2 Modules

2.3 Projective modules and injective modules

1. Let ${}_DM$ be a vector space over a division ring D, since ${}_DM$ has a basis, ${}_DM$ is free, therefore ${}_DM$ is projective. Considering the diagram

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "\lambda"] \arrow[d, "f"'] & T \arrow[dl] \\ & M & \end{tikzcd}$$

where λ is injective, then $T = Im\lambda \oplus T'$ by Lemma 2.2.1. Define $\psi : T \to M$, $\psi(\lambda(a) + b) = f(a)$, thus $\psi \circ \lambda = f$, hence ${}_DM$ is injective.

2. $\Rightarrow$:Suppose P is projective, then there is an index set I such that $f : R^{(I)} \cong P \oplus P'$. Define $f_i = \pi_i \circ f^{-1} \circ \lambda_1 : P \to R$ for each $i \in I$ where $\lambda_1 : P \to P \oplus P'$, $\lambda_1(a) = (a, 0)$, $\pi_i : R^(I) \to R$ the i_{th} projection. Define $x_i = \pi_p f(e_i) \in P$ where $e_i = (\delta_{ij})_{j\in I}$, $\pi_p : P \oplus P' \to P$, $\pi_P((a, b)) = a$. Suppose $f^{-1}\lambda_1(m) = \sum r_i e_i = \pi_j(f^{-1}\lambda_1(m)) = \sum\limits_i r_i\pi_j(e_i) = r_j$ for any $m \in P$, $\pi_P(f^{-1}\lambda_1(m)) = \pi_P\lambda_1(m) = m = \sum\limits_i r_i\pi_P f(e_i) = \sum\limits_i r_i x_i$, thus $f^{-1}\lambda_1(m) = \sum r_i e_i = \sum f_i(m)e_i$, hence $m = \pi_P f(f^{-1}\lambda_1(m)) = \sum f_i(m)\pi_P f(e_i) = \sum f_i(m)x_i$. $\Leftarrow$:Define $\pi : R^{(I)} \to P$, $\pi(e_i) = x_i (i \in I)$. Since for any $\subset i \in I \sum a_i x_i \in P$, there exist $(a_i)_{i\in I} \in R^{(I)}$ such that $\pi((a_i)_{i\in I}) =\subset i \in I \sum a_i x_i$, π is epimorphic. Define $h : P \to R^{(I)}$, $h(m) = (f_i(m))_{i\in I}$, then $\pi \circ h = id_P$, thus $R^{(I)} \cong P \oplus P'$, where $P' = Ker\pi$. In fact, for any $a \in R^{(I)}$, then $h(\pi(a)) \in R^{(I)}$ and $a - h\pi(a)$ satisfies $\pi(h\pi(a)) = 0$, thus $a = h\pi(a) + (a - h\pi(a)$, hence $R^{(I)} = Imh + Ker\pi$. For any $h(a) \in Imh \cap Ker\pi$, then $\pi h(a) = a = 0$, thus $h(a) = 0$. Hence $R^{(I)} = Imh \oplus Ker\pi$ and $h(P) = h\pi(R^{(I)}) = Imh$. For any $a \in P$, $\pi h(a) = a$, then $\pi : Imh \to P$ is surjective. For any $\pi(x) = 0$, there is $y \in P$ such that $x = h(y)$, then $y = \pi h(y) = \pi(x) = 0$, thus $x = 0$, therefore π is injective. Hence $\pi : Imh \to P$ is isomorphism. Therefore $R^{(I)} \cong P \oplus P'$, then P is projective.

3. Since P is a finitely generated projective left R-module, there is P' such that $P \oplus P' = R^n$, then $(P \oplus P')^* \cong P^* \oplus P'^* = Hom_R(R^n, R) \cong R^n_R$. Hence P^* is a projective right R-module.

4. Suppose every submodule of a projective left R-module is projective, N is a submodule of injective module E. For any left ideal I of R, embedding homomorphism $\lambda : I \to R$ and any $f : I \to E/N$, canonical homomorphism $\pi : E \to E/N$. Since I is projective, there is $\varphi : I \to E$ such that $f = \varphi \circ \pi$. Since E is injective, there is $\psi : R \to E$ such that $\psi \circ \lambda = \varphi$, then $(\pi \circ \psi) \circ \lambda = \pi \circ \varphi = f$, i.e. $\pi \circ \psi$ is an extension of f. According to Theorem 2.3.2, E/N is injective. Conversely, for any submodule L of projective module P, any epimorphism $\pi_1 : M \to N$ and any homomorphism $\varphi : L \to N$. Let $\lambda : M \to E$ is injective and E is an injective module,

$K = Ker\pi_1$, then $\lambda(K)$ is a submodule of E, thus $\pi_2 : E \to E/\lambda(K)$ is a canonical homomorphism. Define $\eta : N \to E/\lambda(K)$, $\eta(\pi_1(m)) = \pi_2\lambda(m)$. If $\pi_1(m_1) = \pi_2(m_2)$, then $m_1 - m_2 \in K$, thus $\lambda(m_1 - m_2) \in \lambda(K)$, hence $\pi_2(\lambda(m_1) - \lambda(m_2)) = 0$, therefore η is well-defied. Since $E/\lambda(K)$ is injective, there is $\psi : D \to E/\lambda(K)$ such that $\psi \circ \tau = \eta\varphi$ where $\tau : L \to P$ is an embedding homomorphism. Since P is projective, there is $\xi : P \to E$ such that $\pi_2 \circ \xi = \psi$. Suppose canonical homomorphism $\pi_3 : E \to E/\lambda(M)$, $\lambda(K) \subset \lambda(M)$, then $\pi_4 : (a + \lambda(K)) = a + \lambda(M)$ is a R-module homomorphism and $\pi_4 \circ \pi_2 = \pi_3$, then $\pi_3 \circ \xi \circ \tau = \pi_4\pi_2\xi\tau = \pi_4\psi\tau = \pi_4\eta\varphi$, while $\pi_4\eta\pi_1 = \pi_4\pi_2\lambda = \pi_3\lambda = 0$, while π_1 is surjective, $\pi_4\eta = 0$, then $\pi_3\xi = \pi_4\pi_2\xi = \pi_4\eta\varphi = 0$, thus $Im\xi \subset Ker\pi_3 = \lambda(M)$. Let $\zeta = \lambda^{-1}\xi$, then $\zeta : P \to M$ satisfies $\zeta\tau = \varphi$, this means that L is projective.

5. Example:$I = \mathbb{R}[x]x \leq \mathbb{R}[x]$ is free, but there is not idempotent element e such that $\mathbb{R}[x]x = \mathbb{R}[x]e$.

6. Since $\varphi : Re \to Rf$ is a left R-module isomorphism, $\varphi(e) = rf$, $\varphi^{-1}(f) = se$, then $e = \varphi^{-1}(\varphi(e)) = rse$, $f = \varphi\varphi^{-1}(f) = srf$, and $\begin{cases} erf = rf \\ fse = se \\ erse = e \\ fsrf = f \end{cases}$
   Define right R-module homomorphism $\psi : eR \to fR$, $\psi(ea) = fsea$ and right R-module homomorphism $\psi^{-1} : fR \to eR$, $\psi^{-1}(fa) = erfa$, then $\psi^{-1}\psi(ea) = erfsea = er(fse)a = (erse)a = ea$ and $\psi\psi^{-1}(fa) = fserfa = fs(erf)a = (fsrf)a = fa$, thus ψ is a right R-module isomorphism. Similarly, if $\psi : eR \to fR$ is right R-module isomorphism, then $\varphi : Re \to Rf$ is a left R-module isomorphism.

7. Suppose ${}_RR^n =_R P \oplus_R P'$ where ${}_RP$ is a finitely generated projective module. Suppose $P' \neq 0$(otherwise $P = R^n$ is free), assume that $e \in End_RR^n$ satisfies $e^2 = e, P = R^ne, P' = R^n(1 - e)$, let $e_i \in End_RR^n$ satisfy $\{(0, \cdots, 0, \subseteq i_{th}1, 0, \cdots, 0)|r \in R\} = R^ne_i$(i.e. $e_i = \lambda_i\pi_i$ where $\pi_i : R^n \to R$ and $\lambda_i : R \to R^n$), then $e_1End_RR^ne_1 \cong R$, $e_1 = e_1ee_1 + e_1(1-e)e_1 \in R$ where R is a local ring, then e_1 is identity of $e_1End_RR^ne_1$. Ife_1ee_1 and $e_1(1 - e)e_1$ are not invertible and M is the unique ideal of R, then $< e_1ee_1 >\neq R$ and $< e_1(1 - e)e_1 >\neq R$, thus $< e_1ee_1 >\subseteq M$ and $< e_1(1 - e)e_1 >\subseteq M$, then $e_1 \in M$, thus $M = R$, it is contradiction. Let e_1fe_1 represent invertible element of $\{e_1ee_1, e_1(1 - e)e_1\}$ and $K_1 = Im(e_1f)$, since e_fe_1 and e_1 are isomorphism ${}_RR \to_R R$, then $f : R \to K_1$ and $e_1 : K_1 \to R$ are isomorphism, thus $R^n = K_1 \oplus R^{n-1}$(When $f = e$, $K_1 = R^ne_1e \subseteq P$, then $R^n = K_1 \oplus P_1 \oplus P'$ where $P = K_1 \oplus P_1$, when $f = (1 - e)$, $K_1 = R^ne_1(1 - e) \subseteq P'$, then $R^n = P \oplus K_1 \oplus P''$ where $P' = K_1 \oplus P''$), then by introduction, $P \cong K_1 \oplus \cdots \oplus K_m$ where $K_i \cong_R R$, thus P is free.

8. Suppose ${}_RP$ is projective and $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. If

$f_*(\alpha) = f \circ \alpha = 0$, since f is injective, $\alpha = 0$. Since $g_* \circ f_* = (g \circ f)_* = 0$, $Imf_* \subset Kerg_*$, if $\beta \in Kerg_*$, i.e. $g \circ \beta = 0, \beta \in Hom_R(P, B)$, then $Im\beta \subset Kerg = Imf$, let $f^{-1} : Imf \to A$, $\alpha = f^{-1}\beta \in Hom_R(P, A)$ and $f_*(\alpha) = f(f^{-1} \circ \beta) = \beta \in Imf_*$, then $Imf_* = Kerg_*$. For any $\xi \in Hom_R(P, C)$, since P is projective and $g : B \to C$ is epimorphism, there is $\zeta : P \to B$ such that $g \circ \zeta = \xi = g_*(\zeta)$, thus g_* is epimorphic. Hence $0 \longrightarrow Hom_R(P, A) \xrightarrow{f_*} Hom_R(P, B) \xrightarrow{g_*} Hom_R(P, C) \longrightarrow 0$ is exact. Conversely, let $0 \longrightarrow L \xrightarrow{f} F \xrightarrow{g} P \longrightarrow 0$ is exact where F is free, then $0 \longrightarrow Hom_R(P, L) \xrightarrow{f_*} Hom_R(P, F) \xrightarrow{g_*} Hom_R(P, P) \longrightarrow 0$ is exact, thus there is $h \in Hom_R(P, F)$ such that $g_*(h) = gh = id_P$, hence $0 \longrightarrow L \xrightarrow{f} F \xrightarrow{g} P \longrightarrow 0$ is splitting, therefore P is projective.

9. Suppose ${}_RP$ is injective and $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. If $g^*(\alpha) = \alpha \circ g = 0$, since f is surjective, $\alpha = 0$. Since $f^* \circ g^* = (g \circ f)^* = 0$, $Img^* \subset Kerf^*$, if $\beta \in Kerf^*$, i.e. $\beta \circ f = 0, \beta \in Hom_R(B, E)$, then $Kerg = Imf \subset Ker\beta$, let $g^{-1} : C \to C/Kerg$, $\alpha = \beta g^{-1} \in Hom_R(C, E)$ and $g^*(\alpha) = \beta \circ g \circ g^{-1} = \beta \in Img^*$, then $Img^* = Kerf^*$. For any $\xi \in Hom_R(A, E)$, since E is injective and $f : A \to B$ is monomorphism, there is $\zeta : B \to E$ such that $\zeta \circ f = \xi = f^*(\zeta)$, thus f^* is epimorphic. Hence $0 \longrightarrow Hom_R(C, E) \xrightarrow{g^*} Hom_R(B, E) \xrightarrow{f^*} Hom_R(A, E) \longrightarrow 0$ is exact. Conversely, let $0 \longrightarrow E \xrightarrow{f} J \xrightarrow{g} K \longrightarrow 0$ is exact where F is injective, then $0 \longrightarrow Hom_R(K, E) \xrightarrow{g^*} Hom_R(J, E) \xrightarrow{f^*} Hom_R(E, E) \longrightarrow 0$ is exact, thus there is $h \in Hom_R(J, E)$ such that $f^*(h) = hf = id_E$, hence $0 \longrightarrow E \xrightarrow{f} J \xrightarrow{g} K \longrightarrow 0$ is splitting, therefore E is injective.

10. If ${}_RA$ is injective,
$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\lambda"] \arrow[d, "g"'] & R \arrow[dl, dotted, "\overline{g}"] \\ & A & \end{tikzcd}$$
, then $g(r) = \overline{g}\lambda(r) = \overline{g}(r \cdot 1) = r\overline{g}(1)$, let $a = \overline{g}(1)$, then $g(r) = ra$ for every $r \in R$. Conversely,
$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\lambda"] \arrow[d, "\forall g"'] & R \\ & A & \end{tikzcd}$$
, let $\overline{g} : R \to A, \overline{g}(r) = ra$ for any $r \in R$, then $\overline{g}$ is a R-module homomorphism, and $\overline{g} \circ \lambda = g$, according to Theorem 2.3.2, A is injective.

11. Let $e_i \in R$ satisfy $e_i(\sum\limits_{i=0}^{n} a_k x^k) = a_i x^i$ and $S = \sum\limits_{i=0}^{\infty} e_i R$, for any right ideal I of R_R, then there is I' such that $I \oplus I'$ is an essential submodule of R_R, while $e_i R$ is simple and $e_i R \cap (I \oplus I') \neq 0$, then $e_i R \subseteq I \oplus I'$, thus

$S \subseteq I \oplus I'$, consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r, "\lambda_1"] \arrow[dr, "\varphi\lambda_1"'] & I \oplus I' \arrow[r, "\lambda_2"] \arrow[d, "\forall\varphi"'] & R \arrow[dl, "\overline{\varphi}"] \\ & & R_R & \end{tikzcd},$$

for any right R-module homomorphism $\varphi : I \to R$, $\varphi\lambda : S \to R$ is right R-module homomorphism. Define $\overline{\varphi} : R \to R$, $\overline{\varphi}(f)(\sum a_k x^k) = \sum\limits_{i=0}^{\infty}[\varphi\lambda_1(e_i)](e_i(f(\sum a_k x^k)))$ for any $\sum a_k x^k \in \mathbb{P}[x]$ and any $f \in R$. In particular, $\overline{\varphi}(e_l) = \sum\limits_{i=0}^{\infty} \varphi\lambda_1(e_i)(e_i e_l) = \varphi\lambda_1(e_l)$, thus $\overline{\varphi}\lambda_2\lambda_1 = \varphi\lambda_1$, hence $(\overline{\varphi}\lambda_2 - \varphi)\lambda_1 = 0$, then $Ker(\overline{\varphi}\lambda_2 - \varphi) \supseteq S$, hence for any $a \in I \oplus I'$ there is $r \in R$ such that $ar \neq 0, ar \in S$, therefore $(\overline{\varphi}\lambda_2 - \varphi)(ar) = (\overline{\varphi}\lambda_2 - \varphi)(a)r = 0$. Hence ${}_R R$ is injective.

12. Let $S = \left\{ \left(\begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ 0 & \cdots & 0 \end{array} \right) \middle| a_{1i} \in D \right\}$, then S is an ideal of $\mathfrak{t}(n, D) = R$, $S \cdot M_n(D) \leq R$, $s(a_{ij}) = 0$, thus $e_{1k}(a_{ij}) = 0$ for $1 \leq k \leq n$, hence $(a_{ij}) = 0$. For any submodule $N \neq 0$ of ${}_R M_n(D)$, there is $0 \neq (a_{ij}) \notin N$, then $S(a_{ij}) \neq 0$, $RS(a_{ij}) \cap R = RS(a_{ij}) \neq 0$, thus $N \cap R \neq 0$, hence ${}_R R$ is an essential submodule of ${}_R Q$. Suppose $\varphi : I \to Q$ is left R-module homomorphism where I is an ideal of Q. For any $\sum\limits_{i=1}^{n} q_i a_i \in QI$ where $q_i \in Q$ and $a_i \in I$, define $\psi(\sum\limits_{i=1}^{n} q_i a_i) = \sum\limits_{i=1}^{n} q_i \varphi(a_i)$, if $\sum\limits_{i=1}^{n} q_i a_i = 0$, then for any $s \in S$, $sq_i \in R$, thus $\sum\limits_{i=1}^{n} sq_i a_i = 0$, hence $s \sum\limits_{i=1}^{n} q_i \varphi(a_i) = 0$, therefore $\sum\limits_{i=1}^{n} q_i \varphi(a_i) = 0$. Hence ψ is well-defined Q-module homomorphism. Since ${}_Q Q$ is injective, there is Q-module homomorphism $\phi : Q \to Q$, $\phi|_{QI} = \psi$, $\phi|R : R \to Q$. Since $\phi|_{QI} = \psi$, $\phi_I = (\phi|_{QI})|_{RI} = \psi|_{RI} = \psi_I = \varphi$, $(\phi|_R)|_I = \phi_I$, hence ${}_R Q$ is injective.

13. Suppose $0 \longrightarrow K \xrightarrow{f_1} P \xrightarrow{g_1} M \longrightarrow 0$ and $0 \longrightarrow K' \xrightarrow{f_2} P' \xrightarrow{g_2} M \longrightarrow 0$ are exact, let $a = \{(x, y) \in P \oplus P' | g_1(x) = g_2(y)\}$, consider commutative diagram in Figure 1, $Ker\pi_2 = \{(x, 0)|(x, 0) \in Q\} = \{(x, 0)|g_1(x) = g_2(0) = 0\} = \{(x, 0)|x \in Kerg_1 = K\}$, then $\lambda_1(x) = (x, 0)$, thus there is exact sequence $0 \longrightarrow K \xrightarrow{\lambda_1} Q \xrightarrow{\pi_2} P' \longrightarrow 0$. Similarly, $Ker\pi_1 = \{(0, y)|y \in P'\}$ and there is exact sequence $0 \longrightarrow K' \xrightarrow{\lambda_2} Q \xrightarrow{\pi_1} P \longrightarrow 0$. Since P and P' are projective, $K \oplus P' \cong K' \oplus P$.

14. Consider commutative diagram in Figure 2, let $Q = E \oplus E'/N$ where $N = \{(f_1(m), f_2(m))|m \in M\}$, $\pi_1((x, y) + N) = g_1(x)$, $\pi_2((x, y) + N) = g_2(y)$, $\lambda_1(x) = (x, 0) + N$ and $\lambda_2(y) = (0, y) + N$. For any $x \in Ker\lambda_1$, there is $m \in M$ such that $(x, 0) = (f_1(m), f_2(m))$, since f_2 is injective, $m = 0$, then $x = 0$. If $(x, y) + N \in Ker\pi_2$, then $(x, y) + N = (x, f_2(m)) + N =$

$(x + f_1(m), 0) + (-f_1(m), f_2(m)) + N = (x + f_1(m), 0) + N \in \lambda_1$ for some $m \in M$, while $\pi_2\lambda_1(x) = \pi_2((x, 0) + N) = 0$, then $Ker\pi_2 = Im\lambda_1$. Similarly, $Ker\pi_1 = Im\lambda_2$, since E and E' are injective, $E \oplus L' \cong Q \cong E' \oplus L$.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K' \arrow[r, "="] \arrow[d, "\lambda_2"'] & K' \arrow[d, "f_2"] & \\
0 \arrow[r] & K \arrow[r, "\lambda_1"] \arrow[d, "\|"'] & Q \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & P' \arrow[r] \arrow[d, "g_2"] & 0 \\
0 \arrow[r] & K \arrow[r, "f_1"] & P \arrow[r, "g_1"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Figure 1: Exercise 2.3.13

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M \arrow[r, "f_1"] \arrow[d, "f_2"'] & E \arrow[r, "g_1"] \arrow[d, "\lambda_1"'] & L \arrow[r] & 0 \\
0 \arrow[r] & E' \arrow[r, "\lambda_2"] \arrow[d, "g_2"'] & Q \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & L \arrow[r] & 0 \\
 & L' \arrow[d] & L' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Figure 2: Exercise 2.3.14

15. Let $\varphi : P \oplus P \to R \oplus R$, $\varphi((f_1, f_2)) = (f_1, f_2)\begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}$. Let $F_1 = f_1 \cos x + f_2 \sin x$ and $F_2 = -f_1 \sin x + f_2 \cos x$. When $f_1, f_2 \in P$, then $F_1, F_2 \in R$, it is obvious that φ is a R-module homomorphism. $\varphi^{-1}(F_1, F_2) = (F_1, F_2)\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$ is invertible homomorphism of φ. Hence $P \oplus P \cong R \oplus R$.

16. If $x \in Ker\psi$, then $g'\psi(x) = \phi g(x) = 0$, thus $g(x) \in Ker\phi = 0$, then there is $y \in A$ such that $x = f(y)$, thus $\psi(x) = \psi f(y) = f'\eta(y) = 0$, since f' is injective, $\eta(y) = 0$. While η is injective, $y = 0$, then $x = f(y) = 0$. Hence ψ is injective. For any $x \in B'$, there is $y \in C$ such that $\phi(y) = g'(x)$, thus there is $z \in B$ such that $g(z) = y$, then $\phi(y) = \phi g(z) = g'(x)$, thus $x - \psi(z) \in Kerg' = Imf'$. Hence there is $u \in A'$ such that $x - \psi(z) = f'(u)$, while η is surjective, there is $v \in A$ such that $\eta(v) = u$, then $f'(u) = f'\eta(v) = \psi f(v) = x - \psi(z)$, thus $x = \psi(f(v) + z)$. Hence ψ is surjective.

17. Let $\Omega = \{L \leq E | N \text{ is an essential submodule of } L\}$, it is obvious that $\Omega \neq 0$. Let $L_1 \leq L_2$ if $L_1 \subseteq L_2$, if $\cdots \subseteq L_n \subseteq L_{n+1} \subseteq \cdots$ is an ascending chain of element in Ω, then $L = \cup L_i \leq E$, for any $0 \neq x \in L$, there is i such that $x \in L_i$ and $N \cap Rx \neq$, thus N is an essential submodule of L. According to Zorn's Lemma, there is a maximal element $E(N) \in \Omega$. Suppose $N' \leq E$ is maximal and $E' \cap E(N) = 0$, then $E(N) \oplus N'$ is an essential submodule of E. For any $N' \subsetneqq L \leq E$, if $(E(N) + N') \cap$

$L = N' + (E(N) \cap L) \subseteq N'$, then $E(N) \cap L' = 0$, it is contradiction for N' is maximal. Hence $(E(N) + N')/N'$ is an essential submodule of E/N'. Since $(E(N) \oplus N')/N' \cong E(N) \hookrightarrow E$, there is monomorphism $g : (E(N) \oplus N')/N' \mapsto E$. Since E is injective, there is h such that $h \circ \lambda = g$. Since $Kerg = 0$, $Im\lambda \cap Kerh = 0$, while $Im\lambda$ is an essential submodule of E/N', $Kerh = 0$, i.e. h is injective. While N is an essential submodule of $E(N)$, and $E(N) = Img = h((E(N) + N')/N')$ is an essential submodule of $h(E/N')$, then N is an essential submodule of $h(E/N')$. Since $E(N)$ is maximal, $E(N) = h(E/N') = h((E(N) + N')/N')$, while h is injective, $E/N' = E(N) + N'/N'$, then $E = E(N) + N' = E(N) \oplus N'$. Hence $E(N)$ is injective.

$$\begin{tikzcd} & 0 \arrow[d] & \\ 0 \arrow[r] & (E(N) \oplus N')/N' \arrow[r, "\lambda"] \arrow[d, "g"'] & E/N' \arrow[dl, "h"] \\ & E & \end{tikzcd}$$

18. Suppose $\mathbb{Z}a + K = \mathbb{Q}$. If $a \in K$, then $\mathbb{Z}a \subseteq K$, $\mathbb{Z}a + K = K = \mathbb{Q}$. If $a = \frac{m}{n}, (m, n) = 1, m, n > 0$, then $\frac{1}{n^k} \notin K$ for any $k \in \mathbb{Z}_+$. $\frac{1}{n^k} = \frac{x_k m}{n} + y_k$ where $x_k \in \mathbb{Z}, y_k \in K$, then $\frac{1}{n^{k-1}} = x_k m + n y_k$, thus $y_k = \frac{1}{n^k} - \frac{x_k n^{k-1} m}{n^k} = \frac{1 - x_k n^{k-1} m}{n^k}$, hence $1 = u\frac{m}{n} + v = \frac{um + nv}{n}$ where $u \in \mathbb{Z}$ and $n = um + nv, nv \in K \cap \mathbb{Z}$. If $1 \in K$, then $\frac{1}{n^{k-1}} \in K$ for $\frac{1}{n^{k-1}} = x_k m + n y_k$, it is contradiction. If $1 \notin K$, then for any $p \in \mathbb{Z}$, $\frac{1}{p} \notin K$, thus $\frac{1}{p} \in \mathbb{Z}\frac{1}{n}$, it is impossible. Hence submodule $\mathbb{Z}a \leq \mathbb{Q}$ is superfluous for any $a \in \mathbb{Q}$.

19. $Ker\pi = N = Re_{13}$. If $Re_{13} + M = R$, then $e_{22} = ke_{13} + a$ where $k \in R, a \in M$, thus $e_{22} = e_{22}^2 = e_{22}a \in M$. Similarly, $e_{33} \in M$, then $e_{12} = e12e_{22} \in M$, $e_{13} = e_{13}e_{33} \in M$ and $e_{23} = e_{23}e_{33} \in M$, thus $Re_{13} \subseteq M$, hence Re_{13} is a superfluous submodule of ${}_RR$. Therefore $\pi : R \to R/N$ is a projective cover of R/N.