It is proven in the book.

(3) We just deduce $x + 0 = x$ here, for we have $0 + x = x + 0$ according to (1).

$$a//b + 0//1 = (a1 + b0)//(b1) = a//b$$

(4) We only prove $x + (-x) = 0$ here, for we have $x + (-x) = (-x) + x$ according to (1).

$$a//b + (-a)//b = (ab - ab)//bb = 0//b^2 = 0$$

(5)

$$\begin{aligned} a//b \times c//d &= ac//bd \\ &= ca//db \\ &= c//d \times a//b \end{aligned}$$

(6)

$$\begin{aligned} &(a//b \times c//d) \times e//f \\ &= ac//bd \times e//f \\ &= ace//bdf \\ &= a//b \times ce//df \\ &= a//b \times (c//d \times e//f) \end{aligned}$$

(7) We only prove $x1 = x$ here, for we have $x1 = 1x$ according to (4).

$$a//b \times 1//1 = a1//b1 = a//b$$

(8)

$$\begin{aligned} &a//b(c//d + e//f) \\ &= a//b((cf + ed)//(df)) \\ &= a(cf + ed)//bdf \\ &= ab(cf + ed)//b^2df && \text{(See Lemma 18)} \\ &= ((ac)(bf) + (bd)(ae))//(bd)(bf) \\ &= ac//bd + ae//bf \\ &= (a//b \times c//d) + (a//b + e//f) \end{aligned}$$

(9) This can be deduced from (5) and (8).

(10) We merely conclude $xx^{-1} = 1$ here, since we have $xx^{-1} = x^{-1}x$ from (5).

$$a//b \times b//a = ab//ba = (ab)1//(ab)1 = 1//1$$

The last step is done by Lemma 18. □

Exercise 4.2.4

Proof. For any rational $r = a/b$, a, b are integers. They are either positive, 0, or negative (except that b cannot be 0). When a, b are both positive, then r is also positive. When a is positive but b is negative, then let $b = -p$, where p is positive, thus $a/b = a/(-p) = (-a)/p$ is negative. When $a = 0$, $r = 0$. When a is negative, and b is positive, then by definition r is negative. When a, b are both negative, according to Lemma 16, r is positive.

Therefore, we have iterated through all possible situations and verified that there is and only is one statement for a rational is true. □

Exercise 4.2.5

Proof. Let $x = a/b, y = c/d, z = e/f$. Before proving the following components, we will introduce some useful propositions here.

Lemma 19. *1. $x > 0$ is logically equivalent to x being positive.*

2. $x < 0$ is logically equivalent to x being negative.

Proof.

$$x - 0 = x$$

is itself, so whether x is positive or negative, the same is $x - 0$, then we can deduce $x > 0$ or $x < 0$, and vice versa. □

We can now use simplified notation $x > 0$ to express the same meaning: x is positive.

(a) We check the value of

$$\delta = x - y = a/b + (-c)/d = (ad - bc)/bd$$

δ is also a rational number. According to the previous exercise, it is either positive, negative, or 0. So x either $> y$, $< y$, or $= y$ (We haven't yet proven $x - y = 0 \implies x = y$. Let's prove it now. We can add y to both side of $x - y = 0$ to obtain the result).

(b) According to Lemma 19, $x < y \implies x - y < 0$. Then we multiply $-1/1$ with $x - y$ to obtain (It is easy to see that for rational number r, $-1r = -r$ and $-(-r) = r$)

$$-1/1 \times (x + (-y)) = -x + -(-y) = y - x$$

Since $x - y$ is negative, and the negation of a positive number is negative, so the negation of $x - y$, $y - x$, is positive, which means that $y > x$.

(c) By the hypothesis, $x - y < 0 \wedge y - z < 0$. We are now proving that $i, j < 0 \Longrightarrow i+j < 0$. We can write i, j as $o/p, q/s$ respectively. Let $p, s > 0$, then $o, q < 0$. Then $o/p + q/s = (os + pq)/ps$. We know that $os, pq < 0$ (Write a negative integer as a negation of a positive integer to see that the product of a positive and a negative is also negative).

Now we show that for two positive integers, their sum is still positive. Integers who are positive are also natural number, and their sum remains a natural number. So the sum itself equals to 0 plus itself, which means it is positive. The negation of this sum, which is also $-m+(-n)$, is thus negative. Since that $-m, -n$ can present any negative integer, the fact means that the sum of two negative integers remains a negative integer.

So $os+pq < 0$. But $ps > 0$, so $i+j < 0$. Thus, $(x-y)+(y-z) = x-z < 0$, which means $x < z$.

(d)

$$\begin{aligned} x + z - (y + z) &= x + z + (-)(y + z) \\ &= x + z + (-1)(y + z) \\ &= x + z - z - y \\ &= x - y < 0 \end{aligned}$$

(e) It is easy to verify that the product of two positive rationals is still positive (Writing them as $a/b, c/d$, where $a, b, c, d > 0$, then ac/bd also > 0). Then $xz - yz = z(x - y)$, which is the product of a positive number and a negative number, and is thus a negative number. □

Exercise 4.2.6

Proof. According to (e) of Proposition 4.2.9, we need only to show that $x < y \Longrightarrow -x > -y$. Then we can multiply $xz > yz$ with -1 to obtain what we want.

We know that the negation operation will turn a positive into negative and vice versa. Now we have $x-y < 0$, so the negation $-(x-y) = -x+y = -x-(-y) > 0$, which means that $-x > -y$. □

There are still many properties about rationals that we use for granted (e.g. $x^{-}1$ has the same sign as x; the two definitions of order are the same, that is, $x - y > 0 \equiv x = y + p \equiv x > y$, where $p > 0$). Although they need to be proven prior to being used, we can not cover all of them here. We will prove some of them in the future only if they are used. Also, most of them are not hard to prove. We need not to worry.

We will prove some important ones here:

Proposition 13. *For rational numbers* $x, y, p > 0$

$$x - y > 0 \equiv x = y + p \equiv x > y$$

Proof. $x - y > 0 \equiv x > y$ is the definition of order. We merely need to prove $x - y > 0 \equiv x = y + p$ here.

On one hand, if $x - y > 0$, then p is not others, but the very number $x - y$.

On the other hand, if there exists a $p > 0$ such that $x = y + p$. Then add $-y$ to the both side of the equation, and we can get $x - y = p$, which means $x - y$ is positive. So $x - y > 0$. □

Proposition 14. *We can add two inequalities together. That is,*

$$a < b \wedge c < d \Longrightarrow a + c < b + d$$

Proof. We know that

$$a < b \Longrightarrow a + c < b + c$$

, and

$$c < d \Longrightarrow b + c < d + b$$

According to the transitivity of order, we can derive that

$$a + c < b + c < b + d$$

□

Proposition 15. *We can multiply two inequalities of positives or negatives together. That is,*

$$a, b, c, d > 0 \wedge a < b \wedge c < d \Longrightarrow ac < bd$$

, and

$$a, b, c, d < 0 \wedge a < b \wedge c < d \Longrightarrow ac > bd$$

Proof. When they are all positive, we know that

$$a < b \Longrightarrow ac < bc$$

, and

$$c < d \Longrightarrow bc < bd$$

According to the transitivity of order, we can derive that

$$ac < bc < bd$$

When they are all negative, we know that

$$a < b \Longrightarrow ac > bc$$

, and

$$c < d \Longrightarrow bc > bd$$

According to the transitivity of order, we can derive that

$$ac > bc > bd$$

□

Note that we can already add or multiply equations because of the axiom of substitution, so we can change the $<$ in the inequalities to $\leq$ in the previous two propositions whenever needed.

11 Absolute Value and Exponentiation

Exercise 4.3.1

Proof. (a) $x > 0 \Rightarrow |x| > 0$, $x = 0 \Rightarrow |x| = 0$, $x < 0 \Rightarrow |x| > 0$. So $|x| \geq 0$. And we can see that only when $x = 0$ can $|x| = 0$.

(b) This one is very tedious to prove. Let's enumerate all conditions:

1. $x, y > 0$. On this occasion,

$$|x + y| = x + y = |x| + |y|$$

2. At least one of them is 0. On this occasion, let's just let x be 0, the other situations are similar.

$$|x + y| = |0 + y| = |y| = |0| + |y| = |x| + |y|$$

3. $x = y > 0$. On this occasion, $|x + y| = |2x| = 2x = |x| + |x|$.

4. $x = y < 0$. On this occasion, $|x + y| = |2x| = -2x = |x| + |x|$.

5. $x, y < 0$. On this occasion, $|x + y| = -(x + y) = -x - y = |x| + |y|$.

6. One of them is positive, the other is negative. We specify $x > 0, y < 0$ here. But the other conditions are similar. Under this condition, we further divide the situation into three occasions:

- $x+y>0$ On this occasion, $|x+y|=x+y$, $|x|+|y|=x-y$. Note that $x-y=x+y+2(-y)$, where $2(-y)>0$, so $|x+y|<|x|+|y|$ (See Proposition 13).
- $x+y<0$ On this occasion, $|x+y|=-x-y$, $|x|+|y|=x-y$. Note that $-x-y-(x-y)=2(-x)<0$, so $|x+y|<|x|+|y|$.
- $x+y=0$ On this occasion, $|x+y|=0\leq|x|+|y|$ (Recall that the sum of two positive rationals remains positive).

We have iterated through all conditions.

(c) We shall prove that

$$-|x|\leq x\leq|x|$$

first.

1. If $x>0$, then $x=|x|>0$. And $0>-|x|$, so by the transitivity of order, $-|x|<x$.
2. If $x=0$, then $|x|=-|x|=x=0$.
3. If $x<0$, then $x=-|x|<0$ And $0<|x|$, so $x<|x|$.

This also means that x either equals to $|x|$ or $-|x|$.

Then we prove that $-y\leq x\leq y\equiv y\geq|x|$.

On one hand, if $-y\leq x\leq y$, then when $x=|x|$, we have $|x|\leq y$; when $x=-|x|$, we have $-y\leq-|x|\equiv y\geq|x|$. As stated previously, we know that at least one of the two conditions are satisfied.

On the other hand, if $y\geq|x|$, then $-y\leq-|x|$. But since that $-|x|\leq x\leq|x|$, we can obtain what we want by the transitivity of order.

(d)

1. If $x=y=0$, then $|xy|=0=|x||y|$.
2. If $x,y>0$, then $|xy|=xy=|x||y|$.
3. If $x,y<0$, then $xy>0$, $|xy|=xy=(-x)(-y)=|x||y|$.
4. If one of them is positive, and the other is negative, (say $x>0,y<0$), then $|xy|=-xy=x(-y)=|x||y|$. The other conditions are similar.

Thus, $|-x|=|-1||x|=1|x|=|x|$.

(e) This can be easily conclude from (a).

(f) Since that $|-x|=|x|$, we have $|x-y|=|-(x-y)|=|y-x|$.

(g) Note that $x-z=(x-y)=(y-z)$. Then from (b) we can deduce that $|x-z|\leq|x-y|+|y-z|$, which is $d(x,z)\leq d(x,y)+d(y,z)$. □

Exercise 4.3.2

Proof. (a) If $x = y$, then $|x - y| = 0$. And any positive rational $\varepsilon > 0$, so $|x - y| \leq \varepsilon$.

The other statement is much better easier to prove after we have know the denseness of rationals. We essentially repeat some of the proof work that are done afterwards here. On the other hand, suppose the contradiction, that is, $(\forall \varepsilon > 0)(|x - y| \leq \varepsilon)$, but $x \neq y$. Then $x - y \neq 0$. Let $\delta = |x - y| \neq 0$. We know that $2^{-1} = 1/2 > 0$, so $\delta/2 > 0$. Also we have $\delta/2 + \delta/2 = \delta \Longrightarrow \delta/2 < \delta$. Then let $\varepsilon = \delta/2$. So we have both $|x-y| < \delta/2$ and $|x-y| > \delta/2$, which is impossible.

(b) It is immediately derived from $|x - y| = |y - x|$.

(c)

$$|x - z| \leq |x - y| + |y - z| \leq \varepsilon + \delta$$

(d)

$$|x + z - (y + w)| = |x - y + z - w| \leq |x - y| + |z - w| \leq \varepsilon + \delta$$

$$|x - z - (y - w)| = |x - y + w - z)| \leq |x - y| + |w - z| \leq \varepsilon + \delta$$

(e)

$$|x - y| \leq \varepsilon < \varepsilon'$$

(f) From (c) of Proposition 4.3.3, we can derive that

$$|x - z| \leq \varepsilon \equiv -\varepsilon \leq x - z \leq \varepsilon \equiv z - \varepsilon \leq x \leq z + \varepsilon$$

, and that

$$|x - y| \leq \varepsilon \equiv y - \varepsilon \leq x \leq y + \varepsilon$$

Thus we have

$$y - \varepsilon \leq x \leq z + \varepsilon$$

We will only prove the statement when $z \leq w \leq y$, another one is similar. On this occasion, $-y \leq -w \leq -z$. Add this inequality to $y - \varepsilon \leq x \leq z + \varepsilon$ to obtain that

$$-\varepsilon \leq x - w \leq \varepsilon$$

(g)

$$|xz - yz| = |x - y||z| \leq \varepsilon|z|$$

(h) We will explain why $|a| \leq \varepsilon \wedge |b| \leq \delta$ implies $|a||z| + |b||x| + |a||b| \leq \varepsilon|z| + \delta|x| + \varepsilon\delta$.

First, multiply $|a| \leq \varepsilon$ with $|z|$ to obtain $|a||z| \leq \varepsilon|z|$. Then add both sides of the inequality with $|b||x| + |a||b|$ to gain

$$|a||z| + |b||x| + |a||b| \leq \varepsilon|z| + |b||x| + |a||b| \tag{1}$$

Similarly,

$$|a||z| + |b||x| + |a||b| \leq |a||z| + \delta|x| + |a||b| \tag{2}$$

Finally, we can multiply $|a| \leq \varepsilon$ with $|b| \leq \delta$ as stated by Proposition 15 to derive $|a||b| \leq \varepsilon\delta$. So after some addition we have

$$|a||z| + |b||x| + |a||b| \leq |a||z| + |b||x| + \varepsilon\delta \tag{3}$$

Using Proposition 14, we add (1),(2) and (3) together:

$$3(|a||z| + |b||x| + |a||b|) \leq (\varepsilon|z| + \delta|x| + \varepsilon\delta) + 2(|a||z| + |b||x| + |a||b|)$$

, which can be simplified to

$$|a||z| + |b||x| + |a||b| \leq \varepsilon|z| + \delta|x| + \varepsilon\delta$$

Also note that if we use $x - y$ as a, $z - w$ as b, and derive $|xz - yw|$ from

$$xz = (y + a)(w + b),$$

then what we will get is that xz, yw are $(\delta|y| + \varepsilon|w| + \delta\varepsilon)$ close.

This consequence may seem obvious, but in fact it isn't. And should we change some variables of them, the result may vary. This example tells us that we should be very cautious when dealing with inequalities. What we should do is to carefully derive conclusions from what we have proven instead of taking intuitive things for granted. □

Exercise 4.3.3

Proof. (a)

1. Use induction. We induct on m. First, $x^n x^0 = x^n 1 = x^{n+0}$.

   Suppose that for m, the statement is already true. Then

$$\begin{aligned} x^n x^{m+1} &= x^n(x^m \times x) \\ &= x^n x^m \times x \\ &= x^{n+m} \times x && \text{(The induction hypothesis)} \\ &= x^{n+m+1} \end{aligned}$$

2. Use induction. We induct on m. First, $(x^n)^0 = 1 = x^{n\times 0}$.

   Suppose that for m, the statement is already true. Then

$$\begin{aligned}(x^n)^{m+1} &= (x^n)^m \times x^n \\ &= x^{mn} \times x^n && \text{(The induction hypothesis)} \\ &= x^{mn+n} && \text{(By the previous statement)} \\ &= x^{n(m+1)}\end{aligned}$$

3. Use induction. We induct on n. First, $(xy)^0 = 1 = x^0y^0$.

   Suppose that for m, the statement is already true. Then

$$\begin{aligned}(xy)^{m+1} &= (xy)^m \times xy \\ &= x^m y^m \times xy && \text{(The induction hypothesis)} \\ &= x^m \times x \times y^m \times y \\ &= x^{m+1}y^{m+1}\end{aligned}$$

(b) On one hand, if $x = 0$, then for $n > 0$, $0^n = 0$.

On the other hand, if for $n > 0$, $x^n = 0$, we need to prove $x = 0$. We try to show that $x \neq 0 \Longrightarrow x^n \neq 0$. Use induction. Since that $n \neq 0$, we start from $n = 1$. $x^1 = x^0 \times x = x$.

Suppose that for n, the statement is already true. Then

$$x^{n+1} = x^n \times x,$$

which is the product of two positive rationals, and which is thus positive.

(c) (1) Use induction: $x^0 = y^0 = 1 > 0$.

Suppose that for n, the statement is already true. Then we have two inequalities here:

$$x^n \geq y^n \geq 0$$

and

$$x \geq y \geq 0$$

We can multiply the two because of Proposition 15. Then we have

$$x^{n+1} \geq y^{n+1} \geq 0$$

If $n > 0$, then we induct from 1. The process resembles to what we have just done, so I don't write it here.

(d) Use induction: $|x^0| = |1| = 1 = |1|^0$.

Suppose that for n, the statement is already true. Then

$$|x^{n+1}| = |x^n \times x| = |x^n||x| = |x|^n|x| = |x|^{n+1}$$

□

Definition 4.3.11 We can see that there are now two versions of x^{-1}. Now we try to show that they express the same thing. Write x as a/b. The first version is $x^{-1} = b/a$.

The second version is $x^{-1} = 1/x = 1 \times x^{-1}(\text{version 1}) = x^{-1} = b/a$.

Note that only after we have known this can we say that for the second version of x^{-1}, $(x^{-1})^{-1} = x$.

Now we can also say that $x^{-n} = (x^n)^{-1}$

Exercise 4.3.4

Proof. Except for (3), we have already proven these properties when $m, n \in \mathbb{N}$. Then we will just write m, n as $-m, -n$.

(a) (1)

$$x^{-m}x^{-n} = \frac{1}{x^m}\frac{1}{x^n} = \frac{1}{x^m x^n} = 1/x^{m+n} = x^{-m-n}$$

(2) Before doing this, we must derive that for integers a, b and natural number n, $(a/b)^n = a^n/b^n$. Use induction: $(a/b)^0 = 1 = 1/1 = a^0/b^0$.

Suppose that for n, the statement is already true. Then

$$(a/b)^{n+1} = (a^n/b^n)(a/b) = (a^{n+1}/b^{n+1})$$

We can now close the induction.

Thus,

$$\begin{aligned}
&(x^{-n})^{-m} \\
&= ((1/(x^n))^m)^{-1} && (x^{-n} = (x^n)^{-1}) \\
&= (1/(x^n)^m)^{-1} \\
&= (x^n)^m \\
&= x^{mn}
\end{aligned}$$

(3)

$$\begin{aligned}
(xy)^{-n} &= 1/(xy)^n \\
&= 1/(x^n y^n) \\
&= (1/x^n)(1/y^n) \\
&= x^{-n}y^{-n}
\end{aligned}$$

(b)

Lemma 20. *For $x = (a/b) > y = (c/d) > 0$,*

$$0 < x^{-1} < y{-}1$$

Proof. Let $a, b, c, d > 0$. We know that

$$a \times b^{-1} > c \times d^{-1}$$

Multiply it with $bd > 0$, we have

$$ad > bc$$

. Multiply it with $a^{-1}c^{-1} > 0$, we have

$$d/c > b/a$$

That is,

$$y^{-1} > x^{-1}$$

And they are obviously bigger than 0. □

According to the lemma,

$$x^n \geq y^n \Longrightarrow ((x^n)^{-1} \leq (y^n)^{-1} \equiv x^{-n} \leq y^{-n})$$

(c) We first show that for positive integer n, the statement is true. We assume that $x > y$. Another situation is similar. Use induction, we try to prove that $x \neq y \Longrightarrow x^n \neq y^n$. We need to start from $n = 1$. First, $x^1 = x > y^1 = y$.

Suppose that for n, the statement is already true. Then Multiply $x > y$ with $x^n > y^n$, we have $x^{n+1} > y^{n+1}$.

We can now close the induction.

Then for negative ones, we know that $1/x = 1/y$ iff $x = y$, so $x^n \neq y^n \equiv x^{-n} \neq y^{-n}$.

(d) It is immediately derived since $1/x = 1/y \equiv x = y$. □

Exercise 4.3.5

Proof. Use induction from 1.

$2^1 = 2 > 1$.

Suppose that for N, the statement is already true. Then

$$2^{N+1} = 2 \times 2^N > 2N \geq N + 1$$

(Note that $N \geq 1 \Longrightarrow 2N \geq N + 1$) □

12 Gaps In The Rational Numbers

Exercise 4.4.1

Proof. Existence: We show that when $x \geq 0$, $(\exists n \in \mathbb{Z})(n \leq x < n+1)$. Write x as a/b, where $a \geq 0, b > 0$. If $a = 0$, then $x = 0$, $n = 0$. If $a \neq 0$, then according to Proposition 2.3.9,

$$\exists m \exists r(a = mb + r),$$

where $m, r \in \mathbb{N}$ and $r < b$. Because of this, $mb+b > a$, so $(mb+b)/b > a/b = x$, which means $m+1 > x$. On the other hand, $mb \leq a$, so $mb/b \leq a/b = x$, which means $m \leq x$.

Then when $x < 0$, then $-x > 0$, and $(\exists n \in \mathbb{Z})(n \leq -x < n+1)$, so

$$-n \geq x > -n-1$$

$\geq$ means $>$ or $=$ (exclusive). When $-n > x > -n-1$, let $m = -n-1$, then $m \leq x < m+1$ is true. When $-n = x > -n-1$, let $m = -n$, then $m \leq x < m+1$ is also true. So m is the integer we want if $x < 0$.

Uniqueness: For $n \leq x < n+1$, and $m \neq n$, we try to prove that $m \leq x < m+1$ is not possible. Before doing this, we need some lemmas:

Lemma 21. *(1) For integers i, j, $i < j \equiv i+1 \leq j$. (2) For integer i, there is no integer j such that $i < j < i+1$.*

Proof. (1) It has already been proven for natural numbers. If $i, j < 0$, then $-i, -j > 0$.

$$i < j \equiv -i > -j \equiv -i \geq -j+1 \equiv i \leq j-1 \equiv i+1 \leq j$$

(2) Suppose the contradiction, that there exists a integer j such that $i < j < i+1$. We know that $i < j \equiv i+1 \leq j \equiv j \geq i+1$. But we also have $j < i+1$, which is impossible. □

m either $<$ or $>$ n. On the former case, $m+1 \leq n \leq x$, so $m+1 > x$ is not possible. One the latter case, $x < n+1 \leq m$, so $m \leq x$ is impossible. □

Exercise 4.4.2

Proof. (a) We will use a different approach from the hint the book provided here. After assuming the contradiction, we try to prove that $a_n \leq a_0 - n$. Note that subtraction may results in a overflow (that is, natural numbers

flows to negative integers). So we will first define a_n as integers. And we try to show that no such infinite descent sequences can only lie in $\mathbb{N}$.

Use induction: $a_0 \leq a_0 - 0$.

Suppose the statement for n is already true, then $a_{n+1} < a_n \equiv a_{n+1}+1 \leq a_n \equiv a_{n+1} \leq a_n - 1$. Then we have $a_{n+1} \leq (a_0 - n - 1 = a_n - (n+1))$. We can now close the induction.

However, let $n = a_0$, then $a_{n+1} \leq n - n - 1 = -1$, which means that a_{n+1} does not lie in $\mathbb{N}$.

(b) (1) Yes. For example, $a_n := -n$ satisfies our restrictions.

(2) Yes. Because it is always possible to find a rational between 0 and a_0. □

Exercise 4.4.3

Proof. (1) Suppose the contradiction, that natural number $n = 2k = 2k'+1$, where k, k' are also natural. Then $2k = 2k' + 1$ Then we have $2k > 2k' \implies k > k'$. But $2k + 1 = 2(k' + 1)$, so $2(k' + 1) > 2k \implies k' + 1 > k$. But we know that between $k', k' + 1$ there exists no natural numbers. so it is impossible. (Note that we don't have a proposition saying $ac > bc \implies a > b$ for natural numbers, but we can first deal with them with the range of rationals. Multiply them with c^{-1}, and we will see that the result of the two sides are also natural numbers, so for natural numbers this is true.)

(2)

$$p^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

(3) Treat p, q as rationals. $p^2/2 = q^2 \implies q^2 < p^2$. We show that $q \geq p$ can not be true. It is obvious that $q \neq p$. And when $q > p$, multiply it with itself, $q^2 > p^2$, which is impossible. □

Part IV
The Real Numbers

13 Cauchy Sequences

Exercise 5.1.1

Proof. Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, it is 1-steady for some N. That implies,

$$\forall n \geq N(|a_n - a_N| \leq 1)$$

And we know that $a_1, a_2, \ldots, a_N$ is bounded by some number M, which means $|a_N| \leq M$. Expand $|a_n - a_N| \leq 1$ to obtain

$$a_N - 1 \leq a_n \leq a_N + 1$$

If $a_N \geq 0$, then $a_n \geq a_N - 1 \geq -a_N - 1$, then

$$-(a_N + 1) \leq a_n \leq a_N + 1 \equiv |a_n| \leq |a_N + 1|$$

So

$$|a_n| \leq |a_N + 1| \leq |a_N| + |1| \leq M + 1$$

If $a_N < 0$, then $a_n \leq a_N + 1 < -a_N + 1$, then

$$a_N - 1 \leq a_n \leq -(a_N - 1) \equiv |a_n| \leq |a_N - 1|$$

We can also get

$$|a_n| \leq |a_N - 1| \leq |a_N| + |1| \leq M + 1$$

Therefore, we know that $(a_n)_1^{\infty}$ is bounded by $M + 1$. □

14 Equivalent Cauchy Sequences

Exercise 5.2.1

Proof. Although we need to prove that a_n being a Cauchy sequence is logically equivalent to b_n being a Cauchy sequence, showing that one implies another is enough due to the structure of these statements.

Now we show that a_n being a Cauchy sequence implies that b_n being a Cauchy sequence. We need to show that for any $\varepsilon > 0$, there exists a N such that for all $m, n \geq N$, $|a_m - a_n| \leq \varepsilon$.

First we know that for any $\varepsilon > 0$, there exists a N such that $\forall m, j \geq N(|a_m - b_j| \leq \varepsilon)$. By substituting m with n we have both $|a_m - b_j| \leq \varepsilon$ and $|a_n - b_j| \leq \varepsilon$. Thus,

$$|a_m - a_n| = |(a_m - b_j) - (a_n - b_j)| \leq |a_m - b_j| + |a_n - b_j| \leq 2\varepsilon$$

Then we find the N' such that $\forall m, n \geq N(|a_m - b_n| \leq \varepsilon/2)$, and then for $i, j \geq N$, $|a_i - a_j| \leq \varepsilon$. □

Exercise 5.2.2

Proof. We need only to show that a_n being bounded implies that b_n being so because of the structure of these statements.

Consider Exercise 5.1.1. Since that $(a_n)_{n=1}^{\infty}, (b_n)_{n=1}^{\infty}$ are eventually ε-close, the sequence $(a_n - b_n)_{n=1}^{\infty}$ is eventually ε-steady. Thus, according to the exercise mentioned, it is bounded by some number M. And we say that a_n is bounded by some number N.

So $|a_n - b_n| \leq M$. Again, similar to the proof of $|a_n| \leq |a_N| + 1$ in Exercise 5.1.1, we can obtain that $|b_n| \leq |a_n| + M \leq N + M$. Therefore, b_n is also bounded. □

15 The Construction of the Real Numbers

Exercise 5.3.1

Proof. Reflectivity: It is immediately derived from $|a_n - a_n| = 0$.

Symmetry: It is immediately derived from $|a_n - b_n| = |b_n - a_n|$.

Transitivity: For any $\varepsilon > 0$, we can find M, N such that $\forall n \geq M(|a_n - b_n| \leq \varepsilon)$ and $\forall n \geq N(|b_n - c_n| \leq \varepsilon)$. Let $B = \max(M, N)$. Then for $n \geq B$,

$$|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n| \leq 2\varepsilon$$

This can also be deduced by (c) in Proposition 4.3.7 So a_n and c_n are also equal. □

Exercise 5.3.2

Proof. (1) We need to show that $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence. For any $\varepsilon > 0$, we can find M, N such that $\forall i, j \geq M(|a_i - a_j| \leq \varepsilon)$ and $\forall i, j \geq N(|b_i - b_j| \leq \varepsilon)$. Let $B = \max(M, N)$. Then for $i, j \geq B$, (See (h) in Proposition 4.3.7)

$$|a_i b_i - a_j b_j| \leq \varepsilon(|a_i| + |a_j|) + \varepsilon^2$$

Note that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, so it is bounded by some number M. Thus, $|a_i| + |a_j| \leq 2M$.

For any $\varepsilon' > 0$, we need to find a $\varepsilon > 0$ such that $\varepsilon(|a_i| + |a_j|) + \varepsilon^2 \leq \varepsilon'$. First, if $\varepsilon' \geq 1$, then by setting $\varepsilon < 1$ we can obtain $\varepsilon^2 < 1 \leq \varepsilon'$; if $\varepsilon' < 1$, then we let $\varepsilon < \varepsilon'$, and multiply it with $\varepsilon < 1$, we then have $\varepsilon^2 < \varepsilon'$. After these steps, we can ensure that $\varepsilon' - \varepsilon^2 > 0$.

Consider the number $t = \dfrac{\varepsilon' - \varepsilon^2}{2M} > 0$. If ε already satisfies $\varepsilon < t$, then it is the number we want. If it doesn't, then we can shrink it. That is, let $\varepsilon'' < t \leq \varepsilon$. $\varepsilon'' < \varepsilon$ gives $(\varepsilon'')^2 < (\varepsilon)^2$, then $-(\varepsilon'')^2 > -(\varepsilon)^2$, and finally

$$t'' = \frac{\varepsilon' - (\varepsilon'')^2}{2M} > \frac{\varepsilon' - \varepsilon^2}{2M}$$

So $\varepsilon'' < t < t''$. We can set ε to this ε''. Then

$$\begin{aligned} \varepsilon < \frac{\varepsilon' - \varepsilon^2}{2M} &\Longrightarrow \\ \varepsilon \times 2M < \varepsilon' - \varepsilon^2 &\Longrightarrow \\ \varepsilon \times 2M + \varepsilon^2 < \varepsilon' & \end{aligned}$$

So no matter what $\varepsilon' > 0$ is, we can always find $\varepsilon > 0$ such that

$$\varepsilon(|a_i| + |a_j|) + \varepsilon^2 \leq \varepsilon'$$

. And for this ε', there exists $N \geq 1$ such that

$$\forall i, j \geq N(|a_i b_i - a_j b_j| \leq \varepsilon(|a_i| + |a_j|) + \varepsilon^2 \leq \varepsilon')$$

Then, $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence. So is xy a real number.

(2) For any $\varepsilon > 0$, we can find N such that $\forall n \geq N(|a_n - a'_n| \leq \varepsilon)$. Thus, for such n,

$$|a_n b_n - a'_n b_n| = |b_n||a_n - a'_n| \leq \varepsilon |b_n|$$

Note that $(b_n)_{n=1}^{\infty}$ is bounded by some number M. So $|a_n b_n - a'_n b_n| \leq \varepsilon M$. Therefore, we find the N' such that $\forall n \geq N'(|a_n - a'_n| \leq \varepsilon/M)$. Then for such n, $|a_n b_n - a'_n b_n| \leq \varepsilon$. Thus $(a_n b_n)_{n=1}^{\infty} = (a'_n b_n)_{n=1}^{\infty}$. □

Exercise 5.3.3

Proof. On one hand, if $a = b$, then obviously $a, a, \cdots = b, b, \cdots$.

On the other hand, if $a, a, \cdots \neq b, b, \cdots$, we try to show that $a = b$. Presume the contradiction, that is, $a \neq b$. Then, $|a_n - b_n| = |a - b| \geq |a - b|$. For any $0 < \varepsilon < |a - b|$, the two Cauchy sequences cannot be $\varepsilon-$close, which is impossible. □

Lemma 5.3.14 Here it is asked that why can we deduce $|b_n| \geq \varepsilon/2$ from $|b_{n0} - b_n| \leq \varepsilon/2$ and $|b_{n0} > \varepsilon$. The book says that the triangle inequality can be used. In fact, we use the fact

$$||b_{n0}| - |b_n|| \leq |b_{n0} - b_n|$$

instead of $|b_{n0} - b_n| \leq |b_{n0}| + |b_n|$. Since that $|b_{n0}| - |b_n| \leq ||b_{n0}| - |b_n||$, we have

$$|b_{n0}| - |b_n| \leq \varepsilon/2 \equiv |b_{n0}| \leq \varepsilon/2 + |b_n|$$

But $|b_{n0}| > \varepsilon$, so $\varepsilon/2 + |b_n| > \varepsilon \equiv |b_n| > \varepsilon/2$.

It is not mentioned in (b) of Proposition 4.3.3, but it can be easily proven if we divide conditions, though the process is indeed very tedious.

Exercise 5.3.4

Proof. Since that the two Cauchy sequences are equivalent, they are eventually $\varepsilon-$steadiness for any $\varepsilon > 0$. Then, according to Exercise 5.2.2, $(a_n)_{n=1}^{\infty}$ being bounded implies that $(b_n)_{n=1}^{\infty}$ being so. □

Exercise 5.3.5

Proof. We show that $(\frac{1}{n})_{n=1}^{\infty} = (0)_{n=1}^{\infty}$.

For each $\varepsilon > 0$, we want to find $N \in \mathbb{N}$ such that $n \geq N \longrightarrow |\frac{1}{n} - 0| \leq \varepsilon$. Note that

$$|\frac{1}{n} - 0| \leq \varepsilon \equiv \frac{1}{n} \leq \varepsilon \equiv \frac{1}{\varepsilon} \leq n$$

, which means that we need to find $N \geq \frac{1}{\varepsilon}$. This is always possible as stated by Proposition 4.4.1.

Then the two sequences are equivalent, which proves our proposition. □

Part V
Mathematical Logic

16 Mathematical Statements

Exercise A.1.1 It is ((both X, Y are false) or (both X, Y are true)).

Exercise A.1.2 It is ((Y can be true even if X is false) or (Y can be false even if X is true)).

Exercise A.1.3 Yes. That's the definition of logical equivalent.

Exercise A.1.4 No. It is still possible that (even if X is false, Y is still true).

Consider a statement Y that satisfies:

1. If X, then Y.
2. If X is false, then Y or (exclusively) Y is false.

X, Y satisfy the description in the exercise, but they are not logical equivalent.

Exercise A.1.5 Yes. (Now I'm using the symbols defined in the A.2 for the sake of simplification) $X \iff Y$ means $X \implies Y \wedge \neg X \implies \neg Y$. So does Y and Z. So

$$(X \implies Y \implies Z \wedge \neg X \implies \neg Y \implies \neg Z) \implies (X \implies Z \wedge \neg X \implies \neg Z)$$

, which means X and Z are logical equivalent.

(Note that $A \implies B$ can also be interpreted as a statement, meaning "If A is true, then B is true", just like we did in this example.)

Exercise A.1.6 Yes. $(X \implies Y \implies Z) \implies (X \implies Z)$.

Now we are proving that $Z \implies X \equiv \neg X \implies \neg Z$. Assume that $\neg X \wedge Z$. Since $Z \implies X$, we have a contradiction: $X \wedge \neg X$.

So $X \implies Z \wedge \neg X \implies \neg Z$. Therefore, X, Z are logical equivalent. Besides, we can conclude that $Y \implies X$. Thus X, Y are also logical equivalent.

17 Implication

Why did Tao say

> If X, then Y can also be written as "X can only be true when Y is true"

?

Assume the $X \wedge \neg Y$, but $X \Longrightarrow Y$. So we have a contradiction $Y \wedge \neg Y$.

Define "when $x \neq 2$, $X : x = 2 \Longrightarrow x^2 = 4$ is vacuously true" to ensure that X is always true regardless of the value of x.

My Own Exercise Most of the time, rules of implication are intuitive. But they can be confusing some times. So hereby I introduce an example which I encountered, and which has confused me for a short time.

Proposition 16. *Let P, Q, R be statements, thus*

$$P \Longrightarrow (Q \Longrightarrow R) \equiv (P \wedge Q) \Longrightarrow R$$

Proof. In order to ascertain that two statements in the form of implication are logically equivalent, we must deeply understand what they are. At one time (that is, when all variables have definite value), a statement can only be either true or false, not both. And for a statement in the form of implication: $X \Longrightarrow Y$, it is true iff (*If* X, then Y). We do not need to check it if X is not true.

Now back to the subject. To prove that the two are logically equivalent, we need to show that both (if the former is true, then the latter is true) and (if the latter is true, then the former is true).

Now suppose that $P \Longrightarrow (Q \Longrightarrow R)$ is true. That is, if P, then (if Q, then R). To show that under this condition the latter is true, we need to show that if P, Q are both true, then R is true. Suppose that $P \wedge Q$. Since P is true, (if Q, then R) is true. And we know that Q is true, so R is true. so the latter is true.

Now suppose that the latter is true. We need to verify that the former is also true under this condition. Suppose P is true, then we need to show $Q \Longrightarrow R$ is true, that is, if Q, then R, and we furthermore suppose that Q is true. Now $P \wedge Q$ is true, so we have R is true. □

18 Nested Quantifiers

Exercise A.5.1 **(a)** Let P be $y^2 = x$ is true for each positive number y. And this statement means P is true for each positive number x.

Gaming metaphor: Me and my friend each randomly pick up a positive, say x and y, and check if $y^2 = x$.

The statement is false.

(b) There is at least one positive number x such that for every positive number y, $y^2 = x$.

Gaming metaphor: I have to pick up a positive number x such that whatever positive number y my friend picks up, $y^2 = x$ is always true.

The statement is false.

(c) There is at least two positive numbers x, y such that $y^2 = x$.

Gaming metaphor: Me and my friend each have to pick up a positive number, say x and y, such that $y^2 = x$.

The statement is true. For example, $1^2 = 1$.

(d) The statement $\exists x > 0, y^2 = x$ is true for every $y > 0$.

Gaming metaphor: For each positive number y my friend picks up, I have to pick up a positive number x such that $y^2 = x$.

The statement is true, because y^2 is also positive.

(e) There is at least one positive number y such that for every positive number x, $y^2 = x$ is always true.

Gaming metaphor: I have to find a number $y > 0$ such that regardless of what number x my friend picks up, $y^2 = x$ is always true.

The statement is false.

19 Equality

Exercise A.7.1

Proof. Let $F(x) := x + c$. By axiom 4, $F(a) = F(b)$. That is, $a + c = b + c$. Similarly, by letting $G(x) := a + x$, we have $a + c = a + d$, which, according to axiom 2, becomes $a + d = a + c$. Now we have $a + d = a + c, a + c = b + c$. According to axiom 3, we can conclude that $a + d = b + c$. □

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