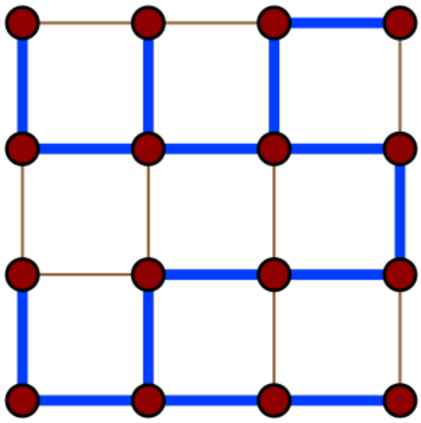
WIKIPEDIA

Zorn's lemma

Zorn's lemma, also known as the **Kuratowski–Zorn lemma**, after mathematicians Max Zorn and Kazimierz Kuratowski, is a proposition of set theory that states that a partially ordered set containing upper bounds for every chain (that is, every totally ordered subset) necessarily contains at least one maximal element.



Zorn's lemma can be used to show that every connected graph has a spanning tree. The set of all sub-graphs that are trees is ordered by inclusion, and the union of a chain is an upper bound. Zorn's lemma says that a maximal tree must exist, which is a spanning tree since the graph is connected.[1] Zorn's lemma is not needed for finite graphs, such as the one pictured here.

Proved by Kuratowski in 1922 and independently by Zorn in 1935,[2] this lemma occurs in the proofs of several theorems of crucial importance, for instance the Hahn–Banach theorem in functional analysis, the theorem that every vector space has a basis,[3] Tychonoff's theorem in topology stating that every product of compact spaces is compact, and the theorems in abstract algebra that in a ring with identity every proper ideal is contained in a maximal ideal and that every field has an algebraic closure.[4]

Zorn's lemma is equivalent to the well-ordering theorem and also to the axiom of choice, in the sense that any one of the three, together with the Zermelo–Fraenkel axioms of set theory, is sufficient to prove the other two.[5] An earlier formulation of Zorn's lemma is Hausdorff's maximum principle which states that every totally ordered subset of a given partially ordered set is contained in a maximal totally ordered subset of that partially ordered set.[6]

Contents

External links

Motivation

Sometimes one wants to prove the existence of a mathematical object (which can be viewed as a maximal element in some poset). One could try proving the existence of such an object by assuming there is no maximal element and using transfinite induction and the assumptions of the situation to get a contradiction. Zorn's lemma tidies up the conditions a situation needs to satisfy in order for such an argument to work. Therefore Zorn's lemma enables mathematicians to not have to repeat the transfinite induction argument by hand each time, but just check the conditions of Zorn's lemma.

> If you are building a mathematical object in stages and find that (i) you have not finished even after infinitely many stages, and (ii) there seems to be nothing to stop you continuing to build, then Zorn's lemma may well be able to help you.
>
> — William Timothy Gowers, "How to use Zorn's lemma"[7]

Statement of the lemma

Zorn's lemma can be stated as:

> **Lemma** — Suppose a partially ordered set P has the property that every chain in P has an upper bound in P. Then the set P contains at least one maximal element.

Variants of this formulation are sometimes used, such as requiring that the set P and the chains be non-empty. See the discussion below.

Example application

Zorn's lemma can be used to show that every nontrivial ring R with unity contains a maximal ideal. In the terminology above, the set P consists of all (two-sided) ideals in R except R itself. Since R is non-trivial, the set P contains the trivial ideal $\{0\}$, and therefore P is non-empty. This set P is partially ordered by set inclusion. Finding a maximal ideal is the same as finding a maximal element in P. The ideal R was excluded because maximal ideals by definition are not equal to R.

To apply Zorn's lemma, take a chain T in P (that is, T is a subset of P that is totally ordered). If T is the empty set, then the trivial ideal $\{0\}$ is an upper bound for T in P. Assume then that T is non-empty. It is necessary to show that T has an upper bound, that is, there exists an ideal $I \subseteq R$ which is bigger than all members of T but still smaller than R (otherwise it would not be in P). Take I to be the union of all the ideals in T. We wish to show that I is an upper bound for T in P. We will first show that I is an ideal of R, and then that it is a proper ideal of R and so is an element of P. Since every element of T is contained in I, this will show that I is an upper bound for T in P, as required.

Because T contains at least one element, and that element contains at least 0, the union I contains at least 0 and is not empty. To prove that I is an ideal, note that if a and b are elements of I, then there exist two ideals $J, K \in T$ such that a is an element of J and b is an element of K. Since T is totally ordered, we know that $J \subseteq K$ or $K \subseteq J$. In the first case, both a and b are members of the ideal K, therefore their sum $a + b$ is a member of K, which shows that $a + b$ is a member of I. In the second case, both a and b are members of the ideal J, and thus $a + b \in I$. Furthermore, if $r \in R$, then ar and ra are elements of J and hence elements of I. Thus, I is an ideal in R.

Now, an ideal is equal to R if and only if it contains 1. (It is clear that if it is equal to R, then it must contain 1; on the other hand, if it contains 1 and r is an arbitrary element of R, then $r1 = r$ is an element of the ideal, and so the ideal is equal to R.) So, if I were equal to R, then it would contain 1, and that means one of the members of T would contain 1 and would thus be equal to R – but R is explicitly excluded from P.

The hypothesis of Zorn's lemma has been checked, and thus there is a maximal element in P, in other words a maximal ideal in R.

Note that the proof depends on the fact that our ring R has a multiplicative unit 1. Without this, the proof wouldn't work and indeed the statement would be false. For example, the ring with $\mathbb{Q}$ as additive group and trivial multiplication (i.e. $ab = 0$ for all a, b) has no maximal ideal (and of course no 1): Its ideals are precisely the additive subgroups. The factor group $\mathbb{Q}/A$ by a proper subgroup A is a divisible group, hence certainly not finitely generated, hence has a proper non-trivial subgroup, which gives rise to a subgroup and ideal containing A.

Proof sketch

A sketch of the proof of Zorn's lemma follows, assuming the axiom of choice. Suppose the lemma is false. Then there exists a partially ordered set, or poset, P such that every totally ordered subset has an upper bound, and that for every element in P there is another element bigger than it. For every totally ordered subset T we may then define a bigger element $b(T)$, because T has an upper bound, and that upper bound has a bigger element. To actually define the function b, we need to employ the axiom of choice.

Using the function b, we are going to define elements $a_0 < a_1 < a_2 < a_3 < \ldots$ in P. This sequence is **really long**: the indices are not just the natural numbers, but all ordinals. In fact, the sequence is too long for the set P; there are too many ordinals (a proper class), more than there are elements in any set, and the set P will be exhausted before long and then we will run into the desired contradiction.

The a_i are defined by transfinite recursion: we pick a_0 in P arbitrary (this is possible, since P contains an upper bound for the empty set and is thus not empty) and for any other ordinal w we set $a_w = b(\{a_v : v < w\})$. Because the a_v are totally ordered, this is a well-founded definition.

This proof shows that actually a slightly stronger version of Zorn's lemma is true:

> If P is a poset in which every well-ordered subset has an upper bound, and if x is any element of P, then P has a maximal element greater than or equal to x. That is, there is a maximal element which is comparable to x.

Alternative formulation

Zorn's lemma is sometimes[8] stated as follows:

> Suppose a non-empty partially ordered set P has the property that every non-empty chain has an upper bound in P. Then the set P contains at least one maximal element.

Although this formulation appears to be formally weaker (since it places on P the additional condition of being non-empty, but obtains the same conclusion about P), in fact the two formulations are equivalent. To verify this, suppose first that P satisfies the condition that every chain in P has an upper bound in P. Then the empty subset of P is a chain, as it satisfies the definition vacuously; so the hypothesis implies that this subset must have an upper bound in P, and this upper bound shows that P is in fact non-empty. Conversely, if P is assumed to be non-empty and satisfies the hypothesis that every non-empty chain has an upper bound in P, then P also satisfies the condition that *every* chain has an upper bound, as an arbitrary element of P serves as an upper bound for the empty chain (that is, the empty subset viewed as a chain).

The difference may seem subtle, but in many proofs that invoke Zorn's lemma one takes unions of some sort to produce an upper bound, and so the case of the empty chain may be overlooked; that is, the verification that all chains have upper bounds may have to deal with empty and non-empty chains separately. So many authors prefer to verify the non-emptiness of the set P rather than deal with the empty chain in the general argument.[9]

History

The Hausdorff maximal principle is an early statement similar to Zorn's lemma.

Kazimierz Kuratowski proved in 1922[10] a version of the lemma close to its modern formulation (it applies to sets ordered by inclusion and closed under unions of well-ordered chains). Essentially the same formulation (weakened by using arbitrary chains, not just well-ordered) was independently given by Max Zorn in 1935,[11] who proposed it as a new axiom of set theory replacing the well-ordering theorem, exhibited some of its applications in algebra, and promised to show its equivalence with the axiom of choice in another paper, which never appeared.

The name "Zorn's lemma" appears to be due to John Tukey, who used it in his book *Convergence and Uniformity in Topology* in 1940. Bourbaki's *Théorie des Ensembles* of 1939 refers to a similar maximal principle as "le théorème de Zorn".[12] The name "Kuratowski–Zorn lemma" prevails in Poland and Russia.

Equivalent forms of Zorn's lemma

Zorn's lemma is equivalent (in ZF) to three main results:

1. Hausdorff maximal principle
2. Axiom of choice
3. Well-ordering theorem.

A well-known joke alluding to this equivalency (which may defy human intuition) is attributed to Jerry Bona: "The Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's lemma?"[13]

Zorn's lemma is also equivalent to the strong completeness theorem of first-order logic.[14]

Moreover, Zorn's lemma (or one of its equivalent forms) implies some major results in other mathematical areas. For example,

1. Banach's extension theorem which is used to prove one of the most fundamental results in functional analysis, the Hahn–Banach theorem
2. Every vector space has a basis, a result from linear algebra (to which it is equivalent[15]). In particular, the real numbers possess a Hamel basis.
3. Every commutative unital ring has a maximal ideal, a result from ring theory
4. Tychonoff's theorem in topology (to which it is also equivalent[16])
5. Every proper filter is contained in an ultrafilter, a result that yields completeness theorem of first-order logic[17]

In this sense, we see how Zorn's lemma can be seen as a powerful tool, especially in the sense of unified mathematics.

In popular culture

The 1970 film, *Zorns Lemma*, is named after the lemma.

This lemma was referenced on *The Simpsons* in the episode "Bart's New Friend".[18]

Notes

1. Serre, Jean-Pierre (2003), *Trees*, Springer Monographs in Mathematics, Springer, p. 23
2. Moore 2013, p. 168
3. Wilansky, Albert (1964). *Functional Analysis*. New York: Blaisdell. pp. 16–17.
4. Jech 2008, ch. 2, §2 Some applications of the Axiom of Choice in mathematics
5. Jech 2008, p. 9

6. Moore 2013, p. 168
7. https://gowers.wordpress.com/2008/08/12/how-to-use-zorns-lemma/
8. For example, Lang, Serge (2002). *Algebra*. Graduate Texts in Mathematics. **211** (Revised 3rd ed.). Springer-Verlag. p. 880. ISBN 978-0-387-95385-4., Dummit, David S.; Foote, Richard M. (1998). *Abstract Algebra* (2nd ed.). Prentice Hall. p. 875. ISBN 978-0-13-569302-5., and Bergman, George M (2015). *An Invitation to General Algebra and Universal Constructions* (https://math.berkeley.edu/~gbergman/245/). Universitext (2nd ed.). Springer-Verlag. p. 162. ISBN 978-3-319-11477-4..
9. Bergman, George M (2015). *An Invitation to General Algebra and Universal Constructions* (https://math.berkeley.edu/~gbergman/245/). Universitext (Second ed.). Springer-Verlag. p. 164. ISBN 978-3-319-11477-4.
10. Kuratowski, Casimir (1922). "Une méthode d'élimination des nombres transfinis des raisonnements mathématiques" (http://matwbn.icm.edu.pl/ksiazki/fm/fm3/fm3114.pdf) [A method of disposing of transfinite numbers of mathematical reasoning] (pdf). *Fundamenta Mathematicae* (in French). **3**: 76–108. Retrieved 24 April 2013.
11. Zorn, Max (1935). "A remark on method in transfinite algebra". *Bulletin of the American Mathematical Society*. **41** (10): 667–670. doi:10.1090/S0002-9904-1935-06166-X (https://doi.org/10.1090%2FS0002-9904-1935-06166-X).
12. Campbell 1978, p. 82.
13. Krantz, Steven G. (2002), "The Axiom of Choice", *Handbook of Logic and Proof Techniques for Computer Science*, Springer, pp. 121–126, doi:10.1007/978-1-4612-0115-1_9 (https://doi.org/10.1007%2F978-1-4612-0115-1_9), ISBN 978-1-4612-6619-8.
14. J.L. Bell & A.B. Slomson (1969). *Models and Ultraproducts*. North Holland Publishing Company. Chapter 5, Theorem 4.3, page 103.
15. Blass, Andreas (1984). *Existence of bases implies the Axiom of Choice*. *Contemp. Math*. Contemporary Mathematics. **31**. pp. 31–33. doi:10.1090/conm/031/763890 (https://doi.org/10.1090%2Fconm%2F031%2F763890). ISBN 9780821850268.
16. Kelley, John L. (1950). "The Tychonoff product theorem implies the axiom of choice". *Fundamenta Mathematicae*. **37**: 75–76.
17. J.L. Bell & A.B. Slomson (1969). *Models and Ultraproducts*. North Holland Publishing Company.
18. "Zorn's Lemma | The Simpsons and their Mathematical Secrets" (http://www.simonsingh.net/Simpsons_Mathematics/zorns-lemma/).

References

- Campbell, Paul J. (February 1978). "The Origin of 'Zorn's Lemma' ". *Historia Mathematica*. **5** (1): 77–89. doi:10.1016/0315-0860(78)90136-2 (https://doi.org/10.1016%2F0315-0860%2878%2990136-2).
- Ciesielski, Krzysztof (1997). *Set Theory for the Working Mathematician*. Cambridge University Press. ISBN 978-0-521-59465-3.
- Jech, Thomas (2008) [1973]. *The Axiom of Choice*. Mineola, New York: Dover Publications. ISBN 978-0-486-46624-8.
- Moore, Gregory H. (2013) [1982]. *Zermelo's axiom of choice: Its origins, development & influence*. Dover Publications. ISBN 978-0-486-48841-7.

External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Zorn lemma" (https://www.encyclopediaofmath.org/index.php?title=p/z099330), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- Zorn's Lemma at ProvenMath (http://www.apronus.com/provenmath/choice.htm) contains a formal proof down to the finest detail of the equivalence of the axiom of choice and Zorn's Lemma.
- Zorn's Lemma (http://us.metamath.org/mpegif/zorn.html) at Metamath is another formal proof. (Unicode version (http://us.metamath.org/mpeuni/zorn.html) for recent browsers.)