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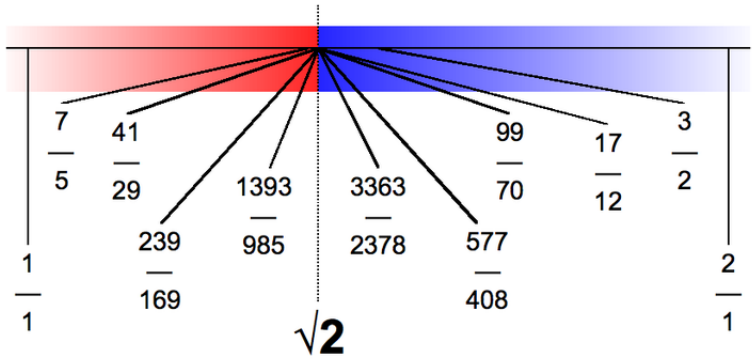
Dedekind cut

In mathematics, **Dedekind cuts**, named after German mathematician Richard Dedekind but previously considered by Joseph Bertrand[1][2], are a method of construction of the real numbers from the rational numbers. A Dedekind cut is a partition of the rational numbers into two non-empty sets *A* and *B*, such that all elements of *A* are less than all elements of *B*, and *A* contains no greatest element. The set *B* may or may not have a smallest element among the rationals. If *B* has a smallest element among the rationals, the cut corresponds to that rational. Otherwise, that cut defines a unique irrational number which, loosely speaking, fills the "gap" between *A* and *B*.[3] In other words, *A* contains every rational number less than the cut, and *B* contains every rational number greater than or equal to the cut. An irrational cut is equated to an irrational number which is in neither set. Every real number, rational or not, is equated to one and only one cut of rationals.



Dedekind used his cut to construct the irrational, real numbers.

Dedekind cuts can be generalized from the rational numbers to any totally ordered set by defining a Dedekind cut as a partition of a totally ordered set into two non-empty parts *A* and *B*, such that *A* is closed downwards (meaning that for all *a* in *A*, $x \leq a$ implies that *x* is in *A* as well) and *B* is closed upwards, and *A* contains no greatest element. See also completeness (order theory).

It is straightforward to show that a Dedekind cut among the real numbers is uniquely defined by the corresponding cut among the rational numbers. Similarly, every cut of reals is identical to the cut produced by a specific real number (which can be identified as the smallest element of the *B* set). In other words, the number line where every real number is defined as a Dedekind cut of rationals is a complete continuum without any further gaps.

Contents

Definition

A Dedekind cut is a partition of the rationals $\mathbb{Q}$ into two subsets A and B such that

1. A is nonempty.
2. $A \neq \mathbb{Q}$.
3. If $x, y \in \mathbb{Q}$, $x < y$, and $y \in A$, then $x \in A$. (A is "closed downwards".)
4. If $x \in A$, then there exists a $y \in A$ such that $y > x$. (A does not contain a greatest element.)

By relaxing the first two requirements, we formally obtain the extended real number line.

Representations

It is more symmetrical to use the (A,B) notation for Dedekind cuts, but each of A and B does determine the other. It can be a simplification, in terms of notation if nothing more, to concentrate on one "half" — say, the lower one — and call any downward closed set A without greatest element a "Dedekind cut".

If the ordered set S is complete, then, for every Dedekind cut (A, B) of S, the set B must have a minimal element b, hence we must have that A is the interval $(-\infty, b)$, and B the interval $[b, +\infty)$. In this case, we say that b *is represented by* the cut (A,B).

The important purpose of the Dedekind cut is to work with number sets that are *not* complete. The cut itself can represent a number not in the original collection of numbers (most often rational numbers). The cut can represent a number b, even though the numbers contained in the two sets A and B do not actually include the number b that their cut represents.

For example if A and B only contain rational numbers, they can still be cut at $\sqrt{2}$ by putting every negative rational number in A, along with every non-negative number whose square is less than 2; similarly B would contain every positive rational number whose square is greater than or equal to 2. Even though there is no rational value for $\sqrt{2}$, if the rational numbers are partitioned into A and B this way, the partition itself represents an irrational number.

Ordering of cuts

Regard one Dedekind cut (A, B) as *less than* another Dedekind cut (C, D) (of the same superset) if A is a proper subset of C. Equivalently, if D is a proper subset of B, the cut (A, B) is again *less than* (C, D). In this way, set inclusion can be used to represent the ordering of numbers, and all other relations (*greater than*, *less than or equal to*, *equal to*, and so on) can be similarly created from set relations.

The set of all Dedekind cuts is itself a linearly ordered set (of sets). Moreover, the set of Dedekind cuts has the least-upper-bound property, i.e., every nonempty subset of it that has any upper bound has a *least* upper bound. Thus, constructing the set of Dedekind cuts serves the purpose of embedding the original ordered set S, which might not have had the least-upper-bound property, within a (usually larger) linearly ordered set that does have this useful property.

Construction of the real numbers

A typical Dedekind cut of the rational numbers $\mathbb{Q}$ is given by the partition (A, B) with

$$A = \{a \in \mathbb{Q} : a^2 < 2 \text{ or } a < 0\},$$
$$B = \{b \in \mathbb{Q} : b^2 \geq 2 \text{ and } b \geq 0\}.$$ [4]

This cut represents the irrational number $\sqrt{2}$ in Dedekind's construction. The essential idea is that we use a set A, which is the set of all rational numbers whose square are less than 2, to "represent" number $\sqrt{2}$, and further, by defining properly arithmetic operators over these sets (addition, subtraction, multiplication, and division), these sets (together with these arithmetic operations) form the familiar real numbers.

To establish this, one must show that A really is a cut (according to the definition) and the square of A, that is $A \times A$ (please refer to the link above for the precise definition of how the multiplication of cuts are defined), is 2 (note that rigorously speaking this is a cut $\{x \mid x \in \mathbb{Q}, x < 2\}$). To show the first part, we show that for any positive rational x with $x^2 < 2$, there is a rational y with $x < y$ and $y^2 < 2$. The choice $y = \frac{2x+2}{x+2}$ works, thus A is indeed a cut. Now armed with the multiplication between cuts, it is easy to check that $A \times A \leq 2$ (essentially, this is because $x \times y \leq 2, \forall x, y \in A, x, y \geq 0$). Therefore to show that $A \times A = 2$, we show that $A \times A \geq 2$, and it suffices to show that for any $r < 2$, there exists $x \in A$, $x^2 > r$. For this we notice that if $x > 0, 2 - x^2 = \epsilon > 0$, then $2 - y^2 \leq \frac{\epsilon}{2}$ for the y constructed above, this means that we have a sequence in A whose square can become arbitrarily close to 2, which finishes the proof.

Note that the equality $b^2 = 2$ cannot hold since $\sqrt{2}$ is not rational.

Generalizations

A construction similar to Dedekind cuts is used for the construction of surreal numbers.

Partially ordered sets

More generally, if S is a partially ordered set, a *completion* of S means a complete lattice L with an order-embedding of S into L. The notion of *complete lattice* generalizes the least-upper-bound property of the reals.

One completion of S is the set of its *downwardly closed* subsets, ordered by inclusion. A related completion that preserves all existing sups and infs of S is obtained by the following construction: For each subset A of S, let A^{u} denote the set of upper bounds of A, and let A^{l} denote the set of lower bounds of A. (These operators form a Galois connection.) Then the Dedekind–MacNeille completion of S consists of all subsets A for which $(A^{\mathrm{u}})^{\mathrm{l}} = A$; it is ordered by inclusion. The Dedekind-MacNeille completion is the smallest complete lattice with S embedded in it.

Notes

1. Bertrand, Joseph (1849). *Traité d'Arithmétique* (https://gallica.bnf.fr/ark:/12148/bpt6k77735p/f209.image.r=%22joseph%20bertrand%22). page 203. "An incommensurable number can be defined only by indicating how the magnitude it expresses can be formed by means of unity. In

what follows, we suppose that this definition consists of indicating which are the commensurable numbers smaller or larger than it ...."

2. Spalt, Detlef (2019). *Eine kurze Geschichte der Analysis*. Springer. doi:10.1007/978-3-662-57816-2 (https://doi.org/10.1007%2F978-3-662-57816-2).
3. Dedekind, Richard (1872). *Continuity and Irrational Numbers* (http://www.math.ubc.ca/~cass/courses/m446-05b/dedekind-book.pdf#page=15) (PDF). Section IV. "Whenever, then, we have to do with a cut produced by no rational number, we create a new *irrational* number, which we regard as completely defined by this cut ... . From now on, therefore, to every definite cut there corresponds a definite rational or irrational number ...."
4. In the second line, $\geq$ may be replaced by $>$ without any difference as there is no solution for $x^2 = 2$ in $\mathbb{Q}$ and $b = 0$ is already forbidden by the first condition. This results in the equivalent expression

$$B = \{b \in \mathbb{Q} : b^2 > 2 \text{ and } b > 0\}.$$

References

- Dedekind, Richard, *Essays on the Theory of Numbers*, "Continuity and Irrational Numbers," Dover: New York, ISBN 0-486-21010-3. Also available (http://www.gutenberg.org/etext/21016) at Project Gutenberg.

External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Dedekind cut" (https://www.encyclopediaofmath.org/index.php?title=p/d030530), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4