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Assignment 3
Subject Code: MAST30027
Lab Time: Thursday 11:00am
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MAST30025 Assingment 3

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Question 1:

#Prequestion

```r
X = scan(file="assignment3_prob1.txt", what=double())
length(X)
```

```
## [1] 100
```

```r
mean(X)
```

```
## [1] 75.726
```

a. Derive the posterior distribution of τ.

Since $x_i|\tau \sim \mathrm{N}(75, \frac{1}{\tau})$ and $\tau \sim \mathrm{Gamma}(2, 1)$ Let $X = (x_1, \ldots, x_{100})$.

The likelihood of $X|\tau$ is given by:

$$
\begin{aligned}
f(X|\tau) &= \prod_{i=1}^{100} f(x_i|\tau) \\
&\propto (\tau)^{50} \exp\left[\frac{-\tau}{2}\sum_{i=1}^{100}(x_i - 75)^2\right]
\end{aligned}
$$

Thus $P(\tau|X)$ is given by:

$$
\begin{aligned}
P(\tau|X) &\propto f(X|\tau)P(\tau) \\
&\propto (\tau)^{50} \exp\left[\frac{-\tau}{2}\sum_{i=1}^{100}(x_i - 75)^2\right](\tau)\exp(-\tau) \\
&\propto (\tau)^{52-1} \exp\left[-\tau(1 + \frac{1}{2}\sum_{i=1}^{100}(x_i - 75)^2)\right]
\end{aligned}
$$

By inspection we can see $\tau|X$ is gamma distribution with $\alpha = 52$ and $\beta = 1 + \frac{1}{2}\sum_{i=1}^{100}(x_i - 75)^2$ or $\tau|X \sim \mathrm{Gamma}(52, 1 + \frac{1}{2}\sum_{i=1}^{100}(x_i - 75)^2)$.

```r
#Calculating therotical beta value
beta = 0.5*sum((X-75)^2)+1
beta
```

```
## [1] 1805.65
```

```r
alpha = 2+0.5*length(X)
alpha
```

```
## [1] 52
```

Thus $\tau|X \sim \text{Gamma}(52, 1805.65)$.

b. Derive the posterior predictive distribution for new score $\tilde{x}$, $P(\tilde{x}|x_1, \ldots, x_{100})$.

From q1 part a we know $\tau|X \sim \text{Gamma}(52, 1805.65) = \text{Gamma}(\alpha, \beta)$ and $\tilde{x}|\tau \sim \text{N}(75, \frac{1}{\tau})$, where $X = (x_1, \ldots, x_{100})$.

$$
\begin{aligned}
P(\tilde{x}|x_1, \ldots, x_{100}) &= \int_0^\infty P(\tilde{x}|\tau)P(\tau|X) \, \mathrm{d}\tau \\
&= \int_0^\infty \sqrt{\frac{\tau}{2\pi}} \exp\left[\frac{-\tau}{2}(\tilde{x}-75)^2\right] \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} \exp(-\beta\tau) \, \mathrm{d}\tau \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}} \int_0^\infty \exp\left[-\tau\left(\frac{(\tilde{x}-75)^2}{2} + \beta\right)\right] \tau^{\frac{2\alpha+1}{2}-1} \, \mathrm{d}\tau \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)\sqrt{2\pi}} \frac{\Gamma(\frac{2\alpha+1}{2})}{\left[\frac{(\tilde{x}-75)^2}{2} + \beta\right]^{\frac{2\alpha+1}{2}}}
\end{aligned}
$$

Using Gamma distribution normalising constant to solve integration.

$$
\begin{aligned}
P(\tilde{x}|x_1, \ldots, x_{100}) &= \frac{\Gamma(\frac{2\alpha+1}{2})}{\Gamma(\frac{\alpha}{2})\sqrt{2\pi\beta}} \left[\frac{(\tilde{x}-75)^2}{2\beta} + 1\right]^{-\frac{-2\alpha+1}{2}} \\
&= \frac{\Gamma(\frac{2\alpha+1}{2})}{\Gamma(\frac{2\alpha}{2})\sqrt{2\alpha\pi\frac{\beta}{\alpha}}} \left[\frac{1}{2\alpha}\frac{(\tilde{x}-75)^2}{\frac{\beta}{\alpha}} + 1\right]^{-\frac{-2\alpha+1}{2}}
\end{aligned}
$$

```r
v=alpha*2
v
```

```
## [1] 104
```

```r
a=75
a
```

```
## [1] 75
```

```r
b=beta/alpha
b
```

```
## [1] 34.72404
```

By inspection we can see the predictive distribution for new $\tilde{x}|(x_1, \ldots, x_{100}) \sim \text{t}(104, 75, 34.72)$, which is a student-t distribution with three parameter of 104, 75, 34.72.

Question 2:

Using the information given we will simulate gamma distribution with parameter $\alpha \geq 1$ and β, as stated below

Let

$$
\begin{aligned}
f(x) = \frac{h(x)^{\alpha-1} e^{-h(x)} h'(x)}{\Gamma(\alpha)} \\
d = \alpha - 1/3, \\
c = \frac{1}{\sqrt{9d}}, \\
h(x) = d(1+cx)^3 \\
g(x) = d\log((1+c*x)^3) - d*(1+c*x)^3 + d
\end{aligned}
$$

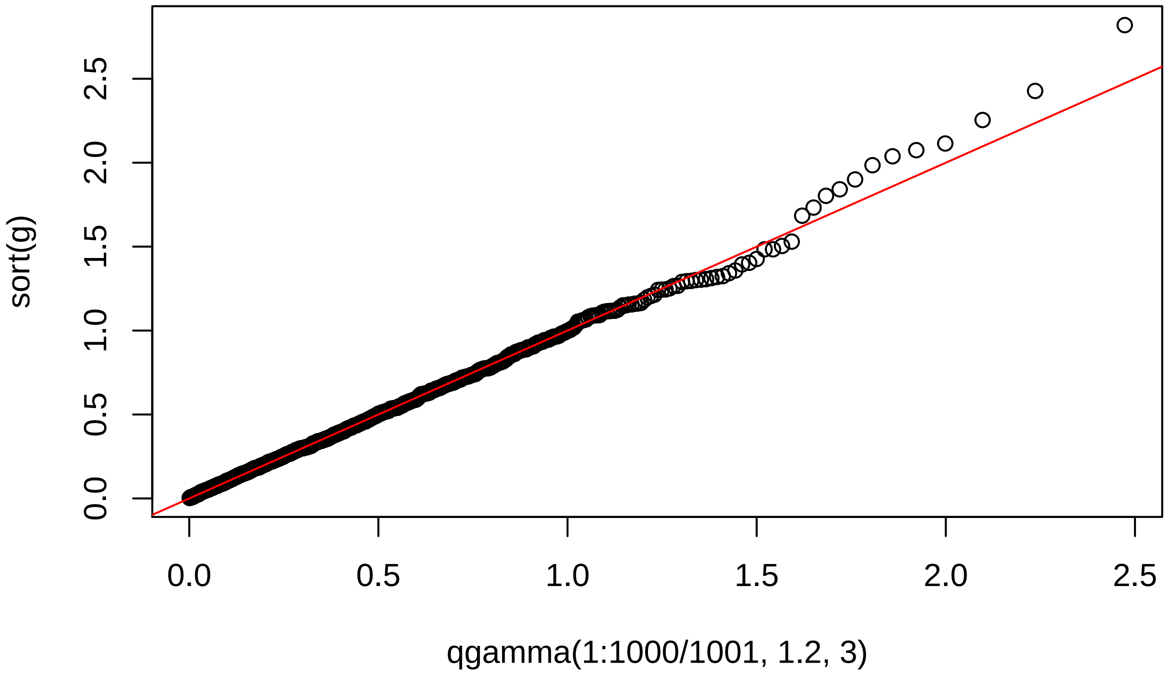
Using $f(x) * \Gamma(\alpha) \propto \exp(g(x))$, we can simulate $f(x)$ by simulating $\exp(g(x))$ using the general rejection method with $p(x) = e^{\frac{-x^2}{2}}$ as an envelope with $X \sim N(0,1)$ with either $X \geq \frac{-1}{c}$ or $h(x) \geq 0$.

The algorithm used follow the step below:

1. Simulate X from $N(0,1)$.
2. If $X \geq \frac{-1}{c}$ return X else start from step 1
3. Generate $Y \sim U(0, p(X))$
4. If $Y <$ exp(g(X)) return $\frac{\exp(g(X))}{\beta}$ else return to step 1
5. Repeat until we get the number of n sample required

```r
#Simulating Gamma Sample with alpha = a, beta = b, number of required sample = n
sim.gamma <- function(a,b,n){
  gamma.sample = c()
  for (i in 1:n){
    gamma.sample[i] = X.Gamma(a)/b
  }
  return(gamma.sample)
}


#Function to Stimulate X and apply general rejection method to g(x) with p(x) as
#an envelope function. Step 1, 2, 3 and Part of 4
X.Gamma <- function(a){
  d=a-1/3
  c=1/sqrt(9*d)
  x.sample=c()
  h <- function(x) d*(1+c*x)^3
  while(TRUE){
    while(TRUE){
      x.point = rnorm(1)
      if (x.point >= -1/c) break
    }
    Y <- runif(1,0,p(x.point))
```

```
    if (Y<exp.g(x.point,a)) return(h(x.point))
  }
}

#Let p(x)=exp(-x^2/2)
p <-function(x){
  x=exp(-x^2/2)
  return(x)
}


#Define exp(g(x))
exp.g <- function(x,a){
  d=a-1/3
  c=1/sqrt(9*d)
  q <- function(x) (1+c*x)^3
  g <- function(x) d*log(q(x))-d*q(x)+d
  x= exp(g(x))
  return(x)
}

#Running the simulation and Plotting qq-plot
g <- sim.gamma(1.2,3,1000)
plot(qgamma(1:1000/1001, 1.2, 3), sort(g))
abline(0, 1, col="red")
```



Looking at the qq-plot we can see that our simulation are close to the pseudo-random of Gamma(1.2,3) for 1000 samples.