1 Groups
1.5 Homomorphisms and normal subgroup

1. Define
$$\begin{array}{rccc} \varphi: & S_n & \to & GL(n,\mathbb{P}) \\ & \sigma & \mapsto & (e_\sigma(1),\cdots,e_\sigma(n)) \end{array}$$
According to Example 1.5.1 φ is a homomorphism. If $(e_\sigma(1),\cdots,e_\sigma(n)) = (e_\tau(1),\cdots,e_\tau(n))$, then $\sigma(i)=\tau(i), i=1,\cdots,n$. So $\sigma=\tau$, hence φ is injective.
$\forall(e_{i_1},\cdots,e_{i_n})\in W$, if we let $\sigma = \begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}$ then $\varphi(\sigma)=(e_{i_1},\cdots,e_{i_n})$.
Hence $\varphi: S_n \to W$ is surjective, and hence an isomorphism.

2. Define
$$\begin{array}{rccc} \varphi: & H & \to & a^{-1}Ha \\ & h & \mapsto & a^{-1}ha \end{array}$$
If $\forall h_1,h_2\in H$ $\varphi(h_1)=\varphi(h_2), i.e., a^{-1}h_1a=a^{-1}h_2a$, then $h_1=h_2$. So φ is injective.
$\forall a^{-1}ha\in a^{-1}Ha$ we have that $\varphi(h)=a^{-1}ha$. So φ is surjective. Hence φ is isomorphism.

3. If H is a normal, according to the definition, $aHa^{-1}\subseteq H = a(a^{-1}Ha)a^{-1}\subseteq aHa^{-1}$, hence $H=aHa^{-1}$. Since $H=aHa^{-1}$, $Ha=aHa^{-1}a=aH$. Suppose $Ha=aH$, for any $x\in H$, then $ax\in aH$, thus there exists $y\in H$ such that $ax=ya$, therefore $axa^{-1}=yaa^{-1}=y\in H$, that is H is normal.

4. $S\neq\emptyset$ for $e\in S$. Let $< S >= \{x_1^{\epsilon_1}x_2^{\epsilon_2}...x_n^{\epsilon_n}|x_i\in S,\epsilon_i=\pm1,n\in\mathbb{Z}\}$, for any $x_1^{\epsilon_1}x_2^{\epsilon_2}...x_n^{\epsilon_n}\in< S >$ and any $g\in G$, then
$$gx_1^{\epsilon_1}x_2^{\epsilon_2}...x_n^{\epsilon_n}g^{-1}=(gx_1g^{-1})^{\epsilon_1}(gx_2g^{-1})^{\epsilon_2}...(gx_ng^{-1})^{\epsilon_n}.$$
Since $gSg^{-1}\subseteq S$ for $g\in G$,
$$gx_1^{\epsilon_1}x_2^{\epsilon_2}...x_n^{\epsilon_n}g^{-1}\in< S > .$$
Hence $< S > \lhd G$.

5. $G=\{\pm1,\pm i,\pm j,\pm k\}$, then $C(G)=\{\pm1\}$ is normal. Hence $< i >= \{\pm1,\pm i\}$, while $[G :< i >]=2$, thus $< i > \lhd G$. Similarly, $< \pm i >$ and $< \pm j >, < \pm k >$ are normal. While the order of subgroup P of G perhaps is $1,2,4,8$. If $|P|=1$, then $P=\{1\}$ is normal. If $|P|=2$, then $P=< -1 >$ is normal. While G has not other element whose order is 2, thus there is only one subgroup whose order is 2. While elements whose order is 4 are $\pm i,\pm j,\pm k$, therefore G has 3 subgroups whose order is 4.

6. For any $a\in G$ and any $b\in H$, $aba^{-1}=baa^{-1}=b\in H$ for $H\leq C(G)$, hence $H\lhd G$. If $G/H=< aH >$, then $G=\{a^kh|h\in H,k\in\mathbb{Z}\}$. For any $a^{k_1}h_1, a^{k_2}h_2\in G$, $a^{k_1}h_1\cdot a^{k_2}h_2=a^{k_1+k_2}h_1h_2=a^{k_2}h_2\cdot a^{k_1}h_1$, therefore G is abelian.

7. Suppose there exists an isomorphism $\varphi : (\mathbb{Z}, +) \to (\mathbb{Q}, +)$, then $\varphi^{-1} : (\mathbb{Q}, +) \to (\mathbb{Z}, +)$ is an isomorphism. Hence for any $n \geq 1$, $\mathbb{Z} = \varphi^{-1}(\mathbb{Q}) = \varphi^{-1}(n\mathbb{Q}) = n\varphi^{-1}(\mathbb{Q}) = n\mathbb{Z}$, it is impossible, therefore any bijection between $\mathbb{Z}$ and $\mathbb{Q}$ is not an isomorphism between the group $(\mathbb{Z}, +)$ and the group $(\mathbb{Q}, +)$.

8. Let $A = \{x_1, x_2, ..., x_{n-1}\}$, construct

$$\begin{tikzcd} A \arrow[rr, "\lambda"] \arrow[dr, "f_1"'] & & F(A) \arrow[dl, "\varphi_1"] \\ & B_n & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "\lambda"] \arrow[dr, "f_2"'] & & F(A) \arrow[dl, "\varphi_2"] \\ & S_n & \end{tikzcd}$$

where $f_1(x_i) = \sigma_i, f_2(x_i) = (i, i+1), \lambda(x_i) = x_i$, then according to Theorem 1.5.2, there exists a unique group homomorphism φ_i such that $\varphi_i \circ \lambda = f_i$. Considering

$$\begin{tikzcd} F(A) \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & B_n \\ & S_n \end{tikzcd}$$

since $Ker\varphi_1$ is a normal group generated by

$$\{x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1} | i = 1, ..., n-1\},$$

while $Ker\varphi_2$ is a normal group generated by

$$\{x_i x_{i+1} x_i x_{i+1}^{-1} x_i^{-1} x_{i+1}^{-1}, x_i^2 | i = 1, ..., n-1\}.$$

Then $Ker\varphi_2 \supseteq Ker\varphi_1$, and φ_i induce isomorphism $\overline{\varphi_1} : F(A)/Ker\varphi_1 \to B_n$ and isomorphism $\overline{\varphi_2} : F(A)/Ker\varphi_2 \to S_n$ respectively. Define $\psi : F(A)/Ker\varphi_1 \to F(A)/Ker\varphi_2$, $\psi(aKer\varphi_1) = aKer\varphi_2$, then ψ is a group isomorphism. Hence $\overline{\varphi}_2 \circ \psi \circ \overline{\varphi}_2^{-1} : B_n \to S_n$ is a group homomorphism. Since ψ is surjective, $\overline{\varphi}_2 \circ \psi \circ \overline{\varphi}_2^{-1}$ is epimorphism.

9. $N \subseteq H \Leftrightarrow NH = H$ or $N \cap H = N$. Since N is a normal subgroup, $NH/N = H/N \cap H$, then $[G : H] = [G : NH][NH : H] = [G : NH][N : H \cap H]$. Since $(|N|, [G : H]) = 1$ and $[N : H \cap N] | [G : H]$, $([N : H \cap N], |N|) = 1$, but $[N : H \cap N] || N|$, $[N : H \cap N] = 1$, i.e. $N \subseteq H$. $N \lhd H$ for $N \lhd G$.

10. (a) $G_1 = G_2 = (\mathbb{Z}, +), H_1 = 2\mathbb{Z}, H_2 = 3\mathbb{Z}$, $H_1 \cong H_2$, but $G_1/H_1 \ncong G_2/H_2$.

    (b) $G_1 = G_2 = (\mathbb{Z}[x], +), H_1 = (\mathbb{Z}, +), H_2 = (\mathbb{Z} + x\mathbb{Z}, +)$, $G_1/H_1 \cong G_2/H_2$, but $H_1 \ncong H_2$.

(c) $G_1 = \mathbb{Z}_6, G_2 = S_3, H_1 = 2\mathbb{Z}_6, H_2 =< (123) >$, $H_1 \cong H_2$ and $G_1/H_1 \cong G_2/H_2 \cong \mathbb{Z}_2$, but $G_1 \ncong G_2$.

11. For any $f(n) \in f(N)$ and any $a \in G$, there exists $a_1 \in G$ such that $f(a_1) = a$ for f is an isomorphism. Then $af(n)a^{-1} = f(a_1)f(n)f(a_1)^{-1} = f(a_1 n a_1{}^{-1}) \in f(N)$, moreover, $f(n)^{-1} = f(n^{-1}) \in f(N)$, $f(N)$ is closed under multiplication, and $e = f(e) \in f(N)$, so $f(N) \lhd G$. Define $\varphi : G/N \to G/f(N)$, $aN \to f(a)f(N)$. If $aN = bN$, then $a^{-1}b \in N$, hence $f(a^{-1}b) = f(a)^{-1}f(b) \in f(N)$, thus φ is well-defined. If $aN \in Ker\varphi$, i.e. $f(a)f(N) = f(N)$, then $f(a) \in f(N)$, thus $f(a) = f(n)$ for some $n \in N$. Since f is injective, $a = n \in N$, then $aN = N$, hence φ is injective. Since f is an isomorphism, φ is surjective. Therefore $G/N \cong G/f(N)$.

12. (1) For any $x, y \in G$, $I_a(xy) = a(xy)a^{-1} = axa^{-1} \cdot aya^{-1} = I_a(x)I_a(y)$. $I_{a^{-1}}I_a(xy) = a^{-1}(axya^{-1})a = xy$, hence $I_{a^{-1}}I_a = id_G$. Similarly, $I_aI_{a^{-1}} = id_G$. Therefore I_a is an automorphism.

    (2) $(I_a)^{-1} = I_{a^{-1}}$, $id_G = I_e$ and $I_a \circ I_b = I_{ab}$, hence $Inn(G)$ is a subgroup of $Aut(G)$. For any $\varphi \in Aut(G)$, $\varphi \circ I_a \circ \varphi^{-1}(x) = \varphi(a\varphi^{-1}(x)a^{-1}) = \varphi(a)x\varphi(a)^{-1} = I_{\varphi(a)}(x)$ for any $x \in G$. Therefore $varphi \circ I_a \circ \varphi^{-1} = I_{\varphi(a)} \in Inn(G)$, then $Inn(G) \lhd Aut(G)$.

    (3) If φ is a nonidentity automorphism of G, since G is an abelian group, $Inn(G) = \{id_G\}$, hence φ is not an automorphism. Therefore φ is an outer automorphism.

    (4) It is obvious that φ is an automorphism of $GL(n, \mathbb{P})$. If $\varphi = I_A$ for $A \in GL(n, \mathbb{P})$, then $\varphi(B = (B^{-1})^T = ABA^{-1}$ for any $B \in GL(n, \mathbb{P})$. Take $B = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & \ddots & \\ & & & n \end{pmatrix}$, then $\varphi(B) = \begin{pmatrix} 1 & & & \\ & \frac{1}{2} & & \\ & & \ddots & \\ & & & \frac{1}{n} \end{pmatrix}$ has eigenvalues:$1, \frac{1}{2}, ..., \frac{1}{n}$. While eigenvalues of ABA^{-1} are $1, 2, ..., n$, hence $(B^{-1})^T \neq ABA^{-1}$. It is contradiction, thus $\varphi \neq I_A$ for any A. Therefore φ is an outer automorphism.

13. If $\varphi : G \to G$, $a \to a^{-1}$, is an automorphism, then $ab = \varphi((ab)^{-1}) = \varphi(b^{-1}a^{-1}) = \varphi(b^{-1})\varphi(a^{-1}) = ba$ for any $a, b \in G$, hence G is abelian. On the contrary, if G is abelian, then $\varphi(ab) = (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1} = \varphi(a)\varphi(b)$ for any $a, b \in G$, hence φ is an endomorphism. At the same time, φ is bijective, therefore φ is an automorphism. If G is abelian, then $\psi(ab) = (ab)^k = a^kb^k = \psi(a)\psi(b)$ for any $a, b \in G$, hence ψ is an endomorphism.

14. Suppose $\lambda : \{x, y, z\} \to G$, $\xi(x) = a^2, \xi(y) = b^2, \xi(z) = ab$, according to Theorem 1.5.2, there exists an unique group homomorphism $\varphi : G \to F(\{a, b\})$, $Im(\varphi) =< x, y, z >$. While $\varphi(x^{k_1}y^{k_2}) = a^{2k_1}b^{2k_2}$, $\varphi(x^{k_1}z^{k_2}) = a^{2k_1+1}(ba)^{k_2-1}b$, $\varphi(y^{k_1}z^{k_2}) = b^{2k_1-1}(ba)^{k_2}b$, $\varphi(y^{k_1}x^{k_2}) = b^{2k_1}(a)^{2k_2}$, $\varphi(z^{k_1}x^{k_2}) =$

$(ab)^{k_1}(a)^{2k_2} = a(ba)^{k_1}a^{2k_2-1}$, $\varphi(z^{k_1}y^{k_2}) = (ab)^{k_1}(b)^{2k_2} = a(ba)^{k_1-1}b^{2k_2+1}$, hence reduced words in G through the action of φ are still reduced. Thus φ is injective, therefore φ is an isomorphism between G and $< x, y, z >$.

15. (1) For any $\varphi \in Aut(\mathbb{Z}^n)$, let $e_i = (0, ..., 1, ..., 0)^T$ where all rows are 0 expect the i_{th}, then $\varphi(e_i) = ((a_{1i}, ..., a_{ni})^T) = \sum_{k=1}^n a_{ik}e_k$. Suppose $\varphi^{-1}(e_i) = (b_{1i}, ..., b_{ni})^T = \sum_{k=1}^n b_{ki}e_k$, then $e_i = \varphi\varphi^{-1}(e_i) = \sum_{k=1}^n b_{ki}\varphi(e_k) = \sum_{k=1}^n \sum_{s=1}^n b_{ki}a_{sk}e_s$, hence $\sum_{k=1}^n a_{sk}b_{ki} = \delta_{is}$, therefore $(\varphi(e_1), \varphi(e_2), ..., \varphi(e_n)) \in GL(n, \mathbb{Z})$ and it's inverse matrix is $(\varphi^{-1}(e_1), ..., \varphi^{-1}(e_n))$. Define $\Phi : Aut(\mathbb{Z}^n) \to GL(n, \mathbb{Z})$ such that $\varphi \to (\varphi(e_1), ..., \varphi(e_n))$. If $\Phi(\varphi) = \Phi(\psi)$, then for any $(a_1, ..., a_n)^T \in \mathbb{Z}^n$,

$$\varphi((a_1, ..., a_n)^T) = \sum_{k=1}^n a_k\varphi(e_k) = \sum_{k=1}^n a_k\psi(e_k) = \psi((a_1, ..., a_n)^T),$$

hence Φ is injective. For any $(\alpha_1, ..., \alpha_n) \in GL(n, \mathbb{Z})$, then $\varphi((a_1, ..., a_n)^T) = \sum_{k=1}^n a_k\alpha_k \in Aut(\mathbb{Z}^n)$, $\varphi^{-1}((a_1, .., a_n)^T) = \sum_{k=1}^n a_k\beta_k$ where $(\beta_1, ..., \beta_n) = (\alpha_1, ..., \alpha_n)^{-1}$. Since $\Phi(\varphi) = (\alpha_1, ..., \alpha_n)$, hence Φ is bijective. We can easily testified that $\Phi(\varphi \circ \psi) = \Phi(\varphi)\Phi(\psi)$. Therefore $Aut(\mathbb{Z}^n) \cong GL(n, \mathbb{Z})$.

(2) Suppose $A = \{x_1, x_2, ..., x_n\}$, $\lambda : A \to F(A)$ which is free group of A. Define $\varphi : A \to \mathbb{Z}_n$, $\varphi(x_i) = e_i$, then there exists an unique group homomorphism $\psi : F(A) \to \mathbb{Z}_n$ such that $\psi(x_i) = e_i$. It is obvious that ψ is epimorphism. Let $N =< \{x_ix_jx_i{}^{-1}x_j{}^{-1} | 1 \leq i, j \leq n\} >$ is a normal subgroup, then $N \subseteq Ker\psi$. For any $x_{i_1}{}^{k_1}, ..., x_{i_m}{}^{k_m} \in Ker\psi$,then

$$x_{i_1}^{k_1}...x_{i_m}^{k_m} = x_{i_1}^{k_1-1}x_{i_1}x_{i_2}x_{i_1}^{-1}x_{i_2}^{-1}x_{i_1}^{1-k_1}x_{i_1}^{k_1-1}x_{i_2}x_{i_1}x_{i_2}^{-1}x_{i_2}^{k_2}...x_{i_m}^{k_m}.$$

$x_{i_1}^{k_1}...x_{i_m}^{k_m} \in N$ iff $x_{i_1}^{k_1-1}x_{i_2}x_{i_1}x_{i_2}^{k_2-1}...x_{i_m}^{k_m} \in N$. While

$$x_{i_1}^{k_1-1}x_{i_2}x_{i_1}x_{i_2}^{k_2-1}...x_{i_m}^{k_m} = x_{i_1}^{k_1-2}x_{i_1}x_{i_2}x_{i_1}^{-1}x_{i_2}^{-1}x_{i_1}^{2-k_1}x_{i-1}^{k_1-2}x_{i_2}x_{i_1}^2x_{i_2}^{k_2-1}...x_{i_m}^{k_m},$$

so $x_{i-1}^{k_1-2}x_{i_2}x_{i_1}^2x_{i_2}^{k_2-1}...x_{i_m}^{k_m} \in N$. Hence iff $x_{i_2}x_{i_1}^{k_1}x_{i_2}^{k_2-1}...x_{i_m}^{k_m} \in N$, and then iff $x_{i_2}^{k_2}x_{i_1}^{k_1}x_{i_3}^{k_3}...x_{i_m}^{k_m} \in N$. Since $\varphi(x_{i_1}^{k_1}...x_{i_m}^{k_m}) = k_1e_{i_1} + ... + k_me_{i_m} = 0$, that is the sum of occurrence number about e_{i_1} is 0. Through above process, we get that x_{i_1} is not occur in $x_{i_2}^{k_2}...x_{i_m}^{k_m}$, therefore $x_{i_1}^{k_1}...x_{i_m}^{k_m} \in N$. Thus $Ker\psi = N$. According to the Fundamental Theorem of Homomorphism, then $F[\{x_1, ...x_n\}]/ < x_ix_jx_i^{-1}x_j^{-1} > \cong \mathbb{Z}^n$.

16. $HN_G(P)/H \cong N_G(P)/H \cap N_G(P) = N_G(P)/N_H(P)$. For any $g \in G$, $gPg^{-1} \subseteq gHg^{-1} \subseteq H$, then there exists $a \in H$ such that $gPg^{-1} = a^{-1}Pa$, hence $agPg^{-1}a^{-1}$, thus $ag \in N_G(P)$, therefore $g = a^{-1}(ag) \in HN_G(P)$.

17. Suppose X is a G-set. Define $\varphi : G \to Sym(X)$, $\varphi(a) = L_a$, the left translation. Since $L_a \circ L_{a^{-1}} = id_X = L_{a^{-1}} L_a$ for any $L_a \in Sym(X)$, $\varphi(ab) = L_{ab}$. For any $x \in X$, we have $L_{ab}(x) = a(bx) = L_a(bx) = L_a \circ L_b(x)$. Thus $L_{ab} = L_a \circ L_b$ and $\varphi(ab) = \varphi(a)\varphi(b)$. On the contrary, suppose $\varphi : G \to Sym(X)$. For any $g \in G$ and $x \in X$, define $G \cdot x := \varphi(g)(x)$. Then $e \cdot x = \varphi(e)(x) = id_x(x) = x$. For any $f, g \in G$ and any $x \in X$, $(fg) \cdot x = \varphi(fg)(x) = (\varphi(f) \cdot \varphi(g))(x) = \varphi(f)(varphi(g)(x)) = f \cdot (g \cdot x)$. Hence X is a G-set.

18. Define $G \times G/H \to G/H$, $(g, aH) \mapsto ga \cdot H$. Then there is a homomorphism $\varphi : G \to Sym(G/H)$. If $\varphi(g) = id$, then $gH = H$ and $g \in H$. Hence $Ker\varphi \subseteq H$, $Ker\varphi \lhd G$. Since $[G : Ker\varphi] = [G : H][H : Ker\varphi]$, $[H : Ker\varphi] | [G : Ker\varphi]$. On the other hand, $G/Ker\varphi \cong Im\varphi \leq Sym(G/H)$, then $|Im\varphi| \mid p!$, hence $[G : Ker\varphi] \mid p!$. If a prime q satisfies $q \mid [G : Ker\varphi]$, then $q \leq p$. By the assumption on p, we have $q = p$. Hence $[G : Ker\varphi] = p$ and $Ker\varphi = H \lhd G$.

19. Without loss of generality, we could Suppose $H =< a_1, ..., a_s >$, $K/H =< b_1H, ..., b_nH >$, then $K = \{b_{i_1}^{s_1} a_{i_1} ... b_{i_n}^{s_n} a_{i_n} | b_{i_j} \in \{b_1, ..., b_n\}, a_{i_j} \in H, s_i \in \mathbb{Z}\}$, thus $K =< b_i a_j | 1 \leq j \leq s, 1 \leq i \leq n >$, hence K is finitely generated.

20. Since $N \lhd G, NP = PN \leq G, PN/N \leq G/N$, $[G/N : PN/N] = [G : PN]$. Since $[G : P] = [G : PN][PN : P]$, $p \nmid [G : PN] = [G/N : PN/N]$. As $NP/N \cong P/N \cap P$, $|NP/N| = p^n$ for some n. Therefore NP/N is a Sylow p-subgroup of G/N.

21. Define $\varphi :< k > / < km > \to \mathbb{Z}_m$, $\varphi(kl+ < km >) = \bar{l}$. For any $kl_1+ < km >, kl_2+ < km > \in < k > / < km >$, then $\varphi(kl_1+ < km > +kl_2+ < km >) = \overline{l_1 + l_2} = \overline{l_1 l_2} = \varphi(kl_1+ < km >) + \varphi(kl_2+ < km >)$. If $kl+ < km > \in Ker\varphi$, i.e. $\varphi(kl+ < km >) = \bar{0}$, thus $l = mn, (n \in \mathbb{Z})$, hence $kl+ < km >=< km >$, therefore φ is injective. If $\bar{l} \in \mathbb{Z}_m$, then there is $kl+ < km > \in < k > / < km >$ such that $\varphi(kl+ < km >) = \bar{l}$, hence φ is surjective. Therefore $< k > / < km > \cong \mathbb{Z}_m$.

22. If $a + tor(H) \in tor(H/tor(H))$, then there is $n_1 > 0$ such that $n_1(a + tor(H)) = n_1 a + tor(H) = 0 + tor(H)$, thus $n_1 a \in tor(H)$, hence there exists $n_2 > 0$ such that $n_2(n_1 a) = 0$, i.e. $(n_2 n_1)a = 0$, therefore $a \in tor(H)$. Whence $a + tor(H) = 0 + tor(H)$, hence $H/tor(H)$ is torsionfree.

23. Let $P := \begin{pmatrix} \frac{1}{\sqrt{2}}E_n & \frac{1}{\sqrt{2}}E_n \\ -\frac{1}{\sqrt{2}}E_n & \frac{1}{\sqrt{2}}E_n \end{pmatrix}$, $J := \begin{pmatrix} 0 & E_n \\ E_n & 0 \end{pmatrix}$, $Q := \begin{pmatrix} \frac{1}{\sqrt{2}}E_n & -\frac{1}{\sqrt{2}}E_n \\ \frac{1}{\sqrt{2}}E_n & \frac{1}{\sqrt{2}}E_n \end{pmatrix}$, $S := \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix}$. Then $PQ = E_{2n}$ Define $\varphi : O(n + n, \mathbb{R}) \to G$, $\varphi(A) = QAP$, then $(QAP)^T J(QAP) = QSP = J$, thus φ is well-defined. Since $Q, P \in GL(2n, \mathbb{R})$, φ is surjective. For any $A, B \in O(n+n, \mathbb{R})$, then $\varphi(AB) = QABP = (QAP)(QBP) = \varphi(A)\varphi(B)$, hence φ is isomorphic.

24. Let $\iota : S \to F(S)$ which is free group of S. Then

$$\begin{tikzcd}
 & S \arrow[r, "\iota"] \arrow[dl, dotted, "\exists\lambda"'] \arrow[dd, "\iota"] \arrow[ddr, dotted, "\lambda"] & F(S) \arrow[dd, "\varphi(Th1.5.2)"] \\
G & & \\
 & F(S) \arrow[ul, "\varphi"] & G \arrow[l, "\psi"]
\end{tikzcd} .$$

$\psi \circ \varphi : F(S) \to F(S)$ and $id_{F(S)} : F(S) \to F(S)$ satisfy $\psi \circ \varphi \circ \iota = id_{F(S)} \circ \iota$, hence $\psi \circ \varphi = id_{F(S)}$. Similarly, $\varphi \circ \psi = id_G$. Therefore $F(S) \cong G$.