P26.

1. (1) $f(x)=x^5-x^3+3x^2-1$, $g(x)=x^3-3x+2$

$$f(x)=x^2(x^3-3x+2)+2x^3+x^2-1$$
$$=x^2(x^3-3x+2)+2(x^3-3x+2)+x^2+6x-5$$
$$=(x^2+2)(x^3-3x+2)+x^2+6x-5$$

$q(x)=x^2+2$, $r(x)=x^2+6x-5$.

2. (2) $f(x)=ax^4+bx^3+1$. $g(x)=(x-1)^2=x^2-2x+1$

$$f(x)=ax^2(x^2-2x+1)+(2a+b)x^3-ax^2+1$$
$$=ax^2(x^2-2x+1)+(2a+b)x(x^2-2x+1)$$
$$+(3a+2b)x^2-(2a+b)x+1$$
$$=ax^2(x^2-2x+1)+(2a+b)x(x^2-2x+1)$$
$$+(3a+2b)(x^2-2x+1)+(4a+3b)x-(3a+2b-1)$$
$$=(ax^2+(2a+b)x+(3a+2b))(x-1)^2$$
$$+(4a+3b)x-(3a+2b-1)$$

$4a+3b=0$, $3a+2b-1=0$, $\to$ $a=3$, $b=-4$.

$\therefore$ $a=3$, $b=-4$时，$f(x)$能被$g(x)$整除.

3. $x^3-3px+2q$

$$=x(x^2+2ax+a^2)-2ax^2-(3p+a^2)x+2q$$
$$=x(x^2+2ax+a^2)-2a(x^2+2ax+a^2)$$
$$+4a^2x+2a^3-(3p+a^2)x+2q$$
$$=(x-2a)(x+a)^2+(3a^2-3p)x+2a^3+2q.$$

$p=a^2$，$q=-a^3$ 时，

$x^3-3px+2q$，可以被 $x^2-2ax+a^2$ 整除。

5. 证明：

"$\Leftarrow$"：$x \mid f(x)$ 时，$f(x)=g(x)\cdot x$，

$f^k(x)=g^k(x)\cdot x^k$，显然成立。

"$\Rightarrow$" 设 $f(x)=g(x)\cdot x+b$。

$$f^k(x)=\sum_{i=0}^{k} g^i(x)\cdot x^i\cdot b^{k-i}\cdot C_k^i$$

展开后只有 b^k 一[illegible]不含因式 x。

$x \mid f^k(x)$，则 $b^k=0$，$b=0$。

$f(x)=g(x)x$。 $x \mid f(x)$。证毕。

7. $f(x)=4x^4-2x^3-16x^2+5x+9$。

$g(x)=2x^3-x^2-5x+4$

$f(x)=2x\cdot g(x)-6x^2-3x+9$。

$r(x)=-6x^2-3x+9$。

$2x^3-x^2-5x+4=-\frac{1}{3}x(-6x^2-3x+9)$

$-2x^2-2x+4$

$=-\frac{1}{3}x(-6x^2-3x+9)+\frac{1}{3}(-6x^2-3x+9)\quad -x+1$

$-6x^2-3x+9=6x(-x+1)-9x+9=(6x+9)(-x+1)$

$\therefore\ (f(x), g(x)) = x-1$

$$-x+1 = 2x^3 - x^2 - 5x + 4 + \left(\tfrac{1}{3}x - \tfrac{1}{3}\right)(-6x^2 - 3x + 9)$$
$$= g(x) + \tfrac{1}{3}(x-1)\left(f(x) - 2x\cdot g(x)\right)$$
$$= \left(-\tfrac{2}{3}x^2 + \tfrac{2}{3}x + 1\right) g(x) + \tfrac{1}{3}(x-1) f(x)$$
$$x-1 = \left(\tfrac{2}{3}x^2 - \tfrac{2}{3}x - 1\right) g(x) - \tfrac{1}{3}(x-1) f(x)$$

取 $u(x) = -\tfrac{1}{3}(x-1)$, $v(x) = \left(\tfrac{2}{3}x^2 - \tfrac{2}{3}x - 1\right)$

8. $f(x) = x^3 + (1+t)x^2 + 2x + 2u$.

$g(x) = x^3 + tx + u$.

$$f(x) = x^3 + tx + u - tx - u + (1+t)x^2 + 2x + 2u$$
$$= g(x) + (1+t)x^2 + (2-t)x + u$$
$$g(x) = \frac{1}{1+t}x\left((1+t)x^2 + (2-t)x + u\right) - \frac{2-t}{1+t}x^2 - \frac{u}{1+t}x + tx + u.$$
$$= \left(\frac{1}{1+t} - \frac{2-t}{(1+t)^2}\right)\left((1+t)x^2 + (2-t)x + u\right) + \frac{(2-t)^2}{(1+t)^2}x + \frac{2-t}{(1+t)^2}u - \frac{u}{1+t}x + tx + u$$

$$\frac{(2-t)^2}{(1+t)^2} - \frac{u}{1+t} + t = 0, \quad \frac{2-t}{(1+t)^2}u + u = 0.$$

$u = 0$, $t^3 + 3t^2 - 3t + 4 = 0$

$(t+4)(t^2 - t + 1) = 0$, $t = -4$

$\therefore\ u = 0$, $t = -4$

9. 证明：$(f_1(x), f_2(x)) = d(x) \neq 0$

存在 $u(x), v(x)$.

s.t. $u(x)f_1(x) + v(x)f_2(x) = d(x)$.

则有 $u(x)\frac{f_1(x)}{d(x)} + v(x)\frac{f_2(x)}{d(x)} = 1$

$\therefore \left(\frac{f_1(x)}{d(x)}, \frac{f_2(x)}{d(x)}\right) = 1$

10. 证明：$s=2$ 时.

$g_1(x) = d(x)h_1(x)$. $g_2(x) = d(x)h_2(x)$

"$\Rightarrow$" 若 $h_1(x), h_2(x)$ 不互素，

则 $h_1(x) = f_1(x)\varphi(x)$. $h_2(x) = f_2(x)\varphi(x)$.

$g_1(x) = d(x)\varphi(x)f_1(x)$. $g_2(x) = d(x)\varphi(x)f_2(x)$.

$d(x)\varphi(x)$ 也是 $g_1(x), g_2(x)$ 的一个公因式.

<u>因</u> $d(x) \mid d(x)\varphi(x)$, $\partial(\varphi(x)) > 0$.

则 $d(x)$ 不是 $g_1(x), g_2(x)$ 的最大公因式，矛盾.

"$\Leftarrow$" 若 $d(x)$ 不是 $g_1(x), g_2(x)$ 的最大公因式.

则. $\partial\left(\frac{g_1(x)}{d(x)}, \frac{g_2(x)}{d(x)}\right) > 0$.

$\partial(h_1(x), h_2(x)) > 0$.

$h_1(x), h_2(x)$ 不互素，矛盾.

综上，$d(x)$ 是 $g_1(x), g_2(x)$ 的最大公因式.

$\Leftrightarrow h_1(x), h_2(x)$ 互素.

11. 证：充分性：

如果 $(f(x), g(x)) = d(x)$，

则 $f(x) = d(x)\varphi(x)$． $g(x) = d(x)\psi(x)$

$$f(x)g(x) = d^2(x)\varphi(x)\psi(x)$$

$$f(x) + g(x) = d(x)(\varphi(x) + \psi(x))$$

$d(x)$ 是 $f(x)g(x)$ 与 $f(x)+g(x)$ 的公因式．

而 $(f(x)g(x), g(x)+f(x)) = 1$， $\therefore \partial(d(x)) = 0$．

$\therefore (f(x), g(x)) = 1$．

~~必~~要性：

设 $(f(x)g(x), f(x)+g(x)) = d(x)$．

$f(x)g(x) = d(x)\varphi(x)$， $f(x)+g(x) = d(x)\psi(x)$．

$$f(x)g(x) + g^2(x) = d(x)\psi(x)g(x)$$

$$f(x)g(x) + f^2(x) = d(x)\psi(x)f(x)$$

$$g^2(x) = d(x)(\psi(x)g(x) - \varphi(x))$$

$$f^2(x) = d(x)(\psi(x)f(x) - \varphi(x))$$

或者， $d(x)$ 是 $g^2(x)$，$f^2(x)$ 的公因式．

而 $(g(x), f(x)) = 1$，$\partial(d(x)) = 0$，

或者， $\psi(x)g(x) - \varphi(x) = \psi(x)f(x) - \varphi(x) = 0$．

$(g(x), f(x)) = 1$， $\partial(\psi(x)) \geq 0$，不成立．

$\therefore (f(x)g(x), f(x)+g(x)) = 1$

综上，得 $(f(x), g(x)) = 1$ 的充要条件是

$(f(x)g(x), f(x)+g(x)) = 1$．

15. 证明：设 $(f(x), g(x)) = d(x)$.

$f(x) = d(x)\varphi(x)$. $g(x) = d(x)\psi(x)$.

得 $\varphi(x)$, $\psi(x)$ 互质.

$d(x)\varphi(x)\psi(x)$ 是 $f(x)$, $g(x)$ 的公倍式.

任取 $f(x)$, $g(x)$ 的公倍式. $h(x)$.

$f(x) \mid h(x)$. $g(x) \mid h(x)$.

$h(x) = d(x)\varphi(x)\psi(x)h'(x)$.

$d(x)\varphi(x)\psi(x) \mid h(x)$.

$\therefore [f(x), g(x)] = d(x)\varphi(x)\psi(x)$.

（显然 $f(x)$, $g(x)$ 首项系数为 1）

$\therefore [f(x), g(x)](f(x), g(x)) = f(x)g(x)$.

证毕.