TOPOLOGY - HOMEWORK 12

Due: 2020 June 11th, 10:00AM

Question 1. Let (X, d) be a complete metric space. Fix $\alpha \in (0,1)$. Let $f_\alpha : X \longrightarrow X$ be a function satisfying

$$d(f_\alpha(x), f_\alpha(y)) \leq \alpha \cdot d(x, y)$$

for every $x, y \in X$. Show that there is exactly one point $x \in X$ such that $f_\alpha(x) = x$.

Question 2. Let (X, d) be a metric space and $A \subset X$ a dense subset such that every Cauchy sequence in A converges to a point in X. Show that (X, d) is complete.

Question 3. (Completeness is not a topological invariant.) Show that there are two metrics d_1 and d_2 on $X = (0, 1)$ such that the following holds:

(i) the topologies on X induced from d_1 and d_2 are equal,
(ii) X is complete with respect to d_1 and incomplete with respect to d_2.

(Hint: $\mathbf{R}$ is complete with respect to the standard metric.)

Question 4. Let $(X, \mathcal{T})$ be a compact topological space. Consider the complete metric space $(C(X, \mathbf{R}), \rho)$, where $\rho(f, g) := \sup_{x \in X} |f(x) - g(x)|$. Let $\{f_i\}_{i \in \mathbf{N}} \subset C(X, \mathbf{R})$ be a seqeunce of continuous functions satisfying the following:

(i) for each $x \in X$, the sequence $\{f_i(x)\}_i$ converges in $\mathbf{R}$, and the function $f(x) := \lim_{i \to \infty} f_i(x)$ thus obtained is continuous;
(ii) $f_i(x) \geq f_{i+1}(x)$ for all $x \in X$ and $i \in \mathbf{N}$.

Show that f_i converges to f with respect to ρ.

Question 5.

(Hint: state a question that you were not able to solve previously and solve it this time – please give yourself a recognition this way. If you have no such a question, give yourself some applause and put "None" there.)