关于商空间

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在介绍商空间前，有必要先介绍一下像集与核的关系.

Sec.1 像集与核的关系

令$f: V \to W$是数域$\mathbb{P}$上的一个线性映射，$\dim V = n$，$\dim W = m$，设$\varepsilon_1, \cdots, \varepsilon_n$是$V$的一组基，$\eta_1, \cdots, \eta_m$是$W$的一组基，令

$$f(\varepsilon_i) = a_{1i}\eta_1 + a_{2i}\eta_2 + \cdots + a_{mi}\eta_m$$

其中$a_{1i}, a_{2i}, \cdots, a_{mi} \in \mathbb{P}\ (i = 1, 2, \cdots, n)$，那么

$$f(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = (f(\varepsilon_1), f(\varepsilon_2), \cdots, f(\varepsilon_n)) = (\eta_1, \eta_2, \cdots, \eta_m)\boldsymbol{A}$$

其中$\boldsymbol{A} = (a_{ij})_{m\times n}$. 对于任一$\xi \in \operatorname{Im} f$，存在$\alpha \in V$,使$f(\alpha) = \xi$.设

$$\alpha = x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n$$

其中$x_i \in \mathbb{P}\ (i = 1, 2, \cdots, n)$，那么

$$\begin{aligned} \xi &= f(x_1\varepsilon_1 + x_2\varepsilon_2 + \cdots + x_n\varepsilon_n) \\ &= x_1 f(\varepsilon_1) + x_2 f(\varepsilon_2) + \cdots + x_n f(\varepsilon_n) \in L(f(\varepsilon_1), f(\varepsilon_2), \cdots, f(\varepsilon_n)) \end{aligned}$$

于是有

命题. 设$fLV \to W$是一个线性映射，$\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$是$V$的任一组基，则

$$\operatorname{Im} f = L(f(\varepsilon_1), f(\varepsilon_2), \cdots, f(\varepsilon_n))$$

再来讨论像空间的维数刻画

定理. 设$f:V\to W$是数域$\mathbb{P}$上的一个线性映射，在V和W的某对基下，f对应的矩阵为$\boldsymbol{A}$，那么有$\dim\operatorname{Im}f=r(\boldsymbol{A})$.

证明. 设f在V的基$\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$与$W$的基$\eta_1,\eta_2,\cdots,\eta_m$下对应的矩阵为$\boldsymbol{A}$，那么，$\dim\operatorname{Im}f=r(\boldsymbol{A})$，这时候就有

$$\operatorname{Im}f=L(f(\varepsilon_1),f(\varepsilon_2),\cdots,f(\varepsilon_n))$$

从而

$$\dim\operatorname{Im}f=r(f(\varepsilon_1),f(\varepsilon_2),\cdots,f(\varepsilon_n))=r((\eta_1,\eta_2,\cdots,\eta_m)\boldsymbol{A})$$

设$\boldsymbol{A}=(\alpha_1,\alpha_2,\cdots,\alpha_n)$，$r(\boldsymbol{A})=r$，则$\boldsymbol{A}$的一个极大线性无关组含有$r$个向量，不妨设为$\alpha_1,\alpha_2,\cdots,\alpha_r$，那么对于任一$r=1,2,\cdots,n$，有

$$\alpha_i=k_1^{(i)}\alpha_2+k_1^{(i)}\alpha_2+\cdots+k_r^{(i)}\alpha_r,\quad k_j^{(i)}\in\mathbb{P}$$

进而可得

$$\begin{aligned}
f(\varepsilon_i)&=(\eta_1,\eta_2,\cdots,\eta_m)\begin{pmatrix}a_{1i}\\a_{2i}\\\vdots\\a_{mi}\end{pmatrix}\\
&=(\eta_1,\eta_2,\cdots,\eta_m)\alpha_i\\
&=(\eta_1,\eta_2,\cdots,\eta_m)(k_1^{(i)}\alpha_1+k_2^{(i)}\alpha_2+\cdots+k_r^{(i)}\alpha_r)\\
&=\sum_{j=1}^{r}k_j^{(i)}(\eta_1,\eta_2,\cdots,\eta_m)\alpha_i\\
&=\sum_{j=1}^{(i)}f(\varepsilon_j)
\end{aligned}$$

即$\operatorname{Im}f$可以由$f(\varepsilon_1),f(\varepsilon_2),\cdots,f(\varepsilon_r)$生成.
设存在$x_1,x_2,\cdots,x_r\in\mathbb{P}$，使$x_1f(\varepsilon_1)+x_2f(\varepsilon_2)+\cdots+x_rf(\varepsilon_r)=\boldsymbol{\theta}$，则

$$(\eta_1,\eta_2,\cdots,\eta_m)(x_1\alpha_1+x_2\alpha_2+\cdots+x_r\alpha_r)=\boldsymbol{\theta}$$

由于$\eta_1,\eta_2,\cdots,\eta_m$是基且$\alpha_1,\alpha_2,\cdots,\alpha_r$线性无关，从而$x_1=x_2=\cdots=x_r=0$，因此$f(\varepsilon_1),f(\varepsilon_2),\cdots,f(\varepsilon_r)$构成$\operatorname{Im}f$的基，于是$\dim\operatorname{Im}f=r=r(\boldsymbol{A})$. □