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Alexandroff extension

In the mathematical field of topology, the **Alexandroff extension** is a way to extend a noncompact topological space by adjoining a single point in such a way that the resulting space is compact. It is named for the Russian mathematician Pavel Alexandroff. More precisely, let X be a topological space. Then the Alexandroff extension of X is a certain compact space X^* together with an open embedding $c : X \to X^*$ such that the complement of X in X^* consists of a single point, typically denoted ∞. The map c is a Hausdorff compactification if and only if X is a locally compact, noncompact Hausdorff space. For such spaces the Alexandroff extension is called the **one-point compactification** or **Alexandroff compactification**. The advantages of the Alexandroff compactification lie in its simple, often geometrically meaningful structure and the fact that it is in a precise sense minimal among all compactifications; the disadvantage lies in the fact that it only gives a Hausdorff compactification on the class of locally compact, noncompact Hausdorff spaces, unlike the Stone–Čech compactification which exists for any topological space, a much larger class of spaces.

Contents

Example: inverse stereographic projection

A geometrically appealing example of one-point compactification is given by the inverse stereographic projection. Recall that the stereographic projection S gives an explicit homeomorphism from the unit sphere minus the north pole (0,0,1) to the Euclidean plane. The inverse stereographic projection $S^{-1} : \mathbb{R}^2 \hookrightarrow S^2$ is an open, dense embedding into a compact Hausdorff space obtained by adjoining the additional point $\infty = (0,0,1)$. Under the stereographic projection latitudinal circles $z = c$ get mapped to planar circles $r = \sqrt{(1+c)/(1-c)}$. It follows that the deleted neighborhood basis of $(1,0,0)$ given by the punctured spherical caps $c \leq z < 1$ corresponds to the complements of closed planar disks $r \geq \sqrt{(1+c)/(1-c)}$. More qualitatively, a neighborhood basis at ∞ is furnished by the sets $S^{-1}(\mathbb{R}^2 \setminus K) \cup \{\infty\}$ as K ranges through the compact subsets of $\mathbb{R}^2$. This example already contains the key concepts of the general case.

Motivation

Let $c : X \hookrightarrow Y$ be an embedding from a topological space X to a compact Hausdorff topological space Y, with dense image and one-point remainder $\{\infty\} = Y \setminus c(X)$. Then $c(X)$ is open in a compact Hausdorff space so is locally compact Hausdorff, hence its homeomorphic preimage X is also locally compact Hausdorff. Moreover, if X were compact then $c(X)$ would be closed in Y and hence not dense. Thus a space can only admit a one-point compactification if it is locally compact, noncompact and Hausdorff. Moreover, in such a one-point compactification the image of a neighborhood basis for x in X gives a neighborhood basis for $c(x)$ in $c(X)$, and—because a subset of a compact Hausdorff space is compact if and only if it is closed—the open neighborhoods of ∞ must be all sets obtained by adjoining ∞ to the image under c of a subset of X with compact complement.

The Alexandroff extension

Put $X^* = X \cup \{\infty\}$, and topologize X^* by taking as open sets all the open subsets U of X together with all sets $V = (X \setminus C) \cup \{\infty\}$ where C is closed and compact in X. Here, $X \setminus C$ denotes setminus. Note that V is an open neighborhood of $\{\infty\}$, and thus, any open cover of $\{\infty\}$ will contain all except a compact subset C of X^*, implying that X^* is compact (Kelley 1975, p. 150).

The inclusion map $c : X \to X^*$ is called the **Alexandroff extension** of X (Willard, 19A).

The properties below all follow from the above discussion:

- The map c is continuous and open: it embeds X as an open subset of X^*.
- The space X^* is compact.
- The image $c(X)$ is dense in X^*, if X is noncompact.
- The space X^* is Hausdorff if and only if X is Hausdorff and locally compact.

The one-point compactification

In particular, the Alexandroff extension $c : X \to X^*$ is a Hausdorff compactification of X if and only if X is Hausdorff, noncompact and locally compact. In this case it is called the **one-point compactification** or **Alexandroff compactification** of X. Recall from the above discussion that any compactification with one point remainder is necessarily (isomorphic to) the Alexandroff compactification.

Let X be any noncompact Tychonoff space. Under the natural partial ordering on the set $\mathcal{C}(X)$ of equivalence classes of compactifications, any minimal element is equivalent to the Alexandroff extension (Engelking, Theorem 3.5.12). It follows that a noncompact Tychonoff space admits a minimal compactification if and only if it is locally compact.

Further examples

Compactifications of discrete spaces

- The one-point compactification of the set of positive integers is homeomorphic to the space consisting of $K = \{0\} \cup \{1/n \mid n$ is a positive integer$\}$ with the order topology.
- A sequence $\{a_n\}$ in a topological space X converges to a point a in X, if and only if the map $f\colon \mathbb{N}^* \to X$ given by $f(n) = a_n$ for n in $\mathbb{N}$ and $f(\infty) = a$ is continuous. Here $\mathbb{N}$ has the discrete topology.
- Polyadic spaces are defined as topological spaces that are the continuous image of the power of a one-point compactification of a discrete, locally compact Hausdorff space.

Compactifications of continuous spaces

- The one-point compactification of n-dimensional Euclidean space $\mathbf{R}^n$ is homeomorphic to the n-sphere S^n. As above, the map can be given explicitly as an n-dimensional inverse stereographic projection.
- The one-point compactification of the product of κ copies of the half-closed interval [0,1), that is, of $[0,1)^\kappa$, is (homeomorphic to) $[0,1]^\kappa$.
- Since the closure of a connected subset is connected, the Alexandroff extension of a noncompact connected space is connected. However a one-point compactification may "connect" a disconnected space: for instance the one-point compactification of the disjoint union of κ copies of the interval (0,1) is a wedge of κ circles.
- Given X compact Hausdorff and C any closed subset of X, the one-point compactification of $X \setminus C$ is X/C, where the forward slash denotes the quotient space.[1]
- If X and Y are locally compact Hausdorff, then $(X \times Y)^* = X^* \wedge Y^*$ where $\wedge$ is the smash product. Recall that the definition of the smash product: $A \wedge B = (A \times B)/(A \vee B)$ where $A \vee B$ is the wedge sum, and again, / denotes the quotient space.[1]

As a functor

The Alexandroff extension can be viewed as a functor from the category of topological spaces with proper continuous maps as morphisms to the category whose objects are continuous maps $c\colon X \to Y$ and for which the morphisms from $c_1\colon X_1 \to Y_1$ to $c_2\colon X_2 \to Y_2$ are pairs of continuous maps $f_X\colon X_1 \to X_2,\ f_Y\colon Y_1 \to Y_2$ such that $f_Y \circ c_1 = c_2 \circ f_X$. In particular, homeomorphic spaces have isomorphic Alexandroff extensions.

See also

- Wallman compactification
- End (topology)
- Riemann sphere
- Normal space
- Stereographic projection
- Pointed set

References

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