2 Modules

2.2 Modules and free modules

1. Let M be a simple $\mathbb{C}[G]$ module, and $0 \neq m \in M$. Then $\varphi : \mathbb{C}[G] \to M, a \mapsto am$, is a homomorphism of left $\mathbb{C}[G]$ modules. Since $Im\varphi = \mathbb{C}[G]m$ is a nonzero submodule of M, $Im = M$. Thus $M \cong \mathbb{C}[G]/Ker\varphi$. In particular, M is a finite-dimensional vector space over $\mathbb{C}$. For any $f \in End_{\mathbb{C}[G]}(M)$, there is a $\lambda \in \mathbb{C}$ and nonzero $v \in M$ such that $f(v) = \lambda v$. Thus $Ker(f - \lambda id_M) \neq 0$ is a submodule of M. Therefore $f = \lambda id_M$. It is obvious that $\lambda id_M \in End_{\mathbb{C}[G]}(M)$ for any $\lambda \in \mathbb{C}$. Hence $End_{\mathbb{C}[G]}(M) = \mathbb{C}id_M \cong \mathbb{C}$ as rings.

2. On the contrary, suppose $\mathbb{Q} = \mathbb{Z}\frac{a_1}{b_1} + \mathbb{Z}\frac{a_2}{b_2} + ... + \mathbb{Z}\frac{a_n}{b_n}$ where $a_i, b_i \in \mathbb{Z}$ and $(a_i, b_i) = 1$. Since $\mathbb{Z}\frac{a_i}{b_i} = \mathbb{Z}(-\frac{a_i}{b_i})$, we assume $a_i \geq 1, b_i \geq 1$ for $1 \leq i \leq n$. Thus there is a prime $p > max\{b_1, b_2, ..., b_n\}$. Since $\frac{1}{p} \in \mathbb{Q}$, there exist $m_i \in \mathbb{Z}$ such that $\frac{1}{p} = \frac{m_1 a_1}{b_1} + \frac{m_2 a_2}{b_2} + ... + \frac{m_n a_n}{b_n}$, then $\frac{b_1 b_2 ... b_n}{p} = (\sum \frac{m_i a_i}{b_i})(b_1 b_2 ... b_n) \in \mathbb{Z}$, this is impossible as $p \nmid b_i$ for any $1 \leq i \leq n$. This contradiction implies that $\mathbb{Q}$ is not finitely generated $\mathbb{Z}$ module.

3. If $M_1 = \sum_{i=1}^{n} Rx_i$ and $M_2 = \sum_{i=n+1}^{m} Rx_i$, then $M_1 + M_2 = \sum_{i=1}^{m} Rx_i$ and $M_1 + M_2/M_1 = (\sum_{i=1}^{m} Rx_i)/M_1 = \sum_{i=1}^{m} R(x_i + M_1)$.

4. Suppose $N = \sum_{i=1}^{m} Rx_i$, $M/N = \sum_{i=m+1}^{n} R(x_i + N)$. Let $M' = \sum_{i=1}^{n} Rx_i \subseteq M$. For any $m \in M$, there are $r_i \in R$ such that $m + N = \sum_{i=m+1}^{n} r_i(x_i + N) = \sum_{i=m+1}^{n} r_i x_i + N$. Then $m - \sum_{i=m+1}^{n} r_i x_i \in N$. Hence

$m - \sum_{i=m+1}^{n} r_i x_i = \sum_{i=1}^{m} r_i x_i$ for some $r_i \in R$. So $m = \sum_{i=1}^{n} r_i x_i \in M'$. That is, $M = M'$ is finitely generated.

5. For any $x|inN$, $fg(x) \in Im(f)$, $g(x - fg(x)) = g(x) - g(x) = 0$, then $x = fg(x) + (x + fg(x)) \in Im(f) + Ker(g)$, therefore $N = Im(f) + Ker(g)$. If $f(a) \in Ker(g)$, then $0 = g(f(a)) = a$. Hence $f(a) = f(0) = 0$. Thus $N = Im(f) \oplus Ker(g)$.

6. (1) Suppose $R = K \oplus L$ is a direct sum of left R-modules. Then K and L are left ideals of R. Let $1 = e + f, e \in K, f \in L$. Then $e = e^2 + ef = e^2 + fe$, hence $ef = fe \in Re \cap Rf \subseteq K \cap L = \{0\}$. Hence $e^2 = e, fe = ef = 0$. Similarly, $f^2 = f$. Hence $Re \subseteq K$ and $Rf \subseteq L$. For any $a \in K$, we have $a = ae + af$, then $a - ae = af \in K \cap L = \{0\}$, $af = 0, a = ae \in Re$. This means $K = Re$, similarly $L = Rf$.

   Conversly, suppose $K \neq 0$ and $L \neq 0$, then $e \neq 0, f \neq 0$. By the assumption, there is a, b not both zero with $ae + bf = 0$. Thus $ae = -bf \in Re \cap Rf = \{0\}$. Since R is an integral domain, $a = b = 0$. This is a contradiction, hence either $K = 0$ or $L = 0$.

   (2) Without loss os generality, we assume $m \leq n$. We prove that $m = n$ by induction on m. If $m = 1$ and $n > 1$, let $\varphi : R^n \to R$ be an isomorphism, $K = \varphi(\{(a, 0, ..., 0)|a \in R\}) \neq 0$, $L = \varphi(\{(0, a_2, ..., a_n)|a_i \in R\}) \neq 0$ and $K \oplus L = R$. This is impossible by (1). Hence $n = 1$ is true. Now let $m \geq 2$, $e_i = (0, ..., 1, ..., 0)$

where the i_{th} element is 1 and the other is 0,$i = 1, 2.., m$,$\varphi : R^m \to R^n$ is a left R-module isomorphism, then φ induces isomorphism $\varphi_1 : Re_1 \to Im\varphi_1$ and $\varphi_2 : Re_2 + ... + Re_m \to Im\varphi_2$, then $R^n = Im\varphi_1 \oplus Im\varphi_2$, $R^{m-1} \simeq Im\varphi/Im\varphi_1 = R^n/Im\varphi_1 = Im\varphi_2$, therefore $m = n$.

7. Define $f_1(x^{2n}) = x^n, f_1(x^{2n-1}) = 0$, and $f_2(x^{2n}) = 0, f_2(x^{2n-1}) = x^n$, if there are $a, b \in End_{\mathbb{P}}(\mathbb{P}[x])$ such that $af_1 + bf_2 = 0$, then $(af_1 + bf_2)(x^{2n}) = a(x^n) = 0$, so $a = 0$; $(af_1 + bf_2)(x^{2n-1}) = b(x^n) = 0$, so $b = 0$. Therefore f_1 and f_2 are linearly-independent over $End_{\mathbb{P}}(\mathbb{P}[x])$. Let $g_1 \in End_{\mathbb{P}}(\mathbb{P}[x])$, $g_1(x^n) = x^{2n}$, $g_2 \in End_{\mathbb{P}}(\mathbb{P}[x])$, $g_2(x^n) = x^{2n-1}$, then $(g_1f_1 + g_2f_2)(x^{2n}) = x^{2n}$, $(g_1f_1 + g_2f_2)(x^{2n-1}) = x^{2n-1}$.Thus $1 = g_1f_1 + g_2f_2 \in Rf_1 + Rf_2$, then for any $r \in R$, $r = (rg_1)f_1 + (rg_2)f_2$, this means $\{f_1, f_2\}$ is a basis of ${}_RR$. Therefore ${}_RR \simeq_R R^2$ $(r \mapsto (rg_1, rg_2))$, hence $R \simeq R^2 \simeq R \oplus R^2 \simeq ... \simeq R \oplus R^{n-2} \simeq R^2 \oplus R^{n-2} = R^n$.

8. Since I is an ideal of R. IM is a submodule of M. Define $R/I \times M/IM \to M/IM$, $(a+I, m+IM) \mapsto am+IM$. If $(a_1+I, m_1+IM) = (a_2+I, m_2+IM)$, then $a_1 - a_2 \in I, m_1 - m_2 \in IM, a_1m_1 - a_2m_2 = (a_1 - a_2)m_1 + a_2(m_1 - m_2) \in IM$, therefore $a_1m_1 + IM = a_2m_2 + IM$. It is easy to check that M/IM is a module with the above action. For any $m + IM \in M/IM$, we have $m + IM = \sum r_ix_i + IM = \sum(r_i+I)(x_i+IM)$. If $\sum(r_i+I)(x_i+IM) = \sum r_ix_i + IM = 0+IM$, then $\sum r_ix_i \in IM$, therefore $\sum r_ix_i = \sum a_jx_j$ where $a_j \in I$. As $\{x_i\}$ is a basis, we have

$r_i = a_i$ for each $i \in I$. This means $r_i + I = 0$, hence $\{x_i + IM | x_i \in B\}$ is a basis of M/IM over R/I.

9. Since $\varphi : R \to S$ is epimorphic, $S \cong R/Ker\varphi$. If $R^m \cong R^n$ as left R-modules, $e_i = (0, ..., 0, 1, 0, ..., 0)$, $i = 1, ..., m$ is a basis of R^m, $\varepsilon_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, $i = 1, \cdots, n$ is a basis of R^n. Then $\{e_i + (Ker\varphi)R^m | 1 \leq i \leq m\}$ is a basis of $R^m/(Ker\varphi R^m)$ over S and $\{\varepsilon_i + (Ker\varphi)R^n | 1 \leq i \leq n\}$ is a basis of $R^n/(Ker\varphi R^n)$ over S. If $\psi : R^m \to R^n$ is an isomorphism of left modules, then $\psi(Ker\varphi R^m) = Ker\varphi \cdot \psi(R^m) = Ker\varphi R^n$. Thus ψ induces a $R/Ker\varphi$-module isomorphism $\overline{\psi} : (R/Ker\varphi)^m \cong R^m/(Ker\varphi R^m) \to R^n/(Ker\varphi R^n) \cong (R/Ker\varphi)^n$, $a + Ker\varphi R^m \mapsto \psi(a) + Ker\varphi R^n$, therefore $m = n$ by the assumption on S.

10. Suppose R is a commutative ring with identity, then R has a maximal ideal I and R/I is a field. Since R/I has invariant dimension property, so is R by Exercise 2.2.9.

11. If ${}_RM \cong_R N$, since ${}_RM \cong R^{|X|}$ and ${}_RN \cong R^{|Y|}$, $R^{|X|} \cong R^{|Y|}$. R has invariant dimension property, so $|X| = |Y|$.

12. Suppose $\{x_i | i \in I\}$ is a basis of S over R and $\{y_j | j \in J\}$ is a basis of T over S. Let $B = \{x_i y_j | i \in I, j \in J\}$, then any $a \in T$, $a = \sum s_j y_j$, $s_j = \sum r_{ij} x_i$, therefore $a = \sum rij x_i y_j$. Suppose $a = \sum rij x_i y_j = 0$, then $\sum r_{ij} x_i = 0$ for each $j \in J$, therefore $r_{ij} = 0$. Thus B is a basis of T over R. Hence $dim_R T = (dim_R S)(dim_S T)$.

13. $(\Rightarrow)$:Suppose $M = \mathbb{Z}/(p^n)$ is indecomposable. If p is

not a prime, then $p = ab$ where a and b are proper divisor of p and $(a, b) = 1$, $\mathbb{Z}/(p^n) \cong \mathbb{Z}/(a^n) \oplus \mathbb{Z}/(b^n)$, $(\varphi : x + (p^n) \mapsto (x + (a^n), x + (b^n)))$. If $x_1 + (p^n) = x_2 + (p^n)$, then $p^n | (x_1 - x_2)$, therefore $a^n | (x_1 - x_2)$ and $b^n | (x_1 - x_2)$, then $(x_1 + (a^n), x_1 + (b^n)) = (x_2 + (a^n), x_2 + (b^n))$, hence φ is well-defined. If $\varphi(x + (p^n)) = 0$, then $a^n | x$ and $b^n | x$, therefore $p^n = a^n b^n | x$, hence φ is injective. For any $(u + (a^n), v + (b^n)) \in \mathbb{Z}/(a^n) \oplus \mathbb{Z}/(b^n)$, there are $\alpha, \beta \in \mathbb{Z}$ such that $\alpha a^n + \beta b^n = 1$, then $\alpha a^n = 1 - \beta b^n$, therefore $\varphi(\alpha a^n) = (\alpha a^n + (a^n), 1 - \beta b^n + (b^n)) = (0, 1 + (b^n))$ and $\varphi(\beta b^n) = (1 + (a^n), 0)$, thus $\varphi(v\alpha a^n + u\beta b^n) = (u + (a^n), v + (b^n))$. Hence φ is surjective. Since $\mathbb{Z} \cong (a^n, b^n)$ and $(a^n) \cap (b^n) = (p^n)$, $\mathbb{Z}/(a^n) \cong (a^n, b^n)/(a^n) \cong (b^n)/(p^n)$. Therefore $\mathbb{Z}/(p^n) \cong (a^n)/(p^n) \oplus (b^n)/(p^n)$, hence p is a prime.

$(\Leftarrow)$:If p is a prime, any submodule of $\mathbb{Z}/(p^n)$ is $(a)/(p^n)$ where $a = a_1 p^s, 0 \leq s \leq n, p \nmid a_1$. If $(a)/(p^n) \neq 0$, then $a = a_1 p^s, 0 \leq s \leq n-1$, in addition, $(b)/(p^n) \neq 0$, then $b = b_1 p^t, 0 \leq t \leq n-1$. If $a = a_1, p \nmid a_1$, then $(p^n, a_1) = 1, vp^n + ua_1 = 1$, therefore $1 + (p^n) = vp^n + ua_1 + (p^n) = u(a_1 + (p^n)) \in (a)/(p^n)$, similarly, $p^s = ua_1 p^s + vp^{s+n}, p^s + (p^n) = u(a + (p^n))$, then $(a)/(p^n) = (p^s)/(p^n)$. If $\mathbb{Z}/(p^n) = (a)/(p^n) \oplus (b)/(p^n)$ and $(a)/(p^n), (b)/(p^n)$ both are proper submodule, then $a = a_1 p^s, 0 \leq s \leq n-1$, $b = b_1 p^t, 0 \leq t \leq n-1$, suppose $s \leq t$, then $(a)/(p^n) \cap (b)/(p^n) \supseteq (p^s)/(p^n) \neq 0$. Hence M is indecomposable.

14. If $R = I_1 \oplus \cdots \bigoplus I_n$ where I_i are indecomposable.

Then there are $e_i \in I_i$ such that $1 = e_1 + \cdots + e_n$. Thus $e_1 = e_1e_1 + \cdots + e_1e_n = e_1e_1 + \cdots + e_ne_1$, then $e_2e_1 + \cdots + e_ne_1 \in I_1$, while $e_2e_1 + \cdots + e_ne_1 \in I_2 + \cdots + I_n$, therefore $e_2e_1 + \cdots + e_ne_1 \in I_1 \cap I_2 + \cdots + I_n = \{0\}$, hence $e_1 = e_1^2$ and $e_1e_i = 0, i = 2, \cdots, n$. Similarly, $e_i = e_i^2, i = 1, \cdots, n$ and $e_ie_j = 0$ for $i \neq j$. Since $Re_i \subseteq I_i$ and $1 \in Re_1 + \cdots + Re_n$, $R = Re_1 + \cdots + Re_n \subseteq I_1 + \cdots + I_n = R$, thus $I_i = Re_i$. If $e_i = e_{i1} + e_{i2}$, $e_{i1}^2 = e_{i1}$, $e_{i2}^2 = e_{i2}$ and $ei1e_{i2} = e_{i2}e_{i1} = 0$, then $I_i = R(e_{i1} + e_{i2}) \subset Re_{i1} + Re_{i2}$, while $e_{i1}(e_{i1} + e_{i2}) = e_{i1} = e_{i1}e_i \in Re_i$, thus $Re_{i1} + Re_{i2} \subset Re_i = I_i$. For any $x = re_{i1} = se_{i2} \in Re_{i1} \cap Re_{i2}$, then $re_{i1} = re_{i1}^2 = se_{i2}e_{i1} = 0$, therefore $I_i = Re_{i1} \oplus Re_{i2}$. While I_i is indecomposable, $e_{i1} = 0$ or $e_{i2} = 0$, i.e. e_i is primitive. Contrary, if $1 = e_1 + \cdots + e_n$ is a sum of pairwise orthogonal primitive idempotent elements, then $R = R1 \subset Re_1 \cdots + Re_n \subset R$, thus $R = Re_1 + \cdots + Re_n$. If $re_1 \in Re_1 \cap Re_2 + \cdots + Re_n$, then $re_1 = r_2e_2 + \cdots + r_ne_n$, thus $re_1 = (r_2e_2 + \cdots + r_ne_n)e_1 = 0$, therefore $Re_1 \cap Re_2 + \cdots + Re_n = \{0\}$. Similarly, $R = Re_1 \oplus \cdots \oplus Re_n$. If $Re_1 = I \oplus J$, then $e_1 = e_{11} + e_{12}$ where $e_{11} \in I$ and $e_{12} \in J$, thus $e_{11} = r_{11}e_1$, $e_{12} = r_{12}e_1$ and $e_{11} = e_{11}e_{11} + e_{11}e_{12}$, while $e_{11}e_{12} \in I \cap J$, then $e_{11} = e_{11}^2$, similarly, $e_{12} = e_{12}^2$, $e_{12}e_{11} = 0$. While e_1 is primitive, then $e_{11} = 0$ or $e_{12} = 0$. If $e_{11} = 0$, then $e_1 = e_{12} \in J$, thus $Re_1 = Re_{12} = J$, therefore $I = 0$, similarly, if $e_{12} = 0$, then $Re_1 = I$. Hence Re_1 is indecomposable. Similarly, Re_i is indecomposable.

15. If $1_M = e + f$ for $1_M \in End_R(M)$ where $e^2 = e, f^2 =$

f and $ef = fe = 0$, then $M = (e+f)M \subset eM + fM \subset M$, thus $M = eM + fM$. If $em \in eM \cap fM$, then $em = fm'$, thus $em = e(em) = e(fm') = 0$, while eM, fM are image of e, f respectively, therefore eM, fM are submodules of M. Hence $M = eM \oplus fM$. Since M is indecomposable, $eM = 0$ or $fM = 0$. If $eM = 0$, then $M = fM$. Thus, for any $m \in M$, $m = fm'$, then $fm = f(fm') = fm'$, therefore $f = id_M$, then $e = 0$. Similarly, If $fM = 0$, then $f = 0$ and $e = id_M$. This means id_M is a primitive idempotent.

16. Let $\Omega = \{P \leq M | P \cap N = 0\}$. $\Omega \neq \emptyset$ for $0 \in \Omega$. Order Ω by $P_1 \leq P_2$ if and only if $P_1 \subset P_2$, suppose $P_1 \leq \cdots \leq P_n \leq \cdots$ is a chain of elements in Ω. Then $P = \bigcup\limits_i P_i \leq M$. If $x \in N \cap (\bigcup\limits_i P_i)$, then there is n such that $x \in N \cap P_n$, thus $x = 0$ for $P_n \in \Omega$, therefore $P \in \Omega$ and $P \geq P_i$. By Zorn's Lemma, there is a maximal element $P \in \Omega$, then $P \oplus N$ is a submodule of M. For any submodule $L \leq M$, if $(P+N) \cap L = 0$, then $N+P+L = N \oplus P \oplus L$, thus $N \cap (P+L) = 0$. Since $P \cap L = 0$, $P+L = P$, then $L = 0$. Hence $N + P \cap \neq 0$ for any nonzero submodule L of M, i.e. $N + P$ is an essential submodule of M.

17. Suppose ${}_{\mathbb{Z}}\mathbb{Q} \subset {}_{\mathbb{Z}}M$ and $\mathbb{Q}$ is essential in M. For any $a \in M$, ${}_{\mathbb{Z}}\mathbb{Q} \cap \mathbb{Z}a \neq 0$, then there is $n \in \mathbb{Z}$ such that $na \in \mathbb{Q}$, thus $na = na'$ where $a' \in \mathbb{Q}$, then $n(a - a') = 0$. Let $M_1 = \{x \in M | nx = 0, n \in \mathbb{Z}, n \neq 0\} \cup \{0\}$, then $M_1 \leq M$, thus $a - a' \in M_1$, therefore $M = \mathbb{Q} + M_1$. If $a \in \mathbb{Q} \cap M_1$, then $na = 0$ for $n \neq 0$, thus $a = 0$. Hence $M = \mathbb{Q} \oplus M_1$, while $M \neq \mathbb{Q}$,

$\mathbb{Q} \cap M_1 = 0$, this means $\mathbb{Q}$ is not essential.

18. Let $I_i = M_n e_{ii}$ where e_{ii} is matrix whose i_{th} row j_{th} column is 1 and else is 0. Then $I_i \leq M_n(D)$, $M_n(D) = I_1 + \cdots + I_n$. For any $0 \neq N \leq I_i$, there is $0 \neq A = \begin{pmatrix} 0 & a_{1i} & 0 \\ \vdots & \vdots & \vdots \\ 0 & a_{ni} & 0 \end{pmatrix} \in N$, assume that $a_{ji} \neq 0$, then $e_{ij}A = a_{ji}e_{ii} \in N$, thus $a_{ji}^{-1}Ea_{ji}e_{ii} = e_{ii} \in N$, therefore $M_n(D)e_{ii} \subset N$, hence $N = I_i$, i.e. I_i is simple. Hence $M_n(D)$ is semisimple.

19. (1) If $R = s_1 \oplus \cdots \oplus s_n$ is a direct sum of simple module, then for any $N \leq R$, according to Lemma 2.2.1, $R = N \oplus s_{i_1} \oplus \cdots \oplus s_{i_m}$ where $\{i_1, \cdots, i_m\} \subset \{1, \cdots, n\}$. Thus $N \cap (s_{i_1} + \cdots + s_{i_m}) = \{0\}$, then N is not an essential submodule when $N \neq 0$ and $N \neq R$. Conversely, R has identity. Let $\Omega = \{I \lhd_l R | 1 \notin I\}$. According to Zorn's Lemma, Ω has a maximal element M. Since M is not essential, there exist $0 \neq I_1 \leq {}_R R$ such that $I_1 \cap M = \{0\}$, while $M + I_1 \supsetneqq M$, then $1 \in M + I_1$, thus $R = M + I_1$, then $1 = e_1 + f$ where $e_1 \in I_1, f \in M$ are orthogonal idempotent. Since M is maximal, $R/M \cong I_1$ is simple, then $I_1 = Re_1$ and $M = Rf$. Consider $\Omega_2 = \{L \leq Rf | f \notin L\}$, similarly, there is maximal submodule $M' \in \Omega_2$ and $I_2 \leq M$ such that $Rf = I_2 \oplus M'$, moreover, $I_2 = Re_2$ and $M' = Rf'$. Let $\Lambda = \{L \leq {}_R R | L is semisimple\}$, there exists maximal element $L \in \Lambda$ and $L' \leq R$, then $R = L \oplus L'$. If $L' \neq 0$, the above proves that there

is simple submodule $S \subset L'$, then $L \oplus S \in \Lambda$. It is contradiction. Hence $L' = 0$, i.e. $R = L$ is semisimple.

(2) If R is semisimple, $I \lhd R$, according to Lemma 2.2.1, ${}_RR = I \oplus J$ where $J = T_1 \oplus \cdots \oplus T_n$ is semisimple left R-module. For any $a \in J$ and any $r = b + c \in R$ where $b \in I, c \in J$, then $ar = ab + ac$, since $ab \in I \cap J = \{0\}$, $ar = ac$, then $J \lhd R$. $R/I \cong J = I_1 \oplus \cdots \oplus T_n$ where T_i are simple R-module, while $J \lhd R$, $IT_i \subset IJ \subset I \cap J = \{0\}$. Thus T_i is R/I module. It is obvious that I_i is R/I simple module. Therefore R/I is semisimple. Conversely, take $I = 0$, then $R \cong R/0$ is semisimple.

20. If R is semisimple, then for any nonzero left ideal I, ${}_RR = I \oplus J = T_1 \oplus \cdots \oplus T_n$ where $J = T_{i_1} \oplus \cdots \oplus T_{i_m}$, thus $I \cong T_{j_1} \oplus \cdots \oplus T_{j_{n-m}}$ is semisimple. Conversely, take $I = R$, then R is semisimple.

21. Let $L = \{\{a_n\}_{n \in \mathbb{N}} | a_n = a_{n+1} = a_{n+2} = \cdots, n \ggg 0\}$, then L is a submodule of $R^{\mathbb{N}}$ and $e_i, w \in L$. Thus $< X \cup \{w\} > \subset L$. For any $\{a_n\}_{n \in \mathbb{N}} \in L$, then there is N_0 such that $a_{N_0} = a_{N_0+1} = \cdots$, therefore $\{a_n\}_{n \in \mathbb{N}} = (a_0 - a_{N_0})e_0 + \cdots + (a_{N_0-1} - a_{N_0})e_{N_0-1} + a_{N_0}w \in < X \cup \{w\} >$, hence $L = < X \cup \{w\} >$.