Topology - Homework 09

Question 1.

Consider a locally compact Hausdorff space X, then for arbitrary $x \in X$ and its arbitrary neighborhood U, there exists a neighborhood V such that $\overline{V}$ is compact and $\overline{V} \subset U$.

Consider a closed set $B \subset X$ not containing x.

Since $x \notin B = \overline{B}$, is not a limit point of B, and then there is a neighbor of x containing no point of B.

Denote this neighborhood by U and find another neighborhood of x, V with $\overline{V} \subset U$. According to the property of locally compact Hausdorff space, we know that can be done.

Then V and $X - \overline{V}$ are two open sets that contain x and B respectively and have no intersection, which means that X is regular.

Question 2.

(i)

Consider arbitrary distinct $x, y \in X$.

Since X is T_3, we know that X is Hausdorff and there are open sets U containing x and V containing y with U and V have no intersection.

Since X is regular, for every $p \in X$ and an open set A containing p, let $B = X - A$ and B is a closed set not containing p. Then there are open set W_1 and W_2 containing p and B respectively and $\overline{W_1} \subset A$.

Then we know there is some open set U_1 containing x such that $\overline{U_1} \subset U$, and there is some open set V_1 containing y such that $\overline{V_1} \subset V$.

Since U and V have no intersection, we know that U_1 and V_1 have no intersection.

And then U_1 and V_1 are what we need.

(ii)

Since X is T_4, we know that X is normal.

Considering arbitrary closed set $A \subset X$ and an open set U containing A, let $U_1 = X - U$ and then U_1 is closed.

For closed set A and U_1, we can find two open sets W_1 and U_2 containing A and U_1 respectively with $W_1 \cap U_2 = \varnothing$.

Then W_1 is an open set containing A and $\overline{W_1} \subset U$.

Similarly, for a closed set B disjoint with A and an open set V containing B, we can find an open set W_2 containing B and $\overline{W_2} \subset V$.

Since X is normal, we can make U and V have no intersection and then W_1 and W_2 will have no intersection.

Then W_1 and W_2 is what we need.

Question 3.

"if":

Assume that d is a metric.

Then for arbitrary distinct $x, y \in X$, there is $r = d(x, y) > 0$.

Then $B(x, r)$ is an open set containing x but not containing y.

This shows that X is T_0.

“only if”:

Assume that X is T_0.

If there are distinct $x, y \in X$ with $d(x, y) = 0$, all open sets containing x will also contain y, which makes contradiction.

Then we know every two distinct points in X have an positive distance, and d is a metric.

Question 4.

(i)

Consider a completely regular space X and arbitrary $x \in X$, arbitrary closed set A not containing x.

There is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$.

Let $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$.

Then U and V have no intersection and are both open since $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$ are open.

Obviously, $x \in U$ and $A \subset V$.

Then we know that X is regular.

(ii)

Consider a pseudo metric space X.

For arbitrary $x \in X$ and arbitrary closed set A not containing x.

Define $d(p, A) \triangleq \inf_{a \in A} d(a, p)$ for $p \in X$.

Since A is closed, if there is some $y \in A$ such that $d(x, y) = 0$, every neighborhood of x will containing some points in A, because a neighborhood of x is also a neighborhood of y. Then we know that x is a limit point of A and $x \in A$, which contradicts with $x \notin A$.

Thus we have $d(x, A) > 0$.

Let $g(p) = \frac{d(p,A)}{d(x,A)}$.

$g(x) = 1, g(A) = \{0\}$.

Since the metric function $d_A(p) \triangleq d(p, A)$ is continuous, we know that $g(p)$ is continuous and $f(p) \triangleq 1 - g(p)$ is also continuous.

$f(x) = 0, f(A) = \{1\}$.

Then we know that X is completely regular.

Metric space implies T_1 and pseudo pseudo metric. So every metric space is Tychonoff.

Question 5.

Let $\mathcal{T}$ be the standard topology on $\mathbb{R}^2$ and we have $\mathcal{T}_{ro} \supset \mathcal{T}$.

So an open set with respect to $\mathcal{T}$ is also open with respect to $\mathcal{T}_{ro}$, and a closed set with respect to $\mathcal{T}$ is also closed with respect to $\mathcal{T}_{ro}$.

Consider the closed set $L : \{(x, x) \in \mathbb{R}^2\}$.

Each subset of L is closed.

Let D be a subset of $\mathbb{R}^2$ with rational coordinates and then D is dense in $\mathbb{R}^2$.

Define $f : \mathcal{P}(L) \to \mathcal{P}(D)$:

$f(A) = D \cap U$, if A is a true subset of L,

$f(\varnothing) = \varnothing$,

$f(L) = D$.

If B is a true subset of D and $B \neq A$, assume $x \in A$ and $x \notin B$.

Then we have $x \in L - B$.

Assume that $\mathbb{R}^2$ is normal with respect to $\mathcal{T}_{ro}$, we can find two open set U_B and V_B containing B and $L - B$ respectively and have no intersection.

Then $x \in U_A \cap V_B$. This nonempty open set must containing some points in D, which belongs to U_A but doesn't belong to U_B. Hence $D \cap U_A \neq D \cap U_B$, and f is an injection.

D is countable and L has the same cardinality with $\mathbb{R}$ (uncountable).

So there is a injection from $\mathcal{P}(D)$ to L.

And then there is a injection from $\mathcal{P}(L)$ to L.

But this is impossible, which means a contradiction.

Then we know that $(\mathbb{R}^2, \mathcal{T}_{ro})$ is not normal.

Since a metrizable space must be normal, we know that $(\mathbb{R}^2, \mathcal{T}_{ro})$ is not metrizable.