2 Modules

2.5 Tensor product and weak dimension

1. (1) Since $0 \to I \to R \to R/I \to 0$ is exact,

$$I \otimes R/J \xrightarrow{f} R/J \to R/I \otimes R/J \to 0$$

   is exact, where $Im(f) = \{\sum_i ar + J | a \in I, r \in R\} = (I+J)/J$, then $R/I \otimes R/J \cong R/J/(I + J/J) \cong R/(I+J)$.

   (2) $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}/(m\mathbb{Z} + n\mathbb{Z}) = \mathbb{Z}/d\mathbb{Z}$ where $d = (m, n)$.

2. Suppose M is a F-vector space generated by $\{v_i | i \in I\}$. Define $\varphi : F \times V \to M$, $\varphi(a, v) = av$, then φ is a K-bilinear mapping. Let $\lambda : F \times V \to F \otimes V$, $\lambda(a, v) = a \otimes v$, then there is a unique F-vector space map Φ satisfied $\Phi \circ \lambda = \varphi$. Since ${}_F F_K$ is a bimodule, $F \otimes V$ is F vector space, for any $a, b_i \in F$ and any $v_i \in V$, $\Phi(a \sum b_i \otimes v_i) = \Phi(\sum ab_i \otimes v_i) = \sum \Phi(ab_i \otimes v_i) = \sum \varphi(ab_i, v_i) = \sum ab_i v_i = a \sum b_i v_i = a \sum \Phi(b_i \otimes v_i) = a\Phi(\sum b_i \otimes v_i)$, thus Φ is a F-linear mapping. Since $\{\Phi(1 \otimes v_i) = v_i | i \in I\}$ is a basis of M over K, $\{1 \otimes v_i | i \in I\}$ is linearly independent over K. It is obvious that $Span(\{1 \otimes v_i | i \in I\}) = F \otimes V$. Hence $\{1 \otimes v_i | i \in I\}$ is a basis of $F \otimes V$ as a vector space over F.

3. It is easy to testify that

$$\iota : M_1 \times \cdots \times M_n \to M_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} M_n,$$

   $\iota(m_1, \cdots, m_n) = m_1 \otimes \cdots \otimes m_n$, is a n-multiplicity middle linear mapping for $m_1 \otimes \cdots \otimes m_{i-1} \otimes (\lambda x_i + \mu y) \otimes m_{i+1} \otimes \cdots \otimes m_n = (\overline{m_1 \otimes \cdots \otimes m_{i-1} \otimes (\lambda x_i + \mu y)}) \otimes (m_{i+1} \otimes \cdots \otimes m_n)$. Let $L(M_1, M_2, \cdots, M_n; G)$ is a set of all middle linear mapping from $M_1 \times \cdots \times M_n$ to abelian group G which constructs an abelian group.

$$\varphi_1 : L(M_1, M_2, \cdots, M_n; G) \cong Hom_{R_{n-1}}(M_n, L(M_1, M_2, \cdots, M_{n-1}; G))$$

   where $L(M_1, M_2, \cdots, M_{n-1}; G)$ is a set of all middle linear mapping whose left R_{n-1}-module is defined as

$$(rf)(m_1, \cdots, m_{n-1}) := f(m_1, \cdots, m_{n-2}, m_{n-1}r).$$

   $(\varphi_1(f)(m_n))(m_1, \cdots, m_{n-1}) := f(m_1, \cdots, m_{n-1}, m_n)$. Then to proof the conclusion of this exercise is equivalent to proof

$$\iota^* : Hom(M_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} M_n; G) \to L(M_1, M_2, \cdots, M_n; G)$$

is isomorphic. By introduction on n, when $n=2$, the conclusion is true for Thm 2.5.1. Define

$\varphi_2 : Hom(M_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} M_n; G) \to Hom_{R_{n-1}}(M_n, Hom_{\mathbb{Z}}(M_1, \cdots, M_{n-1}; G))$,

$\varphi_2(f)(m_n)(m_1 \otimes \cdots \otimes m_{n-1}) = f(m_1 \otimes \cdots \otimes m_n)$.

$\varphi_3 : Hom_{R_{n-1}}(M_n, Hom_{\mathbb{Z}}(M_1, \cdots, M_{n-1}; G)) \to Hom_{R_{n-1}}(M_n, L(M_1, \cdots, M_{n-1}; G))$,

$\varphi_3(f)(m_n)(m_1, \cdots, m_{n-1}) = f(m_n)(m_1 \otimes \cdots \otimes m_{n-1}) = (\iota^*)^{-1}(f(m_n))(m_1 \otimes \cdots \otimes m_{n-1})$ is an isomorphism by introduction. Since φ_1 is isomorphic, φ_1^{-1} is isomorphic. Then $\varphi_2^{-1} \circ \varphi_3^{-1} \circ \varphi_1 : L(M_1, M_2, \cdots, M_n; G) \to Hom(M_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} M_n; G)$, $(\varphi_2^{-1} \circ \varphi_3^{-1} \circ \varphi_1)(f)(a_1 \otimes \cdots \otimes a_n) = [(\varphi_3^{-1} \circ \varphi_1)(f)(a_n)](a_1 \otimes \cdots \otimes a_{n-1}) = \varphi_1(f)(a_n)](a_1, \cdots, a_{n-1}) = f(a_1, \cdots, a_n)$ for any $f \in L(M_1, M_2, \cdots, M_n; G)$ and any $a_1 \otimes \cdots \otimes a_n \in M_1 \underset{R_1}{\otimes} \cdots \underset{R_{n-1}}{\otimes} M_n$, is isomorphic. Thus $(\varphi_2^{-1} \circ \varphi_3^{-1} \circ \varphi_1)(f) \circ \iota = f$ and $(\varphi_2^{-1} \circ \varphi_3^{-1} \circ \varphi_1)(f)$ is unique.

4. Suppose that e_{ij} be matrix identity of $M_s(K)$ and ε_{ij} be matrix identity of $M_t(K)$, then $\{e_{ij} \otimes \varepsilon_{pq} | 1 \le i,j \le s, 1 \le p,q \le t\}$ is a basis of $M_s(K) \underset{K}{\otimes} M_t(K)$. Define $\varphi : M_s(K) \underset{K}{\otimes} M_t(K) \to M_{st}(K)$, $\varphi(e_{ij} \otimes \varepsilon_{pq}) = g_{i+(p-1)s, j+(q-1)s}$ where g_{ij} is matrix identity of $M_{st(K)}$. Thus

$$\begin{aligned}(e_{ij} \otimes \varepsilon_{pq})(e_{i'j'} \otimes \varepsilon_{p'q'}) &= e_{ij}e_{i'j'} \otimes \varepsilon_{pq}\varepsilon_{p'q'} \\ &= \delta_{ji'}e_{ij} \otimes \delta_{qp'}\varepsilon_{pq} \mapsto \delta_{ji'}\delta_{qp'}g_{i+(p-1)s, j'+(q'-1)s}\end{aligned}$$

and $g_{i+(p-1)s, j+(q-1)s} g_{i'+(p'-1)s, j'+(q'-1)s} = \delta_{j+(q-1)s, i'+(p'-1)s} g_{i+(p-1)s, j'+(q'-1)s}$. When $j = i', p = q'$ $\delta_{j+(q-1)s, i'+(p'-1)s} = 1$, inverse, if $j + (q-1)s = i' + (p'-1)s$, then $i' - j = (q - p')s$, thus $j = i', p = q'$ for $1 \le i,j \le s$. Hence φ is isomorphic.

5.

$$\begin{array}{ccccccccc} & & I \otimes L & \xrightarrow{f_1} & I \otimes M & \xrightarrow{f_2} & I \otimes N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3} & & \\ 0 & \longrightarrow & IL & \xrightarrow{g_1} & IM & \xrightarrow{g_2} & IN & \longrightarrow & 0 \end{array}$$

Since L and N are flat, then φ_1 and φ_3 is injective. According to Snack Lemma, $Ker\varphi_1 \xrightarrow{f_1} Ker\varphi_2 \xrightarrow{f_2} Ker\varphi_3$, i.e. φ_2 is injective. According to Proposition 2.5.5, M is flat.

6. $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_r}(\mathbb{C})$, then $T \cong \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_{n_i} \end{pmatrix} \middle| a_i \in \mathbb{C} \right\}$, thus the multiple of T in $\mathbb{C}[G]$ is $n_i = dim_{\mathbb{C}}T$.

7. For any finitely generated left ideal, consider

$$\begin{array}{ccccccccc} & & K\otimes I & \xrightarrow{f} & M\otimes I & \xrightarrow{g} & M/K\otimes I & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle\varphi_3} & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \\ 0 & \longrightarrow & KI & \xrightarrow{f_1} & MI & \xrightarrow{g_1} & (M/K)I(=MI/K) & \longrightarrow & 0 \end{array}$$

When $Kerg_1 = MI \cap K = KI = Imf_1$, for any $a \in Kerg_2$, there is $b \in M{\otimes}I$ such that $g(b) = a$, then $\varphi_2 g(b) = \varphi_2(a) = g_1\varphi_1(b) = 0$, thus $\varphi_1(b) \in Kerg_1 = Imf_1$, therefore there is $c \in KI$ such that $f_1(c) = \varphi_1(b)$ and there is $d \in K{\otimes}I$ such that $\varphi_3(d) = c$, hence $f_1\varphi_3(d) = f_1(c) = \varphi_1(b) = \varphi_1 f(d)$. Since φ_1 is isomorphic, $b = f(d)$, then $a = g(b) = gf(d) = 0$, thus M/K is flat. Contrary, if φ_2 is isomorphic, then φ_1 and φ_2 is isomorphic, while $\varphi_2 g = g_1\varphi_1$, thus $Kerg_1 \cong Kerg = Imf$. For any $x \in KI$, there is $y \in K \otimes I$ such that $\varphi_3(y) = x$, then $f_1\varphi_3(y) = f_1(x) = \varphi_1 f(y)$, thus $f_1(x) \in Kerg_1$, therefore $Imf_1 \subseteq Kerg_1 = MI \cap K$. If $a \in Kerg_1$, there is $b \in M \otimes I$ such that $\varphi_1(b) = a$. Since $g_1\varphi_1(b) = g_1(a) = 0$, $\varphi_2 g(b) = 0$, then $g(b) = 0$ for φ_2 is injective, thus there is $c \in K \otimes I$ such that $b = f(c)$ and $\varphi_1 f(c) = f_1\varphi_3(c) = \varphi_1(b) = a \in Imf_1$. Hence $Kerg_1 = Imf_1$. Therefore $KI = MI \cap K$.

8. If M is injective and $M \leq N$, then $N = M \oplus L$. Thus for any finitely generated right ideal I, there is $IN = IM + IL$, then $IN \cap M = (IM + IL) \cap M = IM + (IL \cap M) = IM$ for $IL \cap M \subseteq L \cap M = 0$.

9. If ${}_RM$ is flat and $\sum\limits_{j=1}^{n} a_j v_j = 0$, let $I = \sum\limits_{j=1}^{n} a_j R$ and F is free which basis is $\{x_1, \cdots, x_n\}$. Consider the exact sequence $0 \longrightarrow K \xrightarrow{f} F \xrightarrow{g} I \longrightarrow 0$ where $g(\sum x_j r_j) = \sum a_j v_j$, $f(a = a)$. Consider the exact sequence

$$0 \longrightarrow K \underset{R}{\otimes} M \xrightarrow{f\otimes id_M} F \underset{R}{\otimes} M \xrightarrow{g\otimes id_M} I \underset{R}{\otimes} M \longrightarrow 0\,,$$

since $\sum\limits_{j=1}^{n} a_j v_j = 0$ and $I \otimes M \to IM$, $\sum r_j \otimes m_j \mapsto \sum r_j m_j$ is isomorphic, $k_i = \sum\limits_{j=1}^{n} x_j c_{ij}$. Since $\sum\limits_{j=1}^{n} a_j v_j = 0$, $\sum\limits_{j=1}^{n} a_j \otimes v_j = 0$ where $a_j 1 I$, then $(g \otimes id_M)(\sum\limits_{j=1}^{n} x_j \otimes v_j) = \sum\limits_{j=1}^{n} a_j \otimes v_j = 0$, thus $\sum\limits_{j=1}^{n} x_j \otimes v_j \in Ker(g \otimes id_M) = Im(f \otimes id_M)$, then there is $\sum\limits_{j=1}^{m} k_j \otimes m_j \in K \otimes M$ such that $\sum\limits_{j=1}^{n} x_j \otimes v_j = \sum\limits_{j=1}^{m} k_j \otimes m_j$. While $k_i \in K \subseteq F$, then $k_i = \sum\limits_{j=1}^{n} x_j c_{ij}$, thus $\sum\limits_{j=1}^{n} x_j \otimes v_j = \sum\limits_{j=1}^{n} x_j \otimes \sum\limits_{i=1}^{m} a_{ij} m_i$,while $x_1, \cdots, x_n$ is a basis, $v_j = \sum\limits_{i=1}^{m} c_{ij} m_i (j = 1, \cdots, n)$

and $\sum\limits_{j=1}^{n} a_j c_{ij} = \sum\limits_{j=1}^{n} f(x_j)c_{ij} = f(\sum x_j c_{ij} = f(k_i) = 0$. Contradiction, since $I \otimes M \to IM$, $\sum a_i \otimes m_i \mapsto \sum a_i m_i$, is injective for any finitely generated right ideal, $_RM$ is flat.

10. According to Exercise 2.3.13, $R^m \oplus K \cong R^n \oplus K'$, then $R^m \oplus K$ is finitely generated, thus $R^n \oplus K'$ is finitely generated. Let $\pi : R^n \oplus K' \to K'$ is a canonical projective map and $x_1, \cdots, x_p$ are generators of $R^n \oplus K'$, then $\pi(x_1), \cdots, \pi(x_n)$ are generators of K'.

11. Suppose that $_RM$ is a finitely presented flat module, then there are m, n and module homomorphism f_0, f_1 such that $_RR^m \xrightarrow{f_1} {_RR^n} \xrightarrow{f_0} {_RM} \longrightarrow 0$ is exact for any finitely generated right ideal I. Consider commute diagram
$$\begin{tikzcd} I \otimes R^m \arrow[r, "id_I \otimes f_1"] \arrow[d, "\varphi_1"'] & I \otimes R^n \arrow[r, "id_I \otimes f_0"] \arrow[d, "\varphi_2"'] & I \otimes M \arrow[r] \arrow[d, "\varphi_3"] & 0 \\ I^m \arrow[r, "g_1"] & I^n \arrow[r, "g_0"] & IM & \end{tikzcd}$$
, for any $a \in Ker\varphi_3$, there is $b \in I \otimes R^n$ such that $a = (id_I \otimes f_0)(b)$, then $g_0\varphi_2(b) = \varphi_3(a) = 0$, thus $\varphi_2(b) \in Kerg_0 = Img_1$, there is $c \in I^m$ such that $g_1(c) = \varphi_2(b)$, therefore there is $d \in I \otimes R^m$ such that $\varphi_1(d) = c$. Hence $\varphi_2(id_I \otimes f_1)(d) = g_1\varphi_1(d) = g_1(c) = \varphi_2(b)$, since φ_2 is isomorphic, $(id_I \otimes f_1)(d) = b$,then $a = (id_I \otimes f_0)(b) = (id_I \otimes f_0)(id_I \otimes f_1)(b) = 0$. Hence φ_3 is isomorphism, then $_RM$ is flat.

12. (1) $\Rightarrow$ (2): Since R_R is flat, R_R^I is flat by (1).
(2) $\Rightarrow$ (3):Suppose that $I \leq_R R$ and F is a free module with free basis $x_1, \cdots, x_n$ that maps onto $_RI$: $F \xrightarrow{f} I \longrightarrow 0$. For each $j = 1, \cdots, n$, let $a_j = f(x_j)$ and let $K = Kerf$, then $k = \sum\limits_{i=1}^{n} a_i x_i, a_i \in R$ for any $k \in Kerf$, thus $\varphi_i : K \to R$, $\varphi_i(\sum\limits_{i=1}^{n} a_i x_i) = a_i$ and $\varphi_i \in Hom_R(K, R), (i = 1, \cdots, n)$. Consider R^K, according to Exercise 1.6.8, there is an isomorphism Ψ from $\prod\limits_{k \in K} R$ to $\{f | f : K \to R\}$. Let $v_i = \Psi^{-1}(\varphi_i), (i = 1, \cdots, n)$, then $v_i \in R^K$ satisfies $k = \sum\limits_{i=1}^{n} a_i x_i = \sum\limits_{i=1}^{n} \varphi_i(k) x_i = \sum\limits_{i=1}^{n} \pi_k(v_i) x_i$ for any $k \in K$, thus $0 = f(k) = \sum\limits_{i=1}^{n} \pi_k(v_i) f(x_i) = \sum\limits_{i=1}^{n} \pi_k(v_i) a_i = \pi_k(\sum\limits_{i=1}^{n} v_i a_i)$ for any $k \in K$, therefore $\sum\limits_{i=1}^{n} v_i a_i = 0 \in R^K$. Since R^K is flat, there are $u_1, \cdots, u_m \in R^K$ and $c_{ij} \in R$ such that $\sum\limits_{j=1}^{n} c_{ij} a_j = 0$ and $v_j = \sum\limits_{i=1}^{m} u_i c_{ij}$. Let $k_i = \sum\limits_{j=1}^{n} c_{ij} x_j$, then $f(k_i) = \sum\limits_{j=1}^{n} a_j f(x_j) = \sum\limits_{j=1}^{n} c_{ij} a_j = 0$, thus $k_1, \cdots, k_n \in K$, while $k =$

$\sum_{j=1}^{n}\pi_k(v_j)x_j = \sum_{j=1}^{n}\pi_k(\sum_{i=1}^{m}u_ic_{ij})x_j = \sum_{i=1}^{m}\pi_k(u_i)\sum_{j=1}^{n}c_{ij}x_i$ for any $k \in K$. Therefore $K = Rk_1 + \cdots + Rk_n$ is finitely generated.

(3) $\Rightarrow$ (1):If $M_\alpha, \alpha \in A$ are flat right R-module, then there is B and surjective map $g_\alpha : R^{(B)} \to M_\alpha$, thus $0 \longrightarrow K_\alpha \longrightarrow R^{(B)} \longrightarrow M_\alpha \longrightarrow 0$, then $0 \longrightarrow \prod_{\alpha\in A} K_\alpha \longrightarrow \prod_{\alpha\in A} R^{(B)} \longrightarrow \prod_{\alpha\in A} M_\alpha \longrightarrow 0$. For any finitely generated left ideal I, $(\prod_{\alpha\in A} K_\alpha)I = \prod_{\alpha\in A} K_\alpha I = \prod_{\alpha\in A}(K_\alpha \cap R^{(B)}I) = \prod_{\alpha\in A} K_\alpha \cap (\prod_{\alpha\in A} R^{(B)}I) = \prod_{\alpha\in A} K_\alpha \cap (\prod_{\alpha\in A} R^{(B)})I$, then $\prod_{\alpha\in A} K_\alpha$ is a pure submodule of $\prod_{\alpha\in A} R^{(B)}$. If $\prod_{\alpha\in A} R^{(B)}$ is flat, then $\prod_{\alpha\in A} M_\alpha$ is flat. If $v_j \in \prod_{\alpha\in A} R^{(B)}, a_j \in R$ such that $\sum_{j=1}^{n} v_ja_j = 0$. Let $f : R^n \to \sum_{j=1}^{n} Ra_j$, $(r_1, \cdots, r_n) \mapsto \sum_{j=1}^{n} r_ja_j$, then $Kerf = Rk_1 + \cdots + Rk_m$ where $k_i = (c_{i_1}, \cdots, c_{i_n}) \in Kerf$, thus $\sum_{j=1}^{n} c_{ij}a_j = f(k_i) = 0$. Let $v_j = (v_{j_\alpha})_{\alpha\in A}$, $v_{j_\alpha} \in R^{(B)}$ for any $\alpha \in A$. Therefore $\sum_{j=1}^{n} v_ja_j = \sum_{j=1}^{n}(v_{j_\alpha})_{\alpha\in A}a_j = 0$, then $\sum_{j=1}^{n} v_{j_\alpha}a_j = 0$ for any $\alpha \in A$, since $v_{1_\alpha}, \cdots, v_{n_\alpha} \in R^{(B)}$, $v_{j_\alpha} = (0, \cdots, 0, v_{j_\alpha\beta_1}, \cdots, v_{j_\alpha\beta_s}, 0, \cdots, 0)(j = 1, \cdots, n)$, then $\sum_{j=1}^{n} v_{j_\alpha\beta_p}a_j = 0$ for all $1 \le p \le s$. Thus $(v_{1_\alpha\beta_p}, \cdots, v_{n_\alpha\beta_p}) \in Kerf$, then $(v_{1_\alpha\beta_p}, \cdots, v_{n_\alpha\beta_p}) = r_{\alpha\beta_{p_1}}k_1 + \cdots + r_{\alpha\beta_{p_n}}k_n$. Let $u_i \neq (0, \cdots, 0, r_{\alpha\beta_{1_i}}, \cdots, r_{\alpha\beta_{s_i}}, 0, \cdots, 0)$, then $v_j = \sum_{i=1}^{n} u_ic_{ij}$. Hence $\prod_{\alpha\in A} R^{(B)}$ is flat.

13. If $\sum_{i\in I} \oplus M_i$ is a faithful flat module, then ${}_RM_i$ is flat for any $i \in I$. For exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, there is

$$0 \longrightarrow A \otimes \sum \oplus M_i \longrightarrow B \otimes \sum \oplus M_i \longrightarrow C \otimes \sum \oplus M_i \longrightarrow 0 .$$

On contrary, if

$$\begin{tikzcd}
0 \arrow[r] & A \otimes \sum \oplus M_i \arrow[r] \arrow[d, "\cong"'] & B \otimes \sum \oplus M_i \arrow[r] \arrow[d, "\cong"'] & C \otimes \sum \oplus M_i \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 \arrow[r] & \sum \oplus A \otimes M_i \arrow[r, "(f_i)_{i\in I}"] & \sum \oplus B \otimes M_i \arrow[r] & \sum \oplus C \otimes M_i \arrow[r] & 0
\end{tikzcd}$$

where $f_i : A \otimes M_i \to B \otimes M_i$ and $\sum \oplus Kerf_i = Ker(f_i)_{i\in I}$, then

$$0 \longrightarrow A \otimes M_i \longrightarrow B \otimes M_i \longrightarrow C \otimes M_i \longrightarrow 0$$

is exact for any $i \in I$. Thus ${}_RM_i$ is faithful flat for any $i \in I$. If ${}_RM_i$ is faithful flat for any $i \in I$, then $\sum \oplus M_i$ is flat. If

$$0 \longrightarrow A \otimes \sum \oplus M_i \longrightarrow B \otimes \sum \oplus M_i \longrightarrow C \otimes \sum \oplus M_i \longrightarrow 0$$

is exact, then $0 \longrightarrow A \otimes M_i \longrightarrow B \otimes M_i \longrightarrow C \otimes M_i \longrightarrow 0$ is exact, thus $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact. Hence $\sum \oplus M_i$ is faithful flat.

14. ($\Rightarrow$) :If ${}_RM$ is a injective cogenerator and $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact, then $0 \longrightarrow Hom_R(C, M) \xrightarrow{g^*} Hom_R(B, M) \xrightarrow{f^*} Hom_R(A, M) \longrightarrow 0$ is exact. On contrary, if

$$0 \longrightarrow Hom_R(C, M) \xrightarrow{g^*} Hom_R(B, M) \xrightarrow{f^*} Hom_R(A, M) \longrightarrow 0$$

is exact, if $0 \neq a \in Kerf$, then there is $\alpha \in Hom_R(A, M)$ such that $\alpha(a) \neq 0$, while f^* is surjective, there is $\beta \in Hom_R(B, M)$ such that $f^*(\beta) = \alpha$,i.e. $\alpha = \beta f$, then $\alpha(a) = \beta f(a) = 0$, it is contradiction. Hence $Kerf = 0$. If $g \circ f \neq 0$, there is $a \in A$ such that $gf(a) \neq 0$, there is $\alpha \in Hom_R(C, M)$ such that $0 \neq \alpha gf(a) = f^*g^*((\alpha))(a) = 0$, it is contradiction. Hence $gf = 0$, i.e. $Imf \subseteq Kerg$. Suppose $\pi : C \to C/Img$, $\pi(x) = x + Img$, then $\pi \circ g : B \to C/Img$ is a zero mapping. $\pi_* : Hom_R(C/Img, M) \to Hom_R(C, M)$ is injective, then $g^*\pi_* : Hom_R(C/Img, M) \to Hom_R(B, M)$ is injective. $g^*\pi_*(\alpha) = \alpha \circ \pi \circ g = 0$ for $\alpha \in Hom_R(C/Img, M)$, then $\alpha = 0$. Since M is a cogenerator, $C/Img = 0$, then g is surjective. Let $\varphi : Kerg \to Kerg/Imf$, $\varphi(a) = a + Imf$, then $\varphi \circ f = 0$, while $\varphi^* : Hom_R(Kerg/Imf, M) \to Hom_R(Kerg, M)$ is injective, $\varphi^*(\alpha) \in Hom_R(Kerg, M)$ for any $\alpha \in Hom_R(Kerg/Imf, M)$. Since $0 \longrightarrow Kerg \longrightarrow B \longrightarrow B/Kerg \longrightarrow 0$ is exact,

$$0 \longrightarrow Hom_R(B/Kerg, M) \longrightarrow Hom_R(B, M) \longrightarrow Hom_R(Kerg, M) \longrightarrow 0$$

is exact, then there is $\Psi \in Hom_R(B, M)$ such that $\Psi|_Kerg = \varphi^*(\alpha)$, thus $\varphi^*(\alpha)(a) = \alpha\varphi(a) = \Psi(a)$. Since $f^*(\Psi)(x) = \Psi(f(x)) = 0$ for any $x \in A$, $\Psi \in Kerf^* = Img^*$, then there is $\eta \in Hom_R(C, M)$ such that $g^*(\eta) = \eta g = \Psi$, thus $\varphi^*(\alpha)(a) = \alpha\varphi(a) = \Psi(a)\eta g(a) = 0$ for any $a \in Kerg$. Therefore $\varphi^*(\alpha) = 0$. Since φ^* is injective, $\alpha = 0$, then $Hom_R(Kerg/Imf, M) = 0$ for α is arbitrary. Since M is cogenerator, $Kerg/Imf = 0$, i.e. $Kerg = Imf$. This means that

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is exact.

($\Leftarrow$) : Consider $0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} N \longrightarrow 0$ is exact where E is injective, then

$$0 \longrightarrow Hom_R(N, M) \xrightarrow{g^*} Hom_R(E, M) \xrightarrow{f^*} Hom_R(M, M) \longrightarrow 0$$

is exact, thus there is $k \in Hom_R(E, M)$ such that $f^*(k) = kf = id_M$, then $0 \longrightarrow M \xrightarrow{f} E \xrightarrow{g} N \longrightarrow 0$ is split, i.e. $E \cong M \oplus N$, then M is injective. For any $0 \neq a \in_R L$, $0 \longrightarrow Ra \xrightarrow{\lambda} L \xrightarrow{\pi} L/Ra \longrightarrow 0$ is exact, then $0 \longrightarrow Hom_R(L/Ra, M) \xrightarrow{\pi^*} Hom_R(L, M) \xrightarrow{\lambda^*} Hom_R(Ra, M) \longrightarrow 0$ is exact. If $Hom_R(Ra, M) = 0$, then

$$0 \longrightarrow Hom_R(L/Ra, M) \xrightarrow{\pi^*} Hom_R(L, M) \xrightarrow{\lambda^*} Hom_R(0, M) \longrightarrow 0$$

is exact,thus $0 \longrightarrow 0 \xrightarrow{\lambda} L \xrightarrow{\pi} L/Ra \longrightarrow 0$ is exact. Therefore π is isomorphic, then $Ker\pi = Ra = 0$, it is contradiction. Hence $Hom_R(Ra, M) \neq 0$, i.e. there is $f \in Hom_R(Ra, M)$ such that $f(a) \neq 0$, that is M is a cogenereator.

15. $(\Rightarrow)$:If ${}_RM$ is a projective generator and $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact, then $0 \longrightarrow Hom_R(M, A) \xrightarrow{f_*} Hom_R(M, B) \xrightarrow{g_*} Hom_R(M, C) \longrightarrow 0$ is exact. On contrary, if $Kerf \neq 0$, then there is $0 \neq \alpha \in Hom_R(M, Kerf)$, $0 \neq \lambda\alpha \in Hom_R(M, A)$ for $M \xrightarrow{\alpha} Kerf \xrightarrow{\lambda} A \longrightarrow 0$, $f_*(\lambda\alpha) = f\lambda\alpha = 0$, it is contradiction for f_* is injective. Hence f is injective. For any $c \in C$, there is $\varphi \in Hom_R(M, C)$ such that $\varphi(a) = c$ for $a \in M$. Since g_* is surjective, there is $\Psi \in Hom_R(M, B)$ such that $g_*\Psi = g\Psi = \varphi$, then $\varphi(a) = g\Psi(a) \in Img$. Hence g is surjective. If $gf \neq 0$, there is $a \in A$ such that $gf(a) \neq 0$, then there is $\varphi \in Hom_R(M, A)$ such that $\varphi(m) = a$ for some $m \in M$, thus $gf\varphi(m) \neq 0$, i.e. $(gf)_* \neq 0$, it is contradiction. Hence $gf = 0$. Since $gf = 0$, then $Imf \subseteq Kerg$. For $A \xrightarrow{f} Kerg \xrightarrow{\pi} Kerg/Imf \longrightarrow 0$, if $Kerg/Imf \neq 0$, there is $0 \neq \eta \in Hom_R(M, Kerg/Imf)$, since M is projective, there is $\zeta : M \to Kerg \subseteq B$ such that $\pi\zeta = \eta$, then $g\zeta = 0 = g_*(\zeta)$, thus $\zeta \in Kerg_* = Imf_*$, therefore there is $\sigma \in Hom_R(M, A)$ such that $\zeta = f\sigma$, then $\pi\zeta = \eta = \pi f\sigma = 0$. It is contradiction for $\eta \neq 0$. Hence $Kerg/Imf = 0$, i.e. $Kerg = Imf$, then $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact.
$(\Leftarrow)$: Let $\varphi :_R F \to_R M$ is epimorphic and F is free, then

$$0 \longrightarrow Ker\varphi \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 0$$

is exact, thus

$$0 \longrightarrow Hom_R(M, Ker\varphi) \longrightarrow Hom_R(M, F) \xrightarrow{\varphi_*} Hom_R(M, M) \longrightarrow 0$$

is exact. Then there is $h \in Hom_R(M, F)$ such that $\varphi_*(h) = \varphi h = id_M$, thus $0 \longrightarrow Ker\varphi \longrightarrow F \xrightarrow{\varphi} M \longrightarrow 0$ is split, i.e. $F \cong M \oplus Ker\varphi$, then M

is projective. For any ${}_RN$ and any $0 \neq x \in_R N$,

$$0 \longrightarrow Rx \xrightarrow{\lambda} N \xrightarrow{\pi} N/Rx \longrightarrow 0$$

is exact, then

$$0 \longrightarrow Hom_R(M, Rx) \xrightarrow{\lambda_*} Hom_R(M, N) \xrightarrow{\pi_*} Hom_R(M, N/Rx) \longrightarrow 0$$

is exact. If there is not $f \in Hom_R(M, N)$ such that $x \in Imf$, then $Hom_R(M, Rx) = 0$,

$$0 \longrightarrow Hom_R(M, 0) \xrightarrow{0_*} Hom_R(M, N) \xrightarrow{\pi_*} Hom_R(M, N/Rx) \longrightarrow 0$$

is exact, thus $0 \longrightarrow 0 \xrightarrow{0} N \xrightarrow{\pi} N/Rx \longrightarrow 0$ is exact. Therefore $Ker\pi = 0$, it is contradiction. Hence there is $f \in Hom_R(M, N)$ such that $x \in Imf$, then M is a generator.

16. Suppose that $R^F = R^{(I)}$ is free, then it is projective. For any $0 \neq x \in_R M$, $Hom_R(R^{(I)}, M) \cong \prod\limits_I Hom_R(R, M)$, while there is a homomorphism $f :_R R \rightarrow_R M$ such that $f(1) = x$, then there is $\varpi \in Hom_R(R^{(I)}, M)$ such that $\varphi(a) = x$. Hence $R^F = R^{(I)}$ is a generator.

17. Since $\mathbb{Q}/\mathbb{Z}$ is injective generator of $\mathbb{Z}$-module, then ${}_RM$ is flat if and only if $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a injective right R-module.(cf. Proof of Projective 2.5.5)

18. For any $f \in Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ and any $n \neq 0$, $f_1(a) = f(\frac{1}{n}a)$ for any $a \in \mathbb{Q}$, then $f_1 \in Hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z})$ such that $nf_1 = f$, i.e. $nHom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = Hom_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$, Therefore $Hom_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$ is a divisible abelian group, hence ${}_{\mathbb{Z}}\mathbb{Q}$ is flat.