Topology - Homework 07

Question 1.

Consider a connected metric space (X, d).

If there are only finite many points in X, let $X = \{x_1, x_2, \cdots, x_n\}, n > 1$.

Let $r_i = \min d(x_i, x_j), 1 \leq j \leq n, j \neq i$.

$B_d(x_1, r_1)$ is an open set and x_1 is the only point in it.

$\bigcup_{i=2}^{n} B_d(x_i, r_i)$ is an open set with other points except x_1.

$X = B_d(x_1, r_1) \bigcup \bigcup_{i=2}^{n} B_d(x_i, r_2)$ is a separation of X and this contradicts with the connectivity of X.

If there are infinite but countable many points in X, let $X = \{x_1, x_2, \cdots\}$.

The set $A = \{d(x_1, x_j) : j \in \mathbb{N}, j > 1\}$ is countable.

But the open interval $(0, d(x_1, x_2))$ is uncountable.

So, there must be some $0 < r < d(x_1, x_2)$ with $r \notin A$.

$X = \{x : d(x, x_1) < r\} \bigcup \{x : d(x, x_1) > r\}$ is a separation of X.

Then we know the metric space having more than one point is uncountable.

Question 2.

Let $B_i = X - A_i$ and then B_i is open and dense since A_i is closed and has an empty interior.

Choose an open set $U \subset X$ and there must be some point $x_1 \in U \cap B_1$ since B_1 is dense.

X is a compact Hausdorff space so that there must be some V_1 with $\overline{V_1} \subset U \cap B_1$.

Similarly, we can obtain $x_2 \in V_1 \cap B_2$ and $\overline{V_2} \subset V_1 \cap B_2$, and so on.

According to the finite intersection property, we know that $\cap_{i=1}^{\infty} \overline{V_i}$ is nonempty and $U \bigcap \cap_{i=1}^{\infty} B_i$ is nonempty.

Since U is chosen arbitrarily, we know that $\cap_{i=1}^{\infty} B_i$ is dense.

Hence $(\cup_i A_i)^\circ = \varnothing$.

Question 3.

(i)

We have

$A_0 = [0, 1]$

$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$

$A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$

...

$A_n = [0, 1] - \cup [\frac{1+3k}{3^n}, \frac{2+3k}{3^n}]$

Each closed interval that constructs A_n is of $\frac{1}{3^n}$ length.

Take arbitrary two distinct x, y from C and there must be some n that makes $\frac{1}{3^n} < |x - y|$. And this shows that x and y belong to different closed intervals.

This means that for every $x \in C$, there must be a neighborhood that containing only x.

Hence C is totally disconnected.

(ii)

A_n is closed for each $n \in \mathbb{N}$.

So $C = \cap A_n$ is also closed and C is a subset of $[0, 1]$, which is compact.

Thus C is compact.

Choose arbitrary $x \in C$ and arbitrary $\epsilon > 0$.

Consider the neighborhood of x, $(x - \epsilon, x + \epsilon)$.

There must be a closed interval containing x and has the length less than 2ϵ as long as we choose $n > \log_3 2\epsilon$.

So there is always some y arbitrarily closed to x and because the arbitrariness of x, X has no isolated points.

As $A_n \subset A_{n-1} \subset \cdots \subset A_0$ and A_i is closed for every i, C is nonempty.

Since C is a Hausdorff space we know that C is uncountable.

Question 4.

(i)

$\mathbb{Q}$ is not locally compact.

Consider 0 and one of its neighborhood $(-\epsilon, \epsilon)$. A compact space containing $(-\epsilon, \epsilon)$ must be closed.

If $[a, b] \cap \mathbb{Q}$ is a space containing $(-\epsilon, \epsilon)$, choose a irrational number c.

$\cup_{i=1}^{\infty}([a, a_i) \cap \mathbb{Q}) \bigcup \cup_{i=1}^{\infty}((b_i, b] \cap \mathbb{Q})$ is a open cover without finite subcover where $\{a_i\}$ is rational number less than c, and $\{b_i\}$ is rational number greater than c.

So $\mathbb{Q}$ is not locally compact.

(ii)

$(\mathbb{R}, \mathcal{T})$ is not locally compact.

Consider an open set U of$(\mathbb{R}, \mathcal{T})$.

If the closure of U is compact, it should be uncountable.

Then there is a strictly increasing infinite sequence, $\{a_i\}_{i \in \mathbb{N}}$.

$\cup_{i \in \mathbb{N}}[a_i, a_{i+1})$ is an open cover without finite subcover,

So no open set has compact closure and $(\mathbb{R}, \mathcal{T})$ is not locally compact.

Question 5.

(i)

Consider a infinite sequence $\{x_n\}_{n=1}^{\infty}$.

If no subsequence of it is convergent, then there must be some index i where $x_n(i)$ keeps oscillating, regardless how many x_n is removed.

This shows that $x_n(i)$ has infinite distinct possible and this contradicts with finiteness of A.

So the product space is sequentially compact.

Since the product space is metrizable, it is compact.

(ii)

The product topology with discrete topology has every subset been open.

So Z_p is closed since its complement is open.

$Z/p^i Z$ is finite, discrete, and compact so that $\prod_{i=1}^{\infty} Z/p^i Z$ is compact.

Z_p is closed subset of $\prod_{i=1}^{\infty} Z/p^i Z$ so that Z_p is compact.

(iii)

Since f_i is a natural projection from $\mathbb{Z}$ to $\mathbb{Z}_p$, there is $\phi_{i+1}(f_{i+1}(a)) = f_i(a)$.

Then we know that f is a surjection and $f(\mathbb{Z})$ is dense in $\mathbb{Z}_p$.