Topology - Homework 08

Question 1.

Consider a function $f : X \to \mathbb{N} \cup \{\infty\}$, where $f(x) = \frac{1}{x}$ for $x > 0$ and $f(0) = \infty$, we know f is a bijection.

Consider an open set $B(0, \epsilon)$, $f(B(0, \epsilon)) = (\frac{1}{\epsilon}, +\infty) \cap \mathbb{N}$ is also an open set.

The open set $B(\frac{1}{n}, \frac{1}{n} - \frac{1}{n+1})$ contains only $\frac{1}{n}$ in X and $\{n\}$ is also an open set in $\mathbb{N} \cup \{\infty\}$.

we know that f maps an open set to an open set.

Then we have f^{-1} been an continuous function. And similarly we can obtain that f is also continuous.

Thus X is homeomorphic to $\mathbb{N} \cup \{\infty\}$.

Question 2.

A space has its one-point compactification if and only if it's a locally compact but not compact Hausdorff space.

But R^ω is not locally compact with the respect to $\mathcal{T}_p$ since its every basis element cannot be contained by some compact subspace.

Consider $B = (a_1, b_1) \times \cdots \times (a_n, b_n) \times \mathbb{R} \times \cdots$.

If there is some compact subspace containing B, the closure of B must be compact, but it isn't.

Thus $(R^\omega, \mathcal{T}_p)$ has no one-point compactification.

Question 3.

Let $i \in \mathbb{N}$.

Consider $A_i \subset X_i$, and $\overline{\prod A_i}$.

Obviously there is $\overline{\prod A_i} \subset \prod \overline{A_i}$.

Considering arbitrary $a \in \prod \overline{A_i}$, and we have $a_i = \pi_i(a) \in \overline{A_i}$.

Then there must be some open set U_i containing a_i and $U_i \cap A_i \neq \varnothing$, since $\overline{A_i} = A_i \cup A_i'$.

Then we have $a \in \cup U_i = U$ and $U \cap \prod A_i \neq \varnothing$, which means that $a \in \overline{\prod A_i}$.

Then we have $\overline{\prod A_i} \supset \prod \overline{A_i}$ and $\overline{\prod A_i} = \prod \overline{A_i}$.

Since X_i is separable, there must be a countable dense subset $A_i \subset X_i$.

The product of A_i is also dense in $\prod X_i$ since $\overline{\prod A_i} = \prod \overline{A_i} = \prod X_i$.

$\prod A_i$ is countable since each A_i is countable.

Then we know that $\prod X_i$ is separable.

Question 4.

Consider $\mathcal{A} = \{[a_\alpha, b_\alpha)\}_{\alpha \in J}$ is an open cover of $\mathbb{R}$ consisting of basis elements.

Let $C = \cup_{\alpha \in J} (a_\alpha, b_\alpha) \subset \mathbb{R}$.

Choose $x \in \mathbb{R} - C$, and we have x not in any interval (a_α, b_α).

Thus, there exists some β s.t. $x = a_\beta$. Choose such β and a rational number q_x in an open interval (a_β, b_β).

$(a_\beta, b_\beta) \subset C$ and $(a_\beta, q_x) = (x, q_x) \subset C$.

Then for $x, y \in \mathbb{R} - C$ with $x < y$, there must be $q_x < q_y$, which means there is an injection from $\mathbb{R} - C$ to $\mathbb{Q}$.

So $\mathbb{R} - C$ is countable.

For every element in $\mathbb{R} - C$, choose a member in $\mathcal{A}$ containing it and we can obtain a countable open cover of $\mathbb{R} - C$, $\mathcal{A}'$.

C has countable basis so that there is a countable open cover of C, (a_α, b_α) where $\alpha = \alpha_1, \alpha_2, \cdots$. Then $\mathcal{A}'' = \{[a_\alpha, b_\alpha) | \alpha = \alpha_1, \alpha_2, \cdots\}$ is a countable open cover of C.

Then we have $\mathcal{A}' \cup \mathcal{A}''$ been an countable subfamily of $\mathcal{A}$ and covering $\mathcal{A}$.

So R_l is a Lindelöf space.

Consider a close set $L = \{x \times (-x) | x \in \mathbb{R}\}$ and an open cover of $\mathbb{R}_l^2$, $(\mathbb{R}_l^2 - L) \cup \{[a, b) \cup [-a, b)\}$.

Every element of this open cover has at most one point in its intersection with L.

So no countable subfamily of this open cover can cover $\mathbb{R}_l^2$ since L is uncountable.

Then we know that $\mathbb{R}_l^2$ is not a Lindelöf space.

Question 5.

(i) $\Rightarrow$ (ii)

If X has countable basis $\mathcal{B} = \{B_i\}_{i \in \mathbb{N}_+}$.

Let $\mathcal{A}$ be an open cover of X.

For every $n \in \mathbb{N}_+$, choose A_n containing B_n if possible.

Then we have $\{A_n\}_{n \in \mathbb{N}_+}$ a countable open cover of X.

(ii) $\Rightarrow$ (i)

Fix $n \in \mathbb{N}_+$.

For an open cover $\{B(x, \frac{1}{n})\}$, there must be a countable subcover $\{B(x_i, \frac{1}{n})\}$.

For every point $y \in X$ there must be some $B(x_i, \frac{1}{n})$ containing y.

For every open set $B(y, d)$ there must be some $B(x_i, \frac{1}{n}) \subset B(y, d)$ as long as choosing suitable n.

Then we know $\{B(x_i, \frac{1}{n})\}$ with $i \in \mathbb{N}_+$ and $n \in \mathbb{N}_+$ forms a countable basis of X.

(i) $\Rightarrow$ (iii)

Let $\{B_i\}_{i \in \mathbb{N}_+}$ be a countable basis of X.

Choose x_i from every nonempty basis element B_i and construct D with such $\{x_i\}$.

For arbitrary $x \in X$, there is a basis element has intersection with D, which means $X \subset \overline{D}$.

So $\overline{D} = X$ and D is dense in X.

(iii) $\Rightarrow$ (i)

Let D be a countable subset of X and $\overline{D} = X$.

Consider $B(x, \frac{1}{n})$ where $x \in D$ and $n \in \mathbb{N}_+$.

For arbitrary open set Y containing y, there must be some $B(x, \frac{1}{n})$ containing y.

If $y \in D$, then we can choose a suitable n s.t. $B(y, \frac{1}{n}) \subset Y$.

If $y \in D'$, there must be some $x \in D$ with any $d(x, y) = \epsilon > 0$.

And then we can choose suitable x and n s.t. $B(x, \frac{1}{n}) \subset Y$ and $\frac{1}{n} > \epsilon$.

Then we know that $\{B(x, \frac{1}{n})\}$ where $x \in D$ and $n \in \mathbb{N}_+$ is a countable basis of X.