Topology - Homework 10

Question 1.

Consider $x \notin (\bigcap_{p>r,p\in D} U_p) - (\bigcup_{q<r,q\in D} U_q)$.

If $x \in \bigcup_{q<r,q\in D} U_q$, then there must be some q such that $q < r$, $q \in D$ and $x \in U_q$,

Thus, $q \in \mathbb{Q}(x)$ since $\mathbb{Q}(x) = \{p|x \in U_p\}$ and then we have $f(x) = \inf \mathbb{Q}(x) < r$.

If $x \notin \bigcap_{p>r,p\in D} U_p$, then there must be some p such that $p > r$, $p \in D$ and $x \notin U_p$.

Thus, $q \notin \mathbb{Q}(x)$ and then we have $f(x) > r$ since p' in $\mathbb{Q}(x)$ must satisfy that $U_{p'} \supset U_p$ which means $p' > p$.

Therefore, we have $x \neq r$ if $x \notin (\bigcap_{p>r,p\in D} U_p) - (\bigcup_{q<r,q\in D} U_q)$.

Consider $x \in (\bigcap_{p>r,p\in D} U_p) - (\bigcup_{q<r,q\in D} U_q)$.

Since $x \in \bigcap_{p>r,p\in D} U_p$, for all $p > r$, $f(x) \leq p$.

Consider some $p > r$, assume that $p = \frac{k}{2^n}$, and there must be some $p' \in [\frac{k}{2^{n+1}}, \frac{k}{2^n}) \cap D$ such that $r < p' < p$.

Then we know that $f(x) \leq r$.

Similarly, we can obtain that $f(x) \geq r$.

Then we know that $f^{-1}(\{r\}) = (\bigcap_{p>r,p\in D} U_p) - (\bigcup_{q<r,q\in D} U_q)$.

Question 2.

If $x \in A$, $d(x, A) = \inf\{d(x, a), a \in A\} = d(x, x) = 0$, and then $f(x) = 0$ since $x \notin B$ and $d(x, B) > 0$.

If $x \in B$, we have $d(x, A) > 0$ since $x \notin A$ and $d(x, B) = 0$, then $f(x) = \frac{d(x,A)}{d(x,A)+0} = 1$.

Thus, we know that $f(A) = 0$ and $f(B) = 1$.

Since A and B are two disjoint closed sets, we have $d(x, A) + d(x, B) > 0$ for all $x \in X$.

If $\inf\{d(x, A) + d(x, B)\} = 0$, there must exist some x be a limit point of both A and B, that is, $x \in A$ and $x \in B$.

This is impossible and we know that $\inf\{d(x, A) + d(x, B)\} > 0$.

Denote $\inf\{d(x, A) + d(x, B)\}$ by d and consider $d(x, y) = r$.

$$\begin{aligned}|f(x) - f(y)| &= |\frac{d(x, A)}{d(x, A) + d(x, B)} - \frac{d(y, A)}{d(y, A) + d(y, B)}| = |\frac{d(x, A)d(y, B) - d(y, A)d(x, B)}{(d(x, A) + d(x, B))(d(y, A) + d(y, B))}| \\ &\leq |\frac{d(x, A)(d(x, B) + r) - (d(x, A) + r)d(x, B)}{d^2}| \\ &\leq |d(x, A) - d(x, B)|\frac{r}{d^2}\end{aligned}$$

This means that, for every given $x \in X$, and a neighbor of $f(x)$, V, we can find a neighbor of x, U, such that $f(U) \subset V$, as long as we choose a enough small r and let V be $B_d(x, r)$.

Then we know that f is continuous.

Question 3.

Consider $\mathbb{R}^\omega$ under the uniform topology.

Obviously, this is a metric space.

If $\mathbb{R}^\omega$ is second countable, assume that $\mathcal{B}$ is a countable basis of it.

Then consider a discret subspace $A \subset \mathbb{R}^\omega$.

Choose $a \in A$, and a basis element B_a has only a in its intersection with A.

For distinct points $a, b \in A$, B_a and B_b must be distinct.

So A must be countable.

But if A consists of $0, 1$ sequences, we know that A is uncountable, which makes a contradiction.

Then we know that $\mathbb{R}^\omega$ is not second countable.

Question 4.

(i)

Consider a uncountable family of nonempty topology spaces $\{(X_i, \mathcal{T}_i)\}_{i\in I}$.

Assume that all $\mathcal{T}_i$s are finer than the trivial topology, that is, has a basis element B_i other than X_i.

Let $X = \prod_{i\in I} X_i$ be a product of uncountably many fisrt countable spaces with the product topology and each $\mathcal{T}_i$ has more than one nonempty open set, that is, has open set been true subset of X_i.

Choose a point, $(x_i)_{i\in I}$ from $\prod_{i\in I} X_i$, and a neighborhood of it, $\{U^n\}_{n<\omega}$.

For every n we have some basis $\prod_i U_i^n \subset U^n$ and $U_i^n = X_i$ for all but finitely many $i \in I$.

Let $J = \{i \in I : \exists n\, U_i^n \neq X_i\}$.

J is countable since $n < \omega$, and hence there is some $i_0 \in I - J$.

$U_{i_0}^n = X_{i_0}$ for all n.

Since $\mathcal{T}_{i_0}$ is finer than the trivial topology and has some nonempty true subset containing x_{i_0}, denote it by V_{i_0}.

If there is no true subset of X_{i_0} containing x_{i_0}, we can adjust the way of choosing x_{i_0} and make it.

Consider the neighborhood of $(x_i)_i$, $U = \prod_i U_i$.

As long as choose $U_{i_0} = V_{i_0}$, U would be disabled to contain any U^n, which contradicts with the first countability of X.

Then we know that if X_i is first countable for every $i \in I$ and $(\prod_i X_i, \mathcal{T}_p)$ is also first countable, then all except countably many of the X_is have the trivial topology.

(ii)

"if":

The product of arbitrarily many singleton sets is also singleton.

Thus the product of countably many metrizable spaces and arbitrarily many singleton spaces is equivalent to the product of countably many metrizable spaces and a singleton space, which is still a product of countably many metrizable spaces.

Then we know that $(\prod_i X_i, \mathcal{T}_p)$ is metrizable since the product of countably many metrizable space is also metrizable with respect to the product topology.

"only if":

Let $X = \prod_{i\in I} X_i$ be a product of uncountably many metric spaces with the product topology and all X_is have at least two points.

If X is metrizable, then it is first countable.

Arbitrarily choose $(x_i)_i \in X$, and $\{U^n\}_{n<\omega}$ be arbitrary countable family of neighborhoods of $(x_i)_i$. Then for every n we have some basis $\prod_i U_i^n \subset U^n$. Since we consider the product topology, we have $U_i^n = X_i$ for all but finitely many $i \in I$.

Let $J = \{i \in I : \exists n\, U_i^n \neq X_i\}$.

J is countable since $n < \omega$, and hence there is some $i_0 \in I - J$.

$U_{i_0}^n = X_{i_0}$ for all n.

Since X_{i_0} is metrizable and has at least two points, there is some true subset of X_{i_0} containing x_{i_0} been open and nonempty. Denote it by V_{i_0}.

Consider the neighborhood of $(x_i)_i$, $U = \prod_i U_i$.

As long as choose $U_{i_0} = V_{i_0}$, U would be disabled to contain any U^n, which contradicts with the first countability of X.

Then we know that $(\prod_i X_i, \mathcal{T}_p)$ is metrizable only if each X_i is metrizable and X_i is a singleton set for all X_i except a countable set of indices.

Question 5.

(i) $\Rightarrow$ (ii)

Since X is metric space, it is Hausdorff, and its one-point compactification is also Hausdorff.

So X^* is regular. In fact, it's normal.

And because X has a countable dense subset, X has a countable basis.

Let $\mathcal{B}$ be a countable basis of X.

Every neigborhood of ∞ is of the form $X^* - C$ where C is a closed set and hence C is compact.

C can be covered by finite elements of $\mathcal{B}$, that is, $C \subset B_{n_1} \cup \cdots \cup B_{n_k}$.

And we have $\infty \in X^* - \overline{B_{n_1}} \cup \cdots \cup \overline{B_{n_k}}$.

Add all open sets of the form $X^* - \overline{B_{n_1}} \cup \cdots \cup \overline{B_{n_k}}$ to $\mathcal{B}$ and obtain $\mathcal{B}^*$.

We have $\mathcal{B}^*$ been a countable basis of X^*.

Thus, the one-point compactification of X also has a countable basis.

Then we know that X^* is metrizable.

Every neighborhood of ∞ is of the form $X^* - C$ where C is a compact subspace of X

(ii) $\Rightarrow$ (i)

Since X^* is metrizable and compact, then X^* has a countable basis.

Since X is a subspace of X^*, X also has a countable basis.

Since X is metrizable, X is separable.