TOPOLOGY - HOMEWORK 11

Due: 2020 June 4th, 10:00AM

Question 1. Show that the metric space $(\mathbf{R}^\omega, d^\omega)$ as defined in HW5 Question 3 is complete.

Question 2. Let (X, d) be a metric space.

(i) Assume there exists $\epsilon > 0$ such that for every $x \in X$ the closure of $B_d(x, \epsilon)$ in X is compact. Show that X is complete.

(ii) Give an example of an incomplete metric space (X, d) satisfying the following property: for every $x \in X$, there exists $\epsilon_x > 0$ such that the closure of $B_d(x, \epsilon_x)$ in X is compact.

Question 3. Fix a metric space (X, d), and define the diameter of every subset $A \subset X$ to be $d(A) := \sup\{d(x, y) : x, y \in A\}$. Show that the following properties are equivalent:

(i) (X, d) is complete,

(ii) for every nested sequence $C_1 \supset C_2 \supset \ldots$ of nonempty closed subsets of X with $\lim_{n\to\infty} d(C_n) = 0$, one has $\bigcap_n C_n \neq \emptyset$.

Question 4. Consider the metric space (H, d) where

$$H = \left\{ (x_i)_{i\in\mathbf{N}} : x_i \in \mathbf{R} \text{ and } \sum_{i\in\mathbf{N}} x_i^2 < \infty \right\}$$

and $d((x_i)_i, (y_i)_i) = \left(\sum_i (x_i - y_i)^2 \right)^{1/2}$. Show that (H, d) is complete.

Question 5. Fix a prime number p. A positive rational number r can be written uniquely in a reduced form $r = p^e b/c$ with $e \in \mathbf{Z}$ and $p, b, c \in \mathbf{N}$ being pairwise co-prime. Define a function

$$|\cdot|_p : \mathbf{Q} \longrightarrow \mathbf{R}$$

by

$$|r|_p = \begin{cases} 0 & \text{if } r = 0, \\ p^{-e} & \text{if } r = p^e b/c \in \mathbf{Q}_{>0} \text{ is in reduced form,} \\ |-r|_p & \text{if } r \in \mathbf{Q}_{<0}. \end{cases}$$

Define $d(-,-) : \mathbf{Q} \times \mathbf{Q} \to \mathbf{R}$ by setting $d(r_1, r_2) = |r_1 - r_2|_p$. Show that $(\mathbf{Q}, d)$ is an incomplete metric space. Can you identify more concretely the completion $(\mathbf{Q}^*, d^*)$ of $(\mathbf{Q}, d)$? (Hint: HW7 Question 5.)