TOPOLOGY - HOMEWORK 10

Due: 2020 May 28th, 10:00AM

Question 1. Retain the notations in our proof of the Urysohn lemma. Show that for any given $r \in (0,1)$, one has

$$f^{-1}(\{r\}) = \left(\bigcap_{p>r, p\in D} U_p \right) - \left(\bigcup_{q<r, q\in D} U_q \right),$$

where $D = \{k/2^n : \ n \in \mathbf{N}, k \in [1, 2^n - 1] \cap \mathbf{N}\}$.

Question 2. Let (X,d) be a metric space and $A, B \subset X$ be two disjoint closed sets. Show that the function

$$f : X \longrightarrow [0,1] \text{ given by } f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)},$$

where $d(x,S) = \inf \{d(x,s) : s \in S\}$ for any $S \subset X$, is continuous and satisfies $f(A) = 0, f(B) = 1$.

Question 3. Give an example of a metric space that is not second countable, and justify your answer. (Note that every metric space is first countable.)

Question 4. Let $\{(X_i, \mathcal{T}_i)\}_{i\in I}$ be a family of nonempty topological spaces.

(i) Show that if X_i is first countable for every $i \in I$ and $(\prod_i X_i, \mathcal{T}_p)$ is also first countable, then all except countably many of the X_i's have the trivial topology, i.e., $\mathcal{T}_i = \{\emptyset, X_i\}$.

(ii) Prove that $(\prod_i X_i, \mathcal{T}_p)$ is metrizable if and only if each X_i is metrizable and X_i is a singleton set for all i except a countable set of indices.

Question 5. Let (X,d) be a locally compact but non-compact metric space. Show that the following two properties are equivalent:

(i) X is seperable,

(ii) the one-point compactification $X^* := X \cup \{\infty\}$ of X is metrizable.