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14. $\begin{pmatrix} 4 & -1 & -3 & 1 & 7 \\ -2 & 5 & -1 & -3 & 3 \\ 2 & 13 & -9 & -5 & 20 \end{pmatrix} \to \begin{pmatrix} 0 & 9 & -5 & -5 & 13 \\ -2 & 5 & -1 & -3 & 3 \\ 0 & 18 & -10 & -8 & 23 \end{pmatrix}$

$\to \begin{pmatrix} 0 & 9 & -5 & -5 & 13 \\ -2 & 5 & -1 & -3 & 3 \\ 0 & 0 & 0 & 2 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 9 & -5 & 0 & \frac{11}{2} \\ -2 & 5 & -1 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 2 & -3 \end{pmatrix}$

$\to \begin{pmatrix} 0 & 9 & -5 & 0 & \frac{11}{2} \\ -2 & \frac{16}{5} & 0 & 0 & -\frac{13}{5} \\ 0 & 0 & 0 & 2 & -3 \end{pmatrix} \to \begin{pmatrix} 0 & 9 & -5 & 0 & \frac{11}{2} \\ 5 & -8 & 0 & 0 & \frac{13}{2} \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix}$

取 $A = \begin{pmatrix} 0 & 9 & -5 & 0 \\ 5 & -8 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ $\quad b = \begin{pmatrix} \frac{11}{2} \\ \frac{13}{2} \\ 3 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 9 & -5 & 0 \\ 5 & -8 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = BC$ 是一个满秩分解

$CC^H = CC^T = \begin{pmatrix} 106 & -72 & 0 \\ -72 & 89 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$

$(CC^H)^{-1} = \begin{pmatrix} \frac{89}{4250} & \frac{36}{2125} & 0 \\ \frac{36}{2125} & \frac{53}{2125} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \qquad C^H(CC^H)^{-1} = \begin{pmatrix} \frac{36}{425} & \frac{53}{425} & 0 \\ \frac{9}{170} & -\frac{4}{85} & 0 \\ -\frac{89}{850} & -\frac{36}{425} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$

$A^+ = C^H(CC^H)^{-1}(B^HB)^{-1}B^H$

$= C^H(CC^H)^{-1}$

$$A^+b = \begin{pmatrix} \frac{36}{425} & \frac{53}{425} & 0 \\ \frac{9}{170} & -\frac{4}{85} & 0 \\ -\frac{89}{850} & -\frac{36}{425} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{11}{2} \\ \frac{13}{2} \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{217}{170} \\ -\frac{1}{68} \\ \frac{383}{340} \\ \frac{3}{2} \end{pmatrix}$$

$$A^+A = \begin{pmatrix} \frac{36}{425} & \frac{53}{425} & 0 \\ \frac{9}{170} & -\frac{4}{85} & 0 \\ -\frac{89}{850} & -\frac{36}{425} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 6 & 9 & -5 & 0 \\ 5 & -8 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{53}{85} & -\frac{4}{17} & -\frac{36}{85} & 0 \\ -\frac{4}{17} & \frac{29}{34} & -\frac{9}{34} & 0 \\ -\frac{36}{85} & -\frac{9}{34} & \frac{89}{170} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$E - A^+A = \begin{pmatrix} \frac{32}{85} & \frac{4}{17} & \frac{36}{85} & 0 \\ \frac{4}{17} & \frac{5}{34} & \frac{9}{34} & 0 \\ \frac{36}{85} & \frac{9}{34} & \frac{81}{170} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

原方程最小二乘解的通解公式为:

$$X = \begin{pmatrix} \frac{217}{170} \\ -\frac{1}{68} \\ \frac{383}{340} \\ \frac{3}{2} \end{pmatrix} + \begin{pmatrix} \frac{32}{85} & \frac{4}{17} & \frac{36}{85} & 0 \\ \frac{4}{17} & \frac{5}{34} & \frac{9}{34} & 0 \\ \frac{36}{85} & \frac{9}{34} & \frac{81}{170} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y$$

$$= \begin{pmatrix} \frac{217}{170} \\ -\frac{1}{68} \\ \frac{383}{340} \\ \frac{3}{2} \end{pmatrix} + k \begin{pmatrix} 8 \\ 5 \\ 9 \\ 0 \end{pmatrix}$$ 其中 k 可在 R 中取任意值.

15. $A=(a_1, a_2, \cdots, a_n)\neq 0$.

A的一个满秩分解 $A=[1](a_1, a_2, \cdots, a_n)$

$B=[1]$. $C=A$.

$$A^+ = C^H(CC^H)^{-1}(B^HB)^{-1}B^H$$

$$= A^H(AA^H)^{-1}$$

$$AA^H = (a_1, a_2, \cdots, a_n)\begin{pmatrix}\bar{a}_1\\ \bar{a}_2\\ \vdots\\ \bar{a}_n\end{pmatrix} = \left[\sum_{i=1}^n a_i\bar{a}_i\right],$$

$$(AA^H)^{-1} = \left[\frac{1}{\sum_{i=1}^n a_i\bar{a}_i}\right]$$

$$A^H(AA^H)^{-1} = \begin{pmatrix}\bar{a}_1\\ \bar{a}_2\\ \vdots\\ \bar{a}_n\end{pmatrix}\left[\frac{1}{\sum_{i=1}^n a_i\bar{a}_i}\right] = \begin{pmatrix}\frac{\bar{a}_1}{\sum_{i=1}^n a_i\bar{a}_i}\\ \vdots\\ \frac{\bar{a}_n}{\sum_{i=1}^n a_i\bar{a}_i}\end{pmatrix}$$

因此 $A^+ = \left(\frac{\bar{a}_1}{\sum_{i=1}^n a_i\bar{a}_i}, \cdots \frac{\bar{a}_n}{\sum_{i=1}^n a_i\bar{a}_i}\right)^T$.

对于 $A^H=\begin{pmatrix}\bar{a}_1\\ \bar{a}_2\\ \vdots\\ \bar{a}_n\end{pmatrix}$, 一个满秩分解 $A^H=\begin{pmatrix}\bar{a}_1\\ \bar{a}_2\\ \vdots\\ \bar{a}_n\end{pmatrix}[1]$

$B=A^H$, $C=[1]$. $(A^H)^+ = C^H(CC^H)^{-1}(B^HB)^{-1}B^H$

$$(A^H)^+ = ((A^H)^HA^H)^{-1}(A^H)^H = (AA^H)^{-1}A = \left(\frac{a_1}{\sum_{i=1}^n a_i\bar{a}_i}, \cdots, \frac{a_n}{\sum_{i=1}^n a_i\bar{a}_i}\right)$$

16. $A=\begin{pmatrix}0&0&1\\0&0&2\\1&1&0\\1&1&1\end{pmatrix}\to\begin{pmatrix}0&0&1\\0&0&0\\1&1&0\\0&0&0\end{pmatrix}$ $r(A)=2$.

A的一组基 $\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$ $\begin{pmatrix}1\\2\\0\\1\end{pmatrix}$ $A=\begin{pmatrix}0&1\\0&2\\1&0\\1&1\end{pmatrix}\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix}$

$B=\begin{pmatrix}0&1\\0&2\\1&0\\1&1\end{pmatrix}$ $C=\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix}$

$$A=C^H(CC^H)^{-1}(B^HB)^{-1}B^H=C^T(CC^T)^{-1}(B^TB)^{-1}B^T$$

$$CC^T=\begin{pmatrix}1&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0\\1&0\\0&1\end{pmatrix}=\begin{pmatrix}2&0\\0&1\end{pmatrix}\quad (CC^T)^{-1}=\begin{pmatrix}\frac{1}{2}&0\\0&1\end{pmatrix}$$

$$B^TB=\begin{pmatrix}0&0&1&1\\1&2&0&1\end{pmatrix}\begin{pmatrix}0&1\\0&2\\1&0\\1&1\end{pmatrix}=\begin{pmatrix}2&1\\1&6\end{pmatrix}\quad (B^TB)^{-1}=\begin{pmatrix}\frac{6}{11}&-\frac{1}{11}\\-\frac{1}{11}&\frac{2}{11}\end{pmatrix}$$

$$C^T(CC^T)^{-1}(B^TB)^{-1}B^T=\begin{pmatrix}1&0\\1&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{1}{2}&0\\0&1\end{pmatrix}\begin{pmatrix}\frac{6}{11}&-\frac{1}{11}\\-\frac{1}{11}&\frac{2}{11}\end{pmatrix}\begin{pmatrix}0&0&1&1\\1&2&0&1\end{pmatrix}$$

$$=\begin{pmatrix}\frac{1}{2}&0\\\frac{1}{2}&0\\0&1\end{pmatrix}\begin{pmatrix}-\frac{1}{11}&-\frac{2}{11}&\frac{6}{11}&\frac{5}{11}\\\frac{2}{11}&\frac{4}{11}&-\frac{1}{11}&\frac{1}{11}\end{pmatrix}=\begin{pmatrix}-\frac{1}{22}&-\frac{1}{11}&\frac{3}{11}&\frac{5}{22}\\-\frac{1}{22}&-\frac{1}{11}&\frac{3}{11}&\frac{5}{22}\\\frac{2}{11}&\frac{4}{11}&-\frac{1}{11}&\frac{1}{11}\end{pmatrix}$$

A的 M-P 逆为 $\begin{pmatrix}-\frac{1}{22}&-\frac{1}{11}&\frac{3}{11}&\frac{5}{22}\\-\frac{1}{22}&-\frac{1}{11}&\frac{3}{11}&\frac{5}{22}\\\frac{2}{11}&\frac{4}{11}&-\frac{1}{11}&\frac{1}{11}\end{pmatrix}$

18. (1) A列满秩时.

A的一个满秩分解 $A=AE$. $B=A, C=E$.

$$A^{+}=C^{H}(CC^{H})^{-1}(B^{H}B)^{-1}B^{H}=(A^{H}A)^{-1}A^{H}$$

$$A^{+}A=(A^{H}A)^{-1}A^{H}A=E_{n}.$$