Mayer-Vietoris sequence

In <u>mathematics</u>, particularly <u>algebraic topology</u> and <u>homology theory</u>, the **Mayer–Vietoris sequence** is an <u>algebraic</u> tool to help compute <u>algebraic</u> invariants of <u>topological</u> spaces, known as their homology and cohomology groups. The result is due to two <u>Austrian</u> mathematicians, <u>Walther Mayer</u> and <u>Leopold Vietoris</u>. The method consists of splitting a space into <u>subspaces</u>, for which the homology or cohomology groups may be easier to compute. The sequence relates the (co)homology groups of the space to the (co)homology groups of the subspaces. It is a <u>natural long exact sequence</u>, whose entries are the (co)homology groups of the whole space, the <u>direct sum</u> of the (co)homology groups of the <u>subspaces</u>, and the (co)homology groups of the <u>intersection</u> of the subspaces.

The Mayer–Vietoris sequence holds for a variety of cohomology and homology theories, including simplicial homology and singular cohomology. In general, the sequence holds for those theories satisfying the Eilenberg–Steenrod axioms, and it has variations for both reduced and relative (co)homology. Because the (co)homology of most spaces cannot be computed directly from their definitions, one uses tools such as the Mayer–Vietoris sequence in the hope of obtaining partial information. Many spaces encountered in topology are constructed by piecing together very simple patches. Carefully choosing the two covering subspaces so that, together with their intersection, they have simpler (co)homology than that of the whole space may allow a complete deduction of the (co)homology of the space. In that respect, the Mayer–Vietoris sequence is analogous to the Seifert–van Kampen theorem for the fundamental group, and a precise relation exists for homology of dimension one.

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Background, motivation, and history

Like the <u>fundamental group</u> or the higher <u>homotopy groups</u> of a space, homology groups are important topological invariants. Although <u>some</u> (co)homology theories are computable using tools of <u>linear algebra</u>, many other important (co)homology theories, especially singular (co)homology, are not computable directly from their definition for nontrivial spaces. For singular (co)homology, the singular (co)chains and (co)cycles groups are often too big to handle directly. More subtle and indirect approaches become necessary. The Mayer–Vietoris sequence is such an approach, giving partial information about the (co)homology groups of any space by relating it to the (co)homology groups of two of its subspaces and their intersection.

The most natural and convenient way to express the relation involves the algebraic concept of <u>exact sequences</u>: sequences of <u>objects</u> (in this case <u>groups</u>) and <u>morphisms</u> (in this case <u>group</u> homomorphisms) between them such that the image of one morphism equals the kernel of the



Leopold Vietoris on his 110th birthday

next. In general, this does not allow (co)homology groups of a space to be completely computed. However, because many important spaces encountered in topology are topological manifolds, simplicial complexes, or CW complexes, which are constructed by piecing together very simple patches, a theorem such as that of Mayer and Vietoris is potentially of broad and deep applicability.

Mayer was introduced to topology by his colleague Vietoris when attending his lectures in 1926 and 1927 at a local university in Vienna. He was told about the conjectured result and a way to its solution, and solved the question for the Betti numbers in 1929. He applied his results to the torus considered as the union of two cylinders. Vietoris later proved the full result for the homology groups in 1930 but did not express it as an exact sequence. The concept of an exact sequence only appeared in print in the 1952 book Foundations of Algebraic Topology by Samuel Eilenberg and Norman Steenrod where the results of Mayer and Vietoris were expressed in the modern form.

Basic versions for singular homology

Let X be a topological space and A, B be two subspaces whose interiors cover X. (The interiors of A and B need not be disjoint.) The Mayer–Vietoris sequence in singular homology for the triad (X, A, B) is a long exact sequence relating the singular homology groups (with coefficient group the integers \mathbb{Z}) of the spaces X, A, B, and the intersection $A \cap B$. There is an unreduced and a reduced version.

Unreduced version

For unreduced homology, the Mayer–Vietoris sequence states that the following sequence is exact:^[9]

$$\cdots
ightarrow H_{n+1}(X) {\stackrel{\partial_*}{\longrightarrow}} H_n(A \cap B) {\stackrel{(i_*,j_*)}{\longrightarrow}} H_n(A) \oplus H_n(B) {\stackrel{k_*-l_*}{\longrightarrow}} H_n(X) {\stackrel{\partial_*}{\longrightarrow}} H_{n-1}(A \cap B)
ightarrow \cdots
ightarrow H_0(A) \oplus H_0(B) {\stackrel{k_*-l_*}{\longrightarrow}} H_0(X)
ightarrow 0.$$

Here $i:A\cap B\hookrightarrow A, j:A\cap B\hookrightarrow B, k:A\hookrightarrow X$, and $l:B\hookrightarrow X$ are <u>inclusion maps</u> and \oplus denotes the <u>direct sum of abelian groups</u>.

Boundary map

The boundary maps ∂_* lowering the dimension may be defined as follows. [10] An element in $H_n(X)$ is the homology class of an n-cycle x which, by <u>barycentric subdivision</u> for example, can be written as the sum of two n-chains u and v whose images lie wholly in A and B, respectively. Thus $\partial x = \partial(u+v) = 0$ so that $\partial u = -\partial v$. This implies that the images of both these boundary (n-1)-cycles are contained in the intersection $A \cap B$. Then $\partial_*([x])$ can be defined to be the class of ∂u in $H_{n-1}(A \cap B)$. Choosing another decomposition x = u' + v' does not affect $[\partial u]$, since $\partial u + \partial v = \partial x = \partial u' + \partial v'$, which implies $\partial u - \partial u' = \partial(v' - v)$, and therefore ∂u and $\partial u'$ lie in the same homology class; nor does choosing a different representative x', since then $\partial x' = \partial x = 0$. Notice that the maps in the Mayer–Vietoris sequence depend on choosing an order for A and B. In particular, the boundary map changes sign if A and B are swapped.

Reduced version

For <u>reduced homology</u> there is also a Mayer–Vietoris sequence, under the assumption that A and B have <u>non-empty</u> intersection.^[11] The sequence is identical for positive dimensions and ends as:

$$\cdots o ilde{H}_0(A\cap B) {\overset{(i_*,j_*)}{\longrightarrow}} ilde{H}_0(A) \oplus ilde{H}_0(B) {\overset{k_*-l_*}{\longrightarrow}} ilde{H}_0(X) o 0.$$

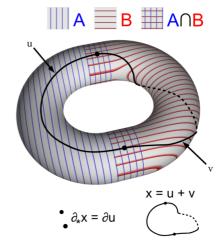


Illustration of the boundary map ∂_* on the torus where the 1-cycle x = u + v is the sum of two 1-chains whose boundary lies in the intersection of A and B.

Analogy with the Seifert-van Kampen theorem

There is an analogy between the Mayer–Vietoris sequence (especially for homology groups of dimension 1) and the <u>Seifert–van</u> Kampen theorem. [10][12] Whenever $A \cap B$ is path-connected, the reduced Mayer–Vietoris sequence yields the isomorphism

$$H_1(X) \cong (H_1(A) \oplus H_1(B))/\mathrm{Ker}(k_* - l_*)$$

where, by exactness,

$$\operatorname{Ker}(k_* - l_*) \cong \operatorname{Im}(i_*, j_*).$$

This is precisely the <u>abelianized</u> statement of the Seifert-van Kampen theorem. Compare with the fact that $H_1(X)$ is the abelianization of the fundamental group $\pi_1(X)$ when X is path-connected. [13]

Basic applications

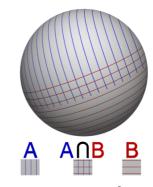
k-sphere

To completely compute the homology of the <u>k</u>-sphere $X = S^k$, let A and B be two hemispheres of X with intersection homotopy equivalent to a (k-1)-dimensional equatorial sphere. Since the <u>k</u>-dimensional hemispheres are <u>homeomorphic</u> to <u>k</u>-discs, which are <u>contractible</u>, the homology groups for A and B are <u>trivial</u>. The Mayer-Vietoris sequence for reduced homology groups then yields

$$\cdots \longrightarrow 0 \longrightarrow \tilde{H}_n \left(S^k \right) \stackrel{\partial_*}{\longrightarrow} \tilde{H}_{n-1} \left(S^{k-1} \right) \longrightarrow 0 \longrightarrow \cdots$$

Exactness immediately implies that the map ∂_* is an isomorphism. Using the <u>reduced</u> homology of the o-sphere (two points) as a base case, it follows^[14]

$$ilde{H}_nig(S^kig)\cong \delta_{kn}\, \mathbb{Z} = egin{cases} \mathbb{Z} & ext{if } n=k \ 0 & ext{if } n
eq k \end{cases}$$



The decomposition for $X = S^2$

where δ is the Kronecker delta. Such a complete understanding of the homology groups for

spheres is in star-bounded and bounded an

Klein bottle

A slightly more homology group Möbius strips A and their intersequence yields^[]

In mathematics, the **Kronecker delta** is a function of two variables, usually just non-negative integers. The function is 1 if the variables are equal, and 0 otherwise:

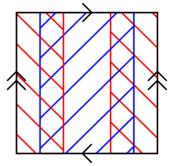
$$\delta_{ij} = \left\{egin{array}{ll} 0 & ext{if } i
eq j, \ 1 & ext{if } i = j. \end{array}
ight.$$

sequence is the calculation of the mposition of X as the union of two illustration on the right). Then A, B ircles, so the nontrivial part of the

$$0 o H_2(X) o \mathbb{Z} \stackrel{lpha}{ o} \mathbb{Z} \oplus \mathbb{Z} o H_1(X) o 0$$

and the trivial part implies vanishing homology for dimensions greater than 2. The central map α sends 1 to (2, -2) since the boundary circle of a Möbius band wraps twice around the core circle. In particular α is injective so homology of dimension 2 also vanishes. Finally, choosing (1, 0) and (1, -1) as a basis for \mathbb{Z}^2 , it follows

$$ilde{H}_{n}\left(X
ight)\cong\delta_{1n}\left(\mathbb{Z}\oplus\mathbb{Z}_{2}
ight)=\left\{egin{array}{ll} \mathbb{Z}\oplus\mathbb{Z}_{2} & ext{if }n=1\ 0 & ext{if }n
eq 1 \end{array}
ight.$$



The Klein bottle (fundamental polygon with appropriate edge identifications) decomposed as two Möbius strips *A* (in blue) and *B* (in red).

Wedge sums

Let X be the <u>wedge sum</u> of two spaces K and L, and suppose furthermore that the identified <u>basepoint</u> is a deformation retract of open neighborhoods $U \subset K$ and $V \subset L$. Letting $A = K \cup V$ and $B = U \cup L$ it follows that $A \cup B = X$ and $A \cap B = U \cup V$, which is contractible by construction. The reduced version of the sequence then yields (by exactness)^[17]

$$ilde{H}_n(Kee L)\cong ilde{H}_n(K)\oplus ilde{H}_n(L)$$

for all dimensions n. The illustration on the right shows X as the sum of two 2-spheres K and L. For this specific case, using the result from above for 2-spheres, one has

$$ilde{H}_n\left(S^2ee S^2
ight)\cong \delta_{2n}\left(\mathbb{Z}\oplus\mathbb{Z}
ight)=\left\{egin{array}{ll} \mathbb{Z}\oplus\mathbb{Z} & ext{if }n=2\ 0 & ext{if }n
eq 2 \end{array}
ight.$$

This decomposition of the wedge sum X of two 2-spheres K and L yields all the homology groups of Y

Suspensions

If X is the <u>suspension</u> SY of a space Y, let A and B be the <u>complements</u> in X of the top and bottom 'vertices' of the double cone, respectively. Then X is the union $A \cup B$, with A and B contractible. Also, the intersection $A \cap B$ is homotopy equivalent to Y. Hence the Mayer–Vietoris sequence

$$\tilde{H}_n(SY) \cong \tilde{H}_{n-1}(Y)$$

yields, for all n, [18]

The illustration on the right shows the 1-sphere X as the suspension of the o-sphere Y. Noting in general that the k-sphere is the suspension of the (k-1)-sphere, it is easy to derive the homology groups of the k-sphere by induction, as above.

This decomposition of the suspension X of the 0-sphere Y yields all the homology groups of X.

Further discussion

Relative form

A relative form of the Mayer–Vietoris sequence also exists. If $Y \subset X$ and is the union of $C \subset A$ and $D \subset B$, then the exact sequence is: 19

$$\cdots o H_n(A\cap B,C\cap D) \overset{(i_*,j_*)}{\longrightarrow} H_n(A,C) \oplus H_n(B,D) \overset{k_*-l_*}{\longrightarrow} H_n(X,Y) \overset{\delta_*}{\longrightarrow} H_{n-1}(A\cap B,C\cap D) o \cdots$$

Naturality

The homology groups are <u>natural</u> in the sense that if $f: X_1 \to X_2$ is a <u>continuous</u> map, then there is a canonical <u>pushforward</u> map of homology groups $f_*: H_k(X_1) \to H_k(X_2)$ such that the composition of pushforwards is the pushforward of a composition: that is, $(g \circ h)_* = g_* \circ h_*$. The Mayer-Vietoris sequence is also natural in the sense that if

$$X_1 = A_1 \cup B_1 \qquad \qquad \text{and} \qquad f(A_1) \subset A_2 \ X_2 = A_2 \cup B_2 \qquad \qquad f(B_1) \subset B_2$$

then the connecting morphism of the Mayer-Vietoris sequence, ∂_* , commutes with f_* .^[20] That is, the following diagram commutes^[21] (the horizontal maps are the usual ones):

Cohomological versions

The Mayer-Vietoris long exact sequence for $\underline{\text{singular cohomology}}$ groups with coefficient $\underline{\text{group}}$ G is $\underline{\text{dual}}$ to the homological version. It is the following: [22]

$$\cdots \rightarrow H^n(X;G) \rightarrow H^n(A;G) \oplus H^n(B;G) \rightarrow H^n(A \cap B;G) \rightarrow H^{n+1}(X;G) \rightarrow \cdots$$

where the dimension preserving maps are restriction maps induced from inclusions, and the (co-)boundary maps are defined in a similar fashion to the homological version. There is also a relative formulation.

As an important special case when G is the group of <u>real numbers</u> \mathbf{R} and the underlying topological space has the additional structure of a smooth manifold, the Mayer–Vietoris sequence for de Rham cohomology is

$$\cdots \to H^n(X) \overset{
ho}{ o} H^n(U) \oplus H^n(V) \overset{\Delta}{ o} H^n(U \cap V) \overset{d^*}{ o} H^{n+1}(X) \to \cdots$$

where $\{U, V\}$ is an <u>open cover</u> of X, ρ denotes the restriction map, and Δ is the difference. The map d^* is defined similarly as the map ∂_* from above. It can be briefly described as follows. For a cohomology class $[\omega]$ represented by <u>closed form</u> ω in $U \cap V$, express ω as a difference of forms $\omega_U - \omega_V$ via a <u>partition of unity</u> subordinate to the open cover $\{U, V\}$, for example. The exterior derivative $d\omega_U$ and $d\omega_V$ agree on $U \cap V$ and therefore together define an n+1 form σ on X. One then has $d^*([\omega]) = [\sigma]$.

Derivation

Consider the long exact sequence associated to the short exact sequences of chain groups (constituent groups of chain complexes)

$$0 o C_n(A \cap B) \overset{lpha}{ o} C_n(A) \oplus C_n(B) \overset{eta}{ o} C_n(A+B) o 0$$

where $\alpha(x) = (x, -x)$, $\beta(x, y) = x + y$, and $C_n(A + B)$ is the chain group consisting of sums of chains in A and chains in B. [9] It is a fact that the singular n-simplices of X whose images are contained in either A or B generate all of the homology group $H_n(X)$. [23] In other words, $H_n(A + B)$ is isomorphic to $H_n(X)$. This gives the Mayer–Vietoris sequence for singular homology.

The same computation applied to the short exact sequences of vector spaces of differential forms

$$0 o \Omega^n(X) o \Omega^n(U) \oplus \Omega^n(V) o \Omega^n(U \cap V) o 0$$

yields the Mayer-Vietoris sequence for de Rham cohomology.^[24]

From a formal point of view, the Mayer–Vietoris sequence can be derived from the <u>Eilenberg–Steenrod axioms</u> for <u>homology</u> theories using the long exact sequence in homology.^[25]

Other homology theories

The derivation of the Mayer–Vietoris sequence from the Eilenberg–Steenrod axioms does not require the dimension axiom, [26] so in addition to existing in ordinary cohomology theories, it holds in extraordinary cohomology theories (such as topological K-theory and cobordism).

Sheaf cohomology

From the point of view of <u>sheaf cohomology</u>, the Mayer–Vietoris sequence is related to <u>Čech cohomology</u>. Specifically, it arises from the <u>degeneration</u> of the <u>spectral sequence</u> that relates <u>Čech cohomology</u> to sheaf cohomology (sometimes called the <u>Mayer–Vietoris spectral sequence</u>) in the case where the open cover used to compute the <u>Čech cohomology</u> consists of two open sets. [27] This spectral sequence exists in arbitrary topoi. [28]

See also

- Excision theorem
- Zig-zag lemma

Notes

- 1. Hirzebruch 1999
- 2. Mayer 1929
- 3. Dieudonné 1989, p. 39
- 4. Mayer 1929, p. 41
- 5. Vietoris 1930
- 6. Corry 2004, p. 345
- 7. Eilenberg & Steenrod 1952, Theorem 15.3
- 8. Eilenberg & Steenrod 1952, §15
- 9. Hatcher 2002, p. 149
- 10. Hatcher 2002, p. 150
- 11. Spanier 1966, p. 187
- 12. Massey 1984, p. 240
- 13. Hatcher 2002, Theorem 2A.1, p. 166
- 14. Hatcher 2002, Example 2.46, p. 150
- 15. Hatcher 2002, p. 384
- 16. Hatcher 2002, p. 151
- 17. Hatcher 2002, Exercise 31 on page 158
- 18. Hatcher 2002, Exercise 32 on page 158
- Hatcher 2002, p. 152
- 20. Massey 1984, p. 208
- 21. Eilenberg & Steenrod 1952, Theorem 15.4
- 22. Hatcher 2002, p. 203
- 23 Hatcher 2002 Proposition 2.21 n. 119

- .o. Hatonor 2002, 1 Topocition 2.21, p. 110
- 24. Bott & Tu 1982, §I.2
- 25. Hatcher 2002, p. 162
- 26. Kono & Tamaki 2006, pp. 25-26
- 27. Dimca 2004, pp. 35-36
- 28. Verdier 1972 (SGA 4.V.3)

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Further reading

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