TOPOLOGY - HOMEWORK 02

Due: 2020 March 19th, 10:00AM

Question 1. Prove:

(i) $\mathrm{card}(\mathbf{Q}) = \aleph_0$, that is, $\mathbf{Q}$ is countably infinite;
(ii) the set of real numbers $\mathbf{R}$ is uncountable.

Question 2. A real number $x \in \mathbf{R}$ is said to be algebraic if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 = 0$$

with $a_i \in \mathbf{Q}$ for all $0 \leq i \leq n-1$; otherwise, it is called transcendental. Assuming that each polynomial equation has only finitely many roots, show that the set $\mathcal{A} \subset \mathbf{R}$ of algebraic numbers is countable. Deduce that the set $\mathbf{R} - \mathcal{A}$ of transcendental numbers is uncountable.

Question 3. Let $X = \{0, 1\}$. Let $X^{\mathbf{N}}$ be the set of all functions $f : \mathbf{N} \longrightarrow X$. We define the support of $f \in X^{\mathbf{N}}$ to be

$$\mathrm{supp}(f) = f^{-1}(\{1\}).$$

Consider set

$$X_0^{\mathbf{N}} = \left\{ f \in X^{\mathbf{N}} : \ \mathrm{supp}(f) \text{ is a finite set} \right\}.$$

Is $X_0^{\mathbf{N}}$ countable or uncountable? Justify your answer.

Question 4. A pseudo-metric space is a pair (S, d), where S is a set and $d : S \times S \longrightarrow \mathbf{R}$ a function satisfying

(1) $d(x, y) \geq 0$ for all $x, y \in S$, and $d(x, x) = 0$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in S$;
(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in S$.

Define a relation $\sim$ on S by $x \sim y$ if and only if $d(x, y) = 0$. Prove that

(i) $\sim$ is an equivalence relation on S (and thus we denote by S^* the set of equivalence classes of S);
(ii) for every $[x], [y] \in S^*$, the function

$$d^*([x], [y]) = d(x, y)$$

gives a well-defined metric on S^*.

Question 5. Let (S, d) be a metric space. Can one always define a metric $d^\natural$ on the power set $\mathcal{P}(S)$ of S such that $d^\natural(\{x\}, \{y\}) = d(x, y)$ for every $x, y \in S$?