Matrices

Appendix D

Revised from Prof. Galen Sasaki's slides

Matrix Introduction

- Reading Assignment
  - Appendix D.1
- Matrices
  - Column and row vectors
    - Dot or scalar product, transpose
  - Matrices
    - Matrix addition and multiplication
    - Transpose
    - Identity matrix

Vectors

Column Vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}.$$

Row Vector

$$\mathbf{x} = [x_1 \; x_2 \; \ldots \; x_n].$$

Dot (or scalar) product

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} = [a_1 \;\; a_2 \;\; a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3,$$

Dyadic product

$$\mathbf{a} \otimes \mathbf{b} = \mathbf{a}\mathbf{b}^{\mathrm{T}} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1 \;\; b_2 \;\; b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix},$$

Transpose of vectors

$$[x_1 \; x_2 \; \ldots \; x_m]^{\mathrm{T}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}^{\mathrm{T}} = [x_1 \; x_2 \; \ldots \; x_m].$$

Multiple operations can be represented with single operation

Matrix

An m x n ***matrix A*** over a set S is a rectangular array of elements of S arranged into m rows and n columns. Such a matrix is said to have ***size m* x *n***

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{R}^{m \times n}.$$

$a_{i,j}$ = element in i^{th} row and j^{th} column

Special cases

- $m = n$: square matrix
- $m = 1$: row vector
- $n = 1$: column vector

Square matrix

$$\begin{bmatrix} 9 & 13 & 5 & 2 \\ 1 & 11 & 7 & 6 \\ 3 & 7 & 4 & 1 \\ 6 & 0 & 7 & 10 \end{bmatrix}$$

Diagonal

Special Square Matrices

Matrix is ***symmetric*** which means for all i,j $a_{i,j} = a_{j,i}$

$$A = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 2 & 0 & 2 \\ 2 & 2 & 0 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 2 & 1 & 1 & 0 \end{array}$$

An **permutation** matrix is a square matrix that has entries that are 0 or 1. Its row sums are 1 and its column sums are 1, i.e., each row has exactly one entry which is 1, and each column has exactly one entry that is 1

$$A = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 & 0 \end{array}$$

An **identity** matrix is a square matrix that is zero everywhere except on its diagonal, which are all 1s

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

More on Matrices

- Suppose A and B are both m x n matrices, and α is a scalar
- $A = B$ iff for all i and j, $a_{i,j} = b_{i,j}$
  - In other words, each element is equal
- (αA) is an m x n matrix, where for all i,j, $(\alpha A)_{i,j} = \alpha a_{i,j}$
  - In other words, multiply each element by α
- $(A + B)$ is an m x n matrix, where for all i,j, $(A + B)_{i,j} = a_{i,j} + b_{i,j}$
  - In other words, the addition is per element
- The ***transpose*** of A, denoted by A^T, is an n x m matrix, where for all i,j, $(A^T)_{i,j} = a_{j,i}$

$$\begin{bmatrix} 1 & -1 \\ 3 & 2 \\ -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -5 \\ -1 & 2 & 7 \end{bmatrix}^T$$

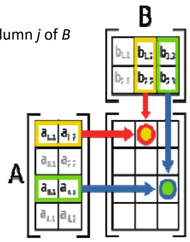
Matrix Multiplication

An ***m x n* matrix *A*** [*m rows of (row vectors of length n)*]
can be **multiplied** with
an ***n x p* matrix *B*** [*p columns of (column vectors of length n)*]

that results in an *m* x *p* matrix *AB*

where $[AB]_{ij}$ = the dot product of row *i* of *A* and column *j* of *B*
i ranges over 1 through *m*
j ranges over 1 through *p*

$$[AB]_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$



Matrix Multiplication

$$AB = \begin{pmatrix} a & b & c \\ p & q & r \\ u & v & w \end{pmatrix} \begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{pmatrix} = \begin{pmatrix} a\alpha+b\lambda+c\rho & a\beta+b\mu+c\sigma & a\gamma+b\nu+c\tau \\ p\alpha+q\lambda+r\rho & p\beta+q\mu+r\sigma & p\gamma+q\nu+r\tau \\ u\alpha+v\lambda+w\rho & u\beta+v\mu+w\sigma & u\gamma+v\nu+w\tau \end{pmatrix},$$

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Matrix Multiplication

Matrix multiplication has the associative property

$$ABC = A(BC) = (AB)C$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$Ax = b \quad \Leftrightarrow \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \end{aligned}$$

Square Matrix Multiplication

- Identity matrix I: special square matrix
  - 1s on the diagonal
  - 0s elsewhere
  - *AI* = *A*
  - I*A* = *A*

$$I_1 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \cdots,\ I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

- If *P* is a permutation matrix and *x* is a column vector then *Px* is a column vector that is basically *x* with its entries permuted
- For any *n* x n matrix *A*, the powers of *A* are defined as follows
  - A^0 = I, where I is the identity matrix
  - $A^n = A\,A^{n-1}$ for all integers $n \geq 1$
    - *A* to the power *n* makes sense due to associativity