TOPOLOGY - HOMEWORK 01

Due: 2020 March 12th, 10:00AM

Question 1. Let A be a nonempty finite set. Let $\mathcal{P}(A) := \{A' : A' \subset A\}$ be the power set of A. Denote by 2^A the set of all functions of the form

$$f : A \longrightarrow \{\pm 1\}.$$

Prove that there is a bijection between $\mathcal{P}(A)$ and 2^A, and compute the cardinality of $\mathcal{P}(A)$.

Question 2. Let A be a nonempty finite set. Denote by $\mathcal{F}(A)$ the set of all bijections

$$f : A \longrightarrow A.$$

Prove that $\mathcal{F}(A)$ is a group with respect to the composite $g \circ f$ for arbitrary $g, f \in \mathcal{F}(A)$. When is $\mathcal{F}(A)$ an abelian group?

Question 3. Let p_n be the n-th prime number in ascending order, e.g., $p_1 = 2, p_2 = 3$ and $p_3 = 5$. Prove (using mathematical induction or otherwise) that

$$p_n < 2^{2^n} \text{ for every } n.$$

Question 4. Let S and S' be the following subsets of $\mathbf{R} \times \mathbf{R}$:

$$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\},$$
$$S' = \{(x, y) : y - x \in \mathbf{Z}\}.$$

(i) Prove that S' is an equivalence relation on $\mathbf{R}$. Describe the equivalence classes of S'.

(ii) Let $\{R_i\}_{i \in I}$ be a set of equivalence relations on $\mathbf{R}$ indexed by a nonempty set I. Prove that

$$\bigcap_{i \in I} R_i$$

is also an equivalence relation on $\mathbf{R}$.

(iii) Describe the equivalence relation T on $\mathbf{R}$ that is the intersection of all equivalence relations on $\mathbf{R}$ that contain S. Describe the equivalence classes of T.

Question 5. Arrange Cantor, Russell, Godel, Zermelo, Fraenkel in a linear order according to their birth dates, i.e., $A_i < A_j$ if A_i was born earlier than A_j.