TOPOLOGY - HOMEWORK 04

Due: 2020 April 2nd, 10:00AM

For questions below, we assume that $\mathbf{R}^n$ is endowed with the standard topology (the one induced from the metric $d(x,y) = (\sum_{i=1}^n (x_i - y_i)^2)^{1/2}$), unless stated otherwise.

Question 1. Let $X = \mathbf{R}^2$ be endowed with the standard topology. Consider the subset

$$A = \{(x, \sin(1/x)) : \ x \in (0,1]\} \subset X.$$

Compute the following four sets:

$$A^o, \ \overline{A}, \ \mathrm{LP}(A), \ \overline{A} \cap \overline{\mathbf{R}^2 - A}.$$

Question 2. (Kuratowski closure operation) Let X be a set and let $f : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ be a function satisfying the following properties (for any $A, B \in \mathscr{P}(X)$):

(i) $f(\emptyset) = \emptyset$,
(ii) $A \subset f(A)$,
(iii) $f(A) = f(f(A))$,
(iv) $f(A \cup B) = f(A) \cup f(B)$.

Prove that

(1) the set
$$\mathcal{T}_f := \{X - A : \ A \in \mathscr{P}(X) \text{ and } f(A) = A\}$$
is a topology on X;
(2) the closue $\overline{A}$ of any $A \subset X$ with respect to the topology $\mathcal{T}_f$ is just $f(A)$.

Question 3. Consider the function $f : [0,1) \longrightarrow S^1$ given by

$$f(t) = (\cos(2\pi t), \sin(2\pi t)).$$

Show that f is not a homeomorphism with respect to the standard subspace topologies on $[0,1)$ and S^1. Is it possible to find a homeomorphism between $[0,1)$ and S^1?

Question 4. Fix a topological space $(X, \mathcal{T})$. A subset $A \subset X$ is called regularly open if $A = (\overline{A})^o$, i.e., A is equal to the interior of its closure.

(i) Give an example of an open set in $\mathbf{R}$ (with respect to the standard topology) that is not regularly open. Can you characterize the regularly open sets in $\mathbf{R}$?
(ii) Prove that for any $A \subset X$, the set $(\overline{A})^o$ is regularly open.

Question 5. Let A be a subset of a topological space $(X, \mathcal{T})$. Prove the equality $\overline{A} = A \cup \mathrm{LP}(A)$.