Topology - Homework 05

Question 1.

(i)

$d^*(x,x) = \min\{d(x,x),1\} = \min\{0,1\} = 0$

$d^*(x,y) = \min\{d(x,y),1\} = \min\{d(y,x),1\} = d^*(y,x)$

$$
\begin{aligned}
&d^*(x,y) + d^*(y,z) \\
&= \min\{d(x,y),1\} + \min\{d(y,z),1\} \\
&\geq \min\{d(x,y)+d(y,z), d(x,y)+1, d(y,z)+1, 2\} \\
&\geq \min\{d(x,z),1\} \\
&= d^*(x,z)
\end{aligned}
$$

Then we know that d^* is a metric on X.

(ii)

Consider arbitrary x in X, every basis element contains x include an ϵ-ball based on x.

So all ϵ-balls with $\epsilon < 1$ constitute a basis of a metric topology.

d and d^* have the same ϵ-balls with $\epsilon < 1$, therefore, they induce the same metric topology.

Then we know that $\mathcal{T}^*$ is equal to $\mathcal{T}$.

Question 2.

For the box topology:

Let (x_n) be a point in R_f and suppose that there is a basic box open set $U = \prod U_n$ such that $(x_n) \in U \subset R_f$, with all U_n open in R and $x_n \in U_n$. This means that for each n there is an $r_n > 0$ such that $(x_n - r_n, x_n + r_n) \subset U_n$.

As (x_n) is eventually zero, there is an index N such that $x_n = 0$ for all $n > N$. Then the sequence (y_n) with $y_n = x_n$ for $n \leq N$ and $y_n = \frac{r_n}{2}$ for $n > N$ is not eventually zero but is in U, which is contradicted with $U \subset R_f$.

Then we know that no point of R_f is an interior point in the box topology, that is, the interior of R_f in the box topology is the empty set.

Every basic open set is of this form:

$U = U_1 \times U_2 \times \cdots \times U_n \times U_{n+1} \times \cdots$.

Take arbitrary $x \in R^\omega - R_f$, and then $x_n \neq 0$ for infinitely many n. When

$$U_n = \begin{cases} (x_n - \frac{|x_n|}{2}, x + \frac{|x_n|}{2}) & x_n \neq 0 \\ (-1, 1) & x_n = 0 \end{cases}$$

if $U = \prod U_i$, then $x \in U$ and $U \bigcap R_f = \varnothing$.

Thus, $R^\omega - R_f$ is an open set in R^ω, that is, R_f is closed, and $\overline{R_f} = R_f$.

For the product topology:

The box topology is finer than the product topology. So there is also no point in R_f being an interior point in the product topology.

Then we know the interior of R_f in the product topology is also the empty set.

Every basic open set of R^ω is of this form:

$U = U_1 \times U_2 \times \cdots \times U_n \times R \times R \times \cdots$

Take arbitrary $x \in R^\omega$, and any basic open set U containing x.

Let $y = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$.

We have $y \in U$ since U only cares about the first n components.

Then we have $x \in \overline{R_f}$, that is $\overline{R_f} = R^\omega$.

Question 3.

$(R^\omega, \mathcal{T}_p)$ is metrizable.

Consider the function $d^\omega((x_i), (y_i)) = \sup\{\frac{d^*(x_i, y_i)}{i} : i \in N\}$ where $d^*(x, y) = \min\{d(x, y), 1\}$.

$\frac{d^*(x_i, z_i)}{i} \leq \frac{d^*(x_i, y_i)}{i} + \frac{d^*(y_i, z_i)}{i} \leq d^\omega((x_i), (y_i)) + d^\omega((y_i), (z_i))$

$d^\omega((x_i), (x_i)) = \sup\{\frac{d^*(x_i, x_i)}{i} : i \in N\} = 0$

$d^\omega((x_i), (y_i)) = \sup\{\frac{d^*(x_i, y_i)}{i} : i \in N\} = \sup\{\frac{d^*(y_i, x_i)}{i} : i \in N\} = d^\omega((y_i), (x_i))$

Then we know that d^ω is a metric on R^ω.

Let U be an open set of the metric topology induced from d^ω and $x \in U$.

Take an open set $V \in \mathcal{T}_p$ s.t. $x \in V \subset U$.

Take an ϵ-ball $B_D(x, \epsilon)$ from U. Choose enough great N with $\frac{1}{N} < \epsilon$.

Let V be the basis element of $\mathcal{T}_p$ and $V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times R \times R \times \cdots$

As for arbitrary $y \in R^\epsilon$ and $i \geq N$, there is $\frac{d^*(x_i, y_i)}{i} \leq \frac{1}{N}$.

Thus, we have $d^\omega(x, y) \leq \max\{\frac{d^*(x_1, y_1)}{1}, \frac{d^*(x_2, y_2)}{2}, \cdots, \frac{d^*(x_N, y_N)}{N}, \frac{1}{N}\}$.

If $y \in V$, then $d^\omega(x, y) < \epsilon$ and $V \subset B_D(x, \epsilon)$.

If $y \notin V$, then consider a basis element of $\mathcal{T}_p$ $U = \prod U_i$ with U_i being a open set in R when $i = \alpha_1, \cdots, \alpha_n$ and $U_i = R$ for other i.

Given $x \in U$, take an open set V from the metric topology s.t. $x \in V \subset U$.

Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ with $i = \alpha_1, \cdots, \alpha_n$.

Let $\epsilon \leq 1$ and define $\epsilon = \min\{\frac{\epsilon}{i} : i = \alpha_1, \cdots, \alpha_n\}$.

Take arbitrary $y \in B_D(x, \epsilon)$, as for every i, we have $\frac{d^*(x_i, y_i)}{i} \leq d^\omega(x, y) < \epsilon$.

Since $i = \alpha_1, \cdots, \alpha_n$, we have $d^\omega(x_i, y_i) < \epsilon_i < \epsilon$, $|x_i - y_i| < \epsilon_i$ and $y \in \prod U_i$.

Then we have $x \in B_D(x, \epsilon) \subset U$.

Now we know d^ω is a metric that induces $\mathcal{T}_p$ and $(R^\omega, \mathcal{T}_p)$ is metrizable.

$(R^\omega, \mathcal{T}_b)$ is not metrizable.

Question 4.

X is disconnected. Since

$\{(x, 0) : x \in R\} \bigcap \{(x, \frac{1}{x}) : x \in R^+\} = \varnothing$ and each of the two set is open in R^2.

Question 5.

Proof:

Consider arbitrary $a \in R^\omega$, and an open set $V = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots$.

If $a \in U$, then V consists of bounded sequences.

If $a \notin U$, then V consists of unbounded sequences.

Thus, U and $R^\omega - U$ form a separation of R^ω with respect to $\mathcal{T}_b$, that is, $(R^\omega, \mathcal{T}_b)$ is disconnected.