Riesz representation theorem

There are several well-known theorems in functional analysis known as the **Riesz representation theorem**. They are named in honor of Frigyes Riesz.

This article will describe his theorem concerning the dual of a Hilbert space, which is sometimes called the Fréchet–Riesz theorem. For the theorems relating linear functionals to measures, see Riesz–Markov–Kakutani representation theorem.

The Hilbert space representation theorem

This theorem establishes an important connection between a Hilbert space and its continuous dual space. If the underlying field is the real numbers, the two are isometrically isomorphic; if the underlying field is the complex numbers, the two are isometrically anti-isomorphic. The (anti-) isomorphism is a particular natural one as will be described next; a natural isomorphism.

Let H be a Hilbert space, and let H^* denote its dual space, consisting of all continuous linear functionals from H into the field $\mathbb{R}$ or $\mathbb{C}$.

If x is an element of H, then the function φ_x, for all y in H defined by

$$\varphi_x(y) = \langle y, x \rangle \,,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of the Hilbert space, is an element of H^*. The Riesz representation theorem states that *every* element of H^* can be written uniquely in this form.

Riesz–Fréchet representation theorem. Let H be a Hilbert space and $\varphi \in H^*$. Then there exists $f \in H$ such that for any $x \in H$, $\varphi(x) = \langle f, x \rangle$. Moreover $\|f\|_H = \|\varphi\|_{H*}$

Proof. Let $M = \{u \in H \mid \varphi(u) = 0\}$. Clearly M is closed subspace of H. If $M = H$, then we can trivially choose $f = 0$. Now assume $M \neq H$. Then $M^\perp$ is one-dimensional. Indeed, let v_1, v_2 be nonzero vectors in $M^\perp$. Then there is nonzero real number λ, such that $\lambda\varphi(v_1) = \varphi(v_2)$. Observe that $\lambda v_1 - v_2 \in M^\perp$ and $\varphi(\lambda v_1 - v_2) = 0$, so $\lambda v_1 - v_2 \in M$. This means that $\lambda v_1 - v_2 = 0$. Now let g be unit vector in $M^\perp$. For arbitrary $x \in H$, let v be the orthogonal projection of x onto $M^\perp$. Then $v = \langle g, x \rangle g$ and $\langle g, x - v \rangle = 0$ (from the properties of orthogonal projections), so that $x - v \in M$ and $\langle g, x \rangle = \langle g, v \rangle$. Thus $\varphi(x) = \varphi(v + x - v) = \varphi(\langle g, x \rangle g) + \varphi(x - v) = \langle g, x \rangle \varphi(g) + 0 = \langle g, x \rangle \varphi(g)$. Hence $f = \varphi(g) g$. We also see $\|f\|_H = \varphi(g)$. From the Cauchy-Bunyakovsky-Schwartz inequality $\varphi(x) \leq \|g\| \|x\| \varphi(g)$, thus for x with unit norm $\varphi(x) \leq \varphi(g)$. This implies that $\|\varphi\|_{H*} = \varphi(g)$.

Given any continuous linear functional g in H^*, the corresponding element $x_g \in H$ can be constructed uniquely by $x_g = g(e_1)e_1 + g(e_2)e_2 + \ldots$, where $\{e_i\}$ is an orthonormal basis of H, and the value of x_g does not vary by choice of basis. Thus, if $y \in H, y = a_1 e_1 + a_2 e_2 + \ldots$, then $g(y) = a_1 g(e_1) + a_2 g(e_2) + \ldots = \langle x_g, y \rangle$.

Theorem. The mapping $\Phi\colon H \to H^*$ defined by $\Phi(x) = \varphi_x$ is an isometric (anti-) isomorphism, meaning that:

- Φ is bijective.
- The norms of x and φ_x agree: $\|x\| = \|\Phi(x)\|$.
- Φ is additive: $\Phi(x_1 + x_2) = \Phi(x_1) + \Phi(x_2)$.
- If the base field is $\mathbb{R}$, then $\Phi(\lambda x) = \lambda\Phi(x)$ for all real numbers λ.
- If the base field is $\mathbb{C}$, then $\Phi(\lambda x) = \bar{\lambda}\Phi(x)$ for all complex numbers λ, where $\bar{\lambda}$ denotes the complex conjugation of λ.

The inverse map of Φ can be described as follows. Given a non-zero element φ of H^*, the orthogonal complement of the kernel of φ is a one-dimensional subspace of H. Take a non-zero element z in that subspace, and set $x = \overline{\varphi(z)} \cdot z/\|z\|^2$. Then $\Phi(x) = \varphi$.

Historically, the theorem is often attributed simultaneously to Riesz and Fréchet in 1907 (see references).

In the mathematical treatment of quantum mechanics, the theorem can be seen as a justification for the popular bra–ket notation. The theorem says that, every bra $\langle\psi|$ has a corresponding ket $|\psi\rangle$, and the latter is unique.

References

- Fréchet, M. (1907). "Sur les ensembles de fonctions et les opérations linéaires" (https://gallica.bnf.fr/ark:/12148/bpt6k3098j/f1414.image). *Les Comptes rendus de l'Académie des sciences* (in French). **144**: 1414–1416.
- Riesz, F. (1907). "Sur une espèce de géométrie analytique des systèmes de fonctions sommables" (https://gallica.bnf.fr/ark:/12148/bpt6k3098j/f1409.image). *Comptes rendus de l'Académie des Sciences* (in French). **144**: 1409–1411.
- Riesz, F. (1909). "Sur les opérations fonctionnelles linéaires" (https://gallica.bnf.fr/ark:/12148/bpt6k3103r/f976.image). *Comptes rendus de l'Académie des Sciences* (in French). **149**: 974–977.
- P. Halmos *Measure Theory*, D. van Nostrand and Co., 1950.
- P. Halmos, *A Hilbert Space Problem Book*, Springer, New York 1982 *(problem 3 contains version for vector spaces with coordinate systems)*.
- Walter Rudin, *Real and Complex Analysis*, McGraw-Hill, 1966, ISBN 0-07-100276-6.
- "Proof of Riesz representation theorem for separable Hilbert spaces" (http://planetmath.org/?op=getobj&from=objects&id=6130). *PlanetMath*.

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