Topology – Homework 04

Question 1:

$\mathring{A} = \{(x, \sin\frac{1}{x}) : x \in (0,1)\}$

$\overline{A} = \{(x, \sin\frac{1}{x}) : x \in (0,1]\} \bigcup \{(0,x) : x \in [-1,1]\}$

$A' = \{(x, \sin\frac{1}{x}) : x \in (0,1]\} \bigcup \{(0,x) : x \in [-1,1]\}$

$\overline{A} \bigcap \overline{\mathcal{R}^2 - A} = \{(0,x) : x \in [-1,1]\}$

Question 2:

Proof:

(i)

When $A = \varnothing$, since $\varnothing \in \mathcal{P}(X)$, $f(\varnothing) = \varnothing$, we know $X - A = X \in \mathcal{T}_f$.

When $A = X$, since $X \subset f(X)$, $f(X) \in \mathcal{P}(X)$, $f(X) \supset X$, we know $f(X) = X$ and $X \in \mathcal{T}_f$.

Consider $(X - A) \bigcap (X - B) = X - A \bigcup B$.

Since $A = f(A)$, $B = f(B)$, $A \bigcup B = f(A) \bigcup f(B) = f(A \bigcup B)$,

we know the intersection of finite elements in $\mathcal{T}_f$ is also an element of $\mathcal{T}_f$.

Consider $(X - A) \bigcup (X - B) = X - A \bigcap B$.

If $x \in A$ and $x \in B$, then $f(x) \in f(A)$ and $f(x) \in f(B)$, that is $f(A \bigcap B) \subset f(A) \bigcap f(B)$.

Since $A = f(A)$, $B = f(B)$, $f(A \bigcap B) \subset f(A) \bigcap f(B) = A \bigcap B$, and $A \bigcap B \subset f(A \bigcap B)$,

we know that $A \bigcap B = f(A \bigcap B)$ and that the union of two elements in $\mathcal{T}_f$ is also in $\mathcal{T}_f$,

and this can be promoted to arbitrary number of elements in $\mathcal{T}_f$.

Above derivations show us that $\mathcal{T}_f$ is a topology on X.

(ii)

From the definition of $\mathcal{T}_f$ we know a set A is closed if and only if $f(A) = A$.

From $f(A) = f(f(A))$ we know that $f(A)$ is closed.

For arbitrary $A \subset X$, we have $A \subset \overline{A}$ and $\overline{A} = A \bigcup \overline{A}$,

$f(\overline{A}) = f(A \bigcup \overline{A}) = f(f(A \bigcup \overline{A})) = f(f(A) \bigcup f(\overline{A})) = f(f(A) \bigcup \overline{A})$.

Since $\overline{A}$ and $f(A)$ are both closed, there should be $\overline{A} = f(A) \bigcup \overline{A}$, which means $f(A) \subset \overline{A}$.

$A \subset f(A)$ and $f(A)$ is closed, so we have $\overline{A} \subset f(A)$.

Then we know $f(A) = \overline{A}$, in other words, the closure $\overline{A}$ of any $A \subset X$ with respect to the topology $\mathcal{T}_f$ is just $f(A)$.

Question 3:

Proof:

f is continuous but f^{-1} is not.

For the open set $U = [0, \frac{1}{4}) \subset [0, 1)$, $f(U)$ is not an open set in S^1, because there doesn't exist an open set V in $\mathcal{R}^2$ that includes $f(0)$, s.t. $V \bigcap S \subset f(U)$.

Thus f is not a homeomorphism with respect to the standard subspace topologies on $[0, 1)$ and S^1.

It's impossible to find a homeomorphism between $[0, 1)$ and S^1.

Question 4:

(i)

$(1, 2) \bigcup (2, 3)$ is an example of an open set in $\mathcal{R}$ that is not regularly open.

The regularly open sets in $\mathcal{R}$ can be characterized by arbitrary (a, b) with $a < b$ and their union with no intervals have the same boundary.

(ii)

Proof:

For arbitrary $A \subset X$, there exist

$$\overline{((\overline{A})^\circ)}^\circ \supset ((\overline{A})^\circ)^\circ = (\overline{A})^\circ$$

and

$$\overline{((\overline{A})^\circ)}^\circ \subset \overline{((\overline{A}))}^\circ = (\overline{A})^\circ.$$

Then we know that $\overline{((\overline{A})^\circ)}^\circ = (\overline{A})^\circ$ and that $(\overline{A})^\circ$ is regularly open.

Question 5:

Proof:

Consider arbitrary $x \in X$.

If $x \in A'$, then every neighborhood of x has points distinct with x in its intersection with A, and this shows $x \in \overline{A}$ and $A' \subset \overline{A}$.

According to the definition of closure, we know $A \subset \overline{A}$ and then we have $A \bigcup A' \subset \overline{A}$.

Let x be a point of $\overline{A}$, if $x \in A$ then we have $x \in A \bigcup A'$.

If $x \notin A$, because of that every neighborhood of x U is intersect with A, U has a point distinct with x, and then $x \in A'$ and $x \in A \bigcup A'$.

Thus we have $\overline{A} \subset A \bigcup A'$.

Then we know that the equality $\overline{A} = A \bigcup A'$ established.