TOPOLOGY - HOMEWORK 03

Due: 2020 March 26th, 10:00AM

Question 1. Let S be a nonempty set.

(i) Consider the discrete metric
$$d : S \times S \longrightarrow \mathbf{R}$$
given by $d(x,x) = 0$ and $d(x,y) = 1$ if $x \neq y$. Show that the topology on S induced from this metric is just $\mathscr{P}(S)$, i.e., every subset of S is an open set in the metric space (S,d).

(ii) Is there always a metric on S such that the induced topology is $\{\emptyset, S\}$?

Question 2. Show that
$$\mathcal{B} = \{(a,b) : a < b \text{ and } a, b \in \mathbf{Q}\}$$
is a basis for the standard topology on $\mathbf{R}$. What is the cardinality of $\mathcal{B}$?

Question 3. Let (S,d) and (M,ρ) be two metric spaces. A function
$$f : S \longrightarrow M$$
is called an isometry if f is bijective and $\rho(f(x), f(y)) = d(x,y)$ for all $x, y \in S$; in this case, $f^{-1} : M \longrightarrow S$ is also an isometry. Prove that there is no isometry between $(\mathbf{R}, d)$ and $(\mathbf{R}^2, \rho)$, where $d(x,y) := |x-y|$ and $\rho(x,y) := \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ are the standard metrics.

Question 4. Fix a set S.

(i) Let $\{\mathcal{T}_i\}_{i \in I}$ be a family of topologies on S index by $i \in I$. Prove that $\bigcap_{i \in I} \mathcal{T}_i$ is a topology on S.

(ii) Give an example of S and two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ such that the set $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on S. Find the unique minimal topology on S which contains $\mathcal{T}_1 \cup \mathcal{T}_2$.

Question 5. A subset of $\mathbf{R}^2$ is called radially open if it contains an open line segment in each direction about each of its points. Consider the set
$$\mathcal{T}_{\mathrm{ro}} = \left\{ S \subset \mathbf{R}^2 : \ S \text{ is radially open} \right\}$$
of radially open sets in $\mathbf{R}^2$.

(i) Prove that $\mathcal{T}_{\mathrm{ro}}$ is a topology for $\mathbf{R}^2$.

(ii) Let $\mathcal{T}$ be the standard topology on $\mathbf{R}^2$ induced from the standard metric, as in Question 3. Is there any relation between $\mathcal{T}$ and $\mathcal{T}_{\mathrm{ro}}$?

(iii) Is there a basis $\mathcal{B} \subset \mathcal{T}_{\mathrm{ro}}$ with $\mathrm{card}(\mathcal{B}) = \aleph_0$? If so, give such a basis.