Topology - Homework 06

Question 1.

Denote $\{(x, \sin x(\frac{1}{x})) : x \in R_+\}$ by S.

Suppose $f : [a, c] \to X$ is a path that connects $(0, 0)$ and arbitrary point in S.

All ts that satisfy $f(t) \in 0 \times R$ construct a closed set that has a largest element b.

Then $f : [b, c] \to X$ is a path that maps b into $0 \times R$ and maps other points distinct with b in $[b, c]$ into S.

Replace $[b, c]$ by $[0, 1]$, and denote that $f(t) = (x(t), y(t))$.

Then we have $x(0) = 0$ and $y(t) = \sin(\frac{1}{x(t)})$ when $x > 0$.

Consider a sequence $t_n \to 0$, such that as for given n, choose u with $0 < u < x(\frac{1}{n})$ to make $\sin(\frac{1}{u}) = (-1)^n$, and according to the intermediate value theorem, there is some t_n satisfying $0 < t_n < \frac{1}{n}$ s.t. $x(t_n) = u$.

Then we have $y(t_n) = (-1)^n$, which contradicts with f's continuity for $y(t_n)$'s non-convergence.

This tells us that there doesn't exit such a continuous f being a path between $(0, 0)$ and a point in S.

Hence X is not path-connected.

Question 2.

(i)

If there exists a ono-to-one map $f : (0, 1) \to (0, 1]$ with its inverse map and itself both continuous.

There must be some x satisfying $0 < x < 1$ and $f(x) = 1$.

If we remove x from $(0, 1)$ and remove 1 from $(0, 1]$, then $(0, x) \bigcup (x, 1)$ and $(0, 1)$ are left.

Obviously $(0, 1)$ is connected while $(0, x) \bigcup (x, 1)$ not.

This contradicts the continuity of f.

With similar derivations we can obtain that none of these three spaces is homeomorphic to the other two.

(ii)

If $f(0) = 0$ or $f(1) = 1$, then the conclusion establishes.

Otherwise, there are $f(0) - 0 > 0$ and $f(1) - 1 < 0$, according to the intermediate value theorem, there must be some with $0 < x < 1$ s.t. $f(x) - x = 0$, the conclusion also establishes.

As for $(0, 1)$, consider $f(x) = x^2 : (0, 1) \to (0, 1)$, obviously for arbitrary $0 < x < 1$ there is $f(x) < x$.

As for $(0, 1]$, consider $f(x) = \frac{1}{2}x : (0, 1] \to (0, 1]$, obviously for arbitrary $0 < x \leq 1$ there is $f(x) < x$.

Thus the conclusion doesn't establish if $[0, 1]$ is replaced by $(0, 1)$ or $(0, 1]$.

Question 3.

Fix $x \in X$ and consider the set S_x of all points in X which are path connected to x.

Let $y \in S_x$, then we can choose an open subset U containing y. For $u \in U$, u is path connected to y which is path connected to x. So by joining paths, we know that u is path connected to x, that is, $U \subset S_x$. Hence, S_x is open.

Consider the closure of S_x. Let $z \in S_x$, and choose an open path connected subset V containing z. Note that $V \bigcap S_x \neq \varnothing$, because V is open. Thus, let $p \in V \bigcap S_x$, then z is path connected to p which is connected to x because $p \in S_x$. Hence p is path connected to x and $p \in S_x$. Then we know $S_x = \overline{S_x}$ and that S_x is closed.

Then $S_x = X$ since S_x is not empty. Then we know that X is path connected.

Question 4.

Consider arbitrary $X \subset R$ and a cover of $X \bigcup (R - A_i)$.

$\bigcup (R - A_i) = R - \bigcap A_i$

$\bigcap A_i$ is finite since A_i is finite.

Thus, there is a finite subgroup of $\{A_i\}$, $\{A_{\alpha_i}\}_{i=1}^{n}$, s.t. $\bigcap_{i=1}^{n} A_{\alpha_i} = \bigcap A_i$.

Then we have a finite subcover of X.

Hence every subset $X \subset R$ is compact.

Question 5.

(i) $\Rightarrow$ (ii):

If $(X, \mathcal{T})$ is compact and suppose all its compact subsets are closed.

Take any topology $\mathcal{T}'$ strictly containing $\mathcal{T}$.

Let $U \in \mathcal{T}' - \mathcal{T}$, and then $X - U$ is not closed in $(X, \mathcal{T})$, that is, it's not compact under $\mathcal{T}$.

So there is a cover $\mathcal{A}$ of $X - U$ by sets in $\mathcal{T}$ that has no finite subcover.

Then $\mathcal{A} \bigcup U$ is a cover of X under $\mathcal{T}'$ that has no subcover and we know that $(X, \mathcal{T}')$ is not compact.

(i) $\Leftarrow$ (ii):

If there is a subset $U \subset X$ that is compact but not closed.

Consider $\mathcal{T}' = \mathcal{T} \bigcup (X - U)$, a subbase for a strictly larger topology than $\mathcal{T}$.

Let $\mathcal{A}$ be a cover of X, if $\mathcal{A} \subset \mathcal{T}$, we have a finite subcover because $(X, \mathcal{T})$ is compact.

Suppose $(X - U) \in \mathcal{A}$. Since U is compact under $\mathcal{T}$, we can obtain a finite subcover as well by picking $X - U$ plus a finite cover of U.

This tells us that X is compact under $\mathcal{T}'$.

Thus, the two statements are equivalent.

Hausdorff space is such an example.