Topology – Homework 03

Question 1:

(i) Proof:

Since an ε - ball $B(x, 1)$ under this metric contains either one element or all elements in S, all single-element-set of S is in the topo basis. According to the method of generating a topo from the topo basis, we know every subset of S is an open set in the metric space (S, d).

(ii) No, since the distance between two elements can be 0 if and only if the two elements are the same.

If the answer is Yes, the topo basis of $\{\varnothing, S\}$ should be $\{S\}$, which means every ε- ball should contains every elements in S, that is , the distance between arbitrary two elements is 0. It's impossible unless there is just one element in S.

Question 2:

Proof:

Consider an open interval (a, b) with $a \in Q$ but b is an irrational number.

We can find a series of open interval $(a, b_1), (a, b_2), \cdots, (a, b_n), \cdots$ with $b_i \in Q$, $b_i < b_{i+1}$ and $\lim_{n\to\infty} b_n = b$.

Then (a, b) can be represented by $\bigcup (a, b_i)$.

Similarly, we can represent any open interval in R.

Then we know $\mathcal{B}$ is a basis for the standard topology on R.

$\mathcal{B}$ is a subset of $Q \times Q$, so $\mathcal{B}$ is countable. Then we know the cardinality of $\mathcal{B}$ is $\aleph_0$.

Question 3:

Proof:

Assume that there is an isometry $f : R \to R^2$ between (R, d) and (R^2, ρ).

Arbitrarily choose $a \in R$ and let $f(a) = (x, y)$.

For a distance $d > 0$, there are only two possible points b make $d(a, b) = d$ while there are infinite possible points (x', y') make $\rho((x, y), (x', y')) = d$.

Since there can't be a bijection between a finite set and an infinite set, we know there doesn't exist an isometry between (R, d) and (R^2, ρ).

Question 4:

(i) Proof:

$\varnothing, S \in \mathcal{T}_i$ for each $i \in I$, then $\varnothing, S \in \bigcap_{i \in I} \mathcal{T}_i$.

Consider arbitrary $A, B \in \bigcap_{i \in I} \mathcal{T}_i$, $A, B \in \mathcal{T}_i$ for each $i \in I$, then $A \bigcup B, A \bigcap B \in \mathcal{T}_i$ for each $i \in I$, which means $A \bigcup B, A \bigcap B \in \bigcup_{i \in I} \mathcal{T}_i$. According to the principle of recursion we know arbitrary union of elements in $\bigcup_{i \in I} \mathcal{T}_i$ and arbitrary intersection of finite elements in $\bigcup_{i \in I} \mathcal{T}_i$ are also in $\bigcup_{i \in I} \mathcal{T}_i$, and this show that $\bigcup_{i \in I} \mathcal{T}_i$ is a topology on S.

(ii)

$S = \{a, b, c\}$

$\mathcal{T}_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$

$\mathcal{T}_2 = \{\varnothing, \{b\}, \{c\}, \{b, c\}, \{a, b, c\}\}$

$\mathcal{T}_1 \bigcup \mathcal{T}_2$ is not a topology on S.

The unique minimal topology on S which contains $\mathcal{T}_1 \bigcup \mathcal{T}_2$ is $\mathcal{P}(S)$.

Question 5:

(i) Proof:

R^2 is radially open and then $R^2 \in \mathcal{T}_{ro}$.

Obviously $\varnothing$ is also in $\mathcal{T}_{ro}$.

Consider $A, B \in \mathcal{T}_{ro}$, each point in $A \bigcup B$ has an open line segment in each direction since the point is in A or B. And a point in $A \bigcap B$ should also has an open line segment in each direction, otherwise there exists a direction where the point doesn't have an open line. In this direction, at least one of A and B just has discrete points starting with that point, so that the point also doesn't have an open line in this direction, which makes contradiction.

According to the principle of recursion we know arbitrary union of elements in $\mathcal{T}_{ro}$ and arbitrary intersection of finite elements in $\mathcal{T}_{ro}$ are also in $\mathcal{T}_{ro}$, which means $\mathcal{T}_{ro}$ is a topology on R^2.

(ii) Yes, that is $\mathcal{T}_{ro} \supset \mathcal{T}$.

(iii) No.