根子空间分解定理

设有数域 P 上的线性空间 V 以及 V 上的线性变换 $\mathcal{A}$.

$\mathcal{A}$ 的特征多项式 $f(\lambda)$ 可以分解为

$$f(\lambda) = \prod_{i=1}^{s} (\lambda - \lambda_i)^{r_i}.$$

其中 $\{\lambda_i\}$ 互素. 令 $f_i(x) = \dfrac{f(x)}{(x-\lambda_i)^{r_i}}$

(1) 根子空间 $\overline{V}_{\lambda_i} = f_i(\mathcal{A})(V) = \mathrm{Im} f_i(\mathcal{A})$

(2) $V = \overline{V}_{\lambda_1} \oplus \overline{V}_{\lambda_2} \oplus \cdots \oplus \overline{V}_{\lambda_s}$

证明: (1) 既然 $f(x) = f_i(x)(x-\lambda_i)^{r_i}$

$$f(\mathcal{A}) = f_i(\mathcal{A})(\mathcal{A} - \lambda_i id)^{r_i}$$

$f(\lambda)$ 是 $\mathcal{A}$ 的特征多项式, $f(\mathcal{A})$ 是零变换,

有 $f_i(\mathcal{A})(\mathcal{A} - \lambda_i id)^{r_i}(V) = f(\mathcal{A})(V) = \{\theta\}$.

$(\mathcal{A} - \lambda_i id)f_i(\mathcal{A})(V) = \{\theta\}$. 由 $\overline{V}_{\lambda_i}$ 定义, $\underline{f_i(\mathcal{A})(V) \subset \overline{V}_{\lambda_i}}$

$f_i(x)$ 与 $(x-\lambda_i)^{r_i}$ 互素, 存在 P 上的多项式 $u(x)$, $v(x)$.

$$u(x)f_i(x) + v(x)(x-\lambda_i)^{r_i} = 1$$

$$u(\mathcal{A})f_i(\mathcal{A}) + v(\mathcal{A})(\mathcal{A} - \lambda_i id)^{r_i} = id.$$

$$(\mathcal{A} - \lambda_i id)^{r_i}(\overline{V}_{\lambda_i}) = \{\theta\}.$$

$\therefore$ $u(\mathcal{A})f_i(\mathcal{A})(\overline{V}_{\lambda_i}) = \overline{V}_{\lambda_i}$. $u(\mathcal{A})(\overline{V}_{\lambda_i}) \subset V$.

$f_i(\mathcal{A})u(\mathcal{A})(\overline{V}_{\lambda_i}) \subset f_i(\mathcal{A})(V)$, $\underline{\overline{V}_{\lambda_i} \subset f_i(\mathcal{A})(V)}$

因此 $\overline{V}_{\lambda_i} = f_i(\mathcal{A})(V)$

(2) 取 $\xi_i \in V_{\lambda_i}$,

对于 $i \neq j$. $(x-\lambda_i)^{r_i} \mid f_j(x)$

有 $g_i(x) \in P[x]$, $f_j(x) = g_i(x)(x-\lambda_i)^{r_i}$.

由于 $\xi_i \in V_{\lambda_i}$, $g_i(A)(A-\lambda_i id)^{r_i}(\xi_i) = \theta$.

$f_j(A)(\xi_i) = \theta$, $j = 1, 2, \cdots, s$.

取 $u(x), v(x)$, 使 $u(x) f_j(x) + v(x)(x-\lambda_j)^{r_j} = 1$.

由 $f_j(x)$ 与 $(x-\lambda_j)^{r_j}$ 互素, 这是可以做到的

令 $\sum_{i=1}^{s} \xi_i = \theta$. 则 $f_i(A)\left(\sum_{i=1}^{s} \xi_i\right) = f_i(A)(\theta) = \theta$.

$i \neq j$ 时, $f_i(A)\xi_j = \theta$

于是有 $f_i(A)(\xi_i) = \theta$.

$u(A) f_i(A) + v(A)(A-\lambda_i id)^{r_i} = id$.

$(u(A) f_i(A) + v(A)(A-\lambda_i id)^{r_i})(\xi_i)$

$= u(A) f_i(A)(\xi_i) + v(A)(A-\lambda_i id)^{r_i}(\xi_i) = \theta + \theta = \theta$.

$\therefore \xi_i = \theta$.

因此 根子空间之间公共元素只有 θ.

$(f_1(x), f_2(x), \cdots, f_s(x)) = 1$,

取 $u_1(x), u_2(x), \cdots, u_s(x)$, s.t. $\sum_{i=1}^{s} u_i f_i(x) = 1$

$\sum_{i=1}^{s} u_i(A) f_i(A) = id$.

$$V = \mathrm{id}(V) = \sum_{i=1}^{s} u_i(\mathcal{A}) f_i(\mathcal{A})(V)$$

$$= \sum_{i=1}^{s} f_i(\mathcal{A}) u_i(\mathcal{A})(V)$$

$$\subset \sum_{i=1}^{s} f_i(\mathcal{A})(V) \qquad (u_i(\mathcal{A})(V) \subset V)$$

$$= \sum_{i=1}^{s} \overline{V}_{\lambda_i}, \quad 即 \quad V \subset \sum_{i=1}^{s} \overline{V}_{\lambda_i}$$

又 $\{\overline{V}_{\lambda_i}\}$ 均为 V 的子空间，$\sum_{i=1}^{s} \overline{V}_{\lambda_i} \subset V$.

$\therefore \quad V = \sum_{i=1}^{s} \overline{V}_{\lambda_i}$. 且此分解是直和分解.

$$V = \bigoplus_{i=1}^{s} \overline{V}_{\lambda_i}.$$