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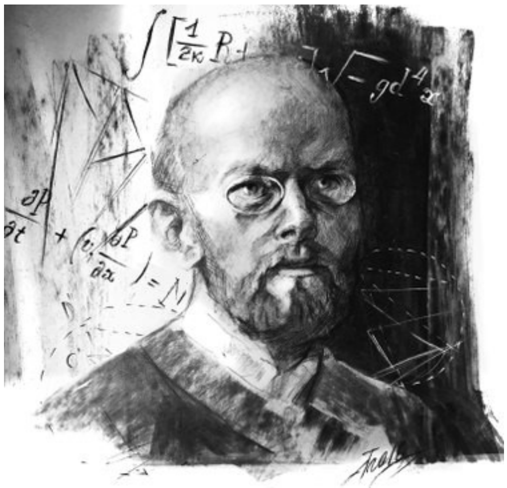
Hilbert's problems



Portrait of David Hilbert in the 1900s, artist Anna Gorban, the cover image of the theme issue. "Hilbert's sixth problem" (http://rsta.royalsocietypublishing.org/content/376/2118). *Phil. Trans. R. Soc. A*. **376**: 2118. 2018.

Hilbert's problems are twenty-three problems in mathematics published by German mathematician David Hilbert in 1900. The problems were all unsolved at the time, and several of them were very influential for 20th-century mathematics. Hilbert presented ten of the problems (1, 2, 6, 7, 8, 13, 16, 19, 21, and 22) at the Paris conference of the International Congress of Mathematicians, speaking on August 8 in the Sorbonne. The complete list of 23 problems was published later, most notably in English translation in 1902 by Mary Frances Winston Newson in the *Bulletin of the American Mathematical Society*.[1]

Contents

Nature and influence of the problems

Hilbert's problems ranged greatly in topic and precision. Some of them are propounded precisely enough to enable a clear affirmative or negative answer, like the 3rd problem, which was the first to be solved, or the 8th problem (the Riemann hypothesis). For other problems, such as the 5th, experts have traditionally agreed on a single interpretation, and a solution to the accepted interpretation has been given, but closely related unsolved problems exist. Sometimes Hilbert's statements were not precise enough to specify a particular problem but were suggestive enough so that certain problems of more contemporary origin seem to apply; for example, most modern number theorists would probably see the 9th problem as referring to the conjectural Langlands correspondence on representations of the absolute Galois group of a number field. Still other problems, such as the 11th and the 16th, concern what are now flourishing mathematical subdisciplines, like the theories of quadratic forms and real algebraic curves.

There are two problems that are not only unresolved but may in fact be unresolvable by modern standards. The 6th problem concerns the axiomatization of physics, a goal that twentieth-century developments of physics (including its recognition as a discipline independent from mathematics) seem to render both more remote and less important than in Hilbert's time. Also, the 4th problem concerns the foundations of geometry, in a manner that is now generally judged to be too vague to enable a definitive answer.

The other twenty-one problems have all received significant attention, and late into the twentieth century work on these problems was still considered to be of the greatest importance. Paul Cohen received the Fields Medal during 1966 for his work on the first problem, and the negative solution of the tenth problem during 1970 by Yuri Matiyasevich (completing work of Martin Davis, Hilary Putnam, and Julia Robinson) generated similar acclaim. Aspects of these problems are still of great interest today.

Ignorabimus

Following Gottlob Frege and Bertrand Russell, Hilbert sought to define mathematics logically using the method of formal systems, i.e., finitistic proofs from an agreed-upon set of axioms.[2] One of the main goals of Hilbert's program was a finitistic proof of the consistency of the axioms of arithmetic: that is his second problem.[a]

However, Gödel's second incompleteness theorem gives a precise sense in which such a finitistic proof of the consistency of arithmetic is provably impossible. Hilbert lived for 12 years after Kurt Gödel published his theorem, but does not seem to have written any formal response to Gödel's work.[b][c]

Hilbert's tenth problem does not ask whether there exists an algorithm for deciding the solvability of Diophantine equations, but rather asks for the *construction* of such an algorithm: "to devise a process according to which it can be determined in a finite number of operations whether the equation is solvable in rational integers." That this problem was solved by showing that there cannot be any such algorithm contradicted Hilbert's philosophy of mathematics.

In discussing his opinion that every mathematical problem should have a solution, Hilbert allows for the possibility that the solution could be a proof that the original problem is impossible.[d] He stated that the point is to know one way or the other what the solution is, and he believed that we always can know this, that in mathematics there is not any "ignorabimus" (statement whose truth can never be known).[e] It seems unclear whether he would have regarded the solution of the tenth problem as an instance of ignorabimus: what is proved not to exist is not the integer solution, but (in a certain sense) the ability to discern in a specific way whether a solution exists.

On the other hand, the status of the first and second problems is even more complicated: there is not any clear mathematical consensus as to whether the results of Gödel (in the case of the second problem), or Gödel and Cohen (in the case of the first problem) give definitive negative solutions or not, since these solutions apply to a certain formalization of the problems, which is not necessarily the only possible one.[f]

The 24th problem

Hilbert originally included 24 problems on his list, but decided against including one of them in the published list. The "24th problem" (in proof theory, on a criterion for simplicity and general methods) was rediscovered in Hilbert's original manuscript notes by German historian Rüdiger Thiele in 2000.[5]

Sequels

Since 1900, mathematicians and mathematical organizations have announced problem lists, but, with few exceptions, these collections have not had nearly as much influence nor generated as much work as Hilbert's problems.

One of the exceptions is furnished by three conjectures made by André Weil during the late 1940s (the Weil conjectures). In the fields of algebraic geometry, number theory and the links between the two, the Weil conjectures were very important. The first of the Weil conjectures was proved by Bernard Dwork, and a completely different proof of the first two conjectures via ℓ-adic cohomology was given by Alexander Grothendieck. The last and deepest of the Weil conjectures (an analogue of the Riemann hypothesis) was proven by Pierre Deligne. Both Grothendieck and Deligne were awarded the Fields medal. However, the Weil conjectures in their scope are more like a single Hilbert problem, and Weil never intended them as a programme for all mathematics. This is somewhat ironic, since arguably Weil was the mathematician of the 1940s and 1950s who best played the Hilbert role, being conversant with nearly all areas of (theoretical) mathematics and having figured importantly in the development of many of them.

Paul Erdős posed hundreds, if not thousands, of mathematical problems, many of them profound. Erdős often offered monetary rewards; the size of the reward depended on the perceived difficulty of the problem.

The end of the millennium, being also the centennial of Hilbert's announcement of his problems, was a natural occasion to propose "a new set of Hilbert problems." Several mathematicians accepted the challenge, notably Fields Medalist Steve Smale, who responded to a request of Vladimir Arnold by proposing a list of 18 problems. Smale's problems have thus far not received much attention from the media, and it is unclear how much serious attention they are getting from the mathematical community.

At least in the mainstream media, the *de facto* 21st century analogue of Hilbert's problems is the list of seven Millennium Prize Problems chosen during 2000 by the Clay Mathematics Institute. Unlike the Hilbert problems, where the primary award was the admiration of Hilbert in particular and mathematicians in general, each prize problem includes a million dollar bounty. As with the Hilbert problems, one of the prize problems (the Poincaré conjecture) was solved relatively soon after the problems were announced.

The Riemann hypothesis is noteworthy for its appearance on the list of Hilbert problems, Smale's list, the list of Millennium Prize Problems, and even in the Weil conjectures, in its geometric guise. Notwithstanding some famous recent assaults from major mathematicians of our day, many experts believe that the Riemann hypothesis will be included in problem lists for many centuries. Hilbert himself declared: "If I were to awaken after having slept for a thousand years, my first question would be: Has the Riemann hypothesis been proven?"[6]

In 2008, DARPA announced its own list of 23 problems that it hoped could cause major mathematical breakthroughs, "thereby strengthening the scientific and technological capabilities of DoD."[7][8]

Summary

Of the cleanly formulated Hilbert problems, problems 3, 7, 10, 11, 14, 17, 19, 20, and 21 have a resolution that is accepted by consensus of the mathematical community. On the other hand, problems 1, 2, 5, 9, 15, 18, and 22 have solutions that have partial acceptance, but there exists some controversy as to whether they resolve the problems.

That leaves 8 (the Riemann hypothesis), 12, 13 and 16[g] unresolved, and 4 and 23 as too vague to ever be described as solved. The withdrawn 24 would also be in this class. Number 6 is deferred as a problem in physics rather than in mathematics.

Table of problems

Hilbert's twenty-three problems are (for details on the solutions and references, see the detailed articles that are linked to in the first column):

Problem	Brief explanation	Status	Year Solved
1st	The continuum hypothesis (that is, there is no set whose cardinality is strictly between that of the integers and that of the real numbers)	Proven to be impossible to prove or disprove within Zermelo–Fraenkel set theory with or without the Axiom of Choice (provided Zermelo–Fraenkel set theory is consistent, i.e., it does not contain a contradiction). There is no consensus on whether this is a solution to the problem.	1940, 1963
2nd	Prove that the axioms of arithmetic are consistent.	There is no consensus on whether results of Gödel and Gentzen give a solution to the problem as stated by Hilbert. Gödel's second incompleteness theorem, proved in 1931, shows that no proof of its consistency can be carried out within arithmetic itself. Gentzen proved in 1936 that the consistency of arithmetic follows from the well-foundedness of the ordinal ε_0.	1931, 1936
3rd	Given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces that can be reassembled to yield the second?	Resolved. Result: No, proved using Dehn invariants.	1900
4th	Construct all metrics where lines are geodesics.	Too vague to be stated resolved or not.[h]	—
5th	Are continuous groups automatically differential groups?	Resolved by Andrew Gleason, depending on how the original statement is interpreted. If, however, it is understood as an equivalent of the Hilbert–Smith conjecture, it is still unsolved.	1953?
6th	Mathematical treatment of the axioms of physics (a) axiomatic treatment of probability with limit theorems for foundation of statistical physics (b) the rigorous theory of limiting processes "which lead from the atomistic view to the laws of motion of continua"	Partially resolved depending on how the original statement is interpreted.[9] Items (a) and (b) were two specific problems given by Hilbert in a later explanation.[1] Kolmogorov's axiomatics (1933) is now accepted as standard. There is some success on the way from the "atomistic view to the laws of motion of continua."[10]	1933–2002?
7th	Is a^b transcendental, for algebraic $a \neq 0,1$ and irrational algebraic b ?	Resolved. Result: Yes, illustrated by Gelfond's theorem or the Gelfond–Schneider theorem.	1934
8th	The Riemann hypothesis ("the real part of any non-trivial zero of the Riemann zeta function is ½") and other prime number problems, among them Goldbach's conjecture and the twin prime conjecture	Unresolved.	—
9th		Partially resolved.[i]	—

Problem	Brief explanation	Status	Year Solved
	Find the most general law of the reciprocity theorem in any algebraic number field.		
10th	Find an algorithm to determine whether a given polynomial Diophantine equation with integer coefficients has an integer solution.	Resolved. Result: Impossible; Matiyasevich's theorem implies that there is no such algorithm.	1970
11th	Solving quadratic forms with algebraic numerical coefficients.	Partially resolved.[11]	—
12th	Extend the Kronecker–Weber theorem on Abelian extensions of the rational numbers to any base number field.	Unresolved.	—
13th	Solve 7th degree equation using algebraic (variant: continuous) functions of two parameters.	Unresolved. The continuous variant of this problem was solved by Vladimir Arnold in 1957 based on work by Andrei Kolmogorov, but the algebraic variant is unresolved.[j]	—
14th	Is the ring of invariants of an algebraic group acting on a polynomial ring always finitely generated?	Resolved. Result: No, a counterexample was constructed by Masayoshi Nagata.	1959
15th	Rigorous foundation of Schubert's enumerative calculus.	Partially resolved.	—
16th	Describe relative positions of ovals originating from a real algebraic curve and as limit cycles of a polynomial vector field on the plane.	Unresolved, even for algebraic curves of degree 8.	—
17th	Express a nonnegative rational function as quotient of sums of squares.	Resolved. Result: Yes, due to Emil Artin. Moreover, an upper limit was established for the number of square terms necessary.	1927
18th	(a) Is there a polyhedron that admits only an anisohedral tiling in three dimensions? (b) What is the densest sphere packing?	(a) Resolved. Result: Yes (by Karl Reinhardt). (b) Widely believed to be resolved, by computer-assisted proof (by Thomas Callister Hales). Result: Highest density achieved by close packings, each with density approximately 74%, such as face-centered cubic close packing and hexagonal close packing.[k]	(a) 1928 (b) 1998
19th	Are the solutions of regular problems in the calculus of variations always necessarily analytic?	Resolved. Result: Yes, proven by Ennio de Giorgi and, independently and using different methods, by John Forbes Nash.	1957
20th	Do all variational problems with certain boundary conditions have solutions?	Resolved. A significant topic of research throughout the 20th century, culminating in solutions for the non-linear case.	?

Problem	Brief explanation	Status	Year Solved
21st	Proof of the existence of linear differential equations having a prescribed monodromic group	Partially resolved. Result: Yes/No/Open depending on more exact formulations of the problem.	?
22nd	Uniformization of analytic relations by means of automorphic functions	Partially resolved.	?
23rd	Further development of the calculus of variations	Too vague to be stated resolved or not.	—

Notes

a. See Nagel and Newman revised by Hofstadter (2001, p. 107),[3] footnote 37: "Moreover, although most specialists in mathematical logic do not question the cogency of [Gentzen's] proof, it is not finitistic in the sense of Hilbert's original stipulations for an absolute proof of consistency." Also see next page: "But these proofs [Gentzen's et al.] cannot be mirrored inside the systems that they concern, and, since they are not finitistic, they do not achieve the proclaimed objectives of Hilbert's original program." Hofstadter rewrote the original (1958) footnote slightly, changing the word "students" to "specialists in mathematical logic". And this point is discussed again on page 109[3] and was not modified there by Hofstadter (p. 108).[3]

b. Reid reports that upon hearing about "Gödel's work from Bernays, he was 'somewhat angry'. ... At first he was only angry and frustrated, but then he began to try to deal constructively with the problem. ... It was not yet clear just what influence Gödel's work would ultimately have" (p. 198–199).[4] Reid notes that in two papers in 1931 Hilbert proposed a different form of induction called "unendliche Induktion" (p. 199).[4]

c. Reid's biography of Hilbert, written during the 1960s from interviews and letters, reports that "Godel (who never had any correspondence with Hilbert) feels that Hilbert's scheme for the foundations of mathematics 'remains highly interesting and important in spite of my negative results' (p. 217). Observe the use of present tense – she reports that Gödel and Bernays among others "answered my questions about Hilbert's work in logic and foundations" (p. vii).[4]

d. This issue that finds its beginnings in the "foundational crisis" of the early 20th century, in particular the controversy about under what circumstances could the Law of Excluded Middle be employed in proofs. See much more at Brouwer–Hilbert controversy.

e. "This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus.*" (Hilbert, 1902, p. 445.)

f. Nagel, Newman and Hofstadter discuss this issue: "The possibility of constructing a finitistic absolute proof of consistency for a formal system such as *Principia Mathematica* is not excluded by Gödel's results. ... His argument does not eliminate the possibility ... But no one today appears to have a clear idea of what a finitistic proof would be like that is *not* capable of being mirrored inside *Principia Mathematica* (footnote 39, page 109). The authors conclude that the prospect "is most unlikely."[3]

g. Some authors consider this problem as too vague to ever be described as solved, although there is still active research on it.

h. According to Gray, most of the problems have been solved. Some were not defined completely, but enough progress has been made to consider them "solved"; Gray lists the fourth problem as too vague to say whether it has been solved.
i. Problem 9 has been solved by Emil Artin in 1927 for Abelian extensions of the rational numbers during the development of class field theory; the non-abelian case remains unsolved, if one interprets that as meaning non-abelian class field theory.
j. It is not difficult to show that the problem has a partial solution within the space of single-valued analytic functions (Raudenbush). Some authors argue that Hilbert intended for a solution within the space of (multi-valued) algebraic functions, thus continuing his own work on algebraic functions and being a question about a possible extension of the Galois theory (see, for example, Abhyankar[12] Vitushkin,[13] Chebotarev,[14] and others). It appears from one of Hilbert's papers[15] that this was his original intention for the problem. The language of Hilbert there is "... Existenz von *algebraischen* Funktionen ...", [existence of *algebraic* functions]. As such, the problem is still unresolved.
k. Gray also lists the 18th problem as "open" in his 2000 book, because the sphere-packing problem (also known as the Kepler conjecture) was unsolved, but a solution to it has now been claimed.

See also

- Landau's problems

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External links

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