TOPOLOGY - MIDTERM TEST

Time: 2020 April 23th, 7:40AM–9:40AM

Question 1. Let $\mathcal{T}$ be the (non-standard) topology on $\mathbf{R}$ generated by the basis

$$\mathcal{B} := \{(a, b] : \ a, b \in \mathbf{R} \text{ and } a < b\}.$$

Compute the closures in $\mathbf{R}$ of the following three subsets with respect to $\mathcal{T}$:

$$\mathbf{Q}, \quad \mathbf{Z}, \quad \{1/n : n \in \mathbf{N}\}.$$

Question 2. Fix a topological space $(X, \mathcal{T})$. A subset $A \subset X$ is called nowhere dense if $(\overline{A})^o = \emptyset$. Prove that a set $A \subset X$ is closed and nonwhere dense if and only if

$$A = \overline{B} \cap \overline{X - B}$$

for some open set $B \subset X$.

Question 3. Let A, B be two disjoint compact subsets in a Hausdorff topological space $(X, \mathcal{T})$. Show that there exist disjoint open sets U and V containing A and B respectively.

Question 4. In a topological space $(X, \mathcal{T})$, a chain of open sets connecting $a \in X$ to $b \in X$ is a sequence

$$U_1, U_2, ..., U_r$$

of open sets such that $a \in U_1$ only, $b \in U_r$ only and $U_i \cap U_{i+1} \neq \emptyset$ for every $1 \leq i \leq r - 1$. Assume that X is connected and let $\mathcal{A} := \{A_j : \ j \in J\}$ be an open cover of X. Show that every two points in X can be connected by a chain of open sets consisting of elements in $\mathcal{A}$.

(Hint: fixing $a \in X$, show that the set of points in X that can be connected to a by chains of open sets is both open and closed.)

Question 5. Denote by $|x|$ the usual absolute value of $x \in \mathbf{R}$. Fix a bijection $r : \mathbf{N} \to \mathbf{Q}$ and write $r_i := r(i)$. Consider the function $d : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$d(x, y) := |x - y| + \sum_{i \in \mathbf{N}} \frac{1}{2^i} \min \left\{ 1, \left| \max_{j \leq i} |x - r_j|^{-1} - \max_{j \leq i} |y - r_j|^{-1} \right| \right\},$$

where by default we take $\infty - \infty = \infty$. Assuming that d is a metric on $\mathbf{R}$, show that $\mathbf{Q} \subset \mathbf{R}$ is an open set with respect to the topology on $\mathbf{R}$ induced from d.

(Hint: show that for every $r \in \mathbf{Q}$, if $\epsilon > 0$ is small enough, then $B_d(r, \epsilon) = \{r\}$.)