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Munkres §13

Ex. 13.1 (Morten Poulsen). Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. The following are equivalent:

(i) $A \in \mathcal{T}$.
(ii) $\forall\, x \in A\, \exists\, U_x \in \mathcal{T} : x \in U_x \subset A$.

Proof. (i) $\Rightarrow$ (ii): If $x \in A$ then $x \in A \subset A$ and $A \in \mathcal{T}$.
(ii) $\Rightarrow$ (i): $A = \bigcup_{x \in A} U_x$, hence $A \in \mathcal{T}$. □

Ex. 13.4 (Morten Poulsen). Note that every collection of topologies on a set X is itself a set: A topologi is a subset of $\mathcal{P}(X)$, i.e. an element of $\mathcal{P}(\mathcal{P}(X))$, hence a collection of topologies is a subset of $\mathcal{P}(\mathcal{P}(X))$, i.e. a set.

Let $\{\mathcal{T}_\alpha\}$ be a nonempty set of topologies on the set X.

(a). Since every $\mathcal{T}_\alpha$ is a topology on X it is clear that the intersection $\bigcap \mathcal{T}_\alpha$ is a topology on X.

The union $\bigcup \mathcal{T}_\alpha$ is in general not a topology on X: Let $X = \{a, b, c\}$. It is straightforward to check that $\mathcal{T}_1 = \{X, \emptyset, \{a\}, \{a, b\}\}$ and $\mathcal{T}_2 = \{X, \emptyset, \{c\}, \{b, c\}\}$ are topologies on X. But $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X, since $\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

(b). The intersection of all topologies that are finer than all $\mathcal{T}_\alpha$ is clearly the smallest topology containing all $\mathcal{T}_\alpha$.

The intersection of all $\mathcal{T}_\alpha$ is clearly the largest topology that is contained in all $\mathcal{T}_\alpha$.

(c). The topology $\mathcal{T}_3 = \mathcal{T}_1 \cap \mathcal{T}_2 = \{X, \emptyset, \{a\}\}$ is the largest topology on X contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

The topology $\mathcal{T}_4 = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ is the smallest topology that contains $\mathcal{T}_1$ and $\mathcal{T}_2$.

Ex. 13.5 (Morten Poulsen). Let $(X, \mathcal{T})$ be a topological space, $\mathcal{A}$ basis for $\mathcal{T}$ and let $\{\mathcal{T}_\alpha\}$ be the set of topologies on X that contains $\mathcal{A}$.

Claim 1. $\mathcal{T} = \bigcap \mathcal{T}_\alpha$.

Proof. "$\subset$": Let $U \in \mathcal{T}$. By lemma 13.1, U is an union of elements of $\mathcal{A}$. Since $\mathcal{T}_\alpha$ is a topology for all α, it follows that $U \in \mathcal{T}_\alpha$ for all α, i.e. $U \in \bigcap \mathcal{T}_\alpha$.
"$\supset$": Clear since $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha \subset \mathcal{T}$. □

Now assume $\mathcal{A}$ is a subbasis.

Claim 2. $\mathcal{T} = \bigcap \mathcal{T}_\alpha$.

Proof. "$\subset$": Let $U \in \mathcal{T}$. By the definition of a subbasis and the remarks at the bottom on page 82, U is an union of finite intersections of elements of $\mathcal{A}$. Since $\mathcal{T}_\alpha$ is a topology for all α, it follows that $U \in \mathcal{T}_\alpha$ for all α, i.e. $U \in \bigcap \mathcal{T}_\alpha$.
"$\supset$": Clear since $\mathcal{A} \subset \bigcap \mathcal{T}_\alpha \subset \mathcal{T}$. □

Ex. 13.6 (Morten Poulsen). The topologies $\mathbf{R}_l$ and $\mathbf{R}_K$ on $\mathbf{R}$ are not comparable:

$\mathbf{R}_l \not\subset \mathbf{R}_K$: Consider $[-1, 0) \in \mathbf{R}_l$. Clearly no basis element $B_K \in \mathbf{R}_K$ satisfy $-1 \in B_K \subset [-1, 0)$, hence $\mathbf{R}_K$ is not finer than $\mathbf{R}_l$, by lemma 13.3.

$\mathbf{R}_K \not\subset \mathbf{R}_l$: Consider $(-1, 1) - K \in \mathbf{R}_K$. Clearly no basis element $B_l \in \mathbf{R}_l$ satisfy $0 \in B_l \subset (-1, 1) - K$, hence $\mathbf{R}_l$ is not finer than $\mathbf{R}_K$, by lemma 13.3.

Ex. 13.7 (Morten Poulsen). We know that $\mathcal{T}_1$ and $\mathcal{T}_2$ are bases for topologies on $\mathbf{R}$. Furthermore $\mathcal{T}_3$ is a topology on $\mathbf{R}$. It is straightforward to check that the last two sets are bases for topologies on $\mathbf{R}$ as well.

The following table show the relationship between the given topologies on $\mathbf{R}$.

	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$
$\mathcal{T}_1$	$=$	$\subset$ (1)	$\not\subset$ (2)	$\subset$ (3)	$\not\subset$ (4)
$\mathcal{T}_2$	$\not\subset$ (5)	$=$	$\not\subset$ (6)	$\subset$ (7)	$\not\subset$ (8)
$\mathcal{T}_3$	$\subset$ (9)	$\subset$ (10)	$=$	$\subset$ (11)	$\not\subset$ (12)
$\mathcal{T}_4$	$\not\subset$ (13)	$\not\subset$ (14)	$\not\subset$ (15)	$=$	$\not\subset$ (16)
$\mathcal{T}_5$	$\subset$ (17)	$\subset$ (18)	$\not\subset$ (19)	$\subset$ (20)	$=$

(1) Lemma 13.3.
(2) Since $\mathbf{R} - (0,1)$ not finite.
(3) Given basis element $(a,b) \in \mathcal{T}_1$ and $x \in (a,b)$ then the basis element $(a,x] \in \mathcal{T}_4$ satisfy $x \in (a,x] \subset (a,b)$, hence $\mathcal{T}_4$ is finer than $\mathcal{T}_1$, by lemma 13.3.
(4) Given a basis element $(a,b) \in \mathcal{T}_1$ and $x \in (a,b)$ then there are clearly no basis element $(-\infty, c) \in \mathcal{T}_5$ such that $x \in (-\infty, c) \subset (a,b)$, hence $\mathcal{T}_5$ is not finer than $\mathcal{T}_1$, by lemma 13.3.
(5) Lemma 13.3.
(6) Since $\mathbf{R} - (0,1)$ not finite.
(7) Given basis element $(a,b) - K \in \mathcal{T}_2$ and $x \in (a,b) - K$. If $x \in (0,1)$ then there exists $m \in \mathbf{Z}_+$ such that $\frac{1}{m} < x < \frac{1}{m-1}$, hence $x \in (\frac{1}{m}, x] \subset (a,b) - K$. If $x \notin (0,1)$ then $x \in (a,x] \subset (a,b) - K$. It follows from (4) and lemma 13.3 that $\mathcal{T}_4$ is finer than $\mathcal{T}_2$.
(8) Since $\mathcal{T}_1 \not\subset \mathcal{T}_5$ and $\mathcal{T}_1 \subset \mathcal{T}_2$.
(9) Let $U \in \mathcal{T}_3$, U nonempty, i.e. $\mathbf{R} - U = \{r_1, \dots, r_n\}$, $r_1 < \cdots < r_n$. Since

$$U = \left(\bigcup_{i=1}^{\infty}(r_1 - i, r_1)\right) \cup \left(\bigcup_{j=1}^{n-1}(r_j, r_{j+1})\right) \cup \left(\bigcup_{k=1}^{\infty}(r_n, r_n + k)\right)$$

it follows that $U \in \mathcal{T}_1$.
(10) Since $\mathcal{T}_3 \subset \mathcal{T}_1 \subset \mathcal{T}_2$.
(11) Let $U \in \mathcal{T}_3$, U nonempty, i.e. $\mathbf{R} - U = \{r_1, \dots, r_n\}$, $r_1 < \cdots < r_n$, and let $x \in U$. If $d = \min\{\, |x - r_i| \mid i \in \{1, \dots, n\}\,\} > 0$ then $x \in (x - \frac{d}{2}, x + \frac{d}{2}] \subset U$. It follows from lemma 13.3 that $\mathcal{T}_4$ is finer than $\mathcal{T}_3$.
(12) Consider $U = \mathbf{R} - \{0\} \in \mathcal{T}_3$. There are no basis element $(-\infty, a) \in \mathcal{T}_5$ such that $1 \in (-\infty, a) \subset U$, hence $\mathcal{T}_5$ is not finer than $\mathcal{T}_4$, by lemma 13.3.
(13) Given basis element $(c,x] \in \mathcal{T}_4$ there is clearly no basis element $(a,b) \in \mathcal{T}_1$ such that $x \in (a,b) \subset (c,x]$, hence $\mathcal{T}_1$ is not finer than $\mathcal{T}_4$, by lemma 13.3.
(14) Given basis element $(c,x] \in \mathcal{T}_4$ there is clearly no basis element $B_K \in \mathcal{T}_2$ such that $x \in B_K \subset (c,x]$, hence $\mathcal{T}_2$ is not finer than $\mathcal{T}_4$, by lemma 13.3.
(15) Since $\mathbf{R} - (0,1]$ not finite.
(16) Given basis element $(c,x] \in \mathcal{T}_4$ there is clearly no basis element $(-\infty, a) \in \mathcal{T}_5$ such that $x \in (-\infty, a) \subset (c,x]$, hence $\mathcal{T}_5$ is not finer than $\mathcal{T}_4$, by lemma 13.3.
(17) Since $(-\infty, a) = \bigcup_{i=1}^{\infty}(a - i, a) \in \mathcal{T}_1$ for all $a \in \mathbf{R}$.
(18) Since $\mathcal{T}_5 \subset \mathcal{T}_1 \subset \mathcal{T}_2$.
(19) Since $\mathbf{R} - (-\infty, 0)$ not finite.
(20) Given basis element $(-\infty, a) \in \mathcal{T}_5$ and $x \in (-\infty, a)$ then clearly $x \in (x - |x-a|, x + \frac{|x-a|}{2}] \subset (-\infty, a)$, hence $\mathcal{T}_4$ is finer than $\mathcal{T}_5$, by lemma 13.3.

Ex. 13.8 (Morten Poulsen).

(a). Let

$$\mathcal{B} = \{\, (a,b) \mid a, b \in \mathbf{Q}, a < b \,\}.$$

It is straightforward to check that $\mathcal{B}$ is a basis. Let $\mathcal{T}$ be the standard topology on $\mathbf{R}$ generated by the basis:

$$\{ (r,s) \mid r,s \in \mathbf{R} \}.$$

Let $U \in \mathcal{T}$ and let $x \in U$. Then (by definition of an open set in a topology generated by a basis) there exists a basis element (r,s), $r,s \in \mathbf{R}$, such that $x \in (r,s)$. Furthermore there exists $a,b \in \mathbf{Q}$ such that $r \leq a < x < b \leq s$, hence $x \in (a,b) \subset (r,s)$. It follows, by lemma 13.2, that $\mathcal{B}$ is a basis for $\mathcal{T}$.

(b). Let

$$\mathcal{C} = \{ [a,b) \mid a,b \in \mathbf{Q}, a < b \}.$$

It is straightforward to check that $\mathcal{C}$ is a basis. Let $\mathcal{T}_{\mathcal{C}}$ be the topology on $\mathbf{R}$ generated by $\mathcal{C}$.

Consider $[\sqrt{2},2) \in \mathbf{R}_l$. There are clearly no basis element $[a,b) \in \mathcal{C}$ such that $\sqrt{2} \in [a,b) \subset [\sqrt{2},2)$, hence $\mathcal{T}_{\mathcal{C}}$ is not finer than $\mathbf{R}_l$, by lemma 13.3.

Since $\mathbf{R}_l$ is clearly finer than $\mathcal{T}_{\mathcal{C}}$, it follows that $\mathbf{R}_l$ is strictly finer than $\mathcal{T}_{\mathcal{C}}$.

References