Topology - Homework 12

Question 1.

Choose arbitrary $x_0 \in X$.

If $f_\alpha(x_0) = x_0$, this case just satisfied the conclusion.

If $f_\alpha(x_0) \neq x_0$, let $x_1 = f_\alpha(x_0)$ and $x_n = f_\alpha(x_{n-1})$ for all $n > 1$.

Then we have $d(x_{n+1}, x_n) \leq \alpha^n d(x_1, x_0)$.

For arbitrary $\epsilon > 0$, choose $N \in \mathbb{N}$ s.t.

$$\alpha^N < \frac{\epsilon(1-\alpha)}{d(x_1, x_0)}$$

Since $\alpha \in (0,1)$, we know this can be done.

Then for arbitrary $m > n > N$, we have

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n) \\
&\leq \alpha^{m-1} d(x_1, x_0) + \alpha^{m-2} d(x_1, x_0) + \cdots + \alpha^n d(x_1, x_0) \\
&= d(x_1, x_0)\alpha^n \sum_{i=0}^{m-n-1} \alpha^i \\
&< d(x_1, x_0)\alpha^n \sum_{i=0}^{\infty} \alpha^i \\
&= d(x_1, x_0)\alpha^n \frac{1}{1-\alpha} \\
&< \epsilon \frac{1-\alpha}{d(x_1, x_0)} \cdot \frac{d(x_1, x_0)}{1-\alpha} \\
&= \epsilon
\end{aligned}$$

This tells us that $\{x_n\}$ is a Cauchy sequence. And since X is complete, there must be some $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

For every $n \in \mathbb{N}$, there is $0 \leq d(x_{n+1}, f_\alpha(x)) = d(f_\alpha(x_n), f_\alpha(x)) \leq \alpha d(x_n, x)$.

Since $\lim_{n\to\infty} \alpha d(x_n, x) = 0$, we know that $\lim_{n\to\infty} d(x_{n+1}, f_\alpha(x)) = 0$

That is, $d(x, f_\alpha(x)) = 0$ and $x = f_\alpha(x)$.

If there is another $y \in X$ satisfying $y = f_\alpha(y)$, then there would exist

$$0 \leq d(x, y) = d(f_\alpha(x), f_\alpha(y)) \leq \alpha d(x, y)$$

Since $\alpha \in (0,1)$, this is impossible.

Hence there is exactly one point $x \in X$ such that $f_\alpha(x) = x$.

Question 2.

Consider arbitrary Cauchy sequence $\{x_n\}$ in X.

For each x_n, choose $a_n \in A$ s.t. $d(a_n, x_n) < \frac{1}{n}$ and obtain a sequence $\{a_n\}$ in A.

We can do this since A is dense in X and x_n is a limit point of A.

$$d(a_n, a_m) \leq d(a_n, x_n) + d(x_n, x_m) + d(x_m, a_m)$$

Since x_n is Cauchy, for arbitrary $\epsilon > 0$, there is some $N_0 \in \mathbb{N}$, for all $m, n > N_0$, $d(x_m, x_n) < \frac{\epsilon}{3}$.

Choose $N_1 > \max\{N_0, \frac{3}{\epsilon}\}$, and then for arbitrary $m, n > N_1$, there is

$$d(a_n, a_m) \leq d(a_n, x_n) + d(x_n, x_m) + d(x_m, a_m) < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

Then we know that $\{a_n\}$ is a Cauchy sequence in A and $\{a_n\}$ converges to some x in X.

That is, for arbitrary $\epsilon > 0$, there exists some $N_2 \in \mathbb{N}$, for every $n > N_2$, $d(a_n, x) < \frac{\epsilon}{2}$.

Choose $N_3 > \max\{N_2, \frac{2}{\epsilon}\}$, then for all $n > N_3$, there is

$$d(x_n, x) \leq d(x_n, a_n) + d(a_n, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This means that $\{x_n\}$ also converges to x.

Then we know that (X, d) is complete.

Question 3.

Consider $d_2(x, y) = |x - y|$.

Then $X = (0, 1)$ is incomplete with respect to d_2 because Cauchy sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}_+}$ is not convergent.

Consider $d_1(x, y) = |\tan \pi(x - \frac{1}{2}) - \tan \pi(y - \frac{1}{2})|$.

Choose arbitrary Cauchy sequence $\{x_n\}$ in $X = (0, 1)$.

For each $n \in \mathbb{N}_+$, let $y_n = \tan \pi(x - \frac{1}{2})$ and we have $d_1(x_m, x_n) = d_2(y_m, y_n)$.

The function $f(x) = \tan \pi(x - \frac{1}{2}) : (0, 1) \to \mathbb{R}$ is bijective.

Since $\mathbb{R}$ is complete with respect to d_2 we know that $\{y_n\}$ converges to some $y \in \mathbb{R}$.

Then we know that $\{x_n\}$ converges to some $x = f^{-1}(y) \in X$ with respect to d_1.

That is, $X = (0, 1)$ is complete with respect to d_1.

Obviously, the topologies on X induced from d_1 and d_2 are both open intervals in $(0, 1)$.

Then we find what we desire.

Question 4.

Consider arbitrary $\epsilon > 0$.

For every i, let $g_i = f_i - f$ and $E_i = \{x \in X : g_i(x) < \epsilon\}$.

Since $(0, \epsilon)$ is open, and f and f_i are both continuous, we know that each E_i is open.

$\{g_i\}$ is monotonically decreasing, hence we have $E_i \subset E_{i+1}$.

$\{f_i\}$ is pointwise converges to f, hence we have $\bigcup_i E_i$ been a open cover of X.

X is compact, hence there must be some $N \in \mathbb{N}$ such $E_N = X$.

Then for all $n > N$ and all $x \in X$, there is

$$0 < g_n(x) = f_n(x) - f(x) < \epsilon$$

That is, for all $n > N$, there is $\rho(f_n, f) = \sup_{x \in X} |f_n(x) - f(x)| < \epsilon$.

Therefore, $\{f_i\}$ converges to f with respect to ρ.

Question 5.

This question is the “Question 3.” in the midterm exam.

Let A, B be two disjoint compact subsets in a Hausdorff topological space $(X, \mathcal{T})$. Show that there exist disjoint open sets U and V containing A and B respectively.

My new attempt:

For each point $a \in A$ and each point $b \in B$ there are disjoint open sets $U_{a,b}$ containing a and $V_{a,b}$ containing b.

Fix $b \in B$, and then the collection $\{U_{a,b} | a \in A\}$ convers A.

By compactness there are points $a_1, \cdots, a_n \in A$ such that $A \subset U_b = \bigcup_{i=1}^n U_{a_i,b}$.

Define $V_b = \bigcap_{i=1}^n V_{a_i,b}$, then V_b is an open neighborhood of b that is disjoint with the open neighborhood of U_b of A. That is, a point and a compact set can be separated by disjoints open sets.

Now B is covered by $\{V_b | b \in B\}$, and again by compactness, finitely many of these open sets suffice to cover B.

Then there are points $b_1, \cdots, b_n \in B$ such that $B \subset V = \bigcup_{i=1}^n V_b$ and the intersection $U = \bigcap_{i=1}^n U_{b_i}$ is a open set containing A.

And then U and V are what we want.