MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 1

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 1.1: If $r \in \mathbf{Q} \setminus \{0\}$ and $x \in \mathbf{R} \setminus \mathbf{Q}$, prove that $r + x$, $rx \notin \mathbf{Q}$.

Solution: We prove this by contradiction. Let $r \in \mathbf{Q} \setminus \{0\}$, and suppose that $r + x \in \mathbf{Q}$. Then, using the field properties of both $\mathbf{R}$ and $\mathbf{Q}$, we have $x = (r + x) - r \in \mathbf{Q}$. Thus $x \notin \mathbf{Q}$ implies $r + x \notin \mathbf{Q}$.

Similarly, if $rx \in \mathbf{Q}$, then $x = (rx)/r \in \mathbf{Q}$. (Here, in addition to the field properties of $\mathbf{R}$ and $\mathbf{Q}$, we use $r \neq 0$.) Thus $x \notin \mathbf{Q}$ implies $rx \notin \mathbf{Q}$. ∎

Problem 1.2: Prove that there is no $x \in \mathbf{Q}$ such that $x^2 = 12$.

Solution: We prove this by contradiction. Suppose there is $x \in \mathbf{Q}$ such that $x^2 = 12$. Write $x = \frac{m}{n}$ in lowest terms. Then $x^2 = 12$ implies that $m^2 = 12n^2$. Since 3 divides $12n^2$, it follows that 3 divides m^2. Since 3 is prime (and by unique factorization in $\mathbf{Z}$), it follows that 3 divides m. Therefore 3^2 divides $m^2 = 12n^2$. Since 3^2 does not divide 12, using again unique factorization in $\mathbf{Z}$ and the fact that 3 is prime, it follows that 3 divides n. We have proved that 3 divides both m and n, contradicting the assumption that the fraction $\frac{m}{n}$ is in lowest terms. ∎

Alternate solution (Sketch): If $x \in \mathbf{Q}$ satisfies $x^2 = 12$, then $\frac{x}{2}$ is in $\mathbf{Q}$ and satisfies $\left(\frac{x}{2}\right)^2 = 3$. Now prove that there is no $y \in \mathbf{Q}$ such that $y^2 = 3$ by repeating the proof that $\sqrt{2} \notin \mathbf{Q}$. ∎

Problem 1.5: Let $A \subset \mathbf{R}$ be nonempty and bounded below. Set $-A = \{-a \colon a \in A\}$. Prove that $\inf(A) = -\sup(-A)$.

Solution: First note that $-A$ is nonempty and bounded above. Indeed, A contains some element x, and then $-x \in A$; moreover, A has a lower bound m, and $-m$ is an upper bound for $-A$.

We now know that $b = \sup(-A)$ exists. We show that $-b = \inf(A)$. That $-b$ is a lower bound for A is immediate from the fact that b is an upper bound for $-A$. To show that $-b$ is the greatest lower bound, we let $c > -b$ and prove that c is not a lower bound for A. Now $-c < b$, so $-c$ is not an upper bound for $-A$. So there exists $x \in -A$ such that $x > -c$. Then $-x \in A$ and $-x < c$. So c is not a lower bound for A. ∎

Problem 1.6: Let $b \in \mathbf{R}$ with $b > 1$, fixed throughout the problem.

Comment: We will assume known that the function $n \mapsto b^n$, from $\mathbf{Z}$ to $\mathbf{R}$, is strictly increasing, that is, that for $m, n \in \mathbf{Z}$, we have $b^m < b^n$ if and only if $m < n$. Similarly, we take as known that $x \mapsto x^n$ is strictly increasing when n is

Date: 1 October 2001.

an *integer* with $n > 0$. We will also assume that the usual laws of exponents are known to hold when the exponents are *integers*. We can't assume anything about fractional exponents, except for Theorem 1.21 of the book and its corollary, because the context makes it clear that we are to assume fractional powers have not yet been defined.

(a) Let $m, n, p, q \in \mathbf{Z}$, with $n > 0$ and $q > 0$. Prove that if $\frac{m}{n} = \frac{p}{q}$, then $(b^m)^{1/n} = (b^p)^{1/q}$.

Solution: By the uniqueness part of Theorem 1.21 of the book, applied to the positive integer nq, it suffices to show that

$$\left[(b^m)^{1/n}\right]^{nq} = \left[(b^p)^{1/q}\right]^{nq}.$$

Now the definition in Theorem 1.21 implies that

$$\left[(b^m)^{1/n}\right]^{n} = b^m \quad \text{and} \quad \left[(b^p)^{1/q}\right]^{q} = b^p.$$

Therefore, using the laws of integer exponents and the equation $mq = np$, we get

$$\begin{aligned}\left[(b^m)^{1/n}\right]^{nq} &= \left[\left[(b^m)^{1/n}\right]^{n}\right]^{q} = (b^m)^q = b^{mq} \\ &= b^{np} = (b^p)^n = \left[\left[(b^p)^{1/q}\right]^{q}\right]^{n} = \left[(b^p)^{1/q}\right]^{nq},\end{aligned}$$

as desired. ∎

By Part (a), it makes sense to define $b^{m/n} = (b^m)^{1/n}$ for $m, n \in \mathbf{Z}$ with $n > 0$. This defines b^r for all $r \in \mathbf{Q}$.

(b) Prove that $b^{r+s} = b^r b^s$ for $r, s \in \mathbf{Q}$.

Solution: Choose $m, n, p, q \in \mathbf{Z}$, with $n > 0$ and $q > 0$, such that $r = \frac{m}{n}$ and $s = \frac{p}{q}$. Then $r + s = \frac{mq+np}{nq}$. By the uniqueness part of Theorem 1.21 of the book, applied to the positive integer nq, it suffices to show that

$$\left[b^{(mq+np)/(nq)}\right]^{nq} = \left[(b^m)^{1/n}(b^p)^{1/q}\right]^{nq}.$$

Directly from the definitions, we can write

$$\left[b^{(mq+np)/(nq)}\right]^{nq} = \left[\left[b^{(mq+np)}\right]^{1/(nq)}\right]^{nq} = b^{(mq+np)}.$$

Using the laws of integer exponents and the definitions for rational exponents, we can rewrite the right hand side as

$$\left[(b^m)^{1/n}(b^p)^{1/q}\right]^{nq} = \left[\left[(b^m)^{1/n}\right]^{n}\right]^{q}\left[\left[(b^p)^{1/q}\right]^{q}\right]^{n} = (b^m)^q (b^p)^n = b^{(mq+np)}.$$

This proves the required equation, and hence the result. ∎

(c) For $x \in \mathbf{R}$, define

$$B(x) = \{b^r \colon r \in \mathbf{Q} \cap (-\infty, x]\}.$$

Prove that if $r \in \mathbf{Q}$, then $b^r = \sup(B(r))$.

Solution: The main point is to show that if $r, s \in \mathbf{Q}$ with $r < s$, then $b^r < b^s$. Choose $m, n, p, q \in \mathbf{Z}$, with $n > 0$ and $q > 0$, such that $r = \frac{m}{n}$ and $s = \frac{p}{q}$. Then also $r = \frac{mq}{nq}$ and $s = \frac{np}{nq}$, with $nq > 0$, so

$$b^r = (b^{mq})^{1/(nq)} \quad \text{and} \quad b^s = (b^{np})^{1/(nq)}.$$

Now $mq < np$ because $r < s$. Therefore, using the definition of $c^{1/(nq)}$,

$$(b^r)^{nq} = b^{mq} < b^{np} = (b^s)^{nq}.$$

Since $x \mapsto x^{nq}$ is strictly increasing, this implies that $b^r < b^s$.

Now we can prove that if $r \in \mathbf{Q}$ then $b^r = \sup(B(r))$. By the above, if $s \in \mathbf{Q}$ and $s \leq r$, then $b^s \leq b^r$. This implies that b^r is an upper bound for $B(r)$. Since $b^r \in B(r)$, obviously no number smaller than b^r can be an upper bound for $B(r)$. So $b^r = \sup(B(r))$. ∎

We now define $b^x = \sup(B(x))$ for every $x \in \mathbf{R}$. We need to show that $B(x)$ is nonempty and bounded above. To show it is nonempty, choose (using the Archimedean property) some $k \in \mathbf{Z}$ with $k < x$; then $b^k \in B(x)$. To show it is bounded above, similarly choose some $k \in \mathbf{Z}$ with $k > x$. If $r \in \mathbf{Q} \cap (-\infty, x]$, then $b^r \in B(k)$ so that $b^r \leq b^k$ by Part (c). Thus b^k is an upper bound for $B(x)$. This shows that the definition makes sense, and Part (c) shows it is consistent with our earlier definition when $r \in \mathbf{Q}$.

(d) Prove that $b^{x+y} = b^x b^y$ for all x, $y \in \mathbf{R}$.

Solution:

In order to do this, we are going to need to replace the set $B(x)$ above by the set

$$B_0(x) = \{b^r \colon r \in \mathbf{Q} \cap (-\infty, x)\}$$

(that is, we require $r < x$ rather than $r \leq x$) in the definition of b^x. (If you are skeptical, read the main part of the solution first to see how this is used.)

We show that the replacement is possible via some lemmas.

Lemma 1. If $x \in [0, \infty)$ and $n \in \mathbf{Z}$ satisfies $n \geq 0$, then $(1 + x)^n \geq 1 + nx$.

Proof: The proof is by induction on n. The statement is obvious for $n = 0$. So assume it holds for some n. Then, since $x \geq 0$,

$$\begin{aligned}(1 + x)^{n+1} &= (1 + x)^n (1 + x) \geq (1 + nx)(1 + x) \\ &= 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x.\end{aligned}$$

This proves the result for $n + 1$. ∎

Lemma 2. $\inf\{b^{1/n} \colon n \in \mathbf{N}\} = 1$. (Recall that $b > 1$ and $\mathbf{N} = \{1, 2, 3, \dots\}$.)

Proof: Clearly 1 is a lower bound. (Indeed, $(b^{1/n})^n = b > 1 = 1^n$, so $b^{1/n} > 1$.) We show that $1+x$ is not a lower bound when $x > 0$. If $1+x$ were a lower bound, then $1 + x \leq b^{1/n}$ would imply $(1 + x)^n \leq (b^{1/n})^n = b$ for all $n \in \mathbf{N}$. By Lemma 1, we would get $1 + nx \leq b$ for all $n \in \mathbf{N}$, which contradicts the Archimedean property when $x > 0$. ∎

Lemma 3. $\sup\{b^{-1/n} \colon n \in \mathbf{N}\} = 1$.

Proof: Part (b) shows that $b^{-1/n} b^{1/n} = b^0 = 1$, whence $b^{-1/n} = (b^{1/n})^{-1}$. Since all numbers $b^{-1/n}$ are strictly positive, it now follows from Lemma 2 that 1 is an upper bound. Suppose $x < 1$ is an upper bound. Then x^{-1} is a lower bound for

$\{b^{1/n} : n \in \mathbf{N}\}$. Since $x^{-1} > 1$, this contradicts Lemma 2. Thus $\sup\{b^{-1/n} : n \in \mathbf{N}\} = 1$, as claimed. ∎

Lemma 4. $b^x = \sup(B_0(x))$ for $x \in \mathbf{R}$.

Proof: If $x \notin \mathbf{Q}$, then $B_0(x) = B(x)$, so there is nothing to prove. If $x \in \mathbf{Q}$, then at least $B_0(x) \subset B(x)$, so $b^x \geq \sup(B_0(x))$. Moreover, Part (b) shows that $b^{x-1/n} = b^x b^{-1/n}$ for $n \in \mathbf{N}$. The numbers $b^{x-1/n}$ are all in $B_0(x)$, and

$$\sup\{b^x b^{-1/n} : n \in \mathbf{N}\} = b^x \sup\{b^{-1/n} : n \in \mathbf{N}\}$$

because $b^x > 0$, so using Lemma 3 in the last step gives

$$\sup(B_0(x)) \geq \sup\{b^{x-1/n} : n \in \mathbf{N}\} = b^x \sup\{b^{-1/n} : n \in \mathbf{N}\} = b^x.$$

∎

Now we can prove the formula $b^{x+y} = b^x b^y$. We start by showing that $b^{x+y} \leq b^x b^y$, which we do by showing that $b^x b^y$ is an upper bound for $B_0(x+y)$. Thus let $r \in \mathbf{Q}$ satisfy $r < x+y$. Then there are $s_0,\, t_0 \in \mathbf{R}$ such that $r = s_0 + t_0$ and $s_0 < x$, $t_0 < y$. Choose $s,\, t \in \mathbf{Q}$ such that $s_0 < s < x$ and $t_0 < t < y$. Then $r < s + t$, so $b^r < b^{s+t} = b^s b^t \leq b^x b^y$. This shows that $b^x b^y$ is an upper bound for $B_0(x+y)$.

(Note that this does not work using $B(x+y)$. If $x + y \in \mathbf{Q}$ but $x,\, y \notin \mathbf{Q}$, then $b^{x+y} \in B(x+y)$, but it is not possible to find s and t with $b^s \in B(x)$, $b^t \in B(y)$, and $b^s b^t = b^{x+y}$.)

We now prove the reverse inequality. Suppose it fails, that is, $b^{x+y} < b^x b^y$. Then

$$\frac{b^{x+y}}{b^y} < b^x.$$

The left hand side is thus not an upper bound for $B_0(x)$, so there exists $s \in \mathbf{Q}$ with $s < x$ and

$$\frac{b^{x+y}}{b^y} < b^s.$$

It follows that

$$\frac{b^{x+y}}{b^s} < b^y.$$

Repeating the argument, there is $t \in \mathbf{Q}$ with $t < y$ such that

$$\frac{b^{x+y}}{b^s} < b^t.$$

Therefore

$$b^{x+y} < b^s b^t = b^{s+t}$$

(using Part (b)). But $b^{s+t} \in B_0(x+y)$ because $s + t \in \mathbf{Q}$ and $s + t < x + y$, so this is a contradiction. Therefore $b^{x+y} \leq b^x b^y$. ∎

Problem 1.9: Define a relation on $\mathbf{C}$ by $w < z$ if and only if either $\mathrm{Re}(w) < \mathrm{Re}(z)$ or both $\mathrm{Re}(w) = \mathrm{Re}(z)$ and $\mathrm{Im}(w) < \mathrm{Im}(z)$. (For $z \in \mathbf{C}$, the expressions $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts of z.) Prove that this makes $\mathbf{C}$ an ordered set. Does this order have the least upper bound property?

Solution: We verify the two conditions in the definition of an order. For the first, let $w,\, z \in \mathbf{C}$. There are three cases.

Case 1: $\mathrm{Re}(w) < \mathrm{Re}(z)$. Then $w < z$, but $w = z$ and $w > z$ are both false.

Case 2: $\mathrm{Re}(w) > \mathrm{Re}(z)$. Then $w > z$, but $w = z$ and $w < z$ are both false.

Case 3: $\mathrm{Re}(w) = \mathrm{Re}(z)$. This case has three subcases.
Case 3.1: $\mathrm{Im}(w) < \mathrm{Im}(z)$. Then $w < z$, but $w = z$ and $w > z$ are both false.
Case 3.2: $\mathrm{Im}(w) > \mathrm{Im}(z)$. Then $w > z$, but $w = z$ and $w < z$ are both false.
Case 3.3: $\mathrm{Im}(w) = \mathrm{Im}(z)$. Then $w = z$, but $w > z$ and $w < z$ are both false.

These cases exhaust all possibilities, and in each of them exactly one of $w < z$, $w = z$, and $w > z$ is true, as desired.

Now we prove transitivity. Let $s < w$ and $w < z$. If either $\mathrm{Re}(s) < \mathrm{Re}(w)$ or $\mathrm{Re}(w) < \mathrm{Re}(z)$, then clearly $\mathrm{Re}(s) < \mathrm{Re}(z)$, so $s < z$. If $\mathrm{Re}(s) = \mathrm{Re}(w)$ and $\mathrm{Re}(w) = \mathrm{Re}(z)$, then the definition of the order requires $\mathrm{Im}(s) < \mathrm{Im}(w)$ and $\mathrm{Im}(w) < \mathrm{Im}(z)$. We thus have $\mathrm{Re}(s) = \mathrm{Re}(z)$ and $\mathrm{Im}(s) < \mathrm{Im}(z)$, so $s < z$ by definition.

It remains to answer the last question. We show that this order does not have the least upper bound property. Let $S = \{z \in \mathbf{C} \colon \mathrm{Re}(z) < 0\}$. Then $S \neq \varnothing$ because $-1 \in S$, and S is bounded above because 1 is an upper bound for S.

We show that S does not have a least upper bound by showing that if w is an upper bound for S, then there is a smaller upper bound. First, by the definition of the order it is clear that $\mathrm{Re}(w)$ is an upper bound for

$$\{\mathrm{Re}(z) \colon z \in S\} = (-\infty, 0).$$

Therefore $\mathrm{Re}(w) \geq 0$. Moreover, every $u \in \mathbf{C}$ with $\mathrm{Re}(u) \geq 0$ is in fact an upper bound for S. In particular, if w is an upper bound for S, then $w - i < w$ and has the same real part, so is a smaller upper bound. ∎

Note: A related argument shows that the set $T = \{z \in \mathbf{C} \colon \mathrm{Re}(z) \leq 0\}$ also has no least upper bound. One shows that w is an upper bound for T if and only if $\mathrm{Re}(w) > 0$.

Problem 1.13: Prove that if $x, y \in \mathbf{C}$, then $||x| - |y|| \leq |x - y|$.

Solution: The desired inequality is equivalent to

$$|x| - |y| \leq |x - y| \quad \text{and} \quad |y| - |x| \leq |x - y|.$$

We prove the first; the second follows by exchanging x and y.

Set $z = x - y$. Then $x = y + z$. The triangle inequality gives $|x| \leq |y| + |z|$. Substituting the definition of z and subtracting $|y|$ from both sides gives the result. ∎

Problem 1.17: Prove that if $x, y \in \mathbf{R}^n$, then

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Interpret this result geometrically in terms of parallelograms.

Solution: Using the definition of the norm in terms of scalar products, we have:

$$\begin{aligned}
\|x + y\|^2 + \|x - y\|^2 &= \langle x + y, x + y\rangle + \langle x - y, x - y\rangle \\
&= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle \\
&\quad + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle \\
&= 2\langle x, x\rangle + 2\langle y, y\rangle = 2\|x\|^2 + 2\|y\|^2.
\end{aligned}$$

The interpretation is that 0, x, y, $x + y$ are the vertices of a parallelogram, and that $\|x+y\|$ and $\|x-y\|$ are the lengths of its diagonals while $\|x\|$ and $\|y\|$ are each the lengths of two opposite sides. Therefore the sum of the squares of the lengths of the diagonals is equal to the sum of the squares of the lengths of the sides. ∎

Note: One can do the proof directly in terms of the formula $\|x\|^2 = \sum_{k=1}^{n} |x_k|^2$. The steps are all the same, but it is more complicated to write. It is also less general, since the argument above applies to any norm that comes from a scalar product.

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 2

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 2.2: Prove that the set of algebraic numbers is countable.

Solution (sketch): For each fixed integer $n \geq 0$, the set P_n of all polynomials with integer coefficients and degree at most n is countable, since it has the same cardinality as the set $\{(a_0, \ldots, a_n) \colon a_i \in \mathbf{N}\} = \mathbf{N}^{n+1}$. The set of all polynomials with integer coefficients is $\bigcup_{n=0}^{\infty} P_n$, which is a countable union of countable sets and so countable. Each polynomial has only finitely many roots (at most n for degree n), so the set of all possible roots of all polynomials with integer coefficients is a countable union of finite sets, hence countable. ∎

Problem 2.3: Prove that there exist real numbers which are not algebraic.

Solution (Sketch): This follows from Problem 2.2, since $\mathbf{R}$ is not countable. ∎

Problem 2.4: Is $\mathbf{R} \setminus \mathbf{Q}$ countable?

Solution (Sketch): No. $\mathbf{Q}$ is countable and $\mathbf{R}$ is not countable. ∎

Problem 2.5: Construct a bounded subset of $\mathbf{R}$ with exactly 3 limit points.

Solution (Sketch): For example, use

$$\left\{\tfrac{1}{n} \colon n \in \mathbf{N}\right\} \cup \left\{1 + \tfrac{1}{n} \colon n \in \mathbf{N}\right\} \cup \left\{2 + \tfrac{1}{n} \colon n \in \mathbf{N}\right\}.$$

∎

Problem 2.6: Let E' denote the set of limit points of E. Prove that E' is closed. Prove that $\overline{E}' = E'$. Is $(E')'$ always equal to E'?

Solution (Sketch): Proving that E' is closed is equivalent to proving that $(E')' \subset E'$. So let $x \in (E')'$ and let $\varepsilon > 0$. Choose $y \in E' \cap (N_\varepsilon(x) \setminus \{x\})$. Choose $\delta = \min(d(x, y), \varepsilon - d(x, y)) > 0$. Choose $z \in E \cap (N_\delta(y) \setminus \{y\})$. The triangle inequality ensures $z \neq x$ and $z \in N_\varepsilon(x)$. This shows x is a limit point of E.

Here is a different way to prove that $(E')' \subset E'$. Let $x \in (E')'$ and $\varepsilon > 0$. Choose $y \in E' \cap (N_{\varepsilon/2}(x) \setminus \{x\})$. By Theorem 2.20 of Rudin, there are infinitely many points in $E \cap (N_{\varepsilon/2}(y) \setminus \{y\})$. In particular there is $z \in E \cap (N_{\varepsilon/2}(y) \setminus \{y\})$ with $z \neq x$. Now $z \in E \cap (N_\varepsilon(x) \setminus \{x\})$.

To prove $\overline{E}' = E'$, it suffices to prove $\overline{E}' \subset E'$. We first claim that if A and B are any subsets of X, then $(A \cup B)' \subset A' \cup B'$. The fastest way to do this is to assume that $x \in (A \cup B)'$ but $x \notin A'$, and to show that $x \in B'$. Accordingly, let $x \in (A \cup B)' \setminus A'$. Since $x \notin A'$, there is $\varepsilon_0 > 0$ such that $N_{\varepsilon_0}(x) \cap A$ contains no

Date: 8 October 2001.

points except possibly x itself. Now let $\varepsilon > 0$; we show that $N_\varepsilon(x) \cap B$ contains at least one point different from x. Let $r = \min(\varepsilon, \varepsilon_0) > 0$. Because $x \in (A \cup B)'$, there is $y \in N_r(x) \cap (A \cup B)$ with $y \neq x$. Then $y \notin A$ because $r \leq \varepsilon_0$. So necessarily $y \in B$, and thus y is a point different from x and in $N_r(x) \cap B$. This shows that $x \in B'$, and completes the proof that $(A \cup B)' \subset A' \cup B'$.

To prove $\overline{E}' \subset E'$, we now observe that

$$\overline{E}' = (E \cup E')' \subset E' \cup (E')' \subset E' \cup E' = E'.$$

An alternate proof that $\overline{E}' \subset E'$ can be obtained by slightly modifying either of the proofs above that $(E')' \subset E'$.

For the third part, the answer is no. Take

$$E = \{0\} \cup \left\{ \tfrac{1}{n} : n \in \mathbf{N} \right\}.$$

Then $E' = \{0\}$ and $(E')' = \varnothing$. (Of course, you must prove these facts.) ∎

Problem 2.8: If $E \subset \mathbf{R}^2$ is open, is every point of E a limit point of E? What if E is closed instead of open?

Solution (Sketch): Every point of an open set $E \subset \mathbf{R}^2$ is a limit point of E. Indeed, if $x \in E$, then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset E$, and it is easy to show that x is a limit point of $N_\varepsilon(x)$.

(Warning: This is *not true* in a general metric space.)

Not every point of a closed set need be a limit point. Take $E = \{(0, 0)\}$, which has no limit points. ∎

Problem 2.9: Let E° denote the set of interior points of a set E, that is, the interior of E.

(a) Prove that E° is open.

Solution (sketch): If $x \in E^\circ$, then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \subset E$. Since $N_\varepsilon(x)$ is open, every point in $N_\varepsilon(x)$ is an interior point of $N_\varepsilon(x)$, hence of the bigger set E. So $N_\varepsilon(x) \in E^\circ$. ∎

(b) Prove that E is open if and only if $E^\circ = E$.

Solution: If E is open, then $E = E^\circ$ by the definition of E°. If $E = E^\circ$, then E is open by Part (a). ∎

(c) If G is open and $G \subset E$, prove that $G \subset E^\circ$.

Solution (sketch): If $x \in G \subset E$ and G is open, then x is an interior point of G. Therefore x is an interior point of the bigger set E. So $x \in E^\circ$. ∎

(d) Prove that $X \setminus E^\circ = \overline{X \setminus E}$.

Solution (sketch): First show that $X \setminus E^\circ \subset \overline{X \setminus E}$. If $x \notin E$, then clearly $x \in \overline{X \setminus E}$. Otherwise, consider $x \in E \setminus E^\circ$. Rearranging the statement that x fails to be an interior point of E, and noting that x itself is not in $X \setminus E$, one gets exactly the statement that x is a limit point of $X \setminus E$.

Now show that $\overline{X \setminus E} \subset X \setminus E^\circ$. If $x \in X \setminus E$, then clearly $x \notin E^\circ$. If $x \notin X \setminus E$ but x is a limit point of $X \setminus E$, then one simply rearranges the definition of a limit point to show that x is not an interior point of E. ∎

(e) Prove or disprove: $\left(\overline{E}\right)^\circ = \overline{E}$.

Solution (sketch): This is false. Example: take $E = (0,1) \cup (1,2)$. We have $E^\circ = E$, $\bar{E} = [0,2]$, and $\left(\overline{E}\right)^\circ = (0,2)$.

Another example is $\mathbf{Q}$. ∎

(f) Prove or disprove: $\overline{E^\circ} = \overline{E}$.

Solution (sketch): This is false. Example: take $E = (0,1) \cup \{2\}$. Then $\overline{E} = [0,1] \cup \{2\}$, $E^\circ = (0,1)$, and $\overline{E}^\circ = [0,1]$.

The sets $\mathbf{Q}$ and $\{0\}$ are also examples: in both cases, $E^\circ = \varnothing$. ∎

Problem 2.11: Which of the following are metrics on $\mathbf{R}$?

(a) $d_1(x,y) = (x-y)^2$.

Solution (Sketch): No. The triangle inequality fails with $x = 0$, $y = 2$, and $z = 4$. ∎

(b) $d_2(x,y) = \sqrt{|x-y|}$.

Solution (Sketch): Yes. Some work is needed to check the triangle inequality. ∎

(c) $d_3(x,y) = |x^2 - y^2|$.

Solution (Sketch): No. $d_3(1,-1) = 0$. ∎

(d) $d_4(x,y) = |x - 2y|$.

Solution (Sketch): No. $d_4(1,1) \neq 0$. Also, $d_4(1,6) \neq d_4(6,1)$. ∎

(e) $d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$.

Solution (Sketch): Yes. Some work is needed to check the triangle inequality. You need to know that $t \mapsto \frac{t}{1+t}$ is nondecreasing on $[0,\infty)$, and that a, $b \geq 0$ implies

$$\frac{a+b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

Do the first by algebraic manipulation. The second is

$$\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

(This is easier than what most people did the last time I assigned this problem.) ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 3

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 2.14: Give an example of an open cover of the interval $(0, 1) \subset \mathbf{R}$ which has no finite subcover.

Solution (sketch): $\{(1/n, 1) \colon n \in \mathbf{N}\}$. (Note that you must show that this works.) ∎

Problem 2.16: Regard $\mathbf{Q}$ as a metric space with the usual metric. Let $E = \{x \in \mathbf{Q} \colon 2 < x^2 < 3\}$. Prove that E is a closed and bounded subset of $\mathbf{Q}$ which is not compact. Is E an open subset of $\mathbf{Q}$?

Solution (sketch): Clearly E is bounded.

We prove E is closed. The fast way to do this is to note that

$$\mathbf{Q} \setminus E = \mathbf{Q} \cap \left[\left(-\infty, -\sqrt{3}\right) \cup \left(-\sqrt{2}, \sqrt{2}\right) \cup \left(\sqrt{3}, \infty\right) \right],$$

and so is open by Theorem 2.30. To do it directly, suppose $x \in \mathbf{Q}$ is a limit point of E which is not in E. Since we can't have $x^2 = 2$ or $x^2 = 3$, we must have $x^2 < 2$ or $x^2 > 3$. Assume $x^2 > 3$. (The other case is handled similarly.) Let $r = |x| - \sqrt{3} > 0$. Then every $z \in N_r(x)$ satisfies

$$|z| \geq |x| - |x - z| > |x| - r > 0,$$

which implies that $z^2 > (|x| - r)^2 = 3$. This shows that $z \notin E$, which contradicts the assumption that x is a limit point of E.

The fast way to see that E is not compact is to note that it is a subset of $\mathbf{R}$, but is not closed in $\mathbf{R}$. (See Theorem 2.23.) To prove this directly, show that, for example, the sets

$$\left\{y \in \mathbf{Q} \colon 2 + \tfrac{1}{n} < y^2 < 3 - \tfrac{1}{n}\right\}$$

form an open cover of E which has no finite subcover.

To see that E is open in $\mathbf{Q}$, the fast way is to write

$$E = \mathbf{Q} \cap \left[\left(-\sqrt{3}, -\sqrt{2}\right) \cup \left(\sqrt{2}, -\sqrt{3}\right) \right],$$

which is open by Theorem 2.30. It can also be proved directly. ∎

Problem 2.19: Let X be a metric space, fixed throughout this problem.

(a) If A and B are disjoint closed subsets of X, prove that they are separated.

Solution (Sketch): We have $A \cap \overline{B} = \overline{A} \cap B = A \cap B = \varnothing$ because A and B are closed. ∎

Date: 15 October 2001.

(b) If A and B are disjoint open subsets of X, prove that they are separated.

Solution (Sketch): $X \setminus A$ is a closed subset containing B, and hence containing $\overline{B}$. Thus $A \cap \overline{B} = \varnothing$. Interchanging A and B, it follows that $\overline{A} \cap B = \varnothing$. ∎

(c) Fix $x_0 \in X$ and $\delta > 0$. Set

$$A = \{x \in X \colon d(x, x_0) < \delta\} \quad \text{and} \quad B = \{x \in X \colon d(x, x_0) > \delta\}.$$

Prove that A and B are separated.

Solution (Sketch): Both A and B are open sets (proof!), and they are disjoint. So this follows from Part (b). ∎

(d) Prove that if X is connected and contains at least two points, then X is uncountable.

Solution: Let x and y be distinct points of X. Let $R = d(x, y) > 0$. For each $r \in (0, R)$, consider the sets

$$A_r = \{z \in X \colon d(z, x) < r\} \quad \text{and} \quad B_r = \{z \in X \colon d(z, x) > r\}.$$

They are separated by Part (c). They are not empty, since $x \in A_r$ and $y \in B_r$. Since X is connected, there must be a point $z_r \in X \setminus (A_r \cup B_r)$. Then $d(x, z_r) = r$.

Note that if $r \neq s$, then $d(x, z_r) \neq d(x, z_s)$, so $z_r \neq z_s$. Thus $r \mapsto z_r$ defines an injective map from $(0, R)$ to X. Since $(0, R)$ is not countable, X can't be countable either. ∎

Problem 2.20: Let X be a metric space, and let $E \subset X$ be a connected subset. Is $\overline{E}$ necessarily connected? Is $\operatorname{int}(E)$ necessarily connected?

Solution to the first question (sketch): The set $\operatorname{int}(E)$ need not be connected. The easiest example to write down is to take $X = \mathbf{R}^2$ and

$$E = \{x \in \mathbf{R}^2 \colon \|x - (1, 0)\| \leq 1\} \cup \{x \in \mathbf{R}^2 \colon \|x - (-1, 0)\| \leq 1\}.$$

Then

$$\operatorname{int}(E) = \{x \in \mathbf{R}^2 \colon \|x - (1, 0)\| < 1\} \cup \{x \in \mathbf{R}^2 \colon \|x - (-1, 0)\| < 1\}.$$

This set fails to be connected because the point $(0, 0)$ is missing. A more dramatic example is two closed disks joined by a line, say

$$E = \{x \in \mathbf{R}^2 \colon \|x - (2, 0)\| \leq 1\} \cup \{x \in \mathbf{R}^2 \colon \|x - (-2, 0)\| \leq 1\} \tag{1}$$
$$\cup \{(\alpha, 0) \in \mathbf{R}^2 \colon -3 \leq \alpha \leq 3\}. \tag{2}$$

Then

$$\operatorname{int}(E) = \{x \in \mathbf{R}^2 \colon \|x - (2, 0)\| < 1\} \cup \{x \in \mathbf{R}^2 \colon \|x - (-2, 0)\| < 1\}.$$

∎

Solution to the second question: If E is connected, then $\overline{E}$ is necessarily connected. To prove this using Rudin's definition, assume $\overline{E} = A \cup B$ for separated sets A and B; we prove that one of A and B is empty. The sets $A_0 = A \cap E$ and $B_0 = B \cap E$ are separated sets such that $E = A_0 \cup B_0$. (They are separated because $\overline{A_0} \subset \overline{A}$ and $\overline{B_0} \subset \overline{B}$.) Because E is connected, one of A_0 and B_0 must be empty; without loss of generality, $A_0 = \varnothing$. Then $A \subset \overline{E} \setminus E$. Therefore $E \subset B$. But then $A \subset \overline{E} \subset \overline{B}$. Because A and B are separated, this can only happen if $A = \varnothing$. ∎

Alternate solution to the second question: If E is connected, we prove that $\overline{E}$ is necessarily connected, using the traditional definition. Thus, assume that $\overline{E} = A \cup B$ for disjoint relatively open sets A and B; we prove that one of A and B is empty. The sets $A_0 = A \cap E$ and $B_0 = B \cap E$ are disjoint relatively open sets in E such that $E = A_0 \cup B_0$. Because E is connected, one of A_0 and B_0 must be empty; without loss of generality, $A_0 = \varnothing$. Then $A \subset \overline{E} \setminus E$ and is relatively open in $\overline{E}$.

Now let $x \in A$. Then there is $\varepsilon > 0$ such that $N_\varepsilon(x) \cap \overline{E} \subset A$. So $N_\varepsilon(x) \cap \overline{E} \subset \overline{E} \setminus E$, which implies that $N_\varepsilon(x) \cap E = \varnothing$. This contradicts the fact that $x \in \overline{E}$. Thus $A = \varnothing$. ∎

Problem 2.22: Prove that $\mathbf{R}^n$ is separable.

Solution (sketch): The subset $\mathbf{Q}^n$ is countable by Theorem 2.13. To show that $\mathbf{Q}^n$ is dense, let $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and let $\varepsilon > 0$. Choose $y_1, \ldots, y_n \in \mathbf{Q}$ such that $|y_k - x_k| < \frac{\varepsilon}{n}$ for all k. (Why is this possible?) Then $y = (y_1, \ldots, y_n) \in \mathbf{Q}^n \cap N_\varepsilon(x)$. ∎

Problem 2.23: Prove that every separable metric space has a countable base.

Solution: Let X be a separable metric space. Let $S \subset X$ be a countable dense subset of X. Let

$$\mathcal{B} = \{N_{1/n}(s) \colon s \in S,\ n \in \mathbf{N}\}.$$

Since $\mathbf{N}$ and S are countable, $\mathcal{B}$ is a countable collection of open subsets of X.

Now let $U \subset X$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since S is dense in X, there is $s \in S \cap N_{1/n}(x)$, that is, $s \in S$ and $d(s, x) < \frac{1}{n}$. Then $x \in N_{1/n}(s)$ and $N_{1/n}(s) \in \mathcal{B}$. It remains to show that $N_{1/n}(s) \subset U$. So let $y \in N_{1/n}(s)$. Then

$$d(x, y) \leq d(x, s) + d(s, y) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $y \in N_\varepsilon(x) \subset U$. ∎

Problem 2.25: Let K be a compact metric space. Prove that K has a countable base, and that K is separable.

The easiest way to do this is actually to prove first that K is separable, and then to use Problem 2.23. However, the direct proof that K has a countable base is not very different, so we give it here. We actually give two versions of the proof, which differ primarily in how the indexing is done. The first version is easier to write down correctly, but the second has the advantage of eliminating some of the subscripts, which can be important in more complicated situations. Note that the second proof is shorter, even after the parenthetical remarks about indexing are deleted from the first proof. Afterwards, we give a proof that every metric space with a countable base is separable.

Solution 1: We prove that K has a countable base. For each $n \in \mathbf{N}$, the open sets $N_{1/n}(x)$, for $x \in K$, form an open cover of K. Since K is compact, this open cover has a finite subcover, say

$$\{N_{1/n}(x_{n,1}), N_{1/n}(x_{n,2}), \ldots, N_{1/n}(x_{n,k_n})\}$$

for suitable $x_{n,1}, x_{n,2}, \ldots, x_{n,k_n} \in K$. (Note: For each n, the collection of x's is different; therefore, they must be labelled independently by both n and a second parameter. The number of them also depends on n, so must be called k_n, $k(n)$, or something similar.)

Now let

$$\mathcal{B} = \{N_{1/n}(x_{n,j}) \colon n \in \mathbf{N},\ 1 \leq j \leq k_n\}.$$

(Note that both subscripts are used here.) Then $\mathcal{B}$ is a countable union of finite sets, hence countable. We show that $\mathcal{B}$ is a base for K.

Let $U \subset K$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since the sets

$$N_{1/n}(x_{n,1}), N_{1/n}(x_{n,2}), \ldots, N_{1/n}(x_{n,k_n})$$

cover K, there is j with $1 \leq j \leq k_n$ such that $x \in N_{1/n}(x_{n,j})$. (Here we see why the double indexing is necessary: the list of centers to choose from *depends on* n, and therefore their names must also depend on n.) By definition, $N_{1/n}(x_{n,j}) \in \mathcal{B}$. It remains to show that $N_{1/n}(x_{n,j}) \subset U$. So let $y \in N_{1/n}(x_{n,j})$. Since x and y are both in $N_{1/n}(x_{n,j})$, we have

$$d(x, y) \leq d(x, x_{n,j}) + d(x_{n,j}, y) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $y \in N_\varepsilon(x) \subset U$. ∎

Solution 2: We again prove that K has a countable base. For each $n \in \mathbf{N}$, the open sets $N_{1/n}(x)$, for $x \in K$, form an open cover of K. Since K is compact, this open cover has a finite subcover. That is, there is a finite set $F_n \subset K$ such that the sets $N_{1/n}(x)$, for $x \in F_n$, still cover K. Now let

$$\mathcal{B} = \{N_{1/n}(x) \colon n \in \mathbf{N},\ x \in F_n\}.$$

Then $\mathcal{B}$ is a countable union of finite sets, hence countable. We show that $\mathcal{B}$ is a base for K.

Let $U \subset K$ be open and let $x \in U$. Choose $\varepsilon > 0$ such that $N_\varepsilon(x) \subset U$. Choose $n \in \mathbf{N}$ such that $\frac{1}{n} < \frac{\varepsilon}{2}$. Since the sets $N_{1/n}(y)$, for $y \in F_n$, cover K, there is $y \in F_n$ such that $x \in N_{1/n}(y)$. By definition, $N_{1/n}(y) \in \mathcal{B}$. It remains to show that $N_{1/n}(y) \subset U$. So let $z \in N_{1/n}(y)$. Since x and z are both in $N_{1/n}(y)$, we have

$$d(x, z) \leq d(x, y) + d(y, z) < \tfrac{1}{n} + \tfrac{1}{n} < \varepsilon,$$

so $z \in N_\varepsilon(x) \subset U$. ∎

It remains to prove the following lemma.

Lemma: Let X be a metric space with a countable base. Then X is separable.

Proof: Let $\mathcal{B}$ be a countable base for X. Without loss of generality, we may assume $\varnothing \notin \mathcal{B}$. For each $U \in \mathcal{B}$, choose an element $x_U \in U$. Let $S = \{x_U \colon U \in \mathcal{B}\}$. Clearly S is (at most) countable. We show it is dense. So let $x \in X$ and let $\varepsilon > 0$. If $x \in S$, there is nothing to prove. Otherwise $N_\varepsilon(x)$ is an open set in X, so there exists $U \in \mathcal{B}$ such that $x \in U \subset N_\varepsilon(x)$. In particular, $x_U \in N_\varepsilon(x)$. Since $x_U \neq x$ and since $\varepsilon > 0$ is arbitrary, this shows that x is a limit point of S. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 4

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 3.1: Prove that if (s_n) converges, then $(|s_n|)$ converges. Is the converse true?

Solution (sketch): Use the inequality $\big||s_n| - |s|\big| \leq |s_n - s|$ and the definition of the limit. The converse is false. Take $s_n = (-1)^n$. (This requires proof, of course.) ∎

Problem 3.2: Calculate $\lim_{n\to\infty} \left(\sqrt{n^2+1} - n\right)$.

Solution (sketch):

$$\sqrt{n^2+1} - n = \frac{n}{\sqrt{n^2+1}+n} = \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} \to \frac{1}{2}.$$

(Of course, the last step requires proof.) ∎

Problem 3.3: Let $s_1 = \sqrt{2}$, and recursively define

$$s_{n+1} = \sqrt{2+\sqrt{s_n}}$$

for $n \in \mathbf{N}$. Prove that (s_n) converges, and that $s_n < 2$ for all $n \in \mathbf{N}$.

Solution (sketch): By induction, it is immediate that $s_n > 0$ for all n, so that s_{n+1} is always defined.

Next, we show by induction that $s_n < 2$ for all n. This is clear for $n = 1$. The computation for the induction step is

$$s_{n+1} = \sqrt{2+\sqrt{s_n}} \leq \sqrt{2+\sqrt{2}} < 2.$$

To prove convergence, it now suffices to show that (s_n) is nondecreasing. (See Theorem 3.14.) This is also done by induction. To start, observe that

$$s_2 = \sqrt{2+\sqrt{s_1}} = \sqrt{2+\sqrt{\sqrt{2}}} > \sqrt{2}.$$

The computation for the induction step is:

$$s_{n+1} - s_n = \sqrt{2+\sqrt{s_n}} - \sqrt{2+\sqrt{s_{n-1}}} = \frac{\sqrt{s_n} - \sqrt{s_{n-1}}}{\sqrt{2+\sqrt{s_n}} + \sqrt{2+\sqrt{s_{n-1}}}} > 0.$$

∎

Date: 22 October 2001.

Problem 3.4: Let $s_1 = 0$, and recursively define

$$s_{n+1} = \begin{cases} \frac{1}{2} + s_n & n \text{ is even} \\ \frac{1}{2} s_n & n \text{ is odd} \end{cases}.$$

for $n \in \mathbf{N}$. Find $\limsup_{n\to\infty} s_n$ and $\liminf_{n\to\infty} s_n$.

Solution (sketch): Use induction to show that

$$s_{2m} = \frac{2^{m-1}-1}{2^m} \quad \text{and} \quad s_{2m+1} = \frac{2^m - 1}{2^m}.$$

It follows that

$$\limsup_{n\to\infty} s_n = 1 \quad \text{and} \quad \liminf_{n\to\infty} s_n = \tfrac{1}{2}.$$

∎

Problem 3.5: Let (a_n) and (b_n) be sequences in $\mathbf{R}$. Prove that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided that the right hand side is defined, that is, not of the form $\infty - \infty$ or $-\infty + \infty$.

Four solutions are presented or sketched. The first is what I presume to be the solution Rudin intended. The second is a variation of the first, which minimizes the amount of work that must be done in different cases. The third shows what must be done if one wants to work directly from Rudin's definition. The fourth is the "traditional" proof of the result, and proceeds via the traditional definition.

Solution 1 (sketch): We give a complete solution for the case

$$\limsup_{n\to\infty} a_n \in (-\infty, \infty) \quad \text{and} \quad \limsup_{n\to\infty} b_n \in (-\infty, \infty).$$

One needs to consider several other cases, but the basic method is the same.

Define

$$a = \limsup_{n\to\infty} a_n \quad \text{and} \quad b = \limsup_{n\to\infty} b_n.$$

Let $c > a + b$. We show that c is not a subsequential limit of $(a_n + b_n)$.

Let $\varepsilon = \frac{1}{3}(c - a - b) > 0$. Use Theorem 3.17 (b) of Rudin to choose $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $a_n < a + \varepsilon$, and also to choose $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $b_n < b + \varepsilon$. For $n \geq \max(N_1, N_2)$, we then have $a_n + b_n < a + b + 2\varepsilon$. It follows that every subsequential limit l of $(a_n + b_n)$ satisfies $l \leq a + b + 2\varepsilon$. Since $c = a + b + 3\varepsilon > a + b + 2\varepsilon$, it follows that c is not a subsequential limit of $(a_n + b_n)$.

We conclude that $a + b$ is an upper bound for the set of subsequential limits of $(a_n + b_n)$. Therefore $\limsup_{n\to\infty}(a_n + b_n) \leq a + b$. ∎

Solution 2: This solution is a variation of Solution 1, designed to handle all cases at once. (You will see, though, that the case breakdown can't be avoided entirely.)

As in Solution 1, define

$$a = \limsup_{n\to\infty} a_n \quad \text{and} \quad b = \limsup_{n\to\infty} b_n,$$

and let $c > a + b$. We show that c is not a subsequential limit of $(a_n + b_n)$.

We first find r, s, $t \in \mathbf{R}$ such that

$$a < r, \quad b < s, \quad c > 0, \quad \text{and} \quad r + s + t \leq c.$$

If $a = \infty$ or $b = \infty$, this is vacuous, since no such c can exist. Next, suppose a and b are finite. If $c = \infty$, then

$$r = a + 1, \quad s = b + 1, \quad \text{and} \quad c = 1$$

will do. Otherwise, let $\varepsilon = \frac{1}{3}(c - a - b) > 0$, and take

$$r = a + \varepsilon, \quad s = b + \varepsilon, \quad \text{and} \quad t = \varepsilon.$$

Finally, suppose at least one of a and b is $-\infty$, but neither is ∞. Exchanging the sequences if necessary, assume that $a = -\infty$. Choose any $s > b$, choose any $t > 0$, and set $r = c - s - t$, which is certainly greater than $-\infty$.

Having r, s, and t, use Theorem 3.17 (b) of Rudin to choose $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies $a_n < r$, and also to choose $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies $b_n < s$. For $n \geq \max(N_1, N_2)$, we then have $a_n + b_n < r + s$. It follows that every subsequential limit l of $(a_n + b_n)$ satisfies $l \leq r + s$. Since $c = r + s + t > r + s$, it follows that c is not a subsequential limit of $(a_n + b_n)$.

We conclude that $a + b$ is an upper bound for the set of subsequential limits of $(a_n + b_n)$. Therefore $\limsup_{n\to\infty}(a_n + b_n) \leq a + b$. ∎

Solution 3 (sketch): We only consider the case that both $a = \limsup_{n\to\infty} a_n$ and $b = \limsup_{n\to\infty} b_n$ are finite. Let $s = \limsup_{n\to\infty}(a_n + b_n)$. Then there is a subsequence $(a_{k(n)} + b_{k(n)})$ of $(a_n + b_n)$ which converges to s. Further, $(a_{k(n)})$ is bounded. (This sequence is bounded above by assumption, and it is bounded below because $(a_{k(n)} + b_{k(n)})$ is bounded and $(b_{k(n)})$ is bounded above.) So there is a subsequence $(a_{l(n)})$ of $(a_{k(n)})$ which converges. (That is, there is a strictly increasing function $n \to r(n)$ such that the sequence $(a_{k\circ r(n)})$ converges, and we let $l = k \circ r \colon \mathbf{N} \to \mathbf{N}$. Note that if we used traditional subsequence notation, we would have the subsequence $j \mapsto a_{n_{k_j}}$ at this point.) Let $c = \lim_{n\to\infty} a_{l(n)}$. By similar reasoning to that given above, the sequence $(b_{l(n)})$ is bounded. Therefore it has a convergent subsequence, say $(b_{m(n)})$. (With traditional subsequence notation, we would now have the subsequence $i \mapsto a_{n_{k_{j_i}}}$. You can see why I don't like traditional notation.) Let $d = \lim_{n\to\infty} b_{m(n)}$. Since $(a_{m(n)})$ is a subsequence of $(a_{l(n)})$, we still have $\lim_{n\to\infty} a_{m(n)} = c$. So $\lim_{n\to\infty}(a_{m(n)} + b_{m(n)}) = c + d$. But also $a_{m(n)} + b_{m(n)}$ is a subsequence of $(a_{k(n)} + b_{k(n)})$, and so converges to s. Therefore $s = c + d$. We have $c \leq a$ and $d \leq b$ by the definition of $\limsup_{n\to\infty} a_n$ and $\limsup_{n\to\infty} b_n$, giving the result. ∎

Solution 4 (sketch): First prove that

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{k\geq n} x_k.$$

(We will probably prove this result in class; otherwise, see Problem A in Homework 6. This formula is closer to the usual definition of $\limsup_{n\to\infty} x_n$, which is

$$\limsup_{n\to\infty} x_n = \inf_{n\in\mathbf{N}} \sup_{k\geq n} x_k,$$

using a limit instead of an infimum.)

Then prove that

$$\sup_{k\geq n}(a_k + b_k) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k,$$

provided that the right hand side is defined. (For example, if both terms on the right are finite, then the right hand side is clearly an upper bound for $\{a_k + b_k \colon k \geq n\}$.) Now take limits to get the result. ∎

Remark: It is quite possible to have

$$\limsup_{n\to\infty}(a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

Problem 3.21: Prove the following analog of Theorem 3.10(b): If

$$E_1 \supset E_2 \supset E_3 \supset \cdots$$

are closed bounded nonempty subsets of a complete metric space X, and if

$$\lim_{n\to\infty} \operatorname{diam}(E_n) = 0,$$

then $\bigcap_{n=1}^{\infty} E_n$ consists of exactly one point.

Solution (sketch): It is clear that $\bigcap_{n=1}^{\infty} E_n$ can contain no more than one point, so we need to prove that $\bigcap_{n=1}^{\infty} E_n \neq \varnothing$.

For each n, choose some $x_n \in E_n$. Then, for each n, we have

$$\{x_n, x_{n+1}, \dots\} \subset E_n,$$

whence

$$\operatorname{diam}(\{x_n, x_{n+1}, \dots\}) \leq \operatorname{diam}(E_n).$$

Therefore (x_n) is a Cauchy sequence. Since X is complete, $x = \lim_{n\to\infty} x_n$ exists in X. Since E_n is closed, we have $x \in E_n$ for all n. So $x \in \bigcap_{n=1}^{\infty} E_n$.

Problem 3.22: Prove the Baire Category Theorem: If X is a complete metric space, and if (U_n) is a sequence of dense open subsets of X, then $\bigcap_{n=1}^{\infty} U_n$ is dense in X.

Note: In this formulation, the statement is true even if $X = \varnothing$.

Solution (sketch): Let $x \in X$ and let $\varepsilon > 0$. We recursively construct points $x_n \in X$ and numbers $\varepsilon_n > 0$ such that

$$d(x, x_1) < \frac{\varepsilon}{3}, \quad \varepsilon_1 < \frac{\varepsilon}{3}, \quad \varepsilon_n \to 0,$$

and

$$\overline{N_{\varepsilon_{n+1}}(x_{n+1})} \subset U_{n+1} \cap \overline{N_{\varepsilon_n}(x_n)}$$

for all n. Problem 3.21 will then imply that

$$\bigcap_{n=1}^{\infty} N_{\varepsilon_n}(x_n) \neq \varnothing.$$

(Note that $\operatorname{diam}\left(\overline{N_{\varepsilon_n}(x_n)}\right) \leq 2\varepsilon_n$.) One easily checks that

$$\bigcap_{n=1}^{\infty} N_{\varepsilon_n}(x_n) \subset N_{\varepsilon}(x) \cap \bigcap_{n=1}^{\infty} U_n.$$

Thus, we will have shown that $\bigcap_{n=1}^{\infty} U_n$ contains points arbitrarily close to x, proving density.

Since U_1 is dense in X, there is $x_1 \in U_1$ such that $d(x, x_1) < \frac{\varepsilon}{3}$. Choose $\varepsilon_1 > 0$ so small that

$$\varepsilon_1 < 1, \quad \varepsilon_1 < \frac{\varepsilon}{3}, \quad \text{and} \quad N_{2\varepsilon_1}(x_1) \subset U_1.$$

Then also

$$\overline{N_{\varepsilon_1}(x_1)} \subset U_1.$$

Given ε_n and x_n, use the density of U_{n+1} in X to choose

$$x_{n+1} \in U_{n+1} \cap N_{\varepsilon_n/2}(x_n).$$

Choose $\varepsilon_{n+1} > 0$ so small that

$$\varepsilon_{n+1} < \frac{1}{n+1}, \quad \varepsilon_{n+1} < \frac{\varepsilon_n}{2}, \quad \text{and} \quad N_{2\varepsilon_{n+1}}(x_{n+1}) \subset U_{n+1}.$$

Then also

$$\overline{N_{\varepsilon_{n+1}}(x_{n+1})} \subset U_{n+1}.$$

This gives all the required properties. (We have $\varepsilon_n \to 0$ since $\varepsilon_n < \frac{1}{n}$ for all n.) ∎

Note: We don't really need to use Problem 3.21 here. If we always require $\varepsilon_n < 2^{-n}$ in the argument above, we will get $d(x_n, x_{n+1}) < 2^{-n-1}$ for all n. This inequality implies that (x_n) is a Cauchy sequence.

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 5

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 3.6: Investigate the convergence or divergence of the following series.

(Note: I have supplied lower limits of summation, which I have chosen for maximum convenience. Of course, convergence is independent of the lower limit, provided none of the individual terms is infinite.)

(a)

$$\sum_{n=0}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right).$$

Solution: The n-th partial sum is

$$\left(\sqrt{n+1} - \sqrt{n}\right) + \left(\sqrt{n} - \sqrt{n-1}\right) + \cdots + \left(\sqrt{1} - \sqrt{0}\right) = \sqrt{n+1}.$$

We have $\lim_{n\to\infty} \sqrt{n} = \infty$, so $\lim_{n\to\infty} \sqrt{n+1} = \infty$. Therefore the series diverges. ∎

Remark: This sort of series is known as a telescoping series. The more interesting cases of telescoping series are the ones that converge.

Alternate solution: We calculate:

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \left(\frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right) \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \geq \frac{1}{\sqrt{n+1} + \sqrt{n+1}} = \frac{1}{2\sqrt{n+1}}.\end{aligned}$$

Now $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by Theorem 3.28 of Rudin. Therefore $\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n}}$ diverges, and hence so does $\sum_{n=0}^{\infty} \frac{1}{2\sqrt{n+1}}$. So the comparison test implies that

$$\sum_{n=0}^{\infty} \left(\sqrt{n+1} - \sqrt{n}\right)$$

diverges. ∎

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{n}.$$

Date: 29 October 2001.

Solution (sketch):

$$\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n\left(\sqrt{n+1}+\sqrt{n}\right)} \leq \frac{1}{n^{3/2}}.$$

Therefore the series converges by the comparison test. ∎

(c)

$$\sum_{n=1}^{\infty} \left(\sqrt[n]{n} - 1 \right)^n.$$

Solution: We use the root test (Theorem 3.33 of Rudin). With $a_n = \left(\sqrt[n]{n} - 1 \right)^n$, we have $\sqrt[n]{a_n} = \sqrt[n]{n} - 1$. Theorem 3.20 (c) of Rudin implies that $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Therefore $\lim_{n\to\infty} \sqrt[n]{a_n} = 0$. Since $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$, convergence follows.

Alternate solution (sketch): Let $x_n = \sqrt[n]{n} - 1$, so that

$$\left(\sqrt[n]{n} - 1 \right)^n = x_n^n \quad \text{and} \quad (1+x_n)^n = n.$$

The binomial formula implies that

$$n = (1+x_n)^n = 1 + nx_n + \frac{n(n-1)}{2} \cdot x_n^2 + \cdots \geq \frac{n(n-1)}{2} \cdot x_n^2,$$

from which it follows that

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$$

for $n \geq 2$. Hence, for $n \geq 4$,

$$\left(\sqrt[n]{n} - 1 \right)^n = x_n^n \leq \left(\frac{2}{n-1} \right)^{n/2} \leq \left(\left(\tfrac{2}{3}\right)^{1/2} \right)^n.$$

Since $\left(\frac{2}{3}\right)^{1/2} < 1$, the series converges by the comparison test. ∎

(d)

$$\sum_{n=0}^{\infty} \frac{1}{1+z^n},$$

for $z \in \mathbf{C}$ arbitrary.

Solution: We show that the series converges if and only if $|z| > 1$.

If $z = \exp(2\pi i r)$, with $r = k/l$, with k an odd integer and l an even integer, then $z^n = -1$ for infinitely many values of n, so that infinitely many of the terms of the series are undefined. Convergence is therefore clearly impossible.

In all other cases with $|z| \leq 1$, we have

$$|1 + z^n| \leq 1 + |z^n| \leq 1 + 1 = 2,$$

which implies that

$$\left| \frac{1}{1+z^n} \right| \geq \frac{1}{2}.$$

The terms thus don't converge to 0, and again the series diverges.

Now let $|z| > 1$. Then $|1 + z^n| \geq |z^n| - 1 = |z|^n - 1$. Choose N such that if $n \geq N$ then $|z|^n > 2$. For such n, we have

$$\frac{|z|^n}{2} > 1,$$

whence

$$|1 + z^n| \geq |z|^n - 1 = \frac{|z|^n}{2} + \left(\frac{|z|^n}{2} - 1 \right) > \frac{|z|^n}{2}.$$

So

$$\left| \frac{1}{1 + z^n} \right| \leq 2 \cdot \frac{1}{|z|^n}$$

for all $n \geq N$. Since $|z| > 1$, the comparison test implies that

$$\sum_{n=0}^{\infty} \frac{1}{1 + z^n}$$

converges. ∎

Problem 3.7: Let $a_n \geq 0$ for $n \in \mathbf{N}$. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Show that $\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{a_n}$ converges.

Solution (sketch): Using the inequality $2ab \leq a^2 + b^2$, we get

$$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2} \left(a_n + \frac{1}{n^2} \right).$$

Since both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converge, $\sum_{n=1}^{\infty} \frac{1}{n} \sqrt{a_n}$ converges by the comparison test. ∎

Problem 3.8: Let (b_n) be a bounded monotone sequence in $\mathbf{R}$, and let $a_n \in \mathbf{C}$ be such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

Solution (sketch): We first reduce to the case $\lim_{n \to \infty} b_n = 0$. Since (b_n) is a bounded monotone sequence, it follows that $b = \lim_{n \to \infty} b_n$ exists. Set $c_n = b_n - b$. Then (c_n) is a bounded monotone sequence with $\lim_{n \to \infty} c_n = 0$. Since $a_n b_n = a_n c_n + a_n b$ and $\sum_{n=1}^{\infty} a_n b$ converges, it suffices to prove that $\sum_{n=1}^{\infty} a_n c_n$ converges. That is, we may assume that $\lim_{n \to \infty} b_n = 0$.

With this assumption, if $b_1 \geq 0$, then $b_1 \geq b_2 \geq \cdots \geq 0$, so $\sum_{n=1}^{\infty} a_n b_n$ converges by Theorem 3.42 in the book. Otherwise, replace b_n by $-b_n$. ∎

Problem 3.9: Find the radius of convergence of each of the following power series:

(a) $\displaystyle\sum_{n=0}^{\infty} n^3 z^n$.

Solution 1: Use Theorem 3.20 (c) of Rudin in the second step to get

$$\limsup_{n \to \infty} \sqrt[n]{n^3} = \left(\lim_{n \to \infty} \sqrt[n]{n} \right)^3 = 1.$$

It now follows from Theorem 3.39 of Rudin that the radius of convergence is 1. ∎

Solution 2: We show that the series converges for $|z| < 1$ and diverges for $|z| > 1$.

For $|z| = 0$, convergence is trivial.

For $0 < |z| < 1$, we use the ratio test (Theorem 3.34 of Rudin). We have

$$\lim_{n\to\infty} \frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} = \lim_{n\to\infty} |z| \left(\frac{n+1}{n}\right)^3 = |z| \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3$$
$$= |z| \left(1 + \lim_{n\to\infty} \frac{1}{n}\right)^3 = |z|.$$

For $|z| < 1$ the hypotheses of Theorem 3.34 (a) of Rudin are therefore satisfied, so that the series converges.

For $|z| > 1$, we use the ratio test. The same calculation as in the case $0 < |z| < 1$ gives

$$\lim_{n\to\infty} \frac{|(n+1)^3 z^{n+1}|}{n^3 z^n|} = |z|.$$

Since $|z| > 1$, it follows that there is N such that for all $n \geq N$ we have

$$\left| \frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} - |z| \right| < \tfrac{1}{2}(|z| - 1).$$

In particular,

$$\frac{|(n+1)^3 z^{n+1}|}{|n^3 z^n|} > 1$$

for $n \geq N$. The hypotheses of Theorem 3.34 (b) of Rudin are therefore satisfied, so that the series diverges.

Theorem 3.39 of Rudin implies that there is some number $R \in [0, \infty]$ such that the series converges for $|z| < R$ and diverges for $|z| > R$. We have therefore shown that $R = 1$. ∎

Solution 3: We calculate

$$\lim_{n\to\infty} \frac{|(n+1)^3|}{|n^3|} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 = \left(1 + \lim_{n\to\infty} \frac{1}{n}\right)^3 = 1.$$

According to Theorem 3.37 of Rudin, we have

$$\liminf_{n\to\infty} \frac{|(n+1)^3|}{|n^3|} \leq \liminf_{n\to\infty} \sqrt[n]{|n^3|} \leq \limsup_{n\to\infty} \sqrt[n]{|n^3|} \leq \limsup_{n\to\infty} \frac{|(n+1)^3|}{|n^3|}.$$

Therefore $\lim_{n\to\infty} \sqrt[n]{|n^3|}$ exists and is equal to 1. It now follows from Theorem 3.39 of Rudin that the radius of convergence is 1. ∎

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot z^n$.

Solution (sketch): Use the ratio test to show that the series converges for all z. (See Solution 2 to Part (a).) So the radius of convergence is ∞. ∎

Remark: Note that $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} \cdot z^n = e^{2z}$.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n^2} \cdot z^n$.

Solution (sketch): Either the root or ratio test gives radius of convergence equal to $\frac{1}{2}$. (Use the methods of any of the three solutions to Part (a).) ∎

(d) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{3^n} \cdot z^n$.

Solution (sketch): Either the root or ratio test gives radius of convergence equal to 3. (Use the methods of any of the three solutions to Part (a).) ∎

Problem 3.10: Suppose the coefficients of the power series $\sum_{n=0}^{\infty} a_n z^n$ are integers, infinitely many of which are nonzero. Prove that the radius of convergence is at most 1.

Solution 1: For infinitely many n, the numbers a_n are nonzero integers, and therefore satisfy $|a_n| \geq 1$. So, if $|z| \geq 1$, then infinitely many of the terms $a_n z^n$ have absolute value $|a_n z^n| \geq |a_n| \geq 1$, and the terms of the series $\sum_{n=0}^{\infty} a_n z^n$ don't approach zero. This shows that $\sum_{n=0}^{\infty} a_n z^n$ diverges for $|z| \geq 1$, and therefore that its radius of convergence is at most 1. ∎

Solution 2 (Sketch): There is a subsequence $(a_{k(n)})$ of (a_n) such that $|a_{k(n)}| \geq 1$ for all n. So $\sqrt[k(n)]{|a_{k(n)}|} \geq 1$ for all n, whence $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$. ∎

Remarks: (1) It is quite possible that infinitely many of the a_n are zero, so that $\liminf_{n\to\infty} \sqrt[n]{|a_n|}$ could be zero. For example, we could have $a_n = 0$ for all odd n.

(2) In Solution 2, the expression $\sqrt[n]{|a_{k(n)}|}$, and its possible limit as $n \to \infty$, have no relation to the radius of convergence.

Problem 3.16: Fix $\alpha > 0$. Choose $x_1 > \sqrt{\alpha}$, and recursively define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Prove that (x_n) is nonincreasing and $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

Solution (sketch): Using the inequality $a^2 + b^2 \geq 2ab$, and assuming $x_n > 0$, we get

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) \geq \sqrt{x_n} \cdot \sqrt{\frac{\alpha}{x_n}} = \sqrt{\alpha}.$$

This shows (using induction) that $x_n > \sqrt{\alpha}$ for all n. Next,

$$x_n - x_{n+1} = \frac{x_n^2 - \alpha}{2x_n} > 0.$$

Thus (x_n) is nonincreasing. We already know that this sequence is bounded below (by α), so $x = \lim_{n\to\infty} x_n$ exists. Letting $n \to \infty$ in the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

gives

$$x = \frac{1}{2}\left(x + \frac{\alpha}{x}\right).$$

This equation implies $x = \pm\sqrt{\alpha}$, and we must have $x = \sqrt{\alpha}$ because (x_n) is bounded below by $\sqrt{\alpha} > 0$. ∎

(b) Set $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}.$$

Further show that, with $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n}.$$

Solution (sketch): Prove all the relations at once by induction on n, together with the statement $\varepsilon_{n+1} > 0$. For $n = 1$, the relation $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}$ is just algebra, the inequality $\frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$ follows from $x_1 > \sqrt{\alpha}$, and the inequality $\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n}$ is just a rewritten form of $\frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$. The statement $\varepsilon_{n+1} > 0$ is clear from $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}$ and $x_1 > 0$.

Now assume all this is known for some value of n. As before, the relation $\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n}$ is just algebra, and implies that $\varepsilon_{n+1} > 0$. (We know that $x_n = \sqrt{\alpha} + \varepsilon_n > \sqrt{\alpha} > 0$.) Since $\varepsilon_n > 0$, we have $x_n > \sqrt{\alpha}$, so the inequality $\frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$ follows. To get the other inequality, write

$$\varepsilon_{n+1} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}} = \frac{\varepsilon_n^2}{\beta} < \left(\frac{1}{\beta} \right) \cdot \left[\beta \cdot \left(\frac{\varepsilon_1}{\beta} \right)^{2^{n-1}} \right]^2 = \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n}.$$

∎

(c) Specifically take $\alpha = 3$ and $x_1 = 2$. show that

$$\frac{\varepsilon_1}{\beta} < \frac{1}{10}, \quad \varepsilon_5 < 4 \cdot 10^{-16}, \quad \text{and} \quad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Solution (sketch): This is just calculation. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 6

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are often close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 3.23: Let X be a metric space, and let $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ be Cauchy sequences in X. Prove that $\lim_{n \to \infty} d(x_n, y_n)$ exists.

Solution (sketch): Since $\mathbf{R}$ is complete, it suffices to show that $(d(x_n, y_n))_{n \in \mathbf{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$. Choose N so large that if $m, n \geq N$, then both $d(x_m, x_n) < \frac{\varepsilon}{2}$ and $d(y_m, y_n) < \frac{\varepsilon}{2}$. Then check that, for such m and n,

$$|d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n) < \varepsilon.$$

∎

Problem 3.24: Let X be a metric space.

(a) Let $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ be Cauchy sequences in X. We say they are equivalent, and write $(x_n)_{n \in \mathbf{N}} \sim (y_n)_{n \in \mathbf{N}}$, if $\lim_{n \to \infty} d(x_n, y_n) = 0$. Prove that this is an equivalence relation.

Solution (sketch): That $(x_n)_{n \in \mathbf{N}} \sim (x_n)_{n \in \mathbf{N}}$, and that $(x_n)_{n \in \mathbf{N}} \sim (y_n)_{n \in \mathbf{N}}$ implies $(y_n)_{n \in \mathbf{N}} \sim (x_n)_{n \in \mathbf{N}}$, are obvious. For transitivity, assume $(x_n)_{n \in \mathbf{N}} \sim (y_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}} \sim (z_n)_{n \in \mathbf{N}}$. Then $(x_n)_{n \in \mathbf{N}} \sim (z_n)_{n \in \mathbf{N}}$ follows by taking limits in the inequality

$$0 \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n).$$

∎

(b) Let X^* be the set of equivalence classes from Part (a). Denote by $[(x_n)_{n \in \mathbf{N}}]$ the equivalence class in X^* of the Cauchy sequence $(x_n)_{n \in \mathbf{N}}$. If $(x_n)_{n \in \mathbf{N}}$ and $(y_n)_{n \in \mathbf{N}}$ are Cauchy sequences in X, set

$$\Delta_0((x_n)_{n \in \mathbf{N}}, (y_n)_{n \in \mathbf{N}}) = \lim_{n \to \infty} d(x_n, y_n).$$

Prove that $\Delta_0((x_n)_{n \in \mathbf{N}}, (y_n)_{n \in \mathbf{N}})$ only depends on $[(x_n)_{n \in \mathbf{N}}]$ and $[(y_n)_{n \in \mathbf{N}}]$. Moreover, show that the formula

$$\Delta([(x_n)_{n \in \mathbf{N}}], [(y_n)_{n \in \mathbf{N}}]) = \Delta_0((x_n)_{n \in \mathbf{N}}, (y_n)_{n \in \mathbf{N}})$$

defines a metric on X^*.

Solution (sketch): It is easy to check that Δ_0 is a semimetric, that is, it satisfies all the conditions for a metric except that possibly

$$\Delta_0((x_n)_{n \in \mathbf{N}}, (y_n)_{n \in \mathbf{N}}) = 0$$

Date: 5 November 2001.

without having $(x_n)_{n\in\mathbf{N}} = (y_n)_{n\in\mathbf{N}}$. (For example,

$$\begin{aligned}\Delta_0((x_n)_{n\in\mathbf{N}},\,(z_n)_{n\in\mathbf{N}}) &= \lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\ &= \Delta_0((x_n)_{n\in\mathbf{N}},\,(y_n)_{n\in\mathbf{N}}) + \Delta_0((y_n)_{n\in\mathbf{N}},\,(z_n)_{n\in\mathbf{N}})\end{aligned}$$

because $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$; the other properties are proved similarly.) We further note that, by definition, $(x_n)_{n\in\mathbf{N}} \sim (y_n)_{n\in\mathbf{N}}$ if and only if $\Delta_0((x_n)_{n\in\mathbf{N}},\,(y_n)_{n\in\mathbf{N}}) = 0$.

Now we prove that $\Delta_0((x_n)_{n\in\mathbf{N}},\,(y_n)_{n\in\mathbf{N}})$ only depends on

$$[(x_n)_{n\in\mathbf{N}}] \quad \text{and} \quad [(y_n)_{n\in\mathbf{N}}].$$

Let $(x_n)_{n\in\mathbf{N}} \sim (r_n)_{n\in\mathbf{N}}$ and $(y_n)_{n\in\mathbf{N}} \sim (s_n)_{n\in\mathbf{N}}$. Then, by the previous paragraph,

$$\begin{aligned}&\Delta_0((x_n)_{n\in\mathbf{N}},\,(y_n)_{n\in\mathbf{N}}) \\ &\leq \Delta_0((x_n)_{n\in\mathbf{N}},\,(r_n)_{n\in\mathbf{N}}) + \Delta_0((r_n)_{n\in\mathbf{N}},\,(s_n)_{n\in\mathbf{N}}) + \Delta_0((s_n)_{n\in\mathbf{N}},\,(y_n)_{n\in\mathbf{N}}) \\ &= 0 + \Delta_0((r_n)_{n\in\mathbf{N}},\,(s_n)_{n\in\mathbf{N}}) + 0 = \Delta_0((r_n)_{n\in\mathbf{N}},\,(s_n)_{n\in\mathbf{N}});\end{aligned}$$

similarly

$$\Delta_0((r_n)_{n\in\mathbf{N}},\,(s_n)_{n\in\mathbf{N}}) \leq \Delta_0((x_n)_{n\in\mathbf{N}},\,(y_n)_{n\in\mathbf{N}}).$$

Thus

$$\Delta_0((r_n)_{n\in\mathbf{N}},\,(s_n)_{n\in\mathbf{N}}) = \Delta_0((x_n)_{n\in\mathbf{N}},\,(y_n)_{n\in\mathbf{N}}).$$

The previous paragraph implies that Δ is well defined. It is now easy to check that Δ satisfies all the conditions for a metric except that possibly

$$\Delta([(x_n)_{n\in\mathbf{N}}], [(y_n)_{n\in\mathbf{N}}]) = 0$$

without having $[(x_n)_{n\in\mathbf{N}}] = [(y_n)_{n\in\mathbf{N}}]$. (For example,

$$\begin{aligned}\Delta([(x_n)_{n\in\mathbf{N}}],\,[(z_n)_{n\in\mathbf{N}}]) &= \Delta_0((x_n)_{n\in\mathbf{N}},\,(z_n)_{n\in\mathbf{N}}) \\ &\leq \Delta_0((x_n)_{n\in\mathbf{N}},\,(y_n)_{n\in\mathbf{N}}) + \Delta_0((y_n)_{n\in\mathbf{N}},\,(z_n)_{n\in\mathbf{N}}) \\ &= \Delta([(x_n)_{n\in\mathbf{N}}],\,[(y_n)_{n\in\mathbf{N}}]) + \Delta_0([(y_n)_{n\in\mathbf{N}}],\,[(z_n)_{n\in\mathbf{N}}]);\end{aligned}$$

the other properties are proved similarly.)

Finally, if $\Delta([(x_n)_{n\in\mathbf{N}}],\,[(y_n)_{n\in\mathbf{N}}]) = 0$ then it follows from the definition of $(x_n)_{n\in\mathbf{N}} \sim (y_n)_{n\in\mathbf{N}}$ that we actually do have $[(x_n)_{n\in\mathbf{N}}] = [(y_n)_{n\in\mathbf{N}}]$. So Δ is a metric. ∎

(c) Prove that X^* is complete in the metric Δ.

The basic idea is as follows. We start with a Cauchy sequence in X^*, which is a sequence of (equivalence classes of) Cauchy sequences in X. The limit is supposed to be (the equivalence class of) another Cauchy sequence in X. This sequence is constructed by taking suitable terms from the given sequences. The choices get a little messy. Afterwards, we will give a different proof.

Solution (sketch): Let $(a_k)_{k\in\mathbf{N}}$ be a Cauchy sequence in X^*; we show that it converges. Each a_k is an equivalence class of Cauchy sequences in X. We may therefore write $a_k = [(x_n^{(k)})_{n\in\mathbf{N}}]$, where each $(x_n^{(k)})_{n\in\mathbf{N}}$ is a Cauchy sequence in X. The limit we construct in X^* will have the form $a = [(x_n^{(f(n))})]$ for a suitable function $f\colon \mathbf{N} \to \mathbf{N}$.

We recursively construct

$$M(1) < M(2) < M(3) < \cdots \quad \text{and} \quad N(1) < N(2) < N(3) < \cdots$$

such that

(1) $\Delta(a_k, a_l) < 2^{-r-2}$ for $k,\, l \geq M(r)$.
(2) For all $k,\, l \leq M(r)$ and $m \geq N(r)$, we have $d(x_m^{(k)},\, x_m^{(l)}) < \Delta(a_k, a_l) + 2^{-r-2}$.
(3) For all $k \leq M(r)$ and $m,\, n \geq N(r)$, we have $d(x_m^{(k)},\, x_n^{(k)}) < 2^{-r-2}$.

To do this, first use the fact that $(a_k)_{k \in \mathbf{N}}$ is a Cauchy sequence to find $M(1)$. Then choose $N(1)$ large enough to satisfy (2) and (3) for $r = 1$; this can be done because $\lim_{m \to \infty} d(x_m^{(k)},\, x_m^{(l)}) = \Delta(a_k, a_l)$ (for (2)) and because $(x_n^{(k)})_{n \in \mathbf{N}}$ is Cauchy (for (3)), and using the fact that there are only finitely many pairs (k, l) to consider in (2) and only finitely many k to consider in (3). Next, use the fact that $(a_k)_{k \in \mathbf{N}}$ is a Cauchy sequence to find $M(2)$, and also require $M(2) > M(1)$. Choose $N(2) > N(1)$ by the same reasoning as used to get $N(1)$. Proceed recursively.

We now take a to be the equivalence class of the sequence

$$x_1^{(1)},\, \ldots,\, x_{N(1)-1}^{(1)},\, x_{N(1)}^{(M(1))},\, \ldots,\, x_{N(2)-1}^{(M(1))},\, x_{N(2)}^{(M(2))},\, \ldots,\, x_{N(3)-1}^{(M(2))},\, x_{N(3)}^{(M(3))},\, \ldots.$$

That is, the function f above is given by $f(n) = M(r)$ for $N(r) \leq n \leq N(r+1) - 1$.

We show that $(x_n^{(f(n))})_{n \in \mathbf{N}}$ is Cauchy. First, estimate:

$$\begin{aligned} d\left(x_{N(r)}^{(M(r))},\, x_{N(r+1)}^{(M(r+1))}\right) &\leq d\left(x_{N(r)}^{(M(r))},\, x_{N(r+1)}^{(M(r))}\right) + d\left(x_{N(r+1)}^{(M(r))},\, x_{N(r+1)}^{(M(r+1))}\right) \\ &< 2^{-r-2} + \Delta(a_{M(r)},\, a_{M(r+1)}) + 2^{-r-3} \\ &< 2^{-r-2} + 2^{-r-2} + 2^{-r-3} < 3 \cdot 2^{-r-2}. \end{aligned}$$

The first term on the second line is gotten from (2) above, because $M(r) \leq M(r)$ and $N(r),\, N(r+1) \geq N(r)$. The other two terms on the second line are gotten from (3) above (for $r+1$)), because $M(r),\, M(r+1) \leq M(r+1)$ and $N(r+1) \geq N(r+1)$. The estimate used to get the third line comes from (1) above. Then use induction to show that $s \geq r$ implies

$$d\left(x_{N(r)}^{(M(r))},\, x_{N(s)}^{(M(s))}\right) \leq 3[2^{-r-2} + 2^{-r-3} + \cdots + 2^{-s-1}].$$

Now let $n \geq N(r)$ be arbitrary. Choose $s \geq r$ such that $N(s) \leq n \leq N(s+1) - 1$. Then $f(n) = M(s)$, so

$$d\left(x_n^{(f(n))},\, x_{N(s)}^{(M(s))}\right) = d\left(x_n^{(M(s))},\, x_{N(s)}^{(M(s))}\right) < 3 \cdot 2^{-s-2},$$

using (3) above with $r = s$ and $k = M(s)$. Therefore

$$d\left(x_{N(r)}^{(M(r))},\, x_n^{(f(n))}\right) < 3[2^{-r-2} + 2^{-r-3} + \cdots + 2^{-s-1} + 2^{-s-2}] < 3 \cdot 2^{-r-1}.$$

Finally, if $m,\, n \geq N(r)$ are arbitrary, then

$$d\left(x_m^{(f(m))},\, x_n^{(f(n))}\right) \leq d\left(x_m^{(f(m))},\, x_{N(r)}^{(M(r))}\right) + d\left(x_{N(r)}^{(M(r))},\, x_n^{(f(n))}\right) < 3 \cdot 2^{-r}.$$

This is enough to prove that $(x_n^{(f(n))})_{n \in \mathbf{N}}$ is Cauchy.

It remains to show that $\Delta(a_k, a) \to 0$. Fix k, choose r with $M(r-1) < k \leq M(r)$, and let $n \geq N(r)$. From the previous paragraph we have

$$d\left(x_{N(r)}^{(M(r))},\, x_n^{(f(n))}\right) < 3 \cdot 2^{-r-1}.$$

Since $n,\, N(r) \geq N(r)$ and $k \leq M(r)$, condition (3) above gives

$$d\left(x_n^{(k)},\, x_{N(r)}^{(k)}\right) < 2^{-r-2}.$$

Furthermore,

$$d\left(x_{N(r)}^{(k)},\, x_{N(r)}^{(M(r))}\right) < \Delta(a_k,\, a_{M(r)}) + 2^{-r-2} < 2^{-r-1} + 2^{-r-2},$$

where the first step uses (2) above and the inequalities k, $M(r) \leq M(r)$ and $N(r) \geq N(r)$, while the second step uses (1) above and the inequality $r \geq M(r-1)$. Combining these estimates using the triangle inequality, we get

$$d(x_n^{(k)},\, x_n^{(f(n))}) < 2^{-r-2} + [2^{-r-1} + 2^{-r-2}] + 3 \cdot 2^{-r-1} < 2^{-r+2}.$$

Therefore

$$\Delta(a_k, a) = \lim_{n\to\infty} d(x_n^{(k)},\, x_n^{(f(n))}) \leq 2^{-r+2}$$

for $M(r-1) < k \leq M(r)$. Since $M(r) \to \infty$, this implies that $\Delta(a_k, a) \to 0$. ∎

Here is a perhaps slicker way to do the same thing, although it isn't any shorter. Essentially, by passing to suitable subsequences, we can take the representative of the limit to be the diagonal sequence, that is, $f(n) = n$ in the proof above. The construction requires the following lemmas.

Lemma 1. Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in a metric space Y. Then there is a subsequence $(x_{k(n)})_{n\in\mathbf{N}}$ of $(x_n)_{n\in\mathbf{N}}$ such that $d(x_{k(n+1)},\, x_{k(n)}) < 2^{-n}$ for all n.

Proof (sketch): Choose $k(n)$ recursively to satisfy $k(n+1) > k(n)$ and $d(x_l, x_m) < 2^{-n}$ for all l, $m \geq k(n)$. ∎

Lemma 2. Let $(x_n)_{n\in\mathbf{N}}$ be a sequence in a metric space Y. Suppose

$$\sum_{k=1}^{\infty} d(x_k,\, x_{k+1})$$

converges. Then $(x_n)_{n\in\mathbf{N}}$ is Cauchy. Moreover, if $n > m$ then

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k,\, x_{k+1}).$$

Proof (sketch): The Cauchy criterion for convergence of a series implies that for all $\varepsilon > 0$, there is N such that if $n > m \geq N$, then $\sum_{k=m}^{n-1} d(x_k,\, x_{k+1}) < \varepsilon$. But the triangle inequality gives $d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k,\, x_{k+1})$. Thus if $n > m \geq N$ then $d(x_m, x_n) < \varepsilon$. The case $m > n \geq N$ is handled by symmetry, and the case $n = m \geq N$ is trivial. ∎

Lemma 3. Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in a metric space Y, and let $(x_{k(n)})_{n\in\mathbf{N}}$ be a subsequence. Then $\lim_{n\to\infty} d(x_n,\, x_{k(n)}) = 0$.

Proof (sketch): Let $\varepsilon > 0$. Choose N such that if m, $n \geq N$ then $d(x_m, x_n) < \varepsilon$. If $n \geq N$, then $k(n) \geq n \geq N$, so $d(x_n,\, x_{k(n)}) < \varepsilon$. ∎

Lemma 4. Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence in a metric space Y. If $(x_n)_{n\in\mathbf{N}}$ has a convergent subsequence, then $(x_n)_{n\in\mathbf{N}}$ converges.

Proof (sketch): Let $(x_{k(n)})_{n\in\mathbf{N}}$ be a subsequence with limit x. Then, using Lemma 3,

$$d(x_n, x) \leq d(x_n,\, x_{k(n)}) + d(x_{k(n)},\, x) \to 0.$$

∎

Proof of the result: Let (a_k) be a Cauchy sequence in X^*; we show that it converges. Use Lemma 1 to choose a subsequence $(a_{r(k)})_{k\in\mathbf{N}}$ such that

$$\Delta(a_{r(k)},\, a_{r(k+1)}) < 2^{-k}$$

for all k. By Lemma 4, it suffices to show that $(a_{r(k)})_{k\in\mathbf{N}}$ converges. Without loss of generality, therefore, we may assume the original sequence $(a_k)_{k\in\mathbf{N}}$ satisfies $\Delta(a_k,\, a_{k+1}) < 2^{-k}$ for all k.

Each a_k is an equivalence class of Cauchy sequences in X. By Lemmas 1 and 3, we may write $a_k = [(x_n^{(k)})_{n\in\mathbf{N}}]$, with $d(x_n^{(k)},\, x_{n+1}^{(k)}) < 2^{-n}$ for all n.

We now estimate $d(x_n^{(k)},\, x_n^{(k+1)})$. For $\varepsilon > 0$, we can find $m > n$ such that

$$\begin{aligned} d(x_m^{(k)},\, x_m^{(k+1)}) &< \Delta([(x_n^{(k)})_{n\in\mathbf{N}}],\, [(x_n^{(k+1)})_{n\in\mathbf{N}}]) + \varepsilon \\ &= \Delta(a_k,\, a_{k+1}) + \varepsilon < 2^{-k} + \varepsilon. \end{aligned}$$

Now, using Lemma 2,

$$d(x_n^{(k)},\, x_m^{(k)}) < 2^{-n} + 2^{-n-1} + \cdots + 2^{-m+1} < 2^{-n+1}.$$

The same estimate holds for $d(x_n^{(k+1)},\, x_m^{(k+1)})$. Therefore

$$\begin{aligned} d(x_n^{(k)},\, x_n^{(k+1)}) &\leq d(x_n^{(k)},\, x_m^{(k)}) + d(x_m^{(k)},\, x_m^{(k+1)}) + d(x_m^{(k+1)},\, x_n^{(k+1)}) \\ &< 2^{-n+1} + \varepsilon + 2^{-n+1}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this gives

$$d(x_n^{(k)},\, x_n^{(k+1)}) \leq 2^{-n+2}$$

for all n and k.

Now define $y_n = x_n^{(n)}$. First, observe that

$$\begin{aligned} d(y_n,\, y_{n+1}) &\leq d(x_n^{(n)},\, x_{n+1}^{(n)}) + d(x_{n+1}^{(n)},\, x_{n+1}^{(n+1)}) \\ &\leq 2^{-n} + 2^{-n+1} < 2^{-n+2}. \end{aligned}$$

Therefore $(y_n)_{n\in\mathbf{N}}$ is Cauchy, by Lemma 2. So $a = [(y_n)_{n\in\mathbf{N}}] \in X^*$.

It remains to show that $\Delta(a_n, a) \to 0$. If $m > n$, then we use the estimates $d(x_n^{(n)},\, x_m^{(n)}) < 2^{-n+1}$ (as above) and $d(y_n, y_m) < 2^{-n+3}$ (obtained similarly, using Lemma 2 again) to get

$$d(x_m^{(n)},\, y_m) \leq d(x_m^{(n)},\, x_n^{(n)}) + d(y_n, y_m) < 2^{-n+4}.$$

In particular,

$$\Delta(a_n, a) = \lim_{m\to\infty} d(x_m^{(n)},\, y_m) \leq 2^{-n+4}.$$

Thus $\Delta(a_n, a) \to 0$, as desired. ∎

(d) Define $f\colon X \to X^*$ by $f(x) = [(x, x, x, \dots)]$. Prove that f is isometric, that is, that $\Delta(f(x), f(y)) = d(x, y)$ for all x, $y \in X$.

Solution (sketch): This is immediate. ∎

(e) Prove that $f(X)$ is dense in X^*, and that $f(X) = X^*$ if X is complete.

Solution (sketch): To prove density, let $[(x_n)_{n\in\mathbf{N}}] \in X^*$, and let $\varepsilon > 0$. Choose N such that if m, $n \geq N$ then $d(x_m, x_n) < \frac{\varepsilon}{2}$. Then

$$\Delta(f(x_N),\, [(x_n)_{n\in\mathbf{N}}]) = \lim_{n\to\infty} d(x_N, x_n),$$

which is at most $\frac{\varepsilon}{2}$ because $d(x_N, x_n) < \frac{\varepsilon}{2}$ for $n \geq N$. (Note that the limit exists by Problem 3.23.) In particular, $\Delta(f(x_N), [(x_n)_{n \in \mathbf{N}}]) < \varepsilon$.

Now assume X is complete. Then $f(X)$ is complete, because f is isometric. Therefore it suffices to prove that a complete subset of a metric space is closed. (A subset of a metric space which is both closed and dense must be equal to the whole space.)

Accordingly, let Y be a metric space, and let $E \subset Y$ be a complete subset. Let $(x_n)_{n \in \mathbf{N}}$ be a sequence in E which converges to some point $y \in Y$; we show $y \in E$. (By a theorem proved in class, this is sufficient to verify that E is closed.) Now $(x_n)_{n \in \mathbf{N}}$ converges, and is therefore Cauchy. Since E is complete, there is $x \in E$ such that $x_n \to x$. By uniqueness of limits, we have $x = y$. Thus $y \in E$, as desired. ∎

Problem A. Prove the equivalence of four definitions of the lim sup of a sequence. That is, prove the following theorem.

Theorem. Let (a_n) be a sequence in $\mathbf{R}$. Let E be the set of all subsequential limits of (a_n) in $[-\infty, \infty]$. Define numbers r, s, t, and $u \in [-\infty, \infty]$ as follows:

(1) $r = \sup(E)$.
(2) $s \in E$ and for every $x > s$, there is $N \in \mathbf{N}$ such that $n \geq N$ implies $a_n < x$.
(3) $t = \inf_{n \in \mathbf{N}} \sup_{k \geq n} a_k$.
(4) $u = \lim_{n \to \infty} \sup_{k \geq n} a_k$.

Prove that s is uniquely determined by (2), that the limit in (4) exists in $[-\infty, \infty]$, and that $r = s = t = u$.

Note: You do not need to repeat the part that is done in the book (Theorem 3.17).

Solution: Theorem 3.17 of Rudin implies that s is uniquely determined by (2) and that $s = r$.

Define $b_n = \sup_{k \geq n} a_k$, which exists in $(-\infty, \infty]$. We clearly have

$$\{a_k : k \geq n + 1\} \subset \{a_k : k \geq n\},$$

so that $\sup_{k \geq n+1} a_k \leq \sup_{k \geq n} a_k$. This shows that the sequence in (4), which has values in $(-\infty, \infty]$, is nonincreasing. Therefore it has a limit $u \in [-\infty, \infty]$, and moreover

$$u = \inf_{n \in \mathbf{N}} b_n = \inf_{n \in \mathbf{N}} \sup_{k \geq n} a_k = t.$$

We finish the proof by showing that $s \leq t$ and $t \leq r$.

To show that $s \leq t$, let $x > s$; we show that $x > t$. Choose y with $x > y > s$. By the definition of s, there is $N \in \mathbf{N}$ such that $n \geq N$ implies $a_n < y$. This implies that $y \geq \sup_{n \geq N} a_n$, so that $x > \sup_{n \geq N} a_n$. It follows that x is not a lower bound for $\{\sup_{n \geq N} a_n : N \in \mathbf{N}\}$. So $x > t$ by the definition of a greatest lower bound.

To show that $t \geq r$, let $x > t$; we show that $x \geq r$. Since $x > t$, it follows that x is not a lower bound for the set $\{\sup_{n \geq N} a_n : N \in \mathbf{N}\}$. Accordingly, there is $N_0 \in \mathbf{N}$ such that $\sup_{n \geq N_0} a_n < x$. In particular, $n \geq N_0$ implies $a_n < x$. Now let $(a_{k(n)})$ be any convergent subsequence of (a_n). Choose $N \in \mathbf{N}$ such that $n \geq N$ implies $k(n) \geq N_0$. Then $n \geq N$ implies $a_{k(n)} < x$, from which it follows that $\lim_{n \to \infty} a_{k(n)} \leq x$. This shows that x is an upper bound for the set E, so that $x \geq \sup(E) = r$. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 7

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 4.1: Let $f \colon \mathbf{R} \to \mathbf{R}$ satisfy $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ for all $x \in \mathbf{R}$. Is f necessarily continuous?

Solution (Sketch): No. The simplest counterexample is

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}.$$

More generally, let $f_0 \colon \mathbf{R} \to \mathbf{R}$ be continuous. Fix $x_0 \in \mathbf{R}$, and fix $y_0 \in \mathbf{R}$ with $y_0 \neq f_0(x_0)$. Then the function given by

$$f(x) = \begin{cases} f_0(x) & x \neq x_0 \\ y_0 & x = x_0 \end{cases}$$

is a counterexample. There are even examples with a nonremovable discontinuity, such as

$$f(x) = \begin{cases} \frac{1}{|x|} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

$\blacksquare$

Problem 4.3: Let X be a metric space, and let $f \colon X \to \mathbf{R}$ be continuous. Let $Z(f) = \{x \in X \colon f(x) = 0\}$. Prove that $Z(f)$ is closed.

Solution 1: The set $Z(f)$ is equal to $f^{-1}(\{0\})$. Since $\{0\}$ is a closed subset of $\mathbf{R}$ and f is continuous, it follows from the Corollary to Theorem 4.8 of Rudin that $Z(f)$ is closed in X. $\blacksquare$

Solution 2: We show that $X \setminus Z(f)$ is open. Let $x \in X \setminus Z(f)$. Then $f(x) \neq 0$. Set $\varepsilon = \frac{1}{2}|f(x)| > 0$. Choose $\delta > 0$ such that $y \in X$ and $d(x, y) < \delta$ imply $|f(x) - f(y)| < \varepsilon$. Then $f(y) \neq 0$ for $y \in N_\delta(x)$. Thus $N_\delta(x) \subset X \setminus Z(f)$ with $\delta > 0$. This shows that $X \setminus Z(f)$ is open. $\blacksquare$

Note that we really could have taken $\varepsilon = |f(x)|$. Also, there is no need to do anything special if $Z(f)$ is empty, or even to mention the that case separately: the argument works (vacuously) just as well in that case.

Solution 3 (sketch): We show $Z(f)$ contains all its limit points. Let x be a limit point of $Z(f)$. Then there is a sequence (x_n) in $Z(f)$ such that $x_n \to x$. Since f is continuous and $f(x_n) = 0$ for all n, we have

$$f(x) = \lim_{n \to \infty} f(x_n) = 0.$$

So $x \in Z(f)$. $\blacksquare$

Date: 12 Nov. 2001.

Again, there is no need to treat separately the case in which $Z(f)$ has no limit points.

Problem 4.4: Let X and Y be metric spaces, and let $f,\, g \colon X \to Y$ be continuous functions. Let $E \subset X$ be dense. Prove that $f(E)$ is dense in $f(X)$. Prove that if $f(x) = g(x)$ for all $x \in E$, then $f = g$.

Solution: We first show that $f(E)$ is dense in $f(X)$. Let $y \in f(X)$. Choose $x \in X$ such that $f(x) = y$. Since E is dense in X, there is a sequence (x_n) in E such that $x_n \to x$. Since f is continuous, it follows that $f(x_n) \to f(x)$. Since $f(x_n) \in f(E)$ for all n, this shows that $x \in \overline{f(E)}$.

Now assume that $f(x) = g(x)$ for all $x \in E$; we prove that $f = g$. It suffices to prove that $F = \{x \in X \colon f(x) = g(x)\}$ is closed in X, and we prove this by showing that $X \setminus F$ is open. Thus, let $x_0 \in F$. Set $\varepsilon = \frac{1}{2} d(f(x_0),\, g(x_0)) > 0$. Choose $\delta_1 > 0$ such that if $x \in X$ satisfies $d(x, x_0) < \delta$, then $d(f(x),\, f(x_0)) < \varepsilon$. Choose $\delta_2 > 0$ such that if $x \in X$ satisfies $d(x, x_0) < \delta$, then $d(g(x),\, g(x_0)) < \varepsilon$. Set $\delta = \min(\delta_1, \delta_2)$. If $d(x, x_0) < \delta$, then (using the triangle inequality several times)

$$\begin{aligned} d(f(x),\, g(x)) &\geq d(f(x_0),\, g(x_0)) - d(f(x),\, f(x_0)) - d(g(x),\, g(x_0)) \\ &> d(f(x_0),\, g(x_0)) - \varepsilon - \varepsilon = 0. \end{aligned}$$

So $f(x) \neq g(x)$. This shows that $N_\delta(x_0) \subset X \setminus F$, so that $X \setminus F$ is open. ∎

The second part is closely related to Problem 4.3. If $Y = \mathbf{R}$ (or $\mathbf{C}^n$, or $\ldots$), then $\{x \in X \colon f(x) = g(x)\} = Z(f - g)$, and $f - g$ is continuous when f and g are. For general Y, however, this solution fails, since $f - g$ won't be defined. The argument given is the analog of Solution 2 to Problem 4.3. The analog of Solution 3 to Problem 4.3 also works the same way: F is closed because if $x_n \to x$ and $f(x_n) = g(x_n)$ for all n, then $\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n)$. The analog of Solution 1 can actually be patched in the following way (using the fact that the product of two metric spaces is again a metric space): Define $h \colon X \to Y \times Y$ by $h(x) = (f(x),\, g(x))$. Then h is continuous and $D = \{(y, y) \colon y \in Y\} \subset Y \times Y$ is closed, so $\{x \in X \colon f(x) = g(x)\} = h^{-1}(D)$ is closed.

Problem 4.6: Let $E \subset \mathbf{R}$ be compact, and let $f \colon E \to \mathbf{R}$ be a function. Prove that f is continuous if and only if the graph $G(f) = \{(x,\, f(x)) \colon x \in E\} \subset \mathbf{R}^2$ is compact.

Remark: This statement is my interpretation of what was intended. Normally one would assume that E is supposed to be a compact subset of an arbitrary metric space X, and that f is supposed to be a function from E to some other metric space Y. (In fact, one might as well assume $E = X$.) The proofs are all the same (with one exception, noted below), but require the notion of the product of two metric spaces. We make $X \times Y$ into a metric space via the metric

$$d((x_1, y_1),\, (x_2, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2};$$

there are other choices which are easier to deal with and work just as well.

We give several solutions for each direction. We first show that if f is continuous then $G(f)$ is compact.

Solution 1 (Sketch): The map $x \mapsto (x, f(x))$ is easily checked to be continuous, and $G(f)$ is the image of the compact set E under this map, so $G(f)$ is compact by Theorem 4.14 of Rudin. ∎

Solution 2 (Sketch): The graph of a continuous function is closed, as can be verified by arguments similar to those of Solutions 2 and 3 to Problem 4.3. The graph is a subset of $E \times f(E)$. This set is bounded (clear) and closed (check this!) in $\mathbf{R}^2$, and is therefore compact. (Note: This does not work for general metric spaces. However, it is true in general that the product of two compact sets, with the product metric, is compact.) Therefore the closed subset $G(f)$ is compact. ∎

Now we show that if $G(f)$ is compact then f is continuous.

Solution 1 (Sketch): We know that the function $g_0 \colon E \times \mathbf{R} \to E$, given by $g_0(x, y) = x$, is continuous. (See Example 4.11 of Rudin.) Therefore $g = g_0|_{G(f)} \colon G(f) \to E$ is continuous. Also g is bijective (because f is a function). Since $G(f)$ is compact, it follows (Theorem 4.17 of Rudin) that $g^{-1} \colon E \to G(f)$ is continuous. Furthermore, the function $h \colon E \times \mathbf{R} \to \mathbf{R}$, given by $h(x, y) = y$, is continuous, again by Example 4.11 of Rudin. Therefore $f = h \circ g^{-1}$ is continuous. ∎

Solution 2 (Sketch): Let (x_n) be a sequence in E with $x_n \to x$. We show that $f(x_n) \to f(x)$. We do this by showing that every subsequence of $(f(x_n))$ has in turn a subsubsequence which converges to $f(x)$. (To see that this is sufficient, let (y_n) be a sequence in some metric space Y, let $y \in Y$, and suppose that (y_n) does not converge to y. Find a subsequence $(y_{k(n)})$ of (y_n) such that $\inf_{n \in \mathbf{N}} d(y_{k(n)}, y) > 0$. Then no subsequence of $(y_{k(n)})$ can converge to y.)

Accordingly, let $(f(x_{k(n)}))$ be a subsequence of $(f(x_n))$. Let $(x_{k(n)})$ be the corresponding subsequence of (x_n). If $(x_{k(n)})$ is eventually constant, then already $f(x_{k(n)}) \to f(x)$. Otherwise, $\{x_{k(n)} \colon n \in \mathbf{N}\}$ is an infinite set, whence so is $\{(x_{k(n)}, f(x_{k(n)})) \colon n \in \mathbf{N}\} \subset G(f)$. Since $G(f)$ is compact, this set has a limit point, say (a, b). It is easy to check that a must equal x. Since $G(f)$ is compact, it is closed, so $b = f(x)$. Since (a, b) is a limit point of $G(f)$, there is a subsequence of $\big((x_{k(n)}, f(x_{k(n)}))\big)$ which converges to (a, b). Using continuity of projection onto the second coordinate, we get a subsequence of $\big(f(x_{k(n)})\big)$ which converges to $f(x)$. ∎

Solution 3: We first observe that the range $Y = \{f(x) \colon x \in E\}$ of f is compact. Indeed, Y is the image of $G(f)$ under the map $(x, y) \mapsto y$, which is continuous by Example 4.11 of Rudin. So Y is compact by Theorem 4.14 of Rudin. It suffices to prove that f is continuous as a function from E to Y, as can be seen, for example, from the sequential criterion for limits (Theorem 4.2 of Rudin).

Now let $x_0 \in E$ and let $V \subset Y$ be an open set containing $f(x_0)$. We must find an open set $U \subset E$ containing x_0 such that $f(U) \subset V$. For each $y \in Y \setminus V$, the point (x_0, y) is not in the closed set $G \subset E \times \mathbf{R}$. Therefore there exist open sets $R_y \subset E$ containing x_0 and $S_y \subset Y$ containing y such that $(R_y \times S_y) \cap G = \varnothing$. Since $Y \setminus V$ is compact, there are n and $y(1), \ldots, y(n) \in Y \setminus V$ such that the sets $S_{y(1)}, \ldots, S_{y(n)}$ cover $Y \setminus V$. Set $U = R_{y(1)} \cap \cdots \cap R_{y(n)}$ to obtain $f(U) \subset V$. ∎

Problem 4.7: Define f, $g\colon \mathbf{R}^2 \to \mathbf{R}$ by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and

$$g(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Prove:

(1) f is bounded.
(2) f is not continuous at $(0,0)$.
(3) The restriction of f to every straight line in $\mathbf{R}^2$ is continuous.
(4) g is not bounded on any neighborhood of $(0,0)$.
(5) g is not continuous at $(0,0)$.
(6) The restriction of g to every straight line in $\mathbf{R}^2$ is continuous.

Solution (Sketch): (1) We use the inequality $2ab \leq a^2 + b^2$ (which follows from $a^2 + b^2 - 2ab = (a-b)^2 \geq 0$). Taking $a = |x|$ and $b = y^2$, we get $2|x|y^2 \leq x^2 + y^4$, which implies $|f(x,y)| \leq \frac{1}{2}$ for all $(x,y) \in \mathbf{R}^2$.

(2) Set $x_n = \frac{1}{n^2}$ and $y_n = \frac{1}{n}$. Then $(x_n, y_n) \to (0,0)$, but $f(x_n, y_n) = \frac{1}{2} \not\to 0 = f(0,0)$.

(3) Clearly f is continuous on $\mathbf{R}^2 \setminus \{(0,0)\}$, so the restriction of f to every straight line in $\mathbf{R}^2$ not going through $(0,0)$ is clearly continuous. Furthermore, the restriction of f to the y-axis is given by $(0,y) \mapsto 0$, which is clearly continuous.

Every other line has the form $y = ax$ for some $a \in \mathbf{R}$. We have

$$f(x,ax) = \frac{a^2 x}{1 + a^4 x^2}$$

for all $x \in \mathbf{R}$, so the restriction of f to this line is given by the continuous function

$$(x,y) \mapsto \frac{a^2 x}{1 + a^4 x^2}.$$

(4) Set $x_n = \frac{1}{n^3}$ and $y_n = \frac{1}{n}$. Then $(x_n, y_n) \to (0,0)$, but $g(x_n, y_n) = n \to \infty$.

(5) This is immediate from (4).

(6) Clearly g is continuous on $\mathbf{R}^2 \setminus \{(0,0)\}$, so the restriction of g to every straight line in $\mathbf{R}^2$ not going through $(0,0)$ is clearly continuous. Furthermore, the restriction of g to the y-axis is given by $(0,y) \mapsto 0$, which is clearly continuous.

Every other line has the form $y = ax$ for some $a \in \mathbf{R}$. We have

$$g(x,ax) = \frac{a^3 x}{1 + a^6 x^4}$$

for all $x \in \mathbf{R}$, so the restriction of g to this line is given by the continuous function

$$(x,y) \mapsto \frac{a^3 x}{1 + a^6 x^4}.$$

∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 8

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 4.8: Let $E \subset \mathbf{R}$ be bounded, and let $f \colon E \to \mathbf{R}$ be uniformly continuous. Prove that f is bounded. Show that a uniformly continuous function on an unbounded subset of $\mathbf{R}$ need not be bounded.

Solution (Sketch): Choose $\delta > 0$ such that if $x, y \in E$ satisfy $|x - y| < \delta$, then $|f(x) - f(y)| < 1$. Choose $n \in \mathbf{N}$ such that $\frac{1}{n} < \delta$. Since E is a bounded subset of $\mathbf{R}$, there are finitely many closed intervals $\left[a_k, \, a_k + \frac{1}{n}\right]$ whose union contains the closed interval $[\inf(E), \sup(E)]$ and hence also E. Let S be the finite set of those k for which $E \cap \left[a_k, \, a_k + \frac{1}{n}\right] \neq \varnothing$. Thus $E \subset \bigcup_{k \in S} \left[a_k, \, a_k + \frac{1}{n}\right]$. Choose $b_k \in E \cap \left[a_k, \, a_k + \frac{1}{n}\right]$. Set $M = 1 + \max_{k \in S} |f(b_k)|$.

We show that $|f(x)| \leq M$ for all $x \in E$. For such x, choose $k \in S$ such that $x \in \left[a_k, \, a_k + \frac{1}{n}\right]$. Then $|x - b_k| \leq \frac{1}{n} < \delta$, so $|f(x) - f(b_k)| < 1$. Thus $|f(x)| \leq |f(x) - f(b_k)| + |f(b_k)| < 1 + M$.

As a counterexample with E unbounded, take $E = \mathbf{R}$ and $f(x) = x$ for all x. ∎

Problem 4.9: Let X and Y be metric spaces, and let $f \colon X \to Y$ be a function. Prove that f is uniformly continuous if and only if for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $E \subset X$ satisfies $\operatorname{diam}(E) < \delta$, then $\operatorname{diam}(f(E)) < \varepsilon$.

Solution: Let f be uniformly continuous, and let $\varepsilon > 0$. Choose $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, then $d(f(x_1), \, f(x_2)) < \frac{1}{2}\varepsilon$. Let $E \subset X$ satisfy $\operatorname{diam}(E) < \delta$. We show that $\operatorname{diam}(f(E)) < \varepsilon$. Let $y_1, y_2 \in f(E)$. Choose $x_1, x_2 \in E$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Then $d(x_1, x_2) < \delta$, so $d(f(x_1), \, f(x_2)) < \frac{1}{2}\varepsilon$. This shows that $d(y_1, y_2) < \frac{1}{2}\varepsilon$ for all $y_1, y_2 \in f(E)$. Therefore

$$\operatorname{diam}(f(E)) = \sup_{y_1, y_2 \in E} d(y_1, y_2) \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

Now assume that for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $E \subset X$ satisfies $\operatorname{diam}(E) < \delta$, then $\operatorname{diam}(f(E)) < \varepsilon$. We prove that f is uniformly continuous. Let $\varepsilon > 0$. Choose $\delta > 0$ as in the hypotheses. Let $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$. Set $E = \{x_1, x_2\}$. Then $\operatorname{diam}(E) < \delta$. So $d(f(x_1), \, f(x_2)) = \operatorname{diam}(f(E)) < \varepsilon$. ∎

Problem 4.10: Use the fact that infinite subsets of compact sets have limit points to give an alternate proof that if X and Z are metric spaces with X compact, and $f \colon X \to Z$ is continuous, then f is uniformly continuous.

Date: 19 Nov. 2001.

Solution: Assume that f is not uniformly continuous. Choose $\varepsilon > 0$ for which the definition of uniform continuity fails. Then for every $n \in \mathbf{N}$ there are $x_n, y_n \in X$ such that $d(x_n, y_n) < \frac{1}{n}$ and $d(f(x_n), f(y_n)) \geq \varepsilon$. Since X is compact, the sequence (x_n) has a convergent subsequence. (See Theorem 3.6 (a) of Rudin.) Let $x = \lim_{n\to\infty} x_{k(n)}$. Since $d(x_{k(n)}, y_{k(n)}) < \frac{1}{k(n)} \leq \frac{1}{n}$, we also have $\lim_{n\to\infty} y_{k(n)} = x$.

If f were continuous at x, we would have

$$\lim_{n\to\infty} f(x_{k(n)}) = \lim_{n\to\infty} f(y_{k(n)}) = f(x).$$

This contradicts $d(f(x_n), f(y_n)) \geq \varepsilon$ for all n. To see this, choose $N \in \mathbf{N}$ such that $n \geq N$ implies

$$d(f(x_{k(n)}), f(x)) < \tfrac{1}{3}\varepsilon \quad \text{and} \quad d(f(y_{k(n)}), f(x)) < \tfrac{1}{3}\varepsilon.$$

Then

$$d(f(x_{k(N)}), f(y_{k(N)}) \leq d(f(x_{k(n)}), f(x)) + d(f(x), f(y_{k(n)})) < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \tfrac{2}{3}\varepsilon,$$

but by construction $d(f(x_{k(N)}), f(y_{k(N)}) \geq \varepsilon$. ∎

Remark: It is not correct to simply claim that the sequence (x_n) has a convergent subsequence $(x_{k(n)})$ and the sequence (y_n) has a convergent subsequence $(y_{k(n)})$. If one chooses convergent subsequences of (x_n) and (y_n), they must be called, say, $(x_{k(n)})$ and $(y_{l(n)})$ for *different* functions $k, l \colon \mathbf{N} \to \mathbf{N}$.

It is nevertheless possible to carry out a proof by passing to convergent subsequences of (x_n) and (y_n). The following solution shows how it can be done. This solution is not recommended here, but in other situations it may be the only way to proceed.

Alternate solution: Assume that f is not uniformly continuous. Choose $\varepsilon > 0$ for which the definition of uniform continuity fails. Then for every $n \in \mathbf{N}$ there are $x_n, y_n \in X$ such that $d(x_n, y_n) < \frac{1}{n}$ and $d(f(x_n), f(y_n)) \geq \varepsilon$. Since X is compact, the sequence (x_n) has a convergent subsequence $(x_{k(n)})$. (See Theorem 3.6 (a) of Rudin.) Then $(y_{k(n)})$ is a sequence in a compact metric space, and therefore, again by Theorem 3.6 (a) of Rudin, it has a convergent subsequence $(y_{k(r(n))})$. Let $l = k \circ r$. Then $(x_{l(n)})$ is a subsequence of the convergent sequence $(x_{k(n)})$, and therefore converges.

Let $x = \lim_{n\to\infty} x_{l(n)}$ and $y = \lim_{n\to\infty} y_{l(n)}$. Now $d(x_{l(n)}, y_{l(n)}) < \frac{1}{l(n)} \leq \frac{1}{n}$. It follows that $d(x, y) = 0$. (To see this, let $\varepsilon > 0$, and choose $N_1, N_2, N_3 \in \mathbf{N}$ so large that $n \geq N_1$ implies $d(x_{l(n)}, x) < \frac{1}{3}\varepsilon$, so large that $n \geq N_2$ implies $d(y_{l(n)}, y) < \frac{1}{3}\varepsilon$, and so large that $n \geq N_3$ implies $\frac{1}{n} < \frac{1}{3}\varepsilon$. Then with $n = \max(N_1, N_2, N_3)$, we get

$$d(x, y) \leq d(x, x_{l(n)}) + d(x_{l(n)}, y_{l(n)}) + d(y_{l(n)}, y) < \tfrac{1}{3}\varepsilon + \tfrac{1}{n} + \tfrac{1}{3}\varepsilon < \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it follows that $d(x, y) = 0$.)

We now know that $\lim_{n\to\infty} y_{l(n)} = x = \lim_{n\to\infty} x_{l(n)}$.

If f were continuous at x, we would have

$$\lim_{n\to\infty} f(x_{l(n)}) = \lim_{n\to\infty} f(y_{l(n)}) = f(x).$$

This contradicts $d(f(x_n), f(y_n)) \geq \varepsilon$ for all n. To see this, choose $N \in \mathbf{N}$ such that $n \geq N$ implies

$$d(f(x_{l(n)}), f(x)) < \tfrac{1}{3}\varepsilon \quad \text{and} \quad d(f(y_{l(n)}), f(x)) < \tfrac{1}{3}\varepsilon.$$

Then

$$d(f(x_{l(N)}),\, f(y_{l(N)}) \leq d(f(x_{l(n)}),\, f(x)) + d(f(x),\, f(y_{l(n)})) < \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \tfrac{2}{3}\varepsilon,$$

but by construction $d(f(x_{l(N)}),\, f(y_{l(N)}) \geq \varepsilon$. ∎

Problem 4.11: Let X and Y be metric spaces, and let $f\colon X \to Y$ be uniformly continuous. Prove that if (x_n) is a Cauchy sequence in X, then $(f(x_n))$ is a Cauchy sequence in Y. Use this result to prove that if Y is complete, $E \subset X$ is dense, and $f_0\colon E \to Y$ is uniformly continuous, then there is a unique continuous function $f\colon X \to Y$ such that $f|_E = f_0$.

Solution: We prove the first statement. Let (x_n) be a Cauchy sequence in X. Let $\varepsilon > 0$. Choose $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(s_1, s_2) < \delta$, then $d(f(s_1),\, f(s_2)) < \varepsilon$. Choose $N \in \mathbf{N}$ such that if $m, n \in \mathbf{N}$ satisfy $m, n \geq N$, then $d(x_m, x_n) < \delta$. Then whenever $m, n \in \mathbf{N}$ satisfy $m, n \geq N$, we have $d(x_m, x_n) < \delta$, so that $d(f(x_m),\, f(x_n)) < \varepsilon$. This shows that $(f(x_n))$ is a Cauchy sequence.

Now we prove the second statement. The neatest arrangement I can think of is to prove the following lemmas first.

Lemma 1. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f\colon E \to Y$ be uniformly continuous. Let (x_n) be a sequence in E which converges to some point in X. Then $\lim_{n\to\infty} f(x_n)$ exists in Y.

Proof: We know that convergent sequences are Cauchy. This is therefore immediate from the first part of the problem and the definition of completeness. ∎

Lemma 2. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f\colon E \to Y$ be uniformly continuous. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, and whenever (r_n) and (s_n) are sequences in E such that $r_n \to x_1$ and $s_n \to x_2$, then

$$d\left(\lim_{n\to\infty} f(r_n),\, \lim_{n\to\infty} f(s_n)\right) < \varepsilon.$$

Note that the limits exist by Lemma 1.

Proof of Lemma 2: Let $\varepsilon > 0$. Choose $\rho > 0$ such that whenever $x_1, x_2 \in E$ satisfy $d(x_1, x_2) < \rho$, then $d(f(x_1),\, f(x_2)) < \frac{1}{2}\varepsilon$. Set $\delta = \frac{1}{2}\rho > 0$. Let $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, and let (r_n) and (s_n) be sequences in E such that $r_n \to x_1$ and $s_n \to x_2$. Let $y_1 = \lim_{n\to\infty} f(r_n)$ and $y_2 = \lim_{n\to\infty} f(s_n)$. (These exist by Lemma 1.) Choose N so large that for all $n \in \mathbf{N}$ with $n \geq N$, the following four conditions are all satisfied:

- $d(r_n, x_1) < \frac{1}{4}\rho$.
- $d(s_n, x_2) < \frac{1}{4}\rho$.
- $d(f(r_n),\, y_1) < \frac{1}{4}\varepsilon$.
- $d(f(s_n),\, y_2) < \frac{1}{4}\varepsilon$.

We then have

$$d(r_N, s_N) \leq d(r_N, x_1) + d(x_1, x_2) + d(x_2, s_N) < \tfrac{1}{4}\rho + \tfrac{1}{2}\rho + \tfrac{1}{4}\rho = \rho.$$

Therefore $d(f(r_N),\, f(s_N)) < \frac{1}{2}\varepsilon$. So

$$d(y_1, y_2) \leq d(y_1,\, f(r_N)) + d(f(r_N),\, f(s_N)) + d(f(s_N),\, y_2) < \tfrac{1}{4}\varepsilon + \tfrac{1}{2}\varepsilon + \tfrac{1}{4}\varepsilon = \varepsilon,$$

as desired. ∎

Theorem. Let X and Y be metric spaces, with Y complete, let $E \subset X$, and let $f\colon E \to Y$ be uniformly continuous. Then there is a unique continuous function $f\colon X \to Y$ such that $f|_E = f_0$.

Proof: If f exists, then it is unique by Problem 4.4, which was in the previous assignment. So we prove existence. For $x \in X$, we want to define $f(x)$ by choosing a sequence (r_n) in E with $\lim_{n\to\infty} r_n = x$ and then setting $f(x) = \lim_{n\to\infty} f_0(r_n)$. We know that such a sequence exists because E is dense in X. We know that $\lim_{n\to\infty} f_0(r_n)$ exists, by Lemma 1. However, we must show that $\lim_{n\to\infty} f_0(r_n)$ only depends on x, not on the sequence (r_n).

To prove this, let (r_n) and (s_n) be sequences in E with

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = x.$$

Let $\varepsilon > 0$; we show that

$$d\left(\lim_{n\to\infty} f_0(r_n),\ \lim_{n\to\infty} f_0(s_n)\right) < \varepsilon.$$

(Since ε is arbitrary, this will give $\lim_{n\to\infty} f_0(r_n) = \lim_{n\to\infty} f_0(s_n)$.) To do this, choose $\delta > 0$ according to Lemma 2. We certainly have $d(x, x) < \delta$. Therefore the conclusion of Lemma 2 gives

$$d\left(\lim_{n\to\infty} f_0(r_n),\ \lim_{n\to\infty} f_0(s_n)\right) < \varepsilon,$$

as desired.

We now get a well defined function $f\colon X \to Y$ by setting $f(x) = \lim_{n\to\infty} f_0(r_n)$, where (r_n) is any sequence in E with $\lim_{n\to\infty} r_n = x$. By considering the constant sequence $x_n = x$ for all n, we see immediately that $f(x) = f_0(x)$ for $x \in E$. We show that f is continuous, in fact uniformly continuous. Let $\varepsilon > 0$. Choose $\delta > 0$ according to Lemma 2. For $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, choose (by density of E, as above) sequences (r_n) and (s_n) in E such that $r_n \to x_1$ and $s_n \to x_2$. Then $d\left(\lim_{n\to\infty} f_0(r_n),\ \lim_{n\to\infty} f_0(s_n)\right) < \varepsilon$. By construction, we have $f(x_1) = \lim_{n\to\infty} f_0(r_n)$ and $f(x_2) = \lim_{n\to\infty} f_0(s_n)$. Therefore we have shown that $d(f(x_1),\ f(x_2)) < \varepsilon$, as desired. ∎

The point of stating Lemma 2 separately is that the proof that f is well defined, and the proof that f is continuous, use essentially the same argument. By putting that argument in a lemma, we avoid repeating it.

Problem 4.12: State precisely and prove the following: "A uniformly continuous function of a uniformly continuous function is uniformly continuous."

Solution: Here is the precise statement:

Proposition. Let X, Y, and Z be metric spaces. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be uniformly continuous functions. Then $g \circ f$ is uniformly continuous.

Proof: Let $\varepsilon > 0$. Choose $\rho > 0$ such that if $y_1, y_2 \in Y$ satisfy $d(y_1, y_2) < \rho$, then $d(g(y_1),\ g(y_2)) < \varepsilon$. Choose $\delta > 0$ such that if $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, then $d(f(x_1),\ f(x_2)) < \rho$. Then whenever $x_1, x_2 \in X$ satisfy $d(x_1, x_2) < \delta$, we have $d(f(x_1),\ f(x_2)) < \rho$, so that $d(g(f(x_1)),\ g(f(x_2))) < \varepsilon$. ∎

Problem 4.14: Let $f\colon [0, 1] \to [0, 1]$ be continuous. Prove that there is $x \in [0, 1]$ such that $f(x) = x$.

Solution: Define $g\colon [0,1] \to \mathbf{R}$ by $g(x) = x - f(x)$. Then g is continuous. Since $f(0) \in [0,1]$, we have $g(0) = -f(0) \leq 0$, while since $g(1) \in [0,1]$, we have $g(1) = 1 - f(1) \geq 0$. If $g(0) = 0$ then $x = 0$ satisfies the conclusion, while if $g(1) = 0$ then $x = 1$ satisfies the conclusion. Otherwise, $g(0) < 0$ and $g(1) > 0$, so Theorem 4.23 of Rudin provides $x \in (0,1)$ such that $g(x) = 0$. This x satisfies $f(x) = x$. ∎

Something much more general is true, namely the Brouwer Fixed Point Theorem:

Theorem. Let $n \geq 1$, and let $B = \{x \in \mathbf{R}^n \colon \|x\| \leq 1\}$. Let $f\colon B \to B$ be continuous. Then there is $x \in B$ such that $f(x) = x$.

The proof requires higher orders of connectedness, and is best done with algebraic topology.

Problem 4.16: For $x \in \mathbf{R}$, define $[x]$ by the relations $[x] \in \mathbf{Z}$ and $x - 1 < [x] \leq x$ (this is called the "integer part of x" or the "greatest integer function"), and define $(x) = x - [x]$ (this is called the "fractional part of x", but the notation (x) is not standard). What discontinuities do the functions $x \mapsto [x]$ and $x \mapsto (x)$ have?

Solution (Sketch): Both functions are continuous at all noninteger points, since $x \in (n, n+1)$ implies $[x] = n$ and $(x) = x - n$; both expressions are continuous on the interval $(n, n+1)$.

Both functions have jump discontinuities at all integers: for $n \in \mathbf{Z}$, we have

$$\lim_{x \to n^+} [x] = \lim_{x \to n^+} n = n = f(n) \quad \text{and} \quad \lim_{x \to n^-} [x] = \lim_{x \to n^-} (n-1) = n - 1 \neq f(n),$$

and also

$$\lim_{x \to n^+} (x) = \lim_{x \to n^+} (x - n) = 0 = f(n)$$

and

$$\lim_{x \to n^-} (x) = \lim_{x \to n^-} [x - (n-1)] = 1 \neq f(n).$$

Problem 4.18: Define $f\colon \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \begin{cases} 0 & x \in \mathbf{R} \setminus \mathbf{Q} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \end{cases}.$$

(By definition, we require $q > 0$. If $x = 0$ we take $p = 0$ and $q = 1$.) Prove that f is continuous at each $x \in \mathbf{R} \setminus \mathbf{Q}$, and that f has a simple discontinuity at each $x \in \mathbf{Q}$.

Solution: We show that $\lim_{x \to 0} f(x) = 0$ for all $x \in \mathbf{R}$. This immediately implies that f is continuous at all points x for which $f(x) = 0$ and has a removable discontinuity at every x for which $f(x) \neq 0$.

Let $x \in \mathbf{R}$, and let $\varepsilon > 0$. Choose $N \in \mathbf{N}$ such that $\frac{1}{N} < \varepsilon$. For $1 \leq n \leq N$, let

$$S_n = \left\{ \frac{a}{n} \colon a \in \mathbf{Z} \text{ and } 0 < \left| \frac{a}{n} - x \right| < 1 \right\}.$$

Then S_n is finite; in fact, $\operatorname{card}(S_n) \leq 2n$. Set

$$S = \bigcup_{n=1}^{N} S_n,$$

which is a finite union of finite sets and hence finite. Note that $x \notin S$. Set

$$\delta = \min\left(1, \ \min_{y \in S} |y - x|\right).$$

Then $\delta > 0$ because $x \notin S$ and S is finite.

Let $0 < |y - x| < \delta$. If $y \notin \mathbf{Q}$, then $|f(y) - 0| = 0 < \varepsilon$. Otherwise, because $y \notin S$, $|y - x| < 1$, and $y \neq x$, it is not possible to write $y = \frac{p}{q}$ with $q \leq N$. Thus, when we write $y = \frac{p}{q}$ in lowest terms, we have $q > N$, so $f(y) = \frac{1}{q} < \frac{1}{N} < \varepsilon$. This shows that $|f(y) - 0| < \varepsilon$ in this case also. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 9

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 4.15: Prove that every continuous open map $f\colon \mathbf{R} \to \mathbf{R}$ is monotone.

Sketches of two solutions are presented. The second is what I expect people to have done. The first is essentially a careful rearrangement of the ideas of the second, done so as to minimize the number of cases. (You will see when reading the second solution why this is desirable.)

Solution (Sketch):

Lemma 1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$. Then $f(a) \neq f(b)$.

Proof (sketch): Suppose $f(a) = f(b)$. Let m_1 and m_2 be the minimum and maximum values of f on $[a, b]$. (These exist because f is continuous and $[a, b]$ is compact.) If $m_1 = m_2$, then $m_1 = m_2 = f(a) = f(b)$, and $f((a, b)) = \{m_1\}$ is not an open set. Since (a, b) is open, this is a contradiction. So suppose $m_1 < m_2$. If $m_1 \neq f(a)$, choose $c \in [a, b]$ such that $f(c) = m_1$. Then actually $c \in (a, b)$. So $f((a, b))$ contains $f(c)$ but contains no real numbers smaller than $f(c)$. This is easily seen to contradict the assumption that $f((a, b))$ is open. The case $m_2 \neq f(a)$ is handled similarly, or by considering $-f$ in place of f. ∎

Note: The last part of this proof is the only place where I would expect a submitted solution to be more complete than what I have provided.

Lemma 2. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbf{R}$ satisfies $x < a$, we have $f(x) < f(a)$.

Proof: We can't have $f(x) = f(a)$, by Lemma 1. If $f(x) = f(b)$, we again have a contradiction by Lemma 1. If $f(x) > f(b)$, then the Intermediate Value Theorem provides $z \in (x, a)$ such that $f(z) = f(b)$. Since $z < b$, this contradicts Lemma 1. If $f(a) < f(x) < f(b)$, then the Intermediate Value Theorem provides $z \in (a, b)$ such that $f(z) = f(x)$. Since $x < z$, this again contradicts Lemma 1. The only remaining possibility is $f(x) < f(a)$. ∎

Lemma 3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbf{R}$ satisfies $b < x$, we have $f(b) < f(x)$.

Proof: Apply Lemma 2 to the function $x \mapsto -f(-x)$. ∎

Lemma 4. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$. If $f(a) < f(b)$, then whenever $x \in \mathbf{R}$ satisfies $a < x < b$, we have $f(a) < f(x) < f(b)$.

Date: 30 Nov. 2001.

Proof: Lemma 1 implies that $f(x)$ is equal to neither $f(a)$ nor $f(b)$. If $f(x) < f(a)$, we apply Lemma 3 to $-f$, with b and x interchanged, to get $f(b) < f(x)$. This implies $f(b) < f(a)$, which contradicts the hypotheses. If $f(x) > f(b)$, we apply Lemma 2 to $-f$, with a and x interchanged, to get $f(a) > f(x)$. This again contradicts the hypotheses. ∎

Corollary 5. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and open. Let $a, b \in \mathbf{R}$ satisfy $a < b$, and suppose that $f(a) < f(b)$. Let $x \in \mathbf{R}$. Then:

(1) If $x \leq a$ then $f(x) \leq f(a)$.
(2) If $x \leq b$ then $f(x) \leq f(b)$.
(3) If $x \geq a$ then $f(x) \geq f(a)$.
(4) If $x \geq b$ then $f(x) \geq f(b)$.

Proof: If we have equality ($x = a$ or $x = b$), the conclusion is obvious. With strict inequality, Part (1) follows from Lemma 2, and Part (4) follows from Lemma 3. Part (2) follows from Lemma 4 if $x > a$, from Lemma 2 if $x < a$, and is trivial if $x = a$. Part (3) follows from Lemma 4 if $x < b$, from Lemma 3 if $x > b$, and is trivial if $x = b$. ∎

I won't actually use Part (2); it is included for symmetry.

Now we prove the result. Choose arbitrary $c, d \in \mathbf{R}$ with $c < d$. We have $f(c) \neq f(d)$ by Lemma 1. Suppose first that $f(c) < f(d)$. Let $r, s \in \mathbf{R}$ satisfy $r \leq s$. We prove that $f(r) \leq f(s)$, and there are several cases. I will try to arrange this to keep the number of cases as small as possible.

Case 1: $r \leq c \leq s$. Then $f(r) \leq f(c) \leq f(s)$ by Parts (1) and (3) of Corollary 5, taking $a = c$ and $b = d$.

Case 2: $r \leq s \leq a$. Then $f(s) \leq f(a)$ by Part (1) of Corollary 5, taking $a = c$ and $b = d$. Further, $f(r) \leq f(s)$ by Part (1) of Corollary 5, taking $a = s$ and $b = d$.

Case 3: $a \leq r \leq s$. Then $f(a) \leq f(r)$ by Part (3) of Corollary 5, taking $a = c$ and $b = d$. Further, $f(r) \leq f(s)$ by Part (4) of Corollary 5, taking $a = c$ and $b = r$.

The case $f(c) > f(d)$ follows by applying the preceding argument to $-f$. ∎

Alternate solution (Brief sketch):

Suppose f is not monotone; we prove that f is not open. Since f isn't nondecreasing, there exist $a, b \in \mathbf{R}$ such that $a < b$ and $f(a) > f(b)$; and since f isn't nonincreasing, there exist $c, d \in \mathbf{R}$ such that $c < d$ and $f(c) < f(d)$. Now there are various cases depending on how a, b, c, and d are arranged in $\mathbf{R}$, and depending on how $f(a)$ and $f(b)$ relate to $f(c)$ and $f(d)$. Specifically, there are 13 possible ways for a, b, c, and d to be arranged in $\mathbf{R}$, namely:

(1) $$a < b < c < d$$

(2) $$a < b = c < d$$

(3) $$a < c < b < d$$

(4) $$a < c < b = d$$

(5) $$a < c < d < b$$

(6) $$a = c < b < d$$

$$a = c < b = d \tag{7}$$

$$a = c < d < b \tag{8}$$

$$c < a < b < d \tag{9}$$

$$c < a < b = d \tag{10}$$

$$c < a < d < b \tag{11}$$

$$c < a = d < b \tag{12}$$

$$c < d < a < b \tag{13}$$

Of these, the arrangement (7) gives an immediate contradiction. For each of the others, we find $x < y < z$ such that $f(y) < f(x), f(z)$ (so that f is not open by Lemma 2 of the previous solution), or such that $f(y) > f(x), f(z)$ (so that f is not open by Lemma 3 of the previous solution). Many cases break down into subcases depending on how the values of f are arranged. We illustrate by treating the arrangement (1).

Suppose $a < b < c < d$ and $f(b) < f(c)$. Set

$$x = a, \quad y = b, \quad \text{and} \quad z = d.$$

Then $x < y < z$ and $f(y) < f(x), f(z)$, so Lemma 2 applies. Suppose, on the other hand, that $a < b < c < d$ and $f(b) \geq f(c)$. Set

$$x = a, \quad y = c, \quad \text{and} \quad z = d.$$

Then again $x < y < z$ and $f(y) < f(x), f(z)$, so Lemma 2 applies. ∎

Problem 5.1: Let $f \colon \mathbf{R} \to \mathbf{R}$ be a function such that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all $x, y \in \mathbf{R}$. Prove that f is constant.

Solution: We first prove that $f'(x) = 0$ for all $x \in \mathbf{R}$. For $h \neq 0$,

$$\left| \frac{f(x+h) - f(x)}{h} \right| = \frac{|f(x+h) - f(x)|}{|h|} \leq \frac{|h^2|}{|h|} = |h|.$$

It follows immediately that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0.$$

It now follows that f is constant. (See Theorem 5.11 (b) of Rudin's book.) ∎

The solution above is the intended solution. However, there is another solution which is nearly as easy and does not use calculus.

Alternate solution: Let $x, y \in \mathbf{R}$ and let $\varepsilon > 0$; we prove that $|f(x) - f(y)| < \varepsilon$. It will clearly follow that f is constant.

Choose $N \in \mathbf{N}$ with $N > \varepsilon^{-1}(x-y)^2$. The hypothesis implies that, for any k, we have

$$\begin{aligned}\left| f\left(x + (k-1)\cdot \tfrac{1}{N}(y-x)\right) - f\left(x + k\cdot \tfrac{1}{N}(y-x)\right)\right| &\leq \left(\tfrac{1}{N}(y-x)\right)^2 \\ &= \left(\frac{1}{N}\right)\left(\frac{(y-x)^2}{N}\right) < \frac{1}{N}\cdot\varepsilon.\end{aligned}$$

Therefore

$$\begin{aligned}|f(x) - f(y)| &\leq \sum_{k=1}^{N} \left| f\left(x + (k-1)\cdot \tfrac{1}{N}(y-x)\right) - f\left(x + k\cdot \tfrac{1}{N}(y-x)\right)\right| \\ &< N\cdot\frac{1}{N}\cdot\varepsilon = \varepsilon.\end{aligned}$$

∎

Problem 5.2: Let $f\colon (a,b) \to \mathbf{R}$ satisfy $f'(x) > 0$ for all $x \in (a,b)$. Prove that f is strictly increasing, that its inverse function g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)}$$

for all $x \in (a,b)$.

Solution: That f is strictly increasing on (a,b) follows from the Mean Value Theorem and the fact that $f'(x) > 0$ for all $x \in (a,b)$.

Define

$$c = \inf_{x\in(a,b)} f(x) \quad \text{and} \quad d = \sup_{x\in(a,b)} f(x).$$

(Note that c could be $-\infty$ and d could be ∞.) Our next step is to prove that f is a bijection from (a,b) to (c,d). Clearly f is injective, and has range contained in $[c,d]$. If $c = f(x)$ for some $x \in (a,b)$, then there is $q \in (a,b)$ with $q < x$. This would imply $f(q) < c$, contradicting the definition of c. So c is not in the range of f. Similarly d is not in the range of f. So the range of f is contained in (c,d). For surjectivity, let $y_0 \in (c,d)$. By the definitions of inf and sup, there are $r,\, s \in (a,b)$ such that $f(r) < y_0 < f(s)$. Clearly $r < s$. The Intermediate Value Theorem provides $x_0 \in (r,s)$ such that $f(x_0) = y_0$. This shows that the range of f is all of (c,d), and completes the proof that f is a bijection from (a,b) to (c,d).

Now we show that $g\colon (c,d) \to (a,b)$ is continuous. Again, let $y_0 \in (c,d)$, and choose r and s as in the previous paragraph. Since f is strictly increasing, and again using the Intermediate Value Theorem, we see that $f|_{[r,s]}$ is a continuous bijection from $[r,s]$ to $[f(r),\, f(s)]$. Since $[r,s]$ is compact, the function $\left(f|_{[r,s]}\right)^{-1} = g|_{[f(r),\, f(s)]}$ is continuous. Since $y_0 \in (f(r),\, f(s))$, it follows that g is continuous at y_0. Thus g is continuous.

Now we find g'. Fix $x_0 \in (a,b)$, and set $y_0 = f(x_0)$. For $y \in (c,d) \setminus \{y_0\}$, we write

$$\frac{g(y) - g(y_0)}{y - y_0} = \left(\frac{f(g(y)) - f(x_0)}{g(y) - x_0}\right)^{-1}.$$

(Note that $g(y) \neq x_0$ because g is injective.) Since g is continuous, we have $\lim_{y \to y_0} g(y) = x_0$. Therefore

$$\lim_{y \to y_0} \frac{f(g(y)) - f(x_0)}{g(y) - x_0} = f'(x_0).$$

Hence

$$\lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} = \frac{1}{f'(x_0)}.$$

That is, $g'(y_0)$ exists and is equal to $\dfrac{1}{f'(x_0)}$, as desired. ∎

Note: I believe, but have not checked, that further use of the Intermediate Value Theorem can be substituted for the use of compactness in the proof that g is continuous.

Problem 5.3: Let $g \colon \mathbf{R} \to \mathbf{R}$ be a differentiable function such that g' is bounded. Prove that there is $r > 0$ such that the function $f(x) = x + \varepsilon g(x)$ is injective whenever $0 < \varepsilon < r$.

Solution: Set $M = \max(0, \sup_{x \in \mathbf{R}}(-g'(x)))$. Set $r = \frac{1}{M}$. (Take $r = \infty$ if $M = 0$.) Suppose $0 < \varepsilon < r$, and define $f(x) = x + \varepsilon g(x)$ for $x \in \mathbf{R}$. For $x \in \mathbf{R}$, we have

$$f'(x) = 1 + \varepsilon g'(x) = 1 - \varepsilon(-g'(x)) \geq 1 - \varepsilon M > 1 - rM = 0$$

(except that $1 - rM = 1$ if $M = 0$). Thus $f'(x) > 0$ for all x, so the Mean Value Theorem implies that f is strictly increasing. In particular, f is injective. ∎

Note: The problem as stated in Rudin's book is slightly ambiguous: it could be interpreted as asking that $f(x) = x + \varepsilon g(x)$ be injective whenever $-r < \varepsilon < r$. To prove this version, take $M = \sup_{x \in \mathbf{R}} |g'(x)|$, and estimate

$$f'(x) = 1 + \varepsilon g'(x) \geq 1 - |\varepsilon| M > 1 - rM = 0.$$

Problem 5.4: Let $C_0, C_1, \ldots, C_n \in \mathbf{R}$. Suppose

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0.$$

Prove that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$$

has at least one real solution in $(0, 1)$.

Solution (Sketch): Define $f \colon \mathbf{R} \to \mathbf{R}$ by

$$f(x) = C_0 x + \frac{C_1 x^2}{2} + \cdots + \frac{C_{n-1} x^n}{n} + \frac{C_n x^{n+1}}{n+1}$$

for $x \in \mathbf{R}$. Then $f(0) = 0$ (this is trivial) and $f(1) = 0$ (this follows from the hypothesis). Since f is differentiable on all of $\mathbf{R}$, the Mean Value Theorem provides $x \in (0, 1)$ such that $f'(x) = 0$. Since

$$f'(x) = C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1} x^{n-1} + C_n x^n,$$

this is the desired conclusion. ∎

Problem 5.5: Let $f \colon (0, \infty) \to \mathbf{R}$ be differentiable and satisfy $\lim_{x \to \infty} f'(x) = 0$. Prove that $\lim_{x \to \infty}[f(x+1) - f(x)] = 0$.

Solution: Let $\varepsilon > 0$. Choose $M \in \mathbf{R}$ such that $x > M$ implies $|f'(x)| < \varepsilon$. Let $x > M$. By the Mean Value Theorem, there is $z \in (x, x+1)$ such that $f(x+1) - f(x) = f'(z)$. Then $|f(x+1) - f(x)| = |f'(z)| < \varepsilon$. This shows that $\lim_{x\to\infty}[f(x+1) - f(x)] = 0$. ∎

Problem 5.9: Let $f \colon \mathbf{R} \to \mathbf{R}$ be continuous. Assume that $f'(x)$ exists for all $x \neq 0$, and that $\lim_{x\to 0} f'(x) = 3$. Does it follow that $f'(0)$ exists?

Solution: We prove that $f'(0) = 3$. Define $g(x) = x$. Then

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 3$$

by assumption. Therefore Theorem 5.13 of Rudin (L'Hospital's rule) applies to the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{g(x)}$$

(because $f - f(0)$ vanishes at 0 and has derivative f'). Thus

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 3.$$

In particular, $f'(0)$ exists. ∎

Note: It is mathematically bad practice (although it is tolerated in freshman calculus courses) to write

$$\lim_{x\to 0} \frac{f(x) - f(0)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} = 3$$

before checking that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

exists, because the equality

$$\lim_{x\to 0} \frac{f(x) - f(0)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

is only known to hold when the second limit exists.

Problem 5.11: Let f be a real valued function defined on a neighborhood of $x \in \mathbf{R}$. Suppose that $f''(x)$ exists. Prove that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by example that the limit might exist even if $f''(x)$ does not exist.

Solution (Sketch): Check using algebra that

$$\begin{aligned}\lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} &= \lim_{h\to 0} \left(\frac{f'(x+h) - f'(x)}{2h} + \frac{f'(x) - f'(x-h)}{2h} \right) \\ &= f''(x).\end{aligned}$$

Now use Theorem 5.13 of Rudin (L'Hospital's rule) to show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = \lim_{h\to 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x).$$

For the counterexample, take

$$f(t) = \begin{cases} 1 & t > x \\ 0 & t = x \\ -1 & t < x \end{cases}.$$

Then

$$\frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = 0$$

for all $h \neq 0$. This shows that the limit can exist even if f isn't continuous at x. ∎

Note 1: I gave a counterexample for an arbitrary value of x, but it suffices to give one at a single value of x, such as $x = 0$.

Note 2: A legitimate counterexample must be defined at x, since it must satisfy all the hypotheses except for the existence of $f''(x)$.

Note 3: Another choice for the counterexample is

$$f(t) = \begin{cases} (t-x)^2 & t \geq x \\ -(t-x)^2 & t < x \end{cases}.$$

This function is continuous at x, and even has a continuous derivative on $\mathbf{R}$, but $f''(x)$ doesn't exist. One can also construct examples which are continuous nowhere on $\mathbf{R}$.

Note 4: It is tempting to use L'Hospital's rule a second time, to get

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} = \lim_{h \to 0} \frac{f''(x+h) + f''(x-h)}{2}.$$

This reasoning is not valid, since the second limit need not exist. (We do not assume that f'' is continuous.)

Problem 5.13: Let a and c be fixed real numbers, with $c > 0$, and define $f = f_{a,c} \colon [-1, 1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} |x|^a \sin\left(|x|^{-c}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Prove the following statements. (You may use the standard facts about the functions $\sin(x)$ and $\cos(x)$.)

Note: The book has

$$f(x) = \begin{cases} x^a \sin\left(|x|^{-c}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

However, unless a is a rational number with odd denominator, this function will not be defined for $x < 0$.

(a) f is continuous if and only if $a > 0$.

Solution (Sketch): Since $x \mapsto \sin(x)$ is continuous, we need only consider continuity at 0. If $a > 0$, then $\lim_{x \to 0} f(x) = 0$ since $|f(x)| \leq |x|^a$ and $\lim_{x \to 0} |x|^a = 0$.

Now define sequences (x_n) and (y_n) by

$$x_n = \frac{1}{\left[\left(2n + \frac{1}{2}\right)\pi\right]^{1/c}} \quad \text{and} \quad y_n = \frac{1}{\left[\left(2n + \frac{3}{2}\right)\pi\right]^{1/c}}.$$

Note that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$$

and

$$\sin\left(|x_n|^{-c}\right) = 1 \quad \text{and} \quad \sin\left(|y_n|^{-c}\right) = -1$$

for all n. (We will use these sequences in other parts of the problem.)

If now $a = 0$, then

$$\lim_{n\to\infty} f(x_n) = 1 \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = -1,$$

so $\lim_{x\to 0} f(x)$ does not exist, and f is not continuous at 0. If $a < 0$, then

$$\lim_{n\to\infty} f(x_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} f(y_n) = -\infty,$$

with the same result. ∎

Note: Since $f(0)$ is defined to be 0, we actually need only consider $\lim_{n\to\infty} f(x_n)$. The conclusion $\lim_{x\to 0} f(x)$ does not exist is stronger, and will be useful later.

(b) $f'(0)$ exists if and only if $a > 1$.

Solution (Sketch): We test for existence of

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h} = \lim_{h\to 0} f_{a-1,\,c}(h),$$

which we saw in Part (a) exists if and only if $a - 1 > 0$. Moreover (for use below), note that if the limit does exist then it is equal to 0. ∎

(c) f' is bounded if and only if $a \geq 1 + c$.

Solution (Sketch): Boundedness does not depend on $f'(0)$ (or even on whether $f'(0)$ exists). So we use the formula

$$f'(x) = ax^{a-1}\sin\left(x^{-c}\right) + cx^{a-c-1}\cos\left(x^{-c}\right)$$

for $x > 0$, and for $x < 0$ we use

$$f'(x) = -f'(-x) = -f'(|x|) = -a|x|^{a-1}\sin\left(|x|^{-c}\right) - c|x|^{a-c-1}\cos\left(|x|^{-c}\right).$$

If $a - c - 1 \geq 0$, then also $a - 1 \geq 0$ (recall that $c > 0$), and f' is bounded (by $c + a$).

Otherwise, we consider the sequences (w_n) and (z_n) given by

$$w_n = \frac{1}{[2n\pi]^{1/c}} \quad \text{and} \quad y_n = \frac{1}{[(2n+1)\,\pi]^{1/c}}.$$

Since

$$\sin\left(w_n^{-c}\right) = \sin\left(z_n^{-c}\right) = 0$$

and

$$\cos\left(w_n^{-c}\right) = 1 \quad \text{and} \quad \cos\left(z_n^{-c}\right) = -1,$$

arguments as in Part (a) show that

$$\lim_{n\to\infty} f'(w_n) = -\infty \quad \text{and} \quad \lim_{n\to\infty} f'(z_n) = \infty,$$

so f' is not bounded. ∎

(d) f' is continuous on $[-1, 1]$ if and only if $a > 1 + c$.

Solution (Sketch): If $a < 1 + c$, then f' is not bounded on $[-1, 1] \setminus \{0\}$ by Part (c), and therefore can't be the restriction of a continuous function on $[-1, 1]$. If $a = 1+c$, then the sequences of Part (c) satisfy

$$\lim_{n\to\infty} f'(w_n) = -c \quad \text{and} \quad \lim_{n\to\infty} f'(z_n) = c,$$

so again f' can't be the restriction of a continuous function on $[-1, 1]$. If $a > 1 + c$, then also $a > 1$, and $\lim_{x\to 0} f'(x) = 0$ by reasoning similar to that of Part (a). Moreover $f'(0) = 0$ by the extra conclusion in the proof of Part (b). So f' is continuous at 0, hence continuous. ∎

$f''(0)$ exists if and only if $a > 2 + c$.

Solution (Sketch): This is reduced to Part (d) in the same way Part (b) was reduced to Part (a). As there, note also that $f''(0) = 0$ if it exists. ∎

(f) f'' is bounded if and only if $a \geq 2 + 2c$.

Solution (Sketch): For $x \neq 0$, we have

$$\begin{aligned} f''(x) = a(a-1)|x|^{a-2} \sin\left(|x|^{-c}\right) + (2ac - c^2 - c)x^{a-c-2} \cos\left(|x|^{-c}\right) \\ - c^2|x|^{a-2c-2} \sin\left(|x|^{-c}\right). \end{aligned}$$

(One handles the cases $x > 0$ and $x < 0$ separately, as in Part (c), but this time the resulting formula is the same for both cases.) Since $c > 0$, if $a - 2c - 2 \geq 0$ then also $a - c - 2 \geq 0$ and $a - 2 \geq 0$, so f'' is bounded on $[-1, 1] \setminus \{0\}$. For $a - 2c - 2 < 0$, consider

$$f''(x_n) = a(a-1)x_n^{a-2} - c^2 x_n^{a-2c-2}.$$

Since $x_n \to 0$ and $a - 2c - 2 < \min(0, a - 2)$, one checks that the term $-c^2 x_n^{a-2c-2}$ dominates and $f''(x_n) \to -\infty$. So f'' is not bounded. ∎

(g) f'' is continuous on $[-1, 1]$ if and only if $a > 2 + 2c$.

Solution (Sketch): Recall from the extra conclusion in Part (e) that $f''(0) = 0$ if it exists. If $a - 2c - 2 > 0$, then also $a - c - 1 > 0$ and $a - 2 > 0$, so $\lim_{x\to 0} f''(x) = 0$ by a more complicated version of the arguments used in Parts (a) and (d). If $a - 2c - 2 < 0$, then f'' isn't bounded on $[-1, 1] \setminus \{0\}$, so f'' can't be continuous on $[-1, 1]$. If $a - 2c - 2 = 0$, then $a - 2 > 0$. Therefore

$$f''(x_n) = a(a-1)x_n^{a-2} - c^2 x_n^{a-2c-2} = a(a-1)x_n^{a-2} - c^2 \to c^2 \neq 0$$

as $n \to \infty$. So f'' is not continuous at 0. ∎

MATH 414 [514] (PHILLIPS) SOLUTIONS TO HOMEWORK 1

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in.

Problem 5.12: Define $f\colon \mathbf{R} \to \mathbf{R}$ by $f(x) = |x|^3$. Compute $f'(x)$ and $f''(x)$ for all real x. Prove that $f'''(0)$ does not exist.

Solution: To make clear exactly what is being done, we first prove a lemma.

Lemma. Let $(a, b) \subset \mathbf{R}$ be an open interval, and let $f, g\colon \mathbf{R} \to \mathbf{R}$ be functions. Let $c \in (a, b)$, and suppose $f'(c)$ exists. Suppose that there is $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subset (a, b)$ and $g(x) = f(x)$ for all $x \in (c - \varepsilon, c + \varepsilon)$. Then $g'(c)$ exists and $g'(c) = f'(c)$.

Proof: We have

$$\frac{g(c+h) - g(c)}{h} = \frac{f(c+h) - f(c)}{h}$$

for all h with $0 < |h| < \varepsilon$. Therefore

$$\lim_{h \to 0} \frac{g(c+h) - g(c)}{h} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

$\blacksquare$

Now we start the calculation. For $x > 0$, we have $f(x) = x^3$. Therefore, by the lemma, $f'(x) = 3x^2$ and $f''(x) = 6x$. Similarly, for $x < 0$, we have $f(x) = -x^3$, so $f'(x) = -3x^2$ and $f''(x) = -6x$.

The lemma is not useful for $x = 0$. So we calculate directly. For $h \neq 0$, we have

$$\left| \frac{f(h) - f(0)}{h} \right| = \left| \frac{|h|^3}{h} \right| = |h|^2.$$

Therefore

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0.$$

With this result in hand, we can calculate $f''(0)$. For $h \neq 0$, we have $|f'(h)| = 3|h|^2$ (regardless of whether h is positive or negative), so

$$\left| \frac{f'(h) - f'(0)}{h} \right| = \left| \frac{f'(h)}{h} \right| = \left| \frac{3|h|^2}{h} \right| = 3|h|.$$

Therefore

$$f''(0) = \lim_{h \to 0} \frac{f'(h) - f'(0)}{h} = 0.$$

Date: 14 January 2002.

Finally, we consider $f'''(0)$. For $h > 0$, we have

$$\frac{f''(h) - f''(0)}{h} = \frac{6h}{h} = 6.$$

For $h < 0$, we have

$$\frac{f''(h) - f''(0)}{h} = \frac{-6h}{h} = -6.$$

So

$$f'''(0) = \lim_{h \to 0} \frac{f''(h) - f''(0)}{h}$$

does not exist. ∎

Problem 5.22: Let $f \colon \mathbf{R} \to \mathbf{R}$ be a function. We say that $x \in \mathbf{R}$ is a *fixed point* of f if $f(x) = x$.

(a) Suppose that f is differentiable and $f'(t) \neq 1$ for all $t \in \mathbf{R}$. Prove that f has at most one fixed point.

Solution: Suppose f has two distinct fixed points r and s. Without loss of generality $r < s$. Apply the Mean Value Theorem on the interval $[r, s]$, to find $c \in (r, s)$ such that $f(s) - f(r) = f'(c)(s - r)$. Since $f(r) = r$ and $f(s) = s$, and since $s - r \neq 0$, this implies that $f'(c) = 1$. This contradicts the assumption that $f'(t) \neq 1$ for all $t \in \mathbf{R}$. ∎

(b) Define f by $f(t) = t + (1 + e^t)^{-1}$ for $t \in \mathbf{R}$. Prove that $0 < f'(t) < 1$ for all $t \in \mathbf{R}$, but that f has no fixed points. (You may use the standard properties of the exponential function from elementary calculus.)

Solution: If x is a fixed point for f, then

$$x = f(x) = x + \frac{1}{1 + e^x},$$

whence

$$\frac{1}{1 + e^x} = 0.$$

This is obviously impossible.

Using the fact from elementary calculus that the derivative of e^t is e^t, and using the differentiation rules proved in Chapter 5 of Rudin's book, we get

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2}.$$

Since $0 < e^t < 1 + e^t < (1 + e^t)^2$, we have

$$0 < \frac{e^t}{(1 + e^t)^2} < 1$$

for all t, from which it is clear that $0 < f'(t) < 1$ for all t. ∎

(c) Suppose there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all $t \in \mathbf{R}$. Prove that f has a fixed point. Prove that if $x_0 \in \mathbf{R}$, and that if the sequence $(x_n)_{n \in \mathbf{N}}$ is defined recursively by $x_{n+1} = f(x_n)$, then $(x_n)_{n \in \mathbf{N}}$ converges to a fixed point of f.

Solution: It suffices to prove the last statement. First, observe that the version of the Mean Value Theorem in Theorem 5.19 of Rudin's book implies that $|f(s) - f(t)| \leq A|s - t|$ for all s, $t \in \mathbf{R}$.

Now let $x_0 \in \mathbf{R}$ be arbitrary, and define the sequence $(x_n)_{n \in \mathbf{N}}$ recursively as in the statement. Using induction and the estimate above, we get

$$|x_{n+1} - x_n| \leq A^n |x_1 - x_0|$$

for all n. Using the triangle inequality in the first step and the formula for the sum of a geometric series at the second step, we get, for $n \in \mathbf{N}$ and $m \in \mathbf{N}$,

$$|x_{n+m} - x_n| \leq \sum_{k=0}^{m-1} A^{n+k} |x_1 - x_0| = \frac{A^n - A^{n+m}}{1 - A} \cdot |x_1 - x_0| \leq \frac{A^n}{1 - A} \cdot |x_1 - x_0|.$$

With this estimate, we can prove that $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$. Since $0 \leq A < 1$, we have $\lim_{N \to \infty} A^N = 0$, so we can choose N so large that

$$\frac{A^N}{1 - A} \cdot |x_1 - x_0| < \varepsilon.$$

For m, $n \geq N$, we then have

$$|x_m - x_n| \leq \frac{A^{\min(m,n)}}{1 - A} \cdot |x_1 - x_0| < \varepsilon.$$

This shows that $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence.

Since $\mathbf{R}$ is complete, $x = \lim_{n \to \infty} x_n$ exists. It is trivial that $\lim_{n \to \infty} x_{n+1} = x$ as well. Using the continuity of f at x in the first step, we get

$$f(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} x_{n+1} = x,$$

that is, x is a fixed point for f. ∎

We can give an alternate proof of the existence of a fixed point, which is of interest, even though I do not see how to use it to show that the fixed point is the limit of the sequence described in the problem without essentially redoing the other solution.

Partial alternate solution (Sketch): Without loss of generality $f(0) \neq 0$. We consider only the case $f(0) > 0$; the proof for $f(0) < 0$ is similar. Define $b = f(0)(1 - A)^{-1} > 0$, and define $g(x) = f(x) - x$. Then g is continuous and $g(0) > 0$. The version of the Mean Value Theorem in Theorem 5.19 of Rudin's book implies that $|f(b) - f(0)| \leq Ab$. Therefore $f(b) \leq f(0) + Ab$, whence

$$g(b) = f(b) - b \leq f(0) + Ab - b = f(0) + (A - 1)f(0)(1 - A)^{-1} = 0.$$

So the Intermediate Value Theorem provides $x \in [0, b]$ such that $g(x) = 0$, that is, $f(x) = x$. ∎

Problem 5.26: Let $f \colon [a, b] \to \mathbf{R}$ be differentiable, and suppose that $f(a) = 0$ and there is a real number A such that $|f'(x)| \leq A|f(x)|$ for all $x \in [a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Hint: Fix $x_0 \in [a, b]$, and define

$$M_0 = \sup_{x \in [a,b]} |f(x)| \quad \text{and} \quad M_1 = \sup_{x \in [a,b]} |f'(x)|.$$

For $x \in [a, b]$, we then have

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

If $A(x_0 - a) < 1$, it follows that $M_0 = 0$. That is, $f(x) = 0$ for all $x \in [a, x_0]$. Proceed.

Solution (Sketch): Choose numbers x_k with

$$a = x_0 < x_1 < \cdots < x_n = b$$

and such that $A(x_k - x_{k-1}) < 1$ for $1 \leq k \leq n$. We prove by induction on k that $f(x) = 0$ for all $x \in [a, x_k]$. This is immediate for $k = 0$. So suppose it is known for some k; we prove it for $k + 1$.

Define

$$M_0 = \sup_{x \in [x_{k-1},\, x_k]} |f(x)| \quad \text{and} \quad M_1 = \sup_{x \in [x_{k-1},\, x_k]} |f'(x)|.$$

The hypotheses imply that $M_1 \leq AM_0$. For $x \in [x_{k-1},\, x_k]$, we have (using the version of the Mean Value Theorem in Theorem 5.19 of Rudin's book at the second step)

$$|f(x)| = |f(x) - f(x_k)| \leq M_1(x - x_k) \leq AM_0(x - x_k).$$

In this inequality, take the supremum over all $x \in [x_{k-1},\, x_k]$, getting

$$M_0 = \sup_{x \in [x_{k-1},\, x_k]} |f(x)| \leq AM_0 \sup_{x \in [x_{k-1},\, x_k]} (x - x_k) = A(x_{k+1} - x_k)M_0.$$

Since $0 \leq A(x_{k+1} - x_k) < 1$ and $M_0 \geq 0$, this can only happen if $M_0 = 0$. Therefore $f(x) = 0$ for all $x \in [x_{k-1},\, x_k]$, and hence for all $x \in [a, x_k]$. This completes the induction step, and the proof. ∎

Problem 6.2: Let $f \colon [a, b] \to \mathbf{R}$ be continuous and nonnegative. Assume that $\int_a^b f = 0$. Prove that $f = 0$.

Solution (Sketch): Assume that $f \neq 0$. Choose $x_0 \in [a, b]$ such that $f(x_0) > 0$. By continuity, there is $\delta > 0$ such that $f(x) > \frac{1}{2} f(x_0)$ for $|x - x_0| < \delta$. Let

$$I = [a, b] \cap \left[x_0 - \tfrac{1}{2}\delta,\, x_0 + \tfrac{1}{2}\delta\right],$$

which is an interval of positive length, say l. It is now easy to construct a partition P such that $L(P, f) \geq l \cdot \frac{1}{2} f(x_0) > 0$. ∎

Alternate solution (Sketch): Let x_0, δ, I, and l be as above. Let χ_I be the characteristic function of I. Check that χ_I is integrable, and $\int_a^b \chi_I = l$, by choosing a partition P of $[a, b]$ such that $L(P, \chi_I) = U(P, \chi_I) = l$. Then $g = \frac{1}{2} f(x_0)\chi_I$ is integrable, and $\int_a^b g = \frac{1}{2} f(x_0) l$. Since $f \geq \chi_I$, we have $\int_a^b f \geq \int_a^b \chi_I > 0$. ∎

Problem 6.4: Let $a, b \in \mathbf{R}$ with $a < b$. Define $f \colon [a, b] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 0 & x \in \mathbf{R} \setminus \mathbf{Q} \\ 1 & x \in \mathbf{Q} \end{cases}.$$

Prove that f is not Riemann integrable on $[a, b]$.

Solution (Sketch): For every partition $P = (x_0, x_1, \ldots, x_n)$ of $[a, b]$, every subinterval $[x_{j-1},\, x_j]$ contains both rational and irrational numbers. Therefore $L(P, f) = 0$ and $U(P, f) = b - a$ for every P. ∎

Problem A: Let X be a complete metric space, and let $f\colon X \to X$ be a function. Suppose that there is a constant k such that $d(f(x),\, f(y)) \leq k d(x,y)$ for all $x,\, y \in X$.

(1) Prove that f is uniformly continuous.

Solution (Sketch): For $\varepsilon > 0$, take $\delta = k^{-1}\varepsilon$. ∎

(2) Suppose that $k < 1$. Prove that f has a unique fixed point, that is, there is a unique $x \in X$ such that $f(x) = x$.

Solution (Sketch): If $r,\, s \in X$ are fixed points, then $d(r,s) = d(f(r),\, f(s)) \leq k d(r,s)$. Since $0 \leq k < 1$, this implies that $d(r,s) = 0$, that is, $r = s$. So f has at most one fixed point.

The proof that f has a fixed point is essentially the same as the proof of Problem 5.22 (c). Choose any $x_0 \in X$. Define a sequence $(x_n)_{n \in \mathbf{N}}$ recursively by $x_{n+1} = f(x_n)$ for $n \geq 1$. The same calculations as there show that

$$d(x_n,\, x_{n+1}) \leq k^n d(x_0, x_1)$$

for all n, that

$$d(x_n,\, x_{n+m}) \leq \frac{k^n}{1-k} \cdot d(x_0, x_1)$$

for $n \in \mathbf{N}$ and $m \in \mathbf{N}$, and hence (using $k < 1$) that $(x_n)_{n \in \mathbf{N}}$ is a Cauchy sequence. Since X is assumed to be complete, $x = \lim_{n \to \infty} x_n$ exists. Using the continuity of f at x, it then follows, as there, that x is a fixed point for f. ∎

(3) Show that the conclusion in Part (2) need not hold if X is not complete.

Solution: Take $X = \mathbf{R} \setminus \{0\}$, with the restriction of the usual metric on $\mathbf{R}$, and define $f\colon X \to X$ by $f(x) = \frac{1}{2}x$ for all $x \in X$. Then $d(f(x),\, f(y)) = \frac{1}{2} d(x,y)$ for all $x,\, y \in X$, but clearly f has no fixed point.

It follows from Part (2) that X is not complete. (This is also easy to check directly: $\left(\frac{1}{n}\right)_{n \in \mathbf{N} \setminus \{0\}}$ is a Cauchy sequence which does not converge.) ∎

MATH 414 [514] (PHILLIPS) SOLUTIONS TO HOMEWORK 2

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A "solution (nearly complete)" is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 6.7: Let $f\colon (0,1] \to \mathbf{R}$ be a function, and suppose that $f|_{[c,1]}$ is Riemann integrable for every $c \in (0,1)$. Define

$$\int_0^1 f = \lim_{c \to 0^+} \int_c^1 f$$

if this limit exists and is finite.

(a) If f is the restriction to $(0,1]$ of a Riemann integrable function on $[0,1]$, show that the new definition agrees with the old one.

Solution: The function $F(x) = \int_0^x f$ is continuous, so that

$$\lim_{c \to 0^+} \int_c^1 f = \lim_{c \to 0^+} \left(\int_0^1 f - F(c) \right) = \int_0^1 f - F(0) = \int_0^1 f.$$

∎

(b) Give an example of a function $f\colon (0,1] \to \mathbf{R}$ such that $\lim_{c \to 0^+} \int_c^1 f$ exists but $\lim_{c \to 0^+} \int_c^1 |f|$ does not exist.

Solution (nearly complete): We will use the fact that the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges but does not converge absolutely.

Define $f\colon (0,1] \to \mathbf{R}$ by setting $f(x) = \frac{1}{n}(-1)^n \cdot 2^n$ for $x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-1}} \right]$ and $n \in \mathbf{N}$. Then $h(c) = \int_c^1 |f|$ increases as c decreases to 0, and is is easy to check that

$$h\left(\frac{1}{2^n} \right) = \sum_{k=1}^{n} \frac{2^k}{k} \cdot \frac{1}{2^k} = \sum_{k=1}^{n} \frac{1}{k}.$$

Therefore $\lim_{c \to 0^+} \int_c^1 |f| = \infty$.

Now we prove that

$$\lim_{c \to 0^+} \int_c^1 f = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

Date: 23 Jan. 2002.

(Note that it is clear that this should be true, but that proving it requires a little care.) First, one checks that

$$\int_{\frac{1}{2^n}}^{1} f = \sum_{k=1}^{n} \frac{(-1)^k}{k}.$$

Second, note that the series converges by the Alternating Series Test. Let its sum be a. Let $\varepsilon > 0$, and choose N such that if $n \geq N$ than

$$\left| a - \sum_{k=1}^{n} \frac{(-1)^k}{k} \right| < \varepsilon.$$

Take $\delta = \frac{1}{2^N}$. Let $0 < c < \delta$, and choose n such that $c \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right)$. Note that $n - 1 \geq N$. If n is even, then

$$\int_{\frac{1}{2^n}}^{1} f \geq \int_{c}^{1} f \geq \int_{\frac{1}{2^{n-1}}}^{1} f.$$

Since $n - 1 \geq N$, we have

$$\left| a - \int_{\frac{1}{2^n}}^{1} f \right| = \left| a - \sum_{k=1}^{n} \frac{(-1)^k}{k} \right| < \varepsilon \quad \text{and} \quad \left| a - \int_{\frac{1}{2^{n-1}}}^{1} f \right| = \left| a - \sum_{k=1}^{n-1} \frac{(-1)^k}{k} \right| < \varepsilon.$$

Therefore $\left| a - \int_c^1 f \right| < \varepsilon$. The case n odd is handled similarly. Thus, whenever $0 < c < \delta$ we have $\left| a - \int_c^1 f \right| < \varepsilon$. This shows that $\lim_{c \to 0^+} \int_c^1 f = a$, as desired. ∎

Problem 6.8: Let $f \colon [a, \infty) \to \mathbf{R}$ be a function, and suppose that $f|_{[a,b]}$ is Riemann integrable for every $b > a$. Define

$$\int_a^{\infty} f = \lim_{b \to \infty} \int_a^b f$$

if this limit exists and is finite. In this case, we say that $\int_a^{\infty} f$ *converges.*

Assume that f is nonnegative and nonincreasing on $[1, \infty)$. Prove that $\int_1^{\infty} f$ converges if and only if $\sum_{n=1}^{\infty} f(n)$ converges.

Solution (nearly complete): First assume that $\sum_{n=1}^{\infty} f(n)$ converges. Since f is nonnegative, $b \mapsto \int_a^b f$ is nondecreasing. It is therefore easy to check (as with sequences) that $\lim_{b \to \infty} \int_1^b f$ exists if and only if $b \mapsto \int_1^b f$ is bounded.

Define $g \colon [1, \infty) \to \mathbf{R}$ by $g(x) = f(n)$ for $x \in [n, n+1)$ and $n \in \mathbf{N}$. Since f is nondecreasing, it follows that $g \geq f$. So if $b \geq 1$ we can choose $n \in N$ with $n + 1 \geq b$, giving

$$\int_1^b f \leq \int_1^n f \leq \int_1^{n+1} g = \sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{\infty} f(k).$$

This shows that $b \mapsto \int_1^b f$ is bounded.

Now assume that $\int_1^{\infty} f$ converges. This clearly implies that the sequence $n \mapsto \int_1^n f$ is bounded (in fact, converges). Define $h \colon [1, \infty) \to \mathbf{R}$ by $h(x) = f(n+1)$ for $x \in [n, n+1)$ and $n \in \mathbf{N}$. Since f is nondecreasing, it follows that $h \leq f$. Therefore

$$\sum_{k=1}^{n} f(k) = f(1) + \sum_{k=2}^{n} f(k) = f(1) + \int_1^n h \leq f(1) + \int_1^n f.$$

This shows that the partial sums $\sum_{k=1}^{n} f(k)$ form a bounded sequence. Since the terms are nonnegative, it follows that $\sum_{n=1}^{\infty} f(n)$ converges. ∎

Problem 6.10: Let $p,\ q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$.

(a) For $u,\ v \in [0, \infty)$ prove that

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q},$$

with equality if and only if $u^p = v^q$.

Comment: The solution below was found by first letting $x = u^p$ and $y = v^q$, so that the inequality is equivalent to $x^{1/p} y^{1/q} = \frac{x}{p} + \frac{y}{q}$ for $x,\ y \in [0, \infty)$; then dividing by y and noting that $\frac{1}{q} - 1 = -\frac{1}{p}$, so that the inequality is equivalent to

$$\left(\frac{x}{y}\right)^{1/p} \leq \left(\frac{1}{p}\right)\left(\frac{x}{y}\right) + \frac{1}{q};$$

then letting $\alpha = \frac{x}{y}$.

Solution: We first claim that $\alpha^{1/p} \leq \frac{1}{p} \cdot \alpha + \frac{1}{q}$ for all $\alpha \geq 0$, with equality if and only if $\alpha = 1$. That we have equality for $\alpha = 1$ is clear. Set

$$f(\alpha) = \alpha^{1/p} - \frac{1}{p} \cdot \alpha - \frac{1}{q}$$

for $\alpha \geq 0$. Then

$$f'(\alpha) = \frac{1}{p} \cdot \alpha^{1/p-1} - \frac{1}{p} = \frac{1}{p}\left(\alpha^{1/p-1} - 1\right)$$

for all $\alpha > 0$. Fix $\alpha > 1$. The Mean Value Theorem provides $\beta \in (1, \alpha)$ such that

$$f(\alpha) - f(1) = f'(\beta)(\alpha - 1).$$

Since $\beta > 1$ and $\frac{1}{p} - 1 < 0$, we have $f'(\beta) < 0$. Since $f(1) = 0$, we therefore get $f(\alpha) < 0$. This proves the claim for $\alpha > 1$. For $0 \leq \alpha < 1$, a similar argument works, using the fact that $f'(\beta) > 0$ for $0 < \beta < 1$.

Now we prove the statement of the problem. If $v^q = 0$, the inequality reduces to $0 \leq \frac{1}{p} u^p$. Clearly this is true for all $u \geq 0$, and equality holds if and only if $u^p = 0$.

Otherwise, set $\alpha = u^p \cdot v^{-q}$. Applying the claim, we get

$$\left(\frac{u^p}{v^q}\right)^{1/p} \leq \left(\frac{1}{p}\right)\left(\frac{u^p}{v^q}\right) + \frac{1}{q}$$

for all $u \geq 0$ and $v > 0$, with equality if and only if $u^p = v^q$. Since $v^q > 0$, we can multiply by v^q, getting

$$uv^{q-q/p} \leq \frac{u^p}{p} + \frac{v^q}{q}$$

for all $u \geq 0$ and $v > 0$, with equality if and only if $u^p = v^q$. Now the relationship $\frac{1}{p} + \frac{1}{q} = 1$ implies that $q - \frac{q}{p} = 1$, completing the proof. ∎

(b) (with the function α taken to be $\alpha(x) = x$). Let $\alpha \colon [a, b] \to \mathbf{R}$ be a nondecreasing function. Let $f,\ g \colon [a, b] \to \mathbf{R}$ be nonnegative functions which are Riemann integrable, and such that

$$\int_a^b f^p = 1 \quad \text{and} \quad \int_a^b g^q = 1.$$

Prove that

$$\int_a^b fg \leq 1.$$

Solution: The function fg is Riemann integrable, by Theorem 6.13 (a) of Rudin.

The inequality in Part (a) implies that $fg \leq \frac{1}{p} f^p + \frac{1}{q} g^q$ on $[a, b]$. Therefore

$$\int_a^b fg \leq \int_a^b \left(\tfrac{1}{p} f^p + \tfrac{1}{q} g^q \right) = \tfrac{1}{p} \int_a^b f^p + \tfrac{1}{q} \int_a^b g^q = \tfrac{1}{p} + \tfrac{1}{q} = 1.$$

∎

(c) (with the function α taken to be $\alpha(x) = x$). Let $f, g \colon [a, b] \to \mathbf{C}$ be Riemann integrable complex valued functions. Prove that

$$\left| \int_a^b fg \right| \leq \left(\int_a^b |f|^p \right)^{1/p} \left(\int_a^b |g|^q \right)^{1/q}.$$

Solution: First, we note that the hypotheses imply that fg, $|f|^p$, and $|g|^q$ are Riemann integrable. For the second two, use Theorems 6.25 and 6.11 of Rudin. For the first, use the decompositions of f and g into real and imaginary parts, and then use Theorems 6.13 (a) and 6.12 (a).

Next observe that

$$\left| \int_a^b fg \right| \leq \int_a^b |f| \cdot |g|.$$

Therefore it is enough to show that

$$\int_a^b |f| \cdot |g| \leq \left(\int_a^b |f|^p \right)^{1/p} \left(\int_a^b |g|^q \right)^{1/q}.$$

Let

$$\alpha = \left(\int_a^b |f|^p \right)^{1/p} \quad \text{and} \quad \beta = \left(\int_a^b |g|^q \right)^{1/q},$$

and apply Part (b) to the functions $f_0 = \alpha^{-1} |f|$ and $g_0 = \beta^{-1} |g|$. ∎

(d) Prove that the result in Part (c) also holds for the improper Riemann integrals defined in Problems 6.7 and 6.8. (Only actually do the case of the improper integral defined in Problems 6.7.)

Solution (nearly complete): We assume that

$$\int_0^1 |f|^p \quad \text{and} \quad \int_0^1 |g|^q$$

exist in the extended sense defined in Problem 6.7, and we need to prove that $\int_0^1 fg$ exists and that

$$\left| \int_0^1 fg \right| \leq \left(\int_0^1 |f|^p \right)^{1/p} \left(\int_0^1 |g|^q \right)^{1/q}.$$

Once existence is proved, the inequality follows immediately from Part (c) by taking the limit as $c \to 0^+$.

Define

$$I(c) = \int_c^1 fg$$

for $c \in (0,1)$. We claim that for every $\varepsilon > 0$ there is $\delta > 0$ such that whenever $c_1, c_2 \in (0, \delta)$, then $|I(c_1) - I(c_2)| < \varepsilon$. This will imply that $\lim_{c \to 0^+} I(c)$ exists. Indeed, restricting to $c = \frac{1}{n}$ with $n \in \mathbf{N} \cap [2, \infty)$ gives a Cauchy sequence in $\mathbf{C}$, which necessarily has a limit α, and it is easy to show that $\lim_{c \to 0^+} I(c) = \alpha$ as c runs through arbitrary values too.

To prove the claim, let $\varepsilon > 0$. Choose $\delta > 0$ so small that if $0 < c < \delta$ then

$$\left| \int_0^1 |f|^p - \int_c^1 |f|^p \right| < \left(\sqrt{\varepsilon} \right)^p$$

and

$$\left| \int_0^1 |g|^q - \int_c^1 |g|^q \right| < \left(\sqrt{\varepsilon} \right)^q.$$

Since $|f|^p$ and $|g|^q$ are nonnegative, it follows that whenever $0 < c_1 < c_2 < \delta$, then

$$\int_{c_1}^{c_2} |f|^p < \left(\sqrt{\varepsilon} \right)^p \quad \text{and} \quad \int_{c_1}^{c_2} |g|^q < \left(\sqrt{\varepsilon} \right)^q.$$

Now apply Part (c) to get

$$\begin{aligned} |I(c_2) - I(c_1)| = \left| \int_{c_1}^{c_2} fg \right| &\leq \left(\int_{c_1}^{c_2} |f|^p \right)^{1/p} \left(\int_{c_1}^{c_2} |g|^q \right)^{1/q} \\ &< \left(\sqrt{\varepsilon} \right) \left(\sqrt{\varepsilon} \right) = \varepsilon. \end{aligned}$$

This takes care of the case $c_1 < c_2$. The reverse case is obtained easily from this one, and the case $c_1 = c_2$ is trivial. ∎

Problem 6.11 (with the function α taken to be $\alpha(x) = x$): For any Riemann integrable function u, define

$$\|u\|_2 = \left(\int_a^b |u|^2 \right)^{1/2}.$$

Prove that if f, g, and h are all Riemann integrable, then

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2.$$

Hint: Use the Schwarz inequality, as in the proof of Theorem 1.37 of Rudin.

Comment: As I read the problem, it is intended that the functions be real. However, the complex case is actually more important, so I will give the proof in that case.

Also, one can use Problem 6.10 (c) at an appropriate stage, but that isn't actually necessary.

Solution (nearly complete): It is enough to prove that

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2.$$

Start by defining, for f, g Riemann integrable,

$$\langle f, g \rangle = \int_a^b f\overline{g}.$$

Note that, by an argument similar to the one given in the proof of Problem 6.10 (c), $f\overline{g}$ is in fact Riemann integrable. It is now immediate to verify the following facts:

(1) The set $R([a, b])$ of Riemann integrable complex functions is a vector space (with the pointwise operations).
(2) For every $g \in R([a, b])$, the function $f \mapsto \langle f, g \rangle$ is linear.
(3) $\langle g, f \rangle = \overline{\langle f, g \rangle}$ for all f, $g \in R([a, b])$.
(4) $\langle f, f \rangle \geq 0$ for all $f \in R([a, b])$.
(5) $\|f\|_2^2 = \langle f, f \rangle$ for all $f \in R([a, b])$.

We next show that

$$|\langle f, g \rangle| \leq \|f\|_2 \|g\|_2$$

for all f, $g \in R([a, b])$. Note that the proof uses only properties (1) through (5) above.

Choose $\alpha \in \mathbf{C}$ with $|af| = 1$ and $\alpha \langle f, g \rangle = |\langle f, g \rangle|$. It is clear that

$$\|\alpha f\|_2 = \|f\|_2 \quad \text{and} \quad \langle \alpha f, g \rangle = |\langle f, g \rangle|.$$

Therefore it suffices to prove the inequality with αf in place of f. That is, without loss of generality we may assume $\langle f, g \rangle \geq 0$.

For $t \in \mathbf{R}$, define

$$p(t) = \langle f + tg,\ f + tg \rangle.$$

Using Properties (2) and (3), and the fact that $\langle f, g \rangle$ is real, we may expand this as

$$p(t) = \langle f, f \rangle + 2t\langle f, g \rangle + t^2 \langle g, g \rangle.$$

Thus, p is a quadratic polynomial. Property (4) implies that its discriminant is nonpositive, that is,

$$4\langle f, g \rangle^2 - 4\langle f, f \rangle \langle g, g \rangle \leq 0.$$

This implies the desired inequality.

Now we prove the inequality

$$\|f + g\|_2 \leq \|f\|_2 + \|g\|_2.$$

Again, the proof depends only on properties (1) through (5) above, and uses nothing about integrals except what went into proving those properties.

We write (using Properties (2) and (3) to expand at the second step, and the previously proved inequality at the third step):

$$\begin{aligned} \|f + g\|_2^2 &= \langle f + g,\ f + g \rangle = \langle f, f \rangle + 2\mathrm{Re}(\langle f, g \rangle) + \langle g, g \rangle \\ &\leq \|f\|_2^2 + \|f\|_2 \|g\|_2 + \|g\|_2^2 = (\|f\|_2 + \|g\|_2)^2. \end{aligned}$$

Now take square roots. ∎

MATH 414 [514] (PHILLIPS) SOLUTIONS TO HOMEWORK 3

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A "solution (nearly complete)" is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 6.15: Let $f\colon [a,b] \to \mathbf{R}$ be a continuously differentiable function satisfying $f(a) = f(b) = 0$ and $\int_a^b f(x)^2\,dx = 1$. Prove that

$$\int_a^b x f(x) f'(x)\,dx = -\tfrac{1}{2} \quad \text{and} \quad \left(\int_a^b [f'(x)]^2\,dx\right)\left(\int_a^b x^2 f(x)^2\,dx\right) > \tfrac{1}{4}.$$

Comments: (1) I do not use the notation "$f^2(x)$", because it could reasonably be interpreted as $f(f(x))$.

(2) This result is related to the Heisenberg Uncertainly Principle in quantum mechanics.

Solution (Sketch): For the equation, use integration by parts (Theorem 6.22 of Rudin),

$$\int_a^b u(x)v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)v(x)\,dx,$$

with

$$v'(x) = f(x)f'(x), \quad v(x) = \tfrac{1}{2}f(x)^2, \quad u(x) = x, \quad \text{and} \quad u'(x) = 1.$$

The inequality

$$\left(\int_a^b [f'(x)]^2\,dx\right)\left(\int_a^b x^2 f(x)^2\,dx\right) \geq \tfrac{1}{4}$$

now follows from the Hölder inequality (Problem 6.10 of Rudin) with $p = q = 2$. (Also see the solution given for Problem 6.11.)

To get strict inequality requires more work. First, we need to know that if

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 = \left(\int_a^b f(x)^2\,dx\right)\left(\int_a^b g(x)^2\,dx\right),$$

which in the notation used in the solution to Problem 6.11 is

$$\langle f, g\rangle^2 = \langle f, f\rangle \cdot \langle g, g\rangle$$

(for real valued f and g), then f and g are linearly dependent, that is, either one of them is zero or there is a constant λ such that $f = \lambda g$. This is proved below. Given this, and taking $g(x) = x f'(x)$, the hypotheses and the first equation proved above

Date: 28 Jan. 2002.

immediately rule out $f = 0$ or $g = 0$. Thus, if the strict inequality fails above, there is a nonzero constant $\lambda \in \mathbf{R}$ such that

$$f'(x) = \lambda x f(x)$$

for all $x \in [a, b]$.

The next step is motivated by the observation that the last equation is a differential equation for f whose solutions have the form

$$f(x) = 0 \quad \text{and} \quad f(x) = \exp\left(\tfrac{1}{2}\lambda x^2 + \gamma\right)$$

for all x (for some constant γ). Let

$$S = \{x \in [a, b] \colon f(x) \neq 0\} \subset [a, b].$$

Then S is open in $[a, b]$ because f is continuous. We know that $S \neq \varnothing$, because $\int_a^b f(x)^2 \, dx = 1$. So choose $x_0 \in S$. Let

$$c = \inf\{t \in [a, x_0] \colon [t, x_0] \subset S\}.$$

Then $c < x_0$ and $(c, x_0] \subset S$. We show that $[c, x_0] \subset S$. On the interval $(c, x_0]$, since $f(x) \neq 0$, we can rewrite the differential equation above as

$$\frac{f'(x)}{f(x)} = \lambda x.$$

Integrating, we get that there is a constant $\gamma \in \mathbf{R}$ such that

$$\log(f(x)) = \tfrac{1}{2}\lambda x^2 + \gamma$$

for $x \in (c, x_0]$. (We have implicitly used Theorem 5.11 (b) of Rudin here: if $f' = 0$ on an interval, then f is constant.) Rewrite the above equation as

$$f(x) = \exp\left(\tfrac{1}{2}\lambda x^2 + \gamma\right)$$

for $x \in (c, x_0]$. Since f is continuous at c, we let x approach zero from above to get

$$f(c) = \exp\left(\tfrac{1}{2}\lambda c^2 + \gamma\right) \neq 0.$$

So $c \in S$, whence $[c, x_0] \subset S$.

Since S is open in $[a, b]$, it is easy to see that this implies that $c = a$. (Otherwise, c is a limit point of $[a, b] \setminus S$ but $c \notin [a, b] \setminus S$.) We saw above that $f(c) \neq 0$. However, this now contradicts the assumption that $f(a) = 0$, thus proving the required strict inequality.

It remains to prove the criterion for $\langle f, g \rangle^2 = \langle f, f \rangle \cdot \langle g, g \rangle$. Referring to the solution to Problem 6.11 of Rudin, and taking $\alpha(x) = x$ there, we see that equality implies that the polynomial

$$p(t) = \langle f, f \rangle + 2t\langle f, g \rangle + t^2 \langle g, g \rangle$$

used there has a real root, which implies the existence of t such that

$$0 = \langle f + tg, \, f + tg \rangle = \int_a^b (f + tg)^2.$$

Since f and g are assumed real and continuous here, and since we are using the Riemann integral, this implies that $f + tg = 0$. ∎

Problem 7.1: Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Solution: Let E be a set, let $f_n \colon E \to \mathbf{C}$ be bounded functions, and assume that $f_n \to f$ uniformly. By the definition of uniform convergence, there is N such that if $n \geq N$ then $\|f_n - f\|_\infty < 1$. Set

$$M = \max\left(\|f_1\|_\infty, \, \|f_2\|_\infty, \ldots, \|f_{N-1}\|_\infty, \, \|f_N\|_\infty + 1\right).$$

Then obviously $\|f_n\|_\infty \leq M$ for $1 \leq n < N$. For $n \geq N$, we have

$$\|f_n\|_\infty \leq \|f_N\|_\infty + \|f_n - f\|_\infty < \|f_N\|_\infty + 1 \leq M.$$

This shows that $\sup_{n \in \mathbf{N}} \|f_n\|_\infty \leq M$, that is, that (f_n) is uniformly bounded. ∎

Problem 7.2: Let $f_n \colon E \to \mathbf{C}$ and $g_n \colon E \to \mathbf{C}$ be uniformly convergent sequences of functions. Prove that $f_n + g_n$ converges uniformly. If, in addition, the functions f_n and g_n are all bounded, prove that $f_n g_n$ converges uniformly.

Solution: Let f and g be the functions (assumed to exist) such that $f_n \to f$ uniformly and $g_n \to g$ uniformly.

For the sum, we show that $f_n + g_n \to f + g$ uniformly. Let $\varepsilon > 0$, choose $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies

$$\sup_{x \in E} |f_n(x) - f(x)| < \tfrac{1}{2}\varepsilon,$$

and choose $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies

$$\sup_{x \in E} |g_n(x) - g(x)| < \tfrac{1}{2}\varepsilon.$$

Set $N = \max(N_1, N_2)$. Let $n \geq N$ and let $x \in E$. Then

$$\begin{aligned} |(f_n + g_n)(x) - (f+g)(x)| &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| \\ &\leq \sup_{y \in E} |f_n(y) - f(y)| + \sup_{x \in E} |g_n(y) - g(y)|. \end{aligned}$$

Therefore $n \geq N$ implies

$$\begin{aligned} &\sup_{x \in E} |(f_n + g_n)(x) - (f+g)(x)| \\ &\qquad \leq \sup_{y \in E} |f_n(y) - f(y)| + \sup_{y \in E} |g_n(y) - g(y)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

This shows that $f_n + g_n \to f + g$ uniformly.

(Here is an alternate way of presenting the computation. Having chosen N as above, for any $n \geq N$ we have

$$\begin{aligned} \sup_{x \in E} |(f_n + g_n)(x) - (f+g)(x)| &\leq \sup_{x \in E} \left[|f_n(x) - f(x)| + |g_n(x) - g(x)|\right] \\ &\leq \sup_{x \in E} |f_n(x) - f(x)| + \sup_{x \in E} |g_n(x) - g(x)| \\ &< \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Note that the second inequality in this computation in general is not an equality.)

Now consider the uniform convergence of the products. Since the functions are bounded, we will use norm notation.

First note that, for any bounded functions f and g, we have $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$. Indeed, for any $x \in E$ we have

$$|f(x)g(x)| = |f(x)| \cdot |g(x)| \leq \|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty,$$

and we take the supremum over $x \in E$ to get the result.

Now let $\varepsilon > 0$. Choose $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies

$$\|f_n(x) - f(x)\|_\infty < \frac{\varepsilon}{2(\|g\|_\infty + 1)},$$

and choose $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies

$$\|g_n(x) - g(x)\|_\infty < \min\left(1, \ \frac{\varepsilon}{2\|f\|_\infty + 1}\right).$$

Set $N = \max(N_1, N_2)$. First, note that $n \geq N$ implies

$$\|g_n\|_\infty \leq \|g\|_\infty + \|g_n - g\|_\infty < \|g\|_\infty + 1.$$

Therefore, using the inequality in the previous paragraph and the triangle inequality for $\|\cdot\|_\infty$ (proved in class), we see that $n \geq N$ implies

$$\begin{aligned}\|f_n g_n - fg\|_\infty &\leq \|f_n g_n - f g_n\|_\infty + \|f g_n - fg\|_\infty \\ &\leq \|f_n - f\|_\infty \|g_n\|_\infty + \|f\|_\infty \|g_n - f\|_\infty \\ &\leq \frac{\varepsilon}{2(\|g\|_\infty + 1)} \cdot (\|g\|_\infty + 1) + \|f\|_\infty \cdot \frac{\varepsilon}{2\|f\|_\infty + 1} < \varepsilon.\end{aligned}$$

This shows that $f_n g_n \to fg$ uniformly. ∎

Problem 7.3: Construct a set E and two uniformly convergent sequences $f_n \colon E \to \mathbf{C}$ and $g_n \colon E \to \mathbf{C}$ of functions such that $f_n g_n$ does not converge uniformly. (Of course, $f_n g_n$ converges pointwise.)

Solution (Sketch): Set $E = \mathbf{R}$, let f_n be the constant function $f_n(x) = \frac{1}{n}$ for all $x \in \mathbf{R}$, and let $f(x) = 0$ for all x. Define g by $g(x) = x$ for all $x \in \mathbf{R}$, and let $g_n = g$ for all n. It is obvious that $f_n \to f$ uniformly and $g_n \to g$ uniformly. Also $f_n g_n \to 0$ pointwise. However, $f_n(n) g_n(n) = 1$ for all n. ∎

Problem 7.4: Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$

For which values of x does the series converge absolutely? On which closed intervals does the series converge uniformly? On which closed intervals does the series fail to converge uniformly? Is the sum continuous wherever the series converges? Is the sum bounded?

Comment: The problem was written in the book with the word "interval". Because of Rudin's strange terminology, I believe "closed interval" is what was meant.

Solution (Sketch):

The situation will be clear with two lemmas. Define

$$f_n(x) = \frac{1}{1 + n^2 x}$$

for all x and n.

Lemma 1. For every $\varepsilon > 0$, we have

$$\sum_{n=1}^{\infty} \|f_n|_{[\varepsilon, \infty)}\|_\infty < \infty.$$

Proof: Observe that if $x \geq \varepsilon$, then

$$0 < f_n(x) = \frac{1}{1 + n^2 x} \leq \frac{1}{1 + n^2 \varepsilon} < \frac{1}{n^2 \varepsilon},$$

so that

$$\| f_n|_{[\varepsilon, \infty)} \|_\infty \leq \frac{1}{n^2 \varepsilon},$$

and

$$\sum_{n=1}^{\infty} \| f_n|_{[\varepsilon, \infty)} \|_\infty \leq \frac{1}{\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

∎

Lemma 2. For every $\varepsilon > 0$, there is $N \in \mathbf{N}$ such that

$$\sum_{n=N}^{\infty} \| f_n|_{(-\infty, -\varepsilon]} \|_\infty < \infty.$$

Proof: Choose $N \in \mathbf{N}$ such that $N^2 > 2\varepsilon^{-1}$. For $x \geq \varepsilon$ and $n \geq N$ we then have

$$n^2 x - 1 \geq n^2 \varepsilon - 1 \geq \frac{2n^2}{N^2} - 1 \geq \frac{n^2}{N^2}.$$

Thus

$$f_n(-x) = -\frac{1}{n^2 x - 1}$$

satisfies

$$0 > f_n(-x) > -\frac{N^2}{n^2}.$$

So

$$\| f_n|_{(-\infty, -\varepsilon]} \|_\infty \leq \frac{N^2}{n^2},$$

and

$$\sum_{n=N}^{\infty} \| f_n|_{(-\infty, -\varepsilon]} \|_\infty \leq N^2 \sum_{n=N}^{\infty} \frac{1}{n^2} < \infty.$$

∎

It is easy to check that $\sum_{n=1}^{\infty} f_n(x)$ fails to converge at $x = 0$, and the series doesn't converge at $x = -\frac{1}{n^2}$ for any $n \in \mathbf{N}$ because one of the functions is not defined there. Thus, the series does not converge even pointwise on any closed interval containing any point of the set

$$S = \{0\} \cup \left\{ -1, \, -\tfrac{1}{4}, \, -\tfrac{1}{9}, \, \dots \right\}.$$

For any closed interval not containing any points of S (even an infinite closed interval), the two lemmas above make it clear that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a bounded continuous function. In particular, the sum is continuous off S.

The sum of the series is, however, not bounded on its domain. From the above, the domain is clearly $\mathbf{R} \setminus S$. We show carefully that if U is any neighborhood of any point $x_0 \in S$, then the sum is not bounded on $U \cap (\mathbf{R} \setminus S)$.

We first consider the case $x_0 = -\frac{1}{n^2}$ for some n. Choose r with

$$0 < r < \frac{1}{n^2} - \frac{1}{(n+1)^2},$$

and set

$$E = U \cap \left[-\frac{1}{n^2} - r, \, \frac{1}{n^2} + r \right] \quad \text{and} \quad E_0 = E \cap (\mathbf{R} \setminus S) = E \setminus \left\{ -\frac{1}{n^2} \right\}.$$

Since $E_0 \subset U \cap (\mathbf{R} \setminus S)$, it suffices to show that the sum is not bounded on E_0. Set $\varepsilon = \frac{1}{(n+1)^2}$, and choose N as in Lemma 2 for this value of ε. We alse require that $N > n$. Then $\sum_{k=N}^{\infty} f_k$ converges to a bounded function on E, by the conclusion of Lemma 2. Moreover, each f_k, for $1 \leq k < N$ but $k \neq n$, is continuous on the compact set $\left[-\frac{1}{n^2} - r, \, \frac{1}{n^2} + r \right]$, hence bounded on its subset E. It follows that $\sum_{k \neq n} f_k$ is bounded on E, hence on E_0. However, f_n is not bounded on E_0, since $-\frac{1}{n^2}$ is a limit point of E_0 and

$$\lim_{x \to \left(-\frac{1}{n^2}\right)^-} f_n(x) = -\infty \quad \text{and} \quad \lim_{x \to \left(-\frac{1}{n^2}\right)^+} f_n(x) = \infty.$$

It follows that

$$\sum_{k=1}^{\infty} f_k = f_n + \sum_{k \neq n} f_k$$

is the sum of a bounded function on E_0 and an unbounded function on E_0, hence is not bounded on E_0.

Now consider the case $x_0 = 0$. Then U also contains $-\frac{1}{n^2}$ for some n. Therefore the sum is not bounded on $U \cap (\mathbf{R} \setminus S)$ by the previous case. ∎

We point out that, when showing that $\sum_{k=1}^{\infty} f_k$ is not bounded on E_0, it is not sufficient to simply show that f_n is unbounded there while all other f_k are bounded there. Here is an example of a series $\sum_{k=0}^{\infty} g_k$ which converges on $\mathbf{R} \setminus \{0\}$ to a bounded function, with g_0 unbounded but with g_k bounded for all $k \neq 0$. Set

$$g_0(x) = -\frac{1}{x^2}$$

and, for $k \geq 1$,

$$h_k(x) = \min\left(k - 1, \, \frac{1}{x^2} \right) \quad \text{and} \quad g_k(x) = h_{k+1}(x) - h_k(x).$$

Then g_0 is not bounded, $|g_k(x)| \leq 1$ for all $k \geq 1$ and all $x \in \mathbf{R} \setminus \{0\}$, but $\sum_{k=0}^{\infty} g_k(x) = 0$ for all $x \in \mathbf{R} \setminus \{0\}$.

MATH 414 [514] (PHILLIPS) SOLUTIONS TO HOMEWORK 4

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A "solution (nearly complete)" is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 6.12 (with $\alpha(x) = x$): Let $\| \cdot \|_2$ be as in Problem 6.11. Let $f \colon [a, b] \to \mathbf{R}$ be Riemann integrable, and let $\varepsilon > 0$. Prove that there is a continuous function $g \colon [a, b] \to \mathbf{R}$ such that $\|f - g\|_2 < \varepsilon$.

Hint: For a suitable partition $P = (x_0, x_1, \ldots, x_n)$ of $[a, b]$, define g on $[x_{j-1}, x_j]$ by

$$g(t) = \frac{x_j - t}{x_j - x_{j-1}} f(x_{j-1}) + \frac{t - x_{j-1}}{x_j - x_{j-1}} f(x_j).$$

Solution (nearly complete): Fix f as in the problem. For any partition P of $[a, b]$, let g_P be the function defined in the hint. Check that g_P is well defined and continuous.

We now choose a suitable partition. Let $M = \sup_{x \in [a,b]} |f(x)|$. Choose $\rho > 0$ so small that

$$\rho^2 (b - a) < \tfrac{1}{2} \varepsilon^2.$$

Choose $\delta > 0$ so small that

$$4 M^2 \rho^{-1} \delta < \tfrac{1}{2} \varepsilon^2.$$

Choose a partition P as in the hint such that

$$U(P, f) - L(P, f) < \delta.$$

Let

$$M_j = \sup_{x \in [x_{j-1}, x_j]} f(x) \quad \text{and} \quad m_j = \inf_{x \in [x_{j-1}, x_j]} f(x).$$

On $[x_{j-1}, x_j]$, we then have both

$$m_j \leq f \leq M_j \quad \text{and} \quad m_j \leq g_P \leq M_j.$$

Therefore

$$\|f - g_P\|_2^2 = \int_a^b |f - g_P|^2 \leq \sum_{j=1}^{n} (M_j - m_j)^2 (x_j - x_{j-1}).$$

Let

$$S = \{ j \in \{1, 2, \ldots, n\} \colon M_j - m_j \geq \rho \}.$$

Date: 17 Feb. 2002.

We have

$$\delta > U(P, f) - L(P, f) = \sum_{j=1}^{n} (M_j - m_j)(x_j - x_{j-1}) \geq \sum_{j \in S} (M_j - m_j)(x_j - x_{j-1}) \geq \rho \sum_{j \in S} (x_j - x_{j-1}),$$

whence

$$\sum_{j \in S} (x_j - x_{j-1}) \leq \rho^{-1}\delta$$

and

$$\sum_{j \in S} (M_j - m_j)^2 (x_j - x_{j-1}) \leq (2M)^2 \rho^{-1}\delta < \tfrac{1}{2}\varepsilon^2.$$

For $j \notin S$, we have $M_j - m_j < \rho$. Therefore

$$\sum_{j \notin S} (M_j - m_j)^2 (x_j - x_{j-1}) \leq \rho^2 \sum_{j \notin S} (x_j - x_{j-1}) \leq \rho^2 \sum_{j=1}^{n} (x_j - x_{j-1}) = \rho^2 (b - a) < \tfrac{1}{2}\varepsilon^2.$$

So

$$\|f - g_P\|_2^2 \leq \sum_{j=1}^{n} (M_j - m_j)^2 (x_j - x_{j-1}) < \tfrac{1}{2}\varepsilon^2 + \tfrac{1}{2}\varepsilon^2 = \varepsilon^2,$$

whence $\|f - g_P\|_2 < \varepsilon$. ∎

Problem 7.5: For $x \in (0, 1)$ and $n \in \mathbf{N}$, define

$$f_n(x) = \begin{cases} 0 & 0 < x < \frac{1}{n+1} \\ \sin^2\left(\frac{\pi}{x}\right) & \frac{1}{n+1} \leq x \leq \frac{1}{n} \\ 0 & \frac{1}{n} < x < 1 \end{cases}.$$

Prove that there is a continuous function f such that $f_n \to f$ pointwise, but that f_n does not converge uniformly to f. Use the series $\sum_{n=1}^{\infty} f_n$ to show that absolute convergence of a series at every point does not imply uniform convergence of the series.

Solution (nearly complete): All functions appearing here are bounded, so we may work in the space $C_{\mathrm{b}}((0, 1))$ with the metric obtained from the norm $\|\cdot\|_\infty$.

It is easy to see that $f_n(x) \to 0$ for all x. However, $\|f_n\|_\infty = 1$ for all n, so f_n does not converge uniformly to 0.

The series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely for all x, because for each fixed x only one term is nonzero. The series $\sum_{n=1}^{\infty} f_n$ does not converge uniformly: letting s_n be the n-th partial sum $s_n = \sum_{k=1}^{n} f_k$, we get $\|s_n - s_{n-1}\|_\infty = \|f_n\|_\infty = 1$ for all n, so the sequence (s_n) is not Cauchy in the metric which determines uniform convergence. ∎

Problem 7.6: Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^2 + n}{n^2}$$

converges uniformly on every bounded interval in $\mathbf{R}$, but does not converge absolutely for any value of x.

Solution (nearly complete): Rewrite:

$$\sum_{n=1}^{\infty}(-1)^n \cdot \frac{x^2+n}{n^2} = x^2 \sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n^2} + \sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n}.$$

For each fixed x, everything on the right converges, so this is legitimate; moreover, it follows that the original series converges for all x.

For each fixed x, the first series on the right converges absolutely but the second does not. Since the difference of absolutely convergent series is absolutely convergent, the series on the left does not converge absolutely.

Now fix a bounded interval $I \subset \mathbf{R}$. We want to show that the original series converges uniformly on I. For $x \in I$ and $n \in \mathbf{N}$, set

$$f_n(x) = \sum_{k=1}^{n}(-1)^k \cdot \frac{x^2+k}{k^2}, \quad g_n(x) = \sum_{k=1}^{n}(-1)^k \cdot \frac{x^2}{k^2}, \quad \text{and} \quad h_n(x) = \sum_{k=1}^{n}(-1)^k \cdot \frac{1}{k}.$$

All functions appearing here are bounded, so we may work in the space $C_{\mathrm{b}}(I)$ with the metric obtained from the norm $\|\cdot\|_\infty$. It is easy to see that the sequence (h_n) converges in this metric to the function h with constant value $\sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n}$, and (because I is bounded) that the sequence (g_n) converges in this metric to the function $g(x) = x^2 \sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n^2}$. Therefore $f_n = g_n + h_n$ converges in this metric to the function $f = g + h$. (The proof is the same as the proof of the convergence of the sum of convergent sequences of complex numbers.) ∎

Alternate proof of the failure of absolute convergence: Fix $x \in \mathbf{R}$. Then

$$\left|(-1)^n \cdot \frac{x^2+n}{n^2}\right| = \frac{x^2+n}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the series

$$\sum_{n=1}^{\infty}\left|(-1)^n \cdot \frac{x^2+n}{n^2}\right|$$

diverges by the comparison test. ∎

Alternate proof of uniform convergence: Define f_n, g_n, h_n, f, g, and h as above. Let $\varepsilon > 0$. Choose M so large that $I \subset [-M, M]$. Using the convergence of $\sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n^2}$, choose N_1 such that if $n \geq N_1$ then

$$\left|\sum_{k=1}^{\infty}(-1)^k \cdot \frac{1}{k^2} - \sum_{k=1}^{n}(-1)^k \cdot \frac{1}{k^2}\right| < \frac{\varepsilon}{2M}.$$

Using the convergence of $\sum_{n=1}^{\infty}(-1)^n \cdot \frac{1}{n}$, choose N_2 such that if $n \geq N_2$ then

$$\left|\sum_{k=1}^{\infty}(-1)^k \cdot \frac{1}{k} - \sum_{k=1}^{n}(-1)^k \cdot \frac{1}{k}\right| < \frac{\varepsilon}{2}.$$

Then $n \geq \max(N_1, N_2)$ implies

$$\|g_n - g\|_\infty \leq \tfrac{1}{2}\varepsilon \quad \text{and} \quad \|h_n - h\|_\infty \leq \tfrac{1}{2}\varepsilon,$$

from which it is easy to conclude that

$$\|f_n - f\|_\infty \leq \varepsilon.$$

∎

Second alternate proof of uniform convergence: We are going to use the Alternating Series Test on a series of functions, but of course this needs justification. We begin with an observation on the Alternating Series Test for a series of numbers $\sum_{k=1}^{\infty}(-1)^k a_k$, where

$$a_1 > a_2 > a_3 > \cdots > 0 \quad \text{and} \quad \lim_{n\to\infty} a_n = 0.$$

We claim (see below) that for any N we have

$$\left| \sum_{k=1}^{\infty}(-1)^k a_k - \sum_{k=1}^{N}(-1)^k a_k \right| \leq a_{N+1}.$$

Assuming this for the moment, consider an interval of the form $[-M, M]$ for $M \in (0, \infty)$. (All bounded intervals are contained in intervals of this form.) For each $x \in [-M, M]$, we have

$$\frac{x^2}{1^2} > \frac{x^2}{2^2} > \frac{x^2}{3^2} > \cdots > 0$$

and

$$\frac{1}{1^2} > \frac{2}{2^2} > \frac{2}{3^2} > \cdots > 0,$$

whence

$$\frac{x^2+1}{1^2} > \frac{x^2+2}{2^2} > \frac{x^2+3}{3^2} > \cdots > 0.$$

Also it is easy to see that

$$\lim_{n\to\infty} \frac{x^2+n}{n^2} = 0.$$

The Alternating Series Test therefore gives pointwise convergence of the series. Moreover, applying the claim at the first step, we get, for $x \in [-M, M]$ and $N \in \mathbf{N}$,

$$\left| \sum_{k=1}^{\infty}(-1)^k \cdot \frac{x^2+k}{k^2} - \sum_{k=1}^{N}(-1)^k \cdot \frac{x^2+k}{k^2} \right| \leq \frac{x^2+N+1}{(N+1)^2} \leq \frac{M^2+N+1}{(N+1)^2}.$$

The right hand side is independent of $x \in [-M, M]$ and has limit 0 as $n \to \infty$, so that the series converges uniformly on $[-M, M]$.

It remains to prove the claim above. Let $s_n = \sum_{k=1}^{n}(-1)^k a_k$ be the partial sums. We can write an even partial sum s_{2n} as

$$s_{2n} = (a_2 - a_1) + (a_4 - a_3) + \cdots + (a_{2n} - a_{2n-1}).$$

Using the analogous formula for s_{2n+2} and the fact that $a_{2n+2} - a_{2n+1} < 0$, we get $s_{2n+2} < s_{2n}$. That is,

$$s_0 > s_2 > s_4 > \cdots .$$

Therefore

$$\sum_{k=1}^{\infty}(-1)^k a_k < s_{2n}$$

for all n. A similar argument on the odd partial sums, using $a_{2n} - a_{2n+1} > 0$, gives

$$s_1 < s_3 < s_5 < \cdots \quad \text{and} \quad \sum_{k=1}^{\infty}(-1)^k a_k > s_{2n+1}$$

In particular, if $N = 2n$ is even then

$$\left| s_N - \sum_{k=1}^{\infty}(-1)^k a_k \right| = s_{2n} - \sum_{k=1}^{\infty}(-1)^k a_k < s_{2n} - s_{2n+1} = a_{2n+1} = a_{N+1},$$

while if $N = 2n$ is odd then

$$\left| s_N - \sum_{k=1}^{\infty}(-1)^k a_k \right| = \sum_{k=1}^{\infty}(-1)^k a_k - s_{2n+1} < s_{2n+2} - s_{2n+1} = a_{2n+2} = a_{N+1}.$$

This proves the claim, and completes the proof of uniform convergence. ∎

Problem 7.7: For $x \in (0,1)$ and $n \in \mathbf{N}$, define

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that there is a function f such that $f_n \to f$ uniformly, and that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ holds for $x \neq 0$ but not for $x = 0$.

Solution: We show that $\|f_n\|_\infty \leq n^{-1/2}$, which will imply that $f_n \to 0$ uniformly. For $|x| \leq n^{-1/2}$, we have $1 + nx^2 \geq 1$, so

$$|f_n(x)| \leq |x| \leq n^{-1/2}.$$

For $|x| \geq n^{-1/2}$, we can write

$$f_n(x) = \frac{x^{-1}}{x^{-2} + n}.$$

Since $x^{-2} + n \geq n$, this gives

$$|f_n(x)| \leq \frac{x^{-1}}{n} \leq \frac{n^{1/2}}{n} = n^{-1/2}.$$

So $|f_n(x)| \leq n^{-1/2}$ for all x, proving the claim. This proves the first statement with $f = 0$.

Now we consider the second part. We calculate:

$$f_n'(x) = \frac{1 - nx^2}{(1 + nx^2)^2}.$$

Clearly $f_n'(0) = 1$ for all n, so $\lim_{n\to\infty} f_n'(0) = 1 \neq f'(0)$. Otherwise, we can rewrite

$$f_n'(x) = \frac{\frac{1}{n^2} - \frac{1}{n}x^2}{\frac{1}{n^2} + \frac{1}{n} \cdot 2x^2 + x^4},$$

and for each fixed $x \neq 0$ we clearly have $\lim_{n\to\infty} f_n'(x) = 0 = f'(0)$. ∎

Alternate solution for the first part (Sketch): Using the formula for f_n' in the solution to the second part above, and standard calculus methods, show that $f_n(x)$ has a global maximum at $n^{-1/2}$, with value $\frac{1}{2}n^{-1/2}$, and a global minimum at $-n^{-1/2}$, with value $-\frac{1}{2}n^{-1/2}$. (Note that this requires consideration of $\lim_{x\to\infty} f_n(x)$ and $\lim_{x\to-\infty} f_n(x)$ as well as those numbers x for which $f_n'(x) = 0$. Forgetting to do so is already a blunder in elementary calculus.) This shows that $\|f_n\|_\infty = \frac{1}{2}n^{-1/2}$, so clearly $\lim_{n\to\infty} \|f_n\|_\infty = 0$. ∎

Second alternate solution for the first part (Sketch): The inequality $(a-b)^2 \geq 0$ implies that $2ab \leq a^2 + b^2$ for all $a, b \in \mathbf{R}$. Apply this result with $a = 1$ and $b = |x|\sqrt{n}$ to get

$$2|x|\sqrt{n} \leq 1 + \left(|x|\sqrt{n}\right)^2 = 1 + nx^2$$

for all $n \in \mathbf{N}$ and $x \in \mathbf{R}$. It follows that

$$|f_n(x)| = \frac{|x|}{1+nx^2} \leq \frac{1}{2\sqrt{n}}$$

for all $n \in \mathbf{N}$ and $x \in \mathbf{R}$. Therefore $f_n \to 0$ uniformly on $\mathbf{R}$. ∎

Problem 7.9: Let X be a metric space, let (f_n) be a sequence of continuous functions from X to $\mathbf{C}$, and let f be a function from X to $\mathbf{C}$. Show that if f_n converges uniformly to f, then for every sequence (x_n) in X with limit x, one has $\lim_{n\to\infty} f_n(x_n) = f(x)$. Is the converse true?

Solution: We prove the direct statement. Let (x_n) be a sequence in X with $x_n \to x$, and let $\varepsilon > 0$. Note that f is continuous because it is the uniform limit of continuous functions. So there is $\delta > 0$ such that whenever $y \in X$ satisfies $d(y, x) < \delta$, then $|f(y) - f(x)| < \frac{1}{2}\varepsilon$. Choose $N_1 \in \mathbf{N}$ so that if $n \geq N_1$ then for all $x \in X$ we have $|f_n(x) - f(x)| < \frac{1}{2}\varepsilon$. Choose $N_2 \in \mathbf{N}$ so that if $n \geq N_2$ then $d(x_n, x) < \delta$. Take $n = \max(N_1, N_2)$. Then for $n \geq N$ we have $d(x_n, x) < \delta$ so that $|f(x_n) - f(x)| < \frac{1}{2}\varepsilon$. Therefore

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

The converse is false in general. Take $X = \mathbf{R}$, $f_n(x) = \frac{1}{n}x$, and $f = 0$. Then $f_n(x) \to f(x)$ for all x, but the convergence is not uniform. Let (x_n) be a sequence in $\mathbf{R}$ such that $\lim_{n\to\infty} x_n = x$. Then (x_n) is bounded, so there is M such that (x_n) and x are all contained in $[-M, M]$. Clearly $f_n|_{[-M, M]}$ converges uniformly to $f|_{[-M, M]}$. The direct statement above therefore gives $\lim_{n\to\infty} f_n(x_n) = f(x)$. (This can also be checked directly:

$$|f_n(x_n) - f(x)| = \left|\tfrac{1}{n}x_n\right| \leq \tfrac{1}{n}M,$$

whence $\lim_{n\to\infty} f_n(x_n) = f(x)$.)

The converse is true if X is compact. (This was, strictly speaking, not asked for.) We first show that the hypotheses imply that f is continuous. Let $x \in X$, and let (x_n) be a sequence in X with $x_n \to x$; we show that $f(x_n) \to f(x)$. Inductively choose $k(1) < k(2) < \cdots$ such that $|f_{k(n)}(x_n) - f(x_n)| < \frac{1}{n}$. Define a new sequence (y_n) in X by taking $y_{k(n)} = x_n$ and $y_n = x$ for $n \not\in \{k(1), k(2), \dots\}$. Clearly $\lim_{n\to\infty} y_n = x$. By hypothesis, we therefore have $\lim_{n\to\infty} f_n(y_n) = f(x)$. Now (x_n) is a subsequence of (y_n), and $(f_{k(n)}(x_n))$ is the corresponding subsequence of $(f_n(y_n))$. Therefore $f_{k(n)}(x_n) \to f(x)$. Since $|f_{k(n)}(x_n) - f(x_n)| < \frac{1}{n}$, it follows easily that $f(x_n) \to f(x)$. So f is continuous.

Now we prove that the convergence is uniform. Suppose not. Then there is $\varepsilon > 0$ such that for all $N \in \mathbf{N}$ there is $n \geq N$ with $\|f_n - f\|_\infty \geq \varepsilon$. Accordingly, there are $k(1) < k(2) < \cdots$ and $x_n \in X$ such that $|f_{k(n)}(x_n) - f(x_n)| > \frac{1}{2}\varepsilon$. Passing to a subsequence, we may assume that there is $x \in X$ such that $x_n \to x$. (This is where we use compactness.) As before, define a new sequence (y_n) in X by taking $y_{k(n)} =$

x_n and $y_n = x$ for $n \notin \{k(1), k(2), \dots\}$. Clearly $\lim_{n\to\infty} y_n = x$. By hypothesis, we therefore have $\lim_{n\to\infty} f_n(y_n) = f(x)$. Also, $\lim_{n\to\infty} f(y_n) = f(x)$ because f is continuous. However, since $y_{k(n)} = x_n$, we have $|f_{k(n)}(y_{k(n)}) - f(y_{k(n)})| > \frac{1}{2}\varepsilon$ for all n. So $(f_{k(n)}(y_{k(n)}))$ and $(f(y_{k(n)}))$ can't have the same limit. This is a contradiction, and therefore we must have $f_n \to f$ uniformly. ∎

Problem 7.10: For $x \in \mathbf{R}$, let (x) denote the fractional part of x, that is, $(x) = x - n$ for $x \in [n, n+1)$. For $x \in \mathbf{R}$, define

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}.$$

Show that the set of points at which f is discontinuous is a countable dense subset of $\mathbf{R}$. Show that nevertheless f is Riemann integrable on every closed bounded interval in $\mathbf{R}$.

Solution (nearly complete): Set

$$f_n(x) = \frac{(nx)}{n^2}.$$

First observe that $\|f_n\|_\infty = \frac{1}{n^2}$, so that $\sum_{n=1}^{\infty} f_n$ converges uniformly by the Weierstrass test.

Lemma. Let X be a metric space, let $f_n \colon X \to \mathbf{C}$ be functions, and suppose that $f_n \to f$ uniformly. Let $x_0 \in X$. If all f_n are continuous at x_0, then f is continuous at x_0.

The proof can be obtained from Theorem 7.11 of Rudin, but we can give a direct proof which imitates the standard proof that the uniform limit of continuous functions is continuous.

Proof: Let $\varepsilon > 0$. Choose $N \in \mathbf{N}$ so that if $n \geq N$ then for all $x \in X$ we have $|f_n(x) - f(x)| < \frac{1}{3}\varepsilon$. Use the continuity of f_N at x_0 to choose $\delta > 0$ such that whenever $x \in X$ satisfies $d(x, x_0) < \delta$, then $|f_N(x) - f_N(x_0)| < \frac{1}{3}\varepsilon$. For all $x \in X$ such that $d(x, x_0) < \delta$, we then have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon. \end{aligned}$$

∎

Our particular functions f_n are continuous at every point of $\mathbf{R} \setminus \mathbf{Q}$, so the lemma implies that f is continuous on $\mathbf{R} \setminus \mathbf{Q}$.

We now show that f is not continuous at any point of $\mathbf{Q}$. So let $x \in \mathbf{Q}$. Write $x = \frac{p}{q}$ in lowest terms. (If $x \in \mathbf{Z}$, take $q = 1$.) One checks that $\lim_{t\to x^+} f_q(t) < \lim_{t\to x^-} f_q(t)$. So define

$$\alpha = \lim_{t\to x^-} f_q(t) - \lim_{t\to x^+} f_q(t) > 0.$$

Choose $N \in \mathbf{N}$ so large that $n \geq N$ implies

$$\sum_{k=n}^{\infty} \frac{1}{k^2} < \tfrac{1}{2}\alpha.$$

For any k, we clearly have $\lim_{t \to x^+} f_k(t) \leq \lim_{t \to x^-} f_k(t)$. Set $n = \max(N, q)$. Then

$$\lim_{t \to x^-} \sum_{k=1}^{n} f_k(t) - \lim_{t \to x^+} \sum_{k=1}^{n} f_k(t) \geq \alpha.$$

Moreover,

$$0 \leq \sum_{k=n+1}^{\infty} f_k(t) \leq \sum_{k=n+1}^{\infty} \frac{1}{k^2} < \tfrac{1}{2}\alpha$$

for all t, whence

$$\liminf_{t \to x^-} \sum_{k=n+1}^{\infty} f_k(t) - \limsup_{t \to x^+} \sum_{k=n+1}^{\infty} f_k(t) \geq -\tfrac{1}{2}\alpha.$$

(We use the obvious definitions of

$$\liminf_{t \to x^-} g(t) \quad \text{and} \quad \limsup_{t \to x^+} g(t)$$

here. Note, however, that these have not been formally defined, so if this is used in a solution that is turned in, it needs justification. To avoid this, simply evaluate everything at particular sequences converging to x from below and above, such as $s_m = x - \frac{1}{m}$ and $t_m = x + \frac{1}{m}$.) Therefore

$$\begin{aligned}
&\liminf_{t \to x^-} f(t) - \limsup_{t \to x^+} f(t) \\
&\qquad = \left(\lim_{t \to x^-} \sum_{k=1}^{n} f_k(t) - \lim_{t \to x^+} \sum_{k=1}^{n} f_k(t) \right) \\
&\qquad\qquad + \left(\liminf_{t \to x^-} \sum_{k=n+1}^{\infty} f_k(t) - \limsup_{t \to x^+} \sum_{k=n+1}^{\infty} f_k(t) \right) \\
&\qquad \geq \tfrac{1}{2}\alpha > 0.
\end{aligned}$$

This is clearly incompatible with continuity of f at x.

The Riemann integrability of f on closed bounded intervals is immediate from Theorem 7.16 of Rudin. ∎

MATH 414 [514] (PHILLIPS) SOLUTIONS TO HOMEWORK 5

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A "solution (nearly complete)" is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 7.16: Let K be a compact metric space, and let (f_n) be a uniformly equicontinuous sequence of functions in $C(X)$. Suppose that (f_n) converges pointwise. Prove that (f_n) converges uniformly.

Solution: We show that (f_n) is a Cauchy sequence in $C(K)$. Since $C(K)$ is complete (Theorem 7.15 of Rudin), this will imply that (f_n) converges uniformly.

Let $\varepsilon > 0$. Choose $\delta > 0$ such that for all $n \in \mathbf{N}$, and for all $x, y \in K$ such that $d(x, y) < \delta$, we have $|f_n(x) - f_n(y)| < \frac{1}{4}\varepsilon$. Since K is compact, there exist $x_1, x_2, \ldots, x_k \in K$ such that the open δ-balls $N_\delta(x_1)$, $N_\delta(x_2)$, $\ldots$, $N_\delta(x_k)$ cover K. Since each sequence $(f_n(x_j))$ converges, there is $N \in \mathbf{N}$ such that whenever $m, n \geq N$ and $1 \leq j \leq k$ we have $|f_m(x_j) - f_n(x_j)| < \frac{1}{4}\varepsilon$. Now let $x \in K$ be arbitrary. Choose j such that $x \in N_\delta(x_j)$. For $m, n \geq N$ we then have

$$\begin{aligned}|f_m(x) - f_n(x)| &\leq |f_m(x) - f_m(x_j)| + |f_m(x_j) - f_n(x_j)| + |f_n(x_j) - f_n(x)| \\ &< \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon = \tfrac{3}{4}\varepsilon.\end{aligned}$$

Since x is arbitrary, this shows that $\|f_m - f_n\|_\infty \leq \frac{3}{4}\varepsilon < \varepsilon$. ∎

Alternate solution: Let $f(x) = \lim_{n\to\infty} f_n(x)$. We show that (f_n) converges uniformly to f.

Let $\varepsilon > 0$. Choose $\delta > 0$ such that for all $n \in \mathbf{N}$, and for all $x, y \in K$ such that $d(x, y) < \delta$, we have $|f_n(x) - f_n(y)| < \frac{1}{4}\varepsilon$. Letting $n \to \infty$, we find that $|f(x) - f(y)| \leq \frac{1}{4}\varepsilon$ whenever $x, y \in K$ satisfy $d(x, y) < \delta$. Since K is compact, there exist $x_1, x_2, \ldots, x_k \in K$ such that the open δ-balls $N_\delta(x_1)$, $N_\delta(x_2)$, $\ldots$, $N_\delta(x_k)$ cover K. Choose $N \in \mathbf{N}$ such that whenever $n \geq N$ and $1 \leq j \leq k$, then $|f_n(x_j) - f(x_j)| < \frac{1}{4}\varepsilon$. Now let $x \in K$ be arbitrary. Choose j such that $x \in N_\delta(x_j)$. For $n \geq N$ we then have

$$\begin{aligned}|f_n(x) - f(x)| &\leq |f_n(x) - f_n(x_j)| + |f_n(x_j) - f(x_j)| + |f(x_j) - f(x)| \\ &< \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon + \tfrac{1}{4}\varepsilon = \tfrac{3}{4}\varepsilon.\end{aligned}$$

Since x is arbitrary, this shows that $\|f_n - f\|_\infty \leq \frac{3}{4}\varepsilon < \varepsilon$. ∎

Problem 7.20: Let $f \in C([0, 1])$, and suppose that

$$\int_0^1 f(x)x^n \, dx = 0$$

Date: 11 Feb. 2002.

for all $n \in \mathbf{N}$. Prove that $f = 0$.

Hint: The hypotheses imply that

$$\int_0^1 f(x)p(x)\,dx = 0$$

for every polynomial p. Use the Weierstrass Theorem to show that

$$\int_0^1 |f(x)|^2\,dx = 0.$$

Note: As is clear from Rudin's hint, which suggested showing that

$$\int_0^1 f(x)^2\,dx = 0,$$

Rudin tacitly assumed that f is real, while I have assumed that f is complex.

Solution (Sketch): It is clear that the hypotheses imply that

$$\int_0^1 f(x)p(x)\,dx = 0$$

for every polynomial p. Use the Weierstrass Theorem to choose polynomials p_n such that $\lim_{n\to\infty} \|p_n - \overline{f}\|_\infty = 0$. Using Theorem 7.16 of Rudin, we get

$$\int_0^1 |f(x)|^2\,dx = \int_0^1 f\overline{f} = \lim_{n\to\infty} \int_0^1 f p_n = 0.$$

Since f is continuous, it is easy to show that this implies $f = 0$. (See Problem 6.2 of Rudin, solved in an earlier solution set.) ∎

Problem 7.21: Let $S^1 = \{z \in \mathbf{C} \colon |z| = 1\}$ be the unit circle in the complex plane. Let $A \subset C(S^1)$ be the subalgebra consisting of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$$

for $\theta \in \mathbf{R}$, with arbitrary $N \in \mathbf{N}$ and $c_0, c_1, \ldots, c_N \in \mathbf{C}$. Then A separates the points of S^1 and vanishes at no point of S^1, but A is not dense in $C(S^1)$.

Hint: For every $f \in A$ we have

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\,d\theta = 0,$$

and this is also true for every $f \in \overline{A}$.

Solution (Sketch): It is easy to check that A is a subalgebra. (Anyway, the problem is worded in such a way that you are to assume this is true.) The subalgebra A separates the points of S^1 because it contains the function $f(z) = z$, and vanishes nowhere because it contains the constant function 1. The verification of the hint for $f \in A$ is direct from the computation, valid for any $n \geq 0$,

$$\int_0^{2\pi} e^{in\theta} \cdot e^{i\theta}\,d\theta = \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = \frac{1}{i(n+1)}\left(e^{i(n+1)\cdot 2\pi} - e^{i(n+1)\cdot 0}\right) = 0.$$

The case $f \in \overline{A}$ now follows from Theorem 7.16 of Rudin.

To complete the proof, it suffices to find $f \in C(S^1)$ such that

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} \, d\theta \neq 0.$$

The function $f(z) = \overline{z}$ (that is, $f(e^{i\theta}) = e^{-i\theta}$) will do. ∎

Problem 7.22 (with $\alpha(x) = x$ for all x): Let $f \colon [a, b] \to \mathbf{C}$ be bounded and Riemann integrable on $[a, b]$. Prove that there are polynomials p_n such that

$$\lim_{n \to \infty} \int_a^b |f - p_n|^2 = 0.$$

(Compare with Problem 6.12 of Rudin.)

Solution (Sketch): In the notation of Problem 6.11 of Rudin (see an earlier solution set), the conclusion is that $\lim_{n\to\infty} \|f - p_n\|_2^2 = 0$. We prove the equivalent statement $\lim_{n\to\infty} \|f - p_n\|_2 = 0$.

Taking real and imaginary parts, without loss of generality f is real. (This reduction uses the triangle inequality for $\| \cdot \|_2$, Problem 6.11 of Rudin, which is solved in an earlier solution set.) Further, it is enough to find, for every $\varepsilon > 0$, a polynomial p such that $\|f - p\|_2 < \varepsilon$.

Use Problem 6.12 of Rudin (solved in an earlier solution set) to find $g \in C_{\mathbf{R}}([0, 1])$ such that $\|f - g\|_2 < \frac{1}{2}\varepsilon$. Use the Weierstrass theorem to find a polynomial p such that

$$\|g - p\|_\infty < \frac{\varepsilon}{\sqrt{2[b - a]}}.$$

Then check that $\|g - p\|_2 < \frac{1}{2}\varepsilon$, so that triangle inequality for $\| \cdot \|_2$, Problem 6.11 of Rudin, implies that $\|f - p\|_2 < \varepsilon$. ∎

Problem 7.24: Let X be a metric space, with metric d. Fix $a \in X$. For each $p \in X$, define $f_p \colon X \to \mathbf{C}$ by

$$f_p(x) = d(x, p) - d(x, a)$$

for $x \in X$. Prove that $|f_p(x)| \leq d(a, p)$ for all $x \in X$, that $f \in C_{\mathrm{b}}(X)$, and that $\|f_p - f_q\| = d(p, q)$ for all p, $q \in X$.

Define $\Phi \colon X \to C_{\mathrm{b}}(X)$ by $\Phi(p) = f_p$ for $p \in X$. Then Φ is an isometry, that is, a distance preserving function, from X to a subset of $C_{\mathrm{b}}(X)$.

Let Y be the closure of $\Phi(X)$ in $C_{\mathrm{b}}(X)$. Prove that Y is complete. Conclude that X is isometric to a dense subset of a complete metric space.

Solution (Sketch): That $|f_p(x)| \leq d(a, p)$ follows from two applications of the triangle inequality for d, namely

$$d(x, p) \leq d(x, a) + d(a, p) \quad \text{and} \quad d(x, a) \leq d(x, p) + d(a, p).$$

This shows that f is bounded.

To prove continuity of f_p, we observe that the map $x \mapsto d(x, w)$ is continuous for every $w \in X$. Indeed, an argument using two applications of the triangle inequality and very similar to the above shows that $|d(x, w) - d(y, w)| \leq d(x, y)$ for all x, $y \in X$. This implies that $x \mapsto d(x, w)$ is continuous (in fact, Lipschitz with constant 1).

The inequality $|d(x, w) - d(y, w)| \leq d(x, y)$ for all $w, x, y \in X$ gives

$$|f_x(w) - f_y(w)| = |d(w, x) - d(w, y)| \leq d(x, y)$$

for all $w \in X$, whence $\|f_x - f_y\| \leq d(x, y)$. Also,

$$f_x(y) - f_y(y) = d(y, x) - d(y, a) - [d(y, y) - d(y, a)] = d(x, y),$$

whence $\|f_x - f_y\| \geq d(x, y)$. It follows that $\|f_x - f_y\| = d(x, y)$, which says exactly that Φ is isometric.

Define $Y = \overline{\Phi(X)}$. The space $C_{\mathrm{b}}(X)$ is complete by Theorem 7.15 of Rudin, so Y is complete by the discussion after Definition 3.12 of Rudin. It follows that X is isometric to a dense subset of the complete metric space Y.

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 6

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A "solution (nearly complete)" is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 8.2: Define

$$a_{jk} = \begin{cases} 0 & j < k \\ -1 & j = k \\ 2^{k-j} & j > k \end{cases}$$

That is, a_{jk} is the number in the j-th row and k-th column of the array:

$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \ldots \\ \frac{1}{2} & -1 & 0 & 0 & \ldots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \ldots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Prove that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = -2 \quad \text{and} \quad \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{jk} = 0.$$

Solution (Sketch): It is immediate from the formula for the sum of a geometric series that the column sums (which are clearly all the same) are all 0. Using the formula for the sum of a finite portion of a geometric series, one sees that the row sums are $-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, \ldots$, and the sum of these is -2. ∎

Problem 8.3: Prove that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{jk}$$

if $a_{j,k} \geq 0$ for all j and k. (The case $+\infty = +\infty$ may occur.)

Comment: We give a solution which combines the finite and infinite cases. This is shorter than a solution using a case breakdown.

Solution: We show that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = \sup\left\{ \sum_{(j,k)\in S} a_{jk} \colon S \subset \mathbf{N} \times \mathbf{N} \text{ finite} \right\}.$$

Since the right hand side is unchanged when j and k are interchanged, this will prove the result.

Date: 18 Feb. 2002.

It is clear that if $S \subset \mathbf{N} \times \mathbf{N}$ is finite, then

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \geq \sum_{(j,k) \in S} a_{jk}.$$

This implies

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk} \geq \sup \left\{ \sum_{(j,k) \in S} a_{jk} \colon S \subset \mathbf{N} \times \mathbf{N} \text{ finite} \right\}.$$

For the reverse inequality, let $s \in \mathbf{R}$ be an arbitrary number satisfying

$$s < \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk};$$

we prove that

$$\sup \left\{ \sum_{(j,k) \in S} a_{jk} \colon S \subset \mathbf{N} \times \mathbf{N} \text{ finite} \right\} > s.$$

Choose $\varepsilon > 0$ such that

$$s + \varepsilon < \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}.$$

Choose $m \in N$ such that

$$\sum_{j=1}^{m} \sum_{k=1}^{\infty} a_{jk} > s + \varepsilon.$$

If for some j_0 with $1 \leq j \leq m$, the sum $\sum_{k=1}^{\infty} a_{j_0 k}$ is infinite, choose n such that $\sum_{k=1}^{n} a_{j_0 k} > s$. Then clearly $S = \{j_0\} \times \{1, 2, \ldots, n\}$ is a subset of $\mathbf{N} \times \mathbf{N}$ such that $\sum_{(j,k) \in S} a_{jk} > s$, and we are done. Otherwise, for $1 \leq j \leq m$ choose $n_j \in \mathbf{N}$ such that

$$\sum_{k=1}^{n_j} a_{jk} > \sum_{k=1}^{\infty} a_{jk} - \frac{\varepsilon}{m}.$$

Set

$$n = \max(n_1, n_2, \ldots, n_m) \quad \text{and} \quad S = \{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}.$$

Then

$$\sum_{(j,k) \in S} a_{jk} > \sum_{j=1}^{m} \left(\sum_{k=1}^{\infty} a_{jk} - \frac{\varepsilon}{m} \right) = \sum_{j=1}^{m} \sum_{k=1}^{\infty} a_{jk} - \varepsilon > s.$$

This proves the desired inequality. ∎

Note: We can even avoid the case breakdown in the last paragraph, as follows. Choose $b_1, b_2, \ldots, b_m$ such that $b_1 + b_2 + \cdots + b_m > s$ and $b_j < \sum_{k=1}^{\infty} a_{jk}$ for $1 \leq j \leq m$. Then choose $n_j \in \mathbf{N}$ such that $\sum_{k=1}^{n_j} a_{jk} > b_j$. However, at this point the case breakdown seems easier.

Problem 8.4: Prove the following limit relations:

(a) $\displaystyle\lim_{x \to 0} \frac{b^x - 1}{x} = \log(b)$ for $b > 0$.

Solution (Sketch): Set $f(x) = b^x = \exp(x \log(b))$. The desired limit is by definition $f'(0)$, which can be gotten from the second expression for f using the chain rule. ∎

(b) $\displaystyle\lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$

Solution (Sketch): This limit is $f'(0)$ with $f(x) = \log(1+x)$. ∎

(c) $\displaystyle\lim_{x\to 0} (1+x)^{1/x} = e.$

Solution (Sketch): By Part (b), we have

$$\lim_{x\to 0} \log\left((1+x)^{1/x}\right) = 1.$$

Apply exp to both sides, using continuity of exp. ∎

(d) $\displaystyle\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x.$

Solution (Sketch): Write

$$\left(1 + \frac{x}{n}\right)^n = \left[\left(1 + \frac{x}{n}\right)^{n/x}\right]^x,$$

note that $\frac{x}{n} \to 0$ as $n \to \infty$, and apply Part (c) (using continuity at the appropriate places). ∎

Problem 8.5: Find the following limits:

(a) $\displaystyle\lim_{x\to 0} \frac{e - (1+x)^{1/x}}{x}.$

Comment: We first do a calculation, which shows what the answer is, and then give a sketch of a solution in the correct logical order.

Calculation (Sketch): Rewrite and then use L'Hospital's Rule to get

$$\begin{aligned}\lim_{x\to 0} \frac{e - (1+x)^{1/x}}{x} &= \lim_{x\to 0} \frac{e - \exp\left(\frac{1}{x}\log(1+x)\right)}{x} \\ &= -\lim_{x\to 0} \frac{(x+1)^{1/x}\left(\frac{1}{x(x+1)} - \frac{\log(x+1)}{x^2}\right)}{1}.\end{aligned}$$

Rewrite:

$$\begin{aligned}&-\lim_{x\to 0} (x+1)^{1/x}\left(\frac{1}{x(x+1)} - \frac{\log(x+1)}{x^2}\right) \\ &\qquad = -\left(\lim_{x\to 0} (x+1)^{1/x}\right)\left(\lim_{x\to 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2}\right) \\ &\qquad = -e \lim_{x\to 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2}.\end{aligned}$$

Use L'Hospital's rule to get

$$-e \lim_{x\to 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} = e \lim_{x\to 0} \frac{\frac{1}{x+1} - \frac{1}{(x+1)^2}}{2x} = e \lim_{x\to 0} \frac{1}{2(x+1)^2} = \frac{e}{2}.$$

Solution (Sketch): We observe that

$$\lim_{x\to 0} \frac{\frac{1}{x+1} - \frac{1}{(x+1)^2}}{2x} = \lim_{x\to 0} \frac{1}{2(x+1)^2}$$

exists (and is equal to $\frac{1}{2}$). Since the other hypotheses of L'Hospital's Rule are also verified, we can use it to show that

$$-\lim_{x\to 0} \frac{\frac{x}{x+1} - \log(x+1)}{x^2} = \lim_{x\to 0} \frac{\frac{1}{x+1} - \frac{1}{(x+1)^2}}{2x} = \frac{1}{2}.$$

Therefore

$$-\lim_{x\to 0} (x+1)^{1/x} \left(\frac{1}{x(x+1)} - \frac{\log(x+1)}{x^2} \right) = \frac{e}{2}.$$

Since the other hypotheses of L'Hospital's Rule are also verified, we can use it again to show that

$$\lim_{x\to 0} \frac{e - (1+x)^{1/x}}{x} = -\lim_{x\to 0} \frac{(x+1)^{1/x} \left(\frac{1}{x(x+1)} - \frac{\log(x+1)}{x^2} \right)}{1} = \frac{e}{2}.$$

∎

(b) $\displaystyle\lim_{n\to\infty} \frac{n}{\log(n)} \left(n^{1/n} - 1 \right)$.

Solution: Rewrite the limit as

$$\lim_{n\to\infty} \frac{n}{\log(n)} \left(n^{1/n} - 1 \right) = \lim_{n\to\infty} \frac{\exp\left(\frac{1}{n}\log(n)\right) - 1}{\frac{1}{n}\log(n)}$$

Now $\lim_{n\to\infty} \frac{1}{n}\log(n) = 0$, so

$$\frac{\exp\left(\frac{1}{n}\log(n)\right) - 1}{\frac{1}{n}\log(n)} = \lim_{h\to 0} \frac{\exp(h) - 1}{h} = \exp'(0) = 1.$$

∎

(c) $\displaystyle\lim_{x\to 0} \frac{\tan(x) - x}{x[1 - \cos(x)]}$.

Solution 1 (Sketch): Write

$$\frac{\tan(x) - x}{x[1 - \cos(x)]} = \left(\frac{1}{\cos(x)} \right) \left(\frac{\sin(x) - x\cos(x)}{x - x\cos(x)} \right).$$

Since $\lim_{x\to 0} \cos(x) = 1$, we get

$$\lim_{x\to 0} \frac{\tan(x) - x}{x[1 - \cos(x)]} = \lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x - x\cos(x)}.$$

Now apply L'Hospital's rule three times (being sure to check that the hypotheses are satisfied!):

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin(x) - x\cos(x)}{x - x\cos(x)} &= \lim_{x\to 0} \frac{x\sin(x)}{1 - \cos(x) + x\sin(x)} \\
&= \lim_{x\to 0} \frac{x\cos(x) + \sin(x)}{x\cos(x) + 2\sin(x)} \\
&= \lim_{x\to 0} \frac{2\cos(x) - x\sin(x)}{3\cos(x) - x\sin(x)} = \frac{2}{3}.
\end{aligned}$$

(See the solution to Part (a) for the logically correct way to write this.) ∎

Solution 2 (Sketch; not recommended): Apply L'Hospital's rule three times to the original expression (being sure to check that the hypotheses are satisfied!):

$$\begin{aligned}\lim_{x\to 0}\frac{\tan(x)-x}{x[1-\cos(x)]} &= \lim_{x\to 0}\frac{[\sec(x)]^2-1}{1-\cos(x)+x\sin(x)}\\ &= \lim_{x\to 0}\frac{2[\sec(x)]^2\tan(x)}{x\cos(x)+2\sin(x)}\\ &= \lim_{x\to 0}\frac{2[\sec(x)]^4+4[\sec(x)]^2[\tan(x)]^2}{3\cos(x)-x\sin(x)} = \frac{2}{3}.\end{aligned}$$

(See the solution to Part (a) for the logically correct way to write this.)

This solution is not recommended because of the messiness of the differentiation. (The results given here were obtained using Mathematica.) ∎

Solution 3 (Sketch; the mathematical justification for the power series manipulations is easy but is not provided here): We compute the limit by substitution of power series. We check easily that the usual power series

$$\sin(x) = x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 - \tfrac{1}{7!}x^7 + \cdots \quad\text{and}\quad \cos(x) = 1 - \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 - \tfrac{1}{6!}x^6 + \cdots$$

follow from the definitions of $\sin(x)$ and $\cos(x)$ in terms of $\exp(ix)$ and from the definition of $\exp(z)$. Start from the equivalent limit given in Solution 1. We have

$$\begin{aligned}\frac{\sin(x)-x\cos(x)}{x-x\cos(x)} &= \frac{\left(x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots\right)-x\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots\right)}{x-x\left(1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots\right)}\\ &= \frac{\left(\frac{1}{2!}-\frac{1}{3!}\right)x^3+\left(\frac{1}{5!}-\frac{1}{4!}\right)x^5+\cdots}{\frac{1}{2!}x^3-\frac{1}{4!}x^5+\cdots}\\ &= \frac{\left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{5!}-\frac{1}{4!}\right)x^2+\cdots}{\frac{1}{2!}-\frac{1}{4!}x^2+\cdots}.\end{aligned}$$

The last expression defines a function of x which is continuous at 0, so

$$\lim_{x\to 0}\frac{\tan(x)-x}{x[1-\cos(x)]} = \lim_{x\to 0}\frac{\sin(x)-x\cos(x)}{x-x\cos(x)} = \frac{\left(\frac{1}{2!}-\frac{1}{3!}\right)}{\left(\frac{1}{2!}\right)} = \frac{2}{3}.$$

∎

(d) $\displaystyle\lim_{x\to 0}\frac{x-\sin(x)}{\tan(x)-x}$.

Solution 1 (Sketch): Write

$$\frac{x-\sin(x)}{\tan(x)-x} = \cos(x)\left(\frac{x-\sin(x)}{\sin(x)-x\cos(x)}\right).$$

Since $\lim_{x\to 0}\cos(x)=1$, we get

$$\lim_{x\to 0}\frac{x-\sin(x)}{\tan(x)-x} = \lim_{x\to 0}\frac{x-\sin(x)}{\sin(x)-x\cos(x)}.$$

Now apply L'Hospital's rule three times (being sure to check that the hypotheses are satisfied!):

$$\lim_{x\to 0} \frac{x - \sin(x)}{\sin(x) - x\cos(x)} = \lim_{x\to 0} \frac{1 - \cos(x)}{x\sin(x)} = \lim_{x\to 0} \frac{\sin(x)}{x\cos(x) + \sin(x)}$$
$$= \lim_{x\to 0} \frac{\cos(x)}{2\cos(x) - x\sin(x)} = \frac{1}{2}.$$

(See the solution to Part (a) for the logically correct way to write this.) ∎

Solution 2 (Sketch): Apply L'Hospital's rule twice to the original expression (being sure to check that the hypotheses are satisfied!):

$$\lim_{x\to 0} \frac{x - \sin(x)}{\tan(x) - x} = \lim_{x\to 0} \frac{1 - \cos(x)}{[\sec(x)]^2 - 1} = \lim_{x\to 0} \frac{\sin(x)}{2[\sec(x)]^2 \tan(x)}$$
$$= \lim_{x\to 0} \frac{1}{2[\sec(x)]^3} = \frac{1}{2}.$$

(See the solution to Part (a) for the logically correct way to write this.) ∎

Solution 3 (Sketch): Use the same method as Solution 3 to Part (c). Details are omitted. ∎

Problem 8.6: Let $f\colon \mathbf{R} \to \mathbf{R}$ be a nonzero function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbf{R}$.

(a) Suppose f is differentiable. Prove that there is $c \in \mathbf{R}$ such that $f(x) = \exp(cx)$ for all $x \in \mathbf{R}$.

Solution (nearly complete): Since $f(0)f(x) = f(x)$ for all x, and since there is some x such that $f(x) \neq 0$, it follows that $f(0) = 1$. The computation

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(h) - f(0)}{h} = f(x)f'(0)$$

shows that $f'(x) = f(x)f'(0)$ for all $x \in \mathbf{R}$. Now let $g(x) = f(x)\exp(-xf'(0))$ for $x \in \mathbf{R}$. Differentiate g using the product rule and the formula above, getting $g'(x) = 0$ for all x. So g is constant. Since $f(0) = 1$, we get $g(0) = 1$. Therefore $f(x) = \exp(xf'(0))$ for all $x \in \mathbf{R}$. ∎

Alternate solution: Use the solution to Part (b). ∎

(b) Suppose f is continuous. Prove that there is $c \in \mathbf{R}$ such that $f(x) = \exp(cx)$ for all $x \in \mathbf{R}$.

Solution (nearly complete): We have $f(0) = 1$ for the same reason as in the first solution to Part (a). By continuity, there is $a > 0$ such that $f(a) > 0$. Next, define $g(x) = f(x)\exp(-a^{-1}x\log(f(a)))$ for $x \in \mathbf{R}$. Then g is continuous, satisfies $g(x+y) = g(x)g(y)$ for all $x, y \in \mathbf{R}$, and $g(a) = 1$. Set

$$S = \inf\{x > 0\colon g(x) = 1\} \quad \text{and} \quad x_0 = \inf S.$$

We have $S \neq \varnothing$ because $a \in S$.

First suppose that $x_0 > 0$. Then $g(x_0) = 1$ by continuity. We have $g\left(\frac{1}{2}x_0\right)^2 = g(x_0) = 1$ but $g\left(\frac{1}{2}x_0\right) \neq 1$, so $g\left(\frac{1}{2}x_0\right) = -1$. Then $g\left(\frac{1}{4}x_0\right)^2 = g\left(\frac{1}{2}x_0\right) = -1$, contradicting the assumption that $g\left(\frac{1}{4}x_0\right)$ is real. So $x_0 = 0$.

We can now show that $g(x) = 1$ for all $x \in \mathbf{R}$. Let $\varepsilon > 0$. Choose $\delta > 0$ such that whenever $y \in \mathbf{R}$ satisfies $|y - x| < \delta$, then $|g(y) - g(x)| < \varepsilon$. By the definition of S and because $x_0 = 0$, there is $z \in S$ such that $0 < z < \delta$. Choose $n \in \mathbf{Z}$ such that $|nz - x| < \delta$. Then $g(nz) = g(z)^n = 1$, so $|g(x) - 1| < \varepsilon$. This shows that $g(x) = 1$.

So $f(x) = \exp(cx)$ with $c = a^{-1} \log(f(a))$. ∎

Alternate solution (Sketch): We have $f(0) = 1$ for the same reason as in the first solution to Part (a). Furthermore, $x \in \mathbf{R}$ implies

$$f(x) = f\left(\tfrac{1}{2}x\right)^2 \geq 0$$

since $f\left(\frac{1}{2}x\right) \in \mathbf{R}$. Moreover, if $f(x) = 0$ then $f\left(\frac{1}{2}x\right) = 0$, and by induction $f\left(2^{-n}x\right) = 0$ for all n. Since f is continuous and $\lim_{n\to\infty} 2^{-n}x = 0$, this contradicts $f(0) = 1$. Therefore $f(x) > 0$ for all x.

Define $g(x) = \exp(x \log(f(1)))$ for $x \in \mathbf{R}$. For $n \in \mathbf{N}$,

$$f\left(\tfrac{1}{n}\right)^n = f(1) \quad \text{and} \quad g\left(\tfrac{1}{n}\right)^n = g(1) = f(1).$$

Since both $f\left(\frac{1}{n}\right)$ and $g\left(\frac{1}{n}\right)$ are positive, and positive n-th roots are unique, it follows that $f\left(\frac{1}{n}\right) = g\left(\frac{1}{n}\right)$ for all $n \in \mathbf{N}$. An easy argument now shows that $f(x) = g(x)$ for all $x \in \mathbf{Q}$. Since f and g are continuous, and since $\mathbf{Q}$ is dense in $\mathbf{R}$, it follows that $f = g$. So $f(x) = \exp(cx)$ with $c = \log(f(1))$. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 7

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A "solution (nearly complete)" is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 8.7: Prove that

$$\frac{2}{\pi} < \frac{\sin(x)}{x} < 1$$

for $0 < x < \frac{1}{2}\pi$.

Solution (Sketch): The inequality $\frac{\sin(x)}{x} < 1$ is the same as $\sin(x) < x$. This is proved by noting that $\sin(0) = 0$ and that the derivative $1 - \cos(x)$ of $x - \sin(x)$ is strictly positive for $0 < x < \frac{1}{2}\pi$.

This also implies that $\frac{\sin(x)}{x} < 1$ for $x = \frac{1}{2}\pi$, so that $\frac{2}{\pi} < 1$.

For the other inequality, suppose there is x_0 with $0 < x_0 < \frac{1}{2}\pi$ such that

$$\frac{2}{\pi} \geq \frac{\sin(x_0)}{x_0}.$$

Using the Intermediate Value Theorem and $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$, there is x_0 with $0 < x_0 < \frac{1}{2}\pi$ such that

$$\frac{2}{\pi} = \frac{\sin(x_0)}{x_0}.$$

Set $f(x) = \sin(x) - \frac{2}{\pi} \cdot x$. For $0 < x \leq \frac{1}{2}\pi$, we have $f''(x) = -\sin(x) < 0$, so that f' is strictly decreasing on $\left[0, \frac{1}{2}\pi\right]$. The Mean Value Theorem gives $z \in (0, x_0)$ such that $f'(z) = 0$, so $f'(x) < 0$ for $x_0 \leq x \leq \frac{1}{2}\pi$. Since $f(x_0) = 0$, we get $f\left(\frac{1}{2}\pi\right) < 0$, a contradiction. ∎

Alternate solution 1 (Sketch): Suppose $f \colon [0, a] \to \mathbf{R}$ is a continuous function such that:

(1) $f(0) = 0$.
(2) $f'(x)$ exists for $x \in (0, a)$.
(3) f' is strictly decreasing on $(0, a)$.

We claim that the function $g(x) = x^{-1} f(x)$ is strictly decreasing on $(0, a]$.

To see that the claim implies the result, take $f(x) = \sin(x)$ and $a = \frac{1}{2}\pi$. We conclude that g is strictly decreasing on $\left(0, \frac{1}{2}\pi\right]$. We get the result by observing that

$$\lim_{x \to 0^+} \frac{\sin(x)}{x} = \sin'(0) = \cos(0) = 1.$$

Date: 27 Feb. 2002.

To prove the claim, let $0 < x_1 < x_2 \leq b$. Use the Mean Value Theorem to choose $c_1 \in (0, x_1)$ and $c_2 \in (x_1, x_2)$ such that

$$\frac{f(x_1)}{x_1} = \frac{f(x_1) - f(0)}{x_1 - 0} = f'(c_1) \quad \text{and} \quad \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c_2).$$

Then $c_1 < c_2$, so $f'(c_1) > f'(c_2)$, whence

$$\frac{f(x_1)}{x_1} > \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Multiply both sides of this inequality by $x_1(x_2 - x_1) > 0$ to get

$$(x_2 - x_1)f(x_1) > x_1(f(x_2) - f(x_1)).$$

Multiply out and cancel $-x_1 f(x_1)$ to get $x_2 f(x_1) > x_1 f(x_2)$, and divide by $x_1 x_2$ to get $g(x_1) > g(x_2)$. ∎

Alternate solution 2 (Sketch): Since

$$\frac{\sin\left(\frac{1}{2}\pi\right)}{\frac{1}{2}\pi} = \frac{2}{\pi} \quad \text{and} \quad \lim_{x \to 0^+} \frac{\sin(x)}{x} = \sin'(0) = \cos(0) = 1,$$

it suffices to prove that the function $g(x) = x^{-1}\sin(x)$ is strictly decreasing on $\left(0, \frac{1}{2}\pi\right)$. We do this by showing that $g'(x) < 0$ on this interval.

Begin by observing that the function $h(x) = x\cos(x) - \sin(x)$ satisfies $h(0) = 0$ and $h'(x) = -\sin(x)$ for all x. Therefore $h'(x) < 0$ for $x \in \left(0, \frac{1}{2}\pi\right]$, and from $h(0) = 0$ we get $h(x) < 0$ for $x \in \left(0, \frac{1}{2}\pi\right]$. Now calculate, for $x \in \left(0, \frac{1}{2}\pi\right)$,

$$g'(x) = \frac{x\cos(x) - \sin(x)}{x^2} = \frac{h(x)}{x^2} < 0.$$

This is the required estimate. ∎

Alternate solution 3 (Sketch): As in the second alternate solution, we set $g(x) = x^{-1}\sin(x)$ and show that $g'(x) < 0$ for $x \in \left(0, \frac{1}{2}\pi\right)$. Define

$$q(x) = \frac{\sin(x)}{\cos(x)} - x$$

for $x \in \left[0, \frac{1}{2}\pi\right)$. Then the quotient rule and the relation $\sin^2(x) + \cos^2(x) = 1$ give

$$q'(x) = \frac{1}{\cos^2(x)} - 1.$$

For $x \in \left(0, \frac{1}{2}\pi\right)$ we have $0 < \cos(x) < 1$, from which it follows that $q'(x) > 0$. Since q is continuous on $\left[0, \frac{1}{2}\pi\right)$ and $q(0) = 0$, we get $q(x) > 0$ for $x \in \left(0, \frac{1}{2}\pi\right)$. It follows that $\sin(x) > x\cos(x)$ for $x \in \left(0, \frac{1}{2}\pi\right)$. As in the second alternate solution, this implies that $g'(x) < 0$ for $x \in \left(0, \frac{1}{2}\pi\right)$. ∎

Problem 8.8: For $n \in \mathbf{N} \cup \{0\}$ and $x \in \mathbf{R}$, prove that

$$|\sin(nx)| \leq n|\sin(x)|.$$

Note that this inequality may be false for n not an integer. For example,

$$\left|\sin\left(\tfrac{1}{2}xm\right)\right| > \tfrac{1}{2}\left|\sin(x)\right|.$$

Solution (Sketch): Combining the formula $\exp(i(x+y)) = \exp(ix)\exp(iy)$ with either the result

$$\cos(x) = \mathrm{Re}(\exp(ix)) \quad \text{and} \quad \sin(x) = \mathrm{Im}(\exp(ix))$$

(for x real) or the definitions

$$\cos(x) = \tfrac{1}{2}[\exp(ix) + \exp(-ix)] \quad \text{and} \quad \sin(x) = \tfrac{1}{2i}[\exp(ix) - \exp(-ix)],$$

prove the addition formula

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

(Using the first suggestion gives this only for real x and y, but that is all that is needed here.)

Now prove the result by induction on n. For $n = 0$ the desired inequality says $0 \leq 0$ for all x, which is certainly true. Assuming it is true for n, we have (using the addition formula in the first step and the induction hypothesis and the inequality $|\cos(a)| \leq 1$ for all real a in the second step)

$$\begin{aligned} |\sin((n+1)x)| &\leq |\sin(x)| \cdot |\cos(nx)| + |\cos(x)| \cdot |\sin(nx)| \\ &\leq |\sin(x)| + n|\sin(x)| = (n+1)|\sin(x)| \end{aligned}$$

for all $x \in \mathbf{R}$. ∎

Alternate solution (Sketch): We first prove the inequality for $0 \leq x \leq \frac{1}{2}\pi$. If $x \in \left[0, \frac{1}{2}\pi\right]$ satisfies $\sin(x) \geq \frac{1}{n}$, then since $|\sin(nx)| \leq 1$ there is nothing to prove. Otherwise, $x < \frac{\pi}{2n}$ by Problem 8.7. Since $t \mapsto \cos(t)$ is nonincreasing on $\left[0, \frac{1}{2}\pi\right]$ (its derivative $-\sin(t)$ is nonpositive there), we get $\cos(nx) \leq \cos(x)$ for $x \in \left[0, \frac{\pi}{2n}\right]$. With $f(x) = n\sin(x) - \sin(nx)$, we therefore have $f(0) = 0$ and

$$f'(x) = n[\cos(x) - \cos(nx)] \leq 0$$

for $x \in \left[0, \frac{\pi}{2n}\right]$. Since also $\sin(nx) \geq 0$ on this interval, the inequality is proved for $0 < x \leq \frac{\pi}{2n}$ and hence $0 \leq x \leq \frac{1}{2}\pi$.

For $-\frac{1}{2}\pi \leq x \leq 0$, the inequality follows from the fact that $t \mapsto \sin(t)$ is an odd function. (This is easily seen from the definition.) For $\frac{1}{2}\pi \leq x \leq \frac{3}{2}\pi$, reduce to $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$ using the identity $\sin(k(x+\pi)) = (-1)^k \sin(kx)$ (which is easily derived from the definition and $\exp(i\pi) = -1$). The inequality now follows for all x by periodicity. ∎

Problem 8.9: (a) For $n \in \mathbf{N}$ set

$$s_n = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \cdots + \tfrac{1}{n}.$$

Prove that

$$\gamma = \lim_{n \to \infty} [s_n - \log(n)]$$

exists. (This limit is called Euler's constant. Numerically, $\gamma \approx 0.5772$. It is not known whether γ is rational or not.)

Solution: We have

$$\tfrac{1}{n} - [\log(n+1) - \log(n)] = \tfrac{1}{n} - \int_n^{n+1} \tfrac{1}{t}\, dt = \int_n^{n+1} \left(\tfrac{1}{n} - \tfrac{1}{t}\right) dt.$$

for $n \in \mathbf{N}$. Since the integrand is between 0 and $\frac{1}{n} - \frac{1}{n+1}$, it follows that

$$0 \leq \tfrac{1}{n} - [\log(n+1) - \log(n)] \leq \tfrac{1}{n} - \tfrac{1}{n+1}.$$

Since

$$\sum_{k=1}^{n} \left(\tfrac{1}{k} - [\log(k+1) - \log(k)]\right) = s_n - \log(n+1),$$

we get by adding up terms

$$s_n - \log(n+1) \leq 1 - \tfrac{1}{n+1} < 1$$

for all n; also,

$$s_n - \log(n+1) = s_{n-1} - \log(n) + \tfrac{1}{n} - [\log(n+1) - \log(n)] \geq s_{n-1} - \log(n).$$

Therefore $(s_n - \log(n+1))$ is a bounded nondecreasing sequence, hence converges.

Next observe that

$$\log(n+1) - \log(n) = \int_n^{n+1} \tfrac{1}{t}\, dt,$$

so that

$$0 \leq \log(n+1) - \log(n) \leq \tfrac{1}{n}.$$

It follows that $\lim_{n\to\infty}[\log(n+1) - \log(n)] = 0$, whence

$$\lim_{n\to\infty}[s_n - \log(n)] = \lim_{n\to\infty}[s_n - \log(n+1)].$$

∎

Alternate solution 1: Let P_n be the partition $P_n = (1, 2, \ldots, n)$ of $[1, n]$. Set $f(x) = x^{-1}$. Since f is nonincreasing, we have

$$L(P_n, f) = \sum_{k=2}^{n} \frac{1}{k} \quad \text{and} \quad U(P_n, f) = \sum_{k=1}^{n-1} \frac{1}{k}.$$

Therefore

$$s_n - \log(n) = 1 + L(P_n, f) - \int_1^n f = \tfrac{1}{n} + U(P_n, f) - \int_1^n f.$$

We prove that $(s_n - \log(n))$ is nonincreasing. We have

$$\begin{aligned} &[s_n - \log(n)] - [s_{n+1} - \log(n+1)] \\ &\qquad = \left[1 + L(P_n, f) - \int_1^n f\right] - \left[1 + L(P_{n+1},\, f) - \int_1^{n+1} f\right] \\ &\qquad = \int_n^{n+1} f - [L(P_{n+1},\, f) - L(P_n, f)] = \int_n^{n+1} f - \tfrac{1}{n+1}. \end{aligned}$$

The last expression is nonnegative because $f(x) \geq \frac{1}{n+1}$ for all $x \in [n,\, n+1]$. So $(s_n - \log(n))$ is nonincreasing.

We prove that $(s_n - \log(n))$ is bounded below (by 0). Indeed,

$$0 \leq U(P_n, f) - \int_1^n f \leq \tfrac{1}{n} + U(P_n, f) - \int_1^n f = s_n - \log(n)$$

for all n.

Since $(s_n - \log(n))$ is nonincreasing and bounded below, $\lim_{n\to\infty}(s_n - \log(n))$ exists. ∎

Alternate solution 2: First prove the following lemma.

Lemma. For every $x \geq 0$, we have $0 \leq x - \log(1+x) \leq \frac{1}{2}x^2$.

Proof: Set $g(x) = x - \log(1 + x)$ and $h(x) = \frac{1}{2}x^2$ for $x \in (-1, \infty)$. Then $g(0) = h(0) = 0$. Furthermore, for all $x \geq 0$ we have

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \geq 0 \quad \text{and} \quad h'(x) - g'(x) = x - \frac{x}{1+x} = \frac{x^2}{1+x} \geq 0.$$

Therefore $0 \leq g(x) \leq h(x)$ for all $x \geq 0$. ∎

Now define

$$b_k = \tfrac{1}{k} - [\log(k) - \log(k+1)] = \tfrac{1}{k} - \log\left(1 + \tfrac{1}{k}\right)$$

for $k \geq 1$. (That the two expressions are equal follows from the algebric properties of the function log. See Equation (40) on Page 181 of Rudin's book.) Since $\log(1) = 0$, we have

$$s_n - \log(n+1) = \sum_{k=1}^{n} b_k.$$

By the lemma, we have $0 \leq b_k \leq \frac{1}{2}k^{-2}$ for $k \geq 1$. Since $\sum_{k=1}^{\infty} k^{-2}$ converges, the Comparison Test shows that $\sum_{k=1}^{\infty} b_k$ converges, whence $\lim_{n\to\infty}[s_n - \log(n+1)]$ exists.

Now show that $\lim_{n\to\infty}[\log(n+1) - \log(n)] = 0$ as in the first solution, and conclude as there that $\lim_{n\to\infty}[s_n - \log(n)]$ exists. ∎

Alternate solution 3 (Outline): This solution differs from the previous one only in the method of proof of the lemma. Instead of comparing derivatives, we use the derivative form of the remainder in Taylor's Theorem (see Theorem 5.15 of Rudin's book) to compare $\log(1 + x)$ with the Taylor polynomials of degrees 1 and 2. ∎

(b) Roughly how large must m be so that $n = 10^m$ satisfies $s_n > 100$?

Solution (Sketch): The proof above gives $0 < s_n - \log(n+1) < 1$ for all n. Therefore $s_n \in (\log(n+1),\, 1 + \log(n+1))$. We have $\log(n+1) \geq 100$ if and only if $n > \exp(100) - 1$. So it suffices to take

$$m = \log_{10}(\exp(100)) = 100 \log_{10}(e) \approx 43.43.$$

∎

Problem 8.10: Prove that

$$\sum_{p \text{ prime}} \frac{1}{p}$$

diverges.

Hint: Given N, let $p_1, p_2, \ldots, p_k$ be those primes that divide at least one integer in $\{1, 2, \ldots, N\}$. Then

$$\sum_{n=1}^{N} \frac{1}{n} \leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) = \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \leq \exp\left(\sum_{j=1}^{k} \frac{1}{p_j}\right).$$

The last inequality holds because $(1 - x)^{-1} \leq \exp(2x)$ for $0 \leq x \leq \frac{1}{2}$.

Solution (Sketch): It suffices to verify the inequalities because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. For the first inequality, let m be the largest power of any prime appearing in the prime factorization of any integer in $\{1, 2, \ldots, N\}$. Then one checks that

$$\prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots + \frac{1}{p_j^m} \right) = \sum_{n \in S} \frac{1}{n},$$

where S is the set of all integers whose prime factorization involves only the primes $p_1, p_2, \ldots, p_k$, and in which no prime appears with multiplicity greater than m. Moreover, $\{1, 2, \ldots, N\} \subset S$. For the other inequality, since $0 \leq \frac{1}{p} \leq \frac{1}{2}$ for all primes p, it suffices to check that $(1-x)^{-1} \leq \exp(2x)$ for $0 \leq x \leq \frac{1}{2}$. Now $1 + 2x \leq \exp(2x)$ for all $x \geq 0$ by any of a number of arguments. The inequality $(1-x)^{-1} \leq 1 + 2x$ for $0 \leq x \leq \frac{1}{2}$ is easily verified by multiplying both sides by $1 - x$. ∎

Problem 8.11: Let $f \colon [0, \infty) \to \mathbf{R}$ be a function such that $\lim_{x \to \infty} f(x) = 1$ and f is Riemann integrable on every interval $[0, a]$ for $a > 0$. Prove that

$$\lim_{t \to 0^+} t \int_0^\infty e^{-tx} f(x) \, dx = 1.$$

Solution (Sketch): Let $\varepsilon > 0$. Choose $a > 0$ such that $|f(x) - 1| < \frac{1}{2}\varepsilon$ for $x \geq a$. Since f is Riemann integrable on $[0, a]$, it is bounded there. Choose M such that $|f(x)| \leq M$ for all $x \in [0, a]$. Choose δ so small that $\delta(M+1) < \frac{1}{2}\varepsilon$. Then $0 < t < \delta$ implies

$$t \int_0^\infty e^{-tx} \cdot 1 \, dx = 1.$$

and

$$\begin{aligned} t \int_0^\infty e^{-tx} |f(x) - 1| \, dx &\leq \delta \int_0^a e^{-tx} (M+1) \, dx + t \int_a^\infty e^{-tx} \tfrac{1}{2}\varepsilon \, dx \\ &\leq \delta(M+1) + \tfrac{1}{2}\varepsilon e^{-at} < \varepsilon. \end{aligned}$$

∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 8

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A "solution (nearly complete)" is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 5.27: Let $R = [a, b] \times [\alpha, \beta] \subset \mathbf{R}^2$ be a rectangle in the plane, and let $\varphi \colon R \to \mathbf{R}$ be a function. A *solution* of the initial value problem $y' = \varphi(x, y)$ and $y(a) = c$ (with $c \in [\alpha, \beta]$) is, by definition, a differentiable function $f \colon [a, b] \to [\alpha, \beta]$ such that $f(a) = c$ and such that $f'(t) = \varphi(t,\, f(t))$ for all $t \in [a, b]$.

Assume that there is a constant A such that

$$|\varphi(x, y_2) - \varphi(x, y_1)| \leq A|y_2 - y_1|$$

for all $x \in [a, b]$ and $y_1,\, y_2 \in [\alpha, \beta]$. Prove that such a problem has at most one solution.

Hint: Apply Problem 5.26 to the difference of two solutions.

Note that this uniqueness theorem does not hold for the initial value problem $y' = y^{1/2}$ and $y(0) = 0$, which has two solutions $f_1(x) = 0$ for all x and $f_2(x) = \frac{1}{4}x^2 0$ for all x. Find all other solutions to this initial value problem.

Solution (Sketch): Let f_1 and f_2 be two solutions to the initial value problem $y' = \varphi(x, y)$ and $y(a) = c$. Then for all $t \in [a, b]$ we have

$$|f_2'(t) - f_1'(t)| = |\varphi(t,\, f(t)) - \varphi(t,\, f(t))| \leq A|f_2'(t) - f_1'(t)|,$$

and also $f_2(c) - f_1(c) = 0$, so Problem 5.26 shows that $f_2 - f_1 = 0$.

Now we investigate the solutions to $y' = y^{1/2}$ and $y(0) = 0$. The intervals are not specified, but it seems reasonable to assume that $[0, \infty) \times [0, \infty)$ is intended.

First, for $r \in [0, \infty)$ define $g_r \colon [0, \infty) \to [0, \infty)$ by

$$g_r(t) = \begin{cases} 0 & 0 \leq t \leq r \\ \frac{1}{4}(t - r)^2 & t > r \end{cases},$$

and define $g_\infty(t) = 0$ for all t. Check that g_r is in fact a solution. (This is trivial everywhere except at $t = r$, where one must calculate $g_r'(t)$ directly from the definition.) We are going to show that these are all solutions.

We claim that if $t_0 \in \mathbf{R}$ and $c > 0$, then the initial value problem $y' = y^{1/2}$ and $y(t_0) = c$ has the solution $f(t) = \frac{1}{4}(t - r)^2$ for $t \geq t_0$, where r is chosen to be $r = t_0 - 2\sqrt{c}$. Applying the uniqueness result in the first part of the problem on $[t_0, b] \times [\alpha, \beta]$, with $\alpha \leq c \leq \frac{1}{4}(b - r)^2 \leq \beta$, and letting $b \to \infty$, $\alpha \to 0$, and $\beta \to \infty$, we find that there are no other solutions $f \colon [t_0, \infty) \to (0, \infty)$. (Be sure to check that the hypotheses hold!)

Date: 5 March 2002.

Now suppose $g \colon [0, \infty) \to [0, \infty)$ is a solution satisfying $g(0) = 0$. We claim that there is $r \in [0, \infty]$ such that $g = g_r$. Without loss of generality $g \neq g_\infty$, so there exists $s > 0$ such that $g(s) > 0$. Let

$$r = \inf(\{s \in [0, \infty) \colon g(s) > 0\}).$$

By continuity, we must have $g(r) = 0$. We will show that $g = g_r$.

Suppose not. There are three cases to consider. First, suppose there is $t \leq r$ such that $g(t) > 0$. Apply the claim above with $t_0 = t$ to obtain

$$g(r) = \tfrac{1}{4}\left(r - t + 2\sqrt{g(t)}\right)^2 \neq 0,$$

a contradiction. Next, suppose there is $t > r$ such that $g(t) = 0$. Choose $s \in (r, t)$ such that $g(s) \neq 0$, and apply the claim above with $t_0 = t$ to obtain

$$g(t) = \tfrac{1}{4}\left(t - s + 2\sqrt{g(s)}\right)^2 \neq 0,$$

a contradiction. Finally, suppose there is $t > r$ such that $g(t) \neq g_r(t)$ and $g(t) \neq 0$. Repeated use of the claim, with t_0 running through a sequence decreasing monotonically to $t - 2\sqrt{g(t)}$, shows that $g(s) = \frac{1}{4}\left(s - t + 2\sqrt{g(t)}\right)^2$ for all $s > t - 2\sqrt{g(t)}$. If $t - 2\sqrt{g(t)} > r$, then continuity gives $g\left(t - 2\sqrt{g(t)}\right) = 0$, and we obtain a contradiction as in the second case. If $t - 2\sqrt{g(t)} < r$, then we obtain a contradiction as in the first case. If $t - 2\sqrt{g(t)} = r$, then one checks that $g = g_r$. ∎

Comment: The techniques for finding solutions found in nonrigorous differential equations courses (such as Math 256 at the University of Oregon) are not proofs of anything, and therefore have no place in a formal proof in this course. (They can, of course, be used to *find* solutions in scratchwork.) One sees this from the fact that these methods do not find most of the solutions g_r given above. Such methods may be used in scratchwork to find solutions, but the functions found this way must be verified to be solutions, and can only be expected to be the only solutions when the hypotheses of the first part of the problem are satisfied.

Problem 8.20: The following simple computation yields a good approximation to Stirling's formula.

Define $f, g \colon [1, \infty) \to \mathbf{R}$ by

$$f(x) = (m + 1 - x)\log(m) + (x - m)\log(m + 1)$$

for $m = 1, 2, \ldots$ and $x \in [m, m + 1]$, and

$$g(x) = \frac{x}{m} - 1 + \log(m)$$

for $m = 1, 2, \ldots$ and $x \in \left[m - \frac{1}{2}, m + \frac{1}{2}\right)$ ($x \in \left[1, m + \frac{1}{2}\right)$ if $m = 1$). Draw the graphs of f and g. Prove that $f(x) \leq \log(x) \leq g(x)$ for $x \geq 1$, and that

$$\int_1^n f(x)\,dx = \log(n!) - \tfrac{1}{2}\log(n) > -\tfrac{1}{8} + \int_1^n g(x)\,dx.$$

Integrate $\log(x)$ over $[1, n]$. Conclude that

$$\tfrac{7}{8} < \log(n!) - \left(n + \tfrac{1}{2}\right)\log(n) + n < 1$$

for $n = 2, 3, 4, \ldots$. Thus

$$e^{7/8} < \frac{n!}{(n/e)^n \sqrt{n}} < e.$$

Solution (Sketch): Note: No graphs are included here.

We need the following lemma, which is essentially a concavity result.

Lemma. Let $[a, b] \subset \mathbf{R}$ be a closed interval, and let $f \colon [a, b] \to \mathbf{R}$ be a continuous function such that $f''(x)$ exists for all $x \in (a, b)$ and $f''(x) \leq 0$ on (a, b). Then

$$f(x) \geq f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

for all $x \in [a, b]$.

Proof (Sketch): Let

$$q(x) = f(x) - f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a)$$

for $x \in [a, b]$. Then q satisfies the same hypotheses as f (note that $q'' = f''$), and $q(a) = q(b) = 0$. It suffices to prove that $q(x) \geq 0$ for all $x \in (a, b)$.

Suppose $x_0 \in (a, b)$ and $q(x_0) < 0$. Use the Mean Value Theorem to choose $c_1 \in (a, x_0)$ and $c_2 \in (x_0, b)$ such that $q'(c_1) < 0$ and $q'(c_2) > 0$. Then $c_1 < c_2$, so the Mean Value Theorem, applied to q', gives $d \in (c_1, c_2)$ such that $q''(d) > 0$. This is a contradiction. ∎

Alternate Proof (Sketch): As in the first proof, let

$$q(x) = f(x) - f(a) + \frac{f(b) - f(a)}{b - a} \cdot (x - a),$$

which satisfies $q''(x) \leq 0$ on (a, b) and $q(a) = q(b) = 0$; it suffices to prove that $q(x) \geq 0$ for all $x \in (a, b)$.

Use the Mean Value Theorem to choose $c \in (a, b)$ such that $q'(c) = 0$. Since $q''(x) \leq 0$ for all $x \in (a, b)$, it follows that q' is nonincreasing. Therefore $q'(x) \geq 0$ for $x \in [a, c]$ and $q'(x) \leq 0$ for $x \in [c, b]$. It follows that q is nondecreasing on $[a, c]$, so for $x \in [a, c]$ we have $q(x) \geq q(a) = 0$. It also follows that q is nonincreasing on $[c, b]$, so for $x \in [c, b]$ we have $q(x) \geq q(b) = 0$. ∎

The lemma can be used directly to show that $f(x) \leq \log(x)$.

To show $g(x) \geq \log(x)$, show that $g(m) = \log(m)$, that $g'(x) \leq \log'(x)$ for $m - \frac{1}{2} \leq x \leq m$, and that $g'(x) \geq \log'(x)$ for $m \leq x \leq m + \frac{1}{2}$.

Next, use the inequality $f(x) \leq \log(x) \leq g(x)$ for $x \geq 1$ to get

$$\int_1^n f(x)\,dx \leq \int_1^n \log(x)\,dx \leq \int_1^n g(x)\,dx.$$

This inequality is actually strict (as is required to solve the problem). To see this, use the fact that there are x_1 and x_2 such that f is continuous at x_1, such that g is continuous at x_2, such that $f(x_1) < \log(x_1)$, and such that $\log(x_2) < g(x_2)$. (This needs proof!)

To calculate $\int_1^n f(x)\,dx$, calculate

$$\int_m^{m+1} f(x)\,dx = \tfrac{1}{2}[\log(m+1) - \log(m)],$$

and add these up. To estimate $\int_1^n g(x)\,dx$, calculate

$$\int_{m-\frac{1}{2}}^{m+\frac{1}{2}} g(x)\,dx = \log(m) \quad \text{and} \quad \int_1^{3/2} g(x)\,dx = \tfrac{1}{8},$$

and estimate

$$\int_{n-\frac{1}{2}}^{n} g(x)\,dx \leq \tfrac{1}{2} g(n) = \tfrac{1}{2}\log(n).$$

(The exact answer for the last one is $\frac{1}{2}\log(n) - \frac{1}{8n}$.) Then add these up. Substituting the resulting values in the strict version of the inequality above, and rearranging, yields

$$\tfrac{7}{8} < \log(n!) - \big(n + \tfrac{1}{2}\big)\log(n) + n < 1,$$

as desired. Then exponentiate. ∎

Problem 8.23: Let $\gamma \colon [a,b] \to \mathbf{C} \setminus \{0\}$ be a continuously differentiable closed curve. Define the *index* of γ to be

$$\operatorname{Ind}(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)}\,dt.$$

Prove that $\operatorname{Ind}(\gamma) \in \mathbf{Z}$.

Compute $\operatorname{Ind}(\gamma)$ for $\gamma(t) = \exp(int)$ and $[a,b] = [0, 2\pi]$.

Explain why $\operatorname{Ind}(\gamma)$ is often called the *winding number* of γ about 0.

Hint for the first part: Find $\varphi \colon [a,b] \to \mathbf{C}$ such that

$$\varphi' = \frac{\gamma'}{\gamma} \quad \text{and} \quad \varphi(a) = 0.$$

Show that $\varphi(b) = 2\pi i \operatorname{Ind}(\gamma)$. Show that $\gamma \exp(-\varphi)$ is constant, so that $\gamma(a) = \gamma(b)$ implies $\exp(\varphi(b)) = \exp(\varphi(a)) = 1$.

Solution (Sketch): The function φ of the hint exists by the Fundamental Theorem of Calculus (Theorem 6.20 of Rudin; *not* Theorem 6.21 of Rudin). That $\varphi(b) = 2\pi i \operatorname{Ind}(\gamma)$ is clear from the construction of φ and the definitions. That $\gamma \exp(-\varphi)$ is constant follows by differentiating it and using the formula for φ' to simplify the derivative to zero. That $\exp(\varphi(b)) = \exp(\varphi(a)) = 1$ is now clear. So $\varphi(b) \in 2\pi i \mathbf{Z}$, whence $\operatorname{Ind}(\gamma) \in \mathbf{Z}$.

The second statement is a simple computation; the answer is n.

Why $\operatorname{Ind}(\gamma)$ is often called the winding number: In the case considered in the second statement, it counts the number of times the curve goes around 0 in the positive sense. (This is true in general. See the comment in the solution to Problem 8.24.) ∎

Problem 8.24: Let γ and $\operatorname{Ind}(\gamma)$ be as in Problem 8.23. Suppose the range of γ does not intersect the negative real axis. Prove that $\operatorname{Ind}(\gamma) = 0$.

Hint: The function $c \mapsto \operatorname{Ind}(\gamma + c)$, defined on $[0, \infty)$, is a continuous integer valued function such that $\lim_{c \to \infty} \operatorname{Ind}(\gamma + c) = 0$.

Solution (Sketch): One checks that

$$c \mapsto \frac{(\gamma + c)'}{\gamma + c} = \frac{\gamma'}{\gamma + c}$$

is a continuous function from $[0, \infty)$ to $C([a, b])$, for example, by checking that $c_n \to c$ implies

$$\frac{(\gamma + c_n)'}{\gamma + c_n} \to \frac{(\gamma + c)'}{\gamma + c}$$

uniformly. Theorem 7.16 of Rudin and the sequential criterion for continuity then imply that $c \mapsto \mathrm{Ind}(\gamma + c)$ is continuous. Furthermore,

$$\frac{(\gamma + c)'}{\gamma + c} \to 0$$

uniformly as $c \to \infty$. Therefore $\lim_{c \to \infty} \mathrm{Ind}(\gamma + c) = 0$. Since $\mathrm{Ind}(\gamma + c) \in \mathbf{Z}$ for all c and since $[0, \infty)$ is connected, we must have $\mathrm{Ind}(\gamma + c) = 0$ for all c, in particular for $c = 0$. ∎

Alternate solution (Sketch): We can give a direct proof of continuity, which is in principle nicer than what was done above.

The first step is to show that

$$r = \inf_{t \in [a,b],\, c \in [0, I)} |\gamma(t) + c| > 0.$$

Since $[a, b]$ is compact, $M = \sup_{t \in [a,b]} |\gamma(t)| < \infty$. Therefore $c \geq 2M$ implies

$$\inf_{t \in [a,b]} |\gamma(t) + c| \geq M.$$

Define $f \colon [a, b] \times [0,\, 2M] \to \mathbf{C}$ by $f(t, c) = |\gamma(t) + c|$. Then f is continuous and never vanishes on $[a, b] \times [0,\, 2M]$, so compactness of $[a, b] \times [0,\, 2M]$ implies that

$$\inf_{t \in [a,b],\, c \in [0,\, 2M]} |\gamma(t) + c| > 0.$$

So

$$\inf_{t \in [a,b],\, c \in [0, I)} |\gamma(t) + c| \geq \min \left(M, \inf_{t \in [a,b],\, c \in [0,\, 2M]} |\gamma(t) + c| \right) > 0.$$

Now let $\varepsilon > 0$. Set

$$\delta = \frac{2\pi r^2 \varepsilon}{1 + (b - a) \sup_{s \in [a,b]} |\gamma'(s)|} > 0.$$

Note that $\sup_{s \in [a,b]} |\gamma'(s)|$ is finite, because γ' is continuous and $[a, b]$ is compact. Let $c,\, d \in [0, \infty)$ satisfy $|c - d| < \delta$. Then $t \in [a, b]$ implies

$$\begin{aligned} \left| \frac{\gamma'(t)}{\gamma(t) + c} - \frac{\gamma'(t)}{\gamma(t) + c} \right| &= \left| \frac{\gamma'(t)(d - c)}{(\gamma(t) + c)(\gamma(t) + d)} \right| \\ &\leq \frac{\delta \sup_{s \in [a,b]} |\gamma'(s)|}{r^2} < \frac{2\pi\varepsilon}{b - a}. \end{aligned}$$

Therefore

$$|\mathrm{Ind}(\gamma + d) - \mathrm{Ind}(\gamma + c)| \leq \frac{1}{2\pi i} \int_a^b \left| \frac{\gamma'(t)}{\gamma(t) + c} - \frac{\gamma'(t)}{\gamma(t) + c} \right| dt < \varepsilon.$$

∎

We give a solution which uses a different method to get a lower bound on $|\gamma(t) + c|$ and also makes explicit the use of the fact that the composite of two continuous functions is continuous. It is possible to combine parts of this solution with parts of the previous solution to obtain two further arrangements of the proof.

Second alternate solution (Sketch): Define $F\colon [0,\infty) \to C([a,b])$ by

$$F(c)(t) = \frac{\gamma'(t)}{\gamma(t)+c},$$

that is, $F(c)$ is the function

$$t \mapsto \frac{\gamma'(t)}{\gamma(t)+c}$$

which is in $C([a,b])$. Further define $I\colon C([a,b]) \to \mathbf{C}$ by

$$I(f) = \frac{1}{2\pi i}\int_a^b f.$$

Then $\operatorname{Ind}(\gamma+c) = I \circ F(c)$. We prove that $c \mapsto \operatorname{Ind}(\gamma+c)$ is continuous by proving that I and F are continuous.

We prove that F is continuous. Let $c_0 \in [0,\infty)$ and let $\varepsilon > 0$. Set $M = \sup_{s\in[a,b]} |\gamma'(s)|$, which is finite because γ' is continuous and $[a,b]$ is compact. Set $r = \inf_{s\in[a,b]} |\gamma(s)+c_0|$, which is strictly positive because $[a,b]$ is compact, γ is continuous, and $\gamma(s)+c_0 \neq 0$ for all $s \in [a,b]$. Choose

$$\delta = \min\left(\frac{r}{2},\ \frac{r^2\varepsilon}{2(1+M)}\right) > 0.$$

Suppose $c \in [0,\infty)$ satisfies $|c-c_0| < \delta$. Since $|\gamma(s)+c_0| > r$ for all $s \in [a,b]$, and since $|c-c_0| < \frac{1}{2}r$, it follows that $|\gamma(s)+c_0| > \frac{1}{2}r$ for all $s \in [a,b]$. Therefore

$$\frac{1}{[\gamma(s)+c_0][\gamma(s)+c]} < \frac{2}{r^2}$$

for all $s \in [a,b]$. So

$$\begin{aligned}
\|F(c)-F(c_0)\|_\infty &= \sup_{s\in[a,b]}\left|\frac{\gamma'(s)}{\gamma(s)+c} - \frac{\gamma'(s)}{\gamma(s)+c_0}\right| = \sup_{s\in[a,b]}\left|\frac{\gamma'(s)(c_0-c)}{[\gamma(s)+c_0][\gamma(s)+c]}\right| \\
&\leq \left(\sup_{s\in[a,b]}|\gamma'(s)|\right)|c-c_0|\left(\sup_{s\in[a,b]}\frac{1}{[\gamma(s)+c_0][\gamma(s)+c]}\right) \\
&\leq M|c-c_0|\cdot\frac{2}{r^2} < \varepsilon.
\end{aligned}$$

This proves continuity of F at c_0.

Now we prove continuity of I. Let $\varepsilon > 0$. Choose

$$\delta = \frac{2\pi\varepsilon}{b-a} > 0.$$

Let $f,\ g \in C([a,b])$ satisfy $\|f-g\|_\infty < \delta$. Then (using Theorem 6.25 of Rudin at the first step)

$$|I(f)-I(g)| \leq \frac{1}{2\pi}\int_a^b |f-g| \leq \frac{\|f-g\|_\infty(b-a)}{2\pi} < \frac{\delta(b-a)}{2\pi} = \varepsilon.$$

So I is continuous. ∎

Problem 8.25: Let γ_1 and γ_2 be closed curves as in Problem 8.23. Suppose that

$$|\gamma_1(t)-\gamma_2(t)| < |\gamma_1(t)|$$

for $t \in [a,b]$. Prove that $\operatorname{Ind}(\gamma_1) = \operatorname{Ind}(\gamma_2)$.

Hint: Put $\gamma(t) = \gamma_1(t)/\gamma_2(t)$. Then $|1 - \gamma(t)| < 1$ for all t, so Problem 8.23 implies that $\mathrm{Ind}(\gamma) = 0$. Also,

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{\gamma_2'(t)}{\gamma_2(t)} - \frac{\gamma_1'(t)}{\gamma_1(t)}$$

for all $t \in [a, b]$.

Comment: In fact, $\mathrm{Ind}(\gamma)$ can be defined for arbitrary continuous closed curves in $\mathbf{C} \setminus \{0\}$, and $\mathrm{Ind}(\gamma)$ is a homotopy invariant of the curve. (Problem 8.26 contains enough to prove this.) Since $\mathrm{Ind}(\gamma) \in \mathbf{Z}$, it follows that two homotopic curves in $\mathbf{C}\setminus\{0\}$ have the same index. Therefore the map $[\gamma] \mapsto \mathrm{Ind}(\gamma)$ defines a function from $\pi_1(\mathbf{C} \setminus \{0\})$ to $\mathbf{Z}$. It is easy to check that this map is a surjective homomorphism. (Injectivity is harder, I think, unless of course you already know the fundamental group.)

Solution (Sketch): The first part of the hint is proved by writing

$$|1 - \gamma(t)| = \left| \frac{\gamma_2(t)}{\gamma_2(t)} - \frac{\gamma_1(t)}{\gamma_2(t)} \right|.$$

It follows that the range of γ does not intersect the negative real axis. So $\mathrm{Ind}(\gamma) = 0$ by Problem 8.23. The equation (gotten from the quotient rule)

$$\frac{\gamma'(t)}{\gamma(t)} = \frac{\gamma_2'(t)}{\gamma_2(t)} - \frac{\gamma_1'(t)}{\gamma_1(t)}$$

shows that $\mathrm{Ind}(\gamma) = \mathrm{Ind}(\gamma_2) - \mathrm{Ind}(\gamma_1)$. ∎

MATH 413 [513] (PHILLIPS) SOLUTIONS TO HOMEWORK 9

Generally, a "solution" is something that would be acceptable if turned in in the form presented here, although the solutions given are usually close to minimal in this respect. A "solution (sketch)" is too sketchy to be considered a complete solution if turned in; varying amounts of detail would need to be filled in. A "solution (nearly complete)" is missing the details in just a few places; it would be considered a not quite complete solution if turned in.

Problem 8.12: Let $\delta \in (0, \pi)$, and let $f_\delta \colon \mathbf{R} \to \mathbf{C}$ be the 2π-periodic function given on $[-\pi, \pi]$ by

$$f_\delta(x) = \begin{cases} 1 & |x| \leq \delta \\ 0 & \delta < |x| \leq \pi \end{cases} .$$

(a) Compute the Fourier coefficients of f_δ.

Solution (Sketch): This is a computation, and gives $c_n = \frac{1}{\pi n} \sin(n\delta)$ for $n \neq 0$, and $c_0 = \frac{1}{\pi}\delta$. ∎

(b) Conclude from Part (a) that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

Solution (Sketch): Apply Theorem 8.14 of Rudin to the function f_δ at $x = 0$ to get

$$1 = \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{\pi n},$$

and rearrange. ∎

(c) Deduce from Parseval's Theorem that

$$\sum_{n=1}^{\infty} \frac{[\sin(n\delta)]^2}{n^2\delta} = \frac{\pi - \delta}{2}.$$

Solution (Sketch): Applying Parseval's Theorem to the result of Part (a) gives

$$\left(\frac{\delta}{\pi}\right)^2 + 2\sum_{n=1}^{\infty} \left(\frac{\sin(n\delta)}{\pi n}\right)^2 = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} |f_\delta|^2 = \frac{\delta}{\pi}.$$

Now multiply by $\pi^2\delta^{-1}$ and rearrange. ∎

(d) Let $\delta \to 0$ and prove that

$$\int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx = \frac{\pi}{2}.$$

Date: 15 March 2002.

Solution (Sketch): Essentially, we want to interpret the sum in Part (c) as a Riemann sum for the improper integral. To to this, we must effectively interchange limit operations.

Let $\varepsilon > 0$. Choose an integer M such that $M > 4\varepsilon^{-1}$. Note that

$$0 \leq \int_M^\infty \left(\frac{\sin(x)}{x}\right)^2 dx \leq \int_M^\infty \left(\frac{1}{x}\right)^2 dx = \frac{1}{M} < \tfrac{1}{4}\varepsilon$$

and, for any integer $n > 0$,

$$0 \leq \frac{1}{n} \sum_{k=nM+1}^{\infty} \left(\frac{\sin\left(\frac{k}{n}\right)}{\left(\frac{k}{n}\right)}\right)^2 \leq \frac{1}{n} \sum_{k=nM+1}^{\infty} \left(\frac{1}{\left(\frac{k}{n}\right)}\right)^2 \leq \int_M^\infty \left(\frac{1}{x}\right)^2 dx = \frac{1}{M} < \tfrac{1}{4}\varepsilon.$$

Using uniform continuity and the Riemann sum interpretation, choose an integer N so large that if $n \geq N$ then

$$\left| \frac{1}{n} \sum_{k=1}^{nM} \left(\frac{\sin\left(\frac{k}{n}\right)}{\left(\frac{k}{n}\right)}\right)^2 - \int_0^M \left(\frac{\sin(x)}{x}\right)^2 dx \right| < \tfrac{1}{4}\varepsilon.$$

Also require that $\frac{1}{2N} < \frac{1}{4}\varepsilon$. Using the triangle inequality several times, this gives, when $n \geq N$,

$$\left| \frac{1}{n} \sum_{k=1}^{\infty} \left(\frac{\sin\left(\frac{k}{n}\right)}{\left(\frac{k}{n}\right)}\right)^2 - \int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx \right| < \tfrac{3}{4}\varepsilon.$$

Part (c) (with $\delta = \frac{1}{n}$) therefore gives, when $n \geq N$,

$$\left| \frac{\pi}{2} - \int_0^\infty \left(\frac{\sin(x)}{x}\right)^2 dx \right| < \varepsilon.$$

∎

(e) Put $\delta = \frac{1}{2}\pi$ in Part (c). What do you get?

Solution (Sketch): Only the terms with odd n appear. Therefore we get

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \left(\frac{\pi}{2}\right)\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8}.$$

∎

Problem 8.13: Let $f \colon \mathbf{R} \to \mathbf{C}$ be the 2π-periodic function given on $[0, 2\pi)$ by $f(x) = x$. Apply Parseval's Theorem to f to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution (Sketch): A computation (integration by parts) shows that f has the Fourier coefficients $c_n = \frac{i}{n}$ for $n \neq 0$. Also, $c_0 = \pi$. So Parseval's Theorem gives

$$\pi^2 + 2\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{2\pi} \int_0^{2\pi} x^2 \, dx = \tfrac{4}{3}\pi^2.$$

Now solve for $\sum_{n=1}^{\infty} \frac{1}{n^2}$. ∎

Problem 8.14: Define $f\colon [-\pi, \pi] \to \mathbf{R}$ by $f(x) = (\pi - |x|)^2$. Prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx)$$

for all x, and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Solution (Sketch): We take f to be the 2π periodic function defined on all of $\mathbf{R}$ by $f(x) = (\pi - |x - 2\pi n|)^2$ for $n \in \mathbf{Z}$ and $x \in [(2n-1)\pi, \, (2n+1)\pi]$. This formula gives two different definitions at each point $(2n+1)\pi$ with $n \in \mathbf{Z}$, but both agree; it is then easy to check that f is continuous, and in particular Riemann integrable over any interval of length 2π.

Next, we find the Fourier coefficients c_n. For this, observe that $f(x) = (\pi - x)^2$ for all $x \in [0, 2\pi]$. For $n \neq 0$, set

$$g_n(x) = \frac{i}{n} \, e^{-inx}(\pi - x)^2 - \frac{2}{n^2} \, e^{-inx}(\pi - x) - \frac{2i}{n^3} \, e^{-inx}$$

for $x \in \mathbf{R}$. Then a calculation shows that $g_n'(x) = e^{-inx}(\pi - x)^2$ for all n and x. (The formula can be found by integrating by parts twice. However, that isn't part of the proof of the problem.) So the Fundamental Theorem of Calculus gives

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} f(x) \, dx = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-inx}(\pi - x)^2 \, dx = \frac{1}{2\pi} \left[g_n(2\pi) - g_n(0) \right] = \frac{2}{n^2}.$$

(When calculating $g_n(2\pi) - g_n(0)$, most of the terms cancel out.) Similarly,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{0}^{2\pi} (\pi - x)^2 \, dx = \frac{\pi^2}{3}.$$

It follows that the partial sum $s_n(f; x)$ (in the notation of Section 8.13 of Rudin's book) is given by

$$s_n(f; x) = \sum_{k=-n}^{n} c_k e^{ikx} = \frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{2}{k^2} \left[e^{ikx} + e^{-ikx} \right] = \frac{\pi^2}{3} + \sum_{k=1}^{n} \frac{4}{k^2} \, \cos(kx).$$

We now verify the hypotheses of Theorem 8.14 of Rudin's book, for every $x \in \mathbf{R}$. For $x \notin 2\pi\mathbf{Z}$, this is easy from the differentiability of f at x. (For $x \in (2n+1)\pi\mathbf{Z}$, use $f(x) = [(2n+1)\pi - x]^2$ for $x \in [2\pi n, \, 2\pi(n+2)]$.) For $x = 0$ we estimate directly. If $|t| < 2\pi$, then

$$f(t) - f(0) = (\pi - |t|)^2 - \pi^2 = |t|(|t| - 2\pi)$$

and $-2\pi \leq |t| - 2\pi \leq 0$, so $|f(t) - f(0)| \leq 2\pi|t|$. This is the required estimate. For other values of $x \in 2\pi\mathbf{Z}$, the required condition follows from periodicity.

It now follows from Theorem 8.14 of Rudin's book that $\lim_{n\to\infty} s_n(f; x) = f(x)$ for all $x \in \mathbf{R}$. That is,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos(nx)$$

for all $x \in \mathbf{R}$.

Putting $x = 0$ gives

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2},$$

from which it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

To get the formula for $\sum_{n=1}^{\infty} \frac{1}{n^4}$, compute $\|f\|_2^2$ two ways: directly and using Parseval's Theorem. Then compare the results. ∎

Problem 8.15: With

$$D_N(x) = \sum_{n=-N}^{N} e^{inx} = \frac{\sin((N+\frac{1}{2})x)}{\sin(\frac{1}{2}x)}$$

as in Section 8.13 of Rudin's book, define

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \left(\frac{1}{N+1}\right)\left(\frac{1-\cos((N+1)x)}{1-\cos(x)}\right)$$

for $x \in \mathbf{R} \setminus 2\pi\mathbf{Z}$.

Solution (Sketch): This may not be the best way, but it will work. Write

$$K_N(x) = \left(\frac{1}{N+1}\right)\left(\frac{1}{\sin(\frac{1}{2}x)}\right) \sum_{n=0}^{N} \frac{1}{2i} \left(\exp(i(n+\tfrac{1}{2})x) - \exp(-i(n+\tfrac{1}{2})x)\right)$$

and use the formula for the sum of a geometric series. Then do a little rearranging. ∎

a. Prove that $K_N(x) \geq 0$.

Solution (Sketch): In the formula above, both the numerator and denominator are nonnegative. ∎

b. Prove that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x)\, dx = 1.$$

Solution (Sketch): This is immediate from the relations

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x) \quad \text{and} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x)\, dx = 1.$$

∎

c. If $\delta > 0$, then

$$K_N(x) \leq \left(\frac{1}{N+1}\right)\left(\frac{2}{1-\cos(\delta)}\right)$$

for $\delta \leq |x| \leq \pi$.

Solution (Sketch): Since K_N is an even function, it suffices to prove this for $\delta \leq x \leq \pi$. Use

$$1 - \cos(x) \geq 1 - \cos(\delta) \quad \text{and} \quad 1 - \cos((N+1)x) \leq 2.$$

∎

Now let $s_n(f; x)$ be as in Section 8.13 of Rudin's book, and define

$$\sigma_N(f; x) = \frac{1}{N+1} \sum_{n=0}^{N} s_n(f; x).$$

Prove that

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t)\, dt,$$

and use this to prove Fejér's Theorem: If $f \colon \mathbf{R} \to \mathbf{C}$ is continuous and 2π periodic, then $\sigma_N(f; x)$ converges uniformly to $f(x)$ on $[-\pi, \pi]$.

Hint: Use (a), (b), and (c) to proceed as in the proof of Theorem 7.26 of Rudin's book.

Solution (Sketch): The formula

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t)\, dt$$

follows from the definitions of $\sigma_N(f; x)$ and $K_N(x)$ and the formula

$$s_n(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_n(t)\, dt.$$

This last formula was proved in Section 8.13 of Rudin's book.

Now assume f is not the zero function. (The result is trivial in this case, and I want to divide by $\|f\|_\infty$.) Let $\varepsilon > 0$. Since f is continuous and periodic, it is uniformly continuous. (Check this!) So there is $\delta_0 > 0$ such that whenever $|t| < \delta_0$ then $|f(x-t) - f(x)| < \frac{1}{2}\varepsilon$. Set $\delta = \frac{1}{2}\delta_0$. Choose N so large that

$$\left(\frac{2\pi \|f\|_\infty}{n+1} \right) \left(\frac{2}{1 - \cos(\delta)} \right) < \varepsilon.$$

For any $n \geq N$ and $x \in \mathbf{R}$, write

$$\begin{aligned}
|\sigma_n(f; x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_n(t)\, dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) K_n(t)\, dt \right| \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(t)|\, dt \\
&= \frac{1}{2\pi} \left(\int_{-\pi}^{-\delta} |f(x-t) - f(t)| K_n(t)\, dt + \int_{-\delta}^{\delta} |f(x-t) - f(t)| K_n(t)\, dt \right. \\
&\qquad \left. + \int_{\delta}^{\pi} |f(x-t) - f(t)| K_n(t)\, dt \right) \\
&\leq \frac{1}{2\pi} \left(2\|f\|_\infty \int_{-\pi}^{-\delta} K_n(t)\, dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} K_n(t)\, dt + 2\|f\|_\infty \int_{\delta}^{\pi} K_n(t)\, dt \right).
\end{aligned}$$

The estimates at the last step are obtained as follows. For the first and third terms,

$$|f(x-t) - f(t)| \leq |f(x-t)| + |f(t)| \leq 2\|f\|_\infty.$$

For the middle term, $|f(x-t) - f(x)| < \frac{1}{2}\varepsilon$ because $\delta < \delta_0$.

Now

$$\int_{-\pi}^{-\delta} K_n(t)\, dt \leq (\pi - \delta)\left(\frac{1}{n+1}\right)\left(\frac{2}{1-\cos(\delta)}\right) < \frac{\varepsilon}{2\|f\|_\infty}.$$

Similarly

$$\int_{\delta}^{\pi} K_n(t)\, dt < \frac{\varepsilon}{2\|f\|_\infty}.$$

Also,

$$\int_{-\delta}^{\delta} K_n(t)\, dt \leq 2\pi.$$

Inserting these estimates, we get

$$\begin{aligned} |\sigma_n(f;x) - f(x)| &\leq \frac{1}{2\pi}\left(2\|f\|_\infty \int_{-\pi}^{-\delta} |K_n(t)\, dt + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} K_n(t)\, dt + 2\|f\|_\infty \int_{\delta}^{\pi} K_n(t)\, dt\right) \\ &\leq \frac{1}{2\pi}\left(\varepsilon + 2\pi \cdot \tfrac{1}{2}\varepsilon + \varepsilon\right) = \left(\frac{2+\pi}{2\pi}\right)\varepsilon < \varepsilon. \end{aligned}$$

This proves uniform convergence. ∎

Problem B:

(1) Let X be a complete metric space, let $x_0 \in X$, let $r > 0$, and let $C < 1$. Let $f\colon N_r(x_0) \to X$ be a function such that $d(f(x),\, f(y)) \leq C d(x,y)$ for all $x,\, y \in N_r(x_0)$ and $d(f(x_0),\, x_0) < (1-C)r$. Prove that f has a unique fixed point z, that is, there is a unique $z \in N_r(x_0)$ such that $f(z) = z$. Further prove that

$$d(z, x_0) \leq \frac{df(x_0),\, x_0)}{1-C}.$$

Solution (sketch): This is essentially the same as Problem A(2). Uniqueness follows easily from the condition $C < 1$.

We prove by induction on n the combined statement:

(1) $f^n(x_0)$ is defined.
(2) $d(f^n(x_0),\, f^{n-1}(x_0)) \leq C^{n-1} d(f(x_0),\, x_0))$.
(3) $d(f^n(x_0),\, x_0) \leq \dfrac{1-C^{n-1}}{1-C}\, d(f(x_0),\, x_0)$.

For $n = 1$, this is immediate. Suppose it is true for n. Condition (3) for n shows that

$$d(f^n(x_0),\, x_0) < \left(\frac{1-C^n}{1-C}\right)(1-C)r = (1-C^{n-1})r < r,$$

so $f^n(x_0) \in N_r(x_0)$ and $f^{n+1}(x_0) = f(f^n(x_0))$ is defined. This is Condition (1) for $n+1$. Then Condition (2) for n and the hypotheses imply

$$\begin{aligned} d(f^{n+1}(x_0),\, f^n(x_0)) &\leq C d(f^n(x_0),\, f^{n-1}(x_0)) \\ &\leq C \cdot C^{n-1} d(f(x_0),\, x_0)) = C^n d(f(x_0),\, x_0)), \end{aligned}$$

which is Condition (2) for $n+1$. Finally

$$\begin{aligned} d(f^{n+1}(x_0),\, x_0) &\leq d(f^{n+1}(x_0),\, f^n(x_0)) + d(f^n(x_0),\, x_0) \\ &\leq C^n d(f(x_0),\, x_0)) + \frac{1-C^{n-1}}{1-C}\, d(f(x_0),\, x_0)) \\ &= \frac{1-C^n}{1-C}\, d(f(x_0),\, x_0)). \end{aligned}$$

This is Condition (3) for $n+1$.

It follows as in previous similar problems that the sequence $(f^n(x_0))$ is Cauchy, and hence has a limit z. Moreover,

$$\begin{aligned} d(z, x_0) \leq \sup_{n \in \mathbf{N}} d(f^n(x_0),\, x_0) &\leq \sup_{n \in \mathbf{N}} \frac{1-C^n}{1-C}\, d(f(x_0),\, x_0)) \\ &\leq \left(\frac{1}{1-C}\right) d(f(x_0),\, x_0)) < r, \end{aligned}$$

so $z \in N_r(x_0)$ and $f(z)$ is defined. It now follows from continuity, as in previous similar problems, that $f(z) = z$. ∎

(2) Let $I,\, J \subset \mathbf{R}$ be open intervals, let $x_0 \in I$, and let $y_0 \in J$. Let $\varphi \colon I \times J \to \mathbf{R}$ be a continuous function and assume that there is a constant M such that.

$$|\varphi(x, y_2) - \varphi(x, y_1)| \leq M|y_2 - y_1|$$

for all $x \in I$ and $y_1,\, y_2 \in J$. Prove that there is $\delta > 0$ such that there is a unique function $f \colon (x_0 - \delta,\, x_0 + \delta) \to J$ satisfying $f(x_0) = y_0$ and $f'(x) = \varphi(x,\, f(x))$ for all $x \in (x_0 - \delta,\, x_0 + \delta)$.

In this problem, you may assume the standard properties of $\int_a^b f$ when $a \leq b$, including the correct version of the Fundamental Theorem of Calculus.

Hint: For a suitable $\delta > 0$ and a suitable subset $N \subset C_{\mathbf{R}}([x_0 - \delta,\, x_0 + \delta])$, define a function $F \colon N \to C_{\mathbf{R}}([x_0 - \delta,\, x_0 + \delta])$ by

$$F(g)(x) = y_0 + \int_{x_0}^{x} \varphi(t,\, g(t))\, dt.$$

If δ and N are chosen correctly, the first part will imply that F has a unique fixed point f. Prove that this fixed point solves the differential equation.

Solution (Sketch): Choose $r > 0$ such that $[y_0 - r,\, y_0 + r] \subset J$. Choose $\delta_0 > 0$ such that $[x_0 - \delta_0,\, x_0 + \delta_0] \subset I$. Set

$$K = \sup_{t \in [x_0 - \delta_0,\, x_0 + \delta_0]} |\varphi(t, y_0)|.$$

Let $\delta > 0$ be any number satisfying

$$\delta \leq \min\left(\frac{1}{2M},\, \frac{r}{3K}\right).$$

In the metric space $C_{\mathbf{R}}([x_0 - \delta,\, x_0 + \delta])$, let g_0 be the constant function $g_0(x) = y_0$ for all $x \in [x_0 - \delta,\, x_0 + \delta]$, and set $N = N_r(g_0)$. Following the hint, define $F \colon N \to C_{\mathbf{R}}([x_0 - \delta,\, x_0 + \delta])$ by

$$F(g)(x) = y_0 + \int_{x_0}^{x} \varphi(t,\, g(t))\, dt.$$

One needs to check that $F(g)(x)$ is defined, that is, that $(t, g(t))$ is in the domain of φ and that $t \mapsto \varphi(t, g(t))$ is Riemann integrable. Both are easy; in fact, $t \mapsto \varphi(t, g(t))$ is continuous. One also needs to observe that $F(g)$ is in fact a continuous function.

We now verify the hypotheses of Part (1) for this function. Let $g_1, g_2 \in N$. For $x \in [x_0 - \delta, x_0 + \delta]$, we then have

$$\begin{aligned} |F(g_1)(x) - F(g_2)(x)| &= \left| \int_{x_0}^{x} [\varphi(t, g_1(t)) - \varphi(t, g_2(t))]\, dt \right| \\ &\leq |x - x_0| \sup_{t \in [x_0 - \delta, x_0 + \delta]} |\varphi(t, g_1(t)) - \varphi(t, g_2(t))| \\ &\leq \delta M \sup_{t \in [x_0 - \delta, x_0 + \delta]} |g_1(t) - g_2(t)| = \delta M \|g_1 - g_2\|_\infty. \end{aligned}$$

Therefore

$$\|F(g_1) - F(g_2)\|_\infty \leq \delta M \|g_1 - g_2\|_\infty \leq \tfrac{1}{2} \|g_1 - g_2\|_\infty.$$

Thus, F is a contraction with constant $C = \frac{1}{2}$. Moreover, for $x \in [x_0 - \delta, x_0 + \delta]$ we have

$$\begin{aligned} |F(g_0)(x) - g_0(x)| &= \left| \int_{x_0}^{x} \varphi(t, y_0)\, dt \right| \\ &\leq |x - x_0| \left(\sup_{t \in [x_0 - \delta, x_0 + \delta]} |\varphi(t, y_0)| \right) \leq \delta K \leq \tfrac{1}{3} r < (1 - C) r. \end{aligned}$$

We can now apply Part (1) to conclude that there is a unique $g \in C_{\mathbf{R}}([x_0 - \delta, x_0 + \delta])$ such that $F(g) = g$, that is,

$$g(x) = y_0 + \int_{x_0}^{x} \varphi(t, g(t))\, dt$$

for all $x \in [x_0 - \delta, x_0 + \delta]$. Since $t \mapsto \varphi(t, g(t))$ is continuous, the Fundamental Theorem of Calculus shows that the right hand side of this equation is differentiable as a function of x, with derivative $x \mapsto \varphi(x, g(x))$. Thus $g'(x) = \varphi(x, g(x))$ for all $x \in [x_0 - \delta, x_0 + \delta]$. That $g(x_0) = y_0$ is immediate.

We have proved existence of a solution on $(x_0 - \delta, x_0 + \delta)$ with

$$\delta = \min\left(\frac{1}{2M}, \frac{r}{3K} \right).$$

Since $g'(x) = \varphi(x, g(x))$ and $g(x_0) = y_0$ imply

$$g(x) = y_0 + \int_{x_0}^{x} \varphi(t, g(t))\, dt,$$

we have also proved uniqueness among all continuous functions $g \colon (x_0 - \delta, x_0 + \delta) \to \mathbf{R}$ satisfying in addition $|g(x) - y_0| < r$ for all x.

We now show that this implies uniqueness among all continuous functions $g \colon (x_0 - \delta, x_0 + \delta) \to J$. Suppose that, with δ as above, there is some solution $h \colon (x_0 - \delta, x_0 + \delta) \to J$, with $|h(x) - y_0| \geq r$ for some x. Set

$$\rho = \inf(\{x \in (x_0 - \delta, x_0 + \delta) \colon |h(x) - y_0| \geq r\}).$$

Then $0 < \rho < \delta$, and at least one of $|h(x_0 + \rho) - y_0| = r$ and $|h(x_0 - \rho) - y_0| = r$ must hold. Uniqueness as proved above applies on the interval $(x_0 - \rho, x_0 + \rho)$, so that $g(x) = h(x)$ for all $x \in (x_0 - \rho, x_0 + \rho)$. By continuity,

$$h(x_0 + \rho) = g(x_0 + \rho) \quad \text{and} \quad h(x_0 - \rho) = g(x_0 - \rho).$$

Since $|g(x_0 + \rho) - y_0| < r$ and $|g(x_0 - \rho) - y_0| < r$, this is a contradiction. ∎