Topology - Homework 01

Question 1:

Proof:

We know that $\mathcal{P}(A)$ and 2^A both has 2^n elements since A is a nonempty finite set and assumed to has n elements.

Let S be a subset of A and a be a element of A, then $S \in \mathcal{P}(A)$,

Consider such a map:
if $a \in S$, the function mapped by S has 1 mapped by a, and if $a \notin S$, the function mapped by S has -1 mapped by a. Obviously such a function belongs to 2^A.

Since the elements in S are distinct, the function mapped by S is also distinct. Then we know the map is injective. The two set has the same number of elements, which means the map is surjective.

Then we know the map is bijective.

Question 2:

Proof:

Closure:
Since A is a nonempty finite set, $g, f : A \to A$, and g, f are bijective, we know $g \circ f : A \to A$.
Then $g \circ f \in \mathcal{F}(A)$.

Associativity:
Consider $f, g, h \in \mathcal{F}(A)$, and arbitrary $a \in A$, $f(a) = b, g(b) = c, h(c) = d$.
$g \circ f(a) = c, h \circ g(b) = d$
$h \circ (g \circ f)(a) = h(c) = d, (h \circ g) \circ f(a) = h \circ g(b) = d$
$h \circ (g \circ f) = (h \circ g) \circ f$

Identity element:
Consider $f \in \mathcal{F}(A)$, and $f(a) = a$.
For arbitrary $g \in \mathcal{F}(A)$, and arbitrary $a \in A$, $g(a) = b$.
$g \circ f(a) = g(a) = b, g \circ f = g$
$f \circ g(a) = f(b) = b, f \circ g = g$
Then we know that f is an identity element.

Inverse element:

Consider $f \in \mathcal{F}(A)$, and denote the identity element of $\mathcal{F}(A)$ by e.

Since f is bijective, there exists an inverse map of f in $\mathcal{F}(A)$, and denoted by f^{-1}.

For arbitrary $a \in A$, $f(a) = b, f^{-1}(b) = a$

$f \circ f^{-1}(b) = f(a) = b,\ f \circ f^{-1} = e$

$f^{-1} \circ f(a) = f^{-1}(b) = a,\ f^{-1} \circ f = e$

Then we know that f^{-1} is an inverse element of f.

Then we know that $\mathcal{F}(A)$ is a group with respect to the composite $g \circ f$ for arbitrary $g, f \in \mathcal{F}(A)$.

$\mathcal{F}(A)$ is an abelian group when $g = f$ in the composite $g \circ f$.

Question 3:

Proof:

When $n = 1, p_n = 2 < 4 = 2^{2^n}$.

Assume that $p_n < 2^{2^n}$ sets up for arbitrary positive integer no larger than n, then we have

$$\prod_{i=1}^{n} p_i < \prod_{i=1}^{n} 2^{2^i} = 2^{\sum_{i=1}^{n} 2^i} = 2^{2^{n+1}-1}$$

and

$$p_{n+1} \leq \prod_{i=1}^{n} p_i + 1 < 2^{2^{n+1}-1} + 1 < 2^{2^{n+1}}$$

Form induction principle, we know that $p_n < 2^{2^n}$ for every n.

Question 4:

(i) Proof:

For $x, y \in R$, if $y - x \in Z$, then $x - y = -(y - x) \in Z$.
For arbitrary $x \in R$, $x - x = 0 \in Z$,
For $x, y, z \in R$, if $y - x \in Z. z - y \in Z$, then $z - x = z - y + y - x \in Z$.
Then we know:

$(x, x) \in S'$,

if $(x, y) \in S'$, then $(y, x) \in S'$,

if $(x, y), (y, z) \in S'$, then $(x, z) \in S'$.
This tells us that S' is an equivalence relation on R.
The elements in a equivalence class of S' have the same decimal part.

(ii) Proof:

For $x \in R$, we have $(x, x) \in R_i$ for each $i \in I$, then $(x, x) \in \bigcap_{i \in I} R_i$.
If $(x, y) \in R_i$, (y, x) should also be in R_i, so if $(x, y) \in \bigcap_{i \in I} R_i$, there should also be $(y, x) \in \bigcap_{i \in I} R_i$.

If $(x, y), (y, z) \in R_i$, then (x, z) should also be in R_i, so if $(x, y), (y, z) \in \bigcap_{i \in I} R_i$, there should also be $(x, z) \in \bigcap_{i \in I} R_i$.
The derivations above tell us that $\bigcap_{i \in I} R_i$ is also an equivalence relation on R.

(iii)
$T = \{(x, y) : y = x + 1 \ or \ y = x \ or \ y = x\}$

A equivalence class of T should be like: $\{\cdots, x - 1, x, x + 1, \cdots\}$ with $x \in R$.

Question 5:

Cantor (1845.3.3~1918.1.6)
Russell (1872.5.18~1970.2.2)
Gödel (1906.4.8~1978.1.14)
Zermelo (1871.7.27~1953.5.21)
Fraenkel (1891.2.17~1965.10.15)

The order should be:
Cantor < Zermelo < Russell < Fraenkel < Gödel