# Topology - Homework 11

## Question 1.

Consider a Cauchy sequence $x_n$ in $\mathbb{R}^\omega$.

$d^\omega(x,y) = \sup\{\frac{d^*(x_i,y_i)}{i} : i \in N\}$.

For a fix $i$, we have $d^*(\pi_i((x_i)), \pi_i((y_i))) \leq i \cdot d^\omega((x_i),(y_i))$.

Sequence $\{\pi_i((x_i)_n)\}$ is a Cauchy sequence in $\mathbb{R}$, and it converges to some $a_i$.

Then we know that $\{\pi_i((x_i)_n)\}$ converge to the point $a = (a_1, a_2, \cdots)$ in $R^\omega$.

Hence the metric space $(\mathbb{R}^\omega, d^\omega)$ is complete.

## Question 2.

(i)

Assume that there is $\epsilon > 0$ such that for every $x \in X$ the closure of $B_d(x,\epsilon)$ in X is compact.

Consider a Cauchy sequence $\{x_n\}$ in $X$.

There is some $N \in \mathbb{N}$ such that for all $m, n > N$, $d(x_m, x_n) < \frac{\epsilon}{2}$.

That is, all points $x_n$ with $n > N$ are all contained in an $\epsilon$-ball.

Since the closure of this $\epsilon$-ball is compact and $X$ is a metric space, the closure of the $\epsilon$-ball is sequentially compact, and the sequence of $x_n$ with $n > N$ is convergent.

Then we know that the sequence $\{x_n\}$ is convergent.

Hence $X$ is complete.

(ii)

The space $\{\frac{1}{n} : n \in \mathbb{N}\}$ is an example.

Obviously it's incomplete, since $0 \notin \{\frac{1}{n} : n \in \mathbb{N}\}$ and 0 is the only limit point of this space.

And this space is compact, so every closed subset of it is also compact.

Hence the closure of every $\epsilon$-ball for every $\frac{1}{n}$ is compact.

## Question 3.

(i) $\Rightarrow$ (ii)

Since $(X,d)$ is complete, every Cauchy sequence in $X$ is convergent.

For every nest sequence $C_1 \supset C_2 \supset \cdots$ of nonempty closed subsets of $X$ with $\lim_{n\to\infty} d(C_n) = 0$, choose $x_i$ s.t. $x_i \in C_i$ and we can obtain a Cauchy sequence $\{x_i\}$.

Assume that $\{x_i\}$ converges to some $x \in X$ and $\lim_{n\to\infty} d(x_n, x) = 0$.

There must be $x \in C_n$ for all $n \in \mathbb{N}_+$.

Otherwise, if $x \notin C_k$, since $C_k$ is closed, $x$ is not a limit point of $C_i$ for all $i > k$. This is contradicted with $\lim_{n\to\infty} d(x_n, x) = 0$.

Then we know that $\bigcap_n C_n \neq \varnothing$.

(i) $\Leftarrow$ (ii)

Consider a Cauchy sequence $\{x_i\}$ in $X$.

Since $X$ is a metric space, we can find a nest sequence $C_1 \supset C_2 \supset \cdots$ of nonempty closed subsets of $X$ with $\lim_{n\to\infty} d(C_n) = 0$, s.t. $x_i \in C_i$ for $i \in \mathbb{N}_+$.

Since $\bigcap_n C_n \neq \varnothing$, assume that $x \in \bigcap_n C_n$.

For arbitrary $\epsilon > 0$, we can find some $N \in \mathbb{N}$, s.t. $d(C_n) < \epsilon$ for all $n > N$.

And $d(x_n, x) < d(C_n) < \epsilon$ since $x_n$ and $x$ are both in $C_n$.

Then we know that $\{x_i\}$ converges to $x$ in $X$.

Hence $X$ is complete.

## Question 4.

Consider a Cauchy sequence $\{(x_i)_n\}_{n\in\mathbb{N}_+}$ in $H$.

Since $|x_i - y_i| \leq (\sum_i (x_i, y_i)^2)^{\frac{1}{2}} = d((x_i), (y_i))$, we have $\{x_i^n\}_n$ is also Cauchy.

Since $\{x_i^n\}_n$ is a Cauchy sequence in $\mathbb{R}$ and $\mathbb{R}$ is complete, it is convergent.

Assume that $\{x_i^n\}_n$ converges to $a_i$.

Then we obtain a point $a = (a_i)$ and $\{(x_i)_n\}$ converges to $(a_i)$.

Choose arbitrary $\epsilon > 0$, there exists some $N \in \mathbb{N}_+$, s.t. $(\sum_i (a_i - x_i^n)^2)^{\frac{1}{2}} < \epsilon$ for all $n > N$, and $\sum_i (a_i - x_i^n)^2 < \epsilon^2$ for all $n > N$. Choose a $n > N$, and we have following relations.

$$\begin{aligned}
\sum_i a_i^2 &= \sum_i (a_i - x_i^n + x_i^n)^2 = \sum_i (a_i - x_i^n)^2 + 2\sum_i (a_i - x_i^n)x_i^n + \sum_i (x_i^n)^2 \\
&\leq \sum_i (a_i - x_i^n)^2 + ((\sum_i (a_i - x_i^n)^2)(\sum_i (x_i^n)^2))^{\frac{1}{2}} + \sum_i (x_i^n)^2 \\
&\leq \epsilon^2 + \epsilon(\sum_i (x_i^n)^2)^{\frac{1}{2}} + \sum_i (x_i^n)^2
\end{aligned}$$

Since $(x_i)_n \in H$ we have $\sum_i (x_i^n)^2 < \infty$ and then $\sum_i a_i^2 < \infty$.

Then we know that $(a_i) \in H$ and $(H, d)$ is complete.

## Question 5.

Consider a sequence $\{x_n\}$ in $\mathbb{Q}$.

For a prime number $p$, consider an integer $a$ and an integer $x_1$ with $x_1^2 \equiv a \mod p$ and $2x_k \not\equiv 0 \mod p$.

Construct $\{x_n\}$ with

$$x_{k+1} = x_k - \frac{x_k^2 - a}{2x_k} = \frac{x_k^2 + a}{2x_k}$$

Assume that $x_k^2 = a + cp^k$.

$$\begin{aligned}
x_{k+1}^2 &= \left(\frac{x_k^2 + a}{2x_k}\right)^2 = \frac{x_k^4 + 2ax_k^2 + a^2}{4x_k^2} \\
&= \frac{4a^2 + 4acp^k + c^2p^{2k}}{4a + 4cp^k} = a + \frac{c^2p^{2k}}{4a + 4cp^k} \\
&= a + c'p^{k+1}
\end{aligned}$$

Then we have $x_k^2 \equiv a \mod p^k$ and $x_{k+m} \equiv x_k \mod p^k$.

This means that $\{x_n\}$ is Cauchy since $d(x_{k+m}, x_k) = -p^k$.

But the limit of $\{x_n\}$ is $\sqrt{a}$.

If $a$ is not perfectly square, then $\{x_n\}$ is not convergent in $\mathbb{Q}$.

Hence $\mathbb{Q}$ is incomplete.

The completion of $(\mathbb{Q}, d)$ is the space of all roots of polynomials with rational coefficients.