$P_{124}-125$.

12. (1). $\begin{bmatrix} \lambda^3-\lambda & 2\lambda^2 \\ \lambda^2+5\lambda & 3\lambda \end{bmatrix} \to \begin{bmatrix} 3\lambda & \lambda^2+5\lambda \\ 2\lambda^2 & \lambda^3-\lambda \end{bmatrix}$

$\to \begin{bmatrix} \lambda & \frac{1}{3}\lambda^2+\frac{5}{3}\lambda \\ \lambda^2 & \frac{1}{2}\lambda^3-\frac{1}{2}\lambda \end{bmatrix} \to \begin{bmatrix} \lambda & \frac{1}{3}\lambda^2+\frac{5}{3}\lambda \\ 0 & \frac{1}{6}\lambda^3-\frac{5}{3}\lambda^2-\frac{1}{2}\lambda \end{bmatrix}$

$\to \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{6}\lambda^3-\frac{5}{3}\lambda^2-\frac{1}{2}\lambda \end{bmatrix} \to \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^3-10\lambda^2-3\lambda \end{bmatrix}$.

(2) $\begin{bmatrix} 1-\lambda & \lambda^2 & \lambda \\ \lambda & \lambda & -\lambda \\ 1+\lambda^2 & \lambda^2 & -\lambda^2 \end{bmatrix} \to \begin{bmatrix} 1 & \lambda^2+\lambda & 0 \\ \lambda & \lambda & -\lambda \\ 1+\lambda^2 & \lambda^2 & -\lambda^2 \end{bmatrix}$

$\to \begin{bmatrix} 1 & \lambda^2+\lambda & 0 \\ \lambda & \lambda & -\lambda \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^2+\lambda \end{bmatrix}$

13. 证. $A(\lambda)$ 与 $B(\lambda)$ 等价.

则 $A(\lambda) = P_1P_2\cdots B(\lambda)Q_1Q_2\cdots$

其中 $P_1, P_2, \cdots, Q_1, Q_2\cdots$ 均为初等矩阵.

三种初等矩阵对应行列式的值均为常数.

$|A(\lambda)| = |P_1||P_2|\cdots|B(\lambda)||Q_1||Q_2|\cdots$

因此 $|A(\lambda)|$ 与 $|B(\lambda)|$ 只差一个非零常数.

15. (1)
$$\begin{pmatrix} 2\lambda & 1 & 0 \\ 0 & -\lambda(\lambda+2) & -3 \\ 0 & 0 & \lambda^2-1 \end{pmatrix} \to \begin{pmatrix} \lambda & 0 & 0 \\ 0 & -\lambda(\lambda+2) & -3 \\ 0 & \frac{1}{3}\lambda^2+\frac{2}{3}\lambda & \lambda^2 \end{pmatrix}$$
$$\to \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda(\lambda+2) & -3 \\ 0 & \frac{1}{3}(\lambda^3-\lambda)(\lambda+2) & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & (\lambda^3-\lambda)(\lambda+2) \end{pmatrix}$$

$D_1(\lambda) = 1.$ $\quad D_2(\lambda) = \lambda.$ $\quad D_3(\lambda) = \lambda^2(\lambda^2-1)(\lambda+2)$

(3)
$$\begin{pmatrix} 1-\lambda & 2\lambda-1 & \lambda \\ \lambda & \lambda^2 & -\lambda \\ 1+\lambda^2 & \lambda^2+\lambda-1 & -\lambda^2 \end{pmatrix} \to \begin{pmatrix} 1 & \lambda^2+2\lambda-1 & 0 \\ \lambda & \lambda^2 & -\lambda \\ 1+\lambda^2 & \lambda^2+\lambda-1 & -\lambda^2 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & \lambda^2+2\lambda-1 & 0 \\ \lambda & \lambda^2 & -\lambda \\ 1 & \lambda^2+\lambda-1-\lambda^3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & \lambda^3+\lambda & 0 \end{pmatrix}$$
$$\to \begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^3+\lambda \end{pmatrix}$$

$D_1(\lambda) = 1.$ $\quad D_2(\lambda) = \lambda,$ $\quad D_3(\lambda) = \lambda^2(\lambda^2+1)$

16. (1) $d_1(\lambda) = 1$, $\quad d_2(\lambda) = \lambda$, $\quad d_3(\lambda) = \lambda(\lambda^2-1)(\lambda+2)$

(3) $d_1(\lambda) = 1$, $\quad d_2(\lambda) = \lambda$, $\quad d_3(\lambda) = \lambda^3+\lambda$

17. (1) λ, λ, (λ^2-1), $(\lambda+2)$.

(3). λ, λ, (λ^2+1).

18. $d_1(\lambda) = 1$

$d_2(\lambda) = \lambda$

$d_3(\lambda) = \lambda^2(\lambda+1)(\lambda-1)$

$d_4(\lambda) = \lambda^2(\lambda-1)(\lambda+1)^3$

$A(\lambda)$ 的标准形为 $\begin{pmatrix} 1 & & & \\ & \lambda & & \\ & & \lambda^2(\lambda+1)(\lambda-1) & \\ & & & \lambda^2(\lambda-1)(\lambda+1)^3 \end{pmatrix}$

19.

$$\lambda E - A = \begin{pmatrix} \lambda & & & a_0 \\ -1 & \ddots & & a_1 \\ & \ddots & \lambda & \vdots \\ & & -1 & \lambda+a_{n-1} \end{pmatrix}$$

$\begin{vmatrix} -1 & \lambda & & \\ & -1 & \ddots & \\ & & \ddots & \lambda \\ & & & -1 \end{vmatrix} = (-1)^{n-1}$ 是 $|\lambda E - A|$ 的一个 $n-1$ 阶子式.

$D_1(\lambda) = D_2(\lambda) = \cdots = D_{n-1}(\lambda) = 1$.

$d_1(\lambda) = d_2(\lambda) = \cdots = d_{n-1}(\lambda) = 1$,

而 $|\lambda E - A| = a_0(-1)^{2n} + \lambda(a_1(-1)^{2n} + \lambda(\cdots + \lambda(a_{n-1}+\lambda)\cdots))$

$= a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + a_n$.

$\therefore D_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$, $d_n(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0$

2). $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix}=1$ 是 $|A(\lambda)|$ 的一个二阶子式.

对于 $A(\lambda)$. $d_1(\lambda)=d_2(\lambda)=1$.

$$|A(\lambda)|=(\lambda-\alpha)(\lambda-\alpha)((\lambda-\alpha)^2+\beta^2)-\begin{vmatrix} 0 & \lambda-\alpha & -1 \\ \beta^2 & 1 & 0 \\ 0 & \beta^2 & \lambda-\alpha \end{vmatrix}$$

$$=(\lambda-\alpha)^4+(\lambda-\alpha)^2\beta^2+\beta^2((\lambda-\alpha)^2+\beta^2)$$

$$=(\lambda-\alpha)^4+\beta^4+2(\lambda-\alpha)^2\beta^2=((\lambda-\alpha)^2+\beta^2)^2$$

$D_4(\lambda)=((\lambda-\alpha)^2+\beta^2)^2$

$|A(\lambda)|$ 的 3 阶子式次数不低于 2. 且小于 4.

且均为实多项式. 且为 $D_4(\lambda)$ 的因式.

因此 $D_3(\lambda)=(\lambda-\alpha)^2+\beta^2$.

$d_3(\lambda)=d_4(\lambda)=(\lambda-\alpha)^2+\beta^2$.

$\therefore$ $A(\lambda)$ 和 $B(\lambda)$ 等价.