Homework 2: PE 2024-2025 [due date: 2024-12-19, 16:00]

Apellidos: **Nombre:**
Grupo:

1	2	3	4	5	6	7	8	T

1. Consider the process $X(t) = Z\sqrt{t}$ for $t \geq 0$ with the same value of $Z \sim N(0,1)$ for all t
   (a) Show that the distribution of the process at time t is the same as that of a Wiener process: $X(t) \sim N(0, \sqrt{t})$
   (b) What is the mathematical property that allows us to prove that this process is not Brownian motion?

2. Consider the Wiener (standard Brownian) process $W(t)$ in $[0,1]$,
   (a) From the property of independent increments,
   $$\mathbb{E}\left[(W(t_2) - W(t_1))(W(s_2) - W(s_1))\right] = \mathbb{E}\left[(W(t_2) - W(t_1))\right]\mathbb{E}\left[(W(s_2) - W(s_1))\right],\ t_2 \geq t_1 \geq s_2 \geq s1 \geq 0,$$
   show that the autocovariances are given by
   $$\gamma(t,s) = \mathbb{E}\left[W(t)W(s)\right] = \min(s,t),$$
   both for $s > t$ and for $t > s$.
   (b) Illustrate this property by simulating a Wiener process in $[0,1]$ and making a plot of the sample estimate and the theoretical values of $\gamma(t, 0.25)$ as a function of $t \in [0,1]$.

3. Consider two independent Wiener processes $W(t)$, $W'(t)$. Show that the following processes have the same covariances as the standard Wiener process:
   (a) $\rho W(t) + \sqrt{1-\rho^2}W'(t) \quad t \geq 0$
   (b) $-W(t) \quad t \geq 0$
   (c) $\sqrt{c}W(t/c); \quad t \geq 0,\ c > 0.$
   (d) $V(0) = 0; V(t) = tW(1/t); \quad t > 0$

   Make a plot of the trajectories of the first three processes to illustrate that they are standard Brownian motion processes. Compare the histogram of the final values of the simulated trajectories with the theoretical density function.

4. Make an animation in Python illustrating the evolution of the distribution of a Brownian motion process starting form x_0:
   $$\mathbb{P}\left(B(t) = x | B(t_0) = x_0\right).$$
   To this end, simulate M trajectories of the process in the interval $[t_0, t0 + T]$ and plot the time evolution of the histogram using as frames a grid of regularly spaced times in that interval. Plot the theoretical form of the density function on the same graph, so that it can be compared with the histogram.

5. Consider the standard Brownian bridge $BB(t)$ in $[0,1]$, such that $BB(0) = BB(1) = 0$, and $\sigma = 1$.
   - Show that the process has zero mean ($\mathbb{E}\left[BB(t)\right] = 0$).
   - Show that the autocovariances follow the equation
   $$\gamma(t,s) = \mathbb{E}\left[BB(t)BB(s)\right] = \min(s,t) - st$$
   .
   - Illustrate this property by simulating the standard Brownian bridge process in $[0,1]$ and making a plot of the estimated and the theoretical $\gamma(t, 0.25)$ as a function of $t \in [0,1]$.