TOPOLOGY - HOMEWORK 05

Due: 2020 April 9th, 10:00AM

Question 1. Let (X, d) be a metric space with induced topology $\mathcal{T}$. Define

$$d^*(x, y) = \min \{d(x, y), 1\}$$

for every $x, y \in X$. Show that

(i) d^* is a metric on X, and

(ii) its induced topology $\mathcal{T}^*$ is equal to $\mathcal{T}$.

Question 2. Consider the Cartesian product

$$\mathbf{R}^\omega := \prod_{i \in \mathbf{N}} \mathbf{R} = \{(x_i)_{i \in \mathbf{N}} \mid x_i \in \mathbf{R}\}.$$

Let $\mathbf{R}_f \subset \mathbf{R}^\omega$ be the subset consisting of all sequences that are "eventually zero", i.e., $(x_i)_{i \in \mathbf{N}}$ such that $x_i \neq 0$ for only finitely many i. What are the interior and closure of $\mathbf{R}_f$ in $\mathbf{R}^\omega$, when the latter is endowed with the box topology $\mathcal{T}_b$, or the product topology $\mathcal{T}_p$?

Question 3. Is the topological space $(\mathbf{R}^\omega, \mathcal{T}_p)$ metrizable? How about $(\mathbf{R}^\omega, \mathcal{T}_b)$?
(Hint: for $\mathcal{T}_p$, consider the function $d^\omega((x_i), (y_i)) = \sup \{d^*(x_i, y_i)/i : \ i \in \mathbf{N}\}$ where d is the standard metric on $\mathbf{R}$.)

Question 4. Consider the following subset of $\mathbf{R}^2$

$$X = \{(x, 0) : x \in \mathbf{R}\} \cup \{(x, 1/x) : x \in \mathbf{R}_{>0}\}$$

endowed with the standard subspace topology from $\mathbf{R}^2$. Is X connected? Justify your answer.

Question 5. Consider the topological space $(\mathbf{R}^\omega, \mathcal{T}_b)$. Let $U \subset \mathbf{R}^\omega$ be the subset consisting of bounded sequences, i.e., those $x = (x_i)_i$ such that there is C_x with $|x_i| \leq C_x$ for all i. Prove that U and $\mathbf{R}^\omega - U$ form a sepration of $\mathbf{R}^\omega$ with respect to $\mathcal{T}_b$, and thus conclude that $(\mathbf{R}^\omega, \mathcal{T}_b)$ is disconnected.