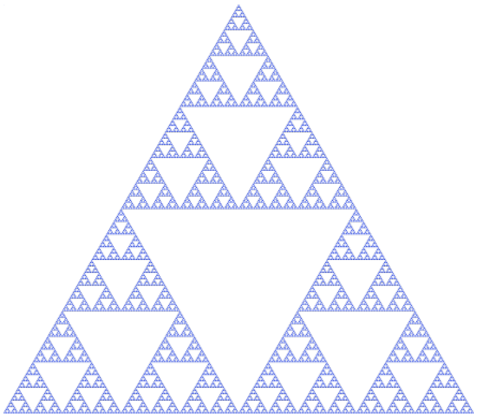
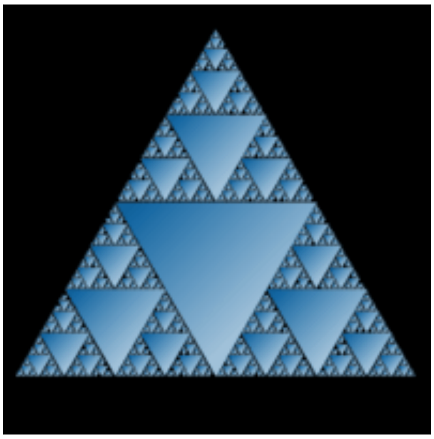
Sierpiński triangle

The **Sierpinski triangle** (also with the original orthography *Sierpiński*), also called the **Sierpinski gasket** or **Sierpinski sieve**, is a fractal attractive fixed set with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles. Originally constructed as a curve, this is one of the basic examples of self-similar sets—that is, it is a mathematically generated pattern that is reproducible at any magnification or reduction. It is named after the Polish mathematician Wacław Sierpiński, but appeared as a decorative pattern many centuries before the work of Sierpiński.[1][2]



Sierpinski triangle



Generated using a random algorithm

Contents

Constructions

Sierpinski triangle in logic: The first 16 conjunctions of lexicographically ordered arguments. The columns interpreted as binary numbers give 1, 3, 5, 15, 17, 51... (sequence A001317 in the OEIS)

There are many different ways of constructing the Sierpinski triangle.

Removing triangles

The Sierpinski triangle may be constructed from an equilateral triangle by repeated removal of triangular subsets:

The evolution of the Sierpinski triangle

1. Start with an equilateral triangle.
2. Subdivide it into four smaller congruent equilateral triangles and remove the central triangle.
3. Repeat step 2 with each of the remaining smaller triangles infinitely.

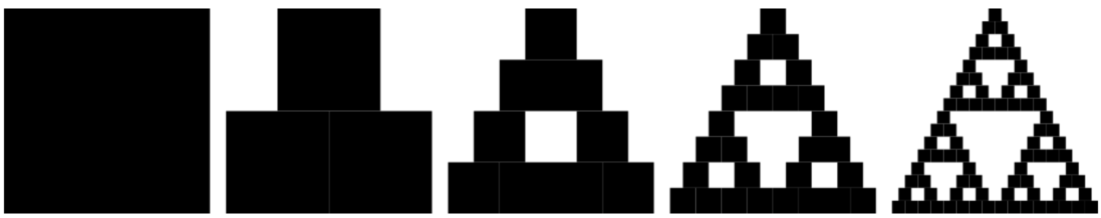
Each removed triangle (a *trema*) is topologically an open set.[3] This process of recursively removing triangles is an example of a finite subdivision rule.

Shrinking and duplication

The same sequence of shapes, converging to the Sierpinski triangle, can alternatively be generated by the following steps:

1. Start with any triangle in a plane (any closed, bounded region in the plane will actually work). The canonical Sierpinski triangle uses an equilateral triangle with a base parallel to the horizontal axis (first image).
2. Shrink the triangle to $\frac{1}{2}$ height and $\frac{1}{2}$ width, make three copies, and position the three shrunken triangles so that each triangle touches the two other triangles at a corner (image 2). Note the emergence of the central hole—because the three shrunken triangles can between them cover only $\frac{3}{4}$ of the area of the original. (Holes are an important feature of Sierpinski's triangle.)
3. Repeat step 2 with each of the smaller triangles (image 3 and so on).

Note that this infinite process is not dependent upon the starting shape being a triangle—it is just clearer that way. The first few steps starting, for example, from a square also tend towards a Sierpinski triangle. Michael Barnsley used an image of a fish to illustrate this in his paper "V-variable fractals and superfractals."[4][5]



Iterating from a square

The actual fractal is what would be obtained after an infinite number of iterations. More formally, one describes it in terms of functions on closed sets of points. If we let d_{A} denote the dilation by a factor of $\frac{1}{2}$ about a point A, then the Sierpinski triangle with corners A, B, and C is the fixed set of the transformation $d_{\mathrm{A}} \cup d_{\mathrm{B}} \cup d_{\mathrm{C}}$.

This is an attractive fixed set, so that when the operation is applied to any other set repeatedly, the images converge on the Sierpinski triangle. This is what is happening with the triangle above, but any other set would suffice.

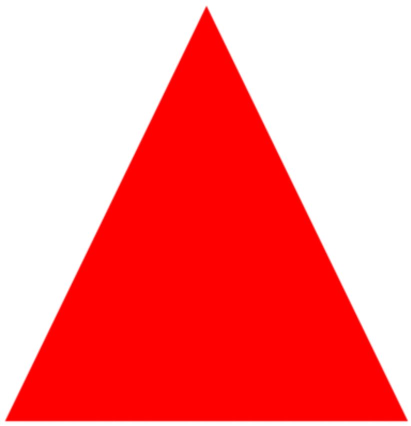
Chaos game

If one takes a point and applies each of the transformations d_A, d_B, and d_C to it randomly, the resulting points will be dense in the Sierpinski triangle, so the following algorithm will again generate arbitrarily close approximations to it:[6]

Animated creation of a Sierpinski triangle using the chaos game

Start by labeling $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ as the corners of the Sierpinski triangle, and a random point $\mathbf{v}_1$. Set $\mathbf{v}_{n+1} = \frac{1}{2}(\mathbf{v}_n + \mathbf{p}_{r_n})$, where r_n is a random number 1, 2 or 3. Draw the points $\mathbf{v}_1$ to $\mathbf{v}_\infty$. If the first point $\mathbf{v}_1$ was a point on the Sierpiński triangle, then all the points $\mathbf{v}_n$ lie on the Sierpinski triangle. If the first point $\mathbf{v}_1$ to lie within the perimeter of the triangle is not a point on the Sierpinski triangle, none of the points $\mathbf{v}_n$ will lie on the Sierpinski triangle, however they will converge on the triangle. If $\mathbf{v}_1$ is outside the triangle, the only way $\mathbf{v}_n$ will land on the actual triangle, is if $\mathbf{v}_n$ is on what would be part of the triangle, if the triangle was infinitely large.

Or more simply:



Animated construction of a Sierpinski triangle

1. Take three points in a plane to form a triangle, you need not draw it.
2. Randomly select any point inside the triangle and consider that your current position.
3. Randomly select any one of the three vertex points.
4. Move half the distance from your current position to the selected vertex.
5. Plot the current position.
6. Repeat from step 3.

This method is also called the chaos game, and is an example of an iterated function system. You can start from any point outside or inside the triangle, and it would eventually form the Sierpinski Gasket with a few leftover points (if the starting point lies on the outline of the triangle, there are no leftover points). With pencil and paper, a brief outline is formed after placing approximately one hundred points, and detail begins to appear after a few hundred. An interactive version of the chaos game can be found here. (https://scratch.mit.edu/projects/170702288/#player)

Arrowhead construction of Sierpinski gasket

Another construction for the Sierpinski gasket shows that it can be constructed as a curve in the plane. It is formed by a process of repeated modification of simpler curves, analogous to the construction of the Koch snowflake:

1. Start with a single line segment in the plane

2. Repeatedly replace each line segment of the curve with three shorter segments, forming 120° angles at each junction between two consecutive segments, with the first and last segments of the curve either parallel to the original line segment or forming a 60° angle with it.

At every iteration, this construction gives a continuous curve. In the limit, these approach a curve that traces out the Sierpenski triangle by a single continuous directed (infinitely wiggly) path, which is called the Sierpinski arrowhead.[8] In fact, the aim of the original article by Sierpinski of 1915, was to show an example of a curve (a Cantorian curve), as the title of the article itself declares.[9][2]

A Sierpinski Triangle is outlined by a fractal tree with three branches forming an angle of 60° between each other. If the angle is reduced, the triangle can be continuously transformed into a fractal resembling a tree.

Cellular automata

The Sierpinski triangle also appears in certain cellular automata (such as Rule 90), including those relating to Conway's Game of Life. For instance, the Life-like cellular automaton B1/S12 when applied to a single cell will generate four approximations of the Sierpinski triangle.[10] A very long one cell thick line in standard life will create two mirrored Sierpinski triangles. The time-space diagram of a replicator pattern in a cellular automaton also often resembles a Sierpinski triangle, such as that of the common replicator in HighLife.[11] The Sierpinski triangle can also be found in the Ulam-Warburton automaton and the Hex-Ulam-Warburton automaton.[12]

Sierpinski triangle using an iterated function system

Arrowhead construction of the Sierpinski gasket

Each subtriangle of the *n*th iteration of the deterministic Sierpinski triangle has an address on a tree with *n* levels (if *n*=∞ then the tree is also a fractal); T=top/center, L=left, R=right, and these sequences can represent both the deterministic form and, "a series of moves in the chaos game"[7]

Pascal's triangle

If one takes Pascal's triangle with 2^n rows and colors the even numbers white, and the odd numbers black, the result is an approximation to the Sierpinski triangle. More precisely, the limit as n approaches infinity of this parity-colored 2^n-row Pascal triangle is the Sierpinski triangle.[13]

A level-5 approximation to a Sierpinski triangle obtained by shading the first 2^5 (32) levels of a Pascal's triangle white if the binomial coefficient is even and black otherwise

Towers of Hanoi

The Towers of Hanoi puzzle involves moving disks of different sizes between three pegs, maintaining the property that no disk is ever placed on top of a smaller disk. The states of an n-disk puzzle, and the allowable moves from one state to another, form an undirected graph, the Hanoi graph, that can be represented geometrically as the intersection graph of the set of triangles remaining after the nth step in the construction of the Sierpinski triangle. Thus, in the limit as n goes to infinity, this sequence of graphs can be interpreted as a discrete analogue of the Sierpinski triangle.[14]

Properties

For integer number of dimensions d, when doubling a side of an object, 2^d copies of it are created, i.e. 2 copies for 1-dimensional object, 4 copies for 2-dimensional object and 8 copies for 3-dimensional object. For the Sierpinski triangle, doubling its side creates 3 copies of itself. Thus the Sierpinski triangle has Hausdorff dimension $\frac{\log(3)}{\log(2)} = \log_2 3 \approx 1.585$, which follows from solving $2^d = 3$ for d.[15]

The area of a Sierpinski triangle is zero (in Lebesgue measure). The area remaining after each iteration is $\frac{3}{4}$ of the area from the previous iteration, and an infinite number of iterations results in an area approaching zero.[16]

The points of a Sierpinski triangle have a simple characterization in barycentric coordinates.[17] If a point has coordinates $(0.u_1u_2u_3\ldots, 0.v_1v_2v_3\ldots, 0.w_1w_2w_3\ldots)$, expressed as binary numerals, then the point is in Sierpinski's triangle if and only if $u_i + v_i + w_i = 1$ for all i.

Generalization to other moduli

A generalization of the Sierpinski triangle can also be generated using Pascal's triangle if a different Modulo is used. Iteration n can be generated by taking a Pascal's triangle with P^n rows and coloring numbers by their value for $x \bmod P$. As n approaches infinity, a fractal is generated.

The same fractal can be achieved by dividing a triangle into a tessellation of P^2 similar triangles and removing the triangles that are upside-down from the original, then iterating this step with each smaller triangle.

Conversely, the fractal can also be generated by beginning with a triangle and duplicating it and arranging $\frac{n(n+1)}{2}$ of the new figures in the same orientation into a larger similar triangle with the vertices of the previous figures touching, then iterating that step.[18]

Analogues in higher dimensions

The **Sierpinski tetrahedron** or **tetrix** is the three-dimensional analogue of the Sierpinski triangle, formed by repeatedly shrinking a regular tetrahedron to one half its original height, putting together four copies of this tetrahedron with corners touching, and then repeating the process.

A Sierpinski square-based pyramid and its 'inverse'

A tetrix constructed from an initial tetrahedron of side-length L has the property that the total surface area remains constant with each iteration. The initial surface area of the (iteration-0) tetrahedron of side-length L is $L^2\sqrt{3}$. The next iteration consists of four copies with side length $\frac{L}{2}$, so the total area is $4(\frac{L}{2})^2\sqrt{3} = 4L^2 \cdot \frac{\sqrt{3}}{4} = L^2\sqrt{3}$ again. Meanwhile the volume of the construction is halved at every step and therefore approaches zero. The limit of this process has neither volume nor surface but, like the Sierpinski gasket, is an intricately connected curve. Its Hausdorff dimension is $\frac{\log(4)}{\log(2)} = 2$. If all points are projected onto a plane that is parallel to two of the outer edges, they exactly fill a square of side length $\frac{L}{\sqrt{2}}$ without overlap.

Sierpinski pyramid recursion progression (7 steps)

History

Wacław Sierpiński described the Sierpinski triangle in 1915. However, similar patterns appear already in the 13th-century Cosmati mosaics in the cathedral of Anagni, Italy,[19] and other places of central Italy, for carpets in many places such as the nave of the Roman Basilica of Santa Maria in Cosmedin,[20] and for isolated triangles positioned in rotae in several churches and basilicas.[1][2] In the case of the isolated triangle, the iteration is at least of three levels.

A medieval triangle, with historically certain dating[2] has been studied recently. It is in porphiry and golden leaf, isolated, level 4 iteration

The Apollonian gasket was first described by Apollonius of Perga (3rd century BC) and further analyzed by Gottfried Leibniz (17th century), and is a curved precursor of the 20th-century Sierpiński triangle.[21]

Etymology

The usage of the word "gasket" to refer to the Sierpinski triangle refers to gaskets such as are found in motors, and which sometimes feature a series of holes of decreasing size, similar to the fractal; this usage was coined by Benoit Mandelbrot, who thought the fractal looked similar to "the part that prevents leaks in motors".[22]

A Sierpiński triangle-based pyramid as seen from above (4 main sections highlighted). Note the self-similarity in this 2-dimensional projected view, so that the resulting triangle could be a 2D fractal in itself.

Animation of a rotating level-4 tetrix showing how some orthographic projections of a tetrix can fill a plane – in this interactive SVG (https://upload.wikimedia.org/wikipedia/commons/1/19/Tetrix_projection_fill_plane.svg), move left and right over the tetrix to rotate the 3D model

See also

- Apollonian gasket, a set of mutually tangent circles with the same combinatorial structure as the Sierpinski triangle
- List of fractals by Hausdorff dimension
- Sierpinski carpet, another fractal named after Sierpinski and formed by repeatedly removing squares from a larger square
- Triforce, a relic in the Legend of Zelda series

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