Topic 11. A Fast Algorithm: Reinsch Algorithm
Spring 2018

Recall we solved

$$\begin{aligned} \hat{f} &= S(\lambda)y \\ &= (I + \lambda Q^T M^{-1} Q)^{-1} y, \end{aligned}$$

where M and Q have been defined before, and $M\sigma = Qf$. To remove the matrix inverse, we have

$$(I + \lambda Q^T M^{-1} Q)\hat{f} = y,$$

which leads to the following:

$$\begin{aligned} \hat{f} &= y - \lambda Q^T M^{-1} Q \hat{f} \\ &= y - \lambda Q^T M^{-1} M \hat{\sigma} \\ &= y - \lambda Q^T \hat{\sigma}. \end{aligned} \tag{1}$$

On the other hand, we have

$$M\hat{\sigma} = Q\hat{f} = Qy - \lambda Q Q^T \hat{\sigma}.$$

To solve for $\hat{\sigma}$, we rewrite the above as

$$(M + \lambda Q Q^T)\hat{\sigma} = Qy. \tag{2}$$

Note that matrix $(M + \lambda Q Q^T)$ is band-limited with no more than five diagonals. Hence $\hat{\sigma}$ can be computed with $O(n)$. The following is called Reinsch algorithm, and its order of complexity should be no more than $O(n)$.

1. Solve $\hat{\sigma}$ via (2);
2. Compute for $\hat{f}$ via (1).