Topology - Homework 02

Question 1:

Proof:

(i) A rational number can be written in the form p/q where p and q are both integers. Then There exist a surjective map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$. As $\mathbb{Z}$ is easy to be proved to be countable, $\mathbb{Z} \times \mathbb{Z}$ is countable too. Then we know that $\mathbb{Q}$ is countable.

(ii) Consider the interval $(0, 1)$. Real numbers in $(0, 1)$ can be written in decimal. Assume that all real numbers in $(0, 1)$ can be permutated in order:

$$\begin{gathered} 0.a_{11}a_{12}a_{13}\cdots \\ 0.a_{21}a_{22}a_{23}\cdots \\ 0.a_{31}a_{31}a_{33}\cdots \end{gathered}$$

Consider a decimal:

$$0.b_1b_2b_3\cdots$$

with $b_1 \neq a_{11}, b_2 \neq a_{22}, b_3 \neq a_{33}, \cdots$, and such a real number is not included in above permutation.

Then we know that there doesn't exist an surjection from $\mathbb{Z}_+$ to $\mathbb{R}$ and $\mathbb{R}$ is uncountable.

Question 2:

Proof:

Since $\mathbb{Q}$ is countable, we can obtain that $\mathbb{Q}, \mathbb{Q}^2, \cdots, \mathbb{Q}^n, \cdots$ are all countable, and then $\bigcup_{n=1}^{\infty} \mathbb{Q}^n$ is countable, too.

As the coefficients are defined, a polynomial equation is defined, which means we can find a surjection from $\bigcup_{n=1}^{\infty} \mathbb{Q}^n$ to the set of all polynomial equations with rational coefficients. This tells us that $\bigcup_{n=1}^{\infty} \mathbb{Q}^n$ is countable. Since every polynomial equation has only finitely many roots, the set of all such roots is countable, too.

Then we know that $\mathcal{A}$ is countable.

If $\mathbb{R} - \mathcal{A}$ is countable, $\mathcal{A} \bigcup \mathbb{R} - \mathcal{A}$ should be countable, which makes contradictions.

Then we obtain that $\mathbb{R} - \mathcal{A}$ is uncountable.

Question 3:

Proof:

Consider the set $\{f \in X^N : card(supp(f)) = i\}$, every function's support contains i different natural numbers. As there are only two elements in the range of f, the elements in $supp(f)$ can identify an unique f. And the support of f can be surjectively mapped by $\mathbb{N}^i$ since it contains i elements.

So there exists a surjection from $\mathbb{N}^i$ to the set $\{f \in X^N : card(supp(f)) = i\}$.

$\bigcup_{i=0}^{\infty} \mathbb{N}^i$ is countable.

Then we obtain that the set $\{f \in X^N : supp(f)\ is\ finite\}$, X_0^N is countable.

Question 4:

Proof:

(i) $d(x, x) = 0 \rightarrow x \sim x$

If $d(x, y) = 0$, then $d(y, x) = 0$. $\rightarrow$ If $x \sim y$, then $y \sim x$.

if $d(x, y) = 0, d(y, z) = 0$, then $d(x, z) \leq d(x, y) + d(y, z) = 0$, $d(x, z) = 0$. $\rightarrow$ If $x \sim y, y \sim z$, then $x \sim z$.

Then we know that $\sim$ is an equivalence relation on S.

(ii) $d(x, y) \geq 0 \rightarrow d^*([x], [y]) \geq 0$

$d(x, y) = d(y, x) \rightarrow d^*([x], [y]) = d^*([y], [x])$

$d(x, y) \leq d(x, z) + d(z, y) \rightarrow d^*([x], [y]) \leq d^*([x], [z]) + d^*([z], [y])$

Then we know that d^* is a well-defined metric on S^*.

Question 5:

Can.

Choose arbitrary element $p \in S$, and define:

$$d^{\natural}(A,B) = \begin{cases} d(a,b),\ if\ A=\{a\}, B=\{b\} \\ 1,\ if\ A\ and\ B\ both\ have\ not\ exactly\ one\ element\ and\ A \neq B \\ 1+d(a,p),\ if\ A=\{a\}\ and\ B\ has\ not\ exactly\ one\ element \\ 0,\ if\ A=B \end{cases}$$

Obviously, this definition satisfies following properties:
$d^{\natural}(A,B) = d^{\natural}(B,A)$
$d^{\natural}(A,B) \geq 0\ and\ d^{\natural}(A,A) = 0$
$d^{\natural}(\{a\},\{b\}) = d(a,b)$

Consider $A, B, C \in \mathcal{P}(S)$,

case 1: $A = \{a\}, B = \{b\}$
If $C = \{c\}$, then $d^{\natural}(A,B) = d(a,b) \leq d(a,c) + d(c,b) = d^{\natural}(A,B) + d^{\natural}(C,B)$.
If $card(C) \neq 1$, then $d^{\natural}(A,B) = d(a,b) \leq 1 + d(p,a) + 1 + d(p,b) = d^{\natural}(A,C) + d^{\natural}(B,C)$.

case 2: $A = \{a\}$, but $card(B) \neq 1$
If $C = \{c\}$, then $d^{\natural}(A,B) = 1 + d(p,a) \leq d(a,c) + 1 + d(p,c) = d^{\natural}(A,C) + d^{\natural}(C,B)$.
If $card(c) \neq 1$, then $d^{\natural}(A,B) = 1 + d(p,a) \leq 1 + d(p,a) + 1 = d^{\natural}(A,C) + d^{\natural}(C,B)$.

case 3: $card(A) \neq 1, card(B) \neq 1$
If $C = \{c\}$, then $d^{\natural}(A,B) = 1 \leq 1 + d(p,c) + 1 + d(p,c) = d^{\natural}(A,C) + d^{\natural}(C,B)$.
If $card(C) \neq 1$, then $d^{\natural}(A,B) = 1 \leq 1 + 1 = d^{\natural}(A,C) + d^{\natural}(C,B)$.

Then we know that $d^{\natural}(A,B) \leq d^{\natural}(A,C) + d^{\natural}(C,B)$ for all $A, B, C \in \mathcal{P}(S)$, which means $d^{\natural}$ is a metric on $\mathcal{P}(S)$.