Arzelà–Ascoli theorem

The **Arzelà–Ascoli theorem** is a fundamental result of mathematical analysis giving necessary and sufficient conditions to decide whether every sequence of a given family of real-valued continuous functions defined on a closed and bounded interval has a uniformly convergent subsequence. The main condition is the equicontinuity of the family of functions. The theorem is the basis of many proofs in mathematics, including that of the Peano existence theorem in the theory of ordinary differential equations, Montel's theorem in complex analysis, and the Peter–Weyl theorem in harmonic analysis and various results concerning compactness of integral operators.

The notion of equicontinuity was introduced in the late 19th century by the Italian mathematicians Cesare Arzelà and Giulio Ascoli. A weak form of the theorem was proven by Ascoli (1883–1884), who established the sufficient condition for compactness, and by Arzelà (1895), who established the necessary condition and gave the first clear presentation of the result. A further generalization of the theorem was proven by Fréchet (1906), to sets of real-valued continuous functions with domain a compact metric space (Dunford & Schwartz 1958, p. 382). Modern formulations of the theorem allow for the domain to be compact Hausdorff and for the range to be an arbitrary metric space. More general formulations of the theorem exist that give necessary and sufficient conditions for a family of functions from a compactly generated Hausdorff space into a uniform space to be compact in the compact-open topology; see Kelley (1991, page 234).

Contents

Statement and first consequences

By definition, a sequence $\{f_n\}_{n\in\mathbf{N}}$ of continuous functions on an interval $I = [a, b]$ is *uniformly bounded* if there is a number M such that

$$|f_n(x)| \le M$$

for every function f_n belonging to the sequence, and every $x \in [a, b]$. (Here, M must be independent of n and x.)

The sequence is said to be *uniformly equicontinuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f_n(x) - f_n(y)| < \varepsilon$$

whenever $|x - y| < \delta$ for all functions f_n in the sequence. (Here, δ may depend on ε, but not x, y or n.)

One version of the theorem can be stated as follows:

> Consider a sequence of real-valued continuous functions $\{f_n\}_{n \in \mathbf{N}}$ defined on a closed and bounded interval $[a, b]$ of the real line. If this sequence is uniformly bounded and uniformly equicontinuous, then there exists a subsequence $\{f_{n_k}\}_{k \in \mathbf{N}}$ that converges uniformly.
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> The converse is also true, in the sense that if every subsequence of $\{f_n\}$ itself has a uniformly convergent subsequence, then $\{f_n\}$ is uniformly bounded and equicontinuous.

(See below for a proof.)

Examples

Differentiable functions

The hypotheses of the theorem are satisfied by a uniformly bounded sequence $\{f_n\}$ of differentiable functions with uniformly bounded derivatives. Indeed, uniform boundedness of the derivatives implies by the mean value theorem that for all x and y,

$$|f_n(x) - f_n(y)| \le K|x - y|,$$

where K is the supremum of the derivatives of functions in the sequence and is independent of n. So, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{2K}$ to verify the definition of equicontinuity of the sequence. This proves the following corollary:

- Let $\{f_n\}$ be a uniformly bounded sequence of real-valued differentiable functions on $[a, b]$ such that the derivatives $\{f_n'\}$ are uniformly bounded. Then there exists a subsequence $\{f_{n_k}\}$ that converges uniformly on $[a, b]$.

If, in addition, the sequence of second derivatives is also uniformly bounded, then the derivatives also converge uniformly (up to a subsequence), and so on. Another generalization holds for continuously differentiable functions. Suppose that the functions f_n are continuously differentiable with derivatives f'_n. Suppose that f_n' are uniformly equicontinuous and uniformly bounded, and that the sequence $\{f_n\}$, is pointwise bounded (or just bounded at a single point). Then there is a subsequence of the $\{f_n\}$ converging uniformly to a continuously differentiable function.

The diagonalization argument can also be used to show that a family of infinitely differentiable functions, whose derivatives of each order are uniformly bounded, has a uniformly convergent subsequence, all of whose derivatives are also uniformly convergent. This is particularly important in the theory of distributions.

Lipschitz and Hölder continuous functions

The argument given above proves slightly more, specifically

- If $\{f_n\}$ is a uniformly bounded sequence of real valued functions on $[a, b]$ such that each *f* is Lipschitz continuous with the same Lipschitz constant K:

$$|f_n(x) - f_n(y)| \leq K|x - y|$$

  for all $x, y \in [a, b]$ and all f_n, then there is a subsequence that converges uniformly on $[a, b]$.

The limit function is also Lipschitz continuous with the same value K for the Lipschitz constant. A slight refinement is

- A set $\mathbf{F}$ of functions f on $[a, b]$ that is uniformly bounded and satisfies a Hölder condition of order α, $0 < \alpha \leq 1$, with a fixed constant M,

$$|f(x) - f(y)| \leq M\,|x - y|^{\alpha}, \qquad x, y \in [a, b]$$

  is relatively compact in $\mathrm{C}([a, b])$. In particular, the unit ball of the Hölder space $\mathrm{C}^{0,\alpha}([a, b])$ is compact in $\mathrm{C}([a, b])$.

This holds more generally for scalar functions on a compact metric space X satisfying a Hölder condition with respect to the metric on X.

Euclidean spaces

The Arzelà–Ascoli theorem holds, more generally, if the functions f_n take values in d-dimensional Euclidean space $\mathbf{R}^d$, and the proof is very simple: just apply the $\mathbf{R}$-valued version of the Arzelà–Ascoli theorem d times to extract a subsequence that converges uniformly in the first coordinate, then a sub-subsequence that converges uniformly in the first two coordinates, and so on. The above examples generalize easily to the case of functions with values in Euclidean space.

Proof

The proof is essentially based on a diagonalization argument. The simplest case is of real-valued functions on a closed and bounded interval:

- Let $I = [a, b] \subset \mathbf{R}$ be a closed and bounded interval. If $\mathbf{F}$ is an infinite set of functions $f : I \to \mathbf{R}$ which is uniformly bounded and equicontinuous, then there is a sequence f_n of elements of $\mathbf{F}$ such that f_n converges uniformly on I.

Fix an enumeration $\{x_i\}_{i \in \mathbf{N}}$ of rational numbers in I. Since $\mathbf{F}$ is uniformly bounded, the set of points $\{f(x_1)\}_{f \in \mathbf{F}}$ is bounded, and hence by the Bolzano–Weierstrass theorem, there is a sequence $\{f_{n_1}\}$ of distinct functions in $\mathbf{F}$ such that $\{f_{n_1}(x_1)\}$ converges. Repeating the same argument for the sequence of points $\{f_{n_1}(x_2)\}$, there is a subsequence $\{f_{n_2}\}$ of $\{f_{n_1}\}$ such that $\{f_{n_2}(x_2)\}$ converges.

By induction this process can be continued forever, and so there is a chain of subsequences

$$\{f_{n_1}\} \supseteq \{f_{n_2}\} \supseteq \cdots$$

such that, for each $k = 1, 2, 3, \ldots$, the subsequence $\{f_{n_k}\}$ converges at $x_1, \ldots, x_k$. Now form the diagonal subsequence $\{f\}$ whose mth term f_m is the mth term in the mth subsequence $\{f_{n_m}\}$. By construction, f_m converges at every rational point of I.

Therefore, given any $\varepsilon > 0$ and rational x_k in I, there is an integer $N = N(\varepsilon, x_k)$ such that

$$|f_n(x_k) - f_m(x_k)| < \tfrac{\varepsilon}{3}, \qquad n, m \geq N.$$

Since the family $\mathbf{F}$ is equicontinuous, for this fixed ε and for every x in I, there is an open interval U_x containing x such that

$$|f(s) - f(t)| < \tfrac{\varepsilon}{3}$$

for all $f \in \mathbf{F}$ and all s, t in I such that $s, t \in U_x$.

The collection of intervals U_x, $x \in I$, forms an open cover of I. Since I is compact, by the Heine-Borel theorem this covering admits a finite subcover $U_1, \ldots, U_J$. There exists an integer K such that each open interval U_j, $1 \leq j \leq J$, contains a rational x_k with $1 \leq k \leq K$. Finally, for any $t \in I$, there are j and k so that t and x_k belong to the same interval U_j. For this choice of k,

$$\begin{aligned} |f_n(t) - f_m(t)| &\leq |f_n(t) - f_n(x_k)| + |f_n(x_k) - f_m(x_k)| + |f_m(x_k) - f_m(t)| \\ &< \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} \end{aligned}$$

for all $n, m > N = \max\{N(\varepsilon, x_1), \ldots, N(\varepsilon, x_K)\}$. Consequently, the sequence $\{f_n\}$ is uniformly Cauchy, and therefore converges to a continuous function, as claimed. This completes the proof.

Generalizations

Compact metric spaces and compact Hausdorff spaces

The definitions of boundedness and equicontinuity can be generalized to the setting of arbitrary compact metric spaces and, more generally still, compact Hausdorff spaces. Let X be a compact Hausdorff space, and let $C(X)$ be the space of real-valued continuous functions on X. A subset

$\mathbf{F} \subset C(X)$ is said to be *equicontinuous* if for every $x \in X$ and every $\varepsilon > 0$, x has a neighborhood U_x such that

$$\forall y \in U_x, \forall f \in \mathbf{F} : \qquad |f(y) - f(x)| < \varepsilon.$$

A set $\mathbf{F} \subset C(X, \mathbf{R})$ is said to be *pointwise bounded* if for every $x \in X$,

$$\sup\{|f(x)| : f \in \mathbf{F}\} < \infty.$$

A version of the Theorem holds also in the space $C(X)$ of real-valued continuous functions on a compact Hausdorff space X (Dunford & Schwartz 1958, §IV.6.7):

> Let X be a compact Hausdorff space. Then a subset $\mathbf{F}$ of $C(X)$ is relatively compact in the topology induced by the uniform norm if and only if it is equicontinuous and pointwise bounded.

The Arzelà–Ascoli theorem is thus a fundamental result in the study of the algebra of continuous functions on a compact Hausdorff space.

Various generalizations of the above quoted result are possible. For instance, the functions can assume values in a metric space or (Hausdorff) topological vector space with only minimal changes to the statement (see, for instance, Kelley & Namioka (1982, §8), Kelley (1991, Chapter 7)):

> Let X be a compact Hausdorff space and Y a metric space. Then $\mathbf{F} \subset C(X, Y)$ is compact in the compact-open topology if and only if it is equicontinuous, pointwise relatively compact and closed.

Here pointwise relatively compact means that for each $x \in X$, the set $\mathbf{F}_x = \{f(x) : f \in \mathbf{F}\}$ is relatively compact in Y.

The proof given can be generalized in a way that does not rely on the separability of the domain. On a compact Hausdorff space X, for instance, the equicontinuity is used to extract, for each $\varepsilon = 1/n$, a finite open covering of X such that the oscillation of any function in the family is less than ε on each open set in the cover. The role of the rationals can then be played by a set of points drawn from each open set in each of the countably many covers obtained in this way, and the main part of the proof proceeds exactly as above.

Necessity

Whereas most formulations of the Arzelà–Ascoli theorem assert sufficient conditions for a family of functions to be (relatively) compact in some topology, these conditions are typically also necessary. For instance, if a set $\mathbf{F}$ is compact in $C(X)$, the Banach space of real-valued continuous functions on a compact Hausdorff space with respect to its uniform norm, then it is bounded in the uniform norm on $C(X)$ and in particular is pointwise bounded. Let $N(\varepsilon, U)$ be the set of all functions in $\mathbf{F}$ whose oscillation over an open subset $U \subset X$ is less than ε:

$$N(\varepsilon, U) = \{f | \operatorname{osc}_U f < \varepsilon\}.$$

For a fixed $x \in X$ and ε, the sets $N(\varepsilon, U)$ form an open covering of $\mathbf{F}$ as U varies over all open neighborhoods of x. Choosing a finite subcover then gives equicontinuity.

Examples

- To every function g that is p-integrable on $[0, 1]$, with $1 < p \leq \infty$, associate the function G defined on $[0, 1]$ by

  $$G(x) = \int_0^x g(t)\,\mathrm{d}t.$$

  Let $\mathbf{F}$ be the set of functions G corresponding to functions g in the unit ball of the space $L^p([0, 1])$. If q is the Hölder conjugate of p, defined by $\frac{1}{p} + \frac{1}{q} = 1$, then Hölder's inequality implies that all functions in $\mathbf{F}$ satisfy a Hölder condition with $\alpha = \frac{1}{q}$ and constant $M = 1$.

  It follows that $\mathbf{F}$ is compact in $C([0, 1])$. This means that the correspondence $g \to G$ defines a compact linear operator T between the Banach spaces $L^p([0, 1])$ and $C([0, 1])$. Composing with the injection of $C([0, 1])$ into $L^p([0, 1])$, one sees that T acts compactly from $L^p([0, 1])$ to itself. The case $p = 2$ can be seen as a simple instance of the fact that the injection from the Sobolev space $H_0^1(\Omega)$ into $L^2(\Omega)$, for Ω a bounded open set in $\mathbf{R}^d$, is compact.

- When T is a compact linear operator from a Banach space X to a Banach space Y, its transpose T^* is compact from the (continuous) dual Y^* to X^*. This can be checked by the Arzelà–Ascoli theorem.

  Indeed, the image $T(B)$ of the closed unit ball B of X is contained in a compact subset K of Y. The unit ball B^* of Y^* defines, by restricting from Y to K, a set $\mathbf{F}$ of (linear) continuous functions on K that is bounded and equicontinuous. By Arzelà–Ascoli, for every sequence $\{y_n^*\}$, in B^*, there is a subsequence that converges uniformly on K, and this implies that the image $T^*\left(y_{n_k}^*\right)$ of that subsequence is Cauchy in X^*.

- When f is holomorphic in an open disk $D_1 = B(z_0, r)$, with modulus bounded by M, then (for example by Cauchy's formula) its derivative f' has modulus bounded by $\frac{2M}{r}$ in the smaller disk $D_2 = B(z_0, \frac{r}{2})$. If a family of holomorphic functions on D_1 is bounded by M on D_1, it follows that the family $\mathbf{F}$ of restrictions to D_2 is equicontinuous on D_2. Therefore, a sequence converging uniformly on D_2 can be extracted. This is a first step in the direction of Montel's theorem.

See also

- Helly's selection theorem
- Fréchet–Kolmogorov theorem

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