PE_stochastic_process_HW_01_Ivan_Sotillo

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1 Procesos Estocásticos HW 1

Este notebook está para acompañar al PDF.

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[21]:
```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import poisson
import math
```

1.1 Ejercicio 1

Illustrate the validity of the derivation by comparing the empirical distribution obtained in a simulation of the Poisson process and the theoretical distribution of P [N (t) = n] given by Eq. (4) for the values = 10, t = 2.

1. A Poisson process with rate λ can be defined as a counting process $\{N(t); t \geq 0\}$ with the following properties

   (a) $N(0) = 0$.

   (b) $N(t)$ has independent and stationary increments.

   (c) Let $\Delta N(t) = N(t + \Delta t) - N(t)$ with $\Delta t \to 0^+$. The following relations hold:

$$\mathbb{P}[\Delta N(t) = 0] = 1 - \lambda \Delta t + o(\Delta t) \tag{1}$$

$$\mathbb{P}[\Delta N(t) = 1] = \lambda \Delta t + o(\Delta t) \tag{2}$$

$$\mathbb{P}[\Delta N(t) \geq 2] = o(\Delta t). \tag{3}$$

From this definition show that

$$\mathbb{P}[N(t) = n] = \frac{1}{n!} \lambda^n t^n e^{-\lambda t}. \tag{4}$$

To this end, set up a system of differential equations for the quantities $\mathbb{P}[N(t) = 0]$, and $\mathbb{P}[N(t) = n]$ with $n \geq 1$. Then verify that Eq. (4) satisfies the differential equations derived.

For instance, the differential equation for $\mathbb{P}[N(t) = 0]$ can be derived from the fact that

$$\mathbb{P}[N(t + \Delta t) = 0] = \mathbb{P}[N(t) = 0]\, \mathbb{P}[\Delta N(t) = 0] \tag{5}$$

Using Eq. (1), we obtain

$$\mathbb{P}[N(t + \Delta t) = 0] = \mathbb{P}[N(t) = 0] - \mathbb{P}[N(t) = 0]\, \lambda \Delta t + o(\Delta t). \tag{6}$$

The corresponding differential equation is obtained in the limit $\Delta t \to 0^+$

$$\frac{d}{dt} \mathbb{P}[N(t) = 0] = -\lambda \mathbb{P}[N(t) = 0]. \tag{7}$$

The solution of this differential equation for the initial condition $\mathbb{P}[N(0) = 0] = 1$ is

$$\mathbb{P}[N(t) = 0] = e^{-\lambda t}. \tag{8}$$

Illustrate the validity of the derivation by comparing the empirical distribution obtained in a simulation of the Poisson process and the theoretical distribution of $\mathbb{P}[N(t) = n]$ given by Eq. (4) for the values $\lambda = 10$, $t = 2$.

$$P[N(t+\Delta t) = n] = P[N(t) = n]\, P[\Delta N(t) = 0] + P[N(t) = n-1]\, P[\Delta N(t) = 1] +$$

$$+ P[N(t) = n-2] \cdot P[\Delta N(t) = 2] + \underbrace{\ldots}_{o(\Delta t)} =$$

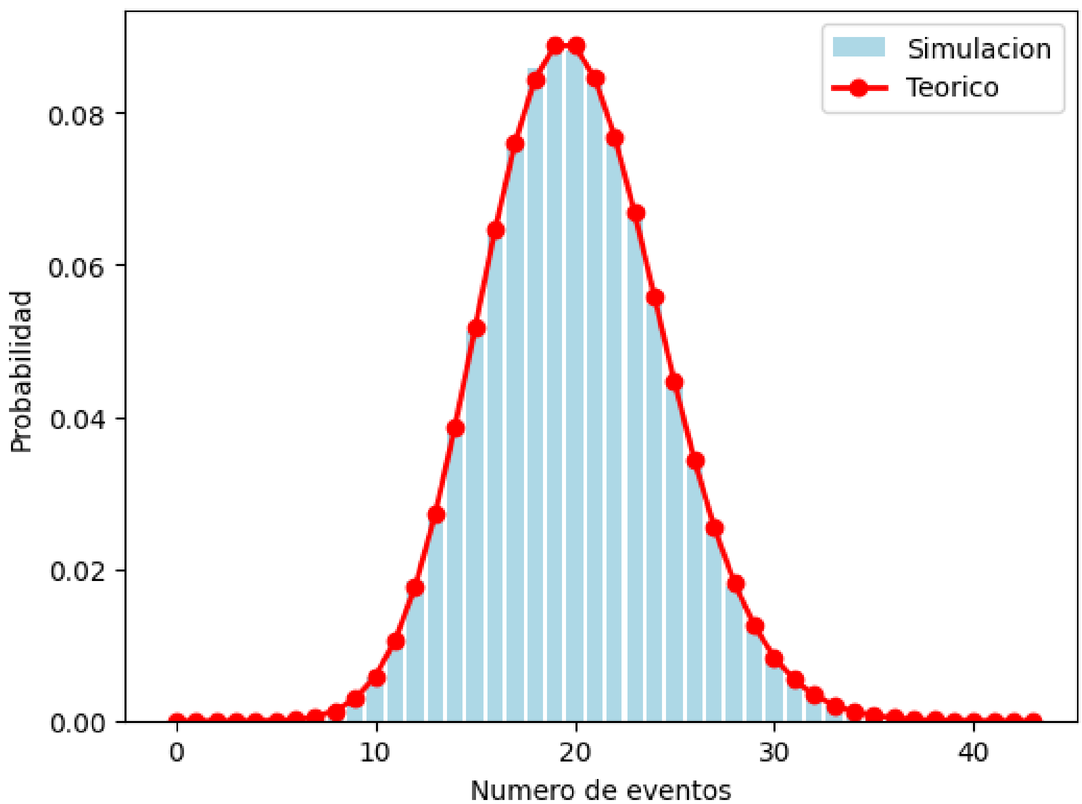
$$= P[N(t)=n] \cdot (1 - \lambda \Delta t + o(\Delta t)) + P[N(t) = n-1] \cdot (\lambda \Delta t + o(\Delta t)) + o(\Delta t)$$

$\Delta t \to 0^+$ $$\frac{d}{dt} P[N(t) = n] = -\lambda P[N(t) = n] + \lambda P[N(t) = n-1]$$

Usando $P[N(t) = n] = \frac{1}{n!} \lambda^n t^n e^{-\lambda t}$ se cumple la condición

[22]:

```
lam = 10
t = 2
N = 100000

# Simulacion
simulation = np.random.poisson(lam * t, N)

# Valores teoricos
n_values = np.arange(0, max(simulation) + 1)
theoretical = poisson.pmf(n_values, lam * t)

# Histograma
_ = plt.hist(simulation, bins=n_values - 0.5, density=True,
             rwidth=0.8, color='lightblue',  label='Simulacion')

# Grafico teorico
_ = plt.plot(n_values, theoretical, 'ro-', linewidth=2, label='Teorico')

plt.xlabel('Numero de eventos')
plt.ylabel('Probabilidad')
plt.legend()
plt.show()
```



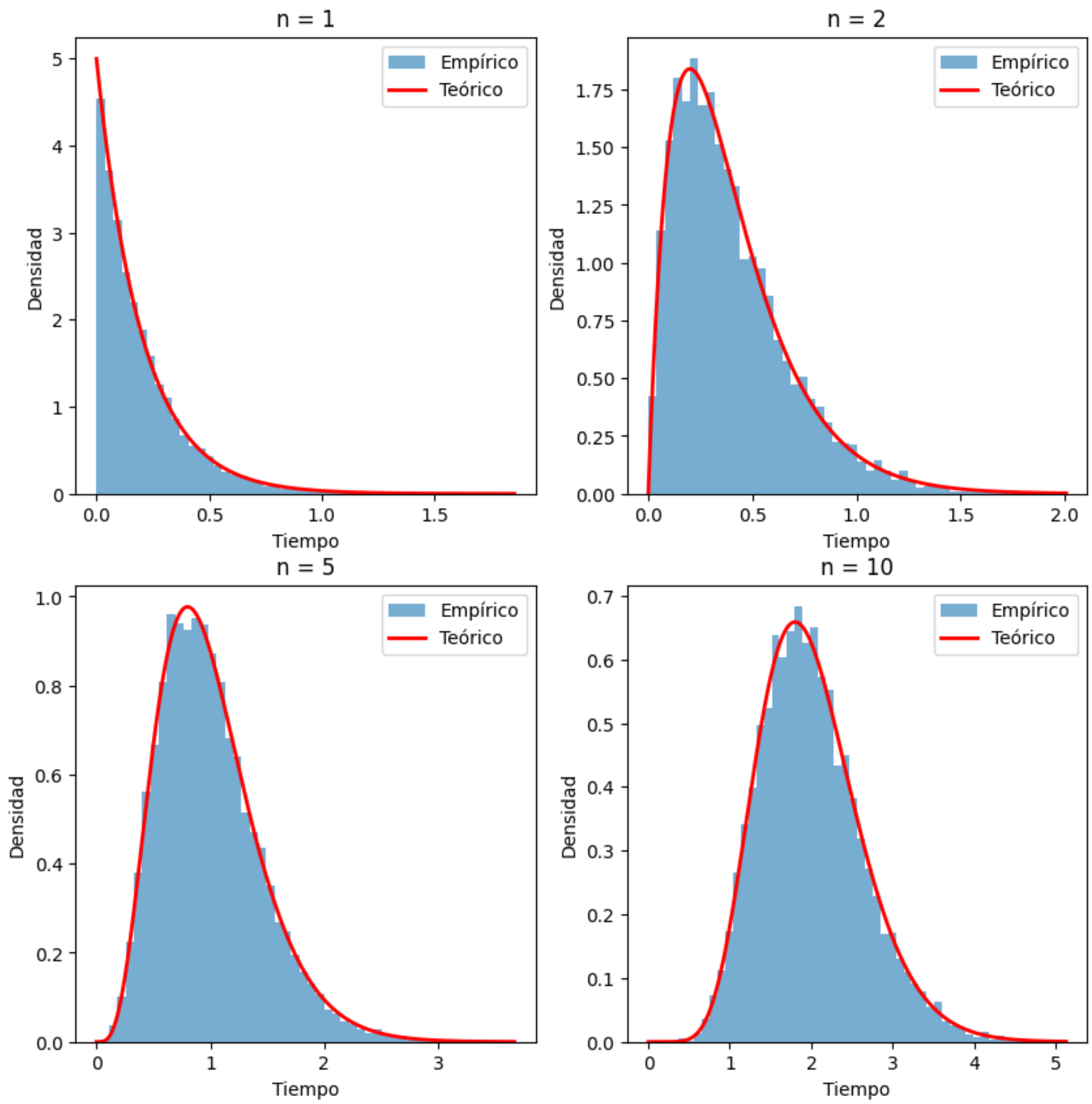
Como se puede ver en la imagen, la distribución obtenida en la simulación es muy similar a la distribución teórica.

1.2 Ejercicio 2

Simulate a Poisson process with $\lambda = 5.0$. From these simulations show for different values of $n = 1, 2, 5, 10$ that the probability density of the nth arrival is

$$f_{S_n}(t) = \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}$$

[23]:
```
lam = 5.0
N = 10000
n_values = np.array([1, 2, 5, 10])

# Diccionario de los valores n-simos
event_times = {n: [] for n in n_values}

# Simulacion
for _ in range(N):
    interarrival_times = np.random.exponential(1 / lam, max(n_values))
    arrival_times = np.cumsum(interarrival_times)

    for n in n_values:
        event_times[n].append(arrival_times[n - 1])

# Histograma
fig, axs = plt.subplots(2, 2, figsize=(10, 10))
axs = axs.ravel()

for i, n in enumerate(n_values):
    axs[i].hist(event_times[n], bins=50, density=True, alpha=0.6,␣
 ↪label="Empírico")

    # Valores teoricos
    t_values = np.linspace(0, max(event_times[n]), 1000)
    theoretical_density = (lam ** n) * (t_values ** (n - 1)) * np.exp(-lam *␣
 ↪t_values) / math.factorial(n - 1)

    axs[i].plot(t_values, theoretical_density, 'r-', linewidth=2,␣
 ↪label="Teórico")
    axs[i].set_title(f'n = {n}')
    axs[i].set_xlabel('Tiempo')
    axs[i].set_ylabel('Densidad')
    axs[i].legend()
```


n = 1
n = 2
n = 5
n = 10
Empírico
Teórico
Densidad
Tiempo

1.3 Ejercicio 3

3. Assume that we have a sample $\{U_i\}_{i=1}^n$ of n iid $U[0,t]$ random variables. The probability density of the order statistics $\{U_{(1)} < U_{(2)} < \ldots < U_{(n)}\}$ is

$$f_{\{U_{(i)}\}_{i=1}^n}\left(\{u_{(i)}\}_{i=1}^n\right) = \frac{n!}{t^n}.$$

Let $\{N(t); t \geq 0\}$ be a Poisson process with rate λ. Show that conditioned on $N(t) = n$, the distribution of arrival times $\{0 < S_1 < S_2 < \ldots < S_n\}$ coincides with the distribution of order statistics of n iid $U[0,t]$ random variables

$$f_{\{S_{(i)}\}_{i=1}^n|N(t)}\left(\{u_{(i)}\}_{i=1}^n |n\right) = \frac{n!}{t^n}.$$

- Use Bayes theorem to calculate the density $f_{\{S_i\}_{i=1}^{n+1}|N(t)}\left(\{s_i\}_{i=1}^{n+1}|n\right)$.

$$f_{\{S_i\}_{i=1}^{n+1}|N(t)}\left(\{s_i\}_{i=1}^{n+1}|n\right) = \frac{f(n|\{s_i\}_{i=1}^{n+1}) \cdot f(\{s_i\}_{i=1}^{n+1})}{f(n)}$$

- Use the fact that $N(t) = n$ if and only if $s_n \leq t < s_{n+1}$

$$f_{N(t)|\{S_i\}_{i=1}^{n+1}}\left(n|\{s_i\}_{i=1}^{n+1}\right) = \begin{array}{ll} 1 & s_n \leq t < s_{n+1} \\ 0 & \text{otherwise.} \end{array}$$

$$f(n|\{s_i\}_{i=1}^{n+1}) = \begin{cases} 1 & \text{if } s_n \leq t \leq s_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

- Focus on the the case $s_n \leq t < s_{n+1}$. $\Rightarrow$ it is always 1

$$f(\{s_i\}_{i=1}^{n+1}|n) = \frac{1 \cdot f(\{s_i\}_{i=1}^{n+1})}{f(n)}$$

- Use the fact that

$$f_{\{S_i\}_{i=1}^{n+1}|N(t)}\left(\{s_i\}_{i=1}^{n+1}|n\right) = f_{S_{n+1}|\{S_i\}_{i=1}^n,N(t)}\left(s_{n+1}|\{s_i\}_{i=1}^n, n\right) f_{\{S_i\}_{i=1}^n|N(t)}\left(\{s_i\}_{i=1}^n |n\right).$$

- Use the memoryless property for $s_{n+1} > t$

$$f_{S_{n+1}|\{S_i\}_{i=1}^n,N(t)}\left(s_{n+1}|\{s_i\}_{i=1}^n, n\right) = f_{S_{n+1}|N(t)}\left(s_{n+1}|n\right)$$

$$f(\{s_i\}_{i=1}^{n+1}|n) = f(s_{n+1}|n) \cdot \overbrace{f(\{s_i\}_{i=1}^{n}|n)}^{\text{Buscamos esto}}$$

- $f(\{s_i\}_{i=1}^{n+1}) = \lambda^{n+1} e^{-\lambda s_{n+1}}$

$$f(\{s_i\}_{i=1}^{n}|n) = \frac{\overset{\text{Sustituimos}}{f(\{s_i\}_{i=1}^{n+1}|n)}}{f(s_{n+1}|n)} = \frac{f(\{s_i\}_{i=1}^{n+1})}{f(n) \cdot f(s_{n+1}|n)} =$$

- $P[N(t)=n] = \frac{1}{n!} \cdot \lambda^n \cdot t^n \cdot e^{-\lambda t}$

$$= \frac{\lambda^{n+1} e^{-\lambda s_{n+1}}}{\frac{1}{n!} \cdot \lambda^n \cdot t^n \cdot e^{-\lambda t} \cdot f(s_{n+1}|n)} = \frac{\lambda^{n+1} e^{-\lambda s_{n+1}}}{\frac{1}{n!} \cdot \lambda^n \cdot t^n \cdot e^{-\lambda t} \cdot \lambda \cdot t \cdot e^{-\lambda(s_{n+1}-t)}} = \frac{n! \cdot \cancel{\lambda^{n+1}} \cdot \cancel{e^{-\lambda s_{n+1}}}}{t^n \cdot \cancel{\lambda^{n+1}} \cdot \underbrace{e^{-\lambda(s_{n+1}-t)} \cdot e^{-\lambda t}}_{\cancel{e^{-\lambda s_{n+1}}} \cdot \cancel{e^{\lambda t}} \cdot \cancel{e^{-\lambda t}} \to 1}} =$$

$$= \boxed{\frac{n!}{t^n}}$$

1.4 Ejercicio 4

Two teams A and B play a soccer match. The number of goals scored by Team A is modeled by a Poisson process $N_A(t)$ with rate $\lambda_A = 0.02$ goals per minute. The number of goals scored by Team B is modeled by a Poisson process $N_B(t)$ with rate $\lambda_B = 0.03$ goals per minute. The two processes are assumed to be independent. Let $N(t)$ be the total number of goals in the game up to and including time t. The game lasts for 90 minutes

4. Two teams A and B play a soccer match. The number of goals scored by Team A is modeled by a Poisson process $N_A(t)$ with rate $\lambda_A = 0.02$ goals per minute. The number of goals scored by Team B is modeled by a Poisson process $N_B(t)$ with rate $\lambda_B = 0.03$ goals per minute. The two processes are assumed to be independent. Let $N(t)$ be the total number of goals in the game up to and including time t. The game lasts for 90 minutes.

(a) Find the probability that no goals are scored.

$$\boxed{\begin{array}{l} P[N(t)=0] = e^{-\lambda \cdot t} \\ P[N(t)=n] = \frac{1}{n!} \cdot \lambda^n \cdot t^n \cdot e^{-\lambda t} \end{array}}$$

Partido $\rightarrow$ 90 mins $\quad t = 90$

$\lambda_A = 0.02$

$\lambda_B = 0.03$

$$P[N_A(90)=0] = e^{-0.02 \cdot 90} = 0.165$$

$$P[N_B(90)=0] = e^{-0.03 \cdot 90} = 0.067$$

$$\boxed{P[N_A(90)=0,\ N_B(90)=0] = e^{-0.02 \cdot 90} \cdot e^{-0.03 \cdot 90} = 0,011}$$

(b) Find the probability that at least two goals are scored in the game.

$N_{AB} = N_A + N_B$ donde $\lambda_{AB} = \lambda_A + \lambda_B = 0.02 + 0.03 = 0.05$

$$P[N_{AB}(90) \geq 2] = 1 - P[N_{AB}(90) < 2] = 1 - \left(P[N_A(90)=0] + P[N_A(90)=1]\right) =$$

$$= 1 - \left(e^{-0.05 \cdot t} + \frac{1}{1!} \cdot 0.05^1 \cdot 90^1 \cdot e^{-0.05 \cdot 90}\right) = \boxed{0.9839}$$

(c) Find the probability of the final score being Team A:1,Team B:2

$$P[N_A(90)=1,\ N_B(90)=2] = P[N_A(90)=1] \cdot P[N_B(90)=2] = \overbrace{\frac{1}{1!} \cdot 0{,}02^1 \cdot 90^1 \cdot e^{-0.02 \cdot 90}}^{0{,}2975} \cdot$$

$$\cdot \underbrace{\frac{1}{2!} \cdot 0.03^2 \cdot 90^2 \cdot e^{-0.03 \cdot 90}}_{0{,}2450} \rightarrow 0{,}0729$$

(d) Find the probability that they draw.

$$P[N_A(90) = N_B(90)] = \sum_{k=0}^{\infty} P[N_A(90)=k] \cdot P[N_B(90)=k] = \sum_{n=0}^{\infty} \frac{1}{k!} \cdot 0.02^k \cdot t^k \cdot e^{-0.02 \cdot t} \cdot$$

$$\cdot \frac{1}{k!} \cdot 0.03^k \cdot t^k \cdot e^{-0.03 \cdot t} = e^{-0.05 \cdot t} \cdot \sum \frac{1}{k! \cdot k!} \cdot (\underbrace{0.02 \cdot t \cdot 0.03 \cdot t}_{})^k = \sum \frac{1}{k! \cdot k!} \cdot \left(\frac{2 \cdot \sqrt{0.03 \cdot 0.02} \cdot t}{2}\right)^{2k} =$$

$$(0.03 \cdot 0.02 \cdot t^2)^k = \left(\sqrt{0.03 \cdot 0.02} \cdot t\right)^{2k}$$

$$I_{\cancel{\nu}\,0}(x) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\cancel{\nu}+1)} \left(\frac{x}{2}\right)^{(2n+\cancel{\nu})} \cdot = \sum_{n=0}^{\infty} \frac{1}{n! \cdot n!} \cdot \left(\frac{x}{2}\right)^{2n} = I_0(x)$$

$\Gamma(n) = (n-1)!$

$$= e^{-0.05\cdot t} \cdot \underbrace{I_0\left(2\cdot\sqrt{0.03\cdot 0.02}\cdot t\right)}_{\text{Calculado con wolfram}} \overset{t=90}{=} 0.0111 \cdot 16.1385 = 0.1793$$

(e) Find the probability that Team B scores the first goal.

Ncesitamos prob A no anota antes de t, y B anota en t.

$$P[N_A(t)=0, S_B=1] = e^{-0.02\cdot t} \cdot 0.03 \cdot e^{-0.03\cdot t}$$

$$\int_0^{90} e^{-0.02\cdot t} \cdot 0.03 \cdot e^{-0.03\cdot t}\, dt = 0.03 \cdot \int_0^{90} e^{-0.02\cdot t} \cdot e^{-0.03\cdot t}\, dt = 0.03 \int_0^{90} e^{-(0.05)\cdot t}\, dt =$$

$$= 0.03 \cdot \left.\frac{e^{-0.05\cdot t}}{-0.05}\right|_0^{90} = \boxed{0.5933}$$

[24]:
```python
# Parámetros
# Tasa de goles de A (goles/minuto)
lambda_A = 0.02
# Tasa de goles de B (goles/minuto)
lambda_B = 0.03
# Duración del partido en minutos
t_game = 90
# Número de partidos simulados
num_simulations = 1000000
```

[25]:
```python
# Inicialización de contadores
no_goals = 0
at_least_two = 0
A_1_B_2 = 0
draws = 0
B_first_goal = 0

for _ in range(num_simulations):
    # Simulamos tiempos de goles para A y B
    times_A = np.cumsum(np.random.exponential(1 / lambda_A, 100))
    times_B = np.cumsum(np.random.exponential(1 / lambda_B, 100))

    # Filtramos los goles dentro del tiempo del partido
    goals_A = times_A[times_A <= t_game]
    goals_B = times_B[times_B <= t_game]

    # Total de goles de cada equipo
    n_A = len(goals_A)
    n_B = len(goals_B)
    total_goals = n_A + n_B

    # 1. No se marcan goles
    if total_goals == 0:
        no_goals += 1

    # 2. Al menos 2 goles
    if total_goals >= 2:
        at_least_two += 1

    # 3. A:1, B:2
    if n_A == 1 and n_B == 2:
        A_1_B_2 += 1

    # 4. Empate
    if n_A == n_B:
        draws += 1
```

```python
    # 5. B anota primero
    if n_B > 0 and (n_A == 0 or goals_B[0] < goals_A[0]):
        B_first_goal += 1
```

(a) Find the probability that no goals are scored.

Theory value: 0.011

[30]:
```python
P_no_goals = no_goals / num_simulations

print(f'P(No se marcan goles) = {P_no_goals:.4f}')
```

```
P(No se marcan goles) = 0.0112
```

(b) Find the probability that at least two goals are scored in the game.

Theory value: 0.9839

[34]:
```python
P_at_least_two = at_least_two / num_simulations

print(f'P(at least two goals) = {P_at_least_two}')
```

```
P(at least two goals) = 0.938796
```

(c) Find the probability of the final score being Team A:1,Team B:2

Theory value: 0.0729

[35]:
```python
P_A_1_B_2 = A_1_B_2 / num_simulations

print(f'P(A:1, B:2) = {P_A_1_B_2:.4f}')
```

```
P(A:1, B:2) = 0.0730
```

(d) Find the probability that they draw.

Theory value: 0.1793

[36]:
```python
P_draws = draws / num_simulations

print(f'P(draw) = {P_draws:.4f}')
```

```
P(draw) = 0.1800
```

(e) Find the probability that Team B scores the first goal.

Theory value: 0.5933

[37]:
```python
P_B_first_goal = B_first_goal / num_simulations

print(f'P(B first goal) = {P_B_first_goal:.4f}')
```

```
P(B first goal) = 0.5935
```