1 Groups

1.8 Nilpotent groups and solvable groups

1. Consider the conjugation on G. $\forall x \in G$

   $|G{\cdot}x| = |\{axa^{-1}|a \in G\}| = |\{axa^{-1}x^{-1}|a \in G\}| \le |G^{(1)}|$.

   But
   $$G_x = \{a \in G|axa^{-1} = x\} = C_G(x),$$
   so
   $$|C_G(x)| = \frac{|G|}{|G \cdot x|} \ge \frac{|G|}{|G^{(1)}|} = [G : G^{(1)}].$$

2. $N =< (1\ 2\ 3) > \lhd S_3, S_3/N \backsimeq \mathbb{Z}_2$ is nilpotent, so S_3 is not nilpotent.

3. Since G/N and G/K are nilpotent, $\exists n \in \mathbb{N}, s.t.,$
   $$\Gamma_n(G/N) = \Gamma_n(G/N) = \{\bar{e}\}.$$
   So $\Gamma_n(G) \subset N \cap K$, hence $\Gamma_n(G/N \cap K) = \{\bar{e}\}$ and hence $G/N \cap K$ is nilpotent.

4. $\forall \{e\} \ne H \lhd G$, $\because G$ is nilpotent, $\therefore \exists k \in \mathbb{N}, s.t.,$ $\Gamma_k(G) = e$. $\because H \lhd G, \therefore H \subset H_1 = [H, G] \subset [G, G] = \Gamma_1(G), \therefore H_2 = [H, G] \subset \Gamma_2(G)$. Repeat this process, we see that
   $$H_k = [H_{k-1}, G] \subset [\Gamma_{k-1}(G), G] = \Gamma_{k-1}(G) = \{e\}.$$
   $\therefore H \supseteq H_{k-1} \subseteq C(G)$, hence $H \cap C(G) \ne \{e\}$.

5. Assume that G is any group with order $p^m q$, and has $np + 1|q$ Sylow $p-subgroups$.
   Since $p > q > 1, kp + 1 = 1$. That is to say Sylow $p-subgroups$ is normal subgroup N. $|N| = p^m$. It is nilpotent and hence solvable. Since G/N is a cyclic group of order q, it is solvable, hence G is solvable.

6. Assume that G is any group with order pqr and $p > q > r$. H is a Sylow $p-subgroup$ of G. K is a Sylow $q - subgroup$, R is a Sylow $r - subgroup$. Denote the number of Sylow $p - subgroup$ of G by k_p and similarly we have k_q and k_r. By Sylow Theorem, $p|k_p - 1$ and $k_p|qr$. So $k_p = 1$ or $k_p = qr$. If $k_p = 1$, H is a normal subgroup of G. Then by Lemma 1.8.1, G is solvable because H is cyclic and G/H is solvable(Exercises 1.8.5).

   If $k_p = qr$, we claim that $k_q = 1$ or $k_r = 1$. Otherwise, the minimum values of k_q and k_r is respectively p and q. Then there are $(qr)(p - 1)$ elements with order p , at least $p(q - 1)$ elements with order q and at least $q(r - 1)$ elements with order r in G. Hence, $(qr)(p - 1) + p(q - 1) + q(r - 1) + 1 \leq pqr$, i.e., $(p - 1)(q - 1) \leq 0$. Contradiction. So $k_q = 1$ or $k_r = 1$. Then we can get the conclusion by analysis similar to the first case.

7. $(\Rightarrow)$
   $H \lhd G \Rightarrow G/H$ is nilpotent.
   $\Gamma_k(G/H) = \{e\}, \Gamma_{k-1}(G/H) \neq \{e\} \Rightarrow \Gamma_{k-1}(G/H) \subseteq C(G/H)$.
   $(\Leftarrow)$
   Since $C(G) \neq \{e\}, C(G/C(G)) \neq \{e\} \Rightarrow C(G) \subsetneq C_1(G) \Rightarrow \cdots \Rightarrow C_n(G) = G$, (since G is finite.) By proposition $1.8.2, G$ is nilpotent.

8. $G = UT(n, \mathbb{P}) \times D$. $\because UT(n, \mathbb{P})$ and D are nilpotent, $\therefore G$ is nilpotent. But D is not nilpotent.

9. Let k_1, k_2, k_3, k_4 denote the subgroups of S_4 gener-

ated by $(1\ 2\ 3), (1\ 2\ 4), (1\ 3\ 4), (2\ 3\ 4)$ resp. $\forall \varphi \in Aut(S_4)$, since k_i is Sylow $3-subgroup$, $\varphi(k_i)$ is also a Sylow $3-subgroup$ of $\varphi(Aut(S_4)) = S_4$. Hence we have that

$$\begin{array}{rccc} \phi: & Aut(S_4) & \to & S_4 \\ & \varphi & \mapsto & \begin{pmatrix} \cdots & i & \cdots \\ \cdots & j & \cdots \end{pmatrix} \end{array}$$

where $\varphi(k_i) = k_j$.

It is clear to see that ϕ is a group homomorphism.

If $\varphi \in ker(\phi)$, since φ^2 preserves $(1\,2\,3), (1\,2\,4), (1\,3\,4)$ and $(2\ 3\ 4)$, it preserves all Sylow $3-subgroups$, hence it preserves any elements of A_4.

Since

$$\begin{array}{rccc} \psi & S_4/A_4 & \to & S_4/A_4 \\ & \sigma A_4 & \mapsto & \varphi^2(\sigma)A_4 \end{array}$$

is group isomorphism, $S_4/A_4 \simeq \mathbb{Z}_2, \varphi = id$. $\forall \sigma \in S_4, \varphi^2(\sigma)A_4 = \sigma A_4$. $\tau \in A_4$,

$$\begin{array}{rcc} \sigma & = & \varphi^2(\sigma)\tau \\ & = & \varphi^2(\varphi^2(\sigma)\tau)\tau \\ & = & \varphi^4(\sigma)\tau^2 \\ & = & \cdots \\ & = & \varphi^{2k}(\sigma). \end{array}$$

10. Since G is nilpotent, $\exists n, s.t., \Gamma_n(G) = \{e\} \subset H$. Assume that k satisfies $\Gamma_k(G) \subset H, \Gamma_{k-1} \not\subseteq H$. Let $a \in \Gamma_{k-1}(G)/H$, then $\forall h \in H, aha^{-1}h^{-1} \in H$. $\therefore aha^{-1} \in H, \therefore a \in N_G(H)$, hence $H \neq N_G(H)$.

11. $(\Rightarrow)$

    If G is nilpotent,H is a maximum subgroup of G,

then $N_G(H) \neq G$. Hence $H \lhd G$.
$(\Leftarrow)$
Any maximum subgroups of G is normal. $\forall$ Sylow $p-subgroup, P$, if P is maximum,then it is normal. Since G is finite, if P is not normal,then $N_G(P) \neq G$, so there is a maximum subgroup $H, s.t., N_G(P) \subset H \subset G$. If $a \in N_G(H)$, then $aPa^{-1} \subset aHa^{-1} \subset H$. So $\exists h \in H, s.t., aPa^{-1} = hPh^{-1} \Rightarrow haP(ha)^{-1} = P \Rightarrow ha \in N_a(P) \Rightarrow a = h^{-1}(ha) \in H$, and $H \lhd G$ so $N_G(H) = G \neq H$. This is a contradiction. Sofar we have shown that all Sylow subgroups are normal, hence G is nilpotent.

12. We show this by induction on i.
If $i = 0$, then $G^{(0)} = G, \varphi(G) \subseteq G = G^{(0)}$.
Assume that $\varphi(G^{(k)}) = G^{(k)}$, then $\varphi(G^{(k+1)}) = \varphi([G^{(k)}, G^{(k)}]) = [\varphi(G^{(k)}), \varphi(G^{(k)})] \subseteq [G^{(k)}, G^{(k)}] \subseteq G^{(k+1)}$.
$\forall a \in G, I_a(G^{(i)}) = aG^{(i)}a^{-1} \subseteq G^{(i)}, \therefore G^{(i)} \lhd G$.

13. Since G is finite and solvable, $\exists G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_{n+1} = \{e\}, s.t., G_i/G_{i+1}$ is cyclic group with order p by Corollary 1.5.2(2). Since $H \neq G$, consider $G/H. \forall aH, bH \in G/H, a^{-1}b^{-1}abH \subset [G,G]H. \because H$ is a maximal subgroup and $[G,G] \subsetneq G, \therefore [G,G] \subseteq H, \therefore abH = baH, \therefore \exists$ a subgroup containing H satisfies thm 1.8.1 and makes G_{i-1}/G_i a cyclic group with order p. Hence $H = G_1$. Hence $[G : H]$ is a prime.

14. For any $a \in G$ and $b \in N$, $aba^{-1}b^{-1} \in N \cap [G : G] = \{e\}$, then $ab = ba$, thus $N \leq C(G)$.

15. Suppose $G = P_1 \times \cdots \times P_r$ where P_i is Sylow p_i-subgroup for $1 \leq i \leq r$. Since every normal subgroup of P_i is also a normal subgroup of G. We assume that $G = P$ is a p-group, N is a minimal normal subgroup of G. For any $a \in G$ and $c, d \in N$, $a[c,d]a^{-1} = [aca^{-1}, ada^{-1}] \in N^(1)$. Thus $N^(1)$ is a normal subgroup of G contained in N. Therefore $N^(1) = \{e\}$ as N is nilpotent and $N \neq N^(1)$. This means that N is abelian. According to Exercise 1.8.4, $N \cap C(G) \neq \{e\}$, since N is minimal, $N \subset C(G)$, while every subgroup of $C(G)$ is normal. N has no trivial subgroup for N is minimal, thus $|N| = p$.