Stone–Weierstrass theorem

In mathematical analysis, the **Weierstrass approximation theorem** states that every continuous function defined on a closed interval $[a, b]$ can be uniformly approximated as closely as desired by a polynomial function. Because polynomials are among the simplest functions, and because computers can directly evaluate polynomials, this theorem has both practical and theoretical relevance, especially in polynomial interpolation. The original version of this result was established by Karl Weierstrass in 1885 using the Weierstrass transform.

Marshall H. Stone considerably generalized the theorem (Stone 1937) and simplified the proof (Stone 1948). His result is known as the **Stone–Weierstrass theorem**. The Stone–Weierstrass theorem generalizes the Weierstrass approximation theorem in two directions: instead of the real interval $[a, b]$, an arbitrary compact Hausdorff space X is considered, and instead of the algebra of polynomial functions, approximation with elements from more general subalgebras of $C(X)$ is investigated. The Stone–Weierstrass theorem is a vital result in the study of the algebra of continuous functions on a compact Hausdorff space.

Further, there is a generalization of the Stone–Weierstrass theorem to noncompact Tychonoff spaces, namely, any continuous function on a Tychonoff space is approximated uniformly on compact sets by algebras of the type appearing in the Stone–Weierstrass theorem and described below.

A different generalization of Weierstrass' original theorem is Mergelyan's theorem, which generalizes it to functions defined on certain subsets of the complex plane.

Contents

Weierstrass approximation theorem

The statement of the approximation theorem as originally discovered by Weierstrass is as follows:

> **Weierstrass Approximation Theorem.** Suppose f is a continuous real-valued function defined on the real interval $[a, b]$. For every $\varepsilon > 0$, there exists a polynomial p such that for all x in $[a, b]$, we have $|f(x) - p(x)| < \varepsilon$, or equivalently, the supremum norm $\|f - p\| < \varepsilon$.

A constructive proof of this theorem using Bernstein polynomials is outlined on that page.

Applications

As a consequence of the Weierstrass approximation theorem, one can show that the space $C[a, b]$ is separable: the polynomial functions are dense, and each polynomial function can be uniformly approximated by one with rational coefficients; there are only countably many polynomials with rational coefficients. Since $C[a, b]$ is metrizable and separable it follows that $C[a, b]$ has cardinality at most $2^{\aleph_0}$. (Remark: This cardinality result also follows from the fact that a continuous function on the reals is uniquely determined by its restriction to the rationals.)

Stone–Weierstrass theorem, real version

The set $C[a, b]$ of continuous real-valued functions on $[a, b]$, together with the supremum norm $\|f\| = \sup_{a \leq x \leq b} |f(x)|$, is a Banach algebra, (that is, an associative algebra and a Banach space such that $\|fg\| \leq \|f\| \cdot \|g\|$ for all f, g). The set of all polynomial functions forms a subalgebra of $C[a, b]$ (that is, a vector subspace of $C[a, b]$ that is closed under multiplication of functions), and the content of the Weierstrass approximation theorem is that this subalgebra is dense in $C[a, b]$.

Stone starts with an arbitrary compact Hausdorff space X and considers the algebra $C(X, \mathbf{R})$ of real-valued continuous functions on X, with the topology of uniform convergence. He wants to find subalgebras of $C(X, \mathbf{R})$ which are dense. It turns out that the crucial property that a subalgebra must satisfy is that it *separates points*: a set A of functions defined on X is said to separate points if, for every two different points x and y in X there exists a function p in A with $p(x) \neq p(y)$. Now we may state:

> **Stone–Weierstrass Theorem (real numbers).** Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X, \mathbf{R})$ which contains a non-zero constant function. Then A is dense in $C(X, \mathbf{R})$ if and only if it separates points.

This implies Weierstrass' original statement since the polynomials on $[a, b]$ form a subalgebra of $C[a, b]$ which contains the constants and separates points.

Locally compact version

A version of the Stone–Weierstrass theorem is also true when X is only locally compact. Let $C_0(X, \mathbf{R})$ be the space of real-valued continuous functions on X which vanish at infinity; that is, a continuous function f is in $C_0(X, \mathbf{R})$ if, for every $\varepsilon > 0$, there exists a compact set $K \subset X$ such that $|f| < \varepsilon$ on $X \setminus K$. Again, $C_0(X, \mathbf{R})$ is a Banach algebra with the supremum norm. A subalgebra A of $C_0(X, \mathbf{R})$ is said to **vanish nowhere** if not all of the elements of A simultaneously vanish at a point; that is, for every x in X, there is some f in A such that $f(x) \neq 0$. The theorem generalizes as follows:

> **Stone–Weierstrass Theorem (locally compact spaces).** Suppose X is a *locally compact* Hausdorff space and A is a subalgebra of $C_0(X, \mathbf{R})$. Then A is dense in $C_0(X, \mathbf{R})$ (given the topology of uniform convergence) if and only if it separates points and vanishes nowhere.

This version clearly implies the previous version in the case when X is compact, since in that case $C_0(X, \mathbf{R}) = C(X, \mathbf{R})$. There are also more general versions of the Stone–Weierstrass that weaken the assumption of local compactness.[1]

Applications

The Stone–Weierstrass theorem can be used to prove the following two statements which go beyond Weierstrass's result.

- If f is a continuous real-valued function defined on the set $[a, b] \times [c, d]$ and $\varepsilon > 0$, then there exists a polynomial function p in two variables such that $|f(x, y) - p(x, y)| < \varepsilon$ for all x in $[a, b]$ and y in $[c, d]$.
- If X and Y are two compact Hausdorff spaces and $f : X \times Y \to \mathbf{R}$ is a continuous function, then for every $\varepsilon > 0$ there exist $n > 0$ and continuous functions $f_1, ..., f_n$ on X and continuous functions $g_1, ..., g_n$ on Y such that $\|f - \sum f_i g_i\| < \varepsilon$.

The theorem has many other applications to analysis, including:

- Fourier series: The set of linear combinations of functions $e_n(x) = e^{2\pi i n x}$, $n \in \mathbf{Z}$ is dense in $C([0, 1]/\{0, 1\})$, where we identify the endpoints of the interval $[0, 1]$ to obtain a circle. An important consequence of this is that the e_n are an orthonormal basis of the space $L^2([0, 1])$ of square-integrable functions on $[0, 1]$.

Stone–Weierstrass theorem, complex version

Slightly more general is the following theorem, where we consider the algebra $C(X, \mathbb{C})$ of complex-valued continuous functions on the compact space X, again with the topology of uniform convergence. This is a C*-algebra with the *-operation given by pointwise complex conjugation.

> **Stone–Weierstrass Theorem (complex numbers).** Let X be a compact Hausdorff space and let S be a separating subset of $C(X, \mathbb{C})$. Then the complex unital *-algebra generated by S is dense in $C(X, \mathbb{C})$.

The complex unital *-algebra generated by S consists of all those functions that can be obtained from the elements of S by throwing in the constant function 1 and adding them, multiplying them, conjugating them, or multiplying them with complex scalars, and repeating finitely many times.

This theorem implies the real version, because if a sequence of complex-valued functions uniformly approximate a given function f, then the real parts of those functions uniformly approximate the real part of f. As in the real case, an analog of this theorem is true for locally compact Hausdorff spaces.

Stone–Weierstrass theorem, quaternion version

Following John C.Holladay (1957) : consider the algebra $C(X, \mathbf{H})$ of quaternion-valued continuous functions on the compact space X, again with the topology of uniform convergence. If a quaternion q is written in the form $q = a + ib;+ jc + kd$ then the **scalar part** a is the **real number** $(q - iqi - jqj - kqk)/4$. Likewise being the **scalar part** of $-qi$, $-qj$ and $-qk$: b,c and d are respectively the **real numbers** $(-qi - iq + jqk - kqj)/4$, $(-qj - iqk - jq + kqi)/4$ and $(-qk + iqj - jqk - kq)/4$. Then we may state :

> **Stone–Weierstrass Theorem (quaternion numbers).** Suppose X is a compact Hausdorff space and A is a subalgebra of $C(X, \mathbf{H})$ which contains a non-zero constant function. Then A is dense in $C(X, \mathbf{H})$ if and only if it separates points.

Stone–Weierstrass theorem, C*-algebra version

The space of complex-valued continuous functions on a compact Hausdorff space X i.e. $C(X, \mathbb{C})$ is the canonical example of a unital commutative C*-algebra $\mathfrak{A}$. The space X may be viewed as the space of pure states on $\mathfrak{A}$, with the weak-* topology. Following the above cue, a non-commutative extension of the Stone–Weierstrass theorem, which has remain unsolved, is as follows:

> **Conjecture.** If a unital C*-algebra $\mathfrak{A}$ has a C*-subalgebra $\mathfrak{B}$ which separates the pure states of $\mathfrak{A}$, then $\mathfrak{A} = \mathfrak{B}$.

In 1960, Jim Glimm proved a weaker version of the above conjecture.

> **Stone–Weierstrass theorem (C*-algebras).**[2] If a unital C*-algebra $\mathfrak{A}$ has a C*-subalgebra $\mathfrak{B}$ which separates the pure state space (i.e. the weak-* closure of the pure states) of $\mathfrak{A}$, then $\mathfrak{A} = \mathfrak{B}$.

Lattice versions

Let X be a compact Hausdorff space. Stone's original proof of the theorem used the idea of lattices in $C(X, \mathbf{R})$. A subset L of $C(X, \mathbf{R})$ is called a lattice if for any two elements $f, g \in L$, the functions $\max\{f, g\}$, $\min\{f, g\}$ also belong to L. The lattice version of the Stone–Weierstrass theorem states:

> **Stone–Weierstrass Theorem (lattices).** Suppose X is a compact Hausdorff space with at least two points and L is a lattice in $C(X, \mathbf{R})$ with the property that for any two distinct elements x and y of X and any two real numbers a and b there exists an element $f \in L$ with $f(x) = a$ and $f(y) = b$. Then L is dense in $C(X, \mathbf{R})$.

The above versions of Stone–Weierstrass can be proven from this version once one realizes that the lattice property can also be formulated using the absolute value $|f|$ which in turn can be approximated by polynomials in f. A variant of the theorem applies to linear subspaces of $C(X, \mathbf{R})$ closed under max (Hewitt & Stromberg 1965, Theorem 7.29):

> **Stone–Weierstrass Theorem.** Suppose X is a compact Hausdorff space and B is a family of functions in $C(X, \mathbf{R})$ such that
>
> 1. B separates points.
> 2. B contains the constant function 1.
> 3. If $f \in B$ then $af \in B$ for all constants $a \in \mathbf{R}$.
> 4. If $f, g \in B$, then $f + g$, $\max\{f, g\} \in B$.
>
> Then B is dense in $C(X, \mathbf{R})$.

More precise information is available:

> Suppose X is a compact Hausdorff space with at least two points and L is a lattice in $C(X, \mathbf{R})$. The function $\varphi \in C(X, \mathbf{R})$ belongs to the closure of L if and only if for each pair of distinct points x and y in X and for each $\varepsilon > 0$ there exists some $f \in L$ for which $|f(x) - \varphi(x)| < \varepsilon$ and $|f(y) - \varphi(y)| < \varepsilon$.

Bishop's theorem

Another generalization of the Stone–Weierstrass theorem is due to Errett Bishop. Bishop's theorem is as follows (Bishop 1961):

> Let A be a closed subalgebra of the Banach space $C(X, \mathbf{C})$ of continuous complex-valued functions on a compact Hausdorff space X. Suppose that $f \in C(X, \mathbf{C})$ has the following property:
>
> $f|_S \in A_S$ for every maximal set $S \subset X$ such that all real functions of A_S are constant.
>
> Then $f \in A$.

Glicksberg (1962) gives a short proof of Bishop's theorem using the Krein–Milman theorem in an essential way, as well as the Hahn–Banach theorem : the process of Louis de Branges (1959). See also Rudin (1973, §5.7).

Nachbin's theorem

Nachbin's theorem gives an analog for Stone–Weierstrass theorem for algebras of complex valued smooth functions on a smooth manifold (Nachbin 1949). Nachbin's theorem is as follows (Llavona 1986):

> Let A be a subalgebra of the algebra $C^\infty(M)$ of smooth functions on a finite dimensional smooth manifold M. Suppose that A separates the points of M and also separates the tangent vectors of M: for each point $m \in M$ and tangent vector v at the tangent space at m, there is a $f \in A$ such that $df(x)(v) \neq 0$. Then A is dense in $C^\infty(M)$.

See also

- Müntz–Szász theorem.
- Bernstein polynomial.
- Runge's phenomenon shows that finding a polynomial P such that $f(x) = P(x)$ for some finely spaced $x = x_n$ is a bad way to attempt to find a polynomial approximating f uniformly. However, as is shown in Walter Rudin's Principles of Mathematical Analysis, one can easily find a polynomial P uniformly approximating f by convolving f with a polynomial kernel.
- Mergelyan's theorem, concerning polynomial approximations of complex functions.

Notes

1. Willard, Stephen (1970). *General Topology* (https://archive.org/details/generaltopology00will_0). Addison-Wesley. p. 293 (https://archive.org/details/generaltopology00will_0/page/293). ISBN 0-486-43479-6.
2. Glimm, James (1960). "A Stone–Weierstrass Theorem for C*-algebras". *Annals of Mathematics*. Second Series. **72** (2): 216–244 [Theorem 1]. doi:10.2307/1970133 (https://doi.org/10.2307%2F1970133). JSTOR 1970133 (https://www.jstor.org/stable/1970133).

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- JG Burkill, *Lectures On Approximation By Polynomials* (http://www.math.tifr.res.in/~publ/ln/tifr16.pdf) (PDF).

Historical works

The historical publication of Weierstrass (in German language) is freely available from the digital online archive of the *Berlin Brandenburgische Akademie der Wissenschaften (http://bibliothek.bbaw.de/)*:

- K. Weierstrass (1885). Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, 1885 (II).

  Erste Mitteilung (http://bibliothek.bbaw.de/bibliothek-digital/digitalequellen/schriften/anzeige/index_html?band=10-sitz/1885-2&seite:int=109) (part 1) pp. 633–639, Zweite Mitteilung (http://bibliothek.bbaw.de/bibliothek-digital/digitalequellen/schriften/anzeige/index_html?band=10-sitz/1885-2&seite:int=272) (part 2) pp. 789–805.

Important historical works of Stone include:

- Stone, M. H. (1937), "Applications of the Theory of Boolean Rings to General Topology", *Transactions of the American Mathematical Society*, Transactions of the American Mathematical Society, Vol. 41, No. 3, **41** (3): 375–481, doi:10.2307/1989788 (https://doi.org/10.2307%2F1989788), JSTOR 1989788 (https://www.jstor.org/stable/1989788).
- Stone, M. H. (1948), "The Generalized Weierstrass Approximation Theorem" (http://dml.cz/bitstream/handle/10338.dmlcz/141501/CommentatMathUnivCarolRetro_52-2011-2_8.pdf) (PDF), *Mathematics Magazine*, **21** (4): 167–184, doi:10.2307/3029750 (https://doi.org/10.2307%2F3029750), hdl:10338.dmlcz/141501 (https://hdl.handle.net/10338.dmlcz%2F141501), JSTOR 3029750 (https://www.jstor.org/stable/3029750); **21** (5), 237–254.

External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Stone–Weierstrass theorem" (https://www.encyclopediaofmath.org/index.php?title=p/s090370), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4