TOPOLOGY - HOMEWORK 08

Due: 2020 May 14th, 10:00AM

Question 1. Consider $X = \{0\} \cup \{1/n : n \in \mathbf{N}\}$ with the subspace topology inherited from $\mathbf{R}$. Show that X is homeomorphic to the one-point compactification $(\mathbf{N} \cup \{\infty\}, \mathcal{T}')$ of $(\mathbf{N}, \mathcal{T})$, where $\mathcal{T}$ is the standard (discrete) topology on $\mathbf{N}$.

Question 2. Show that $(\mathbf{R}^\omega, \mathcal{T}_p)$ has no one-point compactification. (Hint: is $\mathbf{R}^\omega$ locally compact?)

Question 3. Show that if X_i is separable for each $i \in \mathbf{N}$, then the product space $(\prod_{i \in \mathbf{N}} X_i, \mathcal{T}_p)$ is separable.

Question 4. Consider the space $(\mathbf{R}, \mathcal{T}_l)$ with $\mathcal{T}_l$ being generated by the basis

$$\{[a, b) : \ a, b \in \mathbf{R}, a < b\}.$$

Show that $\mathbf{R}$ is Lindelöf with respect to $\mathcal{T}_l$, and that $\mathbf{R} \times \mathbf{R}$ is not Lindelöf with respect to the product topology arising from $\mathcal{T}_l$. (Hint: Munkres' book.)

Question 5. Let (X, d) be a metric space. Show that the following properties are equivalent:

(i) X is second countable,
(ii) X is Lindelöf,
(iii) X is separable.

(Hint: for (ii) $\Rightarrow$ (i), consider $\{B(x, 1/n) : x \in X, n \in \mathbf{N}\}$; fix n and run x first.)