Appendix A: Summations

EE602

Modified from Prof. Galen Sasaki's slides

Summations

- Reading Assignment: Appendix A

A.1 Basic Summation Formulas and Properties

- $\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$
- **Convergence and limits**
  - $\sum_{k=1}^{\infty} a_k$ means $\lim_{n \to \infty} \sum_{k=1}^{n} a_k$
  - If the limit exists then it **converges**, otherwise it **diverges**
- **Linearity**
  - $\sum_{k=1}^{n} (ca_k + b_k) = c \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$
  - $\sum_{k=1}^{n} \Theta(f(k)) = \Theta(\sum_{k=1}^{n} f(k))$???
    - The Θ-notation on the left-hand side applies to the variable k
    - The Θ-notation on the right-hand side applies to the variable n
    - This equality must be proved using definitions on the Θ-notation

Summation Formulas

- **Arithmetic series**
  - $\sum_{k=1}^{n} k = \frac{1}{2} n(n+1) = \Theta(n^2)$
- **Sum of squares and cubes**
  - $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
  - $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$
- **Geometric series**
  - $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$ if $x \neq 1$
  - $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ if $|x| < 1$
  - Assume $0^0 = 1$

Summation Formulas

- **Harmonic series**
  - $\sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$
- **Integrating and differentiating series**
  - $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ if $|x| < 1$
  - Differentiating both sides
  - $\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$ if $|x| < 1$
- **Telescoping series**
  - $\sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0$
  - Example: $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n}$
- **Products**
  - $lg(\prod_{k=1}^{n} a_k) = \sum_{k=1}^{n} \lg(a_k)$

A.2 Bounding Summations

- **Mathematical induction**
  - Please read Appendix A.2
  - We have to be careful with asymptotic notation such as O-notation
  - **Example:** $\sum_{k=1}^{n} k = O(n)$ (which is wrong)
    - Proof: (1) **Base step** is trivial, for n = 1, the sum is O(1)
    - **(2) Induction step**.
    - **Induction hypothesis** $\sum_{k=1}^{n} k = O(n)$
    - What is $\sum_{k=1}^{n+1} k$?
    - $\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + (n+1)$
    - $\sum_{k=1}^{n+1} k = O(n) + (n+1)$ ← wrong here
    - $\sum_{k=1}^{n+1} k = O(n+1)$
    - Problem with the proof is that the “constant” hidden by the big-oh grows with n, which makes it not a constant!

Bounding the Terms with the largest term

- **Simple technique → upper bound**
  - bound each term with the largest term
  - $\sum_{k=0}^{n} k \le \sum_{k=1}^{n} n = n^2$:
  - $\sum_{k=0}^{n} a_k \le \sum_{k=1}^{n} a_{max} = n \cdot a_{max}$
- This is a weak method
- We can bound a series by a geometric series

Bounding the Terms with a geometric series

- **Bound by a geometric series → upper bound**
  - Suppose $\frac{a_{k+1}}{a_k} \le r$ or $a_{k+1} \le r a_k$, 0 < r < 1
    - r must a constant in this case
  - Then $a_k \le a_0 r^k$
  - $\sum_{k=0}^{n} a_k \le \sum_{k=0}^{\infty} a_0 r^k = a_0 \sum_{k=0}^{\infty} r^k = a_0 \frac{1}{1-r}$
  - Example: $\sum_{k=1}^{\infty} (k/3^k)$
  - $\frac{(k+2)/3^{k+2}}{(k+1)/3^{k+1}} = \frac{1}{3}\frac{k+2}{k+1} \le \frac{2}{3}$ for all $k \ge 0$
  - $\sum_{k=1}^{\infty} \frac{k}{3^k} = \sum_{k=0}^{\infty} \frac{k+1}{3^{k+1}} \le \frac{1}{3}\frac{1}{1-2/3} = 1$

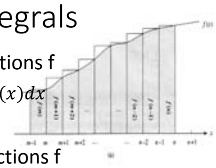
Splitting summations

- Split a summation into multiple parts, each part being easier to bound
- **Example: Lower bound for $\sum_{k=1}^{n} k$**
  - $\sum_{k=1}^{n} k = \sum_{k=1}^{n/2} k + \sum_{\frac{n}{2}+1}^{n} k \ge \sum_{k=1}^{n/2} 0 + \sum_{k=\frac{n}{2}+1}^{n} (\frac{n}{2}) = (n/2)^2 = \Theta(n^2)$
- **Example: Upper bound for $\sum_{k=0}^{\infty} \frac{k^2}{2^k}$**
  - Let's try to bound using a geometric series
  - The ratio of consecutive terms is $\frac{(k+1)^2/2^{k+1}}{k^2/2^k} = \frac{(k+1)^2}{2k^2}$
  - If k = 0, the ratio is 1/0 (!)
  - If k = 1, the ratio is 2.
  - If k = 2, the ratio is 9/8
  - If $k \ge 3$, the ratio is at most 8/9 (finally a value of r < 1)
  - $\sum_{k=0}^{\infty} \frac{k^2}{2^k} = \sum_{k=0}^{2} \frac{k^2}{2^k} + \sum_{k=3}^{\infty} \frac{k^2}{2^k} \le \sum_{k=0}^{2} \frac{k^2}{2^k} + \frac{9}{8}\sum_{k=3}^{\infty} \left(\frac{8}{9}\right)^k$

$$\le \sum_{k=0}^{2} \frac{k^2}{2^k} + \frac{9}{8}\sum_{k=0}^{\infty} \left(\frac{8}{9}\right)^k$$

So, we have $O(1)$

Approximation by Integrals



- For monotonically increasing functions f
  - $\int_{m-1}^{n} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)dx$
- For monotonically decreasing functions f
  - Similar to above except limits on the integrals change
- Example
  - $\int_{1}^{n+1} \frac{dx}{x} \le \sum_{k=1}^{n} \frac{1}{k} \le \int_{1}^{n} \frac{dx}{x}$ +1
  - ln(n+1) $\le \sum_{k=1}^{n} \frac{1}{k} \le \ln(n)$ +1