TOPOLOGY - HOMEWORK 06

Due: 2020 April 16th, 10:00AM

For questions below, we assume that every $X \subset \mathbf{R}^n$ is endowed with the standard subspace topology, unless stated otherwise.

Question 1. Show that the following subspace

$$X = \{(0, y) : y \in \mathbf{R}\} \cup \{(x, \sin(1/x)) : x \in \mathbf{R}_{>0}\}$$

of $\mathbf{R}^2$ is not path-connected.

Question 2. Prove the following:

(i) None of the three spaces $(0,1), (0,1]$ and $[0,1]$ is homeomorphic to the other two.

(ii) Every continuous function $f : [0,1] \longrightarrow [0,1]$ has a fixed point $x \in [0,1]$, i.e., $f(x) = x$.

Is (ii) still true, if $[0,1]$ is replaced by $(0,1)$ or $(0,1]$? Justify your answer.

Question 3. Fix a topological space $(X, \mathcal{T})$. An open neighborhood basis of $x \in X$ is a collection of open sets $\{B_i\}_{i \in I}$ such that

(1) $x \in B_i$ for every i, and

(2) for every open set U containing x, there is a B_i with $B_i \subset U$.

We call X locally path-connected if every point has an open neighborhood basis consisting of path-connected sets. Show that if X is connected and locally path-connected, then it is path-connected.

(Hint: fix $x \in X$ and consider the set S_x of points in X which can be joined to x by a path. Show that S_x is clopen.)

Question 4. Consider $(\mathbf{R}, \mathcal{T})$ with the topology given by

$$\mathcal{T} = \{\emptyset\} \cup \{\mathbf{R} - A : \ A \subset \mathbf{R} \text{ is finite}\}.$$

Show that every subset $X \subset \mathbf{R}$ is compact.

Question 5. Fix a compact topological space $(X, \mathcal{T})$. Prove the equivalence of the following two statements:

(i) every compact subset of X is closed;

(ii) for every topology $\mathcal{T}'$ strictly containing $\mathcal{T}$, the space $(X, \mathcal{T}')$ is non-compact.

Can you give an example of $(X, \mathcal{T})$ satisfying (the equivalent) (i) and (ii) above?