1 Groups
1.7 Simple groups

1. $|SL(2,\mathbb{Z}_2)| = 6$ and $SL(2,\mathbb{Z}_2)$ is a nonabelian group for

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

thus $SL(2,\mathbb{Z}_2) \cong S_3$, $C(SL(2,\mathbb{Z}_2)) = \{E\}$. $PSL(2,\mathbb{Z}_2) = SL(2,\mathbb{Z}_2) \cong S_3$, hence S_3 is not simple. $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = E$, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^3 = E$, then $\left\langle \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \lhd PSL(2,\mathbb{Z}_2)$. Hence $PSL(2,\mathbb{Z}_2)$ is not simple. $|SL(2,\mathbb{Z}_3)| = 24$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$, then $c = 0$ and $a = d$. $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ d & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a & b \\ a & b+d \end{pmatrix}$, then $b = 0$ and $a = d$. Hence

$$C(SL(2,\mathbb{Z}_3)) = \left\{ \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} | k \in \mathbb{Z}_3, k^2 = 1 \right\}.$$

Thus $k = 1, 2$, $|PSL(2,\mathbb{Z}_3)| = 12$. Then $PSL(2,\mathbb{Z}_3)$ has only one Sylow 3-subgroup or 4 Sylow 3-subgroups. If $PSL(2,\mathbb{Z}_3)$ has only one Sylow 3-subgroup P, then $P \lhd PSL(2,\mathbb{Z}_3)$, $PSL(2,\mathbb{Z}_3)$ is not simple. If $PSL(2,\mathbb{Z}_3)$ has 4 Sylow 3-subgroups, then there are 8 elements with order 3, then there is only one Sylow 2-subgroup P, thus $P \lhd PSL(2,\mathbb{Z}_3)$, therefore $PSL(2,\mathbb{Z}_3)$ is not simple.

2. $|SL(3,\mathbb{Z}_2)| = 168$ and $C(SL(3,\mathbb{Z}_2)) = \{E\}$, then $PSL(3,\mathbb{Z}_2) \cong SL(3,\mathbb{Z}_2)$, thus $|PSL(3,\mathbb{Z}_2)| = 168$. According to Exercise 1.2.14, $SL(3,\mathbb{Z}_2) = \langle \{T_{ij}(1) | 1 \leq i \neq j \leq 3\} \rangle$. If $PSL(3,\mathbb{Z}_2)$ not simple, assume $N \lhd PSL(3,\mathbb{Z}_2)$, if $T_{ij} \in N$, $T_{ki}(1)T_{ij}(1)T_{ki}(1)T_{ij}(1) = T_{kj}(1) \in N$, thus $N = PSL(3,\mathbb{Z}_2)$. Hence $PSL(3,\mathbb{Z}_2)$ is simple. (cf. Lang.Algebra.Theorem 9.3)

3. $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/C(SL(2,\mathbb{R}))$ where

$$SL(2,\mathbb{R}) = \langle \{T_{ij}(\lambda) | \lambda \in \mathbb{R}^*, 1 \leq i \neq j \leq 2\} \rangle,$$

then $C(SL(2,\mathbb{R})) = \mathbb{R}E$, hence $PSL(2,\mathbb{R})$ is simple.(cf. Lang.Algebra.Theorem 9.3)

4. According to Lemma 1.7.1, $A_n (n \geq 3)$ generated by 3-cycles, then $A_\infty = \left\langle \underset{i \geq 3}{\cup} A_i \right\rangle$. Let $\{e\} \neq N \lhd A_\infty$. There is a nonidentity $\sigma \in N$. Then there

exists an integer m such that $\sigma \in A_n$ for any $n \geq m$. Thus $N \cap A_n \neq \{e\}$ and $N \cap A_n \lhd A_n$ for any $n \geq m$. According to Theorem 1.7.1, A_n is simple for $n \geq 5$. So for any $n \geq max\{m, 5\}$, $N \cap A_n = A_n$ which means that $A_n \subseteq N$. Since $A_3 \subset A_4 \subset \cdots \subset A_n \subset \cdots$, $N = A_\infty$. Hence A_∞ is simple.

5. (a) If $|G| = 56 = 7 \times 2^3$, according to Sylow Theorem, there are 8 Sylow 7-subgroup or only one Sylow 7-subgroup. If there are only Sylow 7-subgroup P, then $P \lhd G$, thus G is not simple. If there are 8 Sylow-subgroup, then there are 48 elements of order 7, then there are only one Sylow 2-subgroup Q for there are 8 elements domain, thus $Q \lhd G$, therefore G is not simple.

   (b) If $|G| = 148 = 2^2 \times 37$, according to Sylow Theorem, there are only one Sylow 37-subgroup P, then $P \lhd G$, thus G is not simple.

6.
- If $|G| = p$ where p is prime, then G is a cycle group, thus G is abelian, hence $|G| \neq \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59\}$.
- If $|G| = p^r$ where p is prime, according to Example 1.7.1, then G is not simple, thus $|G| \neq \{2^2, 2^3, 2^4, 2^5, 3^2, 3^3, 4^2, 5^2, 6^2, 7^2\}$.
- If $|G| = p^r q (q < p)$ where p, q are prime, then there is only one Sylow p-subgroup, thus G is not simple, hence $|G| \neq \{3^3 \cdot 2, 3^2 \cdot 2, 3 \cdot 2, 5 \cdot 2, 5 \cdot 3, 5^2 \cdot 2, 7 \cdot 2, 7 \cdot 3, 7 \cdot 5, 11 \cdot 2, 11 \cdot 3, 11 \cdot 5, 13 \cdot 2, 13 \cdot 3, 17 \cdot 2, 17 \cdot 3, 19 \cdot 2, 19 \cdot 3, 23 \cdot 2, 29 \cdot 2\}$.
- If $|G| = p^r q$ where $p^r < q$, then there are only one Sylow q-subgroup, thus G is not simple, hence $|G| \neq \{5 \cdot 2^2, 7 \cdot 2^2, 7 \cdot 3 \cdot 2, 11 \cdot 2, 13 \cdot 2\}$. Therefore $|G| \in \{3 \cdot 2^2, 3 \cdot 2^3, 5 \cdot 3 \cdot 2, 5 \cdot 2^3, 5 \cdot 3^2, 3 \cdot 2^4, 7 \cdot 2^3\}$.
- If $|G| = 12$, according to Exercise 1.4.5, G is not simple.
- If $|G| = 56$, according to Exercise 1.7.5, G is not simple.
- If $|G| = 30$, then there are 6 Sylow 5-subgroup or only one Sylow 5-subgroup and 10 Sylow 3-subgroup or only one Sylow 3-subgroup. If there is only one Sylow 5-subgroup, then G is not simple.If there are 6 Sylow 5-subgroup, then there are 24 elements of order 5, then there is only one Sylow 3-subgroup for there are 6 elements domain, thus G is not simple.
- If $|G| = 40$ or 45, there is only one Sylow 5-subgroup, then G is not simple. Hence $|G| = \{24, 48\}$.
- If $|G| = 24$, then there are 3 Sylow 2-subgroup or only one Sylow 2-subgroup. If there is only one Sylow 2-subgroup, then G is not simple. If there are 3 Sylow 2-subgroup P_1, P_2, P_3, let $A = \{P_1, P_2, P_3\}$, G acts on A by conjugation, then this gives a homomorphism $\varphi : G \to S_3$, where $\varphi(g)(i) = j$ if only if $g^{-1} P_i g = P_j$ for $1 \leq i \leq 3$. Since $Im\varphi < S_3$ and $|S_3| < |G|$, thus $\{e\} \neq Ker\varphi \lhd G$, therefore G is not simple.

- If $|G| = 48$, then there are 3 Sylow 2-subgroup or only one Sylow 2-subgroup. If there is only one Sylow 2-subgroup, then G is not simple. If there are 3 Sylow 2-subgroup P_1, P_2, P_3, let $A = \{P_1, P_2, P_3\}$, G acts on A by conjugation, then this gives a homomorphism $\varphi : G \to S_3$, where $\varphi(g)(i) = j$ if only if $g^{-1}P_i g = P_j$ for $1 \leq i \leq 3$. Since $Im\varphi < S_3$ and $|S_3| < |G|$, thus $\{e\} \neq Ker\varphi \lhd G$, therefore G is not simple.

7.

8. According to Exercise 1.3.9, $|C(G)| = p^s$ where $0 < s \leq 3$. If $|C(G)| = p^3$, then G is abelian. According to Fundamental Theorem of Abelian Group, then $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $G \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$ or $G \cong \mathbb{Z}_{p^3}$. If $|C(G)| = p^2$, then $|G/C(G)| = p$, then $G/C(G) =< aC(G) >$, thus $G|congC(G) \rtimes < a >$. If $|C(G)| = p$, then $C(G) =< a >$, $|G/C(G)| = p^2$. Consider $A = G/C(G)$, if $|C(A)| = p$, then $|A/C(A)| = p$, thus $C(A) =< b >$, $A/C(A) =< cC(A) >$, hence $G \cong < a > \rtimes (< b > \rtimes < C >)$.