Appendix B: Sets, Relations, Functions, Tree, an Graphs, etc.

EE602

Modified from Prof. Galen Sasaki's slides

Sets, etc.

- Reading Assignment:
- Read Appendix B.1, B.2, and B.3

Topics

- Sets, relations, and functions
- Predicates and quantifiers
- Proof methods
  - Contradiction (by contrapositive is similar) and
  - Mathematical induction
- Recursion and recursive algorithms

B.1 The language of sets

- Suppose S is a set, x is an element of S
- **$x \in S$** means that x is a member of S
- **$x \notin S$** means that x is not a member of S
- **Set roster notation**: {1, 2, 3}, {1, 2, ..., 10}, or {1, 2, ...}
  - "..." is known as an ellipse, which means "and so forth"
- ***$\varnothing$ is an empty set***
- ***R*** is the set of all real numbers
  - R^{+} is the set of positive real numbers
- ***Z*** is the set of all integers
  - Z^{nonneg} is the set of nonnegative integers, i.e., the natural numbers *N*
- ***Q*** is the set of all rational numbers

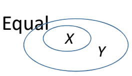
The language of sets

- **Set builder notation**: $\{x \in S \mid P(x)\}$ or $\{x \in S: P(x)\}$
  - Example: $\{x \in N: 11 < x < 100\}$
  - ":" or "|" means "such that"
- *A* is a ***subset*** of *B* ($A \subseteq B$) means every element in *A* is in *B*
- *A* is a ***proper subset*** of *B* ($A \subset B$) means *A* is a subset of *B* but *A* and *B* are not the same sets.
- *Forward slash "/" means "not"*
  - Examples: $\notin$ and $\not\subset$
- **Ordered pair** (x,y) – the order of the elements matters
  - Not the same as (y,x)
  - Not the same as $\{x,y\}$, where order does not matter

Subsets - definitions

- Definitions
  - Subset: $A \subseteq B \leftrightarrow \forall x$, if $x \in A$ then $x \in B$
  - Proper Subset: $A \subset B \leftrightarrow$
    - $A \subseteq B$ and
    - $\exists\, x \in B$ such that $x \notin A$
- Element Argument

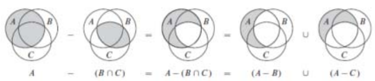
  Basic method for proving that one set is a subset of another
  - Let sets X and Y be given. To prove $X \subseteq Y$
  - 1. Suppose that x is a particular but arbitrarily chosen element of X
  - 2. Show that x is an element of Y
- Basic Method to Proving that Sets are Equal
  - Let sets X and Y be given. To prove X = Y.
  - 1. Prove $X \subseteq Y$
  - 2. Prove $Y \subseteq X$



X is a subset of *Y*

Set Operations and their laws

- The ***intersection*** of sets A and B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- The ***union*** of sets A and B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- The ***difference*** between two sets A and B is the set $A - B = \{x : x \in A \text{ and } x \notin B\}$
- Set operations obey the following laws:
  - **Empty set laws:** $A \cap \emptyset = \emptyset$, $A \cup \emptyset = A$
  - **Idempotency laws:** $A \cap A = A$, $A \cup A = A$
  - **Commutative laws:** $A \cap B = B \cap A$, $A \cup B = B \cup A$
  - **Associative laws:** $A \cap (B \cap C) = (A \cap B) \cap C$; $A \cup (B \cup C) = (A \cup B) \cup C$
  - **Distributive laws:** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cap (B \cup C) = (A \cup B) \cap (A \cup C)$;
  - **Absorption laws:** $A \cap (A \cup B) = A$; $A \cup (A \cap B) = A$;
  - **DeMorgan's laws:** $A - (B \cap C) = (A - B) \cup (A - C)$; $A - (B \cup C) = (A - B) \cap (A - C)$



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Identities

- Commutative laws
  - $A \cup B = B \cup A$
  - $A \cap B = B \cap A$
- Associative laws
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributive laws
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Identity Laws
  - $A \cup \emptyset = A$, here $\emptyset$ is the empty set
  - $A \cap U = A$, here U is the universal set

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Identities

- Complement law
  - $A \cup A^c = U$
  - $A \cap A^c = \emptyset$
- Double complement law
  - $(A^c)^c = A$
- Idempotent law
  - $A \cup A = A$
  - $A \cap A = A$
- Universal bound laws
  - $A \cap \emptyset = \emptyset$
  - $A \cup U = U$

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Identities

- De Morgan's laws
  - $(A \cup B)^c = A^c \cap B^c$
  - $(A \cap B)^c = A^c \cup B^c$
- Absorption laws
  - $(A^c)^c = A$
- Complements of U and $\emptyset$
  - $U^c = \emptyset$
  - $\emptyset^c = U$
- Set difference law
  - $A - B = A \cap B^c$

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Set Operations

- Union $\bigcup_{k=1}^{n} A_k$ $\bigcup_{k=1}^{\infty} A_k$
  - $A \cup B$ = set of elements that are in A or B
- Intersection $\bigcap_{k=1}^{n} A_k$ $\bigcap_{k=1}^{\infty} A_k$
  - $A \cap B$ = set of elements that are in A and B
- Difference (or exclusion)
  - $A - B$ (or $A \setminus B$) = set of elements in A that are not in B
- Complement
  - A^c = set of elements in U that are not in A

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The Universe, the *complement*

- all the sets under consideration are subsets of some larger set U called the ***universe***
- Given a universe U, we define the ***complement*** of a set A as $\sim A = U - A = \{x: x \in U \text{ and } x \notin A\}$.
- For any set $A \subseteq U$, we have the following laws:
  - $\sim\sim A = A$; $A \cap \sim A = \emptyset$; $A \cup \sim A = U$
  - Rewrite DeMorgan's Law: for any two sets $B, C \subseteq U$, we have $\sim(B \cap C) = \sim B \cup \sim C$; $\sim(B \cup C) = \sim B \cap \sim C$

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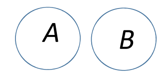
Disjoint Sets

- Two sets A and B are ***disjoint*** if they have no elements in common
  - If $A \cap B = \varnothing$;
- A collection $S = \{S_i\}$ of nonempty sets forms a ***partition*** of a set S if the sets are ***pairwise disjoint***
  - $S_i, S_j \in S$ and $i \neq j$ imply $S_i \cap S_j = \varnothing$;
  - And their union is S, that is, $S = \bigcup_{S_i \in S} S_i$

Disjoint sets

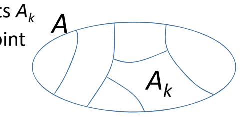


- *A* and *B* are **disjoint** $\leftrightarrow$ $A \cap B = \varnothing$
- Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pair-wise disjoint** or **nonoverlapping**)

  if and only if

  for all $i, k = 1, 2, 3, \ldots$, $A_i \cap A_k = \varnothing$

- A finite or infinite collection $\{A_1, A_2, A_3, \ldots\}$ of nonempty sets is a **partition** of a set *A* if, and only if
  1. A is the union of all the sets A_k
  2. The sets are mutually disjoint



Cardinality

- The number of elements in a set is the ***cardinality*** (or ***size***) of the set, denoted $|S|$
  - $|\varnothing| = 0$
  - $|A|$ = cardinality of set *A*
  - $|\{x, y\}| = 2$
  - If the cardinality of a set is a natural number, we say the set is ***finite***; otherwise, it is ***infinite***.
- For any two finite sets A and B, we have the identity $|A \cup B| = |A| + |B| - |A \cap B|$
  - $|A \cup B| \leq |A| + |B|$
- A finite set of n elements is sometimes called an n-***set***.
  - A 1-set is called a ***singleton***.
  - A subset of k elements of a set is sometimes called a k-***subset***.

Power Set

- Given a set *A*, the **power set** of *A*, is the set of all subsets of *A*
  - $A = \{ x, y \}$
  - Power set of $A = \{ \varnothing, \{ x \}, \{ y \}, \{ x, y \} \}$
- Cardinality of the power set of *A* is $2^{|A|}$

Cartesian product

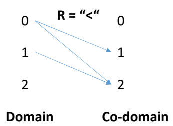
- Given sets *A* and *B*, the **Cartesian product** of *A* and *B* is the set of all ordered pairs (*a*,*b*) where *a* is in *A* and *b* is in *B*.
  - *A* x *B*, read as "*A* cross *B*"
  - $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$
  - $A \times B = \{(a,b) \mid a \in A, b \in B\}$, where "," means "and"

Ordered n-tuples

- Let n be a positive integer
- Let $x_1, x_2, \ldots$ Be the elements of a set
- The ***ordered n-tuple***, $(x_1, x_2, \ldots, x_n)$ consists of $x_1, x_2, \ldots, x_n$
  - The ordering: first x_1, then x_2, and so forth up to x_n
  - A 2-tuple is an ordered pair
  - A 3-tuple is an ordered triple
- Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if, and only if $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$
- Given sets $A_1, A_2, \ldots, A_n$, the ***Cartesian product*** of $A_1, A_2, \ldots, A_n$ denoted by $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$
  - where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$
  - $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$

B.2 Relations

- Let A and B be sets. *a **Relation** R* from A to B is a subset of A x B.
  - $A = B = \{1, 2, 3\}$ and the relation is “<“. Then, $R = \{(0,1), (0,2), (1,2)\}$
  - A = domain and B = co-domain of R
- Given an ordered pair (x,y) in A x B, x is related to y by R if and only if (x,y) is in R
  - Written $x\ R\ y$
  - Example: $1 < 2$
- R is a ***function*** if every element of A maps to one element in B



Note that this relation is not a “function” since
1. Domain member “0” leads to two values “1” and “
   in the co-domain
2. Domain member “0” does not lead to any value in
   the co-domain

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Intervals on the real line

- (a, b)
- (a, b]
- [a, b)
- [a, b]
- (a, ∞), [a, ∞),
- (-∞, b), (-∞, b]

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Properties of Sets: Relations

- Inclusion of intersection
  - $A \cap B \subseteq A$
  - $A \cap B \subseteq B$
- Inclusion in union
  - $A \subseteq A \cup B$
  - $B \subseteq A \cup B$
- Transitive property of subsets
  - If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$

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Statements

- A ***statement*** (or ***proposition***) is a sentence that is true or false but not both
  - Example: The dog is white
- A ***universal statement*** says that a certain property is true for all elements in a set
  - Example: All dogs are animals, or $\forall$ dogs d, d is an animal
- A *conditional statement* says that **if** one thing is true **then** some other thing is true
  - Example: If Z is a dog then Z is an animal
  - If X is true then Y is true
- *Existential statement* says that there is at least one thing for which the property is true
  - Example: There is a polygon that has four sides, or $\exists$ a polygon p, p has 4 sides

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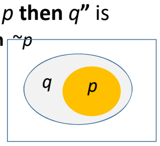
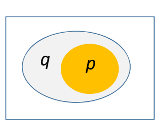
Conditional statements

- *Conditional statement* has the form “If p then q”
  - p is the *hypothesis* (or *antecedent*)
  - q is the *conclusion* (or *consequent*)
- The **contrapositive** of a conditional statement of the form **“if** p **then** q**”** is
  - **If** $\sim q$ **then** $\sim p$



if p **then** q
means p *inside* q



if $\sim q$ **then** $\sim p$
means $\sim q$ *inside* $\sim p$

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Conditional statements

- “r is a *sufficient condition* for s” means “**if** r **then** s”
- “r is a *necessary condition* for s” means “**if** s **then** r”
- “r is a *necessary and sufficient condition* for s” means “s **iff** r”

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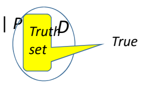
Predicates and quantified statements

- A *predicate* is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
  - Example: Let $P(x)$ be the predicate "$x^2 > x$"
- The *domain* of a predicate variable is the set of all values that may be substituted in place of the variable
  - Example: Domain of $P(x)$ is the set ***R*** of all real numbers
- If $P(x)$ is a predicate and x has the domain D, the *truth set* of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x
  - The *truth set* of $P(x)$ is denoted $\{ x \in D \mid P$



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Predicates and quantified statements I

- A *predicate* is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
- A *statement* is true or false but not both
- Quantifiers
  - A method to convert a predicate to a statement is to add word(s) that refer to quantities of the variables
    - Example: "<u>for all</u> $x \in \boldsymbol{R}$ and $x > 1$, $x < x^2$ "
    - Example: "<u>there exists some</u> value $x \in \boldsymbol{R}$ **such that**, $x > x^2$ "
  - *Universal quantifier* $\forall$ means "for all"
    - Also "for every", "for any", "for each", and "given any"
  - *Existential quantifier* $\exists$ means "there exists"
    - Also, "there is a", "we can find a", "there is at least one", for some", and "for at least one"
    - "such that" is inserted before the predicate

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Predicates and quantified statements

- Let $Q(x)$ be a predicate and D the domain of x
- A *universal* statement has the form "$\forall\ x \in D, Q(x)$"
  - It is **true** iff **$Q(x)$ is true for all x in D**
    - *Example: for all dogs x, x is a mammal*
  - It is **false** iff **$Q(x)$ is false for some x in D**
    - *Example: for all dogs x, x weighs less than four pounds*
    - *Example: counter example, an adult great dane weights at least four pounds*
  - A value of x for which $Q(x)$ is false is called a counterexample to the universal statement
- A *existential* statement has the form "$\exists\ x \in D, Q(x)$"
  - It is **true** iff **$Q(x)$ is true for some x in D**
  - It is **false** iff **$Q(x)$ is false for all x in D**

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Predicates and quantified statements

Negation of quantified statements

- Consider the statement " $\forall\ x \in D, Q(x)$ "
- Its negation is logically equivalent to "$\exists\ x \in D$ such that $\sim Q(x)$ "
- Symbolically $\sim(\ \forall\ \boldsymbol{x \in D, Q(x)}\ ) \equiv \exists\ x \in D$ such that $\sim Q(x)$

- Consider the statement " $\exists\ x \in D$ such that $Q(x)$ "
- Its negation is logically equivalent to "$\forall\ x \in D, \sim Q(x)$ "
- Symbolically $\sim(\ \forall\ \boldsymbol{x \in D, Q(x)}\ ) \equiv \exists\ x \in D$ such that $\sim Q(x)$

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Theorems

- **Theorem** is a statement that has been proven on the basis of previously established statements such as other theorems and or axioms
  - Example: Theorem: Suppose n is even. Then n^2 is even
  - Theorem has assumptions (or hypothesis) and a result
- **Lemma** is like a theorem but its used to prove a theorem
- **Conjecture**: A statement that is asserted to be true but is not proven

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Writing Proofs

- A proof is a convincing argument that a theorem is true
  - Should be clearly understood by someone else
- Step 1. Copy the statement of the theorem to be proved on your paper
- Step 2. Clearly mark the beginning of your proof with the word "Proof"
  - Terminate the proof with
    - "■" (a black box) or
    - "QED", which means "quod erat demonstrandum" – "which had to be proven"

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Writing Proofs

- Step 3. Make your proof self contained
  - At the beginning explain the meaning of each variable
    - Example: "Let x be a positive real number"
    - Example: "Suppose x is a solution to the quadratic equation $x^2 - 2x + 1 = 0$
  - You may introduce new variables
    - Example: "Since x is a solution to the quadratic equation $x^2 - 2x + 1 = 0$
- Step 4. Write your proof in complete, grammatically correct sentences
  - You may use equations, symbols, and shorthand notations
  - As much as possible keep it readable

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Writing Proofs

- Step 5. Keep your reader informed about the status of each statement in the proof
  - If something is assumed, preface it with
    - "Suppose"
    - "Assume"
  - If something is still to be shown, preface it with
    - "We must show that"
    - "In other words, we must show that"
- Step 6. Give a reason for each assertion in your proof
  - Examples:
    - "by hypothesis",
    - "by definition",
    - "by theorem",
    - "by the last equality"

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Writing Proofs

- Step 7. Include the "little words and phrases" that make the logic of your argument clear
  - A new thought that doesn't necessarily follow from the previous statement
    - "Note that",
    - "Observe that",
    - "Note",
    - "But",
    - "Now",
    - "However"
    - In many cases, it's a good idea to start a new paragraph
- Step 8. Display equations and inequalities
  - Examples: each equation and inequality are on separate lines
  - Example:
    - X = (25+2)/3 = (27)/3 = 9

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Getting proofs started

Theorem. Every complete bipartite graph is connected

Proof:

Suppose *G* is a complete bipartite graph

Fill in here

Thus, *G* is connected

■

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Proof methods – some examples

- Direct proof
- Constructive proof
- Nonconstructive proof of existence method
  - Example: pidgeon hole argument
- Disprove by counter example
- Proof by exhaustion
- Proof by cases
  - Divide and conquer
- Proof by contradiction (next slide)
  - Proof by contraposition
- Proof by mathematical induction (later slides)

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Proof by contradiction

Dinosaur Comics
http://www.qwantz.com/index.php

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Proof by Contradiction

- Method of proof by contradiction
  - Step 1. Suppose the statement to prove is false
  - Step 2. Show that this supposition leads logically to a contradiction
  - Step 3. Conclude that the statement to prove is true
- Proof by contrapositive is related

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Contradiction - Example

- Theorem. There is no greatest integer
- Proof
  - **Suppose there is a greatest integer M**
  - Let n = M+1
  - n is an integer and n > M
  - Therefore, **M is not the greatest integer**
  - But we supposed that M is the greatest integer, so we have a contradiction
  - Therefore, there is no greatest integer

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Contradiction – Example 2 – *on your own*

- Theorem. There is no integer that is both odd and even
- Proof
  - Suppose there is an integer n that is odd and even
  - Since n is odd, there is an integer s such that n = 2s+1
  - Since n is even, there is an integer t such that n = 2t
  - Then 2s = 2t+1
  - Then **s = t + (1/2)** by dividing by two on both sides
  - **The left side is integer**
  - The right side is an integer t plus 0.5, so the **right side is not integer**
  - Since the left side is integer but the right side is not, their equality is false
  - We have a contradiction, so the initial assumption is false
  - So there is no integer that is both odd and even

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Contradiction – Example 2 – *on your own*

- Theorem. The sum of a rational and an irrational number is irrational
- Proof
  - **Let m be an arbitrary rational number**
  - **Let n be an arbitrary irrational number**
  - **Suppose m + n is rational**
  - Since m is rational, **m = a/b** for some integers a and b
  - Since m + n is irrational, **m + n = c/d** for some integers c and d
  - **We'll work with m + n = c/d**
    - Now n = c/d – m
    - = c/d – a/b
    - = c/d + (-a/b)
    - **n is the sum of two rational numbers, so it's rational**
    - Contradicts the assumption that n is irrational

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Mathematical induction

- Application
  - Prove something is true of a sequence, e.g., sequence of values, steps of an algorithm
  - Goal: Prove
    - Property P is true for index 1
    - Property P is true for index 2
    - Property P is true for index 3
    - etc
    - In other words, Property P is true over all indices
  - Example property: for all positive integers n, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

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Principle of mathematical induction

- Let P(n) be a property that is defined for integers n
- Let m be a fixed integer, the starting index
- Suppose the following two statements are true
  - P(m) is true – *[starting point]*
  - For all integers k ≥ m, if P(k) is true then P(k+1) is true
- Then the following statement is true
  - for all integers n ≥ m, P(n) is true

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Method of Proof by Mathematical Induction

- **Prove: "For all integers n ≥ m (usually m=0 or 1), a property P(n) is true"**
- **<u>Proof method</u>**
- **Step 1** (basis step): Show that P(m) is true. *[Also called the base case]*
- **Step 2** (induction step): Show that
  - for all integers k ≥ m, if P(k) is true then P(k+1) is true
- [*Step 2 can be done as follows*:]
  - **Suppose** that P(k) is true, where k is any value at least *a*
    - This is called the "induction hypothesis"
  - **Show** that P(k+1) is true

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Example 1 – sum of integers

Prove for all integers n $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Base case: Suppose n = 1 $\sum_{k=1}^{1} k = 1$ Then and $\frac{1(1+1)}{2} = 1$ Therefore $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Induction step:

<u>Induction hypothesis</u>: Suppose for some n >= 1 $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

<u>We now proceed to show that the equality is true for n+1</u>

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Example 1 – sum of integers

<u>Induction hypothesis</u>: Suppose for some n >= 1 $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

<u>We now proceed to show that the equality is true for n+1</u>

Note that $\sum_{k=1}^{n+1} k = (n+1) + \sum_{k=1}^{n} k$

From the **induction hypothesis** $\sum_{k=1}^{n+1} k = (n+1) + \frac{n(n+1)}{2}$

$$(n+1) + \frac{n(n+1)}{2} = \frac{2(n+1)}{2} + \frac{n(n+1)}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

Then $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$

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Strong Mathematical Induction

- Principle of Strong Mathematical Induction
- Let P(n) be a property that is defined for integers n, and let x and y be fixed integers with x ≤ y. Suppose the following two statements are true:
  - P(x), P(x+1), ..., P(y) are true. (basis step)
  - For any integer k ≥ x, if P(x), P(x+1), ..., P(k) is true (inductive hypothesis)
  - Then P(k+1) is true (inductive step)
- Then for all integers n ≥ x, P(n) is true

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Example 2 – sum of geometric series
Go over on your own

Prove any real number r except 1, and any integer n >= 0 $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$

Base case n = 0 $\sum_{k=0}^{0} r^k = r^0 = 1$ and $\frac{r^{n+1}-1}{r-1} = \frac{r-1}{r-1} = 1$

so the left and right hands of the equality are equal to one, which mean

Induction step:
<u>Induction hypothesis</u>: Suppose for n >= 0, $\sum_{k=0}^{n} r^k = \frac{r^{n+1}-1}{r-1}$

<u>We will show that</u> $\sum_{k=0}^{n+1} r^k = \frac{r^{n+2}-1}{r-1}$

[*Expand left side then do some algebra to get the right side*]

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Example 2 – sum of geometric series
Go over on your own

Expand left side then do some algebra to get the right side

$$\sum_{k=0}^{n+1} r^k = r^{n+1} + \sum_{k=0}^{n} r^k$$

$$= r^{n+1} + \frac{r^{n+1}-1}{r-1}$$ **from induction hypothesis**

$$= \frac{r^{n+1}(r-1)}{r-1} + \frac{r^{n+1}-1}{r-1}$$

$$= \frac{r^{n+2} - r^{n+1} + r^{n+1} - 1}{r-1}$$

$$= \frac{r^{n+2}-1}{r-1}$$

So $\sum_{k=0}^{n+1} r^k = \frac{r^{n+2}-1}{r-1}$

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Example: $2^{2n} - 1$ is divisible by 3 *Go over on your own*

- For all integers $n \geq 0$, $2^{2n} - 1$ is divisible by 3
- Proof by mathematical induction
- Base case, n = 0.
  - $2^{2n} - 1 = 2^0 - 1 = 1 - 1 = 0$
  - 0 is divisible by 3, so base case is true
- Induction step
  - Suppose $2^{2n} - 1$ is divisible by 3 for an arbitrary $n \geq 0$
  - We will show that the property is true for n+1, i.e.,
    - $2^{2(n+1)} - 1$ is divisible by 3

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Example: $2^{2n} - 1$ is divisible by 3 – *on your own*

- Induction step
  - Suppose $2^{2n} - 1$ is divisible by 3 for an arbitrary n >= 0
    - Hence, there is an integer r such that $2^{2n} - 1 = 3r$
  - We will show that the property is true for n+1, i.e.,
    - $2^{2(n+1)} - 1$ is divisible by 3
  - We'll start with $2^{2(n+1)} - 1$ and check if it is divisible by 3
  - Note
    - $2^{2(n+1)} - 1 = 2^{2n+2} - 1$ *[Let's make it something we've seen before]*
    - $= 2^{2n} \times 2^2 - 1$
    - $= 2^{2n} \times 4 - 1$
    - $= 4 \times 2^{2n} - 1$
    - $= 4 \times (2^{2n} - 1) + 4 - 1$
    - $= 4 \times (2^{2n} - 1) + 3$
    - $= 4 \times 3r + 3$ *[Induction hypothesis]*
    - $= 3 (4r + 1)$
    - So it's divisible by 3

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Example: $2n+1 < 2^n$ -- *on your own*

- For all integers $n \geq 3$, $2n+1 < 2^n$
- Proof by mathematical induction
- Base case, n = 3.
  - Note 2n+1 = 7
  - Note $2^n = 8$
  - Thus, Since 7 < 8, $2n+1 < 2^n$
  - Induction step
  - Induction hypothesis: Suppose n is arbitrary and n >= 3. Then $2n+1 < 2^n$
  - We want to prove the inequality for n+1

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Example: $2n+1 < 2^n$ - *on your own*

- Induction step
  - Induction hypothesis: Suppose n is arbitrary and n >= 3. Then $2n+1 < 2^n$
  - <u>We want to prove the inequality for n+1</u>
  - Left hand side becomes
    - $2(n+1) + 1 = 2n + 2 + 1$ *[transform it into something we've seen before]*
    - $= 2n + 1 + 1 + 1$
    - $= (2n + 1) + 2$
    - $< 2^n + 2$ *[using induction hypothesis]*
    - $< 2^n + 2^n$ *[using the fact that $2 < 2^k$ for all $k \geq 2$]*
    - $= 2(2^n)$
    - $= 2^{n+1}$
    - Which is the right hand side

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Rooted Trees

- Reading Assignment: Appendix B.5.2 and B.5.3

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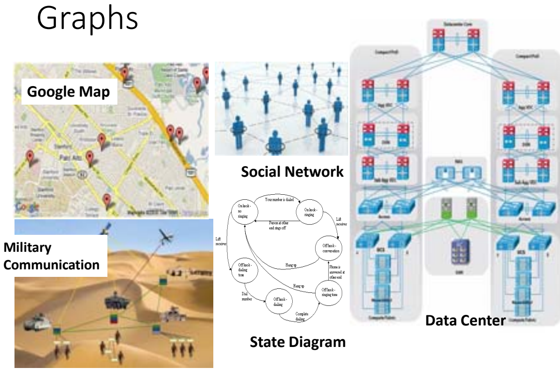
Graphs



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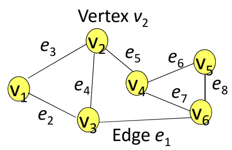
(Undirected) Graph

Visual representation

Graph *G* = (*V*, *E*) or $V(G)$ and $E(G)$
V = Set of *vertices (*or *nodes* or *points)*
E = Set of pairs of vertices
A pair is called an *edge* (or *link*)
"Lines" between vertices



Notation:
Graph *G* = (*V*, *E*) or $V(G)$ and $E(G)$
Common example: $V = \{v_1, v_2, \dots, v_{|V|}\}$
Edge: $e_1, e_2, \dots$
or for *simple* graphs, an ***unordered pair*** (u,v) ,
where u and v are end vertices
which is the same as (v,u)

- $V = \{v_1, v_2, \dots, v_6\}$
- $E = \{e_1, e_2, \dots, e_8\}$

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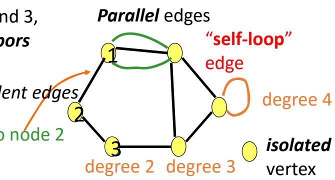
(Undirected) Graph: Definitions

Vertex 2 is ***adjacent*** to 1 and 3,
i.e., are 1 and 3 are ***neighbors***

Degree of a vertex = # *incident edges*

Edge (1,2) is *incident* to node 2



Edges are ***adjacent*** if they have a common incident node, e.g., (1,2) and (2,3) are adjacent
Simple graphs have
- At most one edge between a pair of vertices
- No self-loops

Multigraphs can have parallel edges

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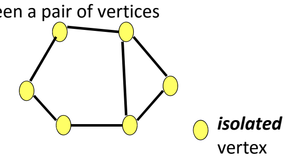
Example simple graph

Simple graphs have
- At most one edge between a pair of vertices
- No self-loops



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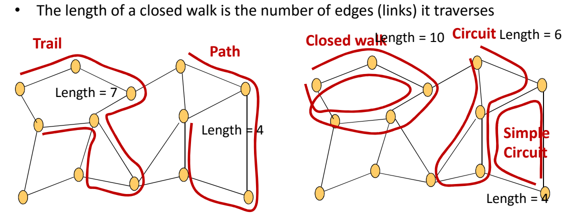
Trails, Paths and Circuits

- A ***trail*** is a walk without a repeated edge
- A ***path*** is a trail that does not contain a repeated vertex
- A ***closed walk*** is a walk that starts and ends at the same vertex
- A ***circuit*** is a closed walk has at least one edge but no repeated edges
- A ***simple circuit*** (or ***cycle***) is a circuit with no repeated vertices except the first and last
- The length of a trail is the number of edges (links) it traverses between its end vertices (nodes), # "hops"
- The length of a closed walk is the number of edges (links) it traverses



Rooted Trees

- A ***rooted tree*** is a graph with the following properties
- Nodes are partitioned into ***levels*** numbered 0, 1, ...
- There is only one vertex at level 0 which is referred as the ***root***
- A node at level *k* is connected by links only to nodes at level *k*-1 and *k*+1
  - Nodes have neighbors only at adjacent levels
- A node at level *k* > 0, is connected to exactly one node at level *k*-1 called its ***parent***
  - The root does not have a parent
  - Nodes are ***children*** of their parents
  - Children are one level further away from the root than their parents
  - Nodes with the same parent are ***siblings***
- The ***height*** of a rooted tree is the maximum level over all the vertices

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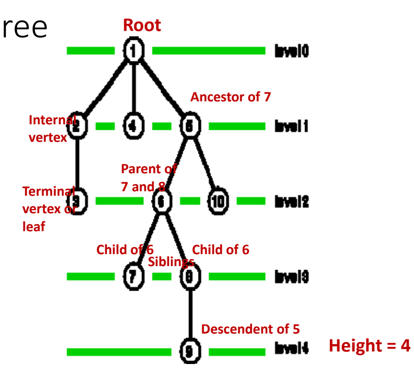
Rooted tree



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Rooted Trees

- Suppose vertex *x* that is along the path from a vertex *y* to the root
  - *x* is an ***ancestor*** of *y*
  - *y* is a ***descendent*** of *x*
  - A node is a descendent and ancestor of itself
  - If $x \neq y$ then *x* is an ***proper ancestor*** of *y*
  - If $y \neq x$ then *y* is an ***proper descendent*** of *x*
- A node with no children is a ***leaf*** or ***external node***
- A node with a child is an ***internal node***
- A node's ***degree*** is the number of its children
  - Different definition of "degree" than for ordinary graphs
- The ***depth*** of a node is it's level
  - The path length from the root to the node
- An ***ordered*** tree has an ordering of its children: child 1, child 2, ...

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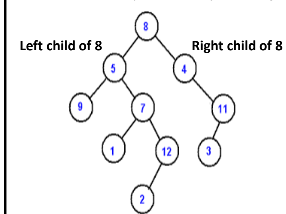
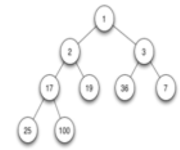
Binary Tree

A ***full binary tree*** is a binary tree where each parent has exactly two children

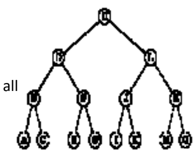
A ***binary tree*** is a rooted tree, where each internal vertex has at most two children: usually labeled ***left*** and ***right***





A ***complete or perfect binary tree*** has all its leaves at the same level and each internal node has exactly two leaves

An ***almost complete binary tree*** is a tree where all levels are filled except possibly the last, and the last level is filled from the left



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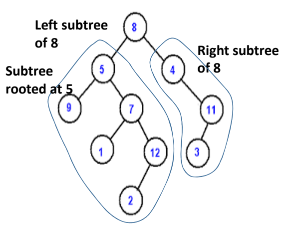
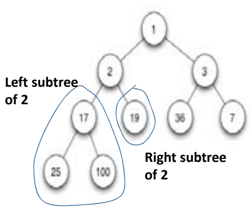
Binary Tree





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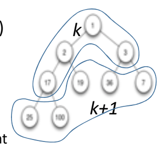
A Result

- **Theorem**. If *T* is a full binary tree with *k* internal vertices, then
  - *T* has *k*+1 terminal vertices (leaves)
  - *T* has a total of 2*k*+1 vertices



- **Proof**
  - Note that $n = p + r$
    - n = number of vertices in T
    - p = number of vertices with a parent
      - $p = 2k$ because only internal vertices are parents, and each have exactly 2 children
    - r = number of vertices without a parent
      - But only the root has no parent, so $r = 1$
  - Then $n = 2k + 1$
  - Next note that $n = k + t$
    - t = number of terminating vertices (leaves)
  - $t + k = 2k + 1$, so $t = k + 1$

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Another Result

- **Theorem**. If *T* is a binary tree with height *h* and *t* leaves then $t \leq 2^h$

  [A tree of height h has at most 2^h leaves]
- **Proof**. *(by strong mathematical induction)*
  - *Let P(k) = "A T is a tree with height k has at most 2^k leaves"*
  - ***Base case***: *P(0)* is true because a tree of height 0 is a single vertex, which is terminal vertex. Thus, the number of terminal vertices = $1 \leq 2^0$
  - ***Induction step***
  - ***Induction hypothesis***: *P(i) is true for i = 0, 1, ..., k*
  - ***We want to show*** P(k+1)
  - *Let r be the root*
    - ***Case 1: r has one child;*** *without loss of generality, assume it's the left child*
    - ***Case 2: r has two children***

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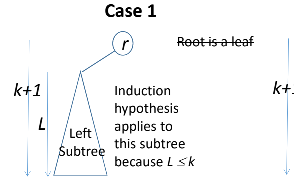
Another Result Continued

Case 1



Left subtree has at most 2^L leaves

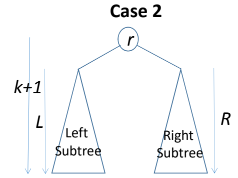
\# leaves = # leaves in left subtree ~~+ root~~

$\leq 2^L \cancel{+1}$

$\leq 2^k \cancel{+2^k}$

$\leq 2^{k+1}$

Case 2



$L \leq k$ $R \leq k$

Induction hypothesis applies to these subtrees

Left subtree has at most 2^L leaves

Right subtree has at most 2^L leave

\# leaves = # leaves in subtrees

$\leq 2^L + 2^R$

$\leq 2^k + 2^k \quad \leq 2^{k+1}$

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