TOPOLOGY - HOMEWORK 07

Due: 2020 May 7th, 10:00AM

Question 1. Show that a connected metric space (X, d) having more than one point is uncountable.
(Hint: pick $a \in X$ and consider the function $d_a : X \longrightarrow \mathbf{R}$ given by $d_a(x) := d(a, x)$.)

Question 2. (Baire category theorem – a special case) Let $(X, \mathcal{T})$ be a compact Hausdorff space. Let $\{A_i : i \in \mathbf{N}\}$ be a collection of closed subsets of X. Assume that $(A_i)^o = \emptyset$ for every i. Show that $\left(\bigcup_i A_i\right)^o = \emptyset$.

Question 3. (Cantor set) Set $A_0 = [0, 1]$, and for each $n \in \mathbf{N}$ we define A_n recursively by

$$A_n = A_{n-1} - \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

Take $C := \bigcap_{n \geq 0} A_n$ with the standard subspace topology from $\mathbf{R}$. Prove the following:

(i) C is totally disconnected, i.e., the connected components of C are singletons.
(ii) C is compact and has no isolated points, and thus conclude that C is uncountable.

Question 4. Are the following two spaces locally compact:

(i) $\mathbf{Q}$ with the standard topology,
(ii) $(\mathbf{R}, \mathcal{T})$ with $\mathcal{T}$ being generated by the basis $\{(a, b] : \ a, b \in \mathbf{R}, a < b\}$?

Justify your answer.

Question 5.(p-adic numbers)

(i) Prove that the product space $(\prod_{i=1}^{\infty} A_i, \mathcal{T}_p)$, where each A_i is a finite set with the discrete topology, is compact with respect to the product topology $\mathcal{T}_p$.
(ii) Fix a prime number p and consider

$$\mathbf{Z}_p := \left\{ (a_i)_{i \in \mathbf{N}} : a_i \in \mathbf{Z}/p^i\mathbf{Z} \text{ and } \phi_{i+1}(a_{i+1}) = a_i \text{ for every } i \right\} \subset \prod_{i=1}^{\infty} (\mathbf{Z}/p^i\mathbf{Z}),$$

where $\phi_i : \mathbf{Z}/p^i\mathbf{Z} \twoheadrightarrow \mathbf{Z}/p^{i-1}\mathbf{Z}, i \geq 2$ is the natural projection map. Endow each $\mathbf{Z}/p^i\mathbf{Z}$ with the discrete topology and $\prod_{i=1}^{\infty}(\mathbf{Z}/p^i\mathbf{Z})$ the product topology. Show that $\mathbf{Z}_p$ is closed and compact with respect to this topology.
(iii) For each $i \geq 1$, let $f_i : \mathbf{Z} \twoheadrightarrow \mathbf{Z}/p^i\mathbf{Z}$ be the natural quotient. Consider the embedding $f : \mathbf{Z} \hookrightarrow \mathbf{Z}_p$ given by $f(a) = (f_i(a))_{i \in \mathbf{N}}$. Prove that $f(\mathbf{Z})$ is dense in $\mathbf{Z}_p$, that is, $\mathrm{Cl}_{\mathbf{Z}_p}(f(\mathbf{Z})) = \mathbf{Z}_p$.

(Hint for (i): consider HW5 Question 3, and use sequential compactness.)