TOPOLOGY - HOMEWORK 09

Due: 2020 May 21st, 10:00AM

Question 1. Show that every locally compact Hausdorff space is regular, i.e., "local compactness + $T_2 \Rightarrow T_3$".

Question 2. Prove the following for a topological space $(X, \mathcal{T})$:

(i) if X is T_3, then for every distinct $x, y \in X$, there exist open sets U, V such that $x \in U, y \in V$ and $\overline{U} \cap \overline{V} = \emptyset$;

(ii) if X is T_4, then for every disjoint closed sets $A, B \subset X$, there exist open sets U, V such that $A \subset U, B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$.

Question 3. Let (X, d) be a pseudometric space. Let $\mathcal{T}_d$ be the topology on X generated by

$$\{B(x, \epsilon) : \; x \in X, \epsilon \in \mathbf{R}_{>0}\},$$

where $B(x, \epsilon) := \{y \in X : d(x, y) < \epsilon\}$. Show that $(X, \mathcal{T}_d)$ is T_0 if and only if d is a metric.

Question 4. A topological space X is called *completely regular* if for every closed set $A \subset X$ and $x \notin A$, there is a continuous function $f : X \longrightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$; it is called *Tychonoff* if it is T_1 and completely regular. Show that

(i) every completely regular space is regular;

(ii) every pseudometric space (X, d) is completely regular, and thus every metric space is Tychonoff.

Question 5. Consider the space $(\mathbf{R}^2, \mathcal{T}_{\text{ro}})$ from Homework 3 Question 5. Show that $(\mathbf{R}^2, \mathcal{T}_{\text{ro}})$ is not normal and thus not metrizable.