Linear Statistical Models Assignment 3

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Question 1: Let A be an $n \times p$ matrix with $n \geq p$.

a. Show directly that $r(A^c A) = r(A)$.

Let $\mathbf{A} = \left[\begin{array}{c|c} \mathbf{M} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$ where $\mathbf{M}$ is a square matrix with $r(\mathbf{A}) \times r(\mathbf{A})$ and $r(\mathbf{A}) = a \leq p$.
By definition $\mathbf{M}$ is a matrix with $r(\mathbf{M}) = r(\mathbf{A})$

Using theorem 6.2, we can find $\mathbf{A}^c$ such that it has the following partition:
$\mathbf{A}^c = \left[\begin{array}{c|c} \mathbf{M}^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]$ where $\mathbf{M}^{-1}$ is the inverse of $\mathbf{M}$.

Hence $\mathbf{A}^c\mathbf{A} = \left[\begin{array}{c|c} I_a & \mathbf{M}^{-1}A_{12} \\ \hline 0 & 0 \end{array}\right]$

All column vectors in $\mathbf{M}^{-1}A_{1,2}$ is a linear combination of the independent column vectors in the identity matrix I_a. Thus $r(\mathbf{A}^c\mathbf{A}) = r(I_a) = r(\mathbf{M}) = a$. Since $r(\mathbf{A}) = r(\mathbf{M})$, this implied $r(\mathbf{A}) = r(\mathbf{A}^c\mathbf{A})$.

b. Show directly that $A^c A$ is idempotent.

Since A^c is a conditional inverse for A then AA^cA=A. Thus,

$$\begin{aligned} (A^c A)^2 &= A^c A A^c A \\ &= A^c A \end{aligned}$$

Hence, $A^c A$ is idempotent

c. Show directly that $A(A^T A)^c A^T$ is unique (invariant to the choice of conditional inverse).

Consider conditional inverse $(A^T A)^c_1$ and an arbitrary conditional inverse $(A^T A)^c_i$ where $i \neq 1$. Now using the properties $A = A(A^T A)^c_i(A^T A)$ and $A^T = (A^T A)^c_1(A^T A)A^T$., we get the following:

$$\begin{aligned} A(A^T A)^c_1 A^T &= A(A^T A)^c_i (A^T A)(A^T A)^c_1 A^T \\ &= A(A^T A)^c_i A^T \end{aligned}$$

Since,$A(A^T A)^c_1 A^T = A(A^T A)^c_i A^T$, this implied it is unique and invariant to the choice of conditional inverse.

Question 3:

$$\text{Let } t = \left[\frac{t_1}{t_2}\right] = \left[\frac{X_1^T X_1 z_1}{X_2^T X_2 z_2}\right] \text{ and } X = \left[\begin{array}{c|c} X_1 & X_2 \end{array}\right]$$

$$X^T X\mathbf{a} = \left[\frac{X_1^T}{X_2^T}\right] \left[\begin{array}{c|c} X_1 & X_2 \end{array}\right] \mathbf{a} = \left[\begin{array}{c|c} X_1^T X_1 & X_1^T X_2 \\ \hline X_2^T X_1 & X_2^T X_2 \end{array}\right] \mathbf{a}$$

We can to rewrite the system of linear equation for $X^T X\mathbf{a} = t$, as an augmented matrix form as $[X^T X|t]$.

$$[X^T X|t] = \left[\begin{array}{cc|c} X_1^T X_1 & X_1^T X_2 & X_1^T X_1 z_1 \\ X_2^T X_1 & X_2^T X_2 & X_2^T X_2 z_2 \end{array}\right] = \left[\begin{array}{cc} X_1^T & 0 \\ 0 & X_2^T \end{array}\right] \left[\begin{array}{cc|c} X_1 & X_2 & X_1 z_1 \\ X_1 & X_2 & X_2 z_2 \end{array}\right]$$

Since, X_2 is continuous factor we can inferred that X_2 is column full rank and since X_1 less then full column rank we can inferred that X_1 can be written as a linear combination X_2.

Thus by theorem 6.2 that $r\left(\left[\begin{array}{cc|c} X_1 & X_2 & X_1 z_1 \\ X_1 & X_2 & X_2 z_2 \end{array}\right]\right) = r(X^T X)$ if and only if $\left[\begin{array}{cc|c} X_1 & X_2 & X_1 z_1 \\ X_1 & X_2 & X_2 z_2 \end{array}\right]$ is a consistent system.

Thus by the fact that X_2 is a linear combination of $X_1$and then $\left[\begin{array}{cc} X_1^T X_1 & X_1^T X_2 \\ X_2^T X_1 & X_2^T X_2 \end{array}\right]$ are linear combination of $\left[\frac{t_1}{t_2}\right]$ if there exist a z_1 such that $X_1^T X_1 z_1 = t_1$ is a consistent system.

Since, $t_1^T \beta_1$ is estimable then $X_1^T X_1 z_1 = t_1$ is a consistent system hence $X^T X z = t$ is also a consistent system. Thus, if $t_1^T \beta_1$ is estimable then $t^T \beta$ where $\beta^T = [\beta_1^T | \beta_2^T]$.